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SHORT RANGE EFFECTS: A SCALAR FIELD-GAUGE FIELD MODEL SUGGESTED BY THE FIBRE BUNDLE METHOD

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SHORT RANGE EFFECTS:

A SCALAR FIELD-GAUGE FIELD MODEL

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APPROVED BY

DISSERTATION COMMITTEE

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INTRODUCTION

The introduction over a decade ago of fibre bundle methods to classical field theory seems to allow us now to develop models of unified fields from more fundamental principals than ever before. Using a Lagrangian based on the curvature of a Principal Fibre Bundle (a mathematical structure which can combine space-time with gauge-groups), we can get field equations which are like those of Utiyama[1] and others, who began their theoretical developments with symmetry and invariance assumptions. The advantage of the fibre bundle method is that these assumptions already appear in the geometry of Principle Fibre Bundles.

The fibre bundle formulation also provides extras like a natural inclusion of scalar fields.

It is already known[2] that the non-linearity of the Yang-Mills type field equations can lead to short-range effects: These field equations have essentially Coulomb type solutions under spherical symmetry conditions, as when two particles are far apart, but when the particles come close together the spherical symmetry is broken, and the

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[1] Ryoyu Utiyama, Physical Review 101, 1597 (1956)

[2] Hendricus G. Loos, Nuclear Physics 72, 677 (1965)

non-linearity of the field equations asserts itself.

In this paper we show, using Abelian models which sidestep the non-linear aspects of the field equations, that the scalar fields predicted by the fibre-bundle method can also produce short range effects.

After developing the field equations in section 1, where the scalar fields are seen as the space-time dependent components of the group metric, we will compare our field equations for U(1) with the field equations of Ehlers[3] and of Gordon[4], who treated the index of refraction as a function of position and velocity. The similarity between the effects of their index of refraction and our scalar fields suggests that the scalar fields might affect the speed of propogation of the gauge fields, and could thus shorten their range.

In the one-dimensional case, unfortunately, trying to force the scalar field to shorten the range of its corresponding gauge field also forces the scalar field to become infinite at large distances. This can be seen at the end of section II where we look at the Yukawa potential and see what is required to produce the same effect from a Coulomb field with the addition of our scalar field.

In section three we begin to develop a rather

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[3]Jurgen Ehlers, Z. Naturforschg. 22a, 1328 (1967)

[4]W. Gordon, Ann. Phys. 72, 421 (1923)

simplistic model from two static charges and a sphere, which will fail for the one-dimensional gauges and then revive when we move up to a two-dimensional gauge. In the one-dimensional case, we will put one of the charges at the center of the sphere and assign the sphere a different electrostatic permitivity from that of the rest of the universe, as though the particle inside was producing a field which altered the permitivity of the vacuum. (When the electromagnetic permeability is constant, this corresponds to altering the index of refraction.) To produce the Yukawa potential, the permitivity only had to be infinite at infinity. Here, to produce the effects we're after, the permitivity must be infinite everywhere except inside the sphere.

Using a two-dimensional Abelian gauge group in sections IV and V, we find that, for certain particles, the $1/r^2$ forces of the gauge fields can vanish outside of the sphere and reappear inside. So, after failing in the case of U(1), this same simplistic model of two particles and a sphere goes on to demonstrate the desired short range effects of the scalar fields.

I.

FIELD EQUATIONS

In this section we will develop the field equations for interacting gravitational, gauge, and scalar fields. The gravitational field is represented by a space-time metric $g_{ab} = g_{ab}(x)$, the gauge fields by vector potentials $A^{\alpha}_{a}(x)$, and the scalar fields by $g_{\alpha\beta}(x)$ (the components of a metric on the gauge group). Latin indices, which run from 1 to 4, refer to space-time components, and Greek indices, which run from 1 to N, refer to Lie group components, where the dimension of the Lie group is N. The space-time metric $g_{ab}(x)$ has the same structure here as in general relativity. The $A^{\alpha}_{a}(x)$ are analagous to the vector potentials of Electricity and Magnetism except that we have a set of N such fields, as indicated by the superscript. Because these fields can interact with each other, their field equations are complicated by non-linear terms involving the Lie Algebra's structure constants. The terms $A^{\alpha}_{ab}(x) = A^{\alpha}_{[a,b]} + \frac{1}{2} C^{\alpha}_{\beta\gamma} A^{\beta}_{a} A^{\gamma}_{b}$ correspond to the electromagnetic field tensor. (Unless stated otherwise, the summation convention is assumed throughout. Symmetric and anti-symmetric permutation sums are indicated by round and square brackets respectively.) The free field Lagrangian for the gauge fields is

$$L_{gauge} = g_{\alpha\beta} A^{\alpha}_{ab} A^{\beta}_{cd} g^{ac} g^{bd}.$$

This equation is the simplest generalization of $L_{E\&M}$ and is found automatically in our Lagrangian for the interacting fields.

If gravity were present by itself, the gravitational field $g_{ab}(x)$ would satisfy the vacuum Einstein equations $G_{ab} = R_{ab} - \frac{1}{2} g_{ab}R = 0$, which come from the Lagrangian density $\sqrt{-g}\,R$, where $R = g^{ab}R_{ab}$ is the Ricci curvature scalar, by means of the method of variations. This equation also appears in our interacting Lagrangian equation.

The terms $g_{\alpha\beta}(x)$ represent a set of scalar fields which we'll ultimately use to shorten the range of the gauge fields. These $g_{\alpha\beta}(x)$ are space-time dependent, but transform as symmetric two-index tensors with respect to changes of the basis of the Lie Algebra.

The Lagrangian we will be using comes from a generalization of the gravitational Lagrangian density in that it comes from the Ricci curvature scalar of a metric on a space with dimension $4 + N$, where N is the dimension of the gauge group. This space contains the usual space-time manifold and the inner space associated with the internal degrees of freedom of the gauge fields.

If we let $\{e_i\}$ be the basis for the space-time manifold, and let $\{\xi_\alpha\}$ be the Lie Algebra basis, the metric for the bundle will operate on the e_i like the space-time metric, and on the ξ_α like the group metric.

The curvature tensor for the bundle is complicated by structure constants $C^A_{BC}e_A = [e_B, e_C]$, where the indices run from 1 to 4+N and $\{e_A\}$ is the basis of the bundle. The curvature for the bundle is then[5]

$$R_{\text{bundle}} = g^{AB}R^C{}_{ACB}$$
$$= g^{AB}(\Gamma^C_{AB,C} - \Gamma^C_{AC,B} + \Gamma^C_{DC}\Gamma^D_{AB} - \Gamma^C_{DB}\Gamma^D_{AC} - \Gamma^C_{AE}C^E_{CB}),$$

and the connection coefficients are given by

$$\Gamma^A_{BC} = \tfrac{1}{2}\, g^{AD}(g_{DB,C} + g_{DC,B} - g_{BC,D} + g_{CE}C^E_{DB} + g_{BE}C^E_{DC}) - \tfrac{1}{2}\, C^A_{BC}.$$

As explained in appendix B, the Lagrangian density[6] for the bundle is

<u>I-1</u>:
$$\mathcal{L} = \sqrt{-g_{ST}}\sqrt{g_G}\,\{R_{ST} - \tfrac{1}{4}g^{\gamma\beta}C^{\alpha}_{\delta\gamma}C^{\delta}_{\alpha\beta} - g_{\alpha\beta}A^{\alpha}_{ab}A^{\beta}_{cd}g^{ac}g^{bd} - \tfrac{1}{2}g^{\alpha[\gamma}g^{\beta]\delta}g_{\alpha\beta,a}g_{\gamma\delta,b}g^{ab}\} - \frac{\partial}{\partial x^b}\{\sqrt{-g_{ST}}\sqrt{g_G}\,(g^{\alpha\beta}g_{\alpha\beta,a})g^{ab}\}$$

but it will be easier to work with if we apply a conformal transformation to change the $\sqrt{-g_{ST}}\sqrt{g_G}\,R_{ST}$ to $\sqrt{-\underline{g}_{ST}}\,\underline{R}_{ST}$.[7] Transforming the space-time metric by $\underline{g}_{ij} = e^{2\sigma}g_{ij}$ gives us

<u>I-1'</u>:
$$\mathcal{L} = \sqrt{g_G}\,e^{-4\sigma}\sqrt{-\underline{g}_{ST}}\,\{e^{2\sigma}[\underline{R}_{ST} + 6\,\underline{\Delta}_2\sigma - 6\,\underline{\Delta}_1\sigma] - \tfrac{1}{4}g^{\gamma\beta}C^{\alpha}_{\delta\gamma}C^{\delta}_{\alpha\beta} - e^{4\sigma}g_{\alpha\beta}A^{\alpha}_{ab}A^{\beta}_{cd}\underline{g}^{ac}\underline{g}^{bd} - \tfrac{1}{2}e^{2\sigma}g^{\alpha[\beta}g^{\gamma]\delta}g_{\alpha\gamma,a}g_{\beta\delta,b}\underline{g}^{ab}\} - \frac{\partial}{\partial x^b}\{e^{-2\sigma}\sqrt{-\underline{g}_{ST}}\sqrt{g_G}\,g^{\alpha\beta}g_{\alpha\beta,a}\underline{g}^{ab}\}$$

where $\underline{\Delta}_2\sigma = \underline{g}^{ab}(\sigma_{,a})_{;b}$; $\underline{\Delta}_1\sigma = \underline{g}^{ab}\sigma_{,a}\sigma_{,b}$; and $\underline{R}_{ST}$ is the

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[5] Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, <u>Gravitation</u> (San Francisco: W. H. Freeman, 1970), p. 277.

[6] see also Y. M. Cho, Journal of Mathematical Physics <u>16</u>, 2029 (1975).

[7] L. P, Eisenhart, <u>Riemannian Geometry</u>, (London, Princeton University Press, 1926), pp. 89-90.

space-time curvature scalar in terms of g_{ij}. Also, the covarient derivatives are now taken with respect to the new metric.

By choosing $e^{2\sigma} = \sqrt{g_G}$ we get

<u>I-1"</u>: $$\mathcal{L} = \sqrt{-g_{ST}}\,\Big[\underline{R}_{ST} + 6\Delta_2\sigma - 6\Delta_1\sigma - \tfrac{1}{4}\sqrt{g_G}\, g^{\gamma\beta} C^{\alpha}_{\delta\gamma} C^{\delta}_{\alpha\beta} - \sqrt{g_G}\, g_{\alpha\beta} A^{\alpha}_{ab} A^{\beta}_{cd}\, \underline{g}^{ac}\underline{g}^{bd} - \tfrac{1}{2} g^{\alpha[\beta} g^{\gamma]\delta} g_{\alpha\gamma,a}\, g_{\beta\delta,b}\, \underline{g}^{ab} \Big\} - \frac{\partial}{\partial x^b}\Big\{ \sqrt{-g_{ST}}\,\big(g^{\alpha\beta} g_{\alpha\beta,a}\big)\, g^{ab} \Big\}$$

The next transformation is performed on the group metric: $\underline{g}_{\alpha\beta} = (g)^{+\frac{1}{2}} g_{\alpha\beta}$, $\underline{g}^{\alpha\beta} = (g)^{-\frac{1}{2}} g^{\alpha\beta}$, and $\underline{g} = g^{(\frac{N+2}{2})}$. Here, g denotes the determinant of the group metric. The scalar field terms of the intermediate Lagrangian, when combined with $-6\underline{\Delta}_1\sigma$ gives us:

$$\begin{aligned} &-6\big[(\tfrac{1}{4})^2 \underline{g}^{ab}(\ln g)_{,a}(\ln g)_{,b}\big] - \tfrac{1}{4}\underline{g}^{\alpha\gamma} g^{\beta\delta} g_{\alpha\beta,a}\, g_{\gamma\delta,b}\, \underline{g}^{ab} + \tfrac{1}{4} g^{\alpha\beta} g_{\alpha\beta,a}\, g^{\gamma\delta} g_{\gamma\delta,b}\, \underline{g}^{ab} \\ &= -\tfrac{1}{4}\big\{ g^{\alpha\gamma} g^{\beta\delta} g_{\alpha\beta,a}\, g_{\gamma\delta,b} + \tfrac{1}{2} g^{\alpha\beta} g_{\alpha\beta,a}\, g^{\gamma\delta} g_{\gamma\delta,b} \big\}\, \underline{g}^{ab} \\ &= -\tfrac{1}{4}\big\{ (-\sqrt{g}\, \underline{g}^{\gamma\delta})_{,a} \big(\tfrac{\underline{g}_{\gamma\delta}}{\sqrt{g}}\big)_{,b} + \tfrac{1}{2}\tfrac{g_{,a}}{g}\tfrac{g_{,b}}{g} \big\}\, \underline{g}^{ab} \\ &= \tfrac{1}{4}\big\{ \tfrac{1}{N+2}\tfrac{\underline{g}_{,a}}{\underline{g}}\tfrac{\underline{g}_{,b}}{\underline{g}} + \underline{g}^{\gamma\delta}{}_{,a}\, \underline{g}_{\gamma\delta,b} \big\}\, \underline{g}^{ab} \end{aligned}$$

Note that $\sqrt{-g_{ST}}\,6\underline{\Delta}_2\sigma = \sqrt{-g_{ST}}\,6\underline{g}^{ab}(\tfrac{1}{4}(\ln g)_{,a})_{;b} = \tfrac{3}{2}(\sqrt{-g_{ST}}\,\underline{g}^{ab}\tfrac{g_{,a}}{g})_{,b}$

The Lagrangian from which we will take our field equations is:

<u>I-2</u>: $$\mathcal{L} = \sqrt{-g_{ST}}\,\Big\{ \underline{R}_{ST} - \tfrac{1}{4}\underline{g}^{\gamma\beta} C^{\alpha}_{\delta\gamma} C^{\delta}_{\alpha\beta} - \underline{g}_{\alpha\beta} A^{\alpha}_{ab} A^{\beta}_{cd}\, \underline{g}^{ac}\underline{g}^{bd} + \tfrac{1}{4}\Big(\tfrac{1}{N+2}\tfrac{\underline{g}_{,a}}{\underline{g}}\tfrac{\underline{g}_{,b}}{\underline{g}} + \underline{g}^{\alpha\beta}{}_{,a}\, \underline{g}_{\alpha\beta,b}\Big)\, \underline{g}^{ab} \Big\} + \tfrac{1}{2}\frac{\partial}{\partial x^b}\Big\{ \sqrt{-g_{ST}}\,\big(g^{\alpha\beta} g_{\alpha\beta,a}\big)\, \underline{g}^{ab} \Big\}$$

The fields we are looking for should extremize:

$$J = \int_V \mathcal{L}\, dx^0 \wedge \ldots \wedge dx^4.$$

We can find these fields by looking at the variation of J in V while requiring that the variation be zero on the boundary of V. Let $\mathcal{L} = \mathcal{L}' + \mathcal{E}^{\gamma}{}_{,\gamma}$. Then:

<u>I-3</u>: $\delta J = \delta \int_V \mathcal{L}\, d^4x + \delta \int_V \mathcal{C}^{\mu}{}_{,\mu}\, d^4x.$

The generalized form of Stokes' theorem says that for a (p+1)-dimensional volume V with a closed p-dimensional boundary ∂V and with a p-form α defined throughout V, the integral of the (p+1)-form $d\alpha$ over the interior of V is equal to the integral of the p-form α over the boundary ∂V.

Thus the integral of $\nabla\cdot\mathcal{C}\ dx^0\wedge\ldots\wedge dx^4$ over V is equal to the integral of $\mathcal{C}\cdot d\mathcal{S}$ over ∂V, where $d\mathcal{S}$ is the (4-1)-dimensional surface element on ∂V. (For instance, for t = constant, $d\mathcal{S} = \pm\, dx^1\, dx^2\, dx^3$ with the sign choosen so that $d\mathcal{S}$ is oriented outwards.)

But, since $\delta \int_V \mathcal{C}^{\beta}{}_{,\beta}\, d^4x = \delta \int_{\partial V} \mathcal{C}^{\beta} (d\mathcal{S})_{\beta}$, and the variation on ∂V is zero, the last term in I-3 will be zero automatically. It thus contributes no information about which fields will give an extremum for J.

The last term of our transformed Lagrangian, as a total divergence, will thus not affect the field equations and will be ignored from now on. Also, the metrics we will be using, or solving for, will be the barred ones in this last Lagrangian; so, as we solve for the field equations, the bars will be dropped.

We will begin with the field equations for the space-time metric. These we will get from:

$$\delta(\sqrt{-g_{ST}}\, R_{ST}) - \tfrac{1}{4} g^{\gamma\beta} C^{\alpha}_{\delta\gamma} C^{\delta}_{\alpha\beta}\, \delta(\sqrt{-g_{ST}}) - (g_{\alpha\beta} A^{\alpha}_{ab} A^{\beta}_{cd})\, \delta(\sqrt{-g_{ST}}\, g^{ac} g^{bd})$$
$$+ \tfrac{1}{4}\left(\frac{1}{N+2} \frac{g_{,a}}{g} \frac{g_{,b}}{g} + g^{\gamma\delta}{}_{,a}\, g_{\gamma\delta,b}\right) \delta(\sqrt{-g_{ST}}\, g^{ab})$$

Evaluation of the individual terms gives:

$$\delta(\sqrt{-g_{sr}}\,R_{sr}) = \sqrt{-g_{sr}}\,R_{ab}(\delta^a_c\delta^b_d - \tfrac{1}{2}g^{ab}g_{cd})\,\delta g^{cd}$$
$$\delta\sqrt{-g_{sr}} = -\tfrac{1}{2}\sqrt{-g_{sr}}\;g_{cd}\;\delta g^{cd}$$
$$\delta(\sqrt{-g_{sr}}\,g^{cd}) = \sqrt{-g_{sr}}\,(\delta^c_a\delta^d_b - \tfrac{1}{2}g^{cd}g_{ab})\,\delta g^{ab}$$
$$\delta(\sqrt{-g_{sr}}\;g^{ec}g^{fd}) = \sqrt{-g_{sr}}\,(g^{ec}\delta^f_a\delta^d_b + \delta^e_a\delta^c_b\,g^{fd} - \tfrac{1}{2}g^{ec}g^{fd}g_{ab})\,\delta g^{ab}$$

Plugging these in gives our field equations.

$$\frac{\delta\mathcal{L}}{\delta g^{ab}} = \sqrt{-g_{sr}}\,\Big\{\underset{sr}{G}_{ab} + \tfrac{1}{4}g^{\gamma\beta}C^{\kappa}_{\lambda\gamma}C^{\lambda}_{\kappa\beta}\,g_{ab}$$
$$- g_{\alpha\beta}[2A^{\alpha}_{ea}A^{\beta}_{cb}\,g^{ec} - \tfrac{1}{2}A^{\alpha}_{ef}A^{\beta}_{cd}\,g^{ec}g^{fd}g_{ab}]$$
$$+ \tfrac{1}{4}\Big[\frac{1}{N+2}\Big(\frac{g_{,a}}{g}\frac{g_{,b}}{g} - \frac{1}{2}\frac{g_{,c}}{g}\frac{g_{,d}}{g}\,g^{cd}g_{ab}\Big) + (g^{\gamma\delta}_{\;\;,a}\,g_{\gamma\delta,b} - \tfrac{1}{2}g^{\gamma\delta}_{\;\;,c}\,g_{\gamma\delta,d}\,g^{cd}g_{ab})\Big]\Big\}$$

These can be grouped into the Einstein tensor, a cosmological term, and the stress energy tensors for the gauge field and the scalar field respectively.

$$\frac{\delta\mathcal{L}}{\delta g^{ab}} = \sqrt{-g_{sr}}\,\Big\{\underset{sr}{G}_{ab} + C_{ab} - \underset{gauge}{T_{ab}} - \underset{scalar}{T_{ab}}\Big\}$$

Next we will consider the gauge fields.

$$-\sqrt{-g_{sr}}\;g_{\alpha\beta}\,g^{ac}g^{bd}\,\delta(A^{\alpha}_{ab}A^{\beta}_{cd}) = -2\sqrt{-g_{sr}}\;g_{\alpha\beta}\,g^{ac}g^{bd}A^{\alpha}_{ab}\,\delta(A^{\beta}_{cd})$$

We want the variation here to be in terms of δA^{α}_{a}.

$$\delta(A^{\beta}_{[c,d]} + \tfrac{1}{2}C^{\beta}_{\kappa\lambda}A^{\kappa}_{c}A^{\lambda}_{d}) = (\delta A^{\beta}_{[c})_{,d]} + \tfrac{2}{2}C^{\beta}_{\kappa\lambda}\,A^{\kappa}_{[c}\delta A^{\lambda}_{d]}$$
$$(\delta A^{\beta}_{[c})_{,d]} = (\delta A^{\beta}_{[c})_{;d]}$$

This will give us:

$$-2\{(\sqrt{-g_{sr}}\;g_{\alpha\beta}g^{ac}g^{bd}A^{\alpha}_{ab}\,\delta A^{\beta}_{[c})_{,d]} - (\sqrt{-g_{sr}}\;g_{\alpha\beta}A^{\alpha}_{ab}\,g^{ac}g^{bd})_{,[d}\,\delta A^{\beta}_{c]}$$
$$+ \sqrt{-g_{sr}}\;g_{\alpha\beta}\,g^{ac}g^{bd}A^{\alpha}_{ab}\,C^{\beta}_{\kappa\lambda}\,A^{\kappa}_{[c}\,\delta A^{\lambda}_{d]}\;\}$$

If M_{ab} is an antisymmetric tensor, then since any tensor $N_{cd} = N_{(cd)}+N_{[cd]}$, we can get $M_{ab}N_{cd}g^{ac}g^{bd} = M_{ab}N_{[cd]}g^{ac}g^{bd}$. Because A^{α}_{ab} is antisymmetric, we won't need to use the antisymmetry brackets on c and d.

The first term is a total divergence and, by reasoning similar to that given earlier, may be deleted in the following.

