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A COMPARISON STUDY BETWEEN A TRADITIONAL AND EXPERIMENTAL FIRST YEAR LINEAR ALGEBRA PROGRAM

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In partial fulfillment of the requirements for the

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A COMPARISON STUDY BETWEEN A TRADITIONAL AND EXPERIMENTAL FIRSRT YEAR LINEAR ALGEBRA PROGRAM

A DISSERTATION
APPROVED FOR THE DEPARTMENT OF MATHEMATICS

By

To
my loving husband, Jeffrey
and my parents
Osman and Fatma

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ABSTRACT

A Comparison Study Between a Traditional

and Experimental First Year Linear Algebra Program (January 2001)

Hamide Dogan

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Chair of Advisory Committee: Dr. Curtis McKnight

This study investigated the effects of use of *Mathematica*, a computer algebra system, in learning basic linear algebra concepts. The study was done by means of comparing two first year linear algebra classes, one traditional and one *Mathematica* implemented class. A total of fifty-five students participated. Each class had a different instructor; The traditional class was taught by a professor in the mathematics department, and the experimental class was taught by the investigator. Students who were already enrolled in each section were used in the study. They were not told the nature of the experiment until after the enrollment was completed.

The traditional section was in lecture format whereas the experimental section was in mostly discovery format; Students in the experimental group discovered definitions of basic abstract concepts mostly through visual-based *Mathematica* notebook demonstrations, whereas the students in the traditional group were given the definitions.

Data was collected through a background questionnaire, post questionnaire, pre-test scores, post-test scores, interviews and observation notes. This study discusses a variety of comparisons between the traditional and the experimental classes. The

data shed light on a range of differences in understanding basic linear algebra concepts.

CHAPTER I

Problem

This study investigates the effects of use of *Mathematica,* a computer algebra system, in learning basic linear algebra concepts. The study is done by means of comparing two first year linear algebra classes, one traditional and one *Mathematica* implemented *class. Mathematica* notebooks contain two and three-dimensional demonstrations of basic linear algebra concepts. By these demonstrations, it is hoped students will construct accurate mental images, and as a result both to have better conceptual understanding and to make transition from computation to abstraction easier. The purpose of the study is to investigate differences in students' understanding of vector space concepts, and to evaluate the strength and weaknesses of both approaches. The study attempts to isolate the two instructional approaches by controlling teaching methodology, homework assignments, quizzes, exams and the textbook.

Brief History

The history of linear algebra goes back to 1800. Contrary to common beliefs, studies on linear algebra started with determinants, not with systems of linear equations. Around 1800, Leibnitz invented determinants in an attempt to solve minima and maxima problems of multivariate functions. Matrix Theory in linear algebra was advanced by Cayley's definition of matrix multiplication. He came up with the definition to define composition of linear transformations. Although matrix theory has a wide variety of applications, its use was not emphasized until the invention of digital computers. With the invention of digital computers, tedious computations of systems of linear equations became much easier, and as a result matrix theory has become the center of attention of variety of disciplines (Tucker, 1993; Almon, 1997; Chang, 1997). The client disciplines' interest on matrix theory has also made research communities, especially mathematics departments, aware of the need for reform in first year linear algebra classes.

As a result of new advancements in technologies such as digital computers and the use of linear algebra in these technologies (Tucker, 1993), linear algebra classes began to attract not only mathematics majors, but variety of students with different backgrounds and different majors such as economics, computer sciences and meteorology. The growing heterogeneity of linear algebra classes brought the question of how one can modify a "first linear algebra curriculum" so that it can respond to the needs of both mathematics and non-mathematics students.

The reform movement in undergraduate linear algebra courses started in a calculus-reform conference in Tulane (Carlson, 1993, 1997; Harel, 1997). In the conference, a linear algebra study group was formed. In 1990, this group started working on a list of recommendations based on results of the surveys and questionnaires collected from faculty members in a variety of colleges, universities and client disciplines. Results of the surveys and questionnaires indicated a high demand from industry and client disciplines for making the first year linear algebra courses matrix-oriented courses. The group made the following recommendations:

1. The syllabus and presentation of the first course in linear algebra must respond to the needs of client disciplines.

2. Mathematics departments should seriously consider making their first course in linear algebra a matrix-oriented course

In addition to the recommendations, there have also been a few studies attempting to investigate possible problems that occur due to the new structure of linear algebra classes. So far, the focus has been on possible correlation between the abstraction level of linear algebra and students' learning difficulties. According to a study done by Dias, Artigue & Didirem (1995), students seem to be having difficulties in recognizing different representations of the same concepts, which is defined as a level of abstraction by Dubinsky (1997). According to Dubinsky and Harel (1997), students can achieve abstraction at this level if flexibility between the representations of the same concepts is established. They also indicated that abstraction can be established if concept images, defined as all mental pictures, properties and processes associated with the concept, and concept definitions, defined as a form of symbols used to specify the concept, are not contradicting each other. They suggested that if abstract definitions are introduced visually, it could help students have better mental images, and as a result better understandings.

Unfortunately, contrary to the expectations, there has not been any scientific study on testing effects of visual instruction on learning and teaching of abstract concepts. This study, through the use of *Mathematica* notebook demonstrations, is one of the first studies attempting to test the possible effects.

Methods

A comparison method was used for the present study. Data was collected from two fall 1999 first year linear algebra classes taught at the University of Oklahoma.

One of the courses was taught traditionally, and the other was taught in a computer laboratory with the use of *Mathematica* notebooks that were created based on two and three-dimensional demonstrations of basic abstract linear algebra concepts. After viewing statements of formal definitions, students were exposed to two or three-dimensional *Mathematica* notebook demonstrations of related definitions. Next, students were asked to experiment on the demonstrations by entering their own examples. They were then asked to interpret the output of each cell, and they were expected to make arguments relating to the formal (textbook) definitions and their relations with the demonstrations. The goal of the *Mathematica* demonstrations was to show students possible connections between formal abstract definitions and their visual images. As a result, students had concept images that were as accurate as possible, which was intended to eliminate possible conflicts between students' mental images and what the formal definitions were stating.

In both classes, the same textbook (Elementary Linear Algebra, R. E. Larson and B. H. Edwards) was used. The same types of homework problems were assigned, and similar quizzes were given. The traditional class had two more students than the *Mathematica* class, which had twenty-six students. Data collection included a background questionnaire including a pre-test, in-class observations, recorded interviews with a few volunteers from both classes, a set of exam and quiz questions, as well as a post-questionnaire. Background questionnaire data were collected to see whether the two classes had students with similar backgrounds. As part of the background questionnaire, a pre-test was given to see if students in both classes started the semester with similar required mathematics knowledge. To test possible differences due to the implementation, students' scores on five common problems from the exams, the final, and from a quiz were used. In an attempt to have a better insight on students' responses, one interview was given during the last week of fall 1999 semester.

The 5-point rubric based on the following general guidelines adapted from Carlson (1998) was used to score students' responses on common problems:

5 Complete response to all aspects of the problem indicates complete mathematical understanding of the problem's concept. Includes only minor computational errors, if any.

4 Responses that falling between 5 and 3

3 Demonstrates understanding of the main idea of the problem. Not totally complete in response to all aspects. Shows some deficiencies in understanding aspects of the problem. Incomplete reasoning.

2 Responses that falling between 3 and 1

1 Attempts, but fails to answer or complete problem. Very limited or no understanding of problem. Contains words, examples, or diagrams that do not reflect the problem.

0 No answer. Written information made no attempt to respond to the problem. Written information was insufficient to allow judgment.

The traditional and *Mathematica* implemented classes *shared* the following:

- Common goal of having students understand vector space concepts, especially related abstract definitions.
- Common duration-1 hour and 15 minute long classes.

- Common time in the school week; Tuesdays and Thursdays.
- Common homework assignments and quizzes.
- Common exam structure- three regular exams and a final.
- Common amounts of help sections, office hours.
- Common semester- both taught in fall 1999.
- Common textbook
- Common structure of lectures-followed the same order in introducing concepts.
- Similar examples given.

Traditional and *Mathematica* implemented classes *differed* in following:

- Experimental group was using graphical representations as well as symbolic representations whereas, the traditional group was using mostly symbolic representations.
- Experimental group was using *Mathematica* to do numerical calculations (use of Mathematica on calculations was not emphasized, but students were not prohibited from using Mathematica to do calculations either) whereas, traditional group was not using any kind of technology.
- Experimental group met in a computer laboratory whereas, traditional group met in a regular classroom.
- Experimental group met at 9am in the morning whereas, traditional group met at 1:30 pm in the afternoon.
- The classes were taught with different instructors-traditional was taught by a professor whereas, experimental was taught by a senior graduate student.

To establish compatibility of the groups, the experimental design included a two-sample statistic, Aspin-Welch-Satterthwaite (AWS), and a non-orthogonal two-way analysis of variance. An AWS will be used on the students' scores from the pre-test and post-test questions. A non-orthogonal analysis of variance will be used on students' scores on five common problems from the exams and quizzes, and on students' attendance, nationality, gender and ability. Quantitative analysis will be supported by a qualitative analysis by the interviews.

Quantitative analysis will examine the following null hypotheses:

Hypothesis 1. There are no statistically significant differences between the control and experimental groups on the conceptual test scores.

Subhypothesis 1a. There are no statistically significant differences between the control and experimental groups on scores for the question addressing whether students can recognize a 2x2 matrix as an object of a given set, and able to write a related proof.

Subhypothesis 1b. There are no statistically significant differences between the control and experimental groups on scores for the question addressing students' concept images of linear independence and spanning set.

Sub hypothesis 1c. There are no statistically significant differences between the control and experimental groups on scores for questions connecting a linearly independent set and a spanning set.

Sub hypothesis 1d. There are no statistically significant differences between the control and experimental groups on question connecting span of a set and writing a proof showing a set is linear dependent.

Hypothesis 2. There are no statistically significant differences between the control and experimental groups on scores for a computational question.

Hypothesis 3. There are no statistically significant differences between the control and experimental groups on scores for questions connecting linear transformations and spanning sets.

The qualitative analysis will focus on understanding possible connection between students' image of the concepts; linear combination, linear independence and spanning set, and their responses on the post-questions. The following questions will guide the qualitative analysis:

Question 1. Are there any differences between the traditional and experimental groups in their concept images of the concepts; linear combination, linear independence and spanning set?

Sub question 1. Are there any differences between the two groups in recognizing the relationship between algebraic equation used as the part of concept definition of linear independence and its implication?

Sub question 2. Are there any differences between the two groups in students' use of their knowledge on linear combination and its relation to span of a set?

Sub question 3. Are there any differences between the two groups in students' methods of recognizing connection between solution of the equation used as part of the concept definition of linear independence, and a linearly independent set?

Sub question 4. Are there any differences between the two groups in students' use of their knowledge of vector space concepts to answer transformation-related problems?

Question 2. Are there any differences between the two groups in students' opinions on the use of technology in first year linear algebra classes?

Sub question1. What does the experimental group think of the use of *Mathematica* in first year linear algebra classes?

CHAPTER II

REVIEW OF LITERATURE

According to Hiebert & Lefevre (1986),

> Essays of the past have treated understandings and skills as instructional outcomes and have dealt with them in the context of advocating instructional programs. The issue has been whether skills, or understandings or both should be emphasized during classroom instruction (p. 2).

They continued: " Today, many of the writings describe the acquisition of knowledge and relationships between different kinds of knowledge" (p.2). Hiebert and Lefevre explained that students are not fully competent in mathematics if either kind of knowledge (procedural and conceptual knowledge) is deficient or if both have been acquired but remain separate entities. When concepts and procedures are not connected, students may have a strong intuitive sense of mathematics, but they are not able to solve the problems. Conceptual knowledge is defined as knowledge that is rich in relationships, and procedural knowledge is defined as knowledge of symbols and syntax of mathematics that implies only awareness of surface features, not a knowledge of meaning. Heilbert and Lefevre said: " Building relationships between conceptual knowledge and the formal symbol system of mathematics is the process that gives meaning to symbols" (p.10). They also added that studies have shown that students from elementary school through college perform successfully on routine paper and pencil problems but lack essential, underlying conceptual knowledge (Erwanger, 1975 ; Rosnick & Clement, 1980; Resnick, 1989).

Harel and Kaput (1991) made similar arguments:

> The perceptual item must somehow come to be integrated with the conceptual one. Otherwise, all one might end up with is an easily reproducible mental

experience of a work or character string with no other mental activity or structure beyond that primitive experience-which is the experience of all together too many students.

Kaput explained that the failure of students to estimate or maintain an order or sense of a calculation could in fact be seen as a failure to cross between symbol and referent (Davis & McKnight, 1980). There is also the aspect of " rote vs. meaningful learning " as well as the aspect of "procedural vs. conceptual knowledge," which are not too different. Rote learning produces knowledge that is absent in relationships, and is tied closely to procedural knowledge. Meaningful learning produces knowledge that is rich in relationships, and is linked to conceptual knowledge.

Many mathematics educators attempt to develop theories like conceptual and procedural knowledge to be able to define the learning of mathematical concepts. One theory called the representational view of mind (Putnam, 1988) defines learning as a process of constructing internal mental representations that accurately mirror the mathematical features of external representations. Then, the question becomes how to introduce the external representations so that the students can see characteristics of the concepts. The idea of instruction in the theory mentioned is to help students construct mental representations that correctly or accurately mirror mathematical relationships located outside the mind in instructional representations. This is opposed to the idea that the overall goal of instruction is to give students representations of relations explicitly. Cobb (1992) argued that instructional materials that were transparent to him might not be transparent to students. This can lead to difficulties in students' learning of the concepts. However, Cobb also added that approaches where the teacher becomes increasingly explicit about what it is that students are supposed to learn can lead to the excessive algorithmatization of mathematics and the disappearance of conceptual meaning.

Symbolization leads to abstraction of mathematical concepts and definitions. This aspect of mathematics brings one to the issue of possible conflicts between students' concept images and concept definitions. Tall and Vinner (1981) agreed that differences between students' concept images and concept definitions are sources of the learning difficulties students are having in higher-level mathematics courses. Zandieh (1996) found a strong connection between her students' image of the derivative concept and its textbook definition. She interviewed nine students in an upper level advanced placement calculus class at a suburban high school. Six of the nine students were National Merit Finalists. Each student was interviewed five times during the academic year. Each interview asked students a diverse set of questions about the derivative concept so that students' responses could be taken as an approximate snapshot of his/her mental image at that point in the course. She explained that the observations indicated several potential obstacles in students' understanding of the formal definitions:

1. The students must understand the processes underlying the concept of derivative.
2. Students must place some value on having a symbolic representation.
3. Students must think of symbolic expressions as having meaning in terms of their experiences in other contexts.

Results of an experiment done by Edwards (1997) in a real analysis class have shown a similar connection between students' understanding and the use of definitions. Edwards said:

> Often definitions are memorized and then the formal words are shoved aside in favor of knowing, as one participant said, "what it really means." However, a robust understanding of the role of mathematical definitions is needed too if students are to be able to use them effectively in more theoretical settings (p. 21).

Edwards stated that the eight undergraduate volunteers interviewed in his study seemed to overlook formal definitions, and to prefer memorizations of these definitions. He also observed that mathematics students that move from more procedural courses into theoretical, proof-intensive courses do not seem to know how to use and understand mathematical definitions. He added:

> Part of the enculturation of college mathematics students into the field of mathematics involves their acceptance and understanding of the role of mathematical definitions-that the words of the formal definition embody the entire meaning of the concept or entity being defined. If the students do not understand the role of the definition in this way they may allow their previous and emerging concept images to dictate the meaning of a definition rather than the words of that definition. Thus, a student's understanding of the role of mathematical definitions is itself part of his or her concept image (p. 19).

He continues with an example of an interviewee's response to a question that required knowing the formal definition of the concept, "supremum". This interviewee seemed to be interpreting the meaning of the formal definition based on her previous conceptual understanding. She was asked to find supremum of the sequence .9, .99, .999,........,.999...... Her answer was .9999..... Even though she could state the definition, she interpreted the definition to fit her prior conceptual understanding. Edwards findings are also supported by Rosnick and Clement (1980). Their study was done at a college. Students in the study were given word problems, and asked to translate those into algebraic equations out loud. They found that many of the college students do not seem to be able to recognize the use of letters as standing for numbers. Rosnick and Clement argue that, these students can write equations correctly but still unsure what these equations mean. As a result of the study, Rosnick and Clement concluded the following:

1. The fundamental concepts of variables and equations should not be treated lightly in high schools and colleges, nor should we assume that our students will develop the appropriate concepts by osmosis.
2. The implications of these are that more attention must be paid to conceptual development in mathematics education.

A similar result is reported in a doctoral dissertation (Moore, 1990). Moore analyzed non-participant observations, and interviews done with 16 students; 8 undergraduate mathematics major, 6 undergraduate mathematics education majors, and 2 graduate mathematics students. Data was collected during the tenth week of fall quarter in 1989 at the University of Georgia. His study aimed at finding possible sources that seem to effect student's ability to learn proofs. His findings show three major sources that seem to affect students' ability to learn proofs;

1. Mathematical Language and Notations (Hadas, 1983).
2. Concept understanding.
3. Getting started on a proof.

He also categorized these findings as seen in figure 1.

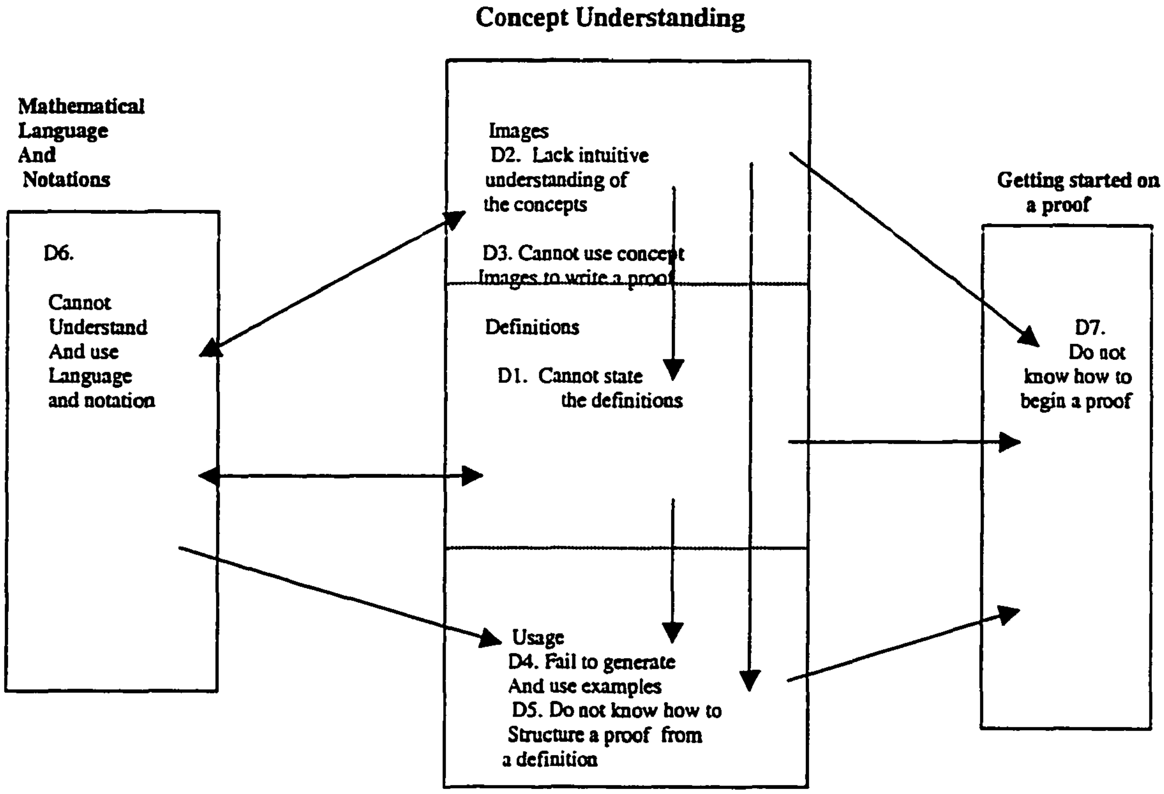


Figure 1. Sources Effecting Students' Ability to Learn Proofs.

He added that mathematical language and notation (see D6) was an obstacle for many students (Galbraith, 1981; Lester, 1975; Harel & Sowder, 1998). Although most of them overcame most of the difficulties in this area by the end of the course, he said, some students had difficulties throughout the course. In Figure 1, the arrows from the mathematical language and notation box indicate that difficulties in this area prevented students from understanding concepts and using definitions. According to Moore, they seem to learn definition by developing their concept images through examples and diagrams they gained an understanding not only of the definition but also of the symbols and words.

First-year linear algebra students are no exception to these issues. There have been similar findings reported in first year linear algebra classes. Most frequently occurring problems are reported as conflicts between students' mental images of basic vector space concepts, (Hillel & Anna Sierpienska, 1994; Arshavsky 1999), and their formal definitions whose structures are based on symbolic representations.

