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I dedicate this dissertation to my wife, Joan, who has been a constant source of encouragement throughout its preparation.

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SECTION I

INTRODUCTION

Let M be an n-dimensional Riemann manifold with positive definite metric tensor g, and let V represent either the set of all tangent vectors of M or the set of all cotangent vectors of M. Each V may be given a vector bundle structure with M as the base space, a linear space R^n as the standard fibre and the general linear group $GL(n)$ as the structural group [1, p.40]. Such a set V of tangent vectors is denoted by $T(M)$ and is called the <u>tangent bundle</u> of M. Similarly, such a set V of cotangent vectors is denoted by $T^*(M)$, and is called the <u>cotangent bundle</u> of M.

S. Sasaki studied the local differential geometry of $T(M)$ by making it into a Riemann manifold with a positive definite metric and relating geometric objects on this Riemann manifold to objects on the base manifold [8, 9]. In this study he defined extended tensors as tensors on $T(M)$ which are obtained from tensors on M in a particular way.

Section II of the present paper is a study of differential geometry of $T(M)$ when it is given a Riemannian structure obtained from the extended tensor $\bar{g}$ of the metric tensor g of M. The pair $(T(M),\bar{g})$ is called the <u>Extended Riemann Manifold</u> of M. It is shown to be isometric with a Riemann extension of M which is defined on $\overset{*}{T}(M)$ by Patterson and Walker [11].

Miron defined in a Riemann manifold a concept of parallelism of vector fields relative to a Myller configuration [5, 6]. In Section III we define a modified Myller configuration in an affinely-connected manifold A, and generalize Miron's concept of parallelism of vector fields relative to this configuration. A particular configuration of interest in the tangent bundle is then examined. Moreover, we define a class of hypersurfaces in the cotangent bundle of M and investigate relationships of parallel displacement among the three spaces involved.

In the research we use standard methods of classical differential geometry and projection tensors associated with two disjoint distributions which span the tangent space of a differentiable manifold. Our study is confined to local properties. When we merely say <u>on</u> <u>M</u>, we mean in a coordinate neighborhood of the C^{∞} n-manifold M. All curves and tensor fields are assumed to be of class C^{∞}, and moreover all curves are regular.

Throughout the paper, when dealing with vector bundles, we shall adhere to the notation that a semicolon represents covariant differentiation relative to the connection on the base space, a colon denotes covariant differentiation relative to the connection on the bundle space, and a comma indicates partial differentiation.

Any real valued function f on the base space M differs from $f \circ \pi$ or $f \circ P$ only in their domains. For convenience of notation we shall denote them simply by f. Their domains will be clear from the context.

Equations, theorems, lemmas and corollaries are numbered in ascending order in each section. For example, equation II.3 will refer to eqüation (3) in Section II. The first three Roman indices will range from 1 to 2n-1, the remaining Roman indices from 1 to n, and Greek indices from 1 to 2n. Repeated indices will as usual indicate summation.

SECTION II

EXTENDED RIEMANN MANIFOLD

1. Extended metric and curvature tensor. Let π denote the projection of $T(M)$ onto M, and (U, x^i) a coordinate neighborhood U of M with coordinate functions x^i. Let (p, ξ) be a tangent vector at $p \in U$ with components ξ^i relative to the induced frame $\{\partial/\partial x^i\}$. Then $\xi^i = dx^i(\xi) = \xi(x^i)$, and $\pi^{-1}(U)$ is a coordinate neighborhood of $T(M)$ with coordinate functions $(\bar{x}^i, \bar{\xi}^i)$ defined by

$$\bar{x}^i = x^i \circ \pi,$$

$$\bar{\xi}^i((p, \xi)) = \xi^i.$$

Let (U', x'^i) be a second neighborhood of M such that $U \cap U' \neq \emptyset$, and let ξ have components ξ'^i relative to $\{\partial/\partial x'^i\}$. Then $\pi^{-1}(U) \cap \pi^{-1}(U') \neq \emptyset$, and to the coordinate transformation

$$x'^i = f'^i(x^1, x^2, \ldots, x^n), \tag{1}$$

on $U \cap U'$, there corresponds the transformation

$$\bar{x}'^i = \bar{f}'^i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n), \quad \bar{\xi}'^i = (\partial \bar{x}'^i / \partial \bar{x}^k)\, \bar{\xi}^k, \tag{2}$$

on $\pi^{-1}(U) \cap \pi^{-1}(U')$. The transformation (2) is said to be the extended transformation of (1), and $\pi^{-1}(U)$ is called an extended coordinate neighborhood of U.

For simplicity of notation we shall use x^i to denote either x^i or $\bar{x}^i$. This should not cause confusion since their respective domains will be clear from the context. In order to further simplify notation, let

$z^i = x^i$ and $z^{n+i} = \xi^i$, so that the extended transformation (2) may be written as

(3) $$z'^{\alpha} = f'^{\alpha}(z^1, z^2, \ldots, z^{2n}).$$

The Jacobian of the extended transformation is given by

(4) $$[\partial z'^{\alpha}/\partial z^{\beta}] = \begin{bmatrix} [\partial z'^i/\partial z^j] & [(\partial^2 z'^i/\partial z^k \partial z^j)\, z^{n+k}] \\ [0] & [\partial z'^i/\partial z^j] \end{bmatrix},$$

which may be used to define tensor fields on $T(M)$.

We shall denote by $(\pi^{-1}(U), z^{\alpha})$ the extended coordinate neighborhood of U with coordinate functions z^{α}.

Let λ be a contravariant (or covariant) vector field on M and let λ^i (or λ_i) be its components with respect to $\{\partial/\partial z^i\}$. Then it is readily verified by use of (4) that

$$\bar{\lambda}^i = \lambda^i, \quad \bar{\lambda}^{n+i} = (\partial\lambda^i/\partial z^k)\, z^{n+k},$$

and

$$\bar{\lambda}_i = (\partial\lambda_i/\partial z^k)\, z^{n+k}, \quad \bar{\lambda}_{n+i} = \lambda_i,$$

define respectively a contravariant vector field and a covariant vector field on $T(M)$ with respect to $\{\partial/\partial z^{\alpha}\}$. The vector field $\bar{\lambda}$ on $T(M)$ thus derived from a contravariant (or covariant) vector field λ on M is called an <u>extended vector field</u> or an <u>extension</u> of λ [8].

In the same way, if λ^{ij} (or λ^i_j or λ_{ij}) are components of a contravariant (or mixed or covariant) tensor field on M with respect to $\{\partial/\partial z^i\}$, then $\bar{\lambda}^{\alpha\beta}$ (or $\bar{\lambda}^{\alpha}_{\beta}$ or $\bar{\lambda}_{\alpha\beta}$) defined by

$$\bar{\lambda}^{ij} = 0, \quad \bar{\lambda}^{i\;n+j} = \bar{\lambda}^{n+i\;j} = \lambda^{ij}, \quad \bar{\lambda}^{n+i\;n+j} = (\partial\lambda^{ij}/\partial z^k) z^{n+k},$$
$$\bar{\lambda}^i_j = \lambda^i_j, \quad \bar{\lambda}^i_{n+j} = 0, \quad \bar{\lambda}^{n+i}_j = (\partial\lambda^i_j/\partial z^k)\, z^{n+k}, \quad \bar{\lambda}^{n+i}_{n+j} = \lambda^i_j,$$
$$\bar{\lambda}_{ij} = (\partial\lambda_{ij}/\partial z^k)\, z^{n+k}, \quad \bar{\lambda}_{i\;n+j} = \bar{\lambda}_{n+i\;j} = \lambda_{ij}, \quad \bar{\lambda}_{n+i\;n+j} = 0,$$

are components of a contravariant (or mixed or covariant) tensor field on $T(M)$ with respect to $\{\partial/\partial z^\alpha\}$. The tensor field $\bar{\lambda}$ on $T(M)$ is called an <u>extension</u> of the contravariant (or mixed or covariant) tensor field λ on M.

Let g_{ij} be components of the metric tensor g on M with respect to $\{\partial/\partial z^i\}$. Then the extension of g given by

$$\begin{aligned} \bar{g}_{ij} &= (\partial g_{ij}/\partial z^k) z^{n+k}, \\ \bar{g}_{i\,n+j} &= \bar{g}_{n+i\,j} = g_{ij}, \\ \bar{g}_{n+i\,n+j} &= 0, \end{aligned} \tag{5}$$

defines an extended metric tensor $\bar{g}$ on the coordinate neighborhood $(\pi^{-1}(U), z^\alpha)$ of $T(M)$. The tangent bundle with this metric is denoted by $(T(M),\bar{g})$, and is called the <u>Extended</u> <u>Riemann</u> <u>Manifold</u> of M.

The element of arc on $T(M)$ with respect to $\bar{g}$ is

$$d\sigma^2 = (\partial g_{ij}/\partial z^k) z^{n+k}\, dz^i\, dz^j + 2 g_{ij}\, dz^i\, dz^{n+j}, \tag{6}$$

from which we note that $\bar{g}$ is not positive definite.