This leaves:

$$\frac{\delta\mathcal{L}}{\delta A^{\beta}_{c}} = 2\sqrt{-g_{ST}}\,\{(g_{\alpha\beta}A^{\alpha}_{ab}\,g^{ac}g^{bd})_{;d} + g_{\kappa\tau}\,g^{ac}g^{bd}\,C^{\tau}_{\kappa\beta}\,A^{\kappa}_{d}\,A^{\alpha}_{ab}\}$$

If sources were present, these would appear in the field equations as a group-space-time source density vector J^{c}_{β}.

$$\frac{\delta\mathcal{L}_{field}}{\delta A^{\beta}_{c}} = \sqrt{-g_{ST}}\;J^{c}_{\beta}$$

The scalar field equations will come from:

$$-\tfrac{1}{4}\sqrt{-g_{ST}}\,C^{\kappa}_{\lambda\alpha}\,C^{\lambda}_{\kappa\beta}\,\delta(g^{\alpha\beta}) - \sqrt{-g_{ST}}\,A^{\alpha}_{ab}\,A^{\beta}_{cd}\,g^{ac}g^{bd}\,\delta(g_{\alpha\beta})$$
$$+\sqrt{-g_{ST}}\,\tfrac{1}{4}\,g^{ab}\,\delta\left(\tfrac{1}{N+2}\,\tfrac{g_{,a}}{g}\,\tfrac{g_{,b}}{g} + g^{\alpha\beta}{}_{,a}\,g_{\alpha\beta,b}\right)$$

So that we can get everything in terms of $\delta(g_{\alpha\beta})$:

a) $$\delta(g^{\alpha\beta}) = -g^{\alpha\gamma}g^{\beta\delta}\,\delta(g_{\gamma\delta})$$

b) $$\sqrt{-g_{ST}}\,\tfrac{1}{2(N+2)}\,g^{ab}(\ln g)_{,a}(\delta\ln g)_{,b} = \sqrt{-g_{ST}}\,\tfrac{1}{2(N+2)}(\ln g)_{,a}\left[\tfrac{1}{g}\,\delta g\right]_{,b}g^{ab}$$
$$= \tfrac{\sqrt{-g_{ST}}}{2(N+2)}\,g^{ab}(\ln g)_{,a}\,g^{\alpha\beta}{}_{,b}\,\delta g_{\alpha\beta} + \left\{\tfrac{\sqrt{-g_{ST}}}{2(N+2)}\,g^{ab}(\ln g)_{,a}\,g^{\alpha\beta}\,\delta g_{\alpha\beta}\right\}_{,b}$$
$$-\left\{\tfrac{\sqrt{-g_{ST}}}{2(N+2)}\,g^{ab}(\ln g)_{,a}\,g^{\alpha\beta}\right\}_{,b}\delta g_{\alpha\beta}$$

c) $$\tfrac{\sqrt{-g_{ST}}}{4}\,g^{ab}\left[(\delta g^{\alpha\beta})_{,a}\,g_{\alpha\beta,b} + g^{\alpha\beta}{}_{,a}(\delta g_{\alpha\beta})_{,b}\right]$$
$$= -\left\{\tfrac{\sqrt{-g_{ST}}}{4}\,g^{ab}g_{\kappa\lambda,a}\,g^{\kappa\alpha}g^{\lambda\beta}\,\delta g_{\alpha\beta}\right\}_{,b} - \left\{\tfrac{\sqrt{-g_{ST}}}{4}\,g^{ab}g_{\kappa\lambda,a}(g^{\kappa\alpha}g^{\lambda\beta})_{,b}\right\}\delta g_{\alpha\beta}$$
$$+\left\{\tfrac{\sqrt{-g_{ST}}}{4}\,g^{ab}g_{\kappa\lambda,a}\,g^{\kappa\alpha}g^{\lambda\beta}\right\}_{,b}\delta g_{\alpha\beta} + \left\{\tfrac{\sqrt{-g_{ST}}}{4}\,g^{ab}g^{\alpha\beta}{}_{,a}\,\delta g_{\alpha\beta}\right\}_{,b}$$
$$-\left\{\tfrac{\sqrt{-g_{ST}}}{4}\,g^{ab}g^{\alpha\beta}{}_{,a}\right\}_{,b}\delta g_{\alpha\beta}$$

Again, the total divergence terms won't contribute to the field equations, so we have:

$$\frac{\delta\mathcal{L}}{\delta g_{\alpha\beta}} = \sqrt{-g_{ST}}\,\{\left[g^{ab}\left(\tfrac{1}{4}g_{\kappa\lambda,a}\,g^{\kappa\alpha}g^{\lambda\beta} - \tfrac{1}{2(N+2)}(\ln g)_{,a}\,g^{\alpha\beta} - \tfrac{1}{4}g^{\alpha\beta}{}_{;a}\right)\right]_{;b}$$
$$-\left[\tfrac{1}{4}C^{\kappa}_{\lambda\mu}C^{\lambda}_{\kappa\nu}\,g^{\mu\alpha}g^{\nu\beta}\right] - \left[A^{\alpha}_{ab}A^{\beta}_{cd}\,g^{ac}g^{bd}\right]$$
$$-\left[\tfrac{1}{4}g^{ab}g_{\kappa\lambda,a}(g^{\kappa\alpha}g^{\lambda\beta})_{,b}\right] + \left[\tfrac{1}{2(N+2)}\,g^{ab}(\ln g)_{,a}\,g^{\alpha\beta}{}_{,b}\right]\}$$

There was considerable change in the form of the scalar field terms of the Lagrangian under the conformal transformations, while the form of the gauge fields remains essentially the same. For this reason we will use the gauge field equations in this paper to investigate the scalar field-gauge field interactions.

Also, although conformal transformations have been used in scalar-tensor theories[8], this appears to be its first use in connection with fibre bundle methods to avoid some scalar-tensor problems[9] caused by the scalar term $\sqrt{g}$ in the original Lagrangian density I-1.

[8]See, for instance, J. O'Hanlon and B. O. J. Tupper, Nuovo Cimento 14B, 190 (1973) and 17B, 1 (1973)

[9]See P. G. Bergmann, Int. Journ. Theor. Phys. 1, 25 (1968)

II.

U(1) AND THE OPTICAL METRIC

Beginning with this section, we will concentrate on the gauge fields and how they may be influenced by the scalar fields.

In order to compare our field equations with those of Ehlers[3] and of Gordon[4], we will choose the group U(1), which is one-dimensional and whose metric is just a scalar g. Here $\sigma^c = \frac{1}{2}J^c$ from section one.

$$\sqrt{-g_{ST}}\,(g\, A_{ab}\, g^{ac} g^{bd})_{;d} = \sqrt{-g_{ST}}\,\sigma^c$$

Now, let's raise the indices on the A and then make it a density by bringing the $\sqrt{-g_{ST}}$ inside the parenthesis.

$$(g\,\mathfrak{A}^{ab})_{,b} = \sqrt{-g_{ST}}\,\sigma^a \equiv \mathfrak{S}^a$$

In Gordon's paper we find the standard electromagnetic field equations for fields in matter:

a) $\quad F_{ij,k} + F_{jk,i} + F_{ki,j} = 0$

b) $\quad \frac{1}{\sqrt{-g_{ST}}}\frac{\partial}{\partial x^k}\left(\sqrt{-g_{ST}}\, H^{ik}\right) = s^i$

c) $\quad H_{ik}\, u^k = \epsilon\, F_{ik}\, u^k$

d) $\quad u_i F_{jk} + u_j F_{ki} + u_k F_{ij} = \mu\,(u_i H_{jk} + u_j H_{ki} + u_k H_{ij})$

where the F_{ij} correspond to the E and B fields and the H_{ij} correspond to the D and H fields, and the u^i is the four-velocity: $u^i = dx^i/\sqrt{-ds^2}$.

The second equations we can rewrite with densities:

$$\mathfrak{H}^{ij} \equiv \sqrt{-g_{ST}}\, H^{ij}, \qquad \mathfrak{S}^i \equiv \sqrt{-g_{ST}}\, s^i,$$

so that we get:

b') $$\frac{\partial}{\partial x^k}\left(\mathfrak{H}^{ik}\right) = \mathfrak{s}^i$$

Multiplying the fourth equation by u^i, taking into consideration the third equation, and noting that $u^i u_i = -1$, we get:

$$-F_{jk} + u_j F_{ki} u^i - u_k F_{ji} u^i = \mu\{-H_{jk} + \epsilon(u_j F_{ki} u^i - u_k F_{ji} u^i)\}$$

or, by rearrainging:

e) $$\mu H_{ij} = F_{ij} + (\epsilon\mu - 1)(u_i F_{jk} u^k - u_j F_{ik} u^k)$$

Let's redefine our metric in terms of ϵ, μ, and the four-velocity u^i as follows:

$$\gamma^{ik} \equiv g^{ik} - (\epsilon\mu - 1)\, u^i u^k$$

$$\gamma_{ik} \equiv g_{ik} + \left(1 - \frac{1}{\epsilon\mu}\right) u_i u_k$$

given one, we can get the other from the requirement $\gamma^{ij}\gamma_{jk} = \delta^i_k$. The index of refraction n for the medium is equal to $\sqrt{\epsilon\mu}$, so that we can substitute n^2 into this new metric, called the "optical" metric. Equation (e) can be rewritten so that the indices of H are raised and both sides are multiplied by $\sqrt{-g_{ST}}$ in preparation for use in the field equation (b).

e') $$\mu \mathfrak{H}^{ij} = \sqrt{-g_{ST}}\,\{g^{il} g^{jk} F_{lk} + (\epsilon\mu - 1)(u^i g^{jl} F_{lk} u^k - u^j g^{il} F_{lk} u^k)\}$$

Because of the antisymmetry of F, we can add the zero term $(\epsilon\mu - 1)^2 u^i u^j u^k u^l F_{kl}$ to (e') without changing the value of the right hand side while arriving at $\sqrt{-g_{ST}}\,\gamma^{ik}\gamma^{jl}F_{kl}$.

If we consider the special case $u^1 = u^2 = u^3 = 0$, we get:

$$-\gamma = \det(\gamma_{ij}) = \det\left(g_{ij} + \left(1 - \frac{1}{\epsilon\mu}\right) u_o u_o\right).$$

That is,

$$\det \begin{bmatrix} g_{00} + (1 - \frac{1}{\varepsilon\mu}) u_0 u_0 & \cdots & g_{03} \\ \vdots & & \vdots \\ g_{30} & \cdots & g_{33} \end{bmatrix}$$

Evaluating the determinant of the matrix gives:

$$-\gamma = -g_{sr} - (1 - \tfrac{1}{\varepsilon\mu}) g_{sr} g^{00} u_0 u_0$$
$$= -g_{sr} + (1 - \tfrac{1}{\varepsilon\mu}) g_{sr}$$

or

$$\gamma = \frac{g_{sr}}{\varepsilon\mu}$$

Putting this together with (e') gives us:

$$\mathfrak{H}^{ij} = \sqrt{\frac{\varepsilon}{\mu}} \sqrt{-\gamma} \;\; \gamma^{ik} \gamma^{jl} F_{kl}$$

Returning now to (b'):

$$\frac{\partial}{\partial x^j} (\mathfrak{H}^{ij}) = \frac{\partial}{\partial x^j} (\sqrt{\varepsilon\mu} \sqrt{-\gamma} \; H^{ij}) = \frac{\partial}{\partial x^j} (\sqrt{\tfrac{\varepsilon}{\mu}} \sqrt{-\gamma} \, \gamma^{ik} \gamma^{jl} F_{kl})$$
$$= \sqrt{\varepsilon\mu} \sqrt{-\gamma} \;\; s^i .$$

Following Gordon's convention, indices raised by the optical metric will be indicated in the next equation by parenthesis around the indices. Also, this raised F will be combined with $\sqrt{\gamma}$ to give a tensor density.

$$\frac{\partial}{\partial x^j} \left(\sqrt{\frac{\varepsilon}{\mu}} \;\; \mathfrak{F}^{(i)(j)} \right) = \sqrt{\varepsilon\mu} \sqrt{-\gamma} \;\; s^i$$

If we identify the $\mathfrak{A}^{ij}$ in our field equation with this $\mathfrak{F}^{(i)(j)}$, then it would seem we should identify our scalar field g, the group metric, with $\sqrt{\frac{\varepsilon}{\mu}}$. This would make g, in essence, n/μ. For those cases where μ is not a function of position, g would be a multiple of n, and where ε is not a function of position, g would be a multiple of 1/n. Under this identification g is related to the relative speed of the propogation of the gauge field. Although this technique won't necessarily give us a weak or strong force model, perhaps by progressively slowing down the propogation of the gauge fields, the influence of these fields would be shortened in a position-dependent index-of-refraction model.

Let's go to an even simpler model by looking at the static case (without the optical metric) in a flat space. Here the source free electrostatic field equation for the electric potential is:

$$\nabla \cdot (\varepsilon \nabla \varphi) = 0 .$$

Our field equations in the static case and flat space develops as follows.

$$[A^{ab}] = [g^{ac} g^{bd} A_{cd}] = [g^{ab}][A_{ab}][g^{ab}]$$

where [ ] indicates the matrix of the quantity inside.

Thus,

$$[A_{ab}] = \frac{1}{2} \left[\begin{array}{c|ccc} 0 & & \nabla\varphi & \\ \hline -\nabla\varphi & 0 & (\partial_2 A_1 - \partial_1 A_2) & (\partial_3 A_1 - \partial_1 A_3) \\ & B_3 & 0 & (\partial_3 A_2 - \partial_2 A_3) \\ & -B_2 & B_1 & 0 \end{array} \right]$$

using the definition $B_i = \varepsilon_{ijk} \partial_j A_k$ (i, j, k run from $1 \to 3$ and ε_{ijk} is the Levi-Civeta tensor), and $A_0 = \varphi$.

Since the forces due to $\underline{B}$ in the electrostatic case are zero, we'll drop the B's from A_{ab}. We get then:

$$[A^{ab}] = \frac{1}{2} \left[\begin{array}{c|c} 0 & -\nabla\varphi \\ \hline \nabla\varphi & 0 \end{array} \right]$$

and $(gA^{ab})_{;b}$ becomes:

$$-\frac{1}{2} \nabla \cdot (g \nabla \varphi) = 0 .$$

Here we would identify g with a multiple of ϵ. With μ constant, ϵ becomes a multiple of n^2, so that g would again be identified with a function of the index of refraction.

This section has shown us that in the one dimensional case our scalar field appears analagous to a function of the index of refraction, and in perhaps the simplest case, the scalar field appears analagous to the permitivity; so the question of whether or not g can shorten the range of the gauge field will become, for the next two sections, the simpler question of whether or not the permitivity can shorten the range of the electrostatic potential.

First, let's look at a classic example of an electrostatic potential with shortened range: the Yukawa type $\varphi = a\frac{e^{-\alpha r}}{r}$. Although this is usually derived from applying static and spherical symmetry conditions to $(\Box - m^2)\varphi = 0$ (plus requiring that φ vanish at infinity), we can at least plug this potential into our field equations and see what the corresponding ϵ would need to be.

The equation

$$\nabla \cdot (\epsilon \nabla \varphi) = \epsilon \nabla^2 \varphi + (\nabla \epsilon) \cdot (\nabla \varphi) = 0$$

becomes

$$\epsilon \frac{1}{r} \frac{\partial^2}{\partial r^2} r\varphi + \left(\frac{\partial}{\partial r}\epsilon\right)\left(\frac{\partial}{\partial r}\varphi\right) = 0$$

From this we get:

$$\frac{d\epsilon}{dr}\left(-\frac{\alpha}{r} - \frac{1}{r^2}\right) + \frac{\alpha^2}{r}\epsilon = 0$$

and then:

$$\varepsilon = \varepsilon_0 \frac{e^{\alpha r}}{\alpha r + 1}$$

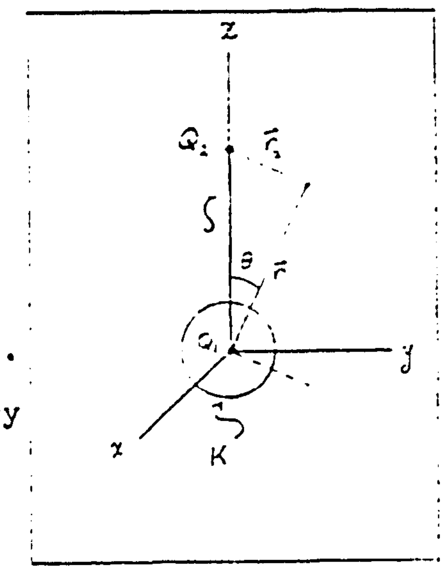
Probably the best reason for going on to other models is that this ε must become infinite for large r. This is like assuming that the default value of ε is infinite and that our scalar field scources must then pull ε down somehow to finite values. In the next section, this type of problem is even more dramatic.

III.

1-DIMENSIONAL MODEL

A relatively simple model from electrostatics is two charges, one of which is at the center of a sphere with a permitivity different from the rest of the universe. (The effect of a second sphere, centered at the other charge, will be discussed later.)

The sphere K (kügel) with radius k and permitivity ϵ_1 is centered at the origin. A charge Q_1 is at the origin, and on the positive z-axis, a distance ζ from the origin, is the second charge Q_2. The distance from Q_1 to an arbitrary point p will be denoted r, and the distance from Q_2 to the same point p will be denoted r_2. The angle θ is that between the positive z-axis and the line from the origin to p. The default value of the permitivity is ϵ_0.



Although the fields have been worked out in several classic texts[10], the methods will be important for our next model.

[10] See, for instance, Julius Adams Stratton, Electromagnetic Theory, (New York, McGraw-Hill, 1941), p204.

In source free regions where the permitivity is constant, the field equation is:

III-1 $$-\varepsilon \nabla^2 \varphi = 0$$

In spherical polar coordinates, the solutions may be written in terms of Legendre's polynomials. Making the usual seperation of radial and angular parts, we will look at the whole equation for $\nabla^2\varphi$, then at the radial part, and finally at the angular solutions. First: $\varphi = R(r)Y(\vartheta,\varphi)$.

III-2 $$\nabla^2\varphi = \frac{1}{r}\frac{\partial^2}{\partial r^2} r(RY) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left[\sin\vartheta \frac{\partial}{\partial\vartheta}(RY)\right] + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}(RY) = 0$$

From this we get:

III-3 $$\frac{1}{R}\frac{\partial}{\partial r}\left[r^2\frac{\partial R}{\partial r}\right] = -\frac{1}{Y}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}(Y)\right) - \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} Y\right] = \kappa$$

The radial equation:

III-4 $$\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} R\right) - \kappa R = 0$$

will have solutions which are regular at $\vartheta = 0$, $\vartheta = \pi$, where Y will have its singularities, if $\kappa = n(n+1)$ with n a real integer. We can write this solution as:

III-5 $$R(r) = a_n r^n + b_n/r^{n+1}.$$

Next, letting $Y(\vartheta,\varphi) = X(\vartheta)W(\varphi)$, and using $x = \cos\vartheta$, we get:

III-6 $$\frac{1}{W}\frac{\partial^2}{\partial\varphi^2} W = -\lambda$$

and

III-7 $$\left(\kappa + \frac{1}{X}\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial}{\partial x} X\right]\right)(1-x^2) = \lambda$$

By requiring that $W(\varphi+2\pi) = W(\varphi)$, and since $W = (\text{constant})\ e^{\pm i\sqrt{\lambda}\varphi}$ is the solution to equation 6, we find that $\sqrt{\lambda}$ must be a real integer m. Because of the axial symmetry of our model, we have m = 0. The equation for the

ϑ function becomes,

III-8 $\quad \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial}{\partial x}X\right] + n(n-1)X = 0.$

The solutions of this equation are Legendre functions of the first and second kind. However, only P_n, the Legendre functions of the first kind, are finite at x = ±1.

The first part of the solutions of $\nabla^2\varphi$ may be written, for this model:

III-9 $\quad \Sigma_{n=0}^{\infty}\left(a_n r^n + \frac{b_n}{r^{n+1}}\right) P_n(\cos\theta)$

We will want φ to be finite for $r\to\infty$ and regular around and through r = 0. This implies that outside of K, the coeffieceients of the r^n will be zero, and that inside of K, the coefficients of the r^{-n-1} will be zero.

Thus far, we haven't considered the effect of the two sources. Let's begin by considering a single point source and spherically symmetric permitivity. We'll go out a distance r and integrate $-\varepsilon\nabla^2\varphi=\rho$, the charge density, over the volume within r. This gives us the charge Q at the center, and by Stoke's theorem we can transform the integral of $-\varepsilon\nabla^2\varphi$ over a sphere to an integral over a spherical surface of radius r.

III-10 $\quad Q = \int_S(-\varepsilon\nabla^2\varphi)\,dv = \int_{\partial S}(-\varepsilon\nabla\varphi)\cdot\underline{d\sigma} = -\left(\varepsilon\frac{\partial}{\partial r}\varphi\right)4\pi r^2$

This equation gives us the familiar potential for a point charge:

III-11 $$\varphi = \frac{Q}{4\pi\varepsilon r}$$

The potential inside of K can now be written as:

III-12 $$\varphi_- = \frac{Q_1}{4\pi\varepsilon_1 r} + \sum_{n=0}^{\infty} a_n r^n P_n(\cos\vartheta)$$

and the potential outside of K,

III-13 $$\varphi_+ = \frac{q_2}{4\pi\epsilon_0 r_2} + \Sigma_{n=0}^{\infty} \frac{b_n}{r^{n+1}} P_n(\cos\vartheta).$$

Depending on whether r is greater or less than ζ, $1/r_2$ can be expressed as one of two series of Legendre polynomials. From the equation

III-14 $$\frac{1}{r_2} = \frac{1}{\sqrt{r^2+\zeta^2-2r\zeta\cos\vartheta}}$$

we get:

III-15 $$\frac{1}{r\sqrt{1+(\zeta/r)^2-2(\zeta/r)x}}$$

III-15' $$\frac{1}{\zeta\sqrt{1+(r/\zeta)^2-2(r/\zeta)x}}$$

where $x = \cos\vartheta$.

For r greater (less) than ζ the expression III-15 (-15') contains the generating function for Legendre Polynomials:

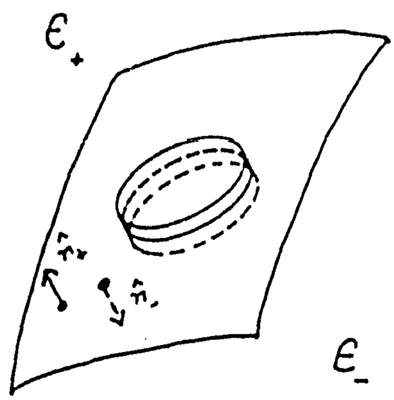
III-16 $$\frac{1}{\sqrt{1+t^2-2tx}} = \Sigma_{n=0}^{\infty} t^n P_n(\cos\vartheta), \quad |t| \le 1.$$

This rewriting of $1/r_2$ will help us to use the boundary conditions at ∂K to evaluate the coefficients a_n and b_n. Those conditons are:

III-17 $$\varphi_+ = \varphi_- \qquad \text{at } r = k,$$

III-18 $$\epsilon_1 \frac{\partial}{\partial r}\varphi_- = \epsilon_0 \frac{\partial}{\partial r}\varphi_+ \qquad \text{at } r = k.$$

The first condition is just the continuity of the potential. The second condition can be seen from integrating $-\nabla\cdot(\epsilon\nabla\varphi)$ over the volume of a small "pillbox" at the boundary between two dielectrics. Half of the pillbox is in one of the dielectrics, half is in the other, and the flatsides are parallel to the boundary surface. The pillbox is to be thin enough so that the area of the edge is negligible relative to the area of the sides.



From the field equation, this integral gives the charge enclosed, which is zero, and by applying Stokes Theorem, we find that this is equal to the area of the sides times their respective normals dotted with $(-\epsilon_{\pm}\nabla\varphi_{\pm})$, where the appropriate permitivity and potential is used. The equation is:

III-19 $$A[\hat{n}_+ \cdot (-\epsilon_+ \nabla \varphi_+)] + A[\hat{n}_- \cdot (-\epsilon_- \nabla \varphi_-)] = 0$$
$$= A\{\hat{n}_+ \cdot (-\epsilon_+ \nabla \varphi_+) - \hat{n}_+ \cdot (-\epsilon_- \nabla \varphi_-)\}.$$

From this we get:

III-20 $$\epsilon_+ \hat{n} \cdot \nabla \varphi_+ = \epsilon_- \hat{n} \cdot \nabla \varphi_-$$

at the boundary.

Applying III-17, and noting that r = k is less than ζ,

III-21 $$\frac{Q_1}{4\pi\epsilon_0}\left\{\frac{1}{\zeta}\Sigma_{n=0}^{\infty}\left(\frac{k}{\zeta}\right)^n P_n(\cos\vartheta)\right\} + \Sigma_{n=0}^{\infty}\frac{b_n}{k^{n+1}}P_n(\cos\vartheta)$$
$$= \Sigma_{n=0}^{\infty} P_n(\cos\vartheta)\left[\frac{Q_1}{4\pi\epsilon_0}\frac{k^n}{\zeta^{n+1}} + \frac{b_n}{k^{n+1}}\right]$$
$$= \frac{Q_1}{4\pi\epsilon_1 k} + \Sigma_{n=0}^{\infty} a_n k^n P_n(\cos\vartheta).$$

Applying III-18':

III-22 $$\epsilon_0\left\{\Sigma_{n=0}^{\infty} P_n(\cos\vartheta)\left[\frac{Q_1}{4\pi\epsilon_0}\frac{nk^{n-1}}{\zeta^{n+1}} - \frac{(n+1)b_n}{k^{n+2}}\right]\right\}$$
$$= \epsilon_1\left\{-\frac{Q_1}{4\pi\epsilon_1 k^2} + \Sigma_{n=1}^{\infty} n a_n k^{n-1} P_n(\cos\vartheta)\right\}$$

We next identify the coefficients of the Legendre polynomials, beginning with n = 0, then n ≥ 1.

III-23 $$\frac{Q_1}{4\pi\epsilon_0\zeta} + \frac{b_0}{k} = \frac{Q_1}{4\pi\epsilon_1 k} + a_0$$

III-24 $$-\epsilon_0\frac{b_0}{k^2} = -\frac{Q_1}{4\pi k^2}$$

From III-24 we find that $b_0 = Q_1/4\pi\epsilon_0$ and this gives us $a_0 = (Q_2/4\pi\epsilon_0\zeta)+(Q_1/4\pi k)(\frac{1}{\epsilon_0} - \frac{1}{\epsilon_1})$. For $n \geq 1$ we have:

III-25 $$\frac{Q_2\ k^n}{4\pi\epsilon_0\ \zeta^{n+1}} + \frac{b_n}{k^{n+1}} = a_n k^n$$

III-26 $$\frac{n\ Q_2\ k^{n-1}}{4\ \pi\ \zeta^{n+1}} - \frac{(n+1)\ b_n\ \epsilon_1}{k^{n+2}} = n\, a_n\, k^{n-1}\epsilon_1$$

Treating this as two equations with two unknowns we find:

III-27 $$a_n = \frac{Q_2}{4\pi\ \zeta^{n+1}}\ \frac{2n+1}{n\epsilon_1 + (n+1)\epsilon_0}$$

III-28 $$b_n = \frac{Q_2}{4\pi}\ \frac{k^{2n+1}}{\zeta^{n+1}}\ \frac{\epsilon_0-\epsilon_1}{\epsilon_0}\ \frac{n}{n\epsilon_1 + (n+1)\epsilon_0}$$

Let's find the forces now on Q_1, Q_2, and K, beginning with Q_1.