Translating between different representations of mathematical ideas is another difficulty that has been reported by several studies. In Dias, Artigue & Didirem (1995), flexibility between different representations of the same vector space concepts was studied, and revealed that students in this study were not capable of recognizing different representations of the same vector space concepts. According to Dubinsky (1997), these conflicts are due to a high level of abstraction occurring in formal definitions.

As a result of these findings, several suggestions have been made towards helping linear algebra students overcome these difficulties. Dubinsky and Harel (1997) suggested the use of visual representations such as computer activities to reduce the conflict between students' concept images and concept definitions. Dubinsky discussed an experiment he had performed in his abstract algebra class. He explained that having students write their own computer programs for searching groups, subgroups and normal groups helped students gain a higher realization of properties of groups, and helped them internalize the members of groups as objects. Dubinsky added:

> It seems that mathematics becomes difficult for students when it concerns topics for which there do not exist simple physical or visual representations. One way in which the use of computers can be helpful is to provide concrete representations for many important mathematical objects and processes (p. 104).

Tall and Winner (1981) mentioned in their article that the difficulty of forming an appropriate concept image, and the coercive effects of an inappropriate one having potential conflicts, can seriously hinder the development of formal theory in the mind of an individual student. Harel and Tall (1991) tied students' difficulties in developing formal theories to difficulties in generalization and abstraction of concepts. Generalization is defined as the process of applying a given argument in a broader context, and is categorized in three groups based on the individual's mental construction:

1. Expansive generalization occurs when the subject expands the applicability range of an existing schema without reconstructing it. For example, algebraic aspects of generalizing vector sum and scalar multiplication from R^2 to R^n can be considered as expansive generalization.
2. Reconstructive generalization occurs when the subject reconstructs an existing schema in order to widen its applicability range. Geometric aspects of generalizing vector sum and scalar multiplication from R^2 to R^n can be considered as reconstructive generalization.
3. Disjunctive generalization occurs when, on moving from a familiar context to a new one, the subject constructs a new disjoint schema to deal with the new context and adds it to the array of schemas available.

Abstraction is defined as a process that occurs when the subject focuses attention on specific properties of a given object and then considers these properties in isolation from the original. Such an application of an abstraction theory can be a case of reconstructive generalization because the abstracted properties are reconstruction of original properties, now applied to broader domain. Abstraction serves two purposes which are particularly attractive to the expert mathematicians:

- Any arguments valid for the abstracted properties apply to all other instances where the abstracted properties hold, so the arguments are more general. Definition of vector space concept is a case of this category.
- Once the abstraction is made, by concentrating on the abstracted properties and ignoring all others, the abstraction should involve less cognitive strain.

As stated above, because of the cognitive reconstruction involved, these two factors may cause great difficulty for the learner. Then, how can one help students pass through the difficult transition, and attain the reconstructive generalization required for the formal abstraction? Harel & Tall's suggestion was that transition can be done more effectively by focusing on a mid-way development in which a specific example is seen by the teacher as a representative, *a generic example,* of the abstract idea. They also warned that students may abstract the wrong properties since students have not yet performed the abstraction. However, if the process is successful, and the students see one or more specific examples as typical of a wider range of examples embodying an abstract concept, then students can attain abstraction, a process of *generic abstraction*. Various computer activities can be instances of generic abstraction. Harel's program (1989) that approached abstraction of vector space concepts by showing specific generic examples in R^2 and in R^3 is an example of a process of generic abstraction. His program was based on the findings of a survey (Harel, 1987). It was found, after a review of various linear algebra books as part of the survey, that there are implicit assumptions on beginning student's background and their attitude towards the course:

1. Beginning students are capable of dealing with abstract structures without extensive preparation.
2. Beginning students can appreciate the economy of thought when particular concept and system are treated through an abstract representation.
    a. Students can understand the idea of representation
    b. Students can deal with situations within a variety of domains.

Findings of the survey also indicated that many of the application domains are not familiar to the students, and also indicated that high school and sophomore university students have serious difficulties understanding algebraic systems which do not have an easily accessible *concrete or visual representation*. Harel adds that many Linear Algebra textbooks contain applications whose domains are not familiar to the students.

He continued with an example of definition of a vector space. He argued that the definition of a vector space concept is, in many linear algebra textbooks, stated and illustrated with one or two models; the concepts of vector dependence, independence and basis, are usually discussed in these models but not elsewhere. Moreover, he said, concepts are always constructed in an *abstract* setting, so students often lack an understanding of the *construction process, e*ven though they may understand the resultant abstract.

As a result, Harel developed a linear algebra program based on gradual introduction of abstraction, and construction of basic notions from a visual basis. His program introduced properties of vector space concepts first in R^2, R^3 and next in R^n. Harel claimed that introducing vector space concepts in this way helped students make the transition to abstraction easily. Results of his study based on a few high school and college students' exam scores seem to support his claim. However, there was no statistical analysis done other than observing the differences between the mean scores and one might question credibility of the study.

Use of Computer Algebra Systems

There have been several computer programs used in teaching first year linear algebra classes. Roberts (1996) said: "Software tools such as MATLAB, Maple, MathCad, and *Mathematica* provide opportunities to enhance the instruction of undergraduate linear algebra." He explained that these software innovations allow students to investigate applications of linear algebra whose computations would be too difficult to perform by hand. He also added that graphics capabilities offered by computer algebra systems allow students to visualize these geometric concepts, thus giving new life to a subjects many students view as theoretical and not much of practical use. As Roberts stated, there have been movements on using computer algebra systems, but unfortunately, there has not been enough scientific evidence supporting the usefulness of using computer algebra systems in learning abstract mathematical definitions. The only scientific study on this issue is the one that investigated effect of using CABRI system in students' mental images of the concepts, both of linear combination and linear independence. The program was based on a geometric model of two-dimensional vector spaces within the dynamic Cabri-geometry II environment. In the study, students' responses from the CABRI class were qualitatively analyzed; it was neither a quantitative nor a comparison study. The study can only indicate possible effects of using CABRI in learning the vector space concepts for a particular group of students. It could not test how much learning occurred due to its implementation. For that, one needs to compare students' gained knowledge by equating all variables, except the implementation, between the two groups. However, there has not been a study on comparing effects of using technology in linear algebra instructions with instructions that do not use technology. The present study will be one of the first studies that intend to compare two instructional approaches; one is the traditional approach, and the other is the *Mathematica* implemented approach.

Chapter III

Research Design and Methodology

The research was conducted at the University of Oklahoma. It was a comparison between two Linear Algebra classes: Math 3333 in fall 1999. One was a traditional linear algebra class and the other was an experimental class using *Mathematica* Notebooks.

Similarities between Classes

The two classes shared the common goal of having students understand the concepts of linear independence, spanning set and linear transformation and the ideas behind the concepts. Both classes were one hour and fifteen minute long. Office hours for the traditional group were from 2 to 3:15 on Mondays and Wednesdays, and for the experimental group were from 11 to 12 on Tuesdays, from 1 to 2 on Thursdays. In both classes, the same set of homework problems was collected once a week on Thursdays. If an exam fell on a Thursday, homework problems were collected on the following Tuesday. There was only one quiz given. The quiz consisted of one problem. Both had three exams covering the same topics. Four out of six questions in the second, the third exams and the final were the same. Both classes had similar and equal number of questions in the first exam.

The investigator observed the classes to make sure topics covered went parallel. Students in both classes saw the concepts in the same order, and similar examples were given. The same text book, Elementary Linear Algebra by Roland E. Larson and

Bruce H. Edwards, third edition, was used in both sections. Both classes covered the same chapters in the same order; chapter one to chapter seven, except chapter five on inner product spaces. There was not enough time to cover inner product spaces in the experimental group. Instructors would come into classroom a few minutes before starting time. They both would write, on the board, an agenda for the day, reminders, announcements, reading assignments, and homework assignments. Classes usually began with one or more of the following: a recap of the material covered the week or day before, going over a homework problem, giving hints on homework problems and/or answering students' questions.

Differences Between Classes

The main difference between the two classes lay in the treatment of *concepts*. A few examples are:

1. Students in the traditional class learned the concepts; linear independence and spanning set through their formal definitions within two class times, whereas students in the experimental group spent one week on discovering characteristics of the concepts through two and three-dimensional *Mathematica* notebooks. They spent the following week discussing connections between the formal definitions and the observed characteristics.
2. In the traditional class, students were responsible only for their homework assignments, whereas students in the experimental group were given *Mathematica*-related activities to work through both in class and outside the class time (see appendix C)

3. The experimental class mostly focused on interpreting the visual representations of concepts. The traditional group mostly focused on interpreting the algebraic (formal) representations of concepts
4. The traditional group followed more formal (abstract) approach to the vector space concepts.
5. The experimental group discussed the connections between span of sets, linearly independent sets and bases of vector spaces in two and three-dimensional *Mathematica* demonstrations, whereas in the traditional class, these concepts were stated through remarks written on the black board.
6. In the experimental class, formal definition of linearly independent sets was introduced by means of tracing arrows on visual *Mathematica* demonstrations, and by means of solving the related homogeneous equations through analyzing visual representations of vectors. One should note that the numerical values of the vectors were not stated. In the traditional class, the formal definition of the concept was stated algebraically.
7. Right after seeing the formal definitions, the traditional class focused on procedural approaches. The experimental class did not focus on procedural solutions until after they have answered related questions through visual representations.
8. The experimental class first focused on the visual characteristics of linear transformations (see appendix C), kernel and image of linear transformations. These concepts were introduced through two and three-dimensional *Mathematica* demonstrations whereas the traditional class

focused on their algebraic characteristics. The kernel and the image of linear transformations were introduced algebraically.

9. The experimental class discussed applications of basic concepts in social sciences such as economics and accounting through interactive *Mathematica* notebooks.

10. The experimental class approach was one of "concept first, techniques later", whereas the traditional group combined concepts and techniques, emphasizing techniques (procedural knowledge).

The manipulative skills were comparatively less important in the experimental group. The traditional group concentrated on these skills for almost the entire semester compared to the experimental group. The experimental class spent one week on algebraic procedures showing whether a given set is linearly independent or a set is a spanning set for a given space. Also, other class time was spent for procedures answering questions of whether a given set is a basis for a vector space. The experimental class mostly focused on understanding concepts through seeing their characteristics on visual representations. The traditional group focused on executing complete and accurate procedures whereas the experimental group focused on interpreting the results of the procedures, comparing them with their visual representations. In short, the experimental class used numerical, symbolic and mostly graphical representations, and the traditional class used mainly symbolic representation, with minimal numerical and graphical representations that were limited to the blackboard and paper-and-pencil drawings.

In the experimental group, students were allowed to use *Mathematica* to do numerical computations such as gauss elimination, determinant of matrices and row reduced echelon form of matrices. As a result, students had more time to interpret the results of these calculations. Even though the use of *Mathematica* on computational problems was not required, the majority chose to use *Mathematica*. In the traditional group, students were required to do computations by hand.

Participants

Both classes were regular size classes; twenty-six in the experimental group, and twenty-nine in the traditional group. Students enrolled in the class that fit their schedules. During the enrollment, the students in the experimental course were not informed of the nature of the class. However, they were told so on the first day of classes after enrollment; they were told that the class was going to be using a computer algebra software called *Mathematica*, and lectures were going to be in a computer laboratory where each student would be assigned to a computer. They also were told that no knowledge of *Mathematica* was required, and they were not expected to learn *Mathematica,* that *Mathematica* was going to be used as a tool to help learn basic linear algebra concepts better. After being informed of the nature of class, they were told that they had the option of switching to one of the other two linear algebra classes. The experimental group started out with 29 students, and the traditional group started out with 34 students. From those 29 students in the experimental group, none switched sections. Two of the 29, three weeks into the semester, dropped the course with the grade W. One of these students had to drop out the course due to a death in his family.

In the traditional class, there were five students who dropped with the grade W. Of the remaining 29, three changed from credit to audit.

In both groups, the instructors informed the students that they were being part of an experiment, and that all information to be gathered for the purpose of the study would be kept confidential. They were also voluntarily asked to sign a consent form. A copy of the consent form can be found in appendix B.

The students in the traditional group were not allowed to use calculators in class or in the exams, whereas the students in the experimental group were encouraged to use technology such as calculators and computer algebra systems. *Mathematica* was the most preferred software among these students.

Class Rooms

The traditional group used a typical classroom with a blackboard, tables and chairs. The room had an overhead projector and one long and two short blackboards. However, the overhead was not used at all. Some of the basic drawings such as two - dimensional coordinate systems and vectors in two-dimensions, were drawn on the blackboard. The distance between the instructor and the students was close enough that each student could hear the instructor with no difficulty. The instructor also could make eye contact with each student.

The experimental group, however, had to change three classrooms within the second week of classes. The group met for the first two class time in a traditional classroom similar to the classroom the traditional group met through out the semester, and met for the next two class time in a computer laboratory located in the same building as the traditional class. However, due to problems on running *Mathematica* on computers in this laboratory, the instructor had to move to a different computer laboratory in a different building. The experimental group met in this lab for the rest of the semester. In this computer lab, there were thirty computers; one for each student. One of which was not working during the first week of the meeting. In addition, there were a white board and an overhead projector with a main computer located up front for the instructor's use. The overhead screen was located in front of the white board, so it was not convenient for the instructor to use the white board and the overhead screen at the same time. The laboratory had a long rectangular shape. The distance between the instructor and the students varied depending on the location of a student. If the student sat up front then the distance was not much and the instructor could make eye contact with the student. If the student sat towards the end of the class, making an eye contact was almost impossible unless the instructor walked towards the back of the classroom.

The computers were loaded with *Mathematica* software as well as Netscape, Instant Messenger, and more. The students had access to the Internet through these computers, which, at times, stole students' attention from lectures.

On the first day in the lab, the students were introduced to *Mathematica*. They were shown the basic *Mathematica* commands as well as how to open *Mathematica*, how to edit in *Mathematica*, how to run cells, and how to save files. They were also told that at the start of each class meeting, they would be provided with a diskette with the current *Mathematica* files on it, and that they would be required to open the files before the lecture started. These diskettes were labeled with students' names for the purpose of keeping track off each student's work on these files. They were asked to write their interpretations of each outcome of the *Mathematica* cells in to the preceding cells.

Mathematica NoteBooks

Mathematica Notebooks were written interactive, guided supplements to the lectures. They were mostly composed of interactive cells of examples and non-examples of the basic linear algebra concepts. Emphases were given mostly to the two and three-dimensional demonstrations of the basic vector space concepts. Each cell in a notebook was labeled as the example corresponding to the example discussed in class. Copies of selected notebooks can be found in Appendix C.

As the concepts were defined in class, and their formal definitions were written on the black board, the corresponding examples on the interactive *Mathematica* cells were run by the students. Students discussed the outcomes of the cells by comparing the characteristics of the demonstrated concepts through the visual demonstrations, and their formal definitions already stated on the board. More of similar interactive cells with different examples and non-examples of the same concept were run by students, and students wrote their interpretations in the proceeding cells.

The students in the experimental group also asked to answer the concept related questions through analyzing visual outputs of the corresponding *Mathematica* cells. For example, in one particular *Mathematica* Notebook (see Appendix C) that covered the concept; linear independence, students were asked to solve; for the coefficients; a, b, and c, related homogeneous equations of the following type:

$a\,\mathbf{v} + b\,\mathbf{w} + c\,\mathbf{u} = \mathbf{0}$, by using the outputs of interactive cells whose outcomes showed the positions of the vectors v, w, and u with the use of different colors for each vector.

In this activity, numerical values of the vectors purposely were not given so that students would be restricted to the two dimensional outcomes of these vectors to be able to solve the equations. The purpose of this activity was to get students have better understanding of the formal definition of a linearly independent set. The textbook uses solution types to the homogeneous equations as part of the formal (abstract) definition of the concept. The formal definition stated the following:

> " *A set of vectors* $S=\{v_1, v_2,\ldots,v_k\}$ *in a vector space V is called* ***linearly independent*** *if the vector equation*
> $c_1 v_1+c_2 v_2+\ldots+c_k v_k=0$
> *has only the trivial solution,* $c_1=0, c_2=0,\ldots,c_k=0$. *If there are also nontrivial solutions, then S is called* ***linearly dependent.*** "

Research Instrument and Data Collection

The data collected for this study consist of a background questionnaire, post questionnaire, pre-test, a quiz, exam, and final scores on the post-questions, interviews, observation notes and recorded lectures.

Background questionnaire

The background questionnaire was given during the first week of the semester. The experimental group took the questionnaire on the first day of classes. The traditional group took the questionnaire on the second day of classes.

The purpose of the questionnaire was to gather information on students' backgrounds, particularly on factors that might influence the results used for comparison in the study.

The questionnaire gathered information on:

1. Previous high school mathematics courses and the years those courses were taken

2. Previous college mathematics courses and the years those classes were taken,
3. Experience with computer algebra systems,
4. Students' opinion on mathematics, algebra, and the use of computer algebra systems in mathematics classes
5. Students' opinion on their learning style and study habits
6. Students' majors
7. Students' course load and other responsibilities
8. Students' ethnicity

A copy of the background questionnaire can be found in Appendix D.

Pretest

As part of the back ground questionnaire, five pretest questions were given. The pretest questions were aimed at testing students' knowledge on basic prerequisite knowledge for the course. A copy of the questions can be found in Appendix D. The purpose of the pretest was to check compatibility of the two groups as well as pointing out any factors that might influence the results. The pretest consisted of basic short algebra questions. They were aimed at testing students' ability to cope with symbolic representations. These questions were all free-response typed questions. The first question was addition of two vectors from a three dimensional space. The second was a question on determining derivative of a composite function at a given point based on the derivatives of the two functions at the same point. The third question was on solving an equation that use 2x2 matrices; the fourth question was on determining function value of a symbolic point "a"; and the fifth question was on solving an equation of variables for one of the variables.

Grading of the pre-test questions were done by the investigator because the questionnaire did not include students' names or anything that would give a hint who each paper belonged to, except the section numbers. To eliminate the bias, before grading started, the investigator covered the section numbers with white tapes.

The correct responses to these questions were counted as one point. If the response did not have the correct answer, zero was assigned. If the response was correct, without paying attention to how the answer was obtained, one point was assigned to the question indicating that the answer for the question was correct. In short, the questions were graded out of one point. One point was for the correct answer, and zero was for incorrect answer.

Interviews

One set of interviews was given during the last week of classes, and during the week before the finals week. Students from both groups volunteered for the interviews. The purpose of the interview was to help answer the research questions, and to have better insight on students' responses on post-questions given in the quiz, the exam and the final.

Course Grades

The study used course grades only to classify students by ability into three groups: AB, CC and DF. Category AB represented those who got letter grade A or B; category CC represented those who got letter grade C; and category DF represented those with letter grade D or F. The variable *ability* was used for statistical purposes to account for some of the within–group variability. The effect of the treatment also was tested by eliminating the differences due to students' abilities.

Nationality

On the variable *nationality*, the two groups were significantly different hence the variable was included in GLM models as a dummy variable. Students' nationalities were categorized as 1=: American, and 0=: International (not American). The effect of the treatment was tested by eliminating the effect due to students' nationalities.

Attendance

The in-class observations indicated that attendance in the traditional group differed due to the fact that the traditional group met at 1:30 AM, and the experimental group met at 9:00 AM. Since *Mathematica* demonstrations mostly were restricted to in-class activities, students attendance in the experimental group might affect the results of the study. As a result, students' attendance from both groups was included in the study. Students' attendance was categorized as 1=: those who attended eighty percent of the time, and 0=: those who attended the class less than eighty percent of the time. The effect of the treatment was tested by eliminating the effects of students' attendance.

Gender

The groups had unequal number of male and female students. The experimental group had more female students than the traditional group. There were only 4 females in the traditional group, and 12 females in the experimental group. To equate the two groups on this variable, students' gender was included in the study. Students' genders were categorized as 1=: Male students, and 0=: Female students. The effect of the treatment was tested by eliminating the effects of students' genders.

Observation Notes

The investigator took notes of relevant aspects to the study either immediately after classes or immediately after informal encounters with students outside the class. Some of the in-class observations were used to make sure that both classes covered the same subjects around the same time with similar examples. They were also used to make sure similar homework problems were assigned, and collected at the same day.

Methodology

Two combined research methodologies, quantitative and qualitative, were chosen to be used in this study in an attempt to determine significance of results and also to discover students' thinking patterns, strengths and weakness. Here, it should be noted that students' final grades were assigned separately by each instructor. The interviewer however made sure the assignments of final letter grades of the two groups did not have much difference by going through students' final letter grades and their numerical grades obtained by adding homework, exam and the final grades (Maximum was 500 points in both classes). For each letter grade, both instructors used almost the same upper and lower bounds varying by 10 points.

Quantitative Analysis

For the pretest questions, since the sample sizes were not equal, an Aspin-Welch-Satterthwaite (AWS) t' statistic (Toothaker, 1996) with the assumption of unequal variances was applied to test the hypothesis that there is no difference between the mean scores: $H_0: m_1=m_2$.

Analysis of the after-treatment scores was performed using two tests, an AWS t' and a non-orthogonal two-way analysis of variance. The independent variables were *treatment* and *controlled variables*. The role of the controlled variables was that of blocking variables used to lessen the variability within each group and between the groups (Maxwell & Delaney, 1990).