The contravariant components of $\bar{g}$ are obtained by solving $\bar{g}^{\alpha\epsilon}\,\bar{g}_{\epsilon\beta} = \delta^\alpha_\beta$ for $\bar{g}^{\alpha\beta}$, where δ^α_β denotes the Kronecker delta. These components are found to be

$$\begin{aligned} \bar{g}^{ij} &= 0, \\ \bar{g}^{i\,n+j} &= \bar{g}^{n+i\,j} = g^{ij}, \\ \bar{g}^{n+i\,n+j} &= (\partial g^{ij}/\partial z^k)\, z^{n+k}. \end{aligned} \tag{7}$$

We denote by L^i_{jk} the coefficients of the connection L induced by g on a coordinate neighborhood of M. Then the coefficients $\Gamma^\alpha_{\beta\gamma}$ of the connection Γ induced by $\bar{g}$ on the extended coordinate neighborhood of $T(M)$ are given by

$$\Gamma^{i}_{jk} = \Gamma^{n+i}_{n+j\ k} = L^{i}_{jk},$$

(8) $$\Gamma^{n+i}_{jk} = (\partial L^{i}_{jk}/\partial z^{p})\, z^{n+p},$$

$$\Gamma^{\alpha}_{n+j\ n+k} = \Gamma^{i}_{n+j\ k} = 0.$$

Γ is called the extension of L to $T(M)$. It is clear from (8) that $\Gamma^{\alpha}_{\beta\gamma} = 0$ if and only if $L^{i}_{jk} = 0$.

If we replace the base manifold M by an affinely-connected differentiable manifold A with an arbitrary connection L^{i}_{jk}, then we can define an affine connection on the tangent bundle $T(A)$ of A by equations (8). Many of the results obtained in this section which depend only on the connection of $T(M)$ will still hold on $T(A)$.

Let R^{i}_{jkl} denote components of the curvature tensor field R of M. The tensor field $\bar{R}$ with components defined by

$$\bar{R}^{\alpha}_{\beta\gamma\delta} = \partial\Gamma^{\alpha}_{\beta\gamma}/\partial z^{\delta} - \partial\Gamma^{\alpha}_{\beta\delta}/\partial z^{\gamma} + \Gamma^{\epsilon}_{\beta\gamma}\Gamma^{\alpha}_{\epsilon\delta} - \Gamma^{\epsilon}_{\beta\delta}\Gamma^{\alpha}_{\epsilon\gamma},$$

in a coordinate neighborhood of $T(M)$ is called the extension of R to $T(M)$. After some tedious calculations we find that

$$\bar{R}^{i}_{jkl} = \bar{R}^{n+i}_{n+j\ kl} = \bar{R}^{n+i}_{j\ n+k\ l} = \bar{R}^{n+i}_{jk\ n+l} = R^{i}_{jkl},$$

(9) $$\bar{R}^{n+i}_{jkl} = (\partial R^{i}_{jkl}/\partial z^{p})\, z^{n+p}.$$

(all other components) = 0,

and

$$\bar{R}_{ijkl} = (\partial R_{ijkl}/\partial z^{p}) z^{n+p},$$

(10) $$\bar{R}_{i\ n+j\ kl} = \bar{R}_{ij\ n+k\ l} = \bar{R}_{ijk\ n+l} = \bar{R}_{n+i\ jkl} = R_{ijkl},$$

(all other components) = 0,

which are components of the covariant curvature tensor field of $(T(M),\bar{g})$

defined by $\bar{R}_{\alpha\beta\gamma\delta} = \bar{g}_{\alpha\epsilon} \bar{R}^{\epsilon}_{\beta\gamma\delta}$. It is clear that $R = 0$ if and only if $\bar{R} = 0$.

The components of the Ricci tensor field of $(T(M),\bar{g})$, defined by $\bar{R}_{\alpha\beta} = \bar{R}^{\epsilon}_{\alpha\beta\epsilon}$, may be expressed in terms of the Ricci tensor field of M by

$$\bar{R}_{ij} = 2\, R_{ij},$$
$$\bar{R}_{n+i\ j} = \bar{R}_{i\ n+j} = \bar{R}_{n+i\ n+j} = 0.$$

Hence, the Ricci tensor field of $(T(M),\bar{g})$ vanishes if and only if the Ricci tensor field of M vanishes.

The scalar curvature of $(T(M),\bar{g})$, given by $\bar{g}^{\alpha\beta}\,\bar{R}_{\alpha\beta}$, is found to vanish identically. Thus, $(T(M),\bar{g})$ is a Riemann manifold of zero scalar curvature.

We now obtain necessary and sufficient conditions for the Extended Riemann Manifold of M to be a Riemann manifold of constant curvature.

<u>Theorem 1</u>. The Extended Riemann Manifold of M is of constant curvature if and only if it is flat, and this is the case if and only if M is flat.

<u>Proof</u>: Necessary and sufficient conditions for $(T(M),\bar{g})$ to be a manifold of constant curvature $\bar{K}$ are given by

$$\bar{R}_{\alpha\beta\gamma\delta} = \bar{K}\,(\bar{g}_{\alpha\gamma}\,\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\,\bar{g}_{\beta\gamma}). \tag{11}$$

By use of (5) and (10) these equations are found to hold if and only if $\bar{K}\ g_{ij}\,g_{kl} = 0$, and consequently if and only if $(T(M),\bar{g})$ is flat. It is clear from (10) that $(T(M),\bar{g})$ is flat if and only if M is flat. Q.E.D.

A Riemann manifold is called a symmetric manifold if the covariant derivative of its curvature tensor field is zero.

The only components of $\bar{R}_{\alpha\beta\gamma\delta:\epsilon}$ which may be different from zero are found to be

$$\bar{R}_{ijkl:n+m} = \bar{R}_{i\ n+j\ kl:m} = \bar{R}_{ij\ n+k\ l:m} = \bar{R}_{n+i\ jkl:m} = R_{ijkl;m},$$

$$(12) \qquad \bar{R}_{ijk\ n+l:m} = R_{ijkl;m},$$

$$\bar{R}_{ijkl:m} = (\partial R_{ijkl;m} / \partial z^h) z^{n+h}.$$

It is clear from equations (12) that $\bar{R}_{\alpha\beta\gamma\delta:\epsilon} = 0$ if and only if $R_{ijkl;m} = 0$. Hence, we have

<u>Theorem</u> <u>2</u>. The Extended Riemann Manifold of M is a symmetric manifold if and only if M is a symmetric manifold.

<u>Corollary</u> <u>1</u>. If M has constant curvature then its Extended Riemann Manifold is a symmetric manifold.

The above corollary follows from the fact that every manifold of constant curvature is also a symmetric manifold.

The manifold $(T(M),\bar{g})$ is an Einstein manifold if its Ricci tensor field satisfies $\bar{R}_{\alpha\beta} = c\bar{g}_{\alpha\beta}$ for some constant c. From equations (5) and the representation above for $\bar{R}_{\alpha\beta}$ this is found to be the case if and only if $c = 0$, and consequently if and only if the Ricci tensor of M is a null tensor.

A Riemann manifold is called a recurrent manifold if it is not a symmetric manifold, and the covariant derivative of its covariant curvature tensor is a product of the curvature tensor and a covariant vector at every point.

<u>Theorem</u> <u>3</u>. The Extended Riemann Manifold of M can not be a recurrent manifold.

Proof: Assume that $(T(M),\bar{g})$ is a recurrent manifold. Then

(13) $$\bar{R}_{\alpha\beta\gamma\delta;\epsilon} = \bar{R}_{\alpha\beta\gamma\delta}\,\lambda_\epsilon\,,\quad \lambda_\epsilon = (\lambda_m, \lambda_{n+m}) \neq 0,$$

which by (10) and (12) reduce to

$$0 = R_{ijkl}\ \lambda_{n+m},$$
$$R_{ijkl;m} = (\partial R_{ijkl}/\partial z^h)\, z^{n+h}\ \lambda_{n+m},$$
$$R_{ijkl;m} = R_{ijkl}\ \lambda_m,$$
$$(\partial R_{ijkl;m}/\partial z^h)\, z^{n+h} = (\partial R_{ijkl}/\partial z^h)\, z^{n+h}\ \lambda_m.$$

The first of these equations implies that at any point of $T(M)$ either $R_{ijkl} = 0$ or $\lambda_{n+m} = 0$, but in either event the second and third equations imply that $R_{ijkl;m} = 0$ and the fourth equation is identically satisfied. Consequently, $(T(M),\bar{g})$ is a symmetric manifold by Theorem 2.
Q.E.D.

2. Horizontal and vertical distributions. The vertical distribution $\mathcal{V}$ of $T(M)$ is the n-dimensional distribution spanned by the vector fields with components $(0, \delta^i_k)$, $k = 1,2, \ldots ,n,$ relative to $\{\partial/\partial z^\alpha\}$. The horizontal distribution $\mathcal{H}$ of $T(M)$ is the n-dimensional distribution spanned by the vector fields with components $(\delta^i_k, -L^i_{jk}\, z^{n+j})$, $k = 1,2, \ldots ,n,$ relative to $\{\partial/\partial z^\alpha\}$.