Using $r \to 0$, we find that the field at the origin is, neglecting the field of Q_1:

III-29 $$E = \frac{Q_2}{4\pi\zeta^2}\ \frac{3}{\epsilon_1 + 2\epsilon_0}$$

This times Q_1 would seem to be the force on particle one. But when we look at the force on Q_2 we find a non-symmetry. Finding the field at the position of Q_2, neglecting its self-field, and multiplying by Q_2:

III-30 $$F_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0\zeta^2} + \frac{Q_2^2(\epsilon_0-\epsilon_1)2k^3}{4\pi\epsilon_0(\epsilon_1+2\epsilon_0)\zeta^5} + \frac{6Q_2^2(\epsilon_0-\epsilon_1)k^5}{4\pi\epsilon_0(2\epsilon_1+3\epsilon_0)\zeta^7} + \cdots$$

The force on K is that on a dielectric sphere near a point charge, which turns out to be the k^n/ζ^{n+2} terms in III-30 and can not solve the problem of non-symmetry.

A somewhat more grueling way to find the forces involved is to first find the field energy and then take the negative of the derivative with respect to the sepera- tion ζ. Through the octopole term:

III-31 $$-\nabla\varphi_+ = \hat{r}\left[-\frac{Q_2}{4\pi\epsilon_0}\frac{\partial}{\partial r}\frac{1}{r_2} + \frac{b_0}{r^2} + \frac{2b_1}{r^3}P_1 + \frac{3b_2}{r^4}P_2 + \frac{4b_3}{r^5}P_3\right] + \hat{\vartheta}\,\frac{1}{r}(\sin\vartheta)\left[\frac{Q}{4\pi\epsilon_0}\frac{\partial}{\partial\cos\vartheta}\frac{1}{r_2} + \frac{b_1}{r^2}P_1' + \frac{b_2}{r^3}P_2' + \frac{b_3}{r^4}P_3'\right]$$

III-32 $$-\nabla\varphi_- = \hat{r}\left[\frac{Q_1}{4\pi\epsilon_1 r^2} - a_1 P_1 - 2a_2 r P_2 - 3a_3 r^2 P_3\right] + \hat{\vartheta}\,\frac{1}{r}(\sin\vartheta)\left[a_1 r P_1' + a_2 r^2 P_2' + a_3 r^3 P_3'\right]$$

with $P_n' = \frac{\partial}{\partial\cos\vartheta} P_n(\cos\vartheta)$.

The field energy density is $\frac{\epsilon}{2}(-\nabla\varphi)\cdot(-\nabla\varphi)$, which must be integrated over three regions when we consider that the region outside of K must be broken up into $k < r < \zeta$ and $\zeta < r$ in order to deal with $1/r_2$.

Beginning with $r < k$, let's let U_- denote the field energy inside K and U_{1-} the self-field energy of Q_1 within K. Then:

III-33 $$\mathcal{U}_- = \frac{\epsilon_1}{2}\int_K r^2\sin\vartheta\,dr\,d\vartheta\,d\phi\left\{\left[\frac{Q_1}{4\pi\epsilon_1 r^2} - a_1P_1 - 2a_2 r P_2 - 3a_3 r^2 P_3\right]^2 + \frac{\sin^2\vartheta}{r^2}\left[a_1 r P_1' + a_2 r^2 P_2' + a_3 r^3 P_3'\right]^2\right\}$$

In converting from ϑ to $\cos\vartheta$ we find

$$\int_0^{2\pi} A(\cos\vartheta)\sin\vartheta\,d\vartheta = \int_{-1}^{1} A(\cos\vartheta)\,d(\cos\vartheta)$$

so we'll use the simpler notation of $x = \cos\vartheta$.

The radial portion of U_- will be:

III-34 $$\mathcal{U}_{1-} + \pi\epsilon_1\left[\frac{2a_1^2k^3}{9} + \frac{8a_2^2k^5}{25} + \frac{18a_3^2k^7}{49}\right]$$

The angular portion requires rewriting terms of the kind $(1-x^2)P_n'P_m'$ with P_lP_k expressions.

The following table will be useful for evaluating the angular parts of the energy in the other regions.

TABLE 1

$$P_0' = 0$$

$(1-x^2)P_1'(x) = \frac{2}{3}(P_0-P_2)$	$P_1' = P_0$
$(1-x^2)P_2'(x) = \frac{6}{5}(P_1-P_3)$	$P_2' = 3P_1$
$(1-x^2)P_3'(x) = \frac{12}{7}(P_2-P_4)$	$P_3' = 5P_2+P_0$

Using this table we can tabulate the expressions we will substitute in the angular portion of III-33.

$$(1-x^2)P_1'P_1' = \frac{2}{3}(P_0P_0-P_2P_0)$$
$$(1-x^2)P_1'P_2' = 2(P_0P_1-P_2P_1)$$
$$(1-x^2)P_1'P_3' = \frac{2}{3}(5P_0P_2+P_0P_0-5P_2P_2-P_2P_0)$$
$$(1-x^2)P_2'P_2' = \frac{6}{5}(3P_1P_1-3P_3P_1)$$
$$(1-x^2)P_2'P_3' = \frac{6}{5}(5P_1P_2-5P_3P_2+P_1P_0-P_3P_0)$$
$$(1-x^2)P_3'P_3' = \frac{12}{7}(5P_2P_2-5P_4P_2+P_2P_0-P_4P_0)$$

Orthogonality will rid us of all but a few of these terms, so that the angular portion becomes:

<u>III-35</u> $$\pi\epsilon_0\left\{\frac{4k^3}{9}a_1^2 + \frac{12k^5}{25}a_2^2 + \frac{24k^7}{49}a_3^2\right\}$$

Combining III-34 and -35 and plugging in the expressions for the a_n:

<u>III-36</u> $$\mathcal{U}_- = \mathcal{U}_{1-} + \pi\epsilon_1\left[\frac{2k^3}{3}a_1^2 + \frac{4k^5}{5}a_2^2 + \frac{6k^7}{7}a_3^2\right]$$
$$= \mathcal{U}_{1-} + \frac{Q_2^2\epsilon_1}{16\pi}\left[\frac{2}{3}\frac{k^3}{\zeta^4}\left(\frac{3}{\epsilon_1+2\epsilon_0}\right)^2 + \frac{4}{5}\frac{k^5}{\zeta^6}\left(\frac{5}{2\epsilon_1+3\epsilon_0}\right)^2 + \frac{6}{7}\frac{k^7}{\zeta^8}\left(\frac{7}{3\epsilon_1+4\epsilon_0}\right)^2\right]$$

The energy U_+ outside of K will include the energy of the self-field of Q_2. This expression for the self-energy of Q_2 outside of K will be a function of ζ, but it will be more convenient to use the total self-energy of Q_2 in a uniform space with permitivity ϵ_0 minus the self-energy of

Q_2 within a sphere of radius k, positioned like K is with respect to Q_2, and with a permitivity ϵ_o. If we denote the self-energy of Q_2 outside of K by U_{2+}, the "total self--energy" mentioned by U_2, and the self-energy within r = k with ϵ_o by U_{2-}, then

III-37
$$\mathcal{U}_{2+} = \mathcal{U}_2 - \mathcal{U}_{2-}$$

Before breaking U_+ down according to whether r is greater or less than ζ, the equation for U_+ is:

III-38
$$\begin{aligned}\mathcal{U}_+ = 2\pi\frac{\epsilon_o}{2}\int_k^\infty\int_{-1}^{1} r^2\,dr\,dx\,\Big\{\Big[&-\frac{2Q_2}{4\pi\epsilon_o}\frac{b_0}{r^2}\frac{\partial}{\partial r}\frac{1}{r_2} - \frac{4Q_2}{4\pi\epsilon_o}\frac{b_1}{r^3}P_1\frac{\partial}{\partial r}\frac{1}{r_2}\\
&-\frac{6Q_2}{4\pi\epsilon_o}\frac{b_2}{r^4}P_2\frac{\partial}{\partial r}\frac{1}{r_2} - \frac{8Q_2}{4\pi\epsilon_o}\frac{b_3}{r^5}P_3\frac{\partial}{\partial r}\frac{1}{r_2} + \frac{4b_1^2}{r^6}P_1P_1 + \frac{9b_2^2}{r^8}P_2P_2\\
&+\frac{16b_3^2}{r^{10}}P_3P_3\Big] + \frac{1}{r^2}(1-x^2)\Big[\frac{2Q_2}{4\pi\epsilon_o}\frac{b_1}{r^3}P_1'\frac{\partial}{\partial x}\frac{1}{r_2} + \frac{2Q_2}{4\pi\epsilon_o}\frac{b_2}{r^4}P_2'\frac{\partial}{\partial x}\frac{1}{r_2}\\
&+\frac{2Q_2}{4\pi\epsilon_o}\frac{b_3}{r^5}P_3'\frac{\partial}{\partial x}\frac{1}{r_2} + \frac{b_1^2}{r^4}P_1'P_1' + \frac{2b_1b_2}{r^5}P_1'P_2' + \frac{2b_1b_3}{r^6}P_1'P_3'\\
&+\frac{b_2^2}{r^6}P_2'P_2' + \frac{2b_2b_3}{r^7}P_2'P_3' + \frac{b_3^2}{r^8}P_3'P_3'\Big]\Big\} - \mathcal{U}_2 - \mathcal{U}_{2-} + \mathcal{U}_{1+}\end{aligned}$$

The term U_{1+} is the self-energy of Q_1 outside of K and comes from the b_o^2 term in the radial part of U_+. The b_nb_m terms will not be affected by $r < \zeta$, $r > \zeta$, since they do not involve $1/r_2$, and can be evaluated seperately. To the radial portion they contribute:

III-39
$$\begin{aligned}\pi\epsilon_o\int_{-1}^{1}\int_k^\infty r^2\,dr\,dx\,\Big\{\frac{4b_1^2}{r^6}P_1P_1 + \frac{9b_2^2}{r^8}P_2P_2 + \frac{16b_3^2}{r^{10}}P_3P_3\Big\}\\
= \pi\epsilon_o\Big\{\frac{8}{9}\frac{b_1^2}{k^3} + \frac{18}{25}\frac{b_2^2}{k^5} + \frac{32}{49}\frac{b_3^2}{k^7}\Big\}\end{aligned}$$

and to the angular portion:

III-40
$$\pi\epsilon_o\int_k^\infty\int_{-1}^{1} r^2\,dr\,dx\,\frac{1}{r^2}(1-x^2)\Big\{\frac{b_1^2}{r^4}P_1'P_1' + \frac{2b_1b_2}{r^5}P_1'P_2' + \frac{2b_1b_3}{r^6}P_1'P_3' + \frac{b_2^2}{r^6}P_2'P_2' + \frac{2b_2b_3}{r^7}P_2'P_3' + \frac{b_3^2}{r^8}P_3'P_3'\Big\}$$
$$= \pi\epsilon_o\Big\{\frac{4}{9}\frac{b_1^2}{k^3} + \frac{12}{25}\frac{b_2^2}{k^5} + \frac{24}{49}\frac{b_3^2}{k^7}\Big\}$$

Although dependent on Q_2 and ζ, III-39 and -40 represent the self-energy of K for r > k. The next parts to be evaluated, then, are those involving the interactions of the self-field of Q_2 and the self-field of K induced by Q_2. The radial portions of this are:

III-41
$$\pi\epsilon_0\int_{-1}^{1}\int_{k}^{\zeta} r^2\,dr\,dx\left\{-\frac{4Q_2}{4\pi\epsilon_0}\frac{b_1}{r^2}\frac{1}{\zeta^2}P_1^2-\frac{6Q_2}{4\pi\epsilon_0}\frac{b_2}{r^3}\frac{2r}{\zeta^3}P_2^2-\frac{8Q_2}{4\pi\epsilon_0}\frac{b_3}{r^4}\frac{3r^2}{\zeta^4}P_3^2\right\}$$
$$=\frac{Q_2}{2}\ln\frac{k}{\zeta}\left\{\frac{4b_1}{3\zeta^2}+\frac{12b_2}{5\zeta^3}+\frac{24b_3}{7\zeta^4}\right\}$$

and

III-42
$$\pi\epsilon_0\int_{-1}^{1}\int_{\zeta}^{\infty} r^2\,dr\,dx\left\{\frac{8Q_2}{4\pi\epsilon_0}\frac{b_1}{r^2}P_1^2\frac{\zeta}{r^3}+\frac{18Q_2}{4\pi\epsilon_0}\frac{b_2}{r^3}P_2^2\frac{\zeta^2}{r^4}+\frac{32Q_2}{4\pi\epsilon_0}\frac{b_3}{r^4}P_3^2\frac{\zeta^3}{r^5}\right\}$$
$$=\frac{Q_2}{2}\left\{\frac{8b_1}{9\zeta^2}+\frac{18b_2}{25\zeta^3}+\frac{32b_3}{49\zeta^4}\right\}$$

The interaction between the self-fields of Q_1 and Q_2 is contained in the radial portion in the term:

III-43
$$\pi\epsilon_0\int_{-1}^{1}\int_{\zeta}^{\infty} r^2\,dr\,dx\,\frac{2Q_2}{4\pi\epsilon_0}\frac{b_0}{r^2}P_0^2\frac{1}{r^2}$$
$$=\frac{Q_2\,b_0}{\zeta}$$

The angular portions are:

III-44
$$\pi\epsilon_0\int_{-1}^{1}\int_{k}^{\zeta} dr\,dx\,(1-x^2)\frac{Q_2}{4\pi\epsilon_0}\left[\frac{b_1}{r^2}P_1'+\frac{b_2}{r^3}P_2'+\frac{b_3}{r^4}P_3'\right]\sum_{n=1}^{\infty}\frac{r^n}{\zeta^{n+1}}P_n'$$
$$=\frac{Q_2}{2}\ln\frac{\zeta}{k}\left\{\frac{4b_1}{3\zeta^2}+\frac{12b_2}{5\zeta^3}+\frac{24b_3}{7\zeta^4}\right\}$$

and

III-45
$$\pi\epsilon_0\int_{-1}^{1}\int_{\zeta}^{\infty} dr\,dx\,\frac{Q_2}{2\pi\epsilon_0}\left[\frac{b_1}{r^2}P_0+\frac{b_2}{r^3}3P_1+\frac{b_3}{r^4}(5P_2+P_0)\right]\left[\frac{2}{3}\frac{\zeta}{r^2}(P_0-P_2)+\frac{6}{5}\frac{\zeta^2}{r^3}(P_1-P_3)+\frac{12}{7}\frac{\zeta^3}{r^4}(P_2-P_4)\right]$$
$$=Q_2\left[\frac{2}{9}\frac{b_1}{\zeta^2}+\frac{6}{25}\frac{b_2}{\zeta^3}+\frac{12}{49}\frac{b_3}{\zeta^4}\right]$$

The remaining ζ dependent term to be evaluated is:

III-46
$$U_{2-}=\pi\epsilon_0\int_{0}^{k}\int_{-1}^{1} r^2\,dr\,dx\left\{\left[\frac{P_1^2}{\zeta^4}+\frac{4r^2P_2^2}{\zeta^6}+\frac{6r^4P_3^2}{\zeta^8}\right]+\frac{1}{r^2}(1-x^2)\left[\frac{rP_1'}{\zeta^2}+\frac{r^2P_2'}{\zeta^3}+\frac{r^3P_3'}{\zeta^4}\right]^2\right\}\frac{Q_2^2}{16\pi^2\epsilon_0^2}$$
$$=\frac{Q_2^2}{4\pi\epsilon_0}\left[\frac{k^3}{6\zeta^4}+\frac{k^5}{5\zeta^6}+\frac{9k^7}{49\zeta^8}\right]$$

Bringing together III-36, -39, -40, -41, -42, -43, -44, and -45, and using III-28 for the b_n, we find, up to the quadropole term,

III-47
$$\begin{aligned}
\mathcal{U} &= \mathcal{U}_+ + \mathcal{U}_- \\
&= \mathcal{U}_2 + \mathcal{U}_{1+} + \mathcal{U}_{1-} \\
&\quad + \frac{Q_1 Q_2}{4\pi\epsilon_0 \zeta} \\
&\quad + \frac{Q_2^2}{4\pi\epsilon_0}\left\{\frac{k^3}{\zeta^4}\left[\frac{(\epsilon_1+2\epsilon_0)(5\epsilon_0-2\epsilon_1)}{6(\epsilon_1+2\epsilon_0)^2}\right] + \frac{k^5}{\zeta^6}\left[\frac{(2\epsilon_1+3\epsilon_0)(8\epsilon_0-3\epsilon_1)}{5(2\epsilon_1+3\epsilon_0)^2}\right]\right\} \\
&\quad + \frac{Q_2^2}{4\pi\epsilon_0}\left\{\frac{k^3}{6\zeta^4} + \frac{k^5}{5\zeta^6}\right\} \\
&= \mathcal{U}_2 + \mathcal{U}_1 \\
&\quad + \frac{Q_1 Q_2}{4\pi\epsilon_0 \zeta} \\
&\quad + \frac{Q_2^2}{4\pi\epsilon_0}\left\{\frac{k^3}{2\zeta^4}\left(\frac{\epsilon_0-\epsilon_1}{\epsilon_1+2\epsilon_0}\right) + \frac{k^5}{\zeta^6}\left(\frac{\epsilon_0-\epsilon_1}{2\epsilon_1+3\epsilon_0}\right)\right\}
\end{aligned}$$

By taking the negative of the derivative of this with respect to ζ we get the force between Q_2 and $\{K, Q_1\}$:

III-48
$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 \zeta^2} + \frac{Q_2^2}{4\pi\epsilon_0}\left\{\frac{2k^3}{\zeta^5}\left(\frac{\epsilon_0-\epsilon_1}{\epsilon_1+2\epsilon_0}\right) + \frac{6k^5}{\zeta^7}\left(\frac{\epsilon_0-\epsilon_1}{2\epsilon_1+3\epsilon_0}\right)\right\}$$

This agrees with the force on Q_2 found earlier. More importantly, it agrees with a requirement in our model that K be tied to Q_1. By taking the derivative with respect to ζ we have effectively tied the motion of K to the motion of Q_1, and if K is to represent a scalar-field produced somehow by the particle Q_1 then K must move when Q_1 does. The introduction of K has not affected the $1/\zeta^2$ part of the force, though, which was something we had been looking for in order to shorten the range.

The introduction of a second sphere K_2 around Q_2 will change the force, but only in terms of the interaction of K_2 with Q_1 and K_2 with K_1. For ζ large with respect to k_1 and k_2, the interaction between K_2 and Q_1 would essentially be like that between K_1 and Q_2. To check the magnitude of the interaction between K_1 and K_2, let's first find the effective dipole induced on K_1 by Q_2. The force on Q_2 due to a dipole at the origin and aligned with Q_2 is

III-49 $$F = \frac{Q_2}{4\pi\epsilon_0}\,\frac{2p}{\zeta^3}$$

implying, from III-48:

III-50 $$p_1 = -\frac{Q_2 k^3(\epsilon_0-\epsilon_1)}{\zeta^2(\epsilon_1+2\epsilon_0)}$$

To the lowest order, the force between p_1 and p_2 (the sphere-sphere force) will be:

III-51 $$-\frac{\partial}{\partial\zeta}\,p_1\left(\frac{2p_1}{4\pi\epsilon_0\zeta^3}\right) = +\frac{7Q_1Q_2k^6(\epsilon_0-\epsilon_1)^2}{2\pi\epsilon_0(\epsilon_1+2\epsilon_0)^2\zeta^8}$$

with the assumption here that K_1 and K_2 are essentially the same.

Since the dipole part of K_1 can induce a dipole part of K_2 which can interact with Q_1 we need the dipole p' induced on a K by a dipole p aligned toward K and at a distance ζ from the center. Starting with $p = qd$, two charges q and $-q$ a distance d apart, the field outside of K is:

$$\varphi_+ = \frac{p}{4\pi\epsilon_0}\begin{cases}\sum_{n=0}^{\infty} nP_n(\cos\theta)\dfrac{r^n}{\zeta^{n+2}} & r<\zeta\\[2mm] -\sum_{n=0}^{\infty} nP_n(\cos\theta)\dfrac{\zeta^{n-1}}{r^{n+1}} & r>\zeta\end{cases} + \frac{p(\epsilon_0-\epsilon_1)}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{n}{n\epsilon_1+(n+1)\epsilon_0}\,\frac{k^{2n+1}}{r^{n+1}\zeta^{n+2}}P_n(\cos\theta)$$

The part of this which would be due to the dipole induced by p is

$$\frac{p'}{4\pi\epsilon_0 r^2} = \frac{p(\epsilon_0-\epsilon_1)}{4\pi\epsilon_0}\,\frac{1}{\epsilon_1+2\epsilon_0}\,\frac{k^3}{r^2\zeta^3}\cos\theta$$

so that

$$p' = \frac{k^3 p}{\zeta^3}\,\frac{(\epsilon_0-\epsilon_1)}{\epsilon_1+2\epsilon_0}$$

The force that p_2' induced by p_1 exerts on Q_1 is

$$F = \frac{Q_1}{4\pi\epsilon_0}\,\frac{2}{\zeta^3}\left(\frac{k^3}{\zeta^3}\,\frac{\epsilon_0-\epsilon_1}{\epsilon_1+2\epsilon_0}\right)\left(-\frac{Q_0 k^3(\epsilon_0-\epsilon_1)}{\zeta^2(\epsilon_1+2\epsilon_0)}\right) = -\frac{Q_1 Q_0}{4\pi\epsilon_0}\,\frac{2k^6}{\zeta^8}\,\frac{(\epsilon_0-\epsilon_1)^2}{(\epsilon_1+2\epsilon_0)^2}\,.$$

These sphere-sphere interactions and induced-induced dipole interactions are weaker than even the quadropole term of the sphere-charge interaction.

The introduction of the second sphere, then, while improving the model conceptually, because it makes both particles act as sources of scalar fields, does not affect the $1/\zeta^2$ portion of the force and therefore does not significantly improve the model.

At this point, if we want the $1/\zeta^2$ portion of the force to be zero because of ϵ_0 when ζ is larger than k_1+k_2, then ϵ_0 must jump to infinity at the boundary. By using a series of spheres we might have been able to get the type of profile needed for the scalar fields in more complex models, but the need for an infinite value in the permitivity in order to shorten the range of the gauge fields seems inherent with the one-dimensional groups.

Still, this particular model of point sources and spherical discontinuities will prove useful in the next section where we consider the two-dimensional Abelian group, and, because of our exercise with the model in this section, we will be ready to face the gymnastics to come in the next section.

IV.

2-DIMENSIONAL MODEL

Before we can develop the model of two point sources and a sphere for the case of the two-dimensional Abelian gauge group, we need to look at the field itself. The gauge field equations in the Abelian case are:

IV-1 $$(g_{\alpha\beta}\Lambda^{\alpha}_{ab}g^{ac}g^{bd})_{;d} = J^{c}_{\beta}$$

since the structure constants are zero. If we look at the static case with the Minkowskian metric, signature -+++, this becomes:

IV-2 $$-\tfrac{1}{2}\nabla\cdot(g_{\alpha\beta}\nabla\varphi^{\alpha}) = J^{0}_{\beta}$$

The vector potentials are zero since all possible sources are stationary. In dealing with sources here, the source density will be denoted by $\rho_{\beta} = 2\ J^{0}_{\beta}$.