Two levels of the independent variable teaching/learning method entered the experiment:

1. A traditional approach, which for this experiment meant predominantly use of symbolic representations and no use of technology, and
2. A reform approach, which for this study meant use of *Mathematica* notebooks to introduce basic linear algebra concepts.

For the non-orthogonal two-way analysis of variance, the concern is to test whether or not the treatment explains a significant amount of the variability. This test is performed with the SAS package, using a General Linear Model (GLM) and Type III Sums of Squares. The model comparison for the test eliminating the effect of each of the controlled variables is:

$y=\mu+\alpha+\beta+\varepsilon$ (Full Model)

$y=\mu+\quad\beta+\varepsilon$ (Reduced Model)

Here, α is the treatment effect, β is the controlled variables to equate the two groups, and to eliminate some of the within group variability, and y is students' scores on the post-questions used to test the research question. Also, μ is the grand mean common to all observations, and ε represents the error.

Grading of each question was done by the interviewer and by one of the four graduate students from the mathematics departments. These graduate students were either about to start in the Ph.D. Program, had successfully completed their master program and passed their qualifying examinations, or they were already working on their doctorate degree. To maintain reliability of the measurement, for each question, a 5-point rubric (Carlson, 1998), was made, and given to each grader before grading started. Also, to eliminate possible bias, and maintain reliability of the grading, the names of the students were covered with white types. The students' papers also were mixed before the grading. The second graders were not associated with either of the groups, and they did not know any of the students, nor their sections. During or after the grading, whenever it was necessary, the interviewer and the second graders got together to discuss grading issues. To test the consistency between the graders, for each question, Pearson correlation coefficients were used to find the correlation between each grader's scores. One should note here that one of the graders for each question was the interviewer. Correlation was evaluated between the interviewer's scores and the scores of one of the four graduate students.

Post-Questions

Five questions were used in the study. One of which was given on a quiz taken right after introducing vector space concepts. Two of them were given on the second exam, one of them was given on the third exam, and the last one was given on the final. The purpose of these questions was to help answer the research question.

Question 1:

This question was a conceptual question for both traditional and experimental groups. Purpose of question 1a was to investigate if students could recognize 2x2 matrices as objects of the given set, and also was aimed at investigating whether students were able to write related proofs (Carlson, 1997). Question stated the following:

Let S be the set of matrices of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ *where a and b are any real numbers.*

a. Show that S is a subspace of $M_{2,2}$

b. Find a basis for S.

This question was aimed at testing whether the students were able to recognize vectors of the set correctly, and to write a complete proof. The question was graded based on the following 5-point rubrics:

5pts:

Complete and correct response.
Use of correct form o f the vectors in S
Use of correct statements of the conditions (closeness under addition and closeness under scalar multiplication).

4pts:

Correct statements of the two conditions (closeness under addition and closeness under scalar multiplication).
Use of correct form of vectors.
Use of the same vector in the sum.
Use of numerical values for entries of matrices (correct form is used) in S.

3pts:

Correct statement of conditions.
Use of incorrect 2x2 form for the matrices.
Represent S as a single matrix in the correct form.
Identifying resulting vectors as vectors of M, not as of S

2pts:

No mention of the two conditions.
Use of correct form of matrices in S.
Defining the set S as set of symmetric matrices.
Mentioning closeness but no explanation on the two conditions.

1pt:

Some work but nothing is correct.

0pt:

No work shown.

Question 1b is a conceptual and mostly visual based question. It was aimed at testing students' knowledge of the concept; basis, and their ability to recognize vectors in the set correctly. The question was graded based on the following 5-point rubric.

5pts:

Have complete and correct response:
A basis should be written as a set. Set notation is used.
Matrices are written correctly, and has numeric entries
The linear independence and spanning set requirements should be shown for the basis.

4 pts:

Basis is written as described above
The linear independence and spanning set requirements should be mentioned
The linear independence and spanning set requirements are not shown.

3pts:

Have correct matrices stated as the vectors of a basis, but not basis is written as a set.
Written in a set but no mention of linear independence and spanning set requirements.
Entries of matrices defined by using variables (correct form used) given in a set or not.

2pts:

One of the matrices given as a basis (with numeric or variable entries)
A basis for M_{22} is given as a basis for S

1pt:

Some work shown, but none is correct.

0 pt:

No work shown

Question 2:

The question investigated if students were able to carry out computational problems involving procedures. The question stated the following:

For the vectors below,
a=(0,-1,1), b=(2,1,1), c=(2,0,2), d=(1,0,1)
i. Is the set {a, b, c} linearly independent (justify your answer)?
ii. What is the dimension of Span {a, b, c}(justify your answer)?
iii. Is the vector (1,2,3) in Span {a, b, c} (Justify your answer)?

The question 2i is a typical procedural (computational) question in linear algebra. This question could be answered through following a procedure that uses row reduce-echelon form of matrices. Or, it could be answered just by observing that the vector c is the sum of the other two vectors; a and b. The purpose of the question was to test how well students answer procedural questions. The following 5-point rubric was used to grade the problem:

5 pts:
Complete and correct response:
Carry out the procedure correctly
Interpret the result of the procedure correctly

4pts:
Responses that falling in between 3 and 5.
Correct answer with correct reasons given for all aspects of the concept, but lack of symbolic representations.

3pts:
Carry out the procedure correctly, but could not have an accurate interpretation.
Have the echelon form of the correct matrix but did not state an answer.

2 pts:
Responses that falling in between 1 and 3.
Signs of understanding of the concept but none of the aspects are stated.
No attempt to use the procedure (Finding row-echelon form of the matrix).

1 pt:
There is some work shown but none is relevant.
No sign of understanding.

0 pt:
No work is shown.

The question 2ii was aimed at testing whether the students were able to combine their knowledge of linear independence and span of a set to find dimension of span of the same set. It was a conceptual question targeting students understanding of the three concepts of linear independence, span of a set, and a basis of a vector space, and their ability to make the connection between them. The question was graded based on the following 5-point rubric:

5 pts:
Correct and complete response:
4 pts:
Correct answer with complete explanation.
Lack of the correct use of symbolic representation.
3 pts:
Correct answer with incomplete explanation,
Lack of the correct use of symbolic representation.
Reference to the response given on part 2i which leads to a wrong answer.
2 pts:
Signs of understanding the concept.
Incorrect answers.
1 pt:
There is some work shown, but none is relevant.
0 pt:
No work is shown.

The question 2iii is both conceptual and procedural. It requires both skills. The question was aimed at testing students' ability to combine their knowledge of span of a set, linear combinations, and ability to carry out the required procedure to reach to the correct and complete response. The question was graded based on the following 5-point rubric:

5 pts:
Correct and complete response:
Statement of linear combination
Statement of span of the set
Carry out the procedure correctly (Solving the vector equation correctly)
4 pts:
Correct and complete response
Lack of the correct use of symbolic representations
Referring back to the responses given on the parts 2i and 2ii of the question
3 pts:
Correct answer with incomplete explanation

Lack of the correct use of symbolic representation.
Correct answer is stated with no explanation.
In correct answer due to minor algebra and calculation mistakes.

2 pts:

Wrong answer,
In complete explanation.
There are some signs of understanding of the concept.

1 pt:

There is some work shown, but none is relevant.

0 pt:

No work was shown.

Question 3:

This question was a conceptual question. It could be answered through memorization however. It investigated students' concept images of the concepts; linear independence and spanning set. It was aimed at comparing the formal definitions of the concepts and how students perceived them. The question stated the following:

Define the following terms:
a. Linearly independent set
b. Spanning set

Question 3a was graded based on the following 5-point rubric:

5pts:

Complete response:
Correct statement of formal definition or informal definition that shows all aspects of the concept.

4pts:

Responses that falling in between 3 and 4

3pts:

Any formal or informal statements which are not complete:
Do not show all aspects of the concept but shows signs of understanding.
Use of the symbol "[" as a set notation or a vector notation

2pts:

Responses that falling in between 3 and 1

1pt:

There is some work shown but none is relevant.

0pt:

No work shown.

The following 5-point rubric was used to grade question 3b:

5 pts:
Complete and correct response:
Correct statement of the definition that shows all aspects of the concept

4 pts:
Responses that falling in between 3 and 4.

3 pts:
Defining the concept by stating that S spans V, or Span(S)=V.

2 pts:
Defining the concept of span of a set incorrectly,
No mention of the statement:
Span(S)=V or S spans V.
Responses that falling in between 3 and 1.

1 pt:
Some work sown but none is relevant.

0 pt:
No work shown.

Question 4:

This question was aimed at investigating whether students were able to apply their knowledge of vector space concepts into linear transformations. The question stated the following:

Given a linear transformation $T: R^3 \text{->} R^2$ by $T(v)=A\, v$ where

$$A=\begin{bmatrix}1 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}$$

a. Find a basis for Ker(T)

b. What is the dim (Image(T)) (Justify your answer)?

The question 4a is both conceptual and procedural. It requires conceptual knowledge to be able to interpret the outcome of the procedure. Knowledge of the concept kernel of a linear transformation is also required. Another aspect of this question is that it requires students to be able to recognize objects of kernel of the transformation as vectors in three dimensional space, but not in two dimensional space. The question was graded based on the following 5-point rubric:

5 pts:

Complete responses showing all aspects of the concept:
Process of row-echelon form is given
A parameter is chosen
Basis is defined as a set that contains one vector with three components.

4 pts:

Correct answer but lacking on some of the symbolic representations:
Not having the set notation written

Using the parameter as part of the definition

such as {t(-2,1,1)}, {(-2t, t, t)} or $t\begin{bmatrix}-2\\1\\1\end{bmatrix}$

Small algebra mistakes (calculation errors) that lead to wrong vectors.

3 pts:

Have the correct procedure (row-reduction).
Have the vector (or slightly different one due to algebra mistakes)
Basis is not stated, instead the following statement is given:
Ker(T)={t (-2,1,1) | t is any real number }.

2 pts:

Have signs of understanding, but nothing is completed:
Starts with the correct procedure (row-reduction) but could not finish it.
Have vectors of the wrong type (vectors of every kind but vectors of three components) as vectors of the basis.

1 pt:

There is some work shown but none is relevant.

0 pt:

No work shown.

The question 4b is a conceptual question. Students were tested to see if they could use their understanding of linear transformations correctly to decide on the dimension of the image of the linear transformation. The following 5-point rubric was used to grade the question:

5-pts:

Complete response showing all aspects of the concept:
Use of the theorem that has the following statement:

# of columns of A – Dim(Ker(A))= Dim(Range(A))
Use of argument of number of leading ones in A based on the theorem that has the following statement:
" Column space of the matrix is the same as the range of the linear transformation "

4 pts

Responses that fall in between 5 and 3

3 pts:

Signs of understanding of the concept
No correct answer is given
Correct answer is given for the wrong reasons
Recalling the theorem stated above for the 5 pts, but does not know dim(Ker(T))
Recalling the # of leading ones in A as the dim(Range (T)) but can not read # of leading ones correctly.

2 pts:

Have signs of understanding
Responses that falling in between 3 and 1.

1 pt:

Some work shown but none is relevant.

0 pt:

No work shown.

Question 5:

This question was purely a conceptual question investigating whether the students were able to make the connection between a linearly dependent set and a spanning set. It also was aimed at investigating whether students were able to use the formal definitions of the concepts to write logical statements. The question stated the following:

Suppose that Span $\{v_1, v_2, v_3, \ldots\ldots\ldots, v_n\} = V$ and w is a vector in V.
Is the set $\{v_1, v_2, v_3, \ldots, v_n, w\}$ linearly independent (Justify your answer)?

The following 5-point rubric was used to grade the question:

5 pts:

Complete arguments on all aspects of the problem:
Statement on span of a set, and on a linearly dependent set
Statement on how span of a set and linearly dependent set are related

4pts:

Correct response:
Incorrect representations of the notations: Representation of a set and a vector.
The relationship between span of a set and a linearly depended set is not stated directly, but there are signs of understanding the relationship.

3pts:

Sign of understanding of the relationship between the two concepts
But the definition of span of a set is not used as part of the argument.
Answers that falling in between 4 and 2.

2pts:

Wrong answer due to not having a good grasp on the definition of a linearly independence set.
Right answer with wrong reasons.
Use of the dimension argument: Since one more vector was added to the number of vectors, the set is linearly dependent.

1pt:

Use of matrices (row reduction operations)
Irrelevant works.

0pt:

No work shown.

Qualitative Analysis

With each volunteer, exactly one interview was conducted lasting approximately one hour. The interview was aimed at examining possible connections between students' mental images of the concepts of linear combination, linear independence and spanning sets, and their ability to answer related questions varying from computational problems to problems of explaining abstract statements.

The goal of the interview was to get more insight on students' responses to the post-questions that were given in the quiz, the exams, and the final, and to see if there were any unanticipated factors that could affect students' responses on the post-questions.

Questions were given in varying orders. For most questions, the order was decided at the time of the interview. In almost all interviews, question number four was given first, and question number two was given next. The rest were given in an order that was decided based on students' responses on the fourth and the second questions.

Interview Questions

Question 1:

Type of solution to the equation $c_1v_1 + c_2v_2 + \ldots\ldots + c_nv_n = 0$ *is an algebraic indication of whether the set is linearly independent. Explain why knowing the solution type to the equation above would indicate whether the set* $\{v_1, v_2, \ldots\ldots, v_n\}$ *is linearly independent.*

Objectives

- Describe students' methods of recognizing the relationship between the algebraic equation and its implication.
- Examine the approaches students use to explain the relationship.

Question 2:

Give a geometric illustration of a linearly independent set.

Objectives:

- Assess how students perceive definition of linear independence
- Examine students' mental images, and its correlation to the text-book definition of the concept
- Examine what sources (Textbook definition, *Mathematica* demos) students use to come up with a graph.

Question 3:

Is (2,3,1) in Span{(1,0,0), (0,1,0)}?
Please explain your answer.

Objective

- Examine how students use their knowledge of linear combination, and its relation to span of a set

Question 4:

Construct a geometric representation of a linear combination of two given vectors $v1$ and $v2$ stemming from the same point.

Objectives

- Examine the correlation between students' mental images, and the text book definition of linear combination.
- Assess how students recall the text book definition to construct a geometric representation of the concept

Question 5:

Given the set $S = \{ f(x) \text{ in } C[0,1] \text{ such that } f(1)=0 \}$

a. Describe a vector in S.

b. Is the set S a subspace of vector space $C[0,1]$? (Explain your answer)

Objectives

- Assess how students perceive a vector in the set S.
- Examine how students use their knowledge of the set S to show the set is a subspace of vector space C[0,1].
- Examine possible connections between students' perception of a vector in the set and their ability to write a related proof.

Question 6:

Describe a vector in Range(T) where $T(v)= Av$ for a matrix

$$A = \begin{bmatrix} 1 & 2 & -7 & 0 \\ 3 & 1 & 6 & 7 \\ 0 & 4 & 0 & 3 \end{bmatrix}$$

Objectives

- Examine how students use their knowledge of vector space concepts to answer transformation-related problems.
- Assess how students make connection between the rows and columns of the matrix and a vector in the range of the linear transformation

Question 7:

Explain why having a nontrivial solution for the equation $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ *implies that the set* $\{v_1, v_2, ..., v_n\}$ *is linearly dependent.*

Objectives

- Examine whether students use their knowledge of linear independent set , and make connection between their knowledge and a nontrivial solution of a related equation
- Assess how students use solution type to the equation to decide whether the set is linearly dependent.
- Describe students' methods of recognizing the connection between solution type of the equation and a linearly dependent set

Question 8:

Would you recommend the use of Mathematica for the first year linear algebra classes?
Please give your reasons to why or why not you would recommend the use of Mathematica for the first year linear algebra classes.

Objective

- Assess what students in the experimental group think of the use of *Mathematica* in first year linear algebra classes.

Question 9:

What is your opinion on using computer activities in first year linear algebra classes?

Objective

- Assess what students in both experimental and control groups think of the use of computer activities in first year linear algebra classes

CHAPTER IV

Presentation and Analysis of Data

This is a comparative study of traditional and experimental first year Linear Algebra groups. Two research methodologies have been combined to analyze the data. This chapter will first examine the compatibility of the two groups by carefully reviewing background information and pretest results. Second, it will present the results of the questions designed to shed light on the research questions of this study.

As stated earlier, each post-test question was graded by two independent graders; one of which was the investigator. Grading was done based on Carlson's 5-point rubric (1998). Reliability between the graders varied from 0.79 to 0.91. The reliability between the graders for the first research question was 0.80, between the graders for the second and the fourth research questions was 0.79, and between the graders for the third and fifth questions was 0.91. Slightly lower reliability between the graders for the second and the fourth questions can be attributed to various reasons. One reason is that 5-point rubric for these questions might have been ill written. Another reason is that it seems to the investigator that these graders may not have followed the outlines of the 5-point rubrics carefully. Over all, there is a high degree of inter-rater reliability.

Background Questionnaire

A copy of the background questionnaire can be found in Appendix D. All questions except question number ten were clear to students. On question number ten, the statements addressing the use of computer algebra systems in mathematics classes were either left out, or students stated on the side of statements that they did not know what the term "computer algebra systems" meant.

Question 1a in the background questionnaire provided information on what mathematics courses students had taken in high school, as well as information on when students had taken these courses. Table 1a summarizes the results.

Twenty-nine (88 percent) students in the traditional group and 26 (89 percent) students in the experimental group had taken high school algebra. Seven students in the traditional group (20 percent) and in the experimental group (24 percent) had taken algebra before 1990. The other 23 students (68 percent) in the traditional group and 19 students (65 percent) in the experimental group had taken algebra either in 1990 or after 1990. Thirty-one students (91 percent) in the traditional group and 27 students (92 percent) in the experimental group had taken a high school geometry course. Five students (15 percent) in the traditional group and 7 students (24 percent) in the experimental group had taken a geometry course before 1990. Seventy-six percent of the students in the traditional group had taken a geometry course either in 1990 or after 1990, whereas only 68 percent of the students in the experimental group had taken a geometry course in year 1990 or after 1990.

None of the students in the traditional group took pre-calculus course before 1990. However, 19 students (56 percent) in the traditional group had taken a pre-calculus in 1990 or after 1990. In the experimental group, 5 students (17 percent) took a pre-calculus before 1990, and 16 students (55 percent) took a pre-calculus course in 1990 or after 1990. Similarly, none in the traditional group has taken advanced mathematics before 1990, but 23 percent had taken one in 1990 or after 1990.

Table 1a

Information on High School Mathematics Courses

Number of traditional-group students = 34

Number of experimental-group students = 29

	Traditional		Experimental	
Courses	<1990	≥1990	<1990	≥1990
Algebra	7 20%	23 68%	7 24%	19 65%
Geometry	5 15%	26 76%	7 24%	20 68%
Pre-Calculus	0 0	19 56%	5 17%	16 55%
Advanced Mathematics	0 0	8 23%	2 6%	10 34%
Calculus	0 0	21 62%	3 10%	17 58%
Others	0 0%	4 12%	1 3%	8 27%

Note. < 1990" stands for the courses taken before the year 1990
≥ 1990 "stands for the courses taken after or during the year 1990.

In the experimental group, 6 percent of the 12 students (40 percent) who had taken advanced mathematics took the class before 1990.

The number of students who took a calculus course in high school differed by one student favoring the traditional group. Of these, 21 students in the traditional group had taken the course after 1990 or in 1990, whereas 3 students (10 percent) in the experimental group had taken the course before 1990. Twelve percent of the traditional group and 30 percent of the experimental group stated that they had taken other mathematics courses when they were in high school.

Special attention should be given to the point that there were students in the experimental group who had taken pre-calculus, advanced mathematics, calculus and other mathematics courses before the year 1990, whereas none of the traditional group has taken these subjects before 1990. Not having current knowledge on these subjects may affect the results of the study.

To check on how current students' knowledge on college mathematics, the data summarized in the table 1b was collected. Over all, table 1b confirms the indications of table 1a. That is the traditional group had taken college mathematics classes more recently than most of the students in the experimental group, which means that most students in the traditional group had fresher knowledge on mathematics concepts than the students in the experimental group. Sixty-five percent of the traditional students had taken calculus I after 1994 or in 1994, and 85 percent of the traditional group had taken calculus II also after 1994 or in 1994. On the other hand, there were only 17 students (58 percent) in the experimental group who took calculus II after 1994 or in 1994. The other 3 students (10 percent) in the experimental group had taken calculus II before 1994. Calculus II and IV was also taken by most of the traditional group after 1994 or in 1994; 85 percent of the students had taken calculus III, and 74 percent of the students had taken calculus IV.

Sixty-two percent and forty four percent of the students in the experimental group had taken calculus III and calculus IV, respectively, either in 1994 or after 1994. These results indicate that the two groups on their calculus background were not equal. The traditional group had more students who had taken calculus

Table 1b

Previous College Course Information

Number of traditional-group students = 34

Number of experimental-group students = 29

	Traditional		Experimental	
Courses	< 1994	≥ 1994	< 1994	≥ 1994
Elementary Functions	0 0%	5 15%	1 3%	2 6%
Calculus I	1 3%	22 65%	4 14%	15 51%
Calculus II	0 0%	29 85%	3 10%	17 58%
Calculus I for Business	0 0 %	1 2%	0 0%	3 10%
Calculus II for Business	0 0%	1 2%	0 0%	1 3%
Calculus III	0 0%	29 85%	2 7%	18 62%
Calculus IV	0 0%	25 74%	2 7%	13 44%
Engineering Mathematics	0 0%	13 38%	2 7%	9 31%
Others	0 0%	5 15%	0 0%	4 14%

classes after or in the year 1994. The experimental group, on the contrary, had fewer students who had taken calculus sequences; fourteen percent of the students took calculus I, 10 percent took calculus II, and 7 percent took calculus III and IV before the year 1994. This may indicate that the experimental group had more students with weaker calculus knowledge.