To every vector field λ on the coordinate neighborhood (U,z^i) of M there corresponds on $\pi^{-1}(U)$, a unique vector field $\tilde{\lambda} \in \mathcal{H}$ defined by

$$\pi_*(\tilde{\lambda}(p)) = \lambda(\pi(p)),\quad \text{for all}\quad p \in \pi^{-1}(U),$$

where π denotes the projection map from $T(M)$ onto M and π_* is its differential. We call $\tilde{\lambda}$ the horizontal lift of λ whose components with respect to $\{\partial/\partial z^\alpha\}$ are given by

(14) $$\tilde{\lambda}^\alpha = (\lambda^i, -L^i_{jk}\,\lambda^j z^{n+k}).$$

Let $[a,b]$ be an interval of R and $\sigma:[a,b]\to M$ a curve in M. A <u>horizontal lift of σ</u> is a curve $\tilde{\sigma}:[a,b]\to T(M)$ in T(M) such that its tangent vector τ belongs to $\mathcal{H}$ and $\pi(\tilde{\sigma}(t)) = \sigma(t)$, $t\in[a,b]$. An arbitrary curve in T(M) is said to be a horizontal curve if it is an integral curve of an element of $\mathcal{H}$, and a vertical curve if it is an integral curve of an element of $\mathcal{V}$.

The tangent plane to T(M) at a point is a direct sum of $\mathcal{H}$ and $\mathcal{V}$ at this point. Thus, if λ is any vector field on T(M) then $\lambda = \lambda' + \lambda''$, where $\lambda'\in\mathcal{H}$ and $\lambda''\in\mathcal{V}$. We define tensor fields a and $\bar{a}$ on T(M) by $a(\lambda) = \lambda'$ and $\bar{a}(\lambda) = \lambda''$. These tensor fields satisfy

$$a\circ a = a^2 = a,\ \bar{a}\circ\bar{a} = \bar{a}^2 = \bar{a},\ a\circ\bar{a} = a\bar{a} = 0,\ \bar{a}\circ a = \bar{a}a = 0,\ a + \bar{a} = I,$$

where 0 denotes the null map and I the identity map. The tensors a and $\bar{a}$ are called <u>projection tensors</u> associated with the disjoint distributions $\mathcal{H}$ and $\mathcal{V}$.

Let $[\lambda^\alpha_\beta]$ denote the 2nx2n matrix

$$[\lambda^\alpha_\beta] = \begin{bmatrix} [\delta^i_j] & [-L^i_{jk}z^{n+k}] \\ [0] & [\delta^i_j] \end{bmatrix},$$

and denote the inverse of $[\lambda^\alpha_\beta]$ by $[\mu^\alpha_\beta]$, so that

$$[\mu^\alpha_\beta] = \begin{bmatrix} [\delta^i_j] & [L^i_{jk}z^{n+k}] \\ [0] & [\delta^i_j] \end{bmatrix}.$$

In $(\pi^{-1}(U), z^\alpha)$ the components of the tensor fields a and $\bar{a}$ are given by $a^\alpha_\beta = \lambda^\alpha_k\mu^k_\beta$ and $\bar{a}^\alpha_\beta = \lambda^\alpha_{n+i}\mu^{n+i}_\beta$, which may be written as

$$(15) \qquad [a^{\alpha}_{\beta}] = \begin{bmatrix} [\delta^i_j] & [-L^i_{jk}\, z^{n+k}] \\ [\,0\,] & [\,0\,] \end{bmatrix},$$

and

$$(16) \qquad [\bar{a}^{\alpha}_{\beta}] = \begin{bmatrix} [\,0\,] & [L^i_{jk}\, z^{n+k}] \\ [\,0\,] & [\delta^i_j] \end{bmatrix}.$$

A distribution on a Riemann manifold is said to be a <u>null distribution</u> if every vector field in the distribution is self-orthogonal, and any two vector fields in the distribution are orthogonal with respect to the Riemann metric. It is proved that the dimension of a null distribution cannot exceed $\frac{1}{2}r$ if r is the dimension of the Riemann space on which the distribution is defined [12, p.241]. It follows directly from equations (5) that $\mathcal{V}$ is a null distribution, and $\mathcal{H}$ is shown to be a null distribution by use of (5) and the identity $\partial g_{ij}/\partial z^k = g_{jm} L^m_{ik} + g_{im} L^m_{jk}$ [2, p.17]. Consequently, $\mathcal{H}$ and $\mathcal{V}$ are null distributions of maximum dimensions on $(T(M),\bar{g})$.

An r-dimensional distribution is parallel with respect to a connection over a coordinate neighborhood if an arbitrary vector in the distribution at a point p is displaced into a vector in the distribution at q by parallel transport along <u>any</u> curve σ in M joining p to q. For a fixed vector at p the final vector at q may vary with the choice of the curve joining p to q. However, if the vector at q is independent of the choice of curve σ, and if this is the case for every vector at p and for every pair of points p,q, then the distribution is said to be <u>strictly parallel</u>.

An r-dimensional distribution of a differentiable manifold is said to be <u>integrable</u> if it is tangent to an r-dimensional submanifold of the specified manifold. A necessary and sufficient condition for the distribution to be integrable is that any two vector fields λ, μ of the distribution implies that the vector field $[\lambda,\mu]$ belongs to the distribution, where $[\lambda,\mu]$ denotes the Lie bracket of the two vector fields defined by

$$[\lambda,\mu]^i = \lambda^k \partial\mu^i/\partial x^k - \mu^k \partial\lambda^i/\partial x^k,$$

relative to the frame induced by the coordinate system.

Necessary and sufficient conditions for integrability and parallelism of distributions in terms of the projection tensors of the distributions are given by the following lemmas, whose proofs may be found in [12, p.254].

<u>Lemma 1</u>. Let H_1 and H_2 be two disjoint distributions which span the tangent space to a differentiable manifold at every point of a coordinate neighborhood. Let a and $\bar{a}$ be the projection tensors associated respectively with H_1 and H_2. A necessary and sufficient condition for H_1 to be integrable is

(17) $$a^{\alpha}_{\epsilon:\delta}\, a^{\epsilon}_{\beta}\, a^{\delta}_{\gamma} = a^{\alpha}_{\epsilon:\delta}\, a^{\epsilon}_{\gamma}\, a^{\delta}_{\beta},$$

and a necessary and sufficient condition for H_2 to be integrable is

(18) $$a^{\alpha}_{\epsilon:\delta}\, \bar{a}^{\epsilon}_{\beta}\, \bar{a}^{\delta}_{\gamma} = a^{\alpha}_{\epsilon:\delta}\, \bar{a}^{\epsilon}_{\gamma}\, \bar{a}^{\delta}_{\beta},$$

where the colon denotes covariant differentiation with respect to any symmetric connection defined on the coordinate neighborhood.

<u>Lemma 2</u>. Let H_1, H_2, a and $\bar{a}$ be defined as in Lemma 1. Let Γ with coefficients $\Gamma^{\alpha}_{\beta\gamma}$ be an arbitrary connection defined on the coordinate neighborhood. H_1 is a parallel distribution with respect

to Γ if and only if

(19) $$a^{\alpha}_{\epsilon:\gamma}\, a^{\epsilon}_{\beta} = 0,$$

and H_2 is a parallel distribution with respect to Γ if and only if

(20) $$a^{\alpha}_{\epsilon:\gamma}\, \bar{a}^{\epsilon}_{\beta} = 0,$$

where the colon denotes covariant differentiation with respect to Γ.

From these two lemmas it is clear that if a distribution is parallel with respect to a symmetric connection then it is an integrable distribution.

By use of equations (8) and (15) the tensor field $a^{\alpha}_{\beta:\gamma}$ on $(T(M),\bar{g})$ is found to have components given by

$$a^{n+i}_{j:k} = R^{i}_{jkp}\, z^{n+p},$$

(21)

$$a^{i}_{j:k} = a^{\alpha}_{n+j:\beta} = a^{\alpha}_{j:n+k} = 0.$$

It is clear from equations (21) that $a^{\alpha}_{\beta:\gamma} = 0$ when $R^{i}_{jkp} = 0$, and since the z^{n+p} are coordinate functions the converse also holds. Moreover, from the relation $a+\bar{a} = I$ we obtain $a^{\alpha}_{\beta:\gamma} = -\bar{a}^{\alpha}_{\beta:\gamma}$. Thus, the projection tensor fields a and $\bar{a}$ associated with $\mathcal{H}$ and $\mathcal{V}$ are covariant constant if and only if $R_{ijkp} = 0$.

<u>Theorem 4</u>. The horizontal distribution $\mathcal{H}$ on $(T(M),\bar{g})$ is integrable if and only if it is a parallel distribution, and this is the case if and only if M is flat.