In order to simplify the notation: g without letter subscripts will denote the group metric operating on the group part of what follows it. (There may be a +,- or 1,2 subscript used to indicate position rather than components.) The determinant of g will be denoted by μ. The potential will be treated like a two-dimensional vector in the group space, and the sources will be treated likewise. That is:

IV-3 $$\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

IV-4 $$Q = \begin{pmatrix}Q_1\\ Q_2\end{pmatrix}$$

Since, with rare exception, the greek letter subscripts won't be used to indicate group components for awhile, the components of g for this two-dimensional Abelian case will be given as

IV-5
$$[g_{\alpha\beta}] = \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$$

Applying this notation we see, for those regions where g is constant,

IV-6
$$-\nabla^2\varphi = -\begin{pmatrix} \nabla^2\varphi_1 \\ \nabla^2\varphi_2 \end{pmatrix} = g^{-1}(-g\,\nabla^2\varphi) = g^{-1}\rho$$

Going to spherical coordinates, we can write part of the solution of φ as

IV-7
$$g\,\varphi = \sum_{n=0}^{\infty}\left(a_n r^n + \frac{b_n}{r^{n+1}}\right) P_n(\cos\vartheta)$$

in almost the same way as in the last section (pages 18, 19), except that we define the a_n and b_n as two-component group vectors:

IV-8
$$a_n = \begin{pmatrix} c_n \\ d_n \end{pmatrix} \qquad\qquad b_n = \begin{pmatrix} f_n \\ g_n \end{pmatrix}$$

and the homogeneous solution of 6 is

IV-7'
$$\varphi = \sum_{n=0}^{\infty}\left[(g^{-1}a_n)\,r^n + (g^{-1}b_n)\,r^{-n-1}\right] P_n(\cos\vartheta)$$

If we have a source Q at the origin, the solution for the potential is, using essentially the same methods as in the last section,

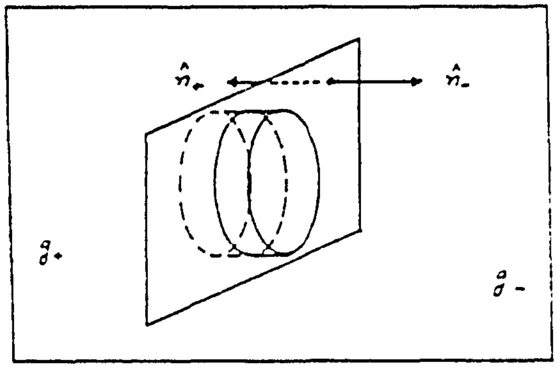
IV-9
$$\varphi = \frac{g^{-1}Q}{4\pi r}$$

being careful only to remember that these are matrices and not scalars. The force then between Q at the center and another charge P at a distance r is

$$\left(-\frac{\partial}{\partial r}\varphi^{\alpha}\right)P_{\alpha} = \frac{\beta P_1 Q_1 - \gamma(P_1 Q_2 + P_2 Q_1) + \alpha(P_2 Q_2)}{4\pi r^2 \mu}$$

We will also need a boundary condition corresponding to III-20. Let's begin with two regions, characterised by g_+ and g_-, and the boundary between them. The field equation is

IV-10 $$-\nabla\cdot(g\,\nabla\varphi) = 0$$



Remember that g is an $n \times n$ matrix and $\nabla\varphi$ is a column matrix with entries $\nabla\varphi^j$, $j = 1,\ldots,n$, where n is the dimension of the Abelian group. Now let's consider a small pill box whose sides are parallel to the boundary, whose curved edge is very small compared to the area A of the sides, and which is placed at the boundary so that one side is in the g_+ region and the other side is in the g_- region. If we integrate $-\nabla\cdot(g\,\nabla\varphi)$ over this volume, we'll get zero. By Stoke's theorem:

IV-11 $$\int_{\text{pillbox}} -\nabla\cdot(g\,\nabla\varphi)\,dv = -\int_{\text{surface}} g\,\nabla\varphi\cdot d\Sigma$$

This surface integral can be written as

IV-12 $$g_+\nabla\varphi_+\cdot\hat{n}_+ A + g_-\nabla\varphi_-\cdot\hat{n}_- A + (\text{negligible})$$

When we take into consideration that this is equal to zero, and that $\hat{n}_+ = -\hat{n}_-$, then we get the boundary conditions

IV-13 $$\hat{n}_+\cdot g_+\nabla\varphi_+ = \hat{n}_+\cdot g_-\nabla\varphi_-$$

When expanded, IV-13 is equivalent, in our two-dimensional case, to the two equations:

IV-13' $\qquad \hat{n}\cdot(\alpha_+\nabla\varphi_+^1 + \gamma_+\nabla\varphi_+^2) = \hat{n}\cdot(\alpha_-\nabla\varphi_-^1 + \gamma_-\nabla\varphi_-^2)$

IV-13" $\qquad \hat{n}\cdot(\gamma_+\nabla\varphi_+^1 + \beta_+\nabla\varphi_+^2) = \hat{n}\cdot(\gamma_-\nabla\varphi_-^1 + \beta_-\nabla\varphi_-^2)$

The additional conditions we will want φ to satisfy for our model are:

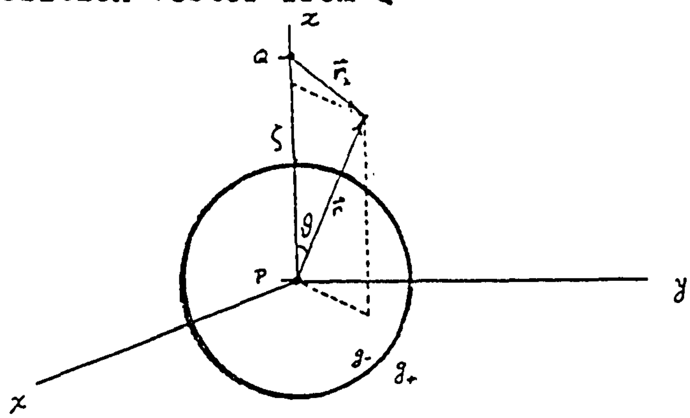
a) φ is continuous everywhere except possibly at the sources,

b) φ is zero at infinity,

c) neglecting the source at the origin, φ is finite through the origin.

Our model is labeled by:

P	the point source at the origin
Q	the point source at a distance ζ along the z-axis
K	the sphere (kügel) of radius k around P
g_+	the group metric value outside of K
g_-	the group metric value inside of K
$\vec{r}$	the position vector from P
$\vec{r}_2$	the position vector from Q



By condition (b) and equations IV-7 and -9:

IV-14 $$g_+ \varphi_+ = \frac{Q}{4\pi r_1} + \sum_{n=0}^{\infty} \frac{b_n}{r^{n+1}} P_n(\cos\vartheta)$$

By condition (c) and equations IV-7 and -9:

IV-15 $$g_- \varphi_- = \frac{P}{4\pi r} + \sum_{n=0}^{\infty} a_n r^n P_n(\cos\vartheta)$$

At r = k we require by condition (a):

IV-16 $$\varphi_+ = \varphi_-$$

and by IV-13:

IV-17 $$g_+ \frac{\partial}{\partial r} \varphi_+ = g_- \frac{\partial}{\partial r} \varphi_-$$

Since $k < \zeta$, we will replace $1/r_2$ at $r = k$ by

IV-18 $$\frac{1}{r_2} = \sum_{n=0}^{\infty} P_n(\cos\vartheta) \frac{k^n}{\zeta^{n+1}}$$

Using IV-16 on -14 and -15 we get

IV-19 $$\sum_{n=0}^{\infty}\left[\frac{g_+^{-1} Q}{4\pi} \frac{k^n}{\zeta^{n+1}} + \frac{g_+^{-1} b_n}{k^{n+1}}\right] P_n(\cos\vartheta) = \frac{g_-^{-1} P}{4\pi k} + \sum_{n=0}^{\infty} g_-^{-1} a_n k^n P_n(\cos\vartheta)$$

Using IV-17

IV-20 $$\sum_{n=0}^{\infty}\left[\frac{Q}{4\pi} \frac{n k^{n-1}}{\zeta^{n+1}} - \frac{(n+1) b_n}{k^{n+2}}\right] = -\frac{P}{4\pi k^2} + \sum_{n=1}^{\infty} n a_n k^{n-1} P_n(\cos\vartheta)$$

By equating the coefficients of the Legendre polynomials we get

(for n = 0)

IV-21 $$g_+^{-1} \frac{Q}{4\pi\zeta} + g_+^{-1} \frac{b_0}{k} = g_-^{-1} \frac{P}{4\pi k} + g_-^{-1} a_0$$

IV-22 $$-\frac{b_0}{k^2} = -\frac{P}{4\pi k^2}$$

(for n > 0)

IV-23 $$g_+^{-1}\left[\frac{Q}{4\pi} \frac{k^n}{\zeta^{n+1}} + \frac{b_n}{k^{n+1}}\right] = g_-^{-1} a_n k^n$$

IV-24 $$\frac{Q\, n\, k^{n-1}}{4\pi\, \zeta^{n+1}} - \frac{(n+1) b_n}{k^{n+2}} = n a_n k^{n-1}$$

From IV-22 we find

IV-25 $$b_0 = \frac{P}{4\pi}$$

Combining this with IV-21 gives

IV-26 $$a_0 = g_-\left[g_+^{-1} \frac{Q}{4\pi\zeta} - (g_+^{-1} - g_-^{-1}) \frac{P}{4\pi k} \right]$$

The equations for $n \geq 1$ may be more easily dealt with when rewritten

IV-23' $$g_-^{-1} a_n k^{2n+1} - g_+^{-1} b_n = g_+^{-1} \frac{Q\, k^{2n+1}}{4\pi\, \zeta^{n+1}}$$

IV-24' $$n\, a_n k^{2n+1} + (n+1)\, b_n = \frac{Q\, n\, k^{2n+1}}{4\pi\, \zeta^{n+1}}$$

Multiplying IV-23' by g_+, dividing IV-24' by (n+1), and adding the two will give us the following equation for a_n:

IV-27 $$\left(g_+ g_-^{-1} + \frac{n}{n+1} \mathbb{1} \right) a_n = \frac{Q}{4\pi\zeta^{n+1}} \left(\frac{2n+1}{n+1} \right)$$

Solving for $a_n = \begin{pmatrix} c_n \\ d_n \end{pmatrix}$ gives us

IV-28 $$a_n = \frac{(n+1)(2n+1)}{\left[(n+1)^2\mu_+ + n^2\mu_- + n(n+1)(\alpha_+\beta_- + \beta_+\alpha_- - 2\gamma_+\gamma_-) \right]} \times \begin{bmatrix} \beta_+\alpha_- - \gamma_+\gamma_- + \frac{n\mu_-}{n+1} & \alpha_+\gamma_- - \gamma_+\alpha_- \\ \beta_+\gamma_- - \gamma_+\beta_- & \alpha_+\beta_- - \gamma_+\gamma_- + \frac{n\mu_-}{n+1} \end{bmatrix} \frac{Q}{4\pi\,\zeta^{n+1}}$$

Individually then

IV-29 $$c_n = \frac{(n+1)(2n+1)\left[Q_1\left(\beta_+\alpha_- - \gamma_+\gamma_- + \frac{n\mu_-}{n+1}\right) + Q_2(\alpha_+\gamma_- - \gamma_+\alpha_-) \right]}{4\pi\,\zeta^{n+1}\left[(n+1)^2\mu_+ + n^2\mu_- + n(n+1)(\alpha_+\beta_- + \beta_+\alpha_- - 2\gamma_+\gamma_-) \right]}$$

IV-30 $$d_n = \frac{(n+1)(2n+1)\left[Q_1(\beta_+\gamma_- - \gamma_+\beta_-) + Q_2\left(\alpha_+\beta_- - \gamma_+\gamma_- + \frac{n\mu_-}{n+1}\right) \right]}{4\pi\,\zeta^{n+1}\left[(n+1)^2\mu_+ + n^2\mu_- + n(n+1)(\alpha_+\beta_- + \beta_+\alpha_- - 2\gamma_+\gamma_-) \right]}$$

Multiplying IV-23' by g_-, dividing IV-24' by n, and adding gives us

IV-31 $$b_n = \left(g_- g_+^{-1} + \left(\tfrac{n+1}{n}\right)\mathbb{1}\right)^{-1}\left(-g_- g_+^{-1} + \mathbb{1}\right)\frac{a}{4\pi}\frac{k^{2n+1}}{\zeta^{n+1}}$$

$$= \left\{\frac{n^2}{\left[(n+1)^2\mu_+ + n^2\mu_- + n(n+1)(\alpha_+\beta_- + \beta_+\alpha_- - 2\gamma_+\gamma_-)\right]}\begin{bmatrix}\alpha_+\beta_- - \gamma_+\gamma_- + \left(\frac{n+1}{n}\right)\mu_+ & \gamma_+\alpha_- - \alpha_+\gamma_- \\ \gamma_+\beta_- - \beta_+\gamma_- & \beta_+\alpha_- - \gamma_+\gamma_- - \left(\frac{n+1}{n}\right)\mu_-\end{bmatrix}\right\}$$

$$\left\{-\frac{1}{\mu_+}\begin{bmatrix}\beta_+\alpha_- - \gamma_+\gamma_- - \mu_+ & \alpha_+\gamma_- - \gamma_+\alpha_- \\ \beta_+\gamma_- - \gamma_+\beta_- & \alpha_+\beta_- - \gamma_+\gamma_- - \mu_+\end{bmatrix}\right\}\left\{\begin{pmatrix}a_1\\a_2\end{pmatrix}\frac{k^{2n+1}}{4\pi\zeta^{n+1}}\right\}$$

$$= \frac{-n k^{2n+1}}{\left[(n+1)^2\mu_+ + n^2\mu_- + n(n+1)(\alpha_+\beta_- + \beta_+\alpha_- - 2\gamma_+\gamma_-)\right]}$$

$$\left\{\begin{bmatrix}n(\mu_- - \alpha_+\beta_- + \gamma_+\gamma_-) - (n+1)(\mu_+ - \beta_+\alpha_- + \gamma_+\gamma_-) & (2n+1)(\alpha_+\gamma_- - \gamma_+\alpha_-) \\ (2n+1)(\beta_+\gamma_- - \gamma_+\beta_-) & n(\mu_- - \beta_+\alpha_- + \gamma_+\gamma_-) - (n+1)(\mu_+ - \alpha_+\beta_- + \gamma_+\gamma_-)\end{bmatrix}\right\}\begin{pmatrix}a_1\\a_2\end{pmatrix}$$

Individually then

IV-32 $$f_n = \frac{-n k^{2n+1}\{a_1[n(\mu_- - \alpha_+\beta_- + \gamma_+\gamma_-) - (n+1)(\mu_+ - \beta_+\alpha_- + \gamma_+\gamma_-)] + a_2[(2n+1)(\alpha_+\gamma_- - \gamma_+\alpha_-)]\}}{4\pi\zeta^{n+1}[(n+1)^2\mu_+ + n^2\mu_- + n(n+1)(\alpha_+\beta_- + \beta_+\alpha_- - 2\gamma_+\gamma_-)]}$$

IV-33 $$g_n = \frac{-n k^{2n+1}\{a_1[(2n+1)(\beta_+\gamma_- - \gamma_+\beta_-)] + a_2[n(\mu_- - \beta_+\alpha_- + \gamma_+\gamma_-) - (n+1)(\mu_+ - \alpha_+\beta_- + \gamma_+\gamma_-)]\}}{4\pi\zeta^{n+1}[(n+1)^2\mu_+ + n^2\mu_- + n(n+1)(\alpha_+\beta_- + \beta_+\alpha_- - 2\gamma_+\gamma_-)]}$$

To find the force between Q and $\{P, h\}$ let's use the field energy method again. First, we'll need the stress-energy tensor for the gauge fields.

IV-34 $$T_{ab} = g_{\alpha\beta}\left\{4 g^{cd} A^\alpha_{ca} A^\beta_{db} - g_{ab} A^\alpha_{cd} A^\beta_{ef} g^{ce} g^{df}\right\}$$

In our case, the energy density is

IV-35 $$T_{00} = g_{\alpha\beta}\left\{4\cdot\tfrac{1}{4}\cdot\nabla\varphi^\alpha\cdot\nabla\varphi^\beta - \tfrac{1}{4}\cdot 2\cdot\nabla\varphi^\alpha\cdot\nabla\varphi^\beta\right\} = \tfrac{1}{2}\, g_{\alpha\beta}\,\nabla\varphi^\alpha\cdot\nabla\varphi^\beta$$

The energy of the fields will be the integral of this over all space. The components of $-\nabla\varphi$ are

IV-36 $$-\hat r\frac{\partial}{\partial r}\varphi_+ = -\sum_{n=0}^{\infty} P_n(\cos\vartheta)\, g_+^{-1}\left[\frac{a}{4\pi}\left(\frac{n r^{n-1}}{\zeta^{n+1}}\right) - \frac{(n+1)\, b_n}{r^{n+2}}\right] \qquad \text{for } k < r < \zeta$$

$$= \sum_{n=0}^{\infty} P_n(\cos\vartheta)\, g_+^{-1}\left[\frac{a\,\zeta^n}{4\pi} + b_n\right]\frac{(n+1)}{r^{n+2}} \qquad \text{for } r > \zeta$$

IV-37 $$-\hat{\vartheta}\,\frac{1}{r}\frac{\partial}{\partial\vartheta}\varphi_0 = \frac{\sqrt{1-x^2}}{r}\sum_{n=1}^{\infty} P_n' g_0^{-1}\left[\frac{Q}{4\pi}\frac{r^n}{\zeta^{n+1}} - \frac{b_n}{r^{n+1}}\right] \text{ for } k < r < \zeta$$

$$= \frac{\sqrt{1-x^2}}{r}\sum_{n=1}^{\infty} P_n' g_0^{-1}\left[\frac{Q}{4\pi}\frac{\zeta^n}{r^{n+1}} - \frac{b_n}{r^{n+1}}\right]_{\text{for } r > \zeta}$$

IV-38 $$-\hat{r}\,\frac{\partial}{\partial r}\varphi_- = g_-^{-1}\left[\frac{P}{4\pi r^2} - \sum_{n=0}^{\infty} n a_n r^{n-1} P_n(\cos\vartheta)\right]$$

IV-39 $$-\hat{\vartheta}\,\frac{1}{r}\frac{\partial}{\partial\vartheta}\varphi_- = \frac{\sqrt{1-x^2}}{r}\sum_{n=1}^{\infty} P_n' g_-^{-1} a_n r^n$$

As was done last time, the terms through $n = 3$ will be carried through the process of finding the energy, which will then be expressed in terms through $n = 2$.

The radial part for $k < r < \zeta$ is:

IV-40 $$\int_{-1}^{1}\int_{k}^{\zeta}\pi g_0\left[\frac{\partial}{\partial r}\varphi_0, \frac{\partial}{\partial r}\varphi_0\right] r^2\,dr\,dx = \pi g_0 \int_{-1}^{1}\int_{k}^{\zeta} r^2\,dr\,dx \left\{ P_0^2 \frac{1}{r^4}\left[g_0^{-1}b_0, g_0^{-1}b_0\right] \right.$$

$$+ P_1^2\left[g_0^{-1}\left(\frac{2b_1}{r^3} - \frac{Q}{4\pi\zeta^2}\right), g_0^{-1}\left(\frac{2b_1}{r^3} - \frac{Q}{4\pi\zeta^2}\right)\right] + P_2^2\left[g_0^{-1}\left(\frac{3b_2}{r^4} - \frac{2rQ}{4\pi\zeta^3}\right), g_0^{-1}\left(\frac{3b_2}{r^4} - \frac{2rQ}{4\pi\zeta^3}\right)\right]$$

$$\left. + P_3^2\left[g_0^{-1}\left(\frac{4b_3}{r^5} - \frac{3r^2Q}{4\pi\zeta^4}\right), g_0^{-1}\left(\frac{4b_3}{r^5} - \frac{3r^2Q}{4\pi\zeta^4}\right)\right] \right\}$$

where $g(A,B) = g_{\alpha\beta}A^{\alpha}B^{\beta}$ (brackets are used when A or B contain parenthesis).

If g is the metric for a region, A and B are vectors in this region, and given a basis S for the vector space, the inner product can be determined from matrices.

$$\begin{aligned} g(g^{-1}A, g^{-1}B) &= [g^{-1}A]_S^T[g]_S[g^{-1}B]_S \\ &= [A]_S^T[g^{-1}]_S^T[g]_S[g^{-1}]_S[B]_S \\ &= [A]_S^T[g^{-1}]_S[B]_S \\ &= g^{-1}(A,B) \end{aligned}$$

where the brackets indicate the matrix representation of the quantity with repect to the basis S.

Applying this to IV-40 we get

<u>IV-40'</u>

$$\int_{-1}^{1}\int_{k}^{\zeta}\pi g_0\left(\frac{\partial}{\partial r}\varphi_0,\frac{\partial}{\partial r}\varphi_0\right)r^2\,dr\,dx = \int_k^{\zeta}2\pi\frac{dr}{r^2}\;g_0^{-1}(b_0,b_0)$$

$$+\int_k^{\zeta}2\pi r\,dr\left\{\frac{4}{3r^4}g_0^{-1}(b_1,b_1)+\frac{9}{5r^6}g_0^{-1}(b_2,b_2)+\frac{16}{7r^8}\;g_0^{-1}(b_3,b_3)\right\}$$

$$-\int_k^{\zeta}2\pi r\,dr\left\{\frac{1}{3\zeta^2 r\pi}\;g_0^{-1}(b_1,a)+\frac{3}{5\pi\zeta^4 r}\,g_0^{-1}(b_2,a)+\frac{6}{7\pi\zeta^6 r}\;g_0^{-1}(b_3,a)\right\}$$

$$+\int_k^{\zeta}\frac{1}{8\pi}\;dr\;g_0^{-1}(a,a)\left\{\frac{r^2}{\zeta^4}+\frac{4r^4}{\zeta^6}+\frac{9r^6}{\zeta^8}\right\}$$

For r > ζ, the radial part is

<u>IV-41</u>

$$\int_{-1}^{1}\int_{\zeta}^{\infty}\pi g_0\left(\frac{\partial}{\partial r}\varphi_0,\frac{\partial}{\partial r}\varphi_0\right)r^2\,dr\,dx = \int_{\zeta}^{\infty}2\pi\;\frac{dr}{r^2}\;g_0^{-1}(b_0,b_0)$$

$$+\int_{\zeta}^{\infty}2\pi\,dr\left\{\frac{4}{3r^4}\,g_0^{-1}(b_1,b_1)+\frac{9}{5r^6}\;g_0^{-1}(b_2,b_2)+\frac{16}{7r^8}\;g_0^{-1}(b_3,b_3)\right\}$$

$$+\int_{\zeta}^{\infty}2\pi r\,dr\left\{\frac{1}{2\pi r^2}\;g_0^{-1}(a,b_0)+\frac{4\zeta}{6\pi r^4}\;g_0^{-1}(a,b_1)+\frac{9\zeta^2}{10\pi r^6}\,g_0^{-1}(a,b_2)+\frac{16\zeta^3}{14\pi r^8}\,g_0^{-1}(a,b_3)\right\}$$

$$+\int_{\zeta}^{\infty}\frac{1}{8\pi}\;dr\;g_0^{-1}(a,a)\left\{\frac{1}{r^2}+\frac{4\zeta^2}{3r^4}+\frac{9\zeta^4}{5r^6}+\frac{16\zeta^6}{7r^8}\right\}$$

The angular part for k < r < ζ is

<u>IV-42</u>

$$\int_k^{\zeta}\frac{1}{8\pi}\;dr\;g_+^{-1}(a,a)\left\{\frac{2r^2}{3\zeta^4}+\frac{6r^4}{5\zeta^6}+\frac{12r^6}{7\zeta^8}\right\}$$

$$+\int_k^{\zeta}2\pi r\,dr\left\{\frac{1}{2\pi r}\left[\frac{2}{3\zeta^2}g_+^{-1}(a,b_1)+\frac{6}{5\zeta^3}\;g_+^{-1}(a,b_2)+\frac{12}{7\zeta^4}\,g_+^{-1}(a,b_3)\right]\right\}$$

$$+\int_k^{\zeta}2\pi r\,dr\left\{\frac{2}{3r^4}\;g_+^{-1}(b_1,b_1)+\frac{6}{5r^6}\,g_+^{-1}(b_2,b_2)+\frac{12}{7r^8}\;g_+^{-1}(b_3,b_3)\right\}$$

For r > ζ the angular part is

IV-43
$$\int_{\varsigma}^{\infty} \frac{1}{8\pi}\, dr\; g_{+}^{-1}(Q,Q) \left\{ \frac{2\varsigma^{2}}{3r^{4}} + \frac{6\varsigma^{4}}{5r^{6}} + \frac{12\varsigma^{6}}{7r^{8}} \right\}$$
$$+ \int_{\varsigma}^{\infty} 2\pi\, dr \left\{ \frac{1}{2\pi} \left[\frac{2\varsigma}{3r^{4}} g_{+}^{-1}(Q,b_1) + \frac{6\varsigma^{3}}{5r^{6}} g_{+}^{-1}(Q,b_2) + \frac{12\varsigma^{5}}{7r^{8}} g_{+}^{-1}(Q,b_3) \right] \right\}$$
$$+ \int_{\varsigma}^{\infty} 2\pi\, dr \left\{ \frac{2}{3r^{4}} g_{+}^{-1}(b_1,b_1) + \frac{6}{5r^{6}} g_{+}^{-1}(b_2,b_2) + \frac{12}{7r^{8}} g_{+}^{-1}(b_3,b_3) \right\}$$

The terms containing $g_{+}^{-1}(b_0,b_0)$ are the part of the self energy of P which is outside of K. Their sum will be denoted U(P,+). Likewise, the terms containing $g_{+}^{-1}(Q,Q)$ are the part of the self energy of Q which is outside of K, and their sum will be denoted by U(Q,+). This last term, though, will again be replaced by

U(Q, everywhere, g_{+}) - U(Q,-,g_{+})

where the first term is the self energy of Q in a space with uniform metric g_{+} and the second is the part of the self energy of Q inside K using g_{+} instead of g_{-}.