The two groups had about the same number of students who had taken an engineering mathematics course, and both had taken the course after or in the year 1994.

Ten percent of the students in the experimental group, and two percent of the students in the traditional group had taken calculus I for businesses. Both groups had taken the course after or in the year 1994. The difference between number of students who had taken business calculus should be noted here. The experimental group actually had more business students than the traditional group. Table 2 also confirms the finding. Table 2 categorized students based on intended majors. The students who were majoring in engineering, computer science, geoscience, mathematics, physics and chemistry were counted as science majors, and those who were majoring in life sciences, business, humanities and education were counted as social science majors. Table 2 shows that 94 percent of the traditional students, and 69 percent of the experimental students were science majors. That indicates that the rest (31 percent) of the experimental group were in social sciences. A few students in both groups were counted as social and science majors due to the fact that these students were double majoring. Here, one should note that social science, mostly business, students at the University of Oklahoma are not required to take intense mathematics courses. They are usually weak on their mathematics knowledge.

Table 3 shows that students in both groups were evenly distributed as the first time takers and repeating students. Nine percent of the traditional and seven percent of the experimental students were repeating the class. These students in both groups indicated that they were repeating the class due to a failing grade from a previous linear algebra course.

Table 2

Students' Majors

Number of traditional-group students = 34

Number of experimental-group students = 29

Majors	Traditional	Experimental
Science	32 94%	20 69%
Social Science	2 6%	9 31%

Both the experimental and the traditional groups had about the same number of full time and part time students. Twenty-seven (93 percent) students in the experimental group and thirty-two (94 percent) students in the traditional group were full time students. The other two students in both classes were enrolled part time.

Table 3

Enrollment status of the students

Number of traditional-group students = 34

Number of experimental-group students = 29

Enrollment Status	Traditional	Experimental
First time Taker	31 91%	27 93%
Repeating	3 9%	2 7%
Full Time	32 94%	27 93%
Part Time More than One course	2 6%	2 7%

The background questionnaire also collected data on students' ethnicity for the purpose of checking diversity of the two groups. The results are summarized on table 4.

Table 4

Students' Ethnicity

Number of traditional-group students = 34

Number of experimental-group students = 29

Ethnicity	Traditional	Experimental
White	20 60%	13 43%
Hispanic	5 14%	2 6%
African American	0 0%	2 6%
American Indian	1 2%	0 0%
None of the above	8 23%	13 45%

Table 4 indicates that the traditional group had slightly more students (60 percent) with white ethnicity than the experimental group (43 percent). The traditional group also had more Hispanics (14 percent) than the experimental group (6 percent). There was no African American student in the traditional group contrary to the two African American students in the experimental group. There was one American Indian in the traditional group, and none in the experimental group.

Twenty-three percent of the traditional group and forty-five percent of the experimental group stated that they were none of the ethnicity stated, that they were either Indian or Asian. A question, addressing students' nationality, in the post questionnaire also revealed that these 13 students (45 percent) in the experimental group were international students, and they have been in the United States for less than two years. The majority of these students stated that they had arrived in the country 6 months to a year ago.

Here, attention should be given to the number of international students in the experimental group. Almost half (45 percent) the class was international students, whereas the traditional class had only 8 (23 percent) international students. Given the abstract nature of linear algebra concepts, this fact about the groups may affect the results of the study.

Since the experimental group intensely used *Mathematica* as part of their lectures, knowing students' experience with computer algebra systems would be helpful to interpret results of analysis of the data correctly. Table 5 shows the distribution of the students in both groups with respect to their experiences with computer algebra systems.

The number of students who had not used any computer algebra systems was higher in the experimental group (69 percent) than the traditional group (56 percent). Also, the traditional group had more students (30 percent) who used a computer algebra system at least once a month or more than once a month, than the experimental group (20 percent). Since the experimental group had used *Mathematica,* a computer algebra system, having students with less experience with computer algebra systems may have affected the results of the study.

Table 5

Experience with Computer Algebra Systems

Number of traditional-group students = 34

Number of experimental-group students = 29

Experience	Traditional	Experimental
Not at all	19 56%	20 69%
Less than once A month	5 14%	3 11%
At least once A month	4 12%	1 3%
More than once A month	6 18%	5 17%

The last question (question # 10) on the questionnaire was an opinion question with fourteen statements. The question stated the following:

Read each statement and then circle the response that matches your feelings. Use the following rating scale:

Strongly Disagree	*Disagree*	*Neutral*	*Agree*	*Strongly Agree*
1	*2*	*3*	*4*	*5*

1. Mathematics is my favorite subject	*1 2 3 4 5*
2. A mathematical topic is of little importance if it has no real world Applications	*1 2 3 4 5*
3. Use of software, such as Mathematica, MathCad, or Derive, enhances learning of college algebra	*1 2 3 4 5*
4. It is necessary to use symbols to define most mathematical concepts	*1 2 3 4 5*
5. The best way to learn mathematics is to study visual representations of given concepts	*1 2 3 4 5*
6. Geometrical demonstrations enhance Learning of mathematical concepts	*1 2 3 4 5*

7. Computation is an important Mathematics skill 1 2 3 4 5

8. In mathematics courses, hard work can make up for having less ability than other students 1 2 3 4 5

9. I like to use technology (calculators, computers etc.) in my classes. 1 2 3 4 5

10. Algebra is my favorite subject 1 2 3 4 5

11. Mathematics is useful 1 2 3 4 5

12. I really need a mathematics textbook with clear explanations to do well in a mathematics course 1 2 3 4 5

13. The best way to learn mathematics is to find good examples of kinds of problems you have to solve and try to follow its pattern. 1 2 3 4 5

14. A student's mathematics program should emphasize theory as well as applications 1 2 3 4 5

For the majority of the statements, two groups' opinions were evenly distributed. Students' opinions for the statements; two, ten and thirteen differed in both groups. The distribution of percentages of students' opinions can be seen on table 6.

Forty seven percent of the traditional group strongly disagreed that a mathematical topic is of little importance if it has no real world applications, whereas there was only 17 percent of the experimental group who strongly disagreed with the same statement.

For the statement 10, students' opinions also differed between the traditional and the experimental group. Those disagreed with statement was 47 percent in the traditional, and 13 percent in the experimental group. Statement 10 stated following:

" Algebra is my favorite subject "

Table 6

Percentage of Students' Opinions

Number of traditional-group students = 34

Number of experimental-group students = 29

Statements		1	2	3	4	5	6	7	8	9	10	11	12	13	14
Strongly	T	5	47	0	2	3	2	2	2	8	5	5	5	14	2
Disagree	E	0	17	3	6	0	0	0	3	13	0	3	3	3	0
Disagree	T	23	29	11	17	14	5	5	5	5	47	0	14	26	5
	E	10	41	0	6	13	6	3	6	0	13	0	10	10	3
Neutral	T	23	20	47	17	38	14	14	8	26	38	2	8	20	8
	E	48	24	44	17	20	13	13	13	17	55	6	24	10	17
Agree	T	32	0	23	47	35	55	41	64	32	5	17	32	17	47
	E	31	17	44	55	55	68	44	41	37	24	34	34	44	58
Strongly	T	14	2	17	14	8	20	35	17	23	2	73	38	20	32
Agree	E	10	0	6	13	10	13	37	34	31	6	55	27	31	24

The experimental group stated that algebra was their favorite subject. This seems to imply that majority of the experimental group were better at algebraic skills than visual skills. They may have been used to solving problems through algebraic manipulations. The visual-based approach in the experimental group may not have responded well to these students' learning styles.

The traditional group dominantly disagreed with the thirteenth statement: 14 percent and 26 percent of the traditional group strongly disagreed and disagreed, respectively. The percentage of those in the experimental group who strongly disagreed and disagreed with the statement was totaling only 13 percent. The statement thirteen stated the following:

> *" The best way to learn mathematics is to find good examples of kinds of problems you have to solve, and try to follow its pattern."*

This may indicate that discovery learning style this study used in the experimental group may not have served well the students' learning style.

Analysis of the Pretest Questions

A copy of the pretest questions can be found in Appendix D. Pretest was given as part of the background questionnaire. The background questionnaire was given on the first day of classes in the experimental group, and in the traditional group, it was given at the second-class time during the first week of classes. On pretest scores, no significant difference for α=0.05, was found between the traditional group and the experimental group. Table 7 shows a summary of the results. There was a non-significant difference of 0.14 of a grading point favoring the traditional group.

Notice should be given to pre-test question 1. The question dealt with addition of two vectors chosen from R^4. Table 8 summarizes the results.

Question 1 stated the following:

Given the following vectors $v=(1,2,3,4)$ and $w=(0,-1,4,5)$. Find $v+w$.

Notice should be given to the high percentages of correct answers in the traditional group. The percentage of correct responses in the experimental group was 65, and in the traditional group was 85. The p-value (0.07) for α=0.1 indicates a significant difference on the mean scores of the two groups. Results indicate that the

Table 7

Results of Pre-test Scores

Total point = 5

Number of traditional-group students = 34

Number of experimental-group students = 29

Group	Mean	SD	Med.	AWS t'	p-value	df
Traditional	3.38	1.18	3.5	0.40	0.69	51.3
Experimental	3.24	1.57	3			

students in the experimental group were not as familiar with vector or vector operations as those in the traditional group.

Table 8

Result of Pre-test Question 1

Total point=1

Number of traditional-group students = 34

Number of experimental-group students = 29

Group	Mean	SD	Med.	AWS t'	p-value	df
Traditional	0.85	0.35	1	1.82	0.07	51
Experimental	0.65	0.47	1			

Analysis of Post-Questionnaire

To examine students' opinions on their courses, a post-questionnaire was given later on in the semester. A copy of the questionnaire can be found in Appendix D. Both groups were given the same questions with an additional question, number five, given only to the experimental group. There was no misunderstanding on most of the questions except question five for the traditional group, and question six for the experimental group. Both questions stated the following:

> *Read each statement and then circle the response that matches your feelings. Use the following choices:*
> *1. Strongly Disagree 2. Somewhat Disagree 3. Disagree*
> *4. Agree 5. Somewhat Agree 6. Strongly Agree*

On these questions, there may have been some confusion due to the ordering of the options; options two and three, and options four and five should have been interchanged. The traditional group had taken the questionnaire a week earlier than the experimental group; the experimental group took the questionnaire on the last meeting of the semester.

Table 9 shows the percentages of number of students who chose each difficulty level of the listed linear algebra concepts.

Forty three percent of the traditional and thirty four percent of the experimental group expressed that learning vector space concepts was very difficult. Students in the traditional group also thought that matrices (four percent) and system of linear equations (eight percent) were very difficult to learn. There was, however, none in the experimental group who thought learning these subjects were very difficult. On the other hand, 39 percent of the traditional group, and 34 percent of the experimental group thought that learning linear transformations was very difficult.

Table 9

Students' opinions on the difficulty level of concepts

Number of students in traditional group = 26

Number of students in experimental group = 23

	Very Difficult		Somewhat Difficult		Not Difficult at all	
	Traditional	Experimental	Traditional	Experimental	Traditional	Experimental
Vector Space	43%	34%	46%	56%	11%	9%
Matrices	4%	0	2%	43%	76%	57%
Linear Equation	8%	0	23%	39%	69%	57%
Linear Transform.	39%	34%	54%	60%	7%	5%

Notice also should be given to the fact that the majority of the experimental group expressed that learning almost all subjects listed on the table were somewhat difficult but not very difficult. However, the traditional group, in general, either thought these subjects were very difficult to learn or not difficult at all. The majority also indicated that learning of the vector space concepts was very difficult. If we consider both vector space and linear transformations, the traditional group had more students (Totaling 82 percent) who indicated that learning these subjects was very difficult than the experimental group (Totaling 68 percent).

Question number 5 on students' opinion on how helpful *Mathematica*-related activities had been in learning basic linear algebra concepts, was given only to the experimental group. 68 percent indicated that *Mathematica* was helpful in learning vector space concepts. Seventy and 71 percent thought *Mathematica* was helpful in learning matrices and system of linear equations respectively. *Mathematica* related activities were chosen as the least helpful for learning linear transformations (53 percent) and for understanding definitions (43 percent), and chosen as the most useful for visualization of the basic concepts (77 percent) and for matrices followed by the numerical calculations (67 percent) and applications (67 percent).

To get students' opinions on how helpful the instructional tools have been for the students in learning linear algebra material, question number 6 for the traditional group and question number 7 for the experimental group were given. Both questions had the same statement. The statement can be found in appendix D. The data analysis indicated that totaling 100 percent of the experimental group found the lectures either somewhat helpful (50 percent) or very helpful (40 percent). There was none with an opinion favoring that lectures were not helpful. One should recall that lectures in the experimental group were integrated with *Mathematica* activities. On the contrary, 12 percent of the traditional group expressed opinions favoring that the lectures were not helpful; 50 percent thought lectures were somewhat helpful, and 38 percent thought lectures were very helpful.

To get students' opinion on more general statements, question number six for the experimental group and question number 5 for the traditional group were given on the post questionnaire. Both questions stated the same problem. The questions stated the following (more detailed copy of the question can be found in Appendix D):

Read each statement and then circle the response that matches your feelings. Use the following choices:

1. Strongly disagree *2. Somewhat disagree* *3.Disagree* *4. Agree*
5. Some what agree *6. Strongly Agree*

Some of the statements and the percentages of number of students who expressed opinions on these are listed below:

Statement 1:

"Technology we used is appropriate for this course"

This was a statement given to the experimental group only: 74 percent of the group agreed with the statement.

Statement 2:

"Computer assisted instructions, such as MATHEMATICA, MathCad, DRIVE, can enhance learning of the material covered in this class."

This statement was given to both groups: 70 percent of the experimental and 79 percent of the traditional group agreed with the statement.

Statement 3:

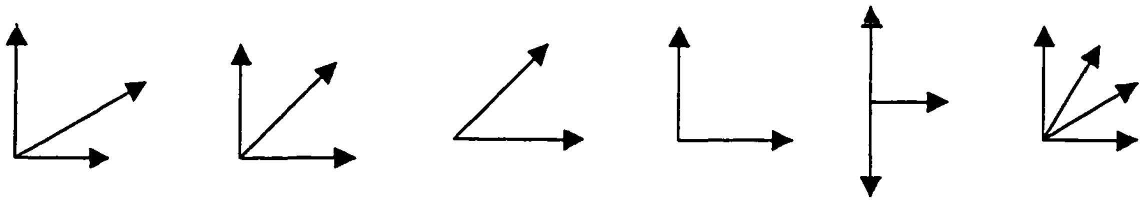
"I have enjoyed the class "

This was given to both groups: 70 percent of the experimental group and 50 percent of the traditional group agreed with the statement. Here, notice should be given to the large difference between the percentages of students in both groups who expressed that they have enjoyed the class. This result indicates that the majority of the experimental group enjoyed the class, contrary to the lower percentages (50 percent) of number of students in the traditional group who expressed the same opinion.

To examine students' ability to answer visual-based problems, the last question on the post questionnaire was given to the students in both groups. A detailed copy of the question can be found in Appendix D. Question was stated as:

*On each graph below, shown a set of vectors originated from (0, 0). Circle the ones that are **linearly independent.***

A. B. C. D. E. F.



The correct response for this question was options C and D; Option C had two vectors with an acute angle in between; Option D had two perpendicular vectors. All the other options had three or more vectors with angles varying from acute to perpendicular. Table 10 shows percentages of students who chose options C and D as the set of linearly independent vectors.

Table 10

Percentages of Students' Responses

Number of students in traditional group = 25

Number of students in experimental group = 21

Options	C	D
Traditional	56%	64%
Experimental	67%	76%

Table 10 indicates that the experimental group seems to be slightly better at answering visual-oriented problems than the traditional group: 56 percent of the traditional group and 67 percent of the experimental group had chosen option C as the set of vectors that are linearly independent. There was 64 percent in the traditional and 76 percent in the experimental group who thought that the vectors in part D were linearly independent. Some students chose other options as well as these two options as linearly independent vectors. In both groups, responses that include all options were around 20 percent within the epsilon difference of plus and minus two. Interestingly, both sections had higher percentages of students who chose option D than option C. The analysis of selected interviews shed light on what bases the students made their decisions. Interviews (see Appendix A) revealed that some students perceived linearly independent vectors as those with different angles in between, and some perceived linearly independent vectors as those that are perpendicular to each other. For instance, student A is more likely to chose all options as his/her answer. According to him/her, a set of vectors is linearly independent if the vectors in the set do not have the same angle between themselves and the x-axis. His understanding of the concept can be detected in the following statement he made during the interview:

> *"A: Okay, I am (pause) I come to apply the same thing. There is no vector in the set that can be produced by adding any of the other two vectors in the set but okay that can be produced by linear combination of any other vectors in the set so I would, if there is like n vectors in the set. <u>I would draw whole bunch of them none of them would be on the same, have the same angle between themselves and x-axis like that so look like that, there will be no vectors that are just shorter versions of each other.</u> "*

Here is a student (interview of student B) in the experimental group describing his/her understanding of the concept:

> *" Umm to represent them, three coming from the same point, I don't think so...Because one of them will always be able to be represented by the, the sum of, of you know scalars times the other two...... Well umm, you had (pause) three*

vectors, and they are all you know coming, passing through zero then umm that you know that they definitely have the trivial solution but you could also may be <u>see that umm if this you know vector was multiplied by something that would bring it this way, and the other was multiplied by something that might bring it umm you this way by a certain amount then you could see that, that this vector</u> could be a result of ...see it looks like ohh, you were just to add these two together but send them in the opposite direction (pause) then you would get opposite of that vector, and then you would get it to be zero. That would say that it is not linearly independent...."

This student's understanding of the concept seems to be more visual-oriented, and seems to be based on his interpretations of *Mathematica* demonstrations used in the experimental group.

The difference of 12 percent between the two groups who chose options C and D does seem to indicate that there is less misinterpretation of the abstract definitions in the experimental group than in the traditional group.

Analysis and Results of Post-Questions

These questions were posed to gather data late in each course. The results indicate understanding after one of the approaches to linear algebra instructions.

Question 1

The purpose of the question was to investigate whether students could recognize 2x2 matrices defined in the question as objects of the set, and be able to write a proof stating the subset as a subspace of the vector space, set of all 2x2 matrices. Question 1 stated the following:

Let S be the set of matrices of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ *where a and b are any real numbers.*

a. Show that S is a subspace of $M_{2,2}$

b. Find a basis for S.

This question, in both groups, was given on a quiz during the week right after vector space and subspace concepts were covered. Students in the traditional group had seen vector space and subspace concepts through abstract definitions written on the blackboard. Subspaces of familiar vector spaces R^2 and R^3 were also stated on the blackboard. They had seen visual descriptions of the concepts at a minimum level; a few drawings for subspaces limited to one-dimensional objects in R^2 were done on the blackboard. In addition to the drawings on the blackboard, students in the experimental group had seen visual *Mathematica* demonstrations of examples and non-examples of two and three dimensional vector spaces and subspaces. These students also experimented with their own examples and non-examples of the concepts on ready *Mathematica* cells. The subject of this question, the set of all nxn matrices as a vector space, was stated on the blackboard. On the other hand, visual examples of subspaces of the vector space, set of all continuous functions, were demonstrated through *Mathematica* notebooks. The objects of subsets of the vector space were drawn on the same graphs through interactive *Mathematica* notebooks, and students were asked to discuss whether the subsets were subspaces or not based on their observations of the graphs. Until after seeing three or four demonstrations, students were not introduced, or expected to use any algebraic procedures that can show whether subsets are subspaces.

Part a of the question mostly required procedural knowledge. Both groups had seen similar problems discussed in class. The traditional group had focused on carrying out the procedures, and the experimental group focused mostly on the conceptual aspects of the concepts through visual demonstrations. The students in the experimental group were given the procedures after seeing related demonstrations but they did not spend as much time on carrying out the procedures. Table 11a and 11b summarize the results of the data analyses on students' scores.

Table 11a

Results on vector spaces and subspaces

Group	n	Mean	SD	Median
T	31	3.77	1.09	4
E	24	3.35	1.37	3

Part b of the question required conceptual knowledge of the concept, *basis*, and also required students to be able to recognize 2x2 matrices with the conditions given in the question as the objects of the set.

The difference between the two groups on the question (part a) requiring conceptual knowledge on vector spaces and subspaces was not significant (t'=1.29, p=0.203, df=43). Table 11a summarizes the results. A non-orthogonal two-way analysis of variance was performed to adjust for the controlled variables, attendance, nationality, gender and ability. Results of these tests were also non-significant. The results of the test of treatment effect eliminating the effect of each of the controlled variables are given on table 11b. (Table 11b shows F-values for the treatment effect eliminating the effect of each of the controlled variables).