<u>Proof</u>: Identify $\mathcal{H}$ and $\mathcal{V}$ respectively with H_1 and H_2 of the preceding lemmas. We shall first show that $\mathcal{H}$ is integrable if and only if $R^{i}_{jkl} = 0$. If $R^{i}_{jkl} = 0$ then a is covariant constant and it follc s from Lemma 1 that $\mathcal{H}$ is integrable. Assume $\mathcal{H}$ is integrable then the

conditions (17) yield $(R^i_{jkp} - R^i_{kjp})\, z^{n+p} = 0$, which implies $R^i_{jkp} = R^i_{kjp}$ since z^{n+p} are coordinate functions, and consequently

$$R^i_{jkp} = -R^i_{kpj} = -R^i_{pkj} = R^i_{pjk} = R^i_{jpk} = -R^i_{jkp}.$$

So we have $R^i_{jkp} = 0$.

If $R^i_{jkl} = 0$ it is clear that $\mathcal{H}$ is a parallel distribution by use of $a^\alpha_{\beta:\gamma} = 0$ and Lemma 2. Assume $\mathcal{H}$ is a parallel distribution on $T(M)$. Then conditions (19) yield $R^i_{jkp}\, z^{n+p} = 0$ for arbitrary z^{n+p}. Hence $R^i_{jkp} = 0$. Q.E.D.

<u>Theorem 5</u>. The vertical distribution $\mathcal{V}$ on $(T(M),\bar{g})$ is a parallel distribution, and is integrable.

<u>Proof</u>: Identify $\mathcal{H}$ and $\mathcal{V}$ respectively with H_1 and H_2 of the preceding lemmas. The tensor

$$a^\alpha_{\epsilon:\gamma}\, \bar{a}^\epsilon_\beta = a^\alpha_{k:\gamma}\, \bar{a}^k_\beta + a^\alpha_{n+k:\gamma}\, \bar{a}^{n+k}_\beta,$$

vanishes on $(T(M),\bar{g})$ since $\bar{a}^k_\beta = a^\alpha_{n+k:\gamma} = 0$. Hence, equations (20) are satisfied. From Lemma 2 it follows that $\mathcal{V}$ is a parallel distribution on $(T(M),\bar{g})$. Since Γ is a symmetric connection it then follows that $\mathcal{V}$ is integrable. Q.E.D.

It is noted that $\mathcal{V}$ is not necessarily strictly parallel, however. Necessary and sufficient conditions for $\mathcal{V}$ to be a strictly parallel distribution on $(T(M),\bar{g})$ are that $\{\partial/\partial z^{n+i}\}$ be parallel vector fields. This would be the case if and only if $\underset{m}{\lambda}{}^\alpha_{:\beta} = 0$, for $m = 1,2, \ldots ,n$, where $\underset{m}{\lambda}{}^\alpha = (0, \delta^i_m)$. Since the only component of $\underset{m}{\lambda}{}^\alpha_{:\beta}$ which is not zero is $\underset{m}{\lambda}{}^{n+i}_{:j} = L^i_{jm}$, we obtain that $\mathcal{V}$ is a strictly parallel distribution on $(T(M),\bar{g})$ if and only if $L^i_{jk} = 0$.

3. Vector fields. Let λ and μ be tangent vectors on M with components λ^i and μ^i relative to $\{\partial/\partial z^i\}$. The extended vector field $\bar{\lambda}$ of λ has components $\bar{\lambda}^\alpha = (\lambda^i, (\partial\lambda^i/\partial z^h)z^{n+h})$ relative to $\{\partial/\partial z^\alpha\}$. Consequently, we obtain

$$\begin{aligned}\bar{g}_{\alpha\beta}\bar{\lambda}^\beta &= ((\partial g_{ij}/\partial z^k)z^{n+k}\ \lambda^j + g_{ij}(\partial\lambda^j/\partial z^k)z^{n+k},\ g_{ij}\ \lambda^j),\\ &= ((\partial\lambda_i/\partial z^k)z^{n+k},\ \lambda_i),\end{aligned}$$

which is the extension of $\lambda_i = g_{ij}\ \lambda^j$. Similarly, we obtain $\bar{g}^{\alpha\beta}\bar{\lambda}_\beta = (\lambda^i, (\partial\lambda^i/\partial z^k)z^{n+k})$ as the extension of $\lambda^i = g^{ij}\ \lambda_j$.

The extension $\overline{[\lambda,\mu]}$ of the Lie bracket $[\lambda,\mu]$ of the vector fields λ and μ has components

$$\overline{[\lambda,\mu]}^\alpha = ([\lambda,\mu]^i, (\partial[\lambda,\mu]^i/\partial z^k)z^{n+k}),$$

which reduces to

$$[\bar{\lambda},\bar{\mu}]^\alpha = \bar{\lambda}^\beta\,\partial\bar{\mu}^\alpha/\partial z^\beta - \bar{\mu}^\beta\,\partial\bar{\lambda}^\alpha/\partial z^\beta,$$

on using the definition of an extended vector field.

These observations prove the following theorem.

Theorem 6. The operation of raising or lowering indices of a vector field or of forming the Lie bracket of two vector fields is commutative with the operation of extending the vector fields.

It follows from the above theorem that integrable distributions on M extend to integrable distributions on $(T(M),\bar{g})$.

Let λ be a contravariant vector field on M with components λ^i relative to $\{\partial/\partial z^i\}$. λ is called a parallel vector field if it is covariant constant, that is, if $\lambda^i_{;j} = 0$ [12, p.239]. Such a field has the property that the whole field on (U,z^i) can be generated by parallel transport, independent of curves, of a vector of the field at an arbitrary point in U. A strictly parallel distribution on M of dimension r is spanned by r parallel vector fields.

Parallel vector fields on $(T(M),\bar{g})$ can similarly be defined, and the following lemma gives necessary and sufficient conditions for an arbitrary vector field defined on $(\pi^{-1}(U), z^{\alpha})$ to be a parallel vector field.

<u>Lemma 3</u>. Let λ be an arbitrary vector field on $(T(M),\bar{g})$ with components $\lambda^{\alpha} = (\lambda^{i}, \lambda^{n+i})$ relative to $\{\partial/\partial z^{\alpha}\}$. Then λ is a parallel vector field if and only if it satisfies the following conditions:

(i) λ^{i} is independent of $z^{n+1}, z^{n+2}, \ldots, z^{2n}$, and satisfies

$$\partial\lambda^{i}/\partial z^{j} = -L^{i}_{jk}\ \lambda^{k};$$

(ii) λ^{n+i} satisfies

$$\partial\lambda^{n+i}/\partial z^{j} = -(\partial L^{i}_{jk}/\partial z^{h})\,\lambda^{k}\,z^{n+h} - L^{i}_{jk}\ \lambda^{n+k},$$

and

$$\partial\lambda^{n+i}/\partial z^{n+j} = -L^{i}_{jk}\ \lambda^{k}.$$

<u>Proof</u>: By use of (8), $\lambda^{\alpha}_{:\beta}$ may be written as

$$\begin{aligned}
&\lambda^{i}_{:j} = \partial\lambda^{i}/\partial z^{j} + L^{i}_{jk}\ \lambda^{k},\\
&\lambda^{i}_{:n+j} = \partial\lambda^{i}/\partial z^{n+j},\\
(22)\qquad &\lambda^{n+i}_{:j} = \partial\lambda^{n+i}/\partial z^{j} + (\partial L^{i}_{jk}/\partial z^{h})\,\lambda^{k}\,z^{n+h} + L^{i}_{jk}\ \lambda^{n+k},\\
&\lambda^{n+i}_{:n+j} = \partial\lambda^{n+i}/\partial z^{n+j} + L^{i}_{jk}\ \lambda^{k}.
\end{aligned}$$

Let $\lambda^{\alpha}_{:\beta} = 0$. Then the first two equations of (22) are equivalent to the conditions (i), and the last two equations are equivalent to the conditions (ii). Conversely, it follows from (22) that $\lambda^{\alpha}_{:\beta} = 0$ holds for any vector field λ on $(T(M),\bar{g})$ which satisfies conditions (i) and (ii) of the lemma. Q.E.D.

It is easily verified by use of equations (4) and conditions (i) of Lemma 3 that λ^i are components of a parallel vector field on M when λ^{α} are components of a parallel vector field on $(T(M),\bar{g})$. Consequently, from the Ricci identity

$$\lambda^i_{;jk} - \lambda^i_{;kj} = R^i_{hkj}\ \lambda^h,$$

we obtain

$$R^i_{hkj}\ \lambda^h = 0,$$

when λ^{α} are components of a parallel vector field on $(T(M),\bar{g})$.

Let μ^{α} denote components of the projection $a(\lambda)$ of λ on $\mathcal{H}$, where λ is an arbitrary vector field on $(\pi^{-1}(U),z^{\alpha})$. Then from (15) we obtain

(23) $$\mu^{\alpha} = (\ \lambda^i,\ -\ L^i_{jk}\ \lambda^j\ z^{n+k}).$$

The following theorem is a consequence of Lemma 3 and properties of the projection tensors.

<u>Theorem 7</u>. A vector field λ on $(T(M),\bar{g})$ is a parallel vector field if and only if both $a(\lambda)$ and $\bar{a}(\lambda)$ are parallel vector fields.

<u>Proof</u>: Let $a(\lambda)$ and $\bar{a}(\lambda)$ be parallel vector fields. Then, from

(24) $$a(\lambda) + \bar{a}(\lambda) = \lambda,$$

and the linearity of covariant differentiation, it follows that λ is a parallel vector field.