Putting together IV-40, -41, -42, and -43

IV-44
$$\mathcal{U}_{+} = \mathcal{U}(P,+) + \mathcal{U}(Q,\text{everywhere},g_{+}) - \mathcal{U}(Q,-,g_{+})$$
$$+ \int_{k}^{\infty} 2\pi\, dr \left\{ \frac{2}{r^{4}} g_{+}^{-1}(b_1,b_1) + \frac{3}{r^{6}} g_{+}^{-1}(b_2,b_2) + \frac{4}{r^{8}} g_{+}^{-1}(b_3,b_3) \right\}$$
$$+ \int_{\varsigma}^{\infty} \frac{dr}{r^{2}}\; g_{+}^{-1}(Q,b_0)$$
$$+ \int_{\varsigma}^{\infty} dr \left\{ \frac{2\varsigma}{r^{4}} g_{+}^{-1}(Q,b_1) + \frac{3\varsigma^{2}}{r^{6}} g_{+}^{-1}(Q,b_2) + \frac{4\varsigma^{3}}{r^{8}} g_{+}^{-1}(Q,b_3) \right\}$$

The first integral here is the self energy of the induced field of K outside of K, the second is the interaction between Q and P, and the third is the interaction between Q and the induced field of K outside of K.

Inside, the radial portion is

IV-45 $$\int_{-1}^{1}\int_{0}^{k}\pi g_{-}\left(-\frac{\partial}{\partial r}\varphi_{-},-\frac{\partial}{\partial r}\varphi_{-}\right)r^{2}dr\,dx=\int_{0}^{k}\frac{1}{8\pi}\frac{dr}{r^{2}}g_{-}^{-1}(P,P)$$

$$+\int_{0}^{k}2\pi\,dr\left\{\frac{r^{2}}{3}g_{-}^{-1}(a_{1},a_{1})+\frac{4r^{4}}{5}g_{-}^{-1}(a_{2},a_{2})+\frac{9r^{6}}{7}g_{-}^{-1}(a_{3},a_{3})\right\}$$

and the angular portion is

IV-46 $$\int_{-1}^{1}\int_{0}^{k}\pi g_{-}\left(-\frac{1}{r}\frac{\partial}{\partial\vartheta}\varphi_{-},-\frac{1}{r}\frac{\partial}{\partial\vartheta}\varphi_{-}\right)r^{2}dr\,dx$$

$$=\int_{0}^{k}2\pi\,dr\left\{\frac{2}{3}r^{2}g_{-}^{-1}(a_{1},a_{1})+\frac{6}{5}r^{4}g_{-}^{-1}(a_{2},a_{2})+\frac{12}{7}r^{6}g_{-}^{-1}(a_{3},a_{3})\right\}$$

which combine to give

IV-47 $$\mathcal{U}_{-}=\int_{0}^{k}\frac{1}{8\pi r^{2}}dr\,g_{-}^{-1}(P,P)$$

$$+\int_{0}^{k}2\pi r\,dr\left\{r^{2}g_{-}^{-1}(a_{1},a_{1})+2r^{4}g_{-}^{-1}(a_{2},a_{2})+3r^{6}g_{-}^{-1}(a_{3},a_{3})\right\}$$

The last part to find before we get the total energy is $U(Q,-,g_{+})$. This we can do by integrating the field energy density of Q over K, using g_{+} instead of g_{-}. Using $\varphi=g_{+}^{-1}Q/4\pi r_{2}$,

IV-48 $$\mathcal{U}(Q,-,g_{+})=\int_{-1}^{1}\int_{0}^{k}\pi r^{2}dr\,dx\,g_{+}^{-1}(Q,Q)\frac{1}{16\pi^{2}}\left\{P_{1}^{2}\frac{1}{\zeta^{4}}+P_{2}^{2}\frac{4r^{2}}{\zeta^{6}}+P_{3}^{2}\frac{9r^{4}}{\zeta^{8}}\right\}$$

$$+\int_{-1}^{1}\int_{0}^{k}\pi r^{2}dr\,dx\,g_{+}^{-1}(Q,Q)\frac{(1-x^{2})}{16\pi^{2}}\left\{P_{1}'P_{1}'\frac{1}{\zeta^{4}}+P_{2}'P_{2}'\frac{r^{2}}{\zeta^{6}}+P_{3}'P_{3}'\frac{r^{4}}{\zeta^{8}}\right\}$$

$$=\int_{0}^{k}\frac{1}{8\pi}dr\,g_{+}^{-1}(Q,Q)\left\{\frac{r^{2}}{\zeta^{4}}+\frac{2r^{4}}{\zeta^{6}}+\frac{3r^{6}}{\zeta^{8}}\right\}$$

$$=\frac{1}{8\pi}g_{+}^{-1}(Q,Q)\left\{\frac{k^{3}}{3\zeta^{4}}+\frac{2k^{5}}{5\zeta^{6}}+\frac{3k^{7}}{7\zeta^{8}}\right\}$$

The total field energy, then, is

IV-49
$$\begin{aligned} \mathcal{U} = {} & \mathcal{U}(P) + \mathcal{U}(Q, \text{everywhere}, g_+) \\ & - \frac{1}{8\pi} g_+^{-1}(Q,Q) \left\{ \frac{k^3}{3\zeta^4} + \frac{2k^5}{5\zeta^6} + \frac{3k^7}{7\zeta^8} \right\} \\ & + \frac{1}{\zeta} g_+^{-1}(Q, b_0) \\ & + \left\{ \frac{2\zeta}{3\zeta^3} g_+^{-1}(Q,b_1) + \frac{3\zeta^2}{5\zeta^6} g_+^{-1}(Q,b_2) + \frac{4\zeta^3}{7\zeta^7} g_+^{-1}(Q,b_3) \right\} \end{aligned}$$

The following is a table evaluating the g(A,B) terms in IV-49. The subscripts 1 and 2 represent - and +, respectively, from the previous notation. In addition, the following abbreviations are used.

$$A_n \equiv [n(\mu_1 - \alpha_2\beta_1 + \gamma_2\gamma_1) - (n+1)(\mu_2 - \beta_2\alpha_1 - \gamma_2\gamma_1)]$$

$$B_n \equiv [n(\mu_1 - \beta_2\alpha_1 + \gamma_2\gamma_1) - (n+1)(\mu_2 - \alpha_2\beta_1 + \gamma_2\gamma_1)]$$

$$\dot{\alpha}_n \equiv [\alpha_2\gamma_1 - \gamma_2\alpha_1](2n+1)$$

$$\dot{\beta}_n \equiv [\beta_2\gamma_1 - \gamma_2\beta_1](2n+1)$$

$$C_n \equiv [\beta_2\alpha_1 - \gamma_2\gamma_1 + \frac{n\mu_1}{n+1}]$$

$$D_n \equiv [\alpha_2\beta_1 - \gamma_2\gamma_1 + \frac{n\mu_1}{n+1}]$$

$$\dot{\alpha} \equiv \dot{\alpha}_0 \qquad \dot{\beta} \equiv \dot{\beta}_0$$

$$g_+^{-1}(Q,Q) = \frac{1}{\mu_2}[\beta_2 Q_1^2 - 2\gamma_2 Q_1 Q_2 + \alpha_2 Q_2^2]$$

$$g_+^{-1}(Q,b_0) = \frac{1}{4\pi\mu_2}[\beta_2 P_1 Q_1 - \gamma_2(P_1 Q_2 + P_2 Q_1) + \alpha_2 P_2 Q_2]$$

$$\begin{aligned} g_+^{-1}(Q,b_n) = {} & -n k^{2n+1}\left(\mu_2 4\pi \zeta^{n+1}[n^2\mu_1 + (n+1)^2\mu_2 + n(n+1)(\alpha_2\beta_1 + \beta_2\alpha_1 - 2\gamma_2\gamma_1)]\right)^{-1} \\ & \{ Q_1^2[\beta_2 A_n - \gamma_2\dot{\beta}_n] + Q_1Q_2[\beta_2\dot{\alpha}_n - \gamma_2(B_n + A_n) + \alpha_2\dot{\beta}_n] \\ & + Q_2^2[-\gamma_2\dot{\alpha}_n + \alpha_2 B_n] \} \end{aligned}$$

$$g_{+}^{-1}(b_n, b_n) = -n^2 k^{4n+2}\left(\mu_2 16\pi \zeta^{2n+2}\left[n^2\mu_1 + (n+1)^2\mu_2 + n(n+1)(\alpha_2\beta_1 + \beta_2\alpha_1 - 2\gamma_2\gamma_1)\right]^2\right)^{-1}$$
$$\{Q_1^2[\beta_2 A_n^2 - 2\gamma_2 A_n\beta_n + \alpha_2\beta_n^2] + 2Q_1Q_2[\beta_2 A_n\dot{\alpha}_n - \gamma_2(A_nB_n + \dot{\alpha}_n\beta_n) + \alpha_2 B_n\beta_n] + Q_2^2[\beta_2\dot{\alpha}_n^2 - 2\gamma\dot{\alpha}_n B_n + \alpha_2 B_n^2]\}$$

$$g_{-}^{-1}(a_n, a_n) = (n+1)^2(2n+1)^2\left(\mu_1 16\pi \zeta^{2n-2}\left[n^2\mu_1 + (n+1)^2\mu_2 + n(n+1)(\alpha_2\beta_1 + \beta_2\alpha_1 - 2\gamma_2\gamma_1)\right]^2\right)^{-1}$$
$$\{Q_1^2[\beta_1 C_n^2 - 2\gamma_1\beta C_n + \alpha_1\beta^2] + 2Q_1Q_2[\beta_1\dot{\alpha}C_n - \gamma_1(C_nD_n + \dot{\alpha}\beta) + \alpha_1\beta C_n] + Q_2^2[\beta_1\dot{\alpha}^2 - 2\gamma_1\dot{\alpha}D_n + \alpha_1 D_n^2]\}$$

From this we get

$$\mathcal{U} = \mathcal{U}(P) + \mathcal{U}(Q, \text{everywhere}, g_+)$$
$$+ \frac{1}{4\pi\mu_2\zeta}[\beta_2 P_1 Q_1 - \gamma_2(P_1Q_2 + P_2Q_1) + \alpha_2 P_2 Q_2]$$
$$+ \frac{k^3}{\pi\zeta^4}\left\{-\left[\frac{\beta_2 Q_1^2 - 2\gamma_2 Q_1Q_2 + \alpha_2 Q_2^2}{24\mu_2}\right] - \left[\frac{Q_1^2[\beta_2 A_1 - 3\gamma_2\beta] + 2Q_1Q_2[3\beta_2\dot{\alpha} - \gamma_2(A_1 + B_1) + 3\alpha_2\beta] + Q_2^2[-3\gamma_2\dot{\alpha} + \alpha_2 B_1]}{6\mu_2[\mu_1 + 4\mu_2 + 2(\alpha_2\beta_1 + \beta_2\alpha_1 - 2\gamma_2\gamma_1)]}\right]\right.$$
$$+ \left[\frac{Q_1^2[\beta_2 A_1^2 - 6\gamma_2\beta A_1 + 9\alpha_2\beta^2] + 2Q_1Q_2[3\beta_2\dot{\alpha}A_1 - \gamma_2(A_1B_1 + 9\dot{\alpha}\beta) + 3\alpha_2\beta B_1] + Q_2^2[9\beta_2\dot{\alpha}^2 - 6\gamma_2\dot{\alpha}B_1 + \alpha_2 B_1^2]}{12\mu_2[\mu_1 + 4\mu_2 + 2(\alpha_2\beta_1 + \beta_2\alpha_1 - 2\gamma_2\gamma_1)]^2}\right]$$
$$\left. + \frac{3}{2}\left[\frac{Q_1^2[\beta_1 C_1^2 - 2\gamma_1\beta C_1 + \alpha_1\beta^2] + 2Q_1Q_2[\beta_1\dot{\alpha}C_1 - \gamma_1(\dot{\alpha}\beta + C_1D_1) + \alpha_1\beta D_1] + Q_2^2[\beta_1\dot{\alpha}^2 - 2\gamma_1\dot{\alpha}D_1 + \alpha_1 D_1^2]}{\mu_1[\mu_1 + 4\mu_2 + 2(\alpha_2\beta_1 + \beta_2\alpha_1 - 2\gamma_2\gamma_1)]^2}\right]\right\}$$
$$+ \frac{k^5}{\pi\zeta^6}\left\{-\left[\frac{\beta_2 Q_1^2 - 2\gamma_2 Q_1Q_2 + \alpha_2 Q_2^2}{20\mu_2}\right] - \frac{6}{20}\left[\frac{Q_1^2[\beta_2 A_2 - 5\gamma_2\beta] + Q_1Q_2[5\beta_2\dot{\alpha} - \gamma_2(A_2 + B_2) + 5\alpha_2\beta] + Q_2^2[-5\gamma_2\dot{\alpha} + \beta_2 B_2]}{\mu_2[4\mu_1 + 9\mu_2 + 6(\alpha_2\beta_1 + \beta_2\alpha_1 - 2\gamma_2\gamma_1)]}\right] + \frac{12}{40}\left[\frac{Q_1^2[\beta_2 A_2^2}{}\right.\right.$$

$$\frac{-10\gamma_1 A_1\dot{\beta} + 25\alpha_1\dot{\beta}^2] + 2Q_1G_1[5\beta_1 A_1\dot{\alpha} - \gamma_1(A_1 C_1 + 25\dot{\alpha}\dot{\beta})}{\mu_1[4\mu_1 + 9\mu_2 + 6(\alpha_1\beta_1 + \beta_1\alpha_1 - 2\gamma_1\gamma_1)]^2}$$

$$\frac{+5\alpha_1 D_1\dot{\beta}] - G_1^2[25\beta_1\dot{\alpha}^2 - 10\gamma_1\dot{\alpha}D_1 + \alpha_1 D_1^2]}{}\Bigg] + \frac{45}{4}\Bigg[\frac{G_1^2[\beta_1 C_1^2}{}$$

$$\frac{-2\gamma_1\dot{\beta}C_1 + \alpha_1\dot{\beta}^2] + 2Q_1G_1[\beta_1\dot{\alpha}C_1 - \gamma_1(\dot{\alpha}\dot{\beta} + C_1 D_1) + \alpha_1\dot{\beta}D_1]}{\mu_1[4\mu_1 + 9\mu_2 + 6(\alpha_1\beta_1 + \beta_1\alpha_1 - 2\gamma_1\gamma_1)]^2}$$

$$\frac{+Q_1^2[\beta_1\dot{\alpha}^2 - 2\gamma_1\dot{\alpha}D_1 + \alpha_1 D_1^2]}{}\Bigg]\Bigg\}$$

(In calculations with numbers, we would try to carry one or two more orders of magnitude than required for our answer, and then drop them at the end. The k^7/ζ^7 term, therefore, now goes into Limbo with its predecessor from section three.)

From the field energy we can now find the force between Q and $\{P,K\}$:

IV-50 $$F = -\frac{\partial}{\partial\zeta}\mathcal{U} = \frac{1}{\zeta^2}f_1 - \frac{4}{\zeta^5}f_4 + \frac{6}{\zeta^7}f_6$$

where f_n denotes the coefficient of $1/\zeta^n$ in 49. Now that we have the form of the force for our model, we can look for conditions in the model which would make the force zero. Let's go on then to our concluding section.

V.

CONCLUSION

As we found in section three, the $1/\zeta^2$ part of the force is dependent on the medium between the spheres, but one of our stated goals is to create a model with a $1/\zeta^2$ term that vanishes outside K (the kügel or sphere) but not inside. Borrowing from electrostatics, we'll quantize gauge field sources, then see what combinations of charges and metrics cause the $1/\zeta^2$ term of F to vanish. To simplify the process, we'll use ±1 and 0 as the source values[11].

First, we'll label the particles predicted by these values.

$$Q: \quad \underset{a^+}{\begin{bmatrix} 0 \\ +1 \end{bmatrix}} \quad \underset{a^-}{\begin{bmatrix} 0 \\ -1 \end{bmatrix}} \quad \underset{b^+}{\begin{bmatrix} +1 \\ 0 \end{bmatrix}} \quad \underset{b^-}{\begin{bmatrix} -1 \\ 0 \end{bmatrix}} \quad \underset{c^+}{\begin{bmatrix} +1 \\ +1 \end{bmatrix}} \quad \underset{c^-}{\begin{bmatrix} -1 \\ -1 \end{bmatrix}} \quad \underset{d^+}{\begin{bmatrix} +1 \\ -1 \end{bmatrix}} \quad \underset{d^-}{\begin{bmatrix} -1 \\ +1 \end{bmatrix}} \quad \underset{n}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

From section four, the $1/\zeta^2$ term is

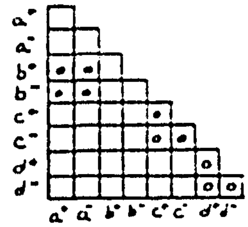
V-1 $$F_0 = \frac{1}{\zeta^2} f_1 = \frac{1}{4\pi\mu_2\zeta^2}\left[\beta_2 P_1 Q_1 - \gamma_2 (P_1 Q_2 + P_2 Q_1) + \alpha_2 P_2 Q_2\right]$$

If we choose $g_2 = \begin{bmatrix} \alpha_2 & \gamma_2 \\ \gamma_2 & \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, we can set up the following table of null/non-null forces F_0 between the particles.

---

[11] If the Q_1 and Q_2 chosen magnitudes are not equal, then they will be related by a constant. The necessary metrics for the equality and inequality cases, if they exist, will be related by a group space transformation involving that constant.

The position on the table is marked with a 0 if F_0 is null and left blank otherwise.



We see from the chart that c-type particles have shortened ranges in interacting with other c-types. The same is true of the d-types.

As to whether or not any combination of α, β, and γ can make all of the pairings null, from V-1 we see that the force will be null if the numerator is. The conditions which must be satisfied are listed here.

Interacting Particles	Numerator = 0
a^+a^+	α
a^+a^-	$-\alpha$
a^+b^+	$-\gamma$
a^+b^-	$+\gamma$
a^+c^+	$-\gamma+\alpha$
a^+c^-	$+\gamma-\alpha$
a^+d^+	$-\gamma-\alpha$
a^+d^-	$\gamma+\alpha$
a^-a^-	α
a^-b^+	γ
a^-b^-	$-\gamma$
a^-c^+	$\gamma-\alpha$
a^-c^-	$-\gamma+\alpha$

Interacting Particles	Numerator
a^-d^+	$+\alpha$
a^-d^-	$-\gamma+\alpha$
b^+b^+	β
b^+b^-	$-\beta$
b^+c^+	$\beta-\gamma$
b^+c^-	$-\beta+\gamma$
b^+d^+	$\beta+\gamma$
b^+d^-	$-\beta-\gamma$
b^-b^-	β
b^-c^+	$-\beta+\gamma$
b^-c^-	$\beta-\gamma$
b^-d^+	$-\beta-\gamma$
b^-d^-	$\beta+\gamma$
c^+c^+	$\beta-2\gamma+\alpha$
c^+c^-	$-\beta+2\gamma-\alpha$
c^+d^+	$\beta-\alpha$
c^+d^-	$-\beta+\alpha$
c^-c^-	$\beta-2\gamma+\alpha$
c^-d^+	$-\beta+\alpha$
c^-d^+	$\beta-\alpha$
d^+d^+	$\beta+2\gamma+\alpha$
d^+d^-	$-\beta-2\gamma-\alpha$
d^-d^-	$\beta+2\gamma+\alpha$

The only numbers for α, β, and γ which make all of these expressions zero is $\alpha=\beta=\gamma=0$, the trivial solution.

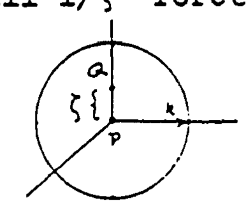
But that violates $\det(g)\neq 0$.

If we look at the c-types by themselves, the expression to be satisfied is $\beta-2\gamma+\alpha=0$. For the d-types, the expression is $\beta+2\gamma+\alpha=0$. To satisfy both requirements, γ must be null, and we get $\beta=-\alpha$. To get both of the c-types and d-types to cancel their F_0 terms we also need $\beta=\alpha$. Thus, we again get $\alpha=\beta=\gamma=0$.

If we choose $\gamma=0$ and $\beta=-\alpha$, allowing F_0 to be zero for the c-type and d-type interactions seperately, but drop the $\beta=\alpha$ requirement for c-type, d-type interactions, then we run into the problem that two particles that don't interact with each other can still interact with a third particle, e.g. two c-type particles and a d-type. This non-transitivity of the forces becomes a questionable alternative to the unacceptable null metric.

Rather than try to hold on to all of the particles that seemed possible, let's just work with the c-type particles and $g_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, since either the c-type or the d-type by itself would give us the desired cancellation of the $1/\zeta^2$ term because of g_2.

Now, since we only want to shorten the range of the $1/\zeta^2$ force, not eliminate it entirely, we should look within K at what can give us a non-null $1/\zeta^2$ force inside. As shown in the figure here, we'll make ζ smaller than k, bringing Q inside K and close



to P. For this setting the fields will be given by

$$g_1\varphi_1 = \frac{P}{4\pi r} + \frac{Q}{4\pi r_1} + \sum_{n=0}^{\infty} a_n r^n P_n(\cos\vartheta), \qquad r<k$$

$$g_2\varphi_2 = \sum_{n=0}^{\infty} \frac{b_n}{r^{n+1}} P_n(\cos\vartheta), \qquad r>k$$

We can now set up our boundary conditions at r = k and solve for the a_n and b_n.

$$\varphi_1 = \varphi_2 \;(r=k) \quad\Rightarrow\quad g_1^{-1}\left\{\frac{P}{4\pi k} + \sum_{n=0}^{\infty}\left[a_n k^n + \frac{Q\zeta^n}{4\pi k^{n+1}}\right]P_n(\cos\vartheta)\right\} = g_2^{-1}\left\{\sum_{n=0}^{\infty}\frac{b_n}{k^{n+1}}P_n(\cos\vartheta)\right\}$$

$$g_1\frac{\partial\varphi_1}{\partial r} = g_2\frac{\partial\varphi_2}{\partial r}\;(r=k) \quad\Rightarrow\quad -\frac{P}{4\pi k^2} + \sum_{n=0}^{\infty}\left[n a_n k^{n-1} - \frac{(n+1)Q\zeta^n}{4\pi k^{n+2}}\right]P_n(\cos\vartheta) = -\sum_{n=0}^{\infty}\frac{(n+1)b_n}{k^{n+2}}P_n(\cos\vartheta)$$

Identifying the coefficients of $P_n(\cos\vartheta)$, for n=0:

$$a_0 = \left[g_1 g_2^{-1} - 1\!\!1\right]\frac{(P+Q)}{4\pi k}$$

$$b_0 = \frac{P+Q}{4\pi}$$

and for n≥1:

$$a_n = -\frac{(n+1)\zeta^n}{4\pi k^{2n+1}}\left[(2n+1)g_1 g_2^{-1} + (2n+3)1\!\!1\right]Q$$

$$b_n = \left[n g_1 g_2^{-1} + (n+1)1\!\!1\right]^{-1}\frac{(2n+1)Q\zeta^n}{4\pi}$$

Using $P_n(1)=1$ and our symmetry, the force on Q is given by

$$F_Q = g_1\left(Q,\; -\frac{\partial}{\partial r}\left\{g_1^{-1}\left[\frac{P}{4\pi r} + \sum_{n=0}^{\infty} a_n r^n\right]\right\}_{r=\zeta}\right)$$

The portion of the force due to the presence of the sphere is $g_1(Q, -\Sigma n a_n \zeta^{n-1})$, and $a_n\zeta^{n-1}$ is of the order ζ^{2n-1}/k^{2n+1}. Thus, if $\zeta \ll k$ and the $1/\zeta^2$ force between Q and P is non-zero, we can essentially ignore the sphere and treat the interaction as though our environment is a uniform universe with $g = g_1$.