Table 11b

Results on vector spaces eliminating the effect of controlled variables.

Controlled Variables	F	p-value	df
Ability	3.44	0.069	(1, 50)
Nationality	2.09	0.154	(1, 51)
Attendance	1.68	0.20	(1,51)
Gender	1.35	0.249	(1,51)

Students, in both groups, who showed the set as a subspace of the vector space $M_{2,2}$, used the same procedure. They first showed that the set was not empty by writing a matrix in the set; next they took symbolic representations of two matrices from the set and showed that their sum was also in the set. They also showed that scalar times any matrix in the set was still in the set. Two students in the traditional and the experimental groups used matrices with numerical entries to show the closeness. In addition, ten students in the traditional and two students in the experimental group showed the closeness by showing that the sum and the product were in the vector space, $M_{2,2}$. And, three students in the traditional group and two students in the experimental group used incorrect objects as the vectors of the subset.

Interviews revealed that those in the traditional group who gave complete and correct proofs might still not have had accurate understanding of the subspace concept (see Appendix A). Even though, student A had a complete and correct proof for an interview problem addressing that the set of all continuous functions whose value at 1 is zero is a subspace of the vector space of all continuous functions, his/her concept image of a subspace was not accurate. To him/her, any subset of a vector space was a subspace as long as the subset contained the zero vector. Here is how he/she described a subspace:

"A: ...It is (pause) in two dimensions. If I have x and y, if my space is that a subspace like a subset looks like that (pointing his drawing on his paper). Something like that, three dimension of this big blub or lesser blub.."

The difference between the traditional group and the experimental group on part b of the question was significant ($t'= -2.38$, $p=0.021$, $df=42.4$). Table 12a summarizes the results.

A non-orthogonal two-way analysis of variance was also performed to adjust for the controlled variables; attendance, nationality, gender and ability. Results of these tests were also significant. The results of the test of treatment effect eliminating the effect of each of the controlled variables are given on table 12b. (Table 12b shows F-values for the treatment effect eliminating the effect of each of the controlled variables).

There were two students in the traditional and one student in the experimental group who left out this part of the question. One student in each group gave the standard basis of the vector space; $M_{2,2}$, as a basis for the subspace, or wrote one

Table 12a

Results on vector spaces and bases

Group	n	Means	SD	Median
T	31	2.12	1.31	3
E	23	3.09	1.56	3

vector of the basis correctly. Eleven students in the traditional group and four students in the experimental group gave a basis with unrelated vectors. Six students in the

Table 12b

Results on vector spaces and bases eliminating the effect of controlled variables.

Controlled Variables	F	p-value	df
Ability	5.89	0.018	(1,50)
Nationality	7.63	0.0079	(1,51)
Attendance	7.36	0.0091	(1,51)
Gender	8.27	0.0059	(1,51)

experimental group wrote a basis with parameters attached to the vectors of the basis. For example, one student gave the following set as a basis for the subspace:

" *A basis must span the set. It also must be linearly independent.*

$$\left\{ a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$ "

Interviews revealed that those who wrote n-tuples (vectors in R^n) as the vectors of their bases seemed to use them because they did not recognize matrices or any other forms such as functions (see appendix A) as vectors. To them, vectors could only be those with components, and they could only be the objects of R^n, n is any positive integer. This can be observed in a student's response to an interview problem:

" Okay I confess that I am not sure how to think of a vector in S. Umm (pause) because S is a set of functions Umm...In Physics, I was given a very restricted definition of vector, (referring to his instructor) has been discussing a very general abstract definition of a vector ... Specifically my problem I think of a vector as look like v one comma v two comma up till v n. When someone say says ohh this set is composed of functions now describe a vector in that set I say okay there is functions in that set I don't think there is vectors in that set......."

Conflicts between the student's previous knowledge of vectors and the abstract definition of the concept seems to be causing learning difficulties (Edwards, 1997; Rosnick & Clement, 1980).

Question 2

Second question was a final exam-question. The traditional and the experimental groups had seen similar problems in class. The traditional group focused mostly on the procedural knowledge of the concept, whereas the experimental group focused on the conceptual knowledge. Students in the experimental group had seen two and three-dimensional *Mathematica* demonstrations of examples and non-examples of linearly independent sets, span of sets and bases of vector spaces. The traditional group spent more time on carrying out the procedures on similar questions than the experimental group. On the other hand, the experimental group spent more time on discussing outcomes of demonstrations, and discovering characteristics of the concepts through observing visual outcomes. On the final, the students in the experimental group, unlike those in the traditional group, were allowed to use calculators or *Mathematica* to carry out the computations or procedures such as row-reduction of matrices. Question 2 stated the following:

Given the following vectors in R^3.
$a=(0,-1,1)$, $b=(2,1,1)$, $c=(2,0,2)$, $d=(1,0,1)$

i. *Is the set $\{a, b, c\}$ linearly independent (justify your answer)?*

ii. *What is the dimension of Span$\{a, b, c\}$(justify your answer)?*

iii. *Is the vector $(1,2,3)$ in Span$\{a, b, c\}$ (Justify your answer)?*

The purpose of question 2i was to examine students' ability to carry out the required procedures, and to see if students could interpret the results of the procedures correctly.

Table 13a

Results on the computational (procedural) problem

Group	n	Means	SD	Median
T	26	3.80	1.44	4
E	25	3.97	1.36	5

Nineteen of the traditional students and ten of the experimental students used the row-reduction procedure to answer the question. In addition, eight students in the experimental group as opposed to two students in the traditional group answered the question by making the observation, without using the row-reduction approach, that the vector c is the sum of the vectors a and b. Furthermore, two students in both groups used the determinant argument (if determinant of the related matrix is zero then the related vectors are linearly dependent).

On the scores of the question, the difference between the two groups was not significant (t'=-0.39, p= 0.7005, df=49). Table 13a summarizes the results. A non-orthogonal two-way analysis of variance was performed to adjust for the controlled variables; attendance, nationality, gender and ability. Results of these tests were not significant either. The results of the test of treatment effect eliminating the effect of each of the controlled variables are given on table 13b. (Table 13b shows F-values for the treatment effect eliminating the effect of each of the controlled variables).

Part 2ii of the question aimed at testing students' ability to combine their conceptual knowledge of linearly independent set and span of a set, with their knowledge of the dimension concept. This question required either conceptual knowledge or procedural knowledge. Students could answer the question by applying the row-reduction procedure of the matrix whose rows consisted of the vectors of the set.

Table 13b

Results on the computational problem eliminating the effect of controlled variables.

Controlled Variables	F	p-value	df
Ability	0.00	0.97	(1,47)
Nationality	0.03	0.85	(1,48)
Attendance	0.26	0.61	(1,48)
Gender	0.51	0.47	(1,48)

Similar problems were discussed in both groups. The traditional group discussed these types of problems through stating, on the blackboard, the connections between the concepts, whereas the experimental group discussed the connections through seeing concrete visual *Mathematica* demonstrations.

No significant difference for $\alpha = 0.05$ was found between the traditional and the experimental groups ($t' = 0.67$, $p = 0.50$, $df = 48.2$). Table 14a summarizes the results.

Table 14a

Results on combining knowledge of linearly independent set, span of a set and dimension

Group	n	Mean	SD	Median
T	26	2.92	1.44	3
E	25	2.70	1.57	2

A non-orthogonal analysis was performed to adjust for the controlled variables; attendance, nationality, gender and ability. Results of these tests were not significant either. The results of the test of treatment effect eliminating the effect of each of the controlled variables are given on table 14b. (Table 14b shows F-values for the treatment effect eliminating the effect of each of the controlled variables).

Seven students in the traditional group and six students in the experimental group used row-reduction process to answer the question. Five and eleven students in traditional and the experimental group respectively, referred to their responses on part i, and six students in the traditional and one student in the experimental group answered the question correctly with no explanation.

The purpose of the question 2iii was to examine students' ability to use their knowledge of span of a set. This question required conceptual and procedural knowledge, and the use of the definition of the concept; span of a set. Students could answer this question by just using their conceptual knowledge or combining their conceptual and procedural knowledge.

Table 14b

Results on combining knowledge of linearly independent set, span of a set and dimension eliminating the effect of controlled variables

Controlled Variables	F	p-value	df
Ability	0.85	0.36	(1,47)
Nationality	0.00	0.98	(1,48)
Attendance	0.29	0.59	(1,48)
Gender	0.40	0.52	(1,48)

The traditional group had seen similar questions through using procedures such as the use of row-reduction process. The experimental group was introduced to procedures after covering the related concepts through two and three-dimensional *Mathematica* demonstrations. Students were first expected to develop their own definition of the concept, and then, they were introduced to the procedures. The traditional group spent more time on learning procedures than the experimental group.

There was no significant difference found between the two groups (t'=0.64, p=0.52, df=46). Table 15a summarizes the results. A non-orthogonal analysis was

Table 15a

Results on using definition of span of a set

Group	n	Mean	SD	Median
T	26	3.50	1.27	3
E	25	3.24	1.58	3

performed to adjust for the controlled variables; attendance, nationality and ability. Results of these tests showed no significance either. The results of the test of treatment effect eliminating the effect of each of the controlled variables are given on table 15b. (Table 15b shows F-values for the treatment effect eliminating the effect of each of the controlled variables).

Table 15b

Results on using definition of span of a set eliminating the effect of controlled variables

Controlled Variables	F	p-value	df
Ability	0.82	0.36	(1,47)
Nationality	0.57	0.45	(1,48)
Attendance	0.25	0.61	(1,48)
Gender	1.15	0.28	(1,48)

Twelve students in the traditional group and six students in the experimental group used row-reduction process to answer the question. Seven students in the experimental group as opposed to one student in the traditional group were able to see that any vector in the span of the set can only be a vector of the form that had non-zero first and second components, but zero third component. Furthermore, three and seven students in the traditional and the experimental groups, respectively, referred back to their responses on part ii to answer the question. Among unrelated responses (i.e. correct answers with wrong reasoning), there were a few responses that seemed to reveal students' interpretation of the concept. For example, a student from the traditional group wrote down the following as his/her reason to why the vector (1,2,3) was not in the span {a, b, c}:

" $\boxed{No}$ *-from part ii) we know the* <u>*dim(span{a,b,c})=2 ≠ dim(1,2,3)=3*</u> "

As the interviews revealed, this student may as well be considering number of components on a vector as the dimension of the vector space. None of the students in the experimental group stated reasons similar to the one above. However, some of them had incorrect answers due to incorrect responses they gave on the previous parts of the question, or due to algebraic mistakes in carrying out the row-reduction process. One should also note that there were six students in the traditional group and three in the experimental group whose responses for part 2i were correct, but had incorrect responses for part 2ii. There were also three students in the traditional group (none in the experimental group) who had correct responses for part 2ii, but had incorrect responses for 2i.

Question 3

Question 3 was given on the second exam. The purpose of the question was to examine students' concept images of linearly independent sets and spanning sets. The question also tested how students were recalling, and interpreting the formal definitions of the concepts. Both groups had seen the formal definitions. The experimental group had seen the definitions while going through various *Mathematica* demonstrations, and discovering some of the characteristics of the concepts through observing, and discussing the outcomes of the demonstrations. The traditional group had first seen statements of the formal definitions, and then, continued with related proofs. These students were expected to discover characteristics of the concepts through proving related statements. Question 3 stated the following:

Define the following terms, and give an example for each term.
a. Linearly independent set
b. Spanning set

The experimental group had seen demonstrations that attempted to show relationships between the solution of the abstract vector equation used in the formal definition of the concept; linearly independent set, and linear independence of the sets. To achieve the goal, students in the experimental group were given two and three dimensional visual representations of vectors whose positions were shown with respect to each other (see Appendix C for a sample demo), and asked to solve related linear vector equations by tracing through the vectors' visual representations.

Difference between the two groups on the definition of a linearly independent set was not significant (t'=-0.26, p=0.792, df=51.9). Table 16a summarizes the results. A non-orthogonal two-way analysis of variance was performed to adjust for the controlled variables; attendance, nationality, gender and ability. Results of these tests were not significant either. The results of the test of treatment effect eliminating the effect of each of the controlled variables are given on table 16b. (Table 16b shows F-values for the treatment effect eliminating the effect of each of the controlled variables).

Table 16a

Results on definition of linearly independent set

Group	n	Mean	SD	Median
T	28	3.75	1.35	4
E	26	3.84	1.31	4

There were two approaches that the students in both groups used to answer this question. Seven students in the traditional group and eight students in the experimental group used the formal definition of the concept, and 14 traditional students and 10

Table 16b

Results on definition of linearly independent set eliminating the effect of

Controlled variables

Controlled Variables	F	p-value	df
Ability	2.34	0.13	(1,49)
Nationality	0.00	0.99	(1,50)
Attendance	0.03	0.85	(1,50)
Gender	0.00	0.96	(1,50)

experimental students used the informal definition of the concept. Here is a typical informal definition given by some students:

" A set of vectors is linearly independent if none of the vectors in the set can be written as linear combinations of the other vectors in the set."

There were also different terms used in stating the informal definition such as "representing a vector in terms of the others", "obtaining one from the others ", and " writing a vector in terms of the others ".

Interviews with a few students from both groups revealed that students may be able to write either formal or informal definition of the concept, however, they may still lack on conceptual understanding of the concept. One student during the interview geometrically interpreted the definition of the concept as those vectors that had nonzero angles in between, and another interpreted it as those vectors whose angles in between were exactly 90 degrees. Both students were, however, able to recall the algebraic definition of the concept correctly. Students in the experimental group had written similar definitions for the concept, but they seemed to have better geometric understanding.

The question 3b examined students' concept image of spanning set. The traditional group had only seen the formal definition of the related concepts, and continued with related proofs. These students were expected to discover characteristics of the concepts through understanding the formal definition and through writing proofs. On the contrary, the students in the experimental group were exposed to two and three dimensional *Mathematica* demonstrations of the concepts (see Appendix C), and they were expected to discover characteristics of the concepts through their visual demonstrations.

Students in both groups had similar responses to this question. A typical response was:

"A spanning set is a subset of a vector space in which all vectors of the vector space can be written as linear combinations of the vectors of the set ".

There were also slightly different statements of the concept such as " A spanning set is a set that spans the vector space", and " a spanning set S is a set such that Span(S)=V" (V was not defined, the investigator assumed that students were using V to represent a vector space).

Table 17a

Results on definition of spanning set

Group	n	Means	SD	Med.
T	28	3.42	1.50	3.5
E	25	3.10	1.39	3

There was no significant difference (t'= 0.98, p=0.334, df=50.9) between the two groups on the question stating definition of a spanning set. Table 17a summarizes the results. After a non-orthogonal two-way analysis of variance was performed to adjust for the controlled variables; attendance, nationality, gender and ability, no significant difference was found between the two groups. The results of the test of treatment effect eliminating the effect of each of the controlled variables are given on table 17b. (Table 17b shows F-values for the treatment effect eliminating the effect of each of the controlled variables).

Table 17b

Results on definition of spanning set eliminating the effect of controlled variables

Controlled Variables	F	p-value	df
Ability	2.82	0.099	(1,49)
Nationality	0.16	0.688	(1,50)
Attendance	0.77	0.385	(1,50)
Gender	0.58	0.45	(1,50)

Question 4

Question 4 was aimed at examining students' ability to apply their knowledge of the concepts; linearly independent set, span of a set and dimension of a vector space, to linear transformations. The question was given on the third exam. Both groups had seen similar questions. The experimental group also had seen two and three-dimensional demonstrations of domain, range and kernel of linear transformations. They discussed dimension and bases of subspaces through the demonstrations. They were given the demonstrations and asked to answer related questions such as dimension of kernel of a transformation based on two and three-dimensional graphs. At that level, students were not given algebraic descriptions of the concepts. Their answers were based only on the graphs. Both groups had seen the effects of elementary matrices as linear transformations on basic geometric shapes such as unit squares and triangles. The traditional group had seen two-dimensional drawings of the shapes on the blackboard, whereas the experimental group had seen two- and three-dimensional colored graphs through *Mathematica* notebooks. They also had chances to observe the effects of their own examples of matrices on the basic geometric shapes.

The experimental group, by tracing through each color, had observed how linear transformations were mapping points on lines. They also, by observing the paths of colors, discussed kernel, domain, range and their dimensions. Through observations, students attempted to write algebraic descriptions of kernel and range of linear transformations.

On the other hand, the traditional group had first seen procedures to write algebraic descriptions of the concepts, and expected to learn more about these concepts through these descriptions. Question 4 stated the following:

Given a linear transformation T: R^3->R^2 by T(v)=A v where $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

a. Find a basis for Ker(T)
b. What is the dim(Image(T)) (Justify your answer)?

Question 4a was aimed at testing students' ability to combine their knowledge of basis with kernel of linear transformations. This question could be done by use of a procedure (row-reduction process), and a basis could be found by interpreting algebraic description of the kernel. Furthermore, an algebraic description for a kernel could be obtained through the row reduction process.

The results of the data analysis did not show any significant difference ($t' = 0.84$, $p=0.407$, $df=42.5$) between the two groups on the question that examined students' ability to write a basis for the kernel of the given linear transformation.

Table 18a

Results on linear transformations and vector spaces

Group	n	Means	SD	Median
T	28	3.20	1.19	3
E	22	3.00	1.34	3

Table 18a summarizes the results. A non-orthogonal analysis of variance performed to adjust for the controlled variables; attendance, nationality, gender and ability showed no significant difference. The results of the test of treatment effect eliminating the effect of each of the controlled variables are given on table 18b. (Table 18b shows F-values for the treatment effect eliminating the effect of each of the controlled variables).

Almost all of the students in the traditional and the experimental groups used the row-reduction procedure mentioned above. In both groups, some arrived at a slightly different answer due to algebra mistakes. Seven students in the traditional group and one student in the experimental group, even though they arrived at a correct description of the kernel, interpreted the description incorrectly. For example, one of

Table 18b

Results on linear transformations and vector spaces eliminating the effect of controlled variables.

Controlled variables	F	p-value	df
Ability	2.94	0.09	(1,46)
Nationality	1.61	0.20	(1,47)
Attendance	0.13	0.72	(1,47)
Gender	0.40	0.52	(1,47)

these students stated the following:

> "...*Let* $z = t$, $y = t$, $x = -2t$. $Ker(T)=\{ t(-2,1), t \in R \}$ $\therefore$ *a basis for* $ker(T) = \{(-2,1)\}$."

There was also wrong use of notations in both groups.

Question 4b had a purpose similar to that of the question 4a. The purpose was to test students' ability to apply their knowledge of vector space concepts to linear transformations. The difference between the two questions is that question 4b required more conceptual knowledge than procedural knowledge. Question 4a could be answered by applying a procedure, whereas there was no procedure given in both sections for question 4b type. Students, however, could use their knowledge of dimension of kernel of the linear transformation whose basis was given in part 4a. Students in the traditional group had seen the concepts; dimension and range of linear transformations, in their abstract forms as stated on the blackboard. The connection between dimension and range had been discussed through the theorem stated below:

" *Let* $T: V \to W$ *be a linear transformation from an n-dimensional vector space V into a vector space W. Then the sum of the dimensions of the range and kernel is equal to the dimension of the domain. That is,* *rank(T)+nullity(T)=n or dim(range)+dim(kernel)=dim(domain)* "

and through an observation:

" *For a linear transformation T: R^n->R^m given by T(x)=Ax for an mxn matrix A, dim(domain(T))=number of columns of A.* "

Students in both groups were expected to combine the two to be able to answer the question. Contrary to the traditional students who were introduced to the theorem through its abstract statement, before the students in the experimental group were given the abstract statement of the theorem, they had gone through visual *Mathematica* activities, and attempted to discover the theorem with the guidance of their instructor. These activities consisted of two- and three-dimensional demonstrations of linear transformations. Furthermore, as each demo was ran, students were asked to decide on the domain, range and kernel of the linear transformations, and their dimensions based on the graphical representations. These students used terms such as " input space " for domain of linear transformations and " target space" for the vector spaces into which the vectors in domains were mapped.

The differences on the students' scores between the two groups were significant (t'=-2.01, p=0.049, df=47.1). Table 19a summarizes the results.

Table 19a

Results on linear transformations, images and dimensions

Group	n	Means	SD	Median
T	28	2.07	1.08	2.5
E	22	2.59	0.73	3

Table 19b

Results on linear transformations, their images and dimension eliminating the effect of controlled variables

Controlled Variables	F	p-value	df
Ability	4.41	0.04	(1,46)
Nationality	4.80	0.033	(1,47)
Attendance	5.02	0.029	(1,47)
Gender	4.41	0.04	(1,47)

A non-orthogonal two-way analysis of variance was performed to adjust for the controlled variables; attendance, nationality, gender and ability. The results of these tests were also significant. The results of the test of treatment effect eliminating the effect of each of the controlled variables are given on table 19b. (Table 19b shows F-values for the treatment effect eliminating the effect of each of the controlled variables).