Let λ be a parallel vector field. Then it follows from (24) that either $a(\lambda)$ and $\bar{a}(\lambda)$ are both parallel vector fields, or neither of them is parallel. We shall show that $a(\lambda)$ with components given by (23) is a parallel vector field. Since $\mu^i = \lambda^i$ and λ is assumed to be a parallel vector field, conditions (i) of Lemma 3 are clearly satisfied. By use of conditions (i) and $R^i_{jkl}\ \lambda^j = 0$ one also finds conditions (ii) are satisfied. Q.E.D.

Let μ be a vector field on M with components μ^i relative to $\{\partial/\partial z^i\}$. Then $\mu^\alpha = (0, \mu^i)$ can be shown to be components of a vertical vector field on $(T(M), \bar{g})$ relative to $\{\partial/\partial z^\alpha\}$. Consequently, Lemma 3 yields the following:

Theorem 8. A vertical vector field μ on $(T(M), \bar{g})$ with components $\mu^\alpha = (0, \mu^i)$ relative to $\{\partial/\partial z^\alpha\}$ is a parallel vector field if and only if μ^i are components of a parallel vector field on M relative to $\{\partial/\partial z^i\}$.

It follows from the definitions of $\tilde{\lambda}$ and $\bar{\lambda}$ that these vector fields are identical if and only if λ is a parallel vector field on M. Moreover, it can easily be shown by Lemma 3 that $\tilde{\lambda}$ is a parallel vector field on $(T(M), \bar{g})$ if and only if λ is a parallel vector field on M. Consequently, the following relationship exists for the vector fields $\tilde{\lambda}$, $\bar{\lambda}$ on $(T(M), \bar{g})$ and λ on M.

Theorem 9. If any one of the vector fields $\tilde{\lambda}$, $\bar{\lambda}$ or λ is a parallel vector field on the manifold for which it is defined, then all three vector fields are parallel.

If λ is a non-trivial vector field on M then $\tilde{\lambda}$ and $\bar{\lambda}$ are non-trivial vector fields on T(M). Theorem 9 then shows that

$(T(M),\bar{g})$ admits a non-trivial parallel vector field if and only if a non-trivial parallel vector field exists on M. We shall obtain a more precise formulation of this relationship, but first we need the following lemma.

Lemma 4. The number of linearly independent, parallel, horizontal vector fields on $(T(M),\bar{g})$ is equal to the number of linearly independent, parallel, vertical vector fields on $(T(M),\bar{g})$. Hence, the strictly parallel distribution of maximum dimension on $(T(M),\bar{g})$ has dimensionality which is a multiple of 2.

Proof: Every parallel, horizontal vector field $(\lambda^i, (\partial\lambda^i/\partial z^k)z^{n+k})$ corresponds to a parallel, vertical vector field $(0, \lambda^i)$, and conversely, every parallel, vertical vector field $(0, \lambda^i)$ gives rise to the parallel, horizontal vector field $(\lambda^i, (\partial\lambda^i/\partial z^k)z^{n+k})$. Moreover, linearly independent horizontal (or vertical) vectors give rise to linearly independent vertical (or horizontal) vectors under this correspondence. Q.E.D.

Theorem 10. The number of linearly independent, parallel vector fields on $(T(M),\bar{g})$ is exactly twice the number of linearly independent, parallel vector fields on M.

Proof: Let r be the dimension of the strictly parallel distribution of maximum dimension on M. A basis of this distribution lifts to r parallel, horizontal vector fields on $(T(M),\bar{g})$. Hence, by Lemma 4 the number of linearly independent, parallel vector fields on $(T(M),\bar{g})$ is at least equal to $2r$. Moreover, this number is at most equal to $2r$ since linearly independent horizontal vectors project by π_* to linearly independent vectors on M, and π_* preserves parallelism. Q.E.D.

It has been shown that every recurrent manifold with a definite metric whose recurrence covector is everywhere non-zero, admits exactly $n-2$ linearly independent, parallel vector fields [10]. Hence, we obtain a corollary to the above theorem.

Corollary 2. If M is a recurrent manifold whose recurrence covector is everywhere non-zero, then $(T(M),\bar{g})$ admits exactly $2n-4$ linearly independent, parallel vector fields.

A vector field λ is called parallel relative to the distribution $\mathcal{H}$ (or $\mathcal{V}$) if for any points p,q on an integral curve of an element of $\mathcal{H}$ (or $\mathcal{V}$), the vectors $\lambda(p)$, $\lambda(q)$ are parallel relative to this curve [12, p.255].

In terms of the projection tensors associated with $\mathcal{H}$ and $\mathcal{V}$, a vector field λ of $(T(M),\bar{g})$ with components λ^α is parallel relative to $\mathcal{H}$ if $\lambda^\alpha_{:\epsilon}\, dz^\epsilon = 0$ for all dz^α satisfying $\bar{a}^\alpha_\epsilon\, dz^\epsilon = 0$. Thus, the condition for λ to be parallel relative to $\mathcal{H}$ is given by $\lambda^\alpha_{:\epsilon}\, a^\epsilon_\beta = 0$. Similarly, the condition for λ to be parallel relative to $\mathcal{V}$ is given by $\lambda^\alpha_{:\epsilon}\, \bar{a}^\epsilon_\beta = 0$. These conditions are each weaker than the condition for parallelism of a vector field, $\lambda^\alpha_{:\beta} = 0$, which is equivalent to both of the conditions above.

Theorem 11. The horizontal lift $\tilde{\lambda}$ of a vector field λ on M is parallel relative to the vertical distribution $\mathcal{V}$ on $(T(M),\bar{g})$.

<u>Proof</u>: The components $\tilde{\lambda}^{\alpha}_{:\epsilon}$ are given by

$$\tilde{\lambda}^{i}_{:j} = \lambda^{i}_{;j},$$

(25)
$$\tilde{\lambda}^{n+i}_{:j} = \lambda^{l} z^{n+m} (-\partial L^{i}_{lm}/\partial z^{j} + \partial L^{i}_{jl}/\partial z^{m} - L^{i}_{jk} L^{k}_{lm}) - L^{i}_{lm}(\partial \lambda^{l}/\partial z^{j}) z^{n+m},$$

$$\tilde{\lambda}^{\alpha}_{:n+j} = 0.$$

Thus, $\tilde{\lambda}^{\alpha}_{:\epsilon}\, \ddot{a}^{\epsilon}_{\beta}$ is seen to vanish by use of (25) and (16). Q.E.D.

Let λ^{i} be components of a contravariant vector field λ on (U,z^{i}) of M, and denote $\lambda_{i} = g_{ij}\lambda^{j}$. λ is called an <u>incompressible vector field</u> if

(26)
$$\lambda^{i}_{;i} = 0;$$

λ is called a <u>Killing vector field</u> if

(27)
$$\lambda_{i;j} + \lambda_{j;i} = 0;$$

and λ is called a <u>harmonic vector field</u> if

(28)
$$\lambda_{i;j} - \lambda_{j;i} = 0, \quad \text{and} \quad \lambda^{i}_{;i} = 0.$$

By use of the definition of $\bar{\lambda}$ and (8), $\bar{\lambda}^{\alpha}_{:\beta}$ and $\bar{\lambda}_{\alpha:\beta}$ may be expressed as

$$\bar{\lambda}^{i}_{:j} = \lambda^{i}_{;j}, \quad \bar{\lambda}^{i}_{:n+j} = 0,$$
$$\bar{\lambda}^{n+i}_{:j} = (\partial\lambda^{i}_{;j}/\partial z^{k})\, z^{n+k}, \quad \bar{\lambda}^{n+i}_{:n+j} = \lambda^{i}_{;j},$$

and

$$\bar{\lambda}_{i:j} = (\partial\lambda_{i;j}/\partial z^{k})\, z^{n+k}, \quad \bar{\lambda}_{i:n+j} = \lambda_{i;j},$$
$$\bar{\lambda}_{n+i:j} = \lambda_{i;j}, \quad \bar{\lambda}_{n+i:n+j} = 0.$$

Consequently, the divergence of $\bar{\lambda}$ is given by

$$\bar{\lambda}^{\alpha}_{:\alpha} = \bar{\lambda}^{i}_{:i} + \bar{\lambda}^{n+i}_{:n+i} = 2\,\lambda^{i}_{;i}.$$

These equations together with the definitions (26), (27), and (28), and the corresponding definitions for these vector fields on $(\pi^{-1}(U), z^{\alpha})$ of $(T(M), \bar{g})$ yield the following theorem.

Theorem 12. The extended vector field $\bar{\lambda}$ of the contravariant vector field λ on M is respectively an incompressible (or Killing or harmonic) vector field on $(T(M), \bar{g})$ if and only if the vector field λ is respectively an incompressible (or Killing or harmonic) vector field on M.