Let's let g_1 be a small change from g_2. That is

$$g_1 = \begin{bmatrix} 1+\sigma & \delta \\ \delta & -1+\rho \end{bmatrix}$$

where σ, ρ, and δ are small displacements from the vacuum value. The force between Q and P is essentially

$$F_- \cong \frac{1}{4\pi\mu_2 \zeta^2}\left[\rho P_1 Q_1 - \delta(P_1 Q_2 + P_2 Q_1) + \sigma P_2 Q_2\right] + F_0$$

$$= -\frac{1}{4\pi \zeta^2}\left[\rho P_1 Q_1 - \delta(P_1 Q_2 + P_2 Q_1) + \sigma P_2 Q_2\right]$$

since $\mu_1 \cong \mu_2 = -1$ and F_0 (from F outside) is null. The energy is essentially

$$E_- \cong -\frac{1}{4\pi\zeta}\left[\rho P_1 Q_1 - \delta(P_1 Q_2 + P_2 Q_1) + \sigma P_2 Q_2\right]$$

For the various c-type interactions we get the force values

c^+c^+: $\qquad -\frac{1}{4\pi\zeta^2}(\rho - 2\delta + \sigma)$

c^+c^-: $\qquad -\frac{1}{4\pi\zeta^2}(-\rho + 2\delta - \sigma)$

c^-c^-: $\qquad -\frac{1}{4\pi\zeta^2}(\rho - 2\delta + \sigma)$

The self-energy-density for any charge P is

$$\frac{1}{2} g_1\left(g_1^{-1}\frac{\partial}{\partial r}\frac{P}{4\pi r},\ g_1^{-1}\frac{\partial}{\partial r}\frac{P}{4\pi r}\right) = \frac{1}{2} g_-^{-1}\left(\frac{-P}{4\pi r^2},\ \frac{-P}{4\pi r^2}\right)$$

$$= -\frac{1}{32\pi^2 r^4}\left[\rho - 2\delta + \sigma\right]$$

Since we want a positive self-energy-density, $(\rho-2\delta+\sigma)$ must be negative. As long as this quantity is strictly negative for r << k, the forces will be repulsive for c^+c^+ and c^-c^- interactions, and attractive for c^+c^- interactions, and the self-energy-densities of the T_{44} term will be positive.

Using the d-types instead of the c-types, the quantity $(\rho+2\delta+\sigma)$ should be negative to insure a positive self--energy-density, and the interaction forces,

d^+d^+	$-\frac{1}{4\pi\zeta^2}(\rho+2\delta-\sigma)$
d^+d^-	$-\frac{1}{4\pi\zeta^2}(-\rho-2\delta-\sigma)$
d^-d^-	$-\frac{1}{4\pi\zeta^2}(\rho+2\delta+\sigma)$

will then follow the repulsive-attractive pattern of the c-type particles. So using either the c-type or d-type particles exclusively, we now have a scalar-field, gauge-field model with a $1/\zeta^2$ force that vanishes at large separation, reappears at small separation, and avoids the non-transitive forces.

But, with or without the non-transitive-force problem, this new model shows that the scalar fields predicted by the fibre bundle method can themselves produce significant short range effects for their attendant gauge fields.

LIST OF REFERENCES

Auslander, Louis, and Robert E. MacKenzie, Introduction to Differentiable Manifolds, (New York: Dover edition, 1977)

Bergmann, Peter G., "Comments on the Scalar-Tensor Theory," Int. Journ. Theor. Phys. 1, 25 (1968)

Chang, Lay Man, Kenneth I. Macrae, and Freydoon Mansouri, Phys Rev D 13, 235 (1976)

Cho, Y. M., "Higher-dimensional Unifications of Gravitation and Gauge Theories," Journal of Mathematical Physics 16, 2029 (1975)

Cho, Y. M., and P. G. O. Freund, Phys Rev D 12, 1711 (1975)

Daniel, M., and C. M. Viallet, Reviews of Modern Physics 52, 175 (1980)

Ehlers, Jürgen, "Zum Ubergang von der Wellenoptik zur geometrischen Optik in der allgemeinen Relativitätstheorie," Z. Naturforschg. 22a, 1328 (1967)

Eisenhart, L. P., Riemannian Geometry, (London: Princeton University Press, 1926)

Gordon, W., "Zur Lichtfortpflanzung nach der Relativitatstheorie," Ann. Phys. 72, 421 (1923)

Kobayashi, S., and K. Nomizu, Foundations of Differential Geometry, Vol. I, (London: Interscience Publishers, 1963)

Loos, Hendricus G., "The Range of Gauge Fields," Nuclear Physics 72, 677 (1965)

Misner, Charles W., Kip S. Thorne, and John Archibald Wheeler, Gravitation (San Francisco: W. H. Freeman, 1970)

Morse, P. M., and H. Feshbach, Methods of Theoretical Physics (New York: McGraw-Hill, 1953)

O'Hanlon, J. and B. O. J. Tupper, "Scalar-Tensor Theories and Conformal Invariance," Nuovo Cimento 14B, 190 (1973)

O'Hanlon, J. and B. O. J. Tupper, "Scalar-Tensor Theories and Conformal Invariance - II," Nuovo Cimento 17B

Trautman, A., "Fibre Bundles Associated with Space-Time," Reports on Mathematical Physics 1, 29 (1970)

Trautman, A., "The Geometry of Gauge Fields," Czech. J. Phys. B 29, 107 (1979)

Utiyama, Ryoyu, "Invarient Theoretical Interpretation of Interaction," Physical Review 101, 1597 (1956)

APPENDIX A

THE SYMMETRY ARGUMENT

This appendix looks at R. Utiyama's 1955 paper which showed how to start with innerspace symmetries (or invariances) and come up with gauge-fields associated with those symmetries. This segment will, for the most part, follow the development in Utiyama's paper, with some notational changes to maintain consistency within this paper.

Let's begin with a system of fields Q^A which is invarient under some transformation group which depends on parameters $\epsilon^1, \epsilon^2, \ldots, \epsilon^n$. That is, given a Lagrangian $L(Q^A, Q^A_{,a})$ and its action integral $I = \int_\Omega L\, d^4x$ where Ω is some arbitrary four-dimensional domain, we start with the invariance of the action integral under the transformation:

A-1: $Q^A \longrightarrow Q^A + Q^A$

$\delta Q^A = T^A_{\alpha B} \epsilon^\alpha Q^B$

ϵ^α = infinitesimal parameter ($\alpha = 1, \ldots, n$)

$T^A_{\alpha B}$ = constant coefficient

and this transformation is assumed to be a Lie group depending on the n parameters ϵ^α with structure constants $f^\alpha_{\beta\gamma}$ defined by $\left[T_\alpha, T_\beta\right]^A_B = T^A_{\alpha C} T^C_{\beta B} - T^A_{\beta C} T^C_{\alpha B} = f^\gamma_{\alpha\beta} T^A_{\gamma B}$. A couple of properties of these structure constants will be useful in our developments. From $[T_\alpha, T_\beta] = -[T_\beta, T_\alpha]$ we get:

A-2: $$f^{\gamma}_{\alpha\beta} = -f^{\gamma}_{\beta\alpha}$$

From the Jacobi identity $[T_\alpha,[T_\beta,T_\gamma]] + [T_\beta,[T_\gamma,T_\alpha]] + [T_\gamma,[T_\alpha,T_\beta]] = 0$ we get:

A-3: $$f^{\zeta}_{\alpha\beta} f^{\eta}_{\zeta\gamma} + f^{\zeta}_{\beta\gamma} f^{\eta}_{\zeta\alpha} + f^{\zeta}_{\gamma\alpha} f^{\eta}_{\zeta\beta} = 0$$

Since I is invarient under A-1 and the domain is arbitrary:

A-4: $$\delta L = \frac{\partial L}{\partial Q^A}\delta Q^A + \frac{\partial L}{\partial Q^A_{,m}}\delta Q^A_{,m} = 0,$$

where the last equality is an identity. This leads to

A-5: $$\frac{\partial L}{\partial Q^A} T^A_{\mu B}\epsilon^\mu Q^B + \frac{\partial L}{\partial Q^A_{,m}} T^A_{\mu B}\epsilon^\mu Q^B_{,m} = 0.$$

Since the ϵ^μ are independent, their coefficients must each be null. If we begin with each of these coefficients being null, we also have sufficient conditions for the invarience of I under the transformation group G.

Let's see what happens when we let the ϵ^μ become functions of position, and require that I also be invarient under these extended conditions. The transformation laws become:

A-1': $$\delta Q^A = T^A_{\mu B}\epsilon^\mu(x) Q^B$$

$$T^A_{\mu B} = \text{constant}$$

$$\epsilon^\mu(x) = \text{infinitesimal arbitrary function.}$$

Now we get an extra term in δL:

A-5': $$\delta L = \frac{\partial L}{\partial Q^A} T^A_{\mu B} Q^B \epsilon^\mu + \frac{\partial L}{\partial Q^A_{,m}} T^A_{\mu B} Q^B_{,m} \epsilon^\mu + \frac{\partial L}{\partial Q^A_{,m}} T^A_{\mu B} Q^B \epsilon^\mu_{,m}$$

The first two terms on the right hand side are the terms we had in A-5, and that identity must still hold, so δL becomes

A-5": $$\delta L = \frac{\partial L}{\partial Q^A_{,m}} T^A_{\mu B} Q^B \epsilon^\mu_{,m}$$

Let's introduce a new field, $A'^J(x)$, $J = 1,\ldots,M$, to try to get back to $\delta L = 0$.

Our new Lagrangian will be denoted $L'(Q^A, Q^A_{,a}, A'^J)$, and our transformations will be

A-6: $$\delta Q^A = T^A_{\mu B} Q^B \epsilon^\mu(x)$$
$$\delta A'^J = U^J_{\mu K} A'^K \epsilon^\mu(x) + C^{Ja}_\mu \epsilon^\mu_{,a}$$

where the U and C are constants to be determined later. The action integral I' for this new Lagrangian is to be invariant under A-6.

There are five questions which can now be answered: 1) What kind of field, A(x), is introduced on account of the invariance? 2) How does A(x) transform under G', the extended Lie group of transformations which depends on the functions $\epsilon^\mu(x)$? 3) What is the form of the interaction between the fields A and Q? 4) How can we determine the new Lagrangian, L'(Q,A), from the original one, L(Q)? 5) What kind of field equations are allowable for A?

As with A-4 for $L(Q^A, Q^A_{,a})$, for $L'(Q^A, Q^A_{,a}, A'^J)$ we have

A-7: $\delta L' = \frac{\partial L'}{\partial Q^A}\delta Q^A + \frac{\partial L'}{\partial Q^A_{,a}}\delta Q^A_{,a} + \frac{\partial L'}{\partial A'^J}\delta A'^J = 0$

Using A-6 and the independence of the ϵ^μ and $\epsilon^\mu_{,a}$, we get from A-7:

A-8: $\frac{\partial L'}{\partial Q^A}T^A_{\mu B}Q^B + \frac{\partial L'}{\partial Q^A_{,a}}T^A_{\mu B}Q^B_{,a} + \frac{\partial L'}{\partial A'^J}U^J_{\mu K}A'^K = 0$

and

A-9: $\frac{\partial L'}{\partial Q^A_{,a}}T^A_{\mu B}Q^B + \frac{\partial L'}{\partial A'^J}C^{Ja}_{\mu} = 0$

From A-9 we see that M must equal 4n in order to determine uniquely the A'^J-dependence of L'. Also, C^{Ja}_{μ} must be nonsingular. Its inverse, $C^{-1\mu}_{aJ}$, is defined by:

$C^{Ja}_{\mu}C^{-1\mu}_{aK} = \delta^J_K$ and $C^{-1\mu}_{aJ}C^{Jb}_{\nu} = \delta^\mu_\nu \delta^b_a$.

If we define A^μ_a as $C^{-1\mu}_{aJ}A'^J$, then

$\frac{\partial L'}{\partial A'^J} = \frac{\partial L'}{\partial A^\mu_a}\frac{\partial A^\mu_a}{\partial A'^J} = \frac{\partial L'}{\partial A^\mu_a}C^{-1\mu}_{aJ}.$

This lets us rewrite A-9 as

A-9': $\frac{\partial L'}{\partial Q^A_{,a}}T^A_{\mu B}Q^B + \frac{\partial L'}{\partial A^\nu_b}C^{-1\nu}_{bJ}C^{Ja}_{\mu} = \frac{\partial L'}{\partial Q^A_{,a}}T^A_{\mu B}Q^B + \frac{\partial L'}{\partial A^\mu_a} = 0$

Notice that this is $\frac{\partial L'}{\partial \nabla_a Q^A}$ when we define the function $\nabla_a Q^A$ by

A-10: $\nabla_a Q^A = Q^A_{,a} - T^A_{\mu B}A^B C^{-1\mu}_{aJ}A'^J = Q^A_{,a} - T^A_{\mu B}Q^B A^\mu_a.$

Our new field should only show up in L' through this $\nabla_a Q^A$.

The transformation property of this A^μ_a is

A-11: $\delta A^\mu_a = C^{-1\mu}_{aJ}U^J_{\nu K}A'^K\epsilon^\nu(x) + C^{-1\mu}_{aJ}C^{Jb}_{\nu}\epsilon^\nu_{,b}$

$$= C^{-1\mu}{}_{aJ} U^J_{\nu K} C^{Kb}{}_{\lambda} (C^{-1\lambda}{}_{bL} A^{,L}) \epsilon^{\nu}(x) + \epsilon^{\mu}_{,a}$$

$$= S^{\mu b}_{\nu a \lambda} (A^{\lambda}_{b}) \epsilon^{\nu} + \epsilon^{\mu}_{,a}$$

It turns out that this new function $S^{\mu b}_{\nu a \lambda}$ is easier to deal with than the U and C functions.

From the requirement $L'(Q^A, Q^A_{,a}, A^{\mu}_a) = L''(Q^A, \nabla_a Q^A)$ we get

$$\frac{\partial L'}{\partial Q^A} = \left.\frac{\partial L''}{\partial Q^A}\right|_{\nabla Q \text{ const}} + \left.\frac{\partial L''}{\partial \nabla_a Q^B}\right|_{Q \text{ const}} (-T^B_{\mu A} A^{\mu}_a)$$

$$\frac{\partial L'}{\partial Q^A_{,a}} = \left.\frac{\partial L''}{\partial \nabla_a Q^A}\right|_{Q \text{ const}}$$

$$\frac{\partial L'}{\partial A^{,J}} = \left.\frac{\partial L''}{\partial \nabla_a Q^A}\right|_{Q \text{ const}} (-T^A_{\mu B} Q^B C^{-1\mu}{}_{aJ})$$

A-8 now becomes

A-12:

$$\left\{ \left.\frac{\partial L''}{\partial Q^A}\right|_{\nabla Q} - \left.\frac{\partial L''}{\partial \nabla_m Q^C}\right|_{Q} T^C_{\beta A} A^{\beta}_m \right\} T^A_{\alpha B} Q^B + \left\{\frac{\partial L''}{\partial \nabla_m Q^A}\right\} T^A_{\alpha B} Q^B_{,m}$$
$$- \left\{ \left.\frac{\partial L''}{\partial \nabla_m Q^A}\right|_{Q} T^A_{\beta B} Q^B C^{-1\beta}{}_{mJ} \right\} U^J_{\alpha K} A^{,K}$$
$$= \left\{ \left.\frac{\partial L''}{\partial Q^A}\right|_{\nabla Q} T^A_{\alpha B} Q^B \right\} + \left\{ \left.\frac{\partial L''}{\partial \nabla_m Q^A}\right|_{Q} T^A_{\alpha B} Q^B_{,m} \right\}$$
$$+ \left\{ \left.\frac{\partial L''}{\partial \nabla_m Q^A}\right|_{Q} Q^B A^{\alpha}_n \left[[T_\alpha, T_\delta]^A_B \delta^n_m - S^{\delta n}_{\alpha m \beta} T^A_{\delta B} \right] \right\}$$
$$= 0$$

Since we want this new Lagrangian to be identical to the old one when A vanishes, we have $L''(Q^A, \nabla_a Q^A) = L(Q^A, \nabla_a Q^A)$ where $Q^A_{,a}$ in the old Lagrangian is replaced by $\nabla_a Q^A$. Because A-5 still holds, the first two terms on the right hand side of A-12 must combine to give zero. The remaining term gives us

<u>A-13</u>: $$\frac{\partial L''}{\partial \nabla_b Q^A}\bigg|_Q Q^B A^\mu_a T^A_{\alpha B}\left[f^\alpha_{\gamma\mu} \delta^a_b - S^{\alpha a}_{\mu b \gamma}\right] = 0$$

which implies that $S^{\delta n}_{\alpha m \beta} = f^{\delta}_{\alpha\beta}\, \delta^n_m$. From this we can get:

<u>A-14</u>: $$\begin{aligned}\delta \nabla_m Q^A &= T^A_{\alpha B}\left[Q^B_{,m}\epsilon^\alpha + Q^B \epsilon^\alpha_{,m}\right] - T^A_{\alpha B}\left\{T^B_{\beta D} Q^D A^\alpha_m \epsilon^\beta \right.\\ &\qquad \left. + Q^B(\delta^n_m f^\alpha_{\gamma\beta} A^\beta_n \epsilon^\gamma + \epsilon^\alpha_{,m})\right\} \\ &= T^A_{\alpha B}\epsilon^\alpha(x)\left[Q^B_{,m} - T^B_{\gamma D} Q^D A^\gamma_m\right] \\ &= T^A_{\alpha B}\epsilon^\alpha(x)\nabla_m Q^B\end{aligned}$$

using $f^\beta_{\gamma\alpha} T^A_{\beta B} = T^A_{\gamma D} T^D_{\alpha B} - T^A_{\alpha D} T^D_{\gamma B}$ to cancel and collect some terms.

The next step is to look at the free-field Lagrangian for A, $L_o(A^\alpha_a, A^\alpha_{a,b})$. Invariance under A-11 gives us, by way of

$$\frac{\partial L_o}{\partial A^\alpha_m}\delta A^\alpha_m + \frac{\partial L_o}{\partial A^\alpha_{m,n}}\delta A^\alpha_{m,n} = 0$$

and the independence of the ϵ^α, $\epsilon^\alpha_{,a}$, and $\epsilon^\alpha_{,ab}$,

$$\frac{\partial L_o}{\partial A^\alpha_a}(f^\alpha_{\gamma\beta} A^\beta_a \epsilon^\gamma + \epsilon^\alpha_{,a}) + \frac{\partial L_o}{\partial A^\alpha_{a,b}}(f^\alpha_{\gamma\beta}(A^\beta_{a,b}\epsilon^\gamma + A^\beta_a \epsilon^\gamma_{,b}) + \epsilon^\alpha_{,ab})$$

must be null, and

<u>A-15</u>: $$\frac{\partial L_o}{\partial A^\alpha_a} f^\alpha_{\gamma\beta} A^\beta_a + \frac{\partial L_o}{\partial A^\alpha_{a,b}} f^\alpha_{\gamma\beta} A^\beta_{a,b} = 0$$

<u>A-16</u>: $$\frac{\partial L_o}{\partial A^\alpha_m} + \frac{\partial L_o}{\partial A^\beta_{n,m}} f^\beta_{\alpha\delta} A^\delta_n = 0$$

<u>A-17</u>: $$\frac{\partial L_o}{\partial A^\alpha_{m,n}} + \frac{\partial L_o}{\partial A^\alpha_{n,m}} = 0$$

A-17 comes from the coefficient of the $\epsilon^{\alpha}_{,mn}$. Since the contraction of a symmetric tensor with an anti-symmetric is automatically null, we can only say that the symmetric portion of $\frac{\partial L_o}{\partial A^{\alpha}_{m,n}}$ must be null. So if $A^{\alpha}_{m,n}$ does show up in L_o, it must be through the combination:

$$A^{\alpha}_{[m,n]} = \partial_m A^{\alpha}_n - \partial_n A^{\alpha}_m .$$

Thus, from A-16 we get

<u>A-16'</u>: $$\frac{\partial L_o}{\partial A^{\alpha}_m} - \frac{\partial L_o}{\partial A^{\beta}_{[n,m]}} f^{\beta}_{\alpha\gamma} A^{\gamma}_n = 0$$

which implies that A^{α}_m and $A^{\alpha}_{m,n}$ appear in L_o only through the particular combination

<u>A-18</u>: $$F^{\alpha}_{mn} = A^{\alpha}_{[m,n]} - \tfrac{1}{2} f^{\alpha}_{\beta\gamma}(A^{\beta}_m A^{\gamma}_n - A^{\beta}_n A^{\gamma}_m)$$

so that we get, looking at the coefficients of ϵ^{α} in $\delta L_o = 0$,

$$\frac{1}{2}\frac{\partial L_o}{\partial F^{\gamma}_{mn}}\left\{ f^{\gamma}_{\alpha\beta} A^{\beta}_{[m,n]} - \frac{1}{2} f^{\gamma}_{\beta\delta}(A^{\beta}_m f^{\delta}_{\alpha\nu} A^{\nu}_n + A^{\delta}_n f^{\beta}_{\alpha\nu} A^{\nu}_m - A^{\beta}_n f^{\delta}_{\alpha\nu} A^{\nu}_m - A^{\delta}_m f^{\beta}_{\alpha\nu} A^{\nu}_n)\right\} = 0$$

$$= \frac{1}{2}\frac{\partial L_o}{\partial F^{\gamma}_{mn}}\left\{ f^{\gamma}_{\alpha\beta} A^{\beta}_{[m,n]} - \frac{1}{2}(A^{\beta}_m A^{\delta}_n - A^{\beta}_n A^{\delta}_m)\left[f^{\gamma}_{\beta\epsilon} f^{\epsilon}_{\alpha\delta} - f^{\gamma}_{\delta\epsilon} f^{\epsilon}_{\alpha\beta}\right]\right\}$$

This, by virtue of A-3, can be shortened to,

<u>A-19</u>: $$\tfrac{1}{2}\frac{\partial L_o}{\partial F^{\gamma}_{mn}} f^{\gamma}_{\alpha\beta}\left\{ A^{\beta}_{[m,n]} - \tfrac{1}{2} f^{\beta}_{\epsilon\delta}(A^{\epsilon}_m A^{\delta}_n - A^{\epsilon}_n A^{\delta}_m)\right\}$$

$$= \frac{\partial L_o}{\partial F^{\gamma}_{mn}} f^{\gamma}_{\alpha\beta} F^{\beta}_{mn} = 0$$

Since we want L_o' to have the same form as L_o, we get the relations:

$$\left.\frac{\partial L_o}{\partial A^{\alpha}_{m,n}}\right|_{A\ \text{const}} = \left.\frac{\partial L'_o}{\partial F^{\alpha}_{mn}}\right|_{A\ \text{const}}$$

$$\left.\frac{\partial L_o}{\partial A^{\alpha}_m}\right|_{\frac{\partial A}{\partial x}\ \text{const}} = \left.\frac{\partial L'_o}{\partial A^{\alpha}_m}\right|_{F\ \text{const}} + \left.\frac{\partial L'_o}{\partial F^{\beta}_{mn}}\right|_{A\ \text{const}}(-f^{\beta}_{\alpha\delta}A_n)$$

Since $A^{\alpha}_{m,n}$ shows up in L'_o only through F^{α}_{mn}, we can substitute $\partial L'_o/\partial F^{\beta}_{nm}$ for $\partial L'_o/\partial A^{\beta}_{[m,n]}$ in A-16' to get

$$\frac{\partial L'_o}{\partial A^{\alpha}_m} + \frac{\partial L'_o}{\partial F^{\beta}_{nm}} f^{\beta}_{\alpha\gamma}A^{\gamma}_n = 0$$

which gives us

$$\left.\frac{\partial L'_o}{\partial A^{\alpha}_m}\right|_{F\ \text{const}} = 0$$

This implies that L_o is a function of F by itself and must satisfy A-19.