Question 5

The purpose of the question 5 was to examine whether students could write proofs for statements that required only the conceptual knowledge of the concepts; linearly independent sets and span of sets. This question was a strictly conceptual question. Students in both groups discussed connections between linearly independent sets and span of sets. The traditional group discussed the connections through algebraic statements on the blackboard, and the experimental group discussed these on their two- and three-dimensional visual *Mathematica* demonstrations. A typical demo started out with either set of linearly independent vectors or dependent vectors, and showed students their spans. Through these, students were expected to make observations on implications such as if the set was linearly independent then span of the set would have the dimension defined by the number of vectors in the set. They also were expected to observe that each vector in a linearly independent set created one dimension of the resulting vector space. Question 5 stated the following:

Suppose that Span $v_1, v_2, v_3, \ldots, v_n\}=V$ and w is a vector in V.
Is the set $\{ v_1, v_2, v_3, \ldots, v_n, w\}$ linearly independent (Justify your answer)?

There was no significant difference (t'=-0.18, p=0.857, df=51.9) between the traditional group and the experimental group on the question that tested students' ability to use their knowledge of basic vector space concepts to write related proofs. However, the results showed that the mean scores differed by 0.09 were favoring the experimental group. Table 20a summarizes the results. Also, there was a non-orthogonal analysis of variance performed to adjust for the controlled variables; attendance, nationality, gender and ability. The results of these tests showed a significant difference between the two groups testing the treatment effect eliminating the effect of the controlled variable *ability*. There was, however, no significant difference found for the tests of treatment effect eliminating the rest of the controlled variables. The results of the test of treatment effect eliminating the effect of each of the controlled variables are given on table 20b. (Table 20b shows F-values for the treatment effect eliminating the effect of each of the controlled variables).

To show that the resulting set was linearly dependent, four students in the traditional group and one student in the experimental group incorrectly used the

Table 20a

Results on vector spaces and proofs

Group	$\underline{n}$	Means	SD	Median
T	28	3.03	1.71	3
E	26	3.12	1.53	4

argument; " Since the set has one more vector, it is linearly dependent." Even though, adding one more vector to an independent set may not, necessarily, result in a linearly dependent set. For example, one may consider V = Span {(1,0,0), (0,1,0)} and adding the vector, (0,0,1), in to the set {(1,0,0), (0,1,0)}, will not make the set linearly

Table 20b

Results on vector space and proof eliminating the effect of controlled variables

Controlled Variables	$\underline{F}$	p-value	df
Ability	5.75	0.02	(1,48)
Nationality	1.03	0.316	(1,49)
Attendance	2.56	0.11	(1,49)
Gender	1.18	0.28	(1,49)

dependent. Those who used the argument of number of vectors in the set did not state the fact that the vector was chosen from the span of the set therefore the resulting set was linearly dependent. Furthermore, eight students in the traditional group and four students in the experimental group attempted to apply the formal definition of linearly independent sets, and one student in the traditional group left out the question.

CHAPTER V

RESULTS AND CONCLUSIONS

The purpose of the study was to compare two ways of teaching first year linear algebra to students.

Conceptual Comparison: Hypothesis 1

Hypothesis 1 stated that there are no statistically significant differences between the control and experimental groups on the conceptual test scores. To investigate the hypothesis, four sub hypotheses were studied.

Sub hypothesis 1a stated that there are no statistically significant differences between the control and experimental groups on scores for the question addressing whether students can recognize a 2x2 matrix as an object of a given set, and be able to write a related proof.

Students in the experimental group did significantly better on the conceptual part of the question (part b) where there was no procedure given. The interviews also indicated that the students in the experimental group were better at recognizing objects of given subsets of vector spaces, which may be the reason for the significant result favoring the experimental group.

The first part (part a) of the question could be done through a procedure given in both groups. The majority in both groups used the procedure to write a proof. The traditional group did slightly better at this question, but, the difference was not significant. The interviews also revealed that students could produce proofs for these types of questions without conceptual understanding of the subspace concept.

Sub hypothesis 1b. There are no statistically significant differences between the control and the experimental groups on scores for the question addressing students' concept images of linear independence and spanning set.

Even though this question was given to test students' concept images of linear independence and spanning set, the interviews indicated that students' responses did not seem to reveal their understanding of the concepts. The interviews showed that it is highly likely for students to produce formal or informal definitions of the concepts through memorization, and they may still not have conceptual understanding.

The two groups seem to answer this question equally well. The results showed no significant difference on students' scores.

Sub hypothesis 1c. There are no statistically significant differences between the control and experimental groups on scores for questions connecting a linearly independent set and a spanning set.

The problems on dimension and basis were aimed at testing students' ability to connect their knowledge of linear independence and spanning set.

Even though for this hypothesis, problems were given to test students' conceptual understanding of the concepts, the investigator observed that the majority of students in both groups used a row reduction approach (a procedure) to answer the problems. These questions also required students to interpret the outcomes of the procedures correctly. Students could interpret the results by recalling similar problems done in class, and still have no conceptual understanding of the concepts. Students in both groups seemed to have done equally well on these problems. The difference between students' scores was not significant. Hence, the hypothesis is retained. There seems to be no significant difference between students' ability to connect the two concepts.

Sub hypothesis 1d. There are no statistically significant differences between the control and experimental groups on question connecting span of a set and writing a proof showing a set is linearly dependent.

The experimental group was slightly better on this question. However, the difference was not significant. The visible difference between responses of the two groups was that many of the students in the traditional group seemed to answer this question by incorrectly recalling theorems covered in class. This also seemed to indicate that some of their recollection was through memorization. On the other hand, students in the experimental group seemed to answer the question based on their understanding, not through memorization

Computational (Procedural) Comparison: Hypothesis 2.

Hypothesis 2 stated that there are no statistically significant differences between the traditional and the experimental groups on scores for computational (procedural) questions.

Computational questions were mostly those that required use of procedures; they required students to be able to carry out procedures correctly. On these questions with the exception of question 2i, the traditional group was slightly better. This can be attributed to the fact that the students in the traditional group, contrary to those in the experimental group, spent more time on carrying out procedures. However, there was no significant difference on students' scores between the two groups. On question 2i, on the other hand, the experimental group was slightly better. No significant difference was found either.

Application of Concepts: Hypothesis 3.

Hypothesis 3 stated that there are no statistically significant differences between the control and experimental groups on scores for questions connecting linear transformations and spanning sets.

The experimental group did significantly better on part b of this question. This question was also a conceptual type question; there was no procedure given in class for this type of questions. It required conceptual understanding of the kernel concept, and required for students to be able to correctly recognize the space where the kernel of the transformation is.

On the scores for part a of the question, students in both groups seemed to do equally well. There was no significant difference between the two groups. Most students in both groups answered the question by using a procedure given in class. Over all, the results indicate that the experimental group did better on the conceptual questions for which no procedure was given in class. On questions that required procedural knowledge, students in both groups did equally well. There were also questions that required both procedural and conceptual knowledge. There was no significant difference between the two groups on these questions either.

Interviews indicated that better research questions should be used to understand students' understanding of basic linear algebra concepts. The questions used in this study may not be revealing students' conceptual understanding. For instance, students may answer the question 2 correctly by carrying out the related procedures, but they may still not have good conceptual understanding. This seemed to appear on responses through memorization. Interviews and students' responses on the post questions indicated that memorization seemed to happen mostly among traditional students. Questions like the last question on the post questionnaire, if given on interviews, might help better understand students' perception of the basic concepts.

Non-Hypothesized Visual-Oriented Question

A non-hypothesized visual question was given on the post questionnaire. It was aimed at testing students' graphical understanding of linear independence. The investigator found that the students in the experimental group did much better on this question than those in the traditional group. Comparing students' performances on the post question tested students' concept image, and on the non-hypothesized visual question, the investigator found that the experimental group, contrary to the traditional group, did equally well on both questions. Even though the traditional group did slightly better on the related post questions than the experimental group, they did poorly on the visual question. The interviews also supported these findings; those who learned the formal definitions symbolically seemed to misinterpret definitions, and also seemed to insist on using their misinterpretations (Edwards, 1997).

Discussion

Previous studies suggested that linear algebra students experience difficulties understanding abstract definitions of basic linear algebra concepts (Hillel and Sierpienska, 1994; Arshavsky 1999). The present study, by including a new learning style, went one step ahead and investigated students' understanding on questions similar to those used in previous studies. *Mathematica* was implemented in one of the sections compared in this study. Students' responses in each group were compared to test the effect of the implementation on students' understanding of basic linear algebra concepts.

The investigator found that the implementation of the technology helped ease some of the learning difficulties. It seemed to help students conceptually understand the abstract definitions better. The investigator also found that students in the experimental group made fewer definition-related errors than those in the traditional group. These students in the experimental group seemed to make better judgments based on abstract definitions whereas the students in the traditional group seemed to repeat what was memorized. These students in the traditional group also seem to insist on using the results of theorems, and mostly recall them incorrectly.

From the interviews and the analysis of the post questions, the investigator found that even though students in the experimental group had better conceptual understanding of the basic concepts and definitions, they were not as good at procedural knowledge. However, they, compared to the students in the traditional group, did equally well on the procedural questions too. Some students in the experimental group expressed that they would like to have little more time on learning procedures. They indicated that they had to spend little more time to learn the procedures by themselves, which, they stated, was frustrating at times.

Use of correct terms among the experimental group seemed to be another issue that should be mentioned here. These students seemed to come up with their own terminology, and use them correctly. To the investigator, this result is not surprising. Since these students were mostly exposed to *Mathematica* notebook demonstrations of the basic definitions, it was expected that the students would adopt terms they saw happening in these demonstrations. These terms seemed to stay with them longer than the book notations that were introduced afterward. Comparing both groups, letting students make their own terminology, as long as it is done correctly, seemed to be more helpful on understanding the concepts involved than asking students learn the terms as the related definitions are introduced.

The experimental group also indicated that the textbook used in this class did not go parallel with the *Mathematica* notebooks. If a right book, they stated, were used, the *Mathematica* activities would be more helpful on learning the basic concepts. Both groups, however, expressed that homework assignments were harder than the examples given in class. The investigator thinks that this should not be considered as an effect of the implementation. Responses to the question on whether there should be a lab section for *Mathematica* activities were mixed. Half the class stated that activities should be covered during class time as the concepts are introduced, and the other half stated that having activities during class time was destructive so they should be covered in a laboratory environment where there is no lecturing involved.

Implementation of *Mathematica* (it should be noted that two different instructors taught the classes. Thus, this fact may have also had influence on students' motivation) seemed to have positive effect on students' motivation. More students in the experimental group indicated that they enjoyed the class than the number of traditional students with the same opinion. The experimental group thought that *Mathematica* activities were more helpful than lectures. The investigator does not find the result surprising since most of the learning was done through *Mathematica* activities, and lecturing was done at a minimum level.

The interviews indicated that *Mathematica* activities may have long-term effect on remembering the basic concepts. The experimental group indicated that they would remember the basic definitions in long-term (The investigator feels that this should be further investigated) whereas the traditional group could not remember the definitions during the interviews even though they had an exam the next day based on these definitions. They indicated that they would, a night before the exam, sit down and memorize them. It should be noted here that these students were mostly "B" students. Here is an outline of some of the indications the investigator observed.

- The notations and symbols do not seem to be affecting students' learning of the basic concepts, assuming the concepts are learned first, not their abstract definitions.
- Students in the group with technology implementation still seem to need more in-class time on learning procedures.
- Over all, technology seems to have positive effect on learning concepts, but not procedures.

Recommendations

The results of this study lead to five main recommendations on teaching linear algebra with technology:

- Enough time should be spent on covering procedures.

- Better computer systems should be used to control students' activities on their computers during lectures.
- One should use laboratory settings that meet basic learning requirements such as allowing eye contact with students, and having less distance between the instructor and the student sitting at the back of the classroom.
- Computers with enough memory to handle visual-based activities should be used.
- Enough emphasis should be given to students' previous misconception of the *vector* concept.

Future Research Questions

While this study addressed several issues in its area of concern, many issues remain to be addressed. These include:

- There is a need for further study with a change of the textbook to one that goes hand in hand with the implementation.
- Some of the questions used in the study did not seem to reflect students' conceptual understanding, hence there is a need to repeat the study with questions better reflecting students' conceptual understanding.
- There is a need to investigate the long-term effect of the implementation. The present study addressed only comparatively short-term effects.

Threats to Validity

There are several issues that can be threats to validity of the research. These threats can be listed as history, maturation, selection, experimental mortality, diffusion of treatments, compensatory equalization of treatments, compensatory rivalry by subjects and resentful demoralization of subjects.

The first threat is **history**. This occurs when an event not part of the treatment takes place between the pretest and the posttest. Both groups experienced the same external events. Internally, the experimental group experienced two changes of classrooms and computer malfunctions throughout the semester. No extraordinary event occurred in the traditional group. The internal events occurred in the experimental group may have affected the results of the study. Hence this is probably one of the serious threats occurring in this study.

The other serious threat is **selection.** This threat occurs when different types of people enter each group. The two groups had unequal number of American and non-American students. Compared to the traditional group, the experimental group had more international and female students. The traditional group on the other hand had more American students. Additionally, the two groups were not equal on number of students who liked algebra, and students with experience in use of technologies. The experimental group had more students who stated that algebra is their favorite subject, and also had more students with less experience with use of technologies. Even though, students were informed of the nature of the experimental group, and told, at the start of the semester, that they could change sections without penalty, the investigator did not observe any transfer between the sections. It should be noted that the two groups did not differ significantly on their pretest scores.

Experimental mortality occurs when subjects do not complete the study.

This threat is plausible in this study because subjects were able to drop the course during the first two-thirds of the study. Students in the experimental group dropped the course during the first week of the semester. In the traditional group, students dropped or changed from credit to audit later in the semester, and those who dropped the course were failing the course at the time of the drop.

Another threat to validity is **diffusion of treatments**. This threat occurs when subjects in two groups communicate with each other. The investigator did not find any indication that there was communication between the two groups.

Compensatory rivalry occurs when subjects in the traditional group become motivated to diminish the expected differences between the experimental and the traditional groups. There was no evidence of this threat found in this study. The traditional group was not told that the experimental treatment was better.

Compensatory equalization of treatments may have occurred because the post questions were discussed with the instructors. It is possible that both instructors chose problems of similar types to discuss in class. The investigator observed that the instructor in the traditional group went out of his/her way to give some graphical demonstrations of concepts to compensate the traditional group for not receiving the treatment. However, these demos were restricted to the blackboard drawings. The magnitude of effect of this compensation can not be determined. This threat was a concern of the study.

The last threat of the list is **resentful demoralization of subjects.** This threat occurs when subjects in the traditional group intentionally perform poorly. No evidence of this threat appeared to the investigator.

In short, there are some threats to the internal validity of this study, specifically, history, selection, mortality, and compensatory equalization of treatments. Selection appears to be the most serious one, and the one most difficult to control in this type of study. One solution to reduce these threats would be to repeat the study in a true experimental format, specifically with random assignment of subjects. However, in this setting, random assignment of subjects would be difficult, while maintaining the other characteristics of the study.

The threats of history and compensatory equalization of treatments can be reduced easily. Not repeating computer-related problems, the threat of history can be reduced. One way to deal with the threat of compensatory equalization of treatments is to make sure the instructors see post questions while given on the test, not before. Assigning one instructor for both groups might be an alternative to reduce the threat. However, it is possible that even this one instructor may, unintentionally, feel for the students in the traditional group, and try to compensate for the treatment.

Conclusion

The present study found evidence that the experimental group performed significantly better than the traditional group in tasks involving only conceptual knowledge. The most noticeable differences in understanding were found in applying basic vector space concepts into linear transformations, also found in writing bases for subspaces by recognizing objects of the subspace as vectors. No significant evidence was found to support the belief that the experimental group performed less well than the traditional group in questions that required procedural knowledge or in questions that required both procedural and conceptual knowledge.

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APPENDIX A

Interviews

The interviews started with explaining students the nature and structure of the interview. The following statement was repeated in each interview:

" Please read questions, and answer each of them to the best of your knowledge. As you answer questions, try to think **aloud**. I should note that this is not to evaluate your performance in your linear algebra class. Your name will NOT be used in the study. I won't be interfering but if I do, it is because I might be trying to understand your thought process. It does not mean your response is incorrect....."

Two selected interviews will be transcribed, and discussed below. The first interview was with a student called "A", from the traditional group, and the second interview with a student called "B", from the experimental group. Students' grades (based on 5 points) on the post questions are summarized on the table 21.

Table 21 Students' grades on the post questions.

	Q1		Q2		Q3	Q4		Q5		
	a	b	a	b		c	d	i	ii	iv
Student A	5	1	5	4	5	2	3	5	5	5
Student B	5	4	5	4	3	4	3	5	5	5

Interview 1

Student A

The interview was conducted on November 17, 1999 at 6 pm. Here, the letter "A" refers to the student and the letter "I" refers to the person conducted the interview.

Question 1:

I: If you think aloud, that will give me a reason how you are thinking, how you are perceiving things..........I won't be judging you with your answer.....Let's go ahead, and start with the first question.

A: Okay. To get things simple, I'll just stay on R3. I have got vector v1 and a vector v2. V sub one represented by v sub a...........I'll use the parallelogram rule to construct the result........Move this vector over, connect to the origin, to the new end point. This vector right here would be v3.

I: what if we use just a single vector, what would be the linear combination of a single vector?

A: It would be the equivalent of okay (Pause) For any combination of v1 For example, It would be (pause). If I have two different coefficients, i would be able to say that ahh, (pause) It would be two. I guess I would just add the coefficients v3=(c1+c2)v1....

I: Picture-wise?

A: Geometrically, that would be (pause), I guess I would just first multiply v1 times first coefficient, draw them both and then take one, move it to the end of the first so I'll assume that the second coefficient results in slightly shorter vector. So I take that vector move it over here so it will look like that so it will look like the resulting vector. (pause) Okay that's not right Umm (Long pause).

I: What was not right to you?

A: Well, (pause) I guess if I assume that the coefficient is greater than one, place it at the tail of the other so it should point in the same direction but it will be twice as long whatever the coefficients to make it. So I like to draw over this, but vector would look like that I think.

Question 2:

I: Okay, well I guess we can go to the next one. This is similar to the first one.....Here I am looking for an illustration of a linearly independent set.

A: Okay (pause) I am not sure what is meant by set. How to represent that Geometrically. I'll just take the element in the set, I make vectors out them, and say that vectors are linearly independent. I don't know how to represent a set geometrically.

Here, this person seemed to be more algebra-oriented. He/she is comfortable with the idea of writing a linearly independent set algebraically, but does not have an idea how these vectors can be represented geometrically.

I: Okay, let's go ahead and do that.

A: If I'll, i'll take like 2x2 matrix say v sub one, v sub 2 v sub 3 v sub four (writing the vectors v1, v2, v3, v3 at the same time), i say that that set is linearly independent...... Then if I take make a vector out of this, out of this and a vector out of that, there is no coefficient by which I can multiply this vector to get that vector. It is how I represent.......

He is still giving an algebraic description of a linearly independent set.

I: Can you draw the picture for that?

A: Ahh

I: Say you have a set which has. Let me just repeat what I understood ..... What are the vectors? Can you repeat that can you repeat that for me one more time?

A: This is a vector (pointing the algebraic description he wrote down) this is a vector. I should have used different notation I guess but v sub c and v sub l equal to v sub 1 and v sub 2. So when I draw v sub k (pause) v sub 1, v sub 2 then v sub l similarly. If they are linearly independent I will get some vector that is not, this is in R^2, that is not, that there is some angle between the two vectors. I (pause) greater than zero, there will be, It won't be like this. It won't be just shorter than this. There will be some, I'll say this v sub k, v sub l, the angle between them, I'll say, did not equal zero so that way if I multiply v sub k by some coefficient there will be no way that I will be able to produce v sub l I think (pause).

I: Did you say there is no way you can produce v sub l from v sub k? ...

A: Right just by multiplying by a coefficient.

I: What if you have a set of three or four vectors?

A: Okay, I am (pause) I come to apply the same thing. There is no vector in the set that can be produced by adding any of the other two vectors in the set but okay that can be produced by linear combination of any other vectors in the set so I would, if there is like n vectors in the set. I would draw whole bunch of them none of them would be on the same, have the same angle between themselves and x-axis like that so look like that, there will be no vectors that are just shorter versions of each other.

Here, his concept image seems to be revealed. Even though he can state the formal definition of the concept, his concept image does not agree with the formal definition. His understanding of a set of linearly independent vectors is that the vectors are linearly independent as long as angles between the vectors and the x-axis *are not equal. He, without noticing the inconsistency between the two, used his both algebraic and mental image of the concept.*

Even though, he is writing both solutions; one based on abstract definition and the other based on his mental image of the concept on the same paper, he still can not see the contradiction between the two.

I: So if I consider those three (pointing out the drawing of three vectors originating from the same point in R2.) , what do you think about them?......

A: (pause) The set is linearly independent, that's what i would say.

He still thinks as long as the angles between the vectors are not zero, they are linearly independent.

I: The reason?

A: Because I stated right in there (pointing out the angle argument) I can't produce v sub j by either adding those two or by a linear combination of these two adding a coefficient multiplied by either one of those I am calling it linearly independent....

I: Okay, can you give me a specific example?

......