From (25) it is clear that $\tilde{\lambda}^{\alpha}_{:\alpha} = \lambda^{i}_{;i}$, and consequently that $\tilde{\lambda}$ is an incompressible vector field if and only if λ is an incompressible vector field. Moreover, from (5) and (25) we obtain

$$\begin{aligned}
\tilde{\lambda}_{n+i:j} &= \lambda_{i;j}, \\
\tilde{\lambda}_{j:n+i} &= \tilde{\lambda}_{n+i:n+j} = 0, \\
\tilde{\lambda}_{i:j} &= z^{n+k}((\partial g_{im}/\partial z^{k})\lambda^{m}_{;j} - g_{il}\, L^{l}_{mk}\, \partial\lambda^{m}/\partial z^{j}) \\
&\quad + g_{il}\,(-\partial L^{l}_{mk}/\partial z^{j} + \partial L^{l}_{jm}/\partial z^{k} - L^{l}_{jr}\, L^{r}_{mk})\ \lambda^{m}\, z^{n+k}.
\end{aligned}$$

These equations prove the following theorem.

Theorem 13. If $\tilde{\lambda}$ is either a harmonic or a Killing vector field on $(T(M), \bar{g})$ then it is the horizontal lift of a parallel vector field on M.

4. Isometries. Let $f: M \rightarrow M'$, where M and M' are n-dimensional Riemann manifolds with positive definite metric tensors g and g', respectively. Denote by (p, λ) a tangent vector λ at $p \in M$, and by f_* the differential of f. The map f induces a map

$$\bar{f}: (T(M), \bar{g}) \rightarrow (T(M'), \bar{g}'),$$

defined by

$$\bar{f}((p, \lambda)) = (f(p), f_*(\lambda)).$$

We call $\bar{f}$ the <u>extended map of f</u>.

Let (U, z^i) and (U', z'^i) be coordinate neighborhoods of M and M', respectively, such that $f(U) \cap U' \neq \emptyset$. Then $\bar{f}: (\pi^{-1}(U), z^\alpha) \longrightarrow (\pi'^{-1}(U'), z'^\alpha)$ is defined by

$$z'^i = f'^i(z^1, z^2, \ldots, z^n), \quad z'^{n+i} = (\partial z'^i / \partial z^k) z^{n+k}, \tag{29}$$

whose Jacobian is

$$[\partial z'^\alpha / \partial z^\beta] = \begin{bmatrix} [\partial z'^i / \partial z^j] & [(\partial^2 z'^i / \partial z^k \partial z^j) z^{n+k}] \\ [0] & [\partial z'^i / \partial z^j] \end{bmatrix}, \tag{30}$$

with inverse

$$\overline{[\partial z^\alpha / \partial z'^\beta]} = \begin{bmatrix} [\partial z^i / \partial z'^j] & [(\partial^2 z^i / \partial z'^k \partial z'^j) z'^{n+k}] \\ [0] & [\partial z^i / \partial z'^j] \end{bmatrix}. \tag{31}$$

It is clear from (30) that the Jacobian of $\bar{f}$ has rank 2n if and only if the Jacobian of f has rank n.

<u>Theorem 14</u>. The map $f: M \longrightarrow M'$ is an isometry if and only if its extended map $\bar{f}$ is an isometry of the Extended Riemann Manifolds of M and M'.

<u>Proof</u>: Let $\bar{f}$ be an isometry. Then the metric tensors of T(M) and T(M') satisfy

$$\bar{g}'_{\alpha\beta} = (\partial z^\epsilon / \partial z'^\alpha)(\partial z^\delta / \partial z'^\beta)\, \bar{g}_{\epsilon\delta},$$

which are equivalent to the following by (31)

$$\begin{aligned} \bar{g}'_{ij} &= z'^{n+k}\, \partial((\partial z^h / \partial z'^i)(\partial z^m / \partial z'^j)\, g_{hm}) / \partial z'^k, \\ \bar{g}'_{i\,n+j} = \bar{g}'_{n+i\,j} &= (\partial z^h / \partial z'^i)(\partial z^m / \partial z'^j)\, g_{hm}, \\ \bar{g}'_{n+i\,n+j} &= 0. \end{aligned} \tag{32}$$

By (5) and (32) we find

$$g'_{ij} = (\partial z^h/\partial z'^i)(\partial z^m/\partial z'^j)g_{hm}. \tag{33}$$

Hence, f is an isometry. Conversely, if f is an isometry then (33) holds, and (32) follows by (5). Thus, $\bar{f}$ is an isometry. Q.E.D.

Let A be a differentiable manifold, P the projection of $T^*(A)$ onto A, and (U,y^i) a coordinate neighborhood of A. Let (p,ω) be a covariant vector at $p \in U$ with components $\omega_i = \omega(\partial/\partial y^i)$. Then $P^{-1}(U)$ is a coordinate neighborhood of $T^*(A)$ with coordinate functions $(\bar{y}^i, \bar{\omega}_i)$ defined by

$$\bar{y}^i = y^i \circ P,$$

$$\bar{\omega}_i((p,\omega)) = \omega_i .$$

As has been our convention throughout for the tangent bundle, $u^i = y^i$ will denote either y^i or $\bar{y}^i$, depending on the domain, and u_{n+i} will denote $\bar{\omega}_i$.

Let A be given a symmetric connection with coefficients L^i_{jk} on (U,u^i). Then $(T^*(A),G)$ is called the Riemann extension of A [11], where G is defined by

$$\begin{aligned} G_{ij} &= -2\, L^k_{ij}\, u_{n+k}, \\ G_{i\ n+j} &= G_{n+j\ i} = \delta^j_i, \\ G_{n+i\ n+j} &= 0, \end{aligned} \tag{34}$$

relative to $\{\partial/\partial u^i, \ \partial/\partial u_{n+i}\}$. Naturally, the connection may be induced by a metric given in A.

Let λ^i be components of a tangent vector at $p \in M$ relative to $\{\partial/\partial z^i\}$. Then the covariant vector ω with components $\lambda_i = g_{ij}\lambda^j$ is called the associate vector to $\lambda = (\lambda^i)$. Let the map

$$f:(T(M),\bar{g}) \longrightarrow (T^*(M),G),$$

be defined by

$$f((p,\lambda)) = (p,\omega).$$

Theorem 15. f is an isometry.

Proof: Let $(\pi^{-1}(U),z^\alpha)$ and $(P^{-1}(U);u^i,u_{n+i})$ be coordinate neighborhoods of $T(M)$ and $T^*(M)$, respectively. Then f may be represented by the functions

$$u^i = z^i,\ u_{n+i} = g_{ik}(z^1,z^2,\ \ldots\ ,z^n)\ z^{n+k},$$

whose inverse is

$$z^i = u^i,\ z^{n+i} = g^{ik}(u^1,u^2,\ \ldots\ ,u^n)\ u_{n+k},$$

with Jacobian

$$(35)\qquad \begin{bmatrix} [\partial z^\alpha/\partial u^j] \\ [\partial z^\alpha/\partial u_{n+j}] \end{bmatrix} = \begin{bmatrix} [\delta^i_j] & [(\partial g^{ik}/\partial u^j)\, u_{n+k}] \\ [0] & [g^{ij}] \end{bmatrix} .$$

By definition, f is an isometry if and only if

$$G_{ij} = (\partial z^\epsilon/\partial u^i)(\partial z^\delta/\partial u^j)\bar{g}_{\epsilon\delta},$$

$$G_{i\ n+j} = (\partial z^\epsilon/\partial u^i)(\partial z^\delta/\partial u_{n+j})\bar{g}_{\epsilon\delta},$$

$$G_{n+i\ n+j} = (\partial z^\epsilon/\partial u_{n+i})(\partial z^\delta/\partial u_{n+j})\bar{g}_{\epsilon\delta},$$

where $G_{\alpha\beta}$ are given by (34). By use of (35) and (5) together with the following well known identities [2, p.17]

$$\partial g_{ij}/\partial z^k = g_{im}\ L^m_{jk} + g_{jm}\ L^m_{ik},$$

$$\partial g^{ij}/\partial z^k = -g^{ih}\ L^j_{hk} - g^{jh}\ L^i_{hk},$$

we conclude that f is an isometry. Q.E.D.

SECTION III

PARALLEL DISPLACEMENT

1. Modified Myller configurations. In an n-dimensional Riemann manifold M with positive definite metric tensor, Miron defined a Myller configuration $M(C,\lambda,T^r)$ to be a triple consisting of a curve C with unit tangent vector τ and arc length s, an r-dimensional distribution T^r on C, and a unit vector field $\lambda \epsilon T^r$. λ is defined to be parallel relative to $M(C,\lambda,T^r)$ if its covariant derivative along C belongs to the orthogonal complement of T^r, and λ is unilaterally conjugate to τ if $D\lambda/ds \ \epsilon \ T^r$ [5,6].

We shall investigate the configuration $A(C,T^r,T^{n-r})$ where A is a C^∞ n-manifold with a symmetric connection, where C is a curve in A, and where T^r and T^{n-r} are disjoint distributions on C of dimensions r and n-r, respectively. We call $A(C,T^r,T^{n-r})$ a modified Myller configuration. Projection tensors associated with T^r and T^{n-r} will be employed to define relative parallelism and relative conjugacy for a vector field $\lambda \epsilon T^r$, generalizing those concepts introduced by Miron.