The transformation property of F^{α}_{mn} is

$$\begin{aligned}
\delta F_{mn} &= \partial_m(f^{\alpha}_{\beta\gamma}\epsilon^{\beta}A^{\gamma}_n + \epsilon^{\alpha}_{,n}) - \partial_n(f^{\alpha}_{\beta\gamma}\epsilon^{\beta}A^{\gamma}_m + \epsilon^{\alpha}_{,m}) \\
&\quad -\tfrac{1}{2}f^{\alpha}_{\beta\gamma}\Big\{(f^{\beta}_{\mu\nu}\epsilon^{\mu}A^{\nu}_m + \epsilon^{\beta}_{,m})A^{\gamma}_n + A^{\beta}_m(f^{\gamma}_{\mu\nu}\epsilon^{\mu}A^{\nu}_n + \epsilon^{\gamma}_{,n}) \\
&\qquad -(f^{\beta}_{\mu\nu}\epsilon^{\mu}A^{\nu}_n + \epsilon^{\beta}_{,n})A^{\gamma}_m - A^{\beta}_n(f^{\gamma}_{\mu\nu}\epsilon^{\mu}A^{\nu}_m + \epsilon^{\gamma}_{,m})\Big\} \\
&= f^{\alpha}_{\beta\gamma}\epsilon^{\beta}(\partial_m A^{\gamma}_n - \partial_n A^{\gamma}_m) + f^{\alpha}_{\beta\gamma}\Big\{A^{\gamma}_n\epsilon^{\beta}_{,m} - A^{\gamma}_m\epsilon^{\beta}_{,n} - \tfrac{1}{2}A^{\gamma}_n\epsilon^{\beta}_{,m} \\
&\qquad - \tfrac{1}{2}A^{\beta}_m\epsilon^{\gamma}_{,n} + \tfrac{1}{2}A^{\gamma}_m\epsilon^{\beta}_{,n} + \tfrac{1}{2}A^{\beta}_n\epsilon^{\gamma}_{,m}\Big\} \\
&\qquad - \tfrac{1}{2}\epsilon^{\mu}\Big\{(A^{\zeta}_m A^{\eta}_n - A^{\zeta}_n A^{\eta}_m)(f^{\alpha}_{\beta\eta}f^{\beta}_{\mu\zeta} + f^{\alpha}_{\zeta\beta}f^{\beta}_{\mu\eta})\Big\} \\
&= f^{\alpha}_{\beta\gamma}\epsilon^{\beta}A^{\gamma}_{[m,n]} - \tfrac{1}{2}\epsilon^{\beta}\Big\{(A^{\zeta}_m A^{\eta}_n - A^{\zeta}_n A^{\eta}_m)(f^{\alpha}_{\beta\gamma}f^{\gamma}_{\zeta\eta} + f^{\alpha}_{\zeta\gamma}f^{\gamma}_{\eta\beta} \\
&\qquad + f^{\alpha}_{\zeta\gamma}f^{\gamma}_{\beta\eta})\Big\} \\
&= f^{\alpha}_{\beta\gamma}\epsilon^{\beta}\Big\{A^{\gamma}_{[m,n]} - \tfrac{1}{2}f^{\gamma}_{\zeta\eta}(A^{\zeta}_m A^{\eta}_n - A^{\zeta}_n A^{\eta}_m)\Big\}
\end{aligned}$$

A-20: $\quad = f^{\alpha}_{\beta\gamma}\epsilon^{\beta}F^{\gamma}_{mn}$

where use has been made of A-2 and A-3.

If we look at $L_T = L_o + L(Q,\nabla Q)$, we can get the variation:

A-21:
$$\begin{aligned}\delta L_T &= \frac{\partial L_T}{\partial Q^A}\delta Q^A + \frac{\partial L_T}{\partial Q^A_{,m}}\delta Q^A_{,m} + \frac{\partial L_T}{\partial A^{\alpha}_m}\delta A^{\alpha}_m + \frac{\partial L_T}{\partial A^{\alpha}_{m,n}}\delta A^{\alpha}_{m,n} \\
&= \left(\frac{\partial L_T}{\partial Q^A} - \frac{\partial}{\partial x^a}\left(\frac{\partial L_T}{\partial Q^A_{,a}}\right)\right)\delta Q^A + \left(\frac{\partial L_T}{\partial A^{\alpha}_m} - \frac{\partial}{\partial x^n}\left(\frac{\partial L_T}{\partial A^{\alpha}_{m,n}}\right)\right)\delta A^{\alpha}_m \\
&\quad + \frac{\partial}{\partial x^m}\left(\frac{\partial L_T}{\partial Q^A_{,m}}\delta Q^A + \frac{\partial L_T}{\partial A^{\alpha}_{n,m}}\delta A^{\alpha}_n\right) \\
&= \frac{\delta L_T}{\delta Q^A}\delta Q^A + \frac{\delta L_T}{\delta A^{\alpha}_m}(f^{\alpha}_{\beta\gamma}\epsilon^{\beta}A^{\gamma}_m + \epsilon^{\alpha}_{,m}) \\
&\quad + \frac{\partial}{\partial x^m}\left\{\frac{\partial L_T}{\partial Q^A_{,m}}\delta Q^A + \frac{\partial L_T}{\partial A^{\alpha}_{n,m}}\delta A^{\alpha}_n\right\} \\
&= \frac{\delta L_T}{\delta Q^A}\delta Q^A + \frac{\delta L_T}{\delta A^{\alpha}_m} f^{\alpha}_{\beta\gamma}A^{\gamma}_m\epsilon^{\beta} - \frac{\partial}{\partial x^m}\left(\frac{\partial L_T}{\partial A^{\alpha}_m}\right)\epsilon^{\alpha} \\
&\quad + \frac{\partial}{\partial x^m}\left\{\frac{\partial L_T}{\partial \nabla_m Q^A}\delta Q^A + \frac{\partial L_T}{\partial F^{\alpha}_{nm}}\delta A^{\alpha}_n + \frac{\partial L_T}{\partial A^{\alpha}_m}\epsilon^{\alpha}\right\}\end{aligned}$$

Since the $\epsilon^{\alpha}(x)$, and their derivatives, should vanish at the boundary of Ω, when we integrate A-21 over Ω the divergence drops out. But the integration of A-21 over Ω is the variance of the action integral, which is zero, and since Ω is arbitrary, A-21 must be zero everywhere. We thus get the identity from A-21:

A-22:
$$\frac{\delta L_T}{\delta Q^A}\delta Q^A + \frac{\delta L_T}{\delta A^{\alpha}_m} f^{\alpha}_{\beta\gamma}\epsilon^{\beta}A^{\gamma}_m - \frac{\partial}{\partial x^m}\left(\frac{\delta L_T}{\delta A^{\alpha}_m}\right)\epsilon^{\alpha} = 0$$

In the case of electromagnetics, we want the Lagrangian to be invarient under the phase transition: $\delta Q^A = i\alpha Q^A$ and $\delta Q^{*A} = -i\alpha Q^{*A}$, where α is a real constant. If we replace α with the function $\lambda(x)$, we should get a vector field $A_m(x)$ with the transformation property $\delta A_m = \partial\lambda/\partial x^m$. (Since there is only one parameter, the structure constants are null.) The new Lagrangian L' has the form $L' = L(Q, \nabla_m Q, Q^*, (\nabla_m Q)^*)$ where $\nabla_m Q^A = Q^A_{,m} - iA_m Q^A$. The free field Lagrangian L_o contains A_m in the form of $F_{mn} = A_{[m,n]}$.

For rotation in isotopic spin space, the transformations are:

$$\delta \psi^a = i \Sigma_{\gamma=1}^3 \epsilon^\gamma \tau_\gamma{}^a{}_b \psi^b$$

and

$$\delta \bar{\psi}_a = -i \Sigma_{\gamma=1}^3 \epsilon^\gamma \bar{\psi}_b \tau_\gamma{}^b{}_a$$

where the τ_γ are the isotopic spin matrices. By replacing the ϵ^α with $\epsilon^\alpha(x)$ we introduce the fields B^α_m (with $\alpha=1,2,3$; $m=1,2,3,4$) which show up in the Lagrangian through

$$\nabla_m \psi^a = \psi^a_{,m} - i\tau_\gamma{}^a{}_b \psi^b B^\gamma_m.$$

With the $f^\alpha_{\beta\gamma}$ defined by $[i\tau_\beta, i\tau_\gamma] = f^\alpha_{\beta\gamma} i\tau_\alpha$, we can get $F^\alpha_{mn} = (\partial_m B^\alpha_n - \partial_n B^\alpha_m) - \frac{1}{2} f^\alpha_{\beta\gamma}(B^\beta_m B^\gamma_n - B^\beta_n B^\gamma_m)$, which is the usual form for Yang-Mills fields.

APPENDIX B

THE FIBRE BUNDLE METHOD

We'll begin this section on the fibre bundle method with some notation conventions. Let's start with h as a differentiable mapping from a differentiable manifold M to a differentiable manifold N.

$$h: M \to N$$

At p in M we have the tangent space, denoted $T_p(M)$, and the corresponding tangent space $T_{h(p)}(N)$ at h(p). The space of all vectors tangent to M is denoted simply T(M). These are related by the induced mapping $h_*: T_p(M) \to T_{h(p)}(N)$, defined by $(h_*X)_{h(p)} \cdot g = X_p(g \cdot h)$, where g is a real-valued function on N and X_p is a tangent vector in $T_p(M)$, and h_* is called the linear differential of h. The dual to the tangent space at p is denoted $T_p^*(M)$, and the relation between the dual spaces is given by $h^*: T^*(N) \to T^*(M)$, defined by $h^*\omega(X) = \omega(h_*X)$, where $\omega \epsilon T^*(N)$ and $X \epsilon T(M)$. In summary:

$$\begin{array}{lcl} M \xrightarrow{h} N & \qquad & p \longrightarrow h(p) \\ T(M) \xrightarrow{h_*} T(N) & \qquad & X \cdot f \longrightarrow (h_*X)g = X(g \cdot h) \\ T^*(M) \xleftarrow{h^*} T^*(N) & \qquad & h^*\omega(X) = \omega(h_*X) \longleftarrow \omega Y \end{array}$$

where f is a function on M, g is a function on N, $X \epsilon T(M)$,

$Y \in T(N)$, $\omega \in T^*(N)$.

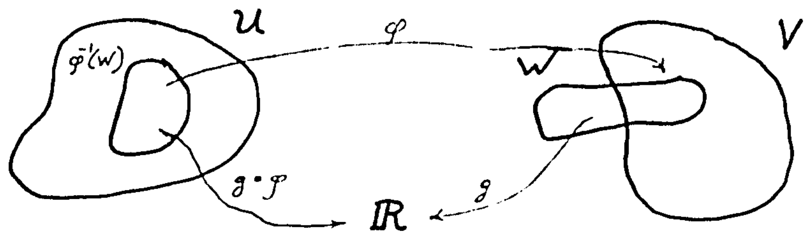
Let M be a Housdorff topological space with a denumerable basis and E^n be an n-dimensional Euclidean space. M is a differentiable manifold[1] if there exists an indexed collection of pairs $\{(W_\alpha, \eta_\alpha)\}$, W_α an open subset of E^n, η_α: $W_\alpha \to M$ a homeomorphism of W_α to an open subset U_α of M, satisfying:

a) for each $m \in M$ there exists α such that $m \in U_\alpha$,

b) for every α and β with $U_\alpha \cap U_\beta \neq \emptyset$, $\eta_\beta^{-1} \cdot \eta_\alpha$ restricted to $\eta_\alpha^{-1}(U_\alpha \cap U_\beta)$ is a differentiable mapping of this set back into E^n.

c) completeness: if η: $W \to U$ is a homeomorphism of an open subset W of E^n to an open subset U of M such that for any α for which $U \cap U_\alpha \neq \emptyset$ the restriction of $\eta^{-1} \cdot \eta_\alpha$ to $\eta_\alpha^{-1}(U \cap U_\alpha)$ and the restriction of $\eta_\alpha^{-1} \cdot \eta$ to $\eta^{-1}(U \cap U_\alpha)$ are differentiable mappings, then there exists an index β such that $(W, \eta) = (W_\beta, \eta_\beta)$.

A mapping φ: $U \to V$, for U open in E^n and V open in E^n is differentiable on U if for all functions g, differentiable on W, open in E^n, the composite function $g \cdot \varphi$ is differentiable on $\varphi^{-1}(W)$.



[1] See Louis Auslander and Robert E. MacKenzie, Introduction to Differentiable Manifolds, (New York, Dover, 1977), Chap. 2.

A vector is defined in terms of the directional derivative: Given a differentiable curve $u:(-1,1)\to M$ which passes through $p = u(0)$, and f, a locally differentiable function, then the vector X tangent to the curve $u(t)$ is defined by

$$X_p(f) = \frac{d}{dt} f(u(t))|_{t=0}, \qquad t \in (-1,1).$$

The set of differentiable curves through p are associated with the vectors tangent to M at p and vice versa. Given the basis $\{u^i\}$ from a coordinate chart (W_α, η_α) at $p \in \eta_\alpha(W_\alpha)$, we define a natural basis for these tangent vectors as $\frac{\partial}{\partial u^i}$, $i = 1,\ldots,n$, defined by $\frac{\partial}{\partial u^i}(u^j) = \delta^j_i$, so that $X = \sum_i f^i \frac{\partial}{\partial u^i}$.

The set of tangent vectors at p is an n-dimensional vector space denoted $T_p(M)$. The space of linear functions $\omega_p : T_p(M)\to R$ is the dual space $T^*_p(M)$. A 1-form is an assignment of duals, or covectors, at each point of M. Given the definition of a total differential of h as $dh(X) = X\cdot h$, and a local neighborhood coordinatized with u^i, a local basis for $T^*_p(M)$ can be developed from the total differentials of the u^i, allowing us to write any 1-form ω (locally) as $\sum_i f_i du^i$. As with the vectors, we assume that all of the vectors and forms are differentiable unless stated otherwise.

The exterior algebra over $T^*_p(M)$ is denoted $\Lambda T^*_p(M)$. For the 1-forms α and β, their wedge product is defined by

$$\alpha\wedge\beta\ (X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X).$$

For γ a p-form and δ a q-form, $\gamma\wedge\delta = (-1)^{pq}\delta\wedge\gamma$. A p-form, using the local basis du^i, can be expressed as

$$\gamma = \frac{1}{r!}\sum_{i_1,\ldots,i_r} du^{i_1}\wedge\ldots\wedge du^{i_r}\; g_{i_1,\ldots,i_r}$$
$$= \sum_{i_1<\cdots<i_r} g_{i_1\ldots i_r}\, du^{i_1}\wedge\ldots\wedge du^{i_r}$$

The notation for the last term needs to be modified a little in order to use the summation convention. One way is to use vertical lines to enclose the subscripts to indicate that the summation, as in $A_{|i_1\cdots i_n|}du^{i_1}\ldots du^{i_n}$, is restricted to $i_1 < i_2 < \cdots < i_n$.

If we use the notation $\mathcal{D}^r(M)$ to denote the set of r-forms on M, $\mathcal{D}^0(M)$ being the set of functions on M, and $\mathcal{D}(M) \equiv \sum_{r=0}^{n}\mathcal{D}^r(M)$, the exterior differentiation can be characterized by:

a) d is an R-linear mapping of $\mathcal{D}(M)$ into itself such that $d(\mathcal{D}^r) \subseteq \mathcal{D}^{r+1}$

b) for $f \in \mathcal{D}^0$, df is the total differential

c) if $\omega \in \mathcal{D}^r$ and $\pi \in \mathcal{D}^s$, then $d(\omega\wedge\pi) = d\omega\wedge\pi + (-1)^r\omega\wedge d\pi$

d) $d^2 = 0$.

In a local coordinate system, if $\omega = \sum_{i_1<\cdots<i_n} f_{i_1\cdots i_n} du^{i_1}\wedge\ldots\wedge du^{i_n}$, then $d\omega = \sum_{i_1<\cdots<i_n} df_{i_1\cdots i_n} du^{i_1}\wedge\ldots\wedge du^{i_n}$.

If we are looking at values in an arbitrary vector space V, rather than just R, as we will be with Lie-algebra--valued forms, then we define a V-valued r-form ω on M as an assignment to each $p \in M$ of a skew-symmetric r-linear mapping of $T_p(M)\times\ldots\times T_p(M)$, r-times, into V. Given a basis $\{e_i\}$ of V, we can express ω as $\sum_i \omega^i e_i$, where the ω^i are the usual r-forms on M. The exterior derivative is now defined by $d\omega = \sum_i d\omega^i e_i$.

A Lie group can be described as a group which is at the same time a differentiable manifold and for which the group operation $\lambda: G\times G \to G$, defined by $\lambda(a,b) = a^{-1}b$, is a differentiable map. We could also say, more by way of example, that it is a continuous group in which one can introduce an n-dimensional co-ordinate system ξ^i with the identity element at the origin, and with the multiplication law given by analytical functions. For instance, given x, y, and z in G with coordinates ξ^α, η^α, and ζ^α respectively, the equation for z = xy can be written as $\zeta^\alpha = f^\alpha(\xi,\eta)$ where the f^α are n analytic functions of the 2n variables ξ^α, η^α.

Let's look for a moment at some differentiable curves through the origin, curves whose coordinates depend differentiably on a parameter ε and chosen so that x(0) = e, the identity of G. If $x(\varepsilon)$ is a subgroup of G, we say that $\left(\frac{\partial x}{\partial \varepsilon}\right)\Big|_{\varepsilon=0}$ generates a one-parameter subgroup, $g(\varepsilon)$ of G. For instance, rotations about the z-axis form a one-parameter subgroup, and ε can be chosen to be ϑ, $\sin\vartheta$, or any other appropriate function of the angle. If $\varepsilon=\vartheta$, then $g(\varepsilon_1)g(\varepsilon_2) = g(\varepsilon_1+\varepsilon_2)$, and all one-parameter subgroups can be expressed (or re-expressed, with a suitable parameter) in this standard form.

Let $g(\varepsilon)$ be a one-parameter subgroup in this standard form. Since $g(\varepsilon) \in G$, there exists $g^{-1}(\varepsilon)$ such that $gg^{-1}=e$, and since this is in standard form, $g^{-1}(\varepsilon) = g(-\varepsilon)$. Since gg^{-1} is a constant, its derivative is null, so

$$g\frac{dg^{-1}}{d\varepsilon} = -\frac{dg}{d\varepsilon}g^{-1}.$$

This gives us

B-1 $$\frac{dg}{d\varepsilon} = -g\frac{dg^{-1}}{d\varepsilon}g.$$

Using $g(0) = e$ and $g^{-1}(\varepsilon) = g(-\varepsilon)$, we find that

$$\begin{aligned} g\frac{dg^{-1}}{d\varepsilon} &= g \lim_{\delta\to 0}\left[\frac{g^{-1}(\varepsilon+\delta)-g^{-1}(\varepsilon)}{\delta}\right] \\ &= \lim_{\delta\to 0}\left[\frac{1}{\delta}(g(\varepsilon)g(-\varepsilon-\delta)-g(\varepsilon)g(-\varepsilon)\right] \\ &= \lim_{\delta\to 0}\left[\frac{1}{\delta}(g(-\delta)-g(0)\right] \\ &= -\frac{dg}{d\varepsilon}\bigg|_{\varepsilon=0} \equiv -a \end{aligned}$$

When we put this back together with B-1 we get the differential equation:

B-2 $$\frac{dg}{d\varepsilon} = ag.$$

Combining B-2 with the initial condition $g(0) = e$, and defining the exponential function by the power series, we get the familiar exponential form for the translations generated by $\frac{\partial g}{\partial\varepsilon}\Big|_{\varepsilon=0} = a$, that is:

$$g(\varepsilon) = \exp[\varepsilon a].$$

If we look now at the left translations of the elements of $T_e(G)$, denoted $L_{a*}(A)$ where $A \in T_e(G)$ and L_{a*} is the linear differential induced by the left translation of G by $a \in G$, we get a left invarient vector field on G. This vector field is the Lie algebra of G, denoted $\mathfrak{G}$. Although the products of elements A, B in $\mathfrak{G}$ are not necessarily in $\mathfrak{G}$, their commutator $[A,B] = AB-BA$ is. Given a basis on $\mathfrak{G}$, this closure property for the commutator implies that

$$[\alpha_i, \alpha_j] = C^k_{ij}$$

These C^i_{jk} are called the structure constants of the Lie algebra. We also have $\mathcal{G}^*$ which is the set of forms ω for which $L^*_a\omega = \omega$. For $A \in \mathcal{G}$ and $\omega \in \mathcal{G}^*$. $\omega(A)$ is a constant on G.

The linear differentials of inner automorphisms of the form $\mathrm{Int}_a(g) = aga^{-1}$ give us the automorphisms of $\mathcal{G}$ called Ad_a. Thus, $\mathrm{Ad}_a(A)f = A(f\cdot\mathrm{Int}_a)$, for $a \in G$, $A \in \mathcal{G}$ and f a function on G. Applied to $b \in G$, $[\mathrm{Ad}_a(A)f]b = A(f(aba^{-1}))$.

Let a be an element of a Lie group G and p be an element of the n-dimensional differentiable manifold P. Let $R_a(p) = R(p,a)$ represent the differentiable mapping $R{:}P\times G\to P$ (i.e. G acts on P on the right). $R_a(p)$ can also be denoted pa. Let M denote the equivalence space of P under G, i.e. if ua = v for some $a \in G$, then ua and v are considered the same element, or are mapped to the same element of M. We will denote this action with π, called the canonical projection, so that for u and v = ua, two points in P related by a, $x = \pi(u) = \pi(v)$ is their projected image in M.

P will be a differentiable principal fibre bundle[2] over M with group G if:

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[2]See Y. M. Cho, J. Math. Phys. 16, 2029 (1975), M. Daniel and C. M. Viallet, Reviews of Modern Physics 52, 175 (1980), and S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. I (London, Interscience Publishers, 1963)

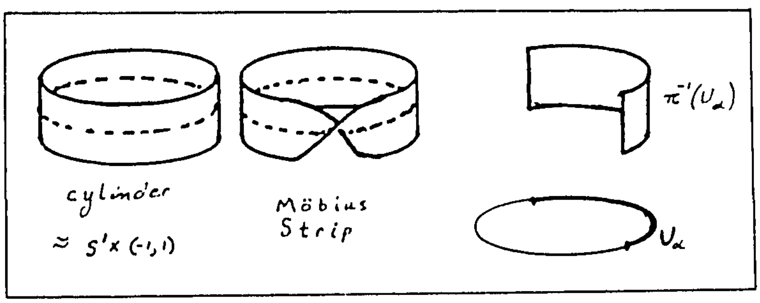
a) G acts freely on P to the right: $ua = u \Leftrightarrow a = e$, the identity of G.

b) π is differentiable.

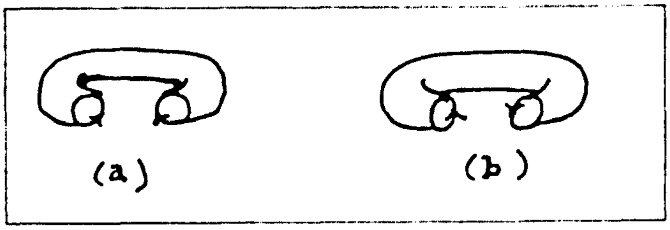
c) P is locally trivial: given $p \in M$, there is a neighborhood U_α of p such that $\pi^{-1}(U_\alpha)$ is isomorphic to $U_\alpha xG$. This is true if there exists a diffeomorphism $\psi: \pi^{-1}(U_\alpha) \to U_\alpha xG$ where $\psi(u) = (\pi(u), \varphi(u))$ for all $u \in \pi^{-1}(U_\alpha)$, with $\varphi_\alpha : \pi^{-1}(U_\alpha) \to G$ a mapping (not necessarily unique) satisfying $\varphi_\alpha(ua) = \varphi_\alpha(u)a$.

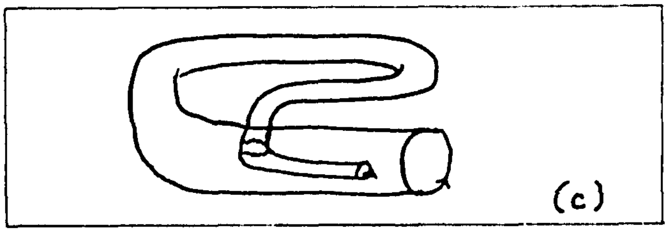
M is called the base space, G the structure group, P the bundle space, or G bundle, over M, and π the projection. $\pi^{-1}(x)$, for $x \in M$, is diffeomorphic to G and is called the fibre over x.