A: Okay first two are fairly easy, I know that these have to be linearly independent, there is a coefficient of which I can multiply one comma zero to get zero comma one (he is also writing the equation) and the next vector I call point five comma point five, I say because (pause) point five comma point seventy five, All right, I can do that to (pause) that the third vector . I wish to construct them such a way that I can't take c sub one times one comma zero plus c sub two times zero comma one 9 (He is writing c1(1,0)+c2(0,1)). (pause).

He does not seem to have complete understanding of the concept. He is not using the dimension argument to see that these vectors are in a two dimensional space, hence any three vectors are always linearly dependent. He is trying to apply the formal definition. He seems to have memorized the definition, but not internalized it.

I: Thinking? ...

A: That I would I can not represent like this. There are no, there exists no two coefficients

I: so you believe that there is one. You are searching for that right?

A: Yes, but I chose my first two vectors poorly because they are basis for R two, so I am going to change one of these, so it is not such a hard problem. I call the first one one and point five for example (pause) so (pause) and,...........well It probably is not that hard but of the top of my head I can't think of three.

He still thinks that there exists a third vector that makes a linearly independent set with his two vectors.

I: Okay (pause) let see (pause) What do you call these variables?.......

A: I call them x and y, and I will say that csub one one comma point five plus c sub two zero comma one (writing the equation), and I will distribute the coefficients to say that I have two vectors c sub one and point five c sub one and zero c sub one and then I will try and find some x and y subset, I will have to say that x, set x such that x is not equal to c sub one. That way there will be no way I can add these two to produce another vector that has a coordinate that equals c sub one. So, If I use that stipulation, and I can so I want to actually put in a real number anything other than one. So two.

I: How about y?

(long pause)

A: Okay, there appears to be a problem with that because I can multiply zero times this two times that (pause) Okay.

He seems to be dependent on a procedure (solving the corresponding equation for the coefficients) to answer the questions. He seems to be using the formal definition of the concept.

I: ......so, there is a problem you said. Because?

A: Because just stipulating that x is not equal to c sub one is not good enough because I can find some scalar by which I can multiply c sub one to produce x.

.....

A: Easy way to do this is to say that angles here, all the angles are different so I would cosine theta equals the coordinate, okay I am just going to call this like v sub one v sub two...say that sin theta equals v sub one over that hypotenuse. I'll say that tan of theta equals opposite over adjacent, so v sub one over v sub 2 and therefore theta equals arc tangent of v sub one over v sub two and then all I have to do is in the set, let's construct

A: set such that arc tangent of v one divided v two produces some data that is not already been produced, so get three different data. And,

I: And that will create the set?

A: I think so.

Question 3:

I: Okay, Let's see. Good. Let's look at this one now. Is the question clear?

A:I think so. These are vectors.

I: v one up to v n, vectors that are coming from this set. And, we say solution to the equation is an algebraic indication of whether the set is linearly independent or not.

A: Okay Umm taking a simple case of this, say that n equals 2 which means I have set with two vectors in it. If (pause) if I can construct a linear combination of this two vectors, so c one times v one plus c two times v two, I can show that equals zero then I know that c one times v one equals minus one times c two times v two

I: Are you assuming, what are we assuming here? ......

A: I am trying to explain the significance of showing that linear combinations of these vectors equals zero. What that means ? since minus one times c two is just another scalar and I know that c one times v one equals c three times v two....both scalars I can divide through and produce some thing that says c four equals c one over c three.., c four times v one equals v two.

I: What is c four?

A: c four equals c one over c three, so I am just dividing through. Equals v two. And that tells me that I can take some scalar divided by v one produce v two so the set is not linearly independent. So Umm..

I: Do we know what the value of c three is?

A: No, It is a real number I assume It is not zero.

I: Why do you assume it is not zero?

A: Because the other vector would not be a vector. If Ohh....if this linear combination I got c three multiplying minus one times c two that means c two must be zero. If I say that c three equals zero, c two must have equal zero then I would have just c one times v one equals zero and therefore either c one or v one must be equal to zero............

I: If v one was a zero vector in the first place, what would you say about the set? ....

A: Well I, I am not sure the specifics of definition, off the top of my head I would say It is linearly dependent, because I can take v two and multiply a scalar zero by it to produce that definition might get around there some how. I don't remember specifically definition (pause) But since I was able to earlier that if c two did not equal zero, the set is linearly (pause) dependent.

He is still using procedural knowledge to answer the question. His understanding the concept, linearly independent vectors, seems to be restricted to the procedure, solving the corresponding equation.

I: If c two equals zero then the set is linearly independent?

A: Umm earlier I said if c two does not equal zero, I can use this reason here to say that the set is linearly independent. I can multiply some scalar by v one Umm (pause) so okay I also say that in the case of c two equals zero if c one is not equal to zero then v one must equal zero so therefore the set is linearly dependent as well so i guess I came to say if I have a condition, in this case if I can have some linear combination in which one of the scalars either c one or c two is not equal to zero, the set is linearly dependent.

I: Okay, if both of them are zero and v one is zero? Is that possible?

A: Yes that's possible. Because the second one is not the zero vector but I am multiplying by the zero coefficient, so I can produce zero vector. But I already know that we stipulated that v one was zero so I already argued that's a linearly dependent set...

I: Okay, can you try to put these two on a picture and see if you can show me the connection between this solution and the positions of those vectors?

A: Okay, Okay first of all I am going to assume that v sub one and v sub two, well I have to, v sub one and v sub two are not the zero vector. So, to make things easy I'll just take them in the first quadrant v sub one, v sub two (he/she is drawing) linear combination of the two is equal to zero says that these two vectors have to lye? On the same line, so I can take some coefficients stretch one and shrink the other such that It looks like that by sum of two, I get the zero vector, so...

I: What if this set is linearly independent? What is the connection between that equation and the positions of those two vectors?

A: Okay, If it is linearly independent then I can't add the two to produce the zero vector. That's what I think when I see this. I can't add this one plus this one to produce zero comma zero so, this does not use any sort of angular geometry.

I: What if we have three vectors in the set. How would you explain the connection between the equation and the set that has three vectors? ......

A: Okay (pause) The answer that I produced says that there is no coefficient that I can think multiply one of the vector by to get the other vector. Umm moving up to three vectors? The way I can think of is just to sum the vectors in my head from

from physics such that I don't get back to the origin. I don't have a loop. And (mumbled a word) I can stretch them or shrink them as much as I want I switch their directions but I can't change their angles from the origin. So, If I have these two vectors here. I know that even if I make this one, these two vectors right here v sub one and v sub two, even if I make this one exact the same length as v sub one, when I sum or if I switch it, there is no way I can get back to the origin. So the resulting vector is not going to be the zero vector. So If I had another, a third vector v sub three, I would do the same thing Umm, Use what I did in the earlier problem, know that the angles are different but it is the same idea. I can't stretch this or shrink this and switch its direction such that I can add it to the other vectors to get back to the origin. If I couldn't do that then I would say the set is linearly independent.

Here, he/she is still insisting on using his knowledge of the formal definition which is based on a procedure. As you read above, even though he/she has a drawing in front of him, he prefers to carry out the procedure (summing the vectors algebraically) first, and get back to the drawing. This implies that he is dependent on his memory of the formal definiti; He will be able to conclude correctly whether the set is linearly independent or not, and have no conceptual understanding.

Question 4:

I: All right, Let's look at this one (problem is written on a paper)

A: Okay when I see span, I think the space of all points in R3 that can be reached by linear combinations these two vectors. So, (pause) to determine the answer to this question I want to say, excuse me

I: Is that two times c sub two?

A: (long pause) I say that that's impossible because when I multiply this coefficients through I get c sub one plus c sub two, when add this I get c sub one comma c sub two, c sub two. And since three did not equal one, I have to, that would imply c sub two does not equal c sub two and that's an absurdity so by contradiction this can't exist in the span of that.

He/she seems to be using a procedure to answer the question. He could have seen directly that no linear combination of the two vectors would result in the vector (1,2,3) since span of the two vectors is a space of set of vectors whose second and third components are the same, and the vector (1,2,3) is not of that kind.

I: What kind of an object would you get out of that span? ...

A: This one specifically?

I: Ya, span of that set (pointing the set)?

A: Umm (pause) x can range any where but Umm z and y have to equal to each other, y equals that z has to equal that so this point right here I would imagine this is plane.............

Question 5:

I: This one, just describe a vector in the range of this linear transformation for me (pause)?

A: Okay, first of all I am attempting to understand the definition. Umm.

I: Definition of what?

A: Umm T, What T entails.

I: T is a linear transformation defined in terms of the matrix.

A: Okay, T is a linear transformation such that I take some vector, a four dimension vector and multiply the matrix A times the vector. So it looks like, I write v like this (pointing how he wrote the vector, a 4 by 1 vector) and the result is (pause)..........resulting vector is going to be three dimensional vector...

I: Okay, If rows zeros, i would just have to multiply one two three seven sand zero, the result would be v sub one times one plus two times v, and that would be the first output so v sub j, second would be this row here times that, so that I can see I am going to perform the number of rows that number of multiplication, so I am only getting one answer so there is going to be three by one matrix. So that also can be read as three dimensional vector....

I: So the range is going to be the set of all vectors Umm (pause) where is the range?

A: If I have domain, I am taking each element in the domain through some transformation the set of all the answers that I could produce. This transformation here produces a co-domain I guess. So the range is set of all possible answers, co-domain is the set of all answers. So, of the top of my head I don't know what the co-domain of this is, so I am going to assume.

I: How would you Umm Just tell me the procedure of getting range of a linear transformation, we don't have to come up with a specific description.

A: Well I know that I am going to be producing a matrix that looks like this and or I can can like that as, and, the combination of those, I guess that is actually....

I: Combination of ?

A: x can be any real number, y can be any real number, z can be any real number, and assuming this linear transformation can produce any x any y, any z in any combination then the range is R3 and infinite...

He/she seems to be able to write an algebraic description by using a related procedure (multiplying the matrix by a general four by one matrix that represents any vector in the domain), but interprets the description incorrectly. This seems to imply that this person does not seem to have complete understanding of the concepts, linear transformations and vector spaces. If he/she had a complete understanding of the concept vector space, he/she would be able to interpret his/her description of the range of the linear transformation correctly. His/her knowledge of the concepts seems to be restricted to the procedural aspects.

I: Let me give you a different matrix (pause). Say we have this one and we are defining a linear transformation with respect to this matrix.

A: Okay

I: What would you say about the range of this transformation?

A: I would say v one ohh plus two, v two (pause)

I: What are you doing now?

A: I am attempting, I think I can find the range for this. I am showing what would result by, by <u>multiplying these.</u> Umm (pause)(performing the matrix multiplication)

I: Are you adding those? So you got these as a result of the multiplication. What are you thinking now?

A: Well I am attempting to take get rid off like three three for example so all is going to be zero.

I: So once you get rid off v three?

A: Then I, my range is restricted so I am trying to, better than being R4 something less than that.

I: Where is range of this is sitting in?......

A: Okay, the vector for this one (second) e defined by this right here so it will be a three by one vector............Assuming that this vector right here (pause) can be composed of any real number v one, v two, v three, if if those can be any real number, and the resulting vector these last two can be any number produced by that plus two times that so I assume that that can be any real number as well. So I say that the range is R3.

I: Can you tell me a basis for that? By just looking at the matrix.

A: A basis for?

I: For the range.

A: Now, I can cheat because I know the standard basis for R3.

I: Can you come up with a different one? (long pause) or can you write a spanning set for the range?

A: I can say (2,0,0) (Long Pause), I can say that set (written on his paper)

I: How did you come up with that set?

A: I believe that I can produce any vector in R3 by linear combination of these (pointing the vectors written on his page) by some coefficient times that some coefficient times that some coefficient times that.

A: ....I decided that the range of the linear transformation is R3, and I decided that I know one of the spanning set is R3. Therefore this is a basis for it........

Question 6:

I: We have this set S defined as the following, and I would like to know how you can describe a vector in that set.

A: Okay, I assume this means continuous functions between 0 and 1.

I: Yes

A: Okay Umm (pause) so (pause) it is any function between that is defined between those these two points I assume such that f(1) equals zero.

I: Any function?

A: Any continuous function. So that f(1)=0. So all functions that are continuous from the the right of this and continuous up to 1 and are zero at one, produce zero at one.

I: Can you give me a specific vector which is in that set?

A: zero comma zero

I: zero comma zero?......

A: f(x) equals zero times x that produces zero function.

I: other than that?

A: f(x) equals x plus one (pause) Umm minus one.

I: Okay let's look at the second part. Is the set a subspace of the vector space?..

A: Okay I confess that I am not sure how to think of a vector in S Umm (pause) because S is a set of functions Umm

I: You don't consider functions as vectors?

A: (puase) No, My thinking is limited. I know that .....

I: what is the problem there? Why you don't?

A: In physics I was given a very restricted definition of vector, (referring to his instructor) has been discussing a very general abstract definition of a vector so...

Here, his/her earlier knowledge of vectors seems to be conflicting with the "new" definition of the same concept. And this seems to be stopping him from learning the most general form of the concept, vector.

I: Which one do you accept usually? ...

A: After reading the linear algebra book tonight (an exam was scheduled for the next day), I will think of (referring to his instructor)'s definition but up till now I think of physics definition what I have got. (reminder: this interview was conducted during the last week of classes).

I: Which is?

A: The vector comes from the Latin word the vectors t carry so I think Umm a set of directions like go at v one unit x direction and then go v two units in the y direction and if you carried a point along those directions and that point will be there and this is I think of as a vector. But a function..

Even though he/she gave a vector that is in the set, he/she rejects to believe that the continuous function he gave is a vector. He thinks that a vector should have a direction.

I: You just wrote me a vector there though when I said describe a vector in this. You don't, even though you are writing it, you don't believe in that? Is that what is going on in your mind?

A: Okay I don't, when I think of a vector I think of I need something with commas like v one comma v 2 whereas a function (pause)………

A: ..Specifically my problem is I think of a vector as look like v one comma v two comma up till v n. When I say, when someone say says ohh this set is composed of functions now describe a vector in that set I say okay there is functions in that set I don't think there is vectors in that set……..

I: Well you gave me this; (writing f(x)=x-1). You said these two are, you were not considering those as vectors?

A: No

I: What are they to you?

A: They are functions…..

I: So you will be more comfortable if that set was given in terms of R^n, n tuples. Right?

A : Okay good thank you for telling me that. If I say that (pause) that this here like more complicated rather than saying just like P_3 (set of all polynomials of degree 2 or less) then I would say a vector is, a represented vector would be (1, x, x square, x cube) This is all continuous functions, so one vector in S, I will say is (pointing (1, x, x square, x cube)).

Question 7:

I: I am more interested in this one actually. This one and one more question. We will be done.

A: (long pause, he is reading the problem) Okay, I assume a non-trivial solution means that of all the coefficients, one of them is not zero. We talked about this

...having a non-trivial solution would be saying that there exists some coefficient that I can multiply for example v three by well okay there is this two coefficients c one and c two by which I can multiply v one and v two respectively to produce v three or something like that then our previous solution means I was able to get a non-zero answer for the coefficients in in that case what I said applies there I can construct a linear combination of v one and v two to produce v three (pause)

Question 8:

I: Okay this one is that, it is partly in linear algebra...I do want to know what your opinion on using computer activities in linear algebra classes.

A: Okay, I took (Referring a course he took in the mathematics department)I took this instructors calculus IV, The instructor used *Mathematica*. I found *Mathematica* to be great as far as visualizing what partial differentials are I was able to understand them immediately the instructor just throw up a *Mathematica* print out, and I would be able to okay I understand this. Umm, linear Algebra, me in, what I found so far, all of linear algebra can be understood in geometric sense so because *Mathematica* can produce umm graph extra, I think it would be helpful ....

I : How are you at proving statements like this one, the one that we just did, Umm (pause, talking about the subset that set of continuous functions that are 1 at x=0)...

A: If you have a few more minutes I like to look at that one I did not.

I: Ohh Ya

A: ....Okay , Umm in general I have had have been really good at proving things. I took discrete mathematics which isn't a very challenging class but half the class was devoted to proofs, so proofs from axioms I found very easy to do. However, this here I have to remember what the definition of a subspace is. I think Umm

I: Is that the problem here, you can not remember what a subspace is?

A: Yes Well, I have an idea of geometric representation of a subspace. It is (pause) in two dimensions. If I have x and y, if my space is that a subspace like a subset looks like that (pointing the drawing on his paper) something like that, three dimensions of this big blub or lesser blub.

He drew a square in x and y coordinate system as a subspace of R^2.

Interview 2

Student B

The interview was conducted on November 29, 1999 at 6 pm. Here the letter "B" refers to the student and the letter "I" refers to the person conducted the interview.

Question 1:

B: (Reading the question) Linear combination of two given vector v one, v two steaming from the same point. Linear combination of two vectors. Okay, so first I am going to draw little coordinates, and then they are coming from the same point. I will make it the origin for simplicity. I would say I am going to draw a vector and then there are two vectors steaming from the same point..................

I: what is the definition to you, definition of a linear combination?

B: Umm, to me, it is if I am thinking correctly, which I have not looked at this very much lately, It is just, you got an equation of a line and then you are manipulating it by adding or by multiplying something to it and then a linear combination would then like adding those two together like that, like equations you know, you got your equations and your variables and then you could do it.......

He/she seems to be recalling the definition of the concept through the Mathematica demonstrations he saw in class.

I: Say it has a w even though we don't have it there, and you are writing w as linear combinations of those two vectors.

B: Ohh Okay, so then, you could do it as something times one of the vectors, the plus something ohh I make this question mark times the other vector u would equal w. Okay, now I am back in the thought process. And, it could be plus or minus and It wouldn't, you know it could just be one or because you could just add them together to get w.

I: Okay, how would you represent that on a picture?

B: Umm

I: Say we have for those question marks say we have c one, we don't know what the value is. Let's represent that question mark as c one, and this one as c two.

B: Okay, well Umm (pause) I mean you just write them in there. But, Umm (Long Pause) and then like your w would be addition of them together, but that's like vector addition that's not necessarily, well I guess that's what we are doing here.

I: What is the definition to you , definition of linear combination?

B: Umm, to me It is, It is If I am thinking correctly, which I haven't look at this very much lately, It is just, you got an equation of a line and then you are manipulating it by adding or by multiplying something to it and then a linear combination would be then like adding those two together like that, like equations you know, you got your equations and your variables and then you could do it.........

He seems to be very nervous. The interviewer is trying to calm him.

I: Like when we have a basis or spanning set, how do we define those?

B: Umm the set where any any like if you have any vector in a whole plane or whatever, it could be represented by by that that set by linear combinations of the vectors in that.

I: Okay, so you are saying linear combinations of the vectors, what do you mean by that?

B: Umm I am meaning like like the four times the adding together. I guess I am using that right....

I: Say we have v and u, you have already chosen v and u but you you wanted to change those to to c one and v. Let's start from just v and u. And can you represent this right there on a picture, how would you draw that?

B: Umm (pause) ya, it is tending to me to draw this way but that's kind of it is not going from the same point necessarily you know Umm

I: Which one is going from the same point ?

B: By drawing it this way and then doing that together that that is not really

I: How would you read that, this equation ?

B: I guess you would say you got this vector and then you will be adding that other one to it, I guess it would be end to end if you'r thinking of it that way as you already have something and then you got four... So I would still I think draw it such that if a and v together and then this resulting vector would be w......

Even though he is able to draw a demo for a linear combination, the statement of the question; the part that says vectors steaming from the same point, seems to confuse him. He seems to be thinking that according to the question, he is supposed to draw all three vectors steaming from the same point that he is not allowed to carry one of the vector to the end of the other vector. He seems to be fighting between what he knows and what he thinks the question is asking.

I: So what would be, w would be, would this w the same as this?

B: What's that? Ohh, this is a, this is the v, I am just transporting it over here, and this w would be the same as as that w.

I: So what happened to c one and c two in this case?

B: Ohh Okay, Well they would still be there. That's still

I: Ohh you are still considering those? What if I have a single vector v and umm say I want to have, ya go ahead

B: If Umm you got you x, y and then you got your vector and then that vector has an x part and a y part, and so then if you wanted to combine these two vectors you have this other vector. You would combine its x parts and its y parts as well. So, that's x one, y one, x two, y two then the resulting vector of those w, x one plus x two Umm y one plus y two, so I guess you will be breaking it down into the x factors and the y factors and then adding them together.....

I: ....Umm what I was asking, well I am after is, let's say we have the vector v, how can you describe or show me c one times v?

B: In the same direction but will have more more magnitude.

I: what if c one is less than 1?

B: If c one is ahh, if it is negative then it will be pointing in the opposite direction.

I: What if it is not negative it is in between zero and 1?

B: then it would just be a smaller one.

I: Say we have now w, and we want to have c two w, how would you draw that?

B: The same, the same fashion I think, except it will be some thing little different.

I: Okay then how would you combine those two (c1v, c2w are pointed)?....

I: You said this is c one v (pointing its graph) and this is c two w, what would be the sum of those two?

B: Umm it will be see I feel like I am saying the same thing It would be this where this is c one (pointing his earlier drawing for v + w).....