Except for their difference in domains, T^r and T^{n-r} as disjoint vector spaces are similar to $\mathcal{H}$ and $\mathcal{V}$ investigated in Section II. Thus, it is obvious that T^r and T^{n-r} on C uniquely determine the associated projection tensors.

Let $a = (a^i_j)$ and $\bar{a} = (\bar{a}^i_j)$ denote the projection tensors on C associated with T^r and T^{n-r}, respectively. Let t be the curve parameter and $\mu = \mu^i \partial/\partial x^i$ any vector field on C. Then we may write

$$\mu^i = a^i_k \mu^k + \bar{a}^i_k \mu^k, \quad \text{or} \quad \mu = a(\mu) + \bar{a}(\mu),$$

where $a(\mu) = (a^i_k \mu^k) \in T^r$ and $\bar{a}(\mu) = (\bar{a}^i_k \mu^k) \in T^{n-r}$. From the definitions of a and $\bar{a}$ one easily establishes

$$\begin{aligned} &Da^i_j/dt = -D\bar{a}^i_j/dt, \\ (1) \qquad &(Da^i_k/dt)\,\bar{a}^k_j = -a^i_k\, D\bar{a}^k_j/dt = a^i_k\, Da^k_j/dt, \\ &(D\bar{a}^i_k/dt)\,a^k_j = -\bar{a}^i_k\, Da^k_j/dt = \bar{a}^i_k\, D\bar{a}^k_j/dt. \end{aligned}$$

<u>Theorem 1</u>. Let b and $\bar{b}$ be (1,1) tensor fields on a curve C in A, which satisfy the following conditions

(2) $\qquad \operatorname{rank}(b) = r, \ \operatorname{rank}(\bar{b}) = n-r, \ b+\bar{b} = I, \ b^2 = b.$

Then there exists a unique configuration $A(C,T^r,T^{n-r})$ for which b is the projection tensor associated with T^r, and $\bar{b}$ is the projection tensor associated with T^{n-r}.

<u>Proof</u>: Consider b and $\bar{b}$ as linear transformations of the tangent space $T(A)$ to A at each point of C. Define

$$T^r = b(T(A)) \quad \text{and} \quad T^{n-r} = \bar{b}(T(A)).$$

It follows from (2) that T^r and T^{n-r} are disjoint distributions on C of dimensions r and $n-r$, respectively. Moreover, from the theory of linear spaces it follows that the last two expressions in (2) are necessary and sufficient conditions for b to be the unique projection onto T^r along T^{n-r}, and for $\bar{b}$ to be the unique projection onto T^{n-r} along T^r [3]. Q.E.D.

Definition 1. A vector field $\lambda \in T^r$ is defined as parallel relative to $A(C,T^r,T^{n-r})$ if $a(D\lambda/dt) = 0$. It will be called briefly as a relative parallel vector field.

Definition 2. A vector field $\lambda \in T^r$ is conjugate to C relative to $A(C,T^r,T^{n-r})$ if $\bar{a}(D\lambda/dt) = 0$. It will be called briefly as a relative conjugate vector field.

One may obtain a set of definitions "dual" to the above by replacing T^r by T^{n-r}, T^{n-r} by T^r, a by $\bar{a}$, and $\bar{a}$ by a wherever they occur.

Let λ_k, $k = 1,2, \ldots ,r$ be base fields on T^r, and $\mu \in T^r$. Then we have

$$\mu = \sum_{k=1}^{r} b^k \lambda_k, D\mu/dt = \sum_{k=1}^{r}((db^k/dt)\lambda_k + b^k D\lambda_k/dt). \tag{3}$$

From (3) it is clear that $\bar{a}(D\mu/dt) = 0$ if $\bar{a}(D\lambda_k/dt) = 0$ for all k. Conversely, T^r is spanned by relative conjugate vector fields if $\bar{a}(D\mu/dt) = 0$ for all $\mu \in T^r$.

It is clear from Definitions 1 and 2 that $\lambda \in T^r$ is both relative parallel and relative conjugate if and only if $D\lambda/dt = 0$.

If $\lambda \in T^r$ then its components λ^i satisfy $a^i_k \lambda^k = \lambda^i$. Hence, we obtain

$$(Da^i_k/dt)\lambda^k + a^i_k D\lambda^k/dt = D\lambda^i/dt,$$

and it follows from $a^2 = a$ that $a^i_m(Da^m_k/dt)\lambda^k = 0$. But this is not a sufficient condition for λ to be an element of T^r. The significance of the necessary condition is given in the following theorem which can be easily proved.

Theorem 2. Let $\lambda = \lambda^i \partial/\partial x^i$ be any vector field on C of $A(C,T^r,T^{n-r})$. Then $\bar{a}(\lambda)$ is relative conjugate if and only if

$$a^i_m (Da^m_k/dt)\ \lambda^k = 0;$$

and $\bar{a}(\lambda)$ is relative parallel if and only if

$$\bar{a}^i_m (D\bar{a}^m_k/dt)\ \lambda^k = -\bar{a}^i_k\ D\lambda^k/dt.$$

From Theorem 2 and its "dual", we obtain the result that $a(\lambda)$ and $\bar{a}(\lambda)$ are both relative conjugate if and only if $(Da^i_k/dt)\ \lambda^k = 0$, and they are both relative parallel if and only if

$$D\lambda^i/dt = (Da^i_k/dt - 2\ a^i_m\ Da^m_k/dt)\ \lambda^k.$$

Let $\mathcal{M} = \mathcal{M}^i\ \partial/\partial x^i$ be any vector field on C. Then necessary and sufficient conditions for T^r to be spanned by relative conjugate vector fields may be written as

$$\bar{a}^i_j\ D(a^j_k \mathcal{M}^k)/dt = \bar{a}^i_j (Da^j_k/dt) \mathcal{M}^k = 0$$

which hold for any $\mathcal{M}$. Consequently we have $\bar{a}^i_j\ Da^j_k/dt = 0$, which by use of (1) yields

Theorem 3. Elements of T^r are relative conjugate as its base fields. Necessary and sufficient conditions for T^r to consist entirely of relative conjugate fields are $\bar{a}^i_k\ D\bar{a}^k_j/dt = 0$.

Let T^r be spanned by relative parallel vector fields λ_k, k = 1,2, ... ,r. Then from (3) we obtain

$$a(D\mathcal{M}/dt) = \sum_{k=1}^{r} (db^k/dt)\ \lambda_k,$$

which implies that any vector field $\mathcal{M} \in T^r$ is relative parallel if and only if it is a constant with respect to a relative parallel base.

From $a + \bar{a} = I$ and (1) we obtain

$$a^i_k\ Da^k_j/dt + \bar{a}^i_k\ Da^k_j/dt = Da^i_j/dt = -D\bar{a}^i_j/dt.$$

It is then easy to show that T^r and T^{n-r} are both spanned by relative conjugate vector fields if and only if Da^i_j/dt or $D\bar{a}^i_j/dt$ is a zero tensor field.

Let $\mathcal{T} \in T^r$ denote the tangent to C of the configuration $A(C,T^r,T^{n-r})$. We call C a <u>relative auto-parallel curve</u> if $\mathcal{T}$ is a relative parallel vector field.

By II.15 and Theorem 2 we have

$$\mathcal{M}^{n+i} = -L^i_{jk}\, \mathcal{M}^j\, z^{n+k} \quad \text{and} \quad \dot{\mathcal{M}}^i + L^i_{jk}\, \mathcal{M}^j\, \dot{z}^k = 0, \tag{4}$$

if $\mathcal{M} \in \mathcal{H}$ is relative parallel.

Consider the configuration $T(M)(C,\mathcal{H},\mathcal{V})$ where $\mathcal{H}$ and $\mathcal{V}$ are the horizontal and vertical distributions of $T(M)$. If C is a horizontal curve then $\dot{z}^i$ does not vanish for all i at any point. Consequently, $\pi(C)$ is a curve in M. It then follows from (4) that C is the horizontal lift of a geodesic in M if it is a relative auto-parallel curve. Conversely, (4) establishes the fact that the horizontal lift $\tilde{C}$ of a geodesic C in M is a relative auto-parallel curve. Hence, we have

<u>Theorem 4</u>. Let $\mathcal{T} \in \mathcal{H}$. Then C is a relative auto-parallel curve if and only if it is the horizontal lift of a geodesic in M.

In case that $\mathcal{T} \in \mathcal{V}$, then C is a relative auto-parallel curve if and only if it is a minimal geodesic in $(T(M),\bar{g})$. This can be easily proved by Theorem 2.

2. <u>Hypersurface in cotangent bundle</u>. Let G be a symmetric 2-co tensor on $(P^{-1}(U);u^i,u_{n+i})$ defined by

$$\begin{aligned} G_{ij} &= g_{ij} + g^{hk} L^l_{hi} L^m_{kj} u_{n+l} u_{n+m}, \\ G_{i\,n+j} &= G_{n+j\,i} = -g^{jh} L^k_{hi} u_{n+k}, \\ G_{n+i\,n+j} &= g^{ij}. \end{aligned} \tag{5}$$

Since for any tangent vector $\lambda^{\alpha} = (\lambda^{i}, \lambda^{n+i})$ on $(T^{*}(M), G)$

$$\|\lambda\| = G_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} = g_{ij}\lambda^{i}\lambda^{j} + g^{ij}(\lambda^{n+i} - L^{k}_{ih}\lambda^{h}u_{n+k})(\lambda^{n+j} - L^{l}_{jm}\lambda^{m}u_{n+l}),$$

it is clear that G is positive definite.