The first pair of examples starts with the circle, S^1, for M and the interval (-1,1) for the fibres. Locally, this will look like the cross product of two intervals, but there are two possible global structures: the cylinder and the Möbius strip. The cylinder is isomorphic to the cross product of the circle and the interval (-1,1), but the Möbius strip is not isomorphic to any cross product. Thus, although they look alike locally, they are very different globally.



The next pair takes M to be the circle again, but the fibre is also a circle, instead of an interval. Locally, $\pi^{-1}(U_\alpha)$, for a neighborhood U_α, will look like a tube or cylinder. When we try to put things together globally, we again get two possible surfaces. The simpler is the torus, which is isomorphic to $S^1 x S^1$. Here, the second S^1 represents the fibre, and since the continuous group U(1) can be thought of as a circle, we could look at the torus as $S^1 xU(1)$. The other figure can't exist in three space. It's sometimes referred to as the Klein Bottle or Klein Jar. Suppose we take a tube, assign directions to the edges of each end, as shown, and then try to bring the openings together so that the directions match. Figure (a) goes together rather nicely to make the torus, but figure (b) will require passing one end through the wall (if you're stuck in three space) and then lining it up with the other end, as shown in figure (c).





The next concept is that of a cross-section. A global cross-section is a differentiable mapping of the base space M into the bundle space P in such a way that $\pi\cdot\sigma$ is the identity map on M. A local cross-section over a neighborhood U_α is defined the same way: $\sigma_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ and

$\pi\sigma_\alpha$ is the identity map on U_α. A convenient notation for trivial cross-sections is given by using the φ_α from part three of the definition of the principle fibre bundle. Given the definition $\sigma_\alpha = u[\varphi_\alpha(u)]^{-1}$, where $[\varphi_\alpha(u)]^{-1}$ is the inverse of the image of u in G under φ_α, σ_α turns out to be independent of u. Suppose v is another point in the fibre through u. Then there is an a in G such that v = ua. Now

$$v[\varphi_\alpha(v)]^{-1} = ua[\varphi_\alpha(ua)]^{-1} = ua[\varphi_\alpha(u)a]^{-1} = ua\cdot a^{-1}[\varphi_\alpha(u)]^{-1}$$
$$= u[\varphi_\alpha(u)]^{-1}$$

With this notation, $\psi_\alpha(\sigma_\alpha(x)) = (x,e)$, so that $\sigma_\alpha(U_\alpha)$ corresponds to $U_\alpha x\{e\}$ under the diffeomorphism $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha xG$. We could also express the trivial cross-sections as $\sigma_\alpha = \{(x,a) /$ a is constant$\}$, where we confuse the difference between $\pi^{-1}(U_\alpha)$ and $U_\alpha xG$.

These cross-sections also have transition functions in the areas where their neighborhoods overlap. That is, given σ_α over U_α and σ_β over U_β, with $U_\alpha \cap U_\beta \neq \emptyset$, there exists $\psi_{\alpha\beta} : (U_\alpha \cap U_\beta) \to G$, such that

$$\sigma_\beta(x) = \sigma_\alpha(x)\psi_{\alpha\beta}(x)$$

for x in $(U_\alpha \cap U_\beta)$.

In order to develop the concept of a connection[3], we'll begin by developing the idea of vertical vectors in

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[3]See Y. M. Cho, J. Math. Phys. 16, 2029 (1975), M. Daniel and C. M. Viallet, Reviews of Modern Physics 52, 175 (1980), as well as A. Trautman, Rep. Math. Phys. 1, 29 (1970).

$T_u(P)$ for each u in P. Let $u_t = R_{\exp[tA]}u$ be the differentiable curve through u induced by A in $\mathfrak{G}$. The tangent at u of this curve is called the fundamental vector $\Sigma(A)$, in $T_u(P)$, associated with A. Given a function f on P, then

$$\Sigma(A)_u f = \frac{d}{dt} f(u_t)|_{t=0},$$

in accordance with our original definition of vectors. Because all of the points of u_t are generated from u by the curve $\exp[tA]$ in G, all of the points of u_t lie in the fibre through u, so that $\Sigma(A)$ is tangent to the fibre. The set of all such fundamental vectors, G_u, is isomorphic to $\mathfrak{G}$ and forms a special subspace of $T_u(P)$ called the verticle subspace. Note that $\pi_*(\Sigma(A)) = 0$, and an alternate definition for G_u could be the space of all vectors Z for which $\pi_*(Z) = 0$ (i.e. the kernal of $\pi_*: T_u(P) \to T_{\pi(u)}(M)$).

A connection Γ on P can now be defined as a choice of a supplementary set of vectors Q_u in $T_u(P)$ such that

a) $T_u(P) = Q_u \oplus G_u$

b) $Q_{ua} = (R_a)_* Q_u$

c) Q_u depends differentiably on u.

This subspace Q_u is called the horizontal subspace of $T_u(P)$.

A Lie-algebra-valued one-form ω may now be defined as the pull back of $\Sigma(A)$ to A. If we apply R_a^* to ω, we get a shift by means of $Ad_{a^{-1}}$. Note that

$$(R_a^*\omega)[\Sigma(A)] = \omega[R_{a*}\Sigma(A)]$$

and

$$[R_{a*}\Sigma(A)]_u f = \Sigma(A)_{R_{a^{-1}}u}(f \cdot R_a) = \frac{d}{dt} f \cdot R_a[R_{\exp tA}(R_{a^{-1}}u)]\Big|_{t=0}.$$

$$= \frac{d}{dt} f(R_{a^{-1}[\exp tA]a}\, u)|_{t=0}$$
$$= \frac{d}{dt} f(R_{\mathrm{Int}_{a^{-1}}[\exp tA]}\, u)|_{t=0}\,.$$

The tangents of the curves $\mathrm{Int}_{a^{-1}}[\exp tA]$ and $[\exp tA]$ are related by the linear differential of $\mathrm{Int}_{a^{-1}}$ (i.e. $\mathrm{Ad}_{a^{-1}}$). Thus,

$$[R_{a*}\Sigma(A)]_u f = \Sigma(Ad_{a^{-1}}A)_u f$$

giving us

<u>B-3</u> $$R_a^* \omega = Ad_{a^{-1}}\omega.$$

Since $\omega(X) = 0$ for X in Q_u, this relation is trivially true for Q_u, so B-3 is true for all of $T_u(P)$. This ω is called the connection form for the connection Γ.

This connection form also gives an alternate definition of Q_u as the kernal of $\omega: T_u(P) \to \mathfrak{G}$.

Once we are given the connection form ω, we can use a local cross-section $\sigma_\alpha: U_\alpha \to P$ to obtain a 1-form $\omega_\alpha = \sigma_\alpha^* \omega$ on U_α with values in $\mathfrak{G}$. Given the basis $\{x^i\}$ on U_α and the basis $\{e^\mu\}$ on $\mathfrak{G}$, we can write, for $X = \xi^i \frac{\partial}{\partial x^i}$ in $T(U_\alpha)$, $\omega_\alpha(X) = \omega_\alpha(\xi^i \frac{\partial}{\partial x^i}) = ((\omega_{\alpha\mu} e^\mu)_j\, dx^j)(\xi^i \frac{\partial}{\partial x^i}) = (\omega_{\alpha\mu} e^\mu)_i \xi^i = \omega_{\alpha i\mu} \xi^i e^\mu$. The $\omega_{\alpha i\mu}\xi^i$ are real-valued functions (for real--valued vector spaces $\mathfrak{G}$). These $\omega_{\alpha i\mu}$ may be considered the gauge-fields corresponding to the connection form ω. The connection form has the advantage of being defined for all of P, while $\omega_{\alpha i\mu}$, which is dependent on the cross--section σ_α, is defined only locally for nontrivial fibre bundles. The choice of a cross-section here corresponds to a choice of gauge.

Given the cross-sections σ_α and σ_β, and their transition function $\psi_{\alpha\beta}: U_\alpha \cap U_\beta \to G$, the transformations for the ω_α is

$$\omega_\beta = Ad_{\psi_{\alpha\beta}^{-1}}(\omega_\alpha) + \psi_{\alpha\beta}^{-1}\, d_M\, \psi_{\alpha\beta}\,.$$

where d_M is the exterior derivative on M.

For example, given the trivial bundle $R^4 xG$, let σ_1 and σ_2 be related by g in G, and ω be the connection form.

$$\omega_1 = \sigma_1^*(\omega) \qquad\qquad \omega_2 = \sigma_2^*(\omega)$$

and
$$\omega_2 = Ad_{g^{-1}}(\omega_1) + g^{-1}dg\,.$$

Writing ω_α as $A_{\alpha i}\, dx^i$, we find

$$A_{2i}dx^i = Ad_{g^{-1}}(A_{1j}\,dx^j) + g^{-1}dg = Ad_{g^{-1}}(A_{1j}dx^j) + g^{-1}\left(\frac{\partial}{\partial x^k}g\right)dx^k$$

This will give us

$$A_{2n} = g^{-1}A_{1n}\,g + g^{-1}\frac{\partial}{\partial x^n}g\,,$$

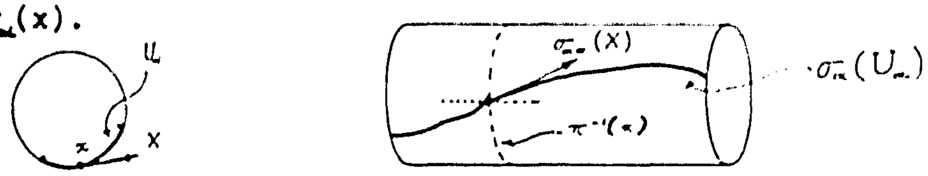
which is the gauge transformation formula for gauge potentials[4]. If the group G is the one for isospin space, this could be written:

$$\vec{A}_{2n} = g^{-1}\vec{A}_{1n}\,g + g^{-1}\partial_n g\,.$$

To better see the relationship between the connection form and the gauge potentials, let's look at U_α, a neighborhood of x on the circle, and P can be either the torus $S^1 xU(1)$ or the Klein Jar, since we will be looking

[4]See Y. M. Cho, J. Math. Phys. 16, 2029 (1975), and M. Daniel and C. M. Viallet, Reviews of Modern Physics 52, 175 (1980).

here at just the local properties. Let σ_α be a cross-section mapping U_α into $\pi^{-1}(U_\alpha)$. Let X be a tangent vector at x, and $\sigma_{\alpha *}X$ be the corresponding tangent vector to $\sigma_\alpha(U_\alpha)$ at $\sigma_\alpha(x)$.



$M = S^1$ $\qquad$ $\pi^{-1}(U_\alpha)$

We can identify points on $\pi^{-1}(U_\alpha)$ with their counterparts (x,a) in U_αxG, since $\pi^{-1}(U_\alpha)$ and U_αxG are equivalent by the definition of a fibre bundle. Using π and φ_α, where φ_α was chosen in the construction of P, we can identify p with $(\pi(p),\varphi_\alpha(p))$. U_α has a basis u^i and G has a basis ξ^μ, so we can use the product basis $x^A = \{u^i$ for $A = 1,\ldots,n$; ξ^μ for $A = n+1,\ldots,n+m$ with $\mu = A-n\}$, where n and m are the dimensions of M and G respectively. Notice that $\sigma_{\alpha *}(x)$ could have both a vertical and a horizontal component, so that we can use the expression

$$\sigma_{\alpha *}(X) = hor\ \sigma_{\alpha *}(X) + vert\ \sigma_{\alpha *}(X)$$

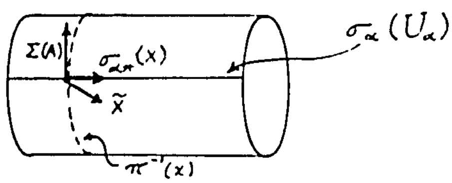
where hor(Y) and vert(Y) are the horizontal and vertical components of a vector Y in T(P). If we now apply ω to $\sigma_{\alpha *}(x)$, we lose the horizontal part and find a vector A in $\mathfrak{G}$ for which vert $\sigma_{\alpha *}(x) = \Sigma(A)$.

The lift of a vector X in $T_x(M)$ to u in $\pi^{-1}(x)$ is that element of Q_u whose image under π_* is X. The lift of X will be denoted $\widetilde{X}$.

For $u = \sigma_\alpha(x)$, the lift of $\partial_a \equiv \frac{\partial}{\partial x^a}$ to u is horizontal, but as we've seen, $\sigma_{\alpha *}(\partial_a)$ might not be. In fact, if we consider the local expression of the connection ω: $\omega_\alpha = \sigma_\alpha^*(\omega) = A_{\alpha a} dx^a = (A_{\alpha a \mu} e^\mu) dx^a$, where $\{e^\mu\}$ is a basis of $\mathcal{G}$, then $\omega_\alpha(\partial_a) = \omega(\sigma_{\alpha *}\partial_a) = B_a = [A_{\alpha b} dx^b]\partial_a = A_{\alpha a}$. This means that the vertical part of $\sigma_{\alpha *}\partial_a$ is, in essence, the connection coefficient. If we subtract $\Sigma(A_{\alpha a})$ from $\sigma_{\alpha *}(\partial_a)$ we will get a horizontal vector, since $\omega(\sigma_{\alpha *}\partial_a - \Sigma(A_{\alpha a})) = 0$. However, the lift of a vector is unique, so

$$\tilde{\partial}_a \Big|_u = \sigma_{\alpha *}\partial_a - \Sigma(A_{\alpha a})$$

In comparing this with the covarient derivative, $\mathcal{D}_a = \partial_a - A_a$, we can see that $\mathcal{D}_a$ corresponds to $\tilde{\partial}_a\big|_u$ when we identify $\sigma_{\alpha *}\partial_a$ and $\Sigma(A_{\alpha a})$ with ∂_a and $A_{\alpha a}$ respectively[5]. Looking back at our notation for the trivial cross-section $\sigma_{\alpha *} = u\varphi_\alpha^{-1}(u)$, and the result that $\sigma_\alpha(U_\alpha) \approx U_\alpha \times \{e\}$, the figure for $\pi^{-1}(U_\alpha)$ could have been drawn:



$$\tilde{X} = \sigma_{\alpha *}(X) - \Sigma(A)$$
$$= \text{hor}\ \sigma_{\alpha *}(X)$$
$$\text{where } A = \omega(\sigma_{\alpha *}(X)).$$

and we can then justify identifying $\sigma_{\alpha *}(\partial_a)$ with ∂_a and $\Sigma(A_{\alpha a})$ with $A_{\alpha a}$. Although $\mathcal{D}_a = \partial_a - A_{\alpha a}$ doesn't look horizontal according to this picture, remember that "horizontal" is defined by the connection Γ, not by the tangents of $\sigma_\alpha(U_\alpha)$.

If we take the commutator of the covarient derivatives,

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[5]See M. Daniel and C. M. Viallet, Reviews of Modern Physics 52, 175 (1980)

we get the gauge field $F_{\alpha ab}$,

$$[\mathcal{D}_a, \mathcal{D}_b] = -\{(\partial_a A_{\alpha b} - \partial_b A_{\alpha a}) - [A_{\alpha a}, A_{\alpha b}]\} = -F_{\alpha ab} .$$

When we express the $A_{\alpha b}$ in terms of coordinates $\partial_\mu = \frac{\partial}{\partial t_\mu}$ in $\mathcal{G}$, and using[6] $\partial_\mu A^{\nu}_{\alpha b} = -C^{\nu}_{\mu\lambda} A^{\lambda}_{\alpha b}$, then the commutator becomes

$$\begin{aligned}[\mathcal{D}_a, \mathcal{D}_b] &= -\{(\partial_a A^{\mu}_{\alpha b}\partial_\mu - \partial_b A^{\mu}_{\alpha a}\partial_\mu) - [A^{\mu}_{\alpha a}\partial_\mu, A^{\nu}_{\alpha b}\partial_\nu]\} \\ &= -\{(\partial_a A^{\mu}_{\alpha b} - \partial_b A^{\mu}_{\alpha a}) + C^{\mu}_{\kappa\lambda} A^{\kappa}_{a} A^{\lambda}_{b}\}\partial_\mu \\ &= -F^{\mu}_{\alpha ab}\,\partial_\mu\end{aligned}$$

Overlooking the cross-section dependence indicated by α, our $F^{\mu}_{\alpha ab}$ is the same object we found in Utiyama's work, and when we develop our Lagrangian from the curvature of the principle fibre bundle, we will see that the gauge fields are present only in terms of the F^{μ}_{ab}.

First we will establish a basis for our principle fibre bundle, with M as space-time, and G as our transformation group. For this calculation of R, let's use the horizontal lift basis for T(P), i.e. $\partial_A = \{\tilde{\partial}_a$ for A=1,2,3,4; and ∂_μ for A=4+1,...,4+m, with μ=A-4$\}$, where m is the dimension of G.

The metric on $\mathcal{G}$ will be the bi-invarient form[7]

$$g_{\mu\nu} = g(\partial_\mu, \partial_\nu) = f\, C^{\alpha}_{\mu\beta}\, C^{\beta}_{\nu\alpha}$$

where f is a function of space-time. If $\mathcal{G}$ is semi-simple,

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[6] See Y. M. Cho, J. Math Phys 16, 2029 (1975)

[7] A right-invarient form based on the Cartan-Killing form was used in Y. M. Cho and P. G. O. Freund, Phys. Rev. D 12, 1711 (1975). See also: A. Trautman, Czech. J. Phys. B 29, 107 (1979) and L. N. Chang, K. I. Macrae, and F. Mansouri, Phys. Rev. D 13, 235 (1976)

as with U(1)xSU(2), then the metric can be considered a combination of subordinate parts, each with its own f. For example, with U(1)xSU(2), the metric can be written

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & B \end{bmatrix}$$

where A and B are functions of space-time. For the Abelian groups, since the commutators are null, each component of $g_{\mu\nu}$ is an arbitrary function of space-time.

The metric for P should preserve the actions of the space-time and group metrics, while making the vertical and horizontal parts orthogonal. When we use the horizontal lift basis, these requirements give us the metric

$$\left[\gamma_{AB}\right] = \begin{bmatrix} g_{ab} & 0 \\ 0 & g_{\mu\nu} \end{bmatrix}$$

and its inverse

$$\left[\gamma^{AB}\right] \quad \begin{bmatrix} g^{ab} & \\ & g^{\mu\nu} \end{bmatrix}$$

The connection coefficients are given by[8]

$$\Gamma^{A}_{BC} = \tfrac{1}{2}\gamma^{AD}\left[\gamma_{DB,C} + \gamma_{DC,B} - \gamma_{BC,D} + \gamma_{CE}C^{E}_{DB} + \gamma_{BE}C^{E}_{DC}\right] - \tfrac{1}{2}C^{A}_{BC},$$

where $C^{A}_{BC}e_{A} = [e_{B}, e_{C}]$ for the basis $\{e_{A}\}$ of P. For the horizontal lift basis, these structure constants are

$$[\partial_{\mu}, \partial_{\nu}] = C^{\lambda}_{\mu\nu}\,\partial_{\lambda}$$

$$[\partial_{\mu}, \partial_{a}] = 0$$

$$[\partial_{a}, \partial_{b}] = -F^{\mu}_{ab}\,\partial_{\mu} = C^{\mu}_{ab}\,\partial_{\mu},$$

[8] See C.W. Misner, K.S. Thorne, and J.A. Wheeler, Gravitation, (San Francisco, W. H. Freeman, 1973), p. 314

so that the only nonnull C's are of the form $C^{\lambda}_{\mu\nu}$ and C^{λ}_{ab}.

This gives us:

$$\Gamma^{\alpha}_{\beta\gamma} = -\tfrac{1}{2} C^{\alpha}_{\beta\gamma}$$

$$\Gamma^{\alpha}_{\beta a} = \tfrac{1}{2} g^{\alpha\mu} g_{\mu\beta,a}$$

$$\Gamma^{\alpha}_{ab} = \tfrac{1}{2} F^{\alpha}_{ab}$$

$$\Gamma^{a}_{\alpha\beta} = -\tfrac{1}{2} g^{ab} g_{\alpha\beta,b}$$

$$\Gamma^{a}_{b\alpha} = -\tfrac{1}{2} g^{ac} g_{\alpha\beta} F^{\beta}_{cb} = \Gamma^{a}_{\alpha b}$$

Γ^{a}_{bc} will be left as is.

Note that $\Gamma^{\mu}_{\nu\lambda}$ and Γ^{μ}_{cd} are antisymmetric, while the rest are symmetric.

The formula for R is[9]

$$R = \gamma^{AB} R^{C}{}_{ACB}$$
$$= \gamma^{AB}\left\{\Gamma^{C}_{AB,C} - \Gamma^{C}_{AC,B} + \Gamma^{C}_{DC}\Gamma^{D}_{AB} - \Gamma^{C}_{DB}\Gamma^{D}_{AC} - \Gamma^{C}_{AE} C^{E}_{CB}\right\}$$

Plugging in terms and collecting and cancelling gives us

$$R = R_{st} - \tfrac{1}{4} g^{\alpha\beta} C^{\gamma}_{\alpha\delta} C^{\delta}_{\beta\gamma} - \tfrac{1}{4} g^{ab} g^{cd} g_{\alpha\beta} F^{\alpha}_{ac} F^{\beta}_{bd}$$
$$+ \tfrac{1}{4} g^{ab} g^{\alpha\beta} g^{\gamma\delta}\left[g_{\alpha\gamma,a} g_{\beta\delta,b} - g_{\alpha\beta,a} g_{\gamma\delta,b}\right]$$
$$- \tfrac{1}{2} g^{ab}\left[g^{\alpha\beta} g_{\alpha\beta,a}\right]_{;b} - \tfrac{1}{2} g^{\alpha\beta}\left[g^{ab} g_{\alpha\beta,a}\right]_{;b}$$

The action integral is[10]

$$I = \int R \; dx^1 \wedge \dots \wedge dx^{n+m}$$
$$= \int \sqrt{\gamma}\; R \; d^4x \; d^m\xi$$
$$= \int \sqrt{-g_{st}}\,\sqrt{g}\; R \; d^4x \; d^m\xi$$

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[9] Misner, Thorne, and Wheeler, p. 277

[10] Cf. Y. M. Cho, J. Math. Phys. 16, 2029 (1975) and P. M. Morse and H. Feshbach, Methods of Theoretical Physics, (New York, McGraw-Hill, 1953), p.275

where $dx^1 \wedge \ldots \wedge dx^{4+m}$ is the volume element, which is equal to $\sqrt{\gamma}\, dx^1 \ldots dx^{4+m}$, and $\sqrt{\gamma} = \sqrt{-g_{ST}}\sqrt{g}$ with $-g_{ST}$ the determinant of the space-time metric and g the determinant of the group metric. d^4x and $d^m\xi$ are the ST and group volume elements. By using the $\sqrt{-g_{ST}}\sqrt{g}$ in $\mathcal{L} = \sqrt{\gamma}\, R$, we can write $\mathcal{L}$ in the form[11]

$$\begin{aligned}
\mathcal{L} = \sqrt{-g_{ST}}\sqrt{g}\, R_{ST} &- \tfrac{1}{4}\sqrt{-g_{ST}}\sqrt{g}\; g^{\alpha\beta} C^{\gamma}_{\alpha\delta} C^{\delta}_{\beta\gamma} \\
&- \tfrac{1}{4}\sqrt{-g_{ST}}\sqrt{g}\; g_{\alpha\beta} F^{\alpha}_{ab} F^{\beta}_{cd}\, g^{ac} g^{bd} \\
&- \tfrac{1}{4}\sqrt{-g_{ST}}\sqrt{g}\; g^{ab} g^{\alpha\beta} g^{\gamma\delta} [g_{\alpha\gamma,a} g_{\beta\delta,b} - g_{\alpha\beta,a} g_{\gamma\delta,b}] \\
&- \frac{\partial}{\partial x^b}\left\{\sqrt{-g_{ST}}\sqrt{g}\; g^{\alpha\beta} g_{\alpha\beta,a}\, g^{ab}\right\}
\end{aligned}$$

In conclusion, this Lagrangian density contains the F^{α}_{ab} demanded by the symmetry arguments of Utiyama, although they are derived here from geometric arguments; and we also have an explicit Lagrangian, rather than just the restrictions which Utiyama's symmetry arguments placed on the Lagrangian. In addition the method has given us the $g_{\alpha\beta}$ as scalar fields. From here we can develop the field equations as in section one.

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[11] Cf. Y. M. Cho, J. Math. Phys. <u>16</u>, 2029 (1975) and Y. M. Cho and P. G. O. Freund, Phys. Rev. D <u>12</u>, 1711 (1975).