The contravariant components $G^{\alpha\beta}$ of G are given by $G^{ij} = G_{n+i\ n+j}$, $G^{i\ n+j} = -G_{j\ n+i}$, $G^{n+i\ n+j} = G_{ij}$, and the coefficients $\Gamma^{\alpha}_{\beta\gamma}$ of the connection Γ induced by G are found to be

$$\Gamma^{i}_{jk} = L^{i}_{jk} + \tfrac{1}{2}g^{im}g^{pr}(L^{v}_{rk}R^{u}_{pmj} + L^{v}_{rj}R^{u}_{pmk})\,u_{n+u}\,u_{n+v},$$

$$(6)\qquad \begin{aligned}\Gamma^{n+i}_{jk} &= \tfrac{1}{2}g^{hm}g^{pr}L^{w}_{hi}(L^{v}_{rk}R^{u}_{pmj} + L^{v}_{rj}R^{u}_{pmk})\,u_{n+u}\,u_{n+v}\,u_{n+w}\\ &\quad + (L^{h}_{jk}L^{v}_{hi} + \tfrac{1}{2}(-L^{v}_{ij,k} - L^{v}_{ik,j} + L^{h}_{ik}L^{v}_{hj} + L^{h}_{ij}L^{v}_{hk})\,u_{n+v},\end{aligned}$$

$$\Gamma^{n+i}_{n+j\ k} = \tfrac{1}{2}g^{hm}g^{jp}L^{u}_{ih}R^{v}_{pkm}\,u_{n+u}\,u_{n+v} - L^{j}_{ik},$$

$$\Gamma^{i}_{n+j\ k} = \tfrac{1}{2}g^{im}g^{jp}R^{v}_{pkm}\,u_{n+v},$$

$$\Gamma^{\alpha}_{n+j\ n+k} = 0.$$

Denote by $\mathcal{V}^{*}$ the vertical distribution on $(T^{*}(M), G)$ spanned by the vector fields with components $(0, \delta^{i}_{k})$, $k=1,2, \ldots ,n$ and by $\mathcal{H}^{*}$ the horizontal distribution spanned by the fields with components $(\delta^{i}_{k}, L^{m}_{ik}u_{n+m})$, $k = 1,2, \ldots ,n$ relative to $\{\partial/\partial u^{i}, \partial/\partial u_{n+i}\}$. Let λ be a vector field on M. Its horizontal lift $\tilde{\lambda}$ has components

$$\tilde{\lambda}^{\alpha} = (\lambda^{i}, L^{k}_{im}\lambda^{m}u_{n+k})$$

relative to $\{\partial/\partial u^{i}, \partial/\partial u_{n+i}\}$.

In $(T^{*}(M), G)$ we define a hypersurface S with its unit normal $N \in \mathcal{V}^{*}$ at every point. Then S may be represented by

$$u^{i} = f^{i}(y^{1}, y^{2}, \ldots, y^{2n-1}),\quad u_{n+i} = f^{n+i}(y^{1}, y^{2}, \ldots, y^{2n-1}),$$

where y^a are coordinate functions on $S \cap P^{-1}(U)$. The vector field $\lambda = \lambda^a \partial/\partial y^a$ on S has in $(T^*(M),G)$ the following components

$$\lambda^i = u^i_{,a} \lambda^a \quad \text{and} \quad \lambda^{n+i} = u_{n+i,a} \lambda^a.$$

Let $\bar{C}$, with tangent $\bar{\tau}$, be a curve in $(T^*(M),G)$ that is parametrized by arc length σ. We denote by $\mathcal{A}_i$ and $\mathcal{B}_i$ the following expressions

$$\mathcal{A}_i = \dot{u}_{n+i} - L^k_{ij} \dot{u}^j u_{n+k},$$

and

$$\mathcal{B}_i = d\mathcal{A}_i/d\sigma - L^k_{ij} \dot{u}^j \mathcal{A}_k.$$

From (6) $\bar{C}$ is a geodesic in $(T^*(M),G)$ if and only if

$$\text{(7)} \quad \begin{aligned} \bar{D}\bar{\tau}^i/d\sigma &= \ddot{u}^i + L^i_{jk} \dot{u}^j \dot{u}^k + g^{im} g^{pr} R^h_{pjm} u_{n+h} \dot{u}^j \mathcal{A}_r = 0, \\ \bar{D}\bar{\tau}^{n+i}/d\sigma &= \mathcal{B}_i = 0. \end{aligned}$$

If $\bar{C}$ is a geodesic in S, then

$$\bar{D}\bar{\tau}^\alpha/d\sigma = G_{\epsilon\delta} N^\epsilon (\bar{D}\bar{\tau}^\delta/d\sigma) N^\alpha,$$

which are equivalent to

$$\text{(8)} \quad \begin{aligned} &\ddot{u}^i + L^i_{jk} \dot{u}^j \dot{u}^k + g^{im} g^{pr} R^h_{pjm} u_{n+h} \dot{u}^j \mathcal{A}_r = 0, \\ &\mathcal{B}_i = g^{jm} N^{n+m} \mathcal{B}_j N^{n+i}. \end{aligned}$$

By (5) and the definition of a horizontal vector one may show that arc length of horizontal curves in $(T^*(M),G)$ is invariant under projections P.

<u>Theorem 5</u>. Let $\tilde{C}$ be a lift in $(T^*(M),G)$ of a curve C in M. Then the following statements are equivalent:

(i) $\tilde{C}$ is a geodesic in S.

(ii) $\tilde{C}$ is a geodesic in $(T^*(M),G)$.

(iii) C is a geodesic in M.

<u>Proof</u>: $\tilde{C}$ is horizontal if and only if $\mathcal{A}_i = 0$. Then (7) and (8) for $\tilde{C}$ are equivalent to

$$\bar{D}\dot{u}^i/d\sigma = \ddot{u}^i + L^i_{jk}\,\dot{u}^j\,\dot{u}^k = 0,$$

which proves the theorem. Q.E.D.

Temporarily denote the coordinate functions on a neighborhood of $(T^*(M),G)$ by $x^\alpha = (u^i, u_{n+i})$. Then the components of the second fundamental form on S relative to $\{\partial/\partial y^a\}$ are given by [2, p.148]

$$\Omega_{ab} = -G_{\alpha\epsilon}\,N^\epsilon_{:\beta}\,x^\alpha_{,a}\,x^\beta_{,b}.$$

Going back to our usual notation for the coordinate functions we obtain by use of (5) and (6)

$$\Omega_{ab} = -g^{ij}(N^{n+i}_{,a} - L^k_{ih}\,N^{n+k}\,u^h_{,a})(u_{n+j,b} - L^h_{jk}\,u_{n+h}\,u^k_{,b})$$

$$+ \tfrac{1}{2}g^{mp}\,R^s_{pjh}\,u_{n+s}\,N^{n+m}\,u^h_{,a}\,u^j_{,b}.$$

Denote $\theta_{ia} = N^{n+i}_{,a} - L^k_{ih}\,N^{n+k}\,u^h_{,a}$ and $\Phi_{ia} = u_{n+i,a} - L^r_{is}\,u_{n+r}\,u^s_{,a}$. Since Ω is a symmetric tensor on S and R is skew-symmetric in j and h, the expression for Ω_{ab} may be written as

$$\Omega_{ab} = -\tfrac{1}{2}g^{ij}(\theta_{ia}\,\Phi_{jb} + \theta_{ib}\,\Phi_{ja}).$$

Let $\bar{C}$ be a curve in S parametrized by arc length, and $\lambda = \lambda^a\,\partial/\partial y^a$ a unit vector field on S. T. K. Pan defined the normal curvature vector of λ with respect to $\bar{C}$ to be the normal component of $\bar{D}\lambda/d\sigma$ [7]. If the normal curvature vector of λ is denoted by $_\lambda K N$, then $_\lambda K$ is given by

$$_\lambda K = e\,\Omega_{ab}\,\lambda^a\,\dot{y}^b,$$

where $e = \pm 1$. Pan called $\bar{C}$ an asymptotic line of λ if

${}_{\lambda}K = 0$ on $\bar{C}$; an indicatrix of λ if λ is parallel on $\bar{C}$; and, a curve of λ if λ is tangent to $\bar{C}$.

Theorem 6. All horizontal curves in S are asymptotic lines of horizontal vector fields on S.

Proof: If $\bar{C}$ is a horizontal curve and λ is a horizontal vector field we have $\Phi_{ia}\lambda^a = \Phi_{ib}\dot{y}^b = 0$. Hence ${}_{\lambda}K = 0$. Q.E.D.

Corollary 3. A horizontal curve in S is an indicatrix of a horizontal vector field if and only if it is so in $(T^*(M),G)$.

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