I: ....Would it be possible to have this w and, and v drawn, and take a look at c one v plus c two w, starting from v and w.

It seems that the question may have been misunderstood by the student. Hence the interviewer restated the question.

B: Ya, it would, you just be taking out the part with multiplying by a scalar quantity.

I: Okay, so do you feel more comfortable having this setting then this one (pointing the drawing of c1v+c2w where c1v and c2w are drawn)?......which one is easier for you to consider, and why?

B: Umm I mean initially looking at you would think this is easier to consider because there is, you are deleting that a factor, you are not thinking about other factor. So, this will be you know may be easier to look at .....

I: what would be the resulting vector here? How would you draw that?

B: Umm, like I have been...you kind of like imagine, yet another point you know or of origin, and then you kind of see one, again and then the sum will be this new vector because you are starting from the point, you know they are both starting from the same point .....

I: If this, this is the vector w. Is vector w the same as c two w?

B: No

I: How come you are writing it that way then? What would c two w be?

B: It would be either beyond it or .

I: Let's say c two is greater than one.

B: Okay then c two w would be (drawing the vector) something that's bigger.....

I: So what was it? Did you misunderstood the problem? Did you want the third vector, or every vector steaming from the same point?

B: Ya Ya ....

I: Okay so when you say linear combinations of two vectors, you are just considering v one plus v two? Is that what you are considering?.....

B: Umm I was, initially I was I was thinking some vec like some scalar times the vector plus some scalar times the other vector. That's what I was thinking.

I: Okay when you put those here in the picture you tend to put comma v and c two u, you don't put v or u. And then draw the picture from there...you are starting from the , the scalar multiplication of those two vectors, you are not starting from the vectors...

B: Okay, I see what you are saying.

I: Is there a reason for that?

B: Ya because (pause) if you are thinking of it as, if if multiplying them is allowable Umm then you get to do that because if you don't multiply you know if you don't have result of the product or what ever you are going to call it. If you don't have that already then it is not, you are not going to have the same answer because you can't multiply them afterward, I would not think or I find it easier to multiply and, afterward, I guess you could find the resultant may be and then multiply it by some scalar but I think since you can multiply them by different scalars. It would be best to do it that way.

It appears that there was miscommunication between the student and the interviewer.

I: Okay say I have u and v here and I have this (pointing a linear combination of the two vectors, and pointing a drawing of the two vectors u and v steaming from the same point) How can you draw that (the linear combination)?

B: Ya, ya, umm one half will be cut that in half, and the two v will be two times the vector that is there (pause) and then to add them together then you could just, I just have on the end of the other (pause) say you got your one half leave there, and then the resultant is when you add the xs any ys together, and so it all end up, it all end up there (pointing the end of the second vector). That's one half.

He/she seems to know what to do, how to represent linear combination of vectors geometrically, but He/she explained his response after a sequence of questions asked by the interviewer. One explanation for this could be that the question was misleading.

Question 2:

I: Okay how about number two?

B: Give a geometric illustration of a linearly independent set. Okay (Long Pause) See linearly independent means that it can't be (Long Pause) It can't be created by (Pause)

I: Can you give a specific example? Example of a set that is linearly independent.

B: ....Umm (pause) I am thinking, What I am thinking is that it is like, I, In my head like a vector or something that can't be produced by linear combinations of other vectors. But, but I would think that everything that could happen to. That could happen for anything.

I: Anything, what do you mean by anything?

B: Well, under that definition of, if that is what linearly independent is, I would think every thing could or you know nothing would be linearly independent....Well, because like if you had, I know that's not true, because I know I am I am getting that wrong like if you had zero zero zero , zero zero zero could be illustrated by you know zero times, but that would be only the (pause) trivial solution, so may be it is a (pause) set that can only be (pause) can only (pause) It has something to do with the trivial solution.

I: Okay, Let's say we have a set which has two vectors u and v and this set is linearly independent.

B: Okay, so umm I would say that the only point that these two are equal is when they are both being multiplied by zero.

I: Okay (pause) say you have three vectors, and the set is linearly independent; u, v and w.

B: Umm (pause) Then I would I, I (pause) umm I am thinking that it would be the same case but umm it kind of seems like it just one of them like if these could be written as a linear combination you know of everything except zero but zero, but for w it has to be you know zero times zero but but as far as you know u you could have other things multiplied by them and then added together to get u. I think as long, I think there just has to be one of them where the others have to be zero scalars multiplied to them to be added together, to to be in w. I think that's okay......

I: Okay, umm what you have, say 2x here, x and x square plus two, what would you say about the set ? The set that contains those three vectors.

B: I would say it is (pause) I think I would say it is still linearly independent. Because, umm yes this one could be written as a result of this, but you can't have this involve well, I guess you, this one can be written as as sum of you know these others but umm this one, It can't get umm x square so I would say this is still linearly independent. Because, there is one of them that still falls into that problem of only the trivial solution.

His/her recollection of the formal definition seems to incomplete. He/she partially has the right idea, but it is not complete.

I: Can you draw a set of two vectors that are linearly independent in R two? Or in any space, you have two vectors u and v that are linearly independent.

B: Umm, see just two vectors?

I: Yes

B: Then like if you had a vector going this way (Drawing) and then a vector heading out this way, then those two would be linearly independent of each other because they only intersect or have the same solution at the zero point.

I: Okay how about umm three points ohh three vectors that are linearly independent, u, v and w?

B: It would be the same case. The umm, the only point that they are all three equal would be at the origin........

I: Can you draw those three?

....

B: I guess not, now that I think about it, because you can get this resulting vector by multiplying the others by something other than zero to get that so that would mean that what I am saying here is incorrect. Because you might be able to with three of them, to, you know, get two of them to had together to be one of the others so far three........

I: Okay, what if we really want to have three vectors that are steaming from the same point, zero, zero. Is it possible to have three vectors that are linearly independent?

B: (pause) Umm to represent them, three coming from the same point, I don't think so....Because one of them will always be able to be represented by the, the sum of, of you know scalars times the other two.

Even though his/her recollection of the definition was not complete, he was able to correct his/her misunderstanding of the definition by observing his/her three vectors drawn by him/her. After drawing three vectors steaming from the same point, he/her realized that one of the three vectors could be written as sum of the other two (his vectors were chosen from R^2) ......

Question 3:

I: Okay let's take a look at that one.

B: Umm solution to the equation c one two times v one (reading the question) c two v two .... c n v n equals zero is an algebraic indication of whether the set v one v two (pause) is linearly independent,.........Umm if you knew the solution to this, if you knew every c in here and if you knew that one of them could not be zero (pause) and and you would still get zero as an answer and you would know that umm that the set of these is not linear, or it is linearly dependent......Because you would not have trivial solution for all the cs, all c times vs you wouldn't, you wouldn't they wouldn't all be equal to zero and so, you know one of the cs would not be zero and so you could know that you could have a linear combination to create a zero vector.

He/she seems to be using Mathematica demonstrations on showing connection between type of solutions to vector equations, and vectors being linearly independent or dependent. Demonstrations were on solving vector equations through tracing vectors, and counting units back to the zero vector.

......

B: Okay (Pause) Well umm, you had (pause) three vectors, and they are all you know coming, passing through zero (in two dimension) then umm that you know that they definitely have the trivial solution but you could also may be see that umm If this you know vector was multiplied by something that would bring it this way, and the other was multiplied by something that might bring it umm you know this way by a certain amount then you could see that, that this vector could be a result of (pause) the other..........See it looks like ohh, you know if you were just to add these two together but send them in the opposite direction (pause) then you would get opposite of that vector, and then <u>you would get it to be zero.</u> That would say that it is not linearly independent........

I: Okay umm what if we are just considering the vectors that are originating from zero in R^3 (pause).

B: in R^3, (long Pause) umm okay umm (pause) if if two of the vectors , if this okay, let say two of the vectors are in just x and z plane, and y value is nothing but then you have a vector that does have a y value, then it would be impossible to get those three to add up to be zero. Because of that y factor there.

I: And those three would be?

B: Those three would be linearly independent.

He/she seems to be analyzing questions based on his/her visual understanding and observations; He/she seems to be thinking more visually than formally.

I: (The interviewer has drawn vectors in R^2 and asked him/her whether they were linearly independent) What do you think of those three?

B: Umm, they are umm linearly dependent. If If these Umm are at the opposite (two of three vectors were drawn as opposite of each other) end of each other. Because you could multiply this by some factor less than one and add them together and you could get zero, and then you'll have to, you know, have zero times the other one. But you would be multiplying this by some other factor other than zero, and you could still get zero .

It did not take long for him to fix his misunderstanding of the definition, a linearly independent set. His response above shows that he, through his drawings, discovered his misunderstanding and fixed it. Even if he lacks on remembering of formal definitions, it seems that he is capable of getting the statements from his own visual understandings of the concepts.

Question 4:

I: Okay this one.

B: Okay is three two one (referring to the vector (2,3,1)) in span of this, please explain your answer. Let see, umm.

I: Can you define span of a set first?

B: Yes, Umm the the span of a set is all possible linear combinations of the vectors of the set. And, ahh this not possible because umm the only way well, linear combinations of these two, the x and the y value are always going to be the same. And, since these are different, it is not possible.

I: Okay, Umm can you tell me what the span of these two vectors geometrically?

B: Umm, Ya, tell you or show you? Okay, It would be Umm (pause), it would be some two vectors going out going out Umm (pause) ya, like you know opposite ways (pause) I am thinking, you know I want to think it is like a plane..........like let see. If you had like this times anything or everything then you just keep getting this this you know, long long answers here, long line (he/she seems to be recalling a *Mathematica* demonstration. See Appendix C) and this you get a line but it is going you know straight on the x or whatever and then when add those together umm I guess you would get a plane because this can go any which way this way two, and then you will be adding this two factors of that and so you will be getting umm just all kinds of different lines, you you could get a full spectrum that way, and then if this is going that way then adding those together you could get a full spectrum, and like so I guess it would span a plan.

He/she seems to be remembering the concept visually , span of a set through Mathematica Demonstrations. He/she is even recalling visual constructions of span of a set.

I: Okay, what would be the dimension of that ?..

B: Ohh, two dimensional.

I: Can you give me a basis?

B: A basis for the plane?

I: Span of that?

B: I think that's it.

I: what is it?

B: ......ya the span, those two vectors are, is the basis.

I: The set that has those two vectors?

B: Ya

....

Question 5:

B: Okay given set S, f, f of x is an element of continuous functions from zero to one such that f of one equals zero. Okay, describe a vector in S. Umm (pause) It would be ahh it would be anything going through here as long as umm at at x equals one it is umm (pause) at zero.

I: Okay, can you give me a specific one, a specific vector in that set?

B: Ya, umm (pause) let see y equals ahh x minus one I think because if you plug in one you get zero.

He/she does not seem to have any problem with recognizing objects of the set as continuous functions.

I: Okay another one, one more?

B: You could have (he/she wrote the function $(x-1)^2$)

I: How about part b?

.....

B: Okay umm you have to show that umm closed under addition and closed under scalar multiplication and you have to show that there is at least one answer to it (He/she means the subset is not empty, there is at least one vector in the set).

I: Okay, what is that answer? For this case

B: Well umm we know that that would work (pointing out the function, x-1, he wrote down earlier)............

I: Can you draw a subspace for me?.....

B: Ya umm it has to pass through zero, so like this is our whole plane and this line passing through zero would be a subspace.

.....

I: Okay, umm how about a two dimensional subspace?

B: Two dimensional (pause) ohh a plane that is a subspace of something

I: Any plane? Can you draw me one?

B: Okay see I okay okay if if your space is all of R three then umm then any subspace of R three would be a plane passing through zero.

I: Any subspace of R three?

...

B: I am thinking It would be a plane passing through through the origin.

I: how about a circle ?

B: Umm that passes through zero?

I: No

B: A circle would not be because it is not passing through zero.

I: What was it passing through zero?

B: it is not closed under umm addition stuff.

Question 6:

I: Okay let's take a look at that one .

B: Okay, describe a vector in Range T where T v equals a v for a matrix ....Umm so I can just think off a ahh well see this is a three by four ahh I had to have a four by umm something umm

I: What is that something?

B: Ahh (he/she wrote down 1)

I: okay four by one?

B: Ya four by one. So ahh and I can just think of any v or so if we just did one one one

I: why don't you write it in general form?

...

B: Ohh Ohh okay umm (he/she wrote he general form of the vector) and then so the range would be you multiply those together, so you would get umm what is it, it is going to be a three by one , so it would be a x plus two y and then it would be umm wait a minute (pause) I am sorry it is four by one. Okay, but you would, still same, and then you would have y and then umm let see and so this would be your umm (long pause)

I: So this would be what?

....

B: Ohh well, umm I am taking these zeros are are inconsequential, or something they they serve no purpose but umm (pause) but it seems weird because I am getting, I should be getting ahh I should be getting three by one, I am fine. Ya, I was thinking I was getting ahh ahh wrong size as my answer, and I was wondering why is that, but ya this ahh (pause) that, that should be a description of of a vector, depending on what ys and xs and zs you can choose.........

.....

I: What is the dimension of the range of the linear transformation?

B: Umm (pause) dimension of the range. I think it is three because you've got three vectors there.

......

B: Describe a vector in range. Ohh okay, okay, okay, ya this is, so it is, it would be the number of ordered pairs or your, you know this is R three because you have got one two three parts to your vector and so its dimension would be three.

I: Okay, what if I give you this matrix (wrote down a new matrix with two zero rows) instead of that?

B: (pause) umm and if I am remembering correctly its dimension would be umm two. Because if you look at (pause) is that zero as well?

I: yes the last row.

B: Umm it will still be two.

I: what is a typical vector, for this case, in the range?

B: Umm a typical vector in the range (pause)umm ohh just well if you were to do ahh the same thing (referring to matrix multiplication he/she performed) then so I am going to have those four by one so (pause) (writing out the vector) and then just the y, and then zero, two like because you know the third...

I: Okay what happened to zero?

B: Umm you don't have to worry about it because you know zero, zero. That does not tell you anything.

I: Does this set in R two ? (pause) Is this a pair in R two or R three?

B: R two

I: Okay but you get, you entered four components and get three components.....

B: I see, so you need to say that it is, that that last point is zero, because if it is not then (It could be anything.........

Question 7:

I: Okay well we are at the last question. This is about *Mathematica* and the demos we have been using. What's your opinion on the demos and the software *Mathematica*?

B: Umm I like I like seeing the geometric you know ahh representations of what we are doing ....you know (referring to his/her instructor) did give the assignments with just us looking at it and I thought It was helpful with umm (pause) deciding when things are linearly independent. Stuff you know you know that they cross at zero.....

I: How about learning definitions and the basic, basic concepts like spanning set, span of a set or linear independence, dependence.

B: I say, ya I think it is helpful for the initial you know getting familiar with the new topic....

I: Umm Did it help you to remember those definitions?..

B: Ya Umm like when we were doing these things here I kept trying to see those things in my mine (referring to *Mathematica* Demonstrations) Because that is how I remember them before you know for test and stuff I would try to, try to think of you know of of definitions and things. I think it is helpful for that...

....

I: Would you be able to remember those like linear independence, dependence after the class over?

B: I think *Mathematica* helps with that just for getting the mental picture in your mind rather thatn just words. I think in the long run, I guess, It would be helpful.

I: So you are saying you would prefer those demonstrations right after definitions are given..

B: Ya, I kind of would want them just like you look at it for little bit and then you see how it applies to the book and may be to the problems kind of applied.

APPENDIX B

INFORMED CONSENT FORM

Student Consent for Participation in Research Project Conducted under the auspices of the University of Oklahoma

This study called "A Comparison Study Between a Traditional and a *Mathematica* Implemented First Year Linear Algebra Programs" is conducted by Hamide Dogan, and sponsored by Dr. McKnight, a faculty member at the Mathematics Department, University of Oklahoma.

Purposes of the study are to investigate differences in students' understanding of vector space concepts, and evaluate the strength and weaknesses of the two instructional approaches.

In the experimental class, *Mathematica* Notebooks are used. The notebooks are based on 3Dimensional demonstrations of basic abstract concepts. Students are asked to run cells on their *Mathematica* notebooks, and also asked to practice on these notebooks. Students in both classes are getting similar homework assignments, and the same number of quizzes as well as exams. The experimental group is getting less lecturing and more practice than the control group.

There are no reasonably foreseeable risks or discomforts to students who participate in this study. One benefit to the student is that a volunteer for the interview may get free tutoring at the same time. The main benefit of the study is that students may appreciate that their participation in the study may contribute to our understanding of how students learn and whether the use of technology improve students' learning.

The students' participation in this study is voluntary. Refusal to participate will involve no penalty to the student. Refusal in the study does not mean that the student would have to change classes or drop the course. It only means that the researchers would not use that student's data as part of their study. The student may stop participating in the study at any time without penalty.

The students participating in the study will not be named in any published results. The results of the students' work will remain confidential. If the student has any questions, they may contact either Hamide Dogan at (405)-573-9731 or
Dr. McKnight at (405)-325-2728.

Student Name (PRINT) : ______________________________

Student Signature : ______________________________

APPENDIX C

Applications:

Example 1

Monthly price of the stock A, starting December 1998.

Month	1	2	3	4	5	6	7	8	9	10
Price	35	35	37	40	45	60	65	60	89	90

```
lpts = Table[ {{ 1, 35}, {2, 35}, {3, 37},
    {4, 40}, {5, 45 }, {6, 60}, {7, 65}, {8, 60}, {9, 89}, {10, 90}}
    ];
```

```
p1 = ListPlot[lpts, PlotStyle -> Hue[1]]
```

Need to find values of a, b and c first:

```
p2 = Plot[a + b x + c x^2, {x, -1, 10}]
```

```
Show[p1, p2]
```

```
a = {{1, 1, 1, 35}, {1, 6, 36, 60}, {1, 9, 81, 89}}
```

```
MatrixForm[RowReduce[a]]
```

Stock's predicted price after 3 months:

```
x = 1; 1 + x + x^2
```

PRACTICE:

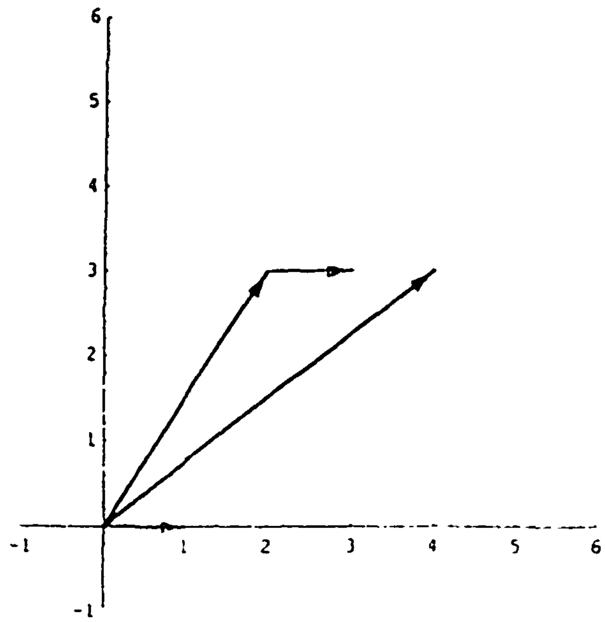
What would be the solution for the following equation?

a u+ b v+ c w=0 where u =(1,0), v=(2,3), w= (4,3), and a, b , c are unknowns.

Use the following cell:

In[92]:=

```
u1 = Vector[{1, 0}, Color → Hue[.5],
    TailWidth → .01, HeadScale → .05];
u2 = Vector[{2, 3}, Color → Hue[.7], TailWidth → .01,
   HeadScale → .05]; w = MyArrow[{2, 3},
   {3, 3}, HeadScale → .05, Color → Hue[.5]];
u3 = Vector[{4, 3}, Color → Hue[1],
    TailWidth → .01, HeadScale → .05];
Show[u1, u2, u3, w, PlotRange → {{-1, 6}, {-1, 6}},
 Axes → True, AspectRatio → Automatic]
```



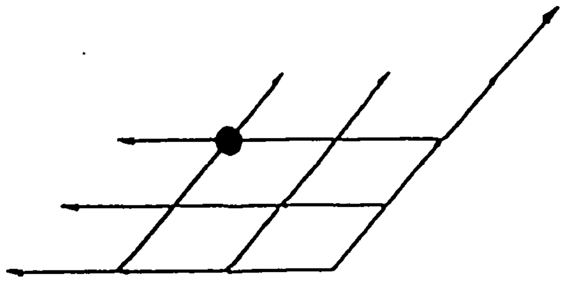
Out[95]= - Graphics -

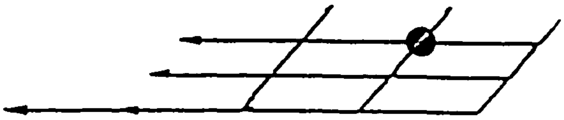
Take home Quiz II

Due Next Tuesday

Given graphs of W= Span({ (1,-1,0), (0,1,1) }) and V=Span({ (0,1,1), (2,-2,0) }). The red dot on each graph is the position of vector v in vector spaces W and V. Let
B= { (1,-1,0), (0,1,1) } , and
S={ (0,1,1), (2,-2,0) }

Graph W | **Graph of V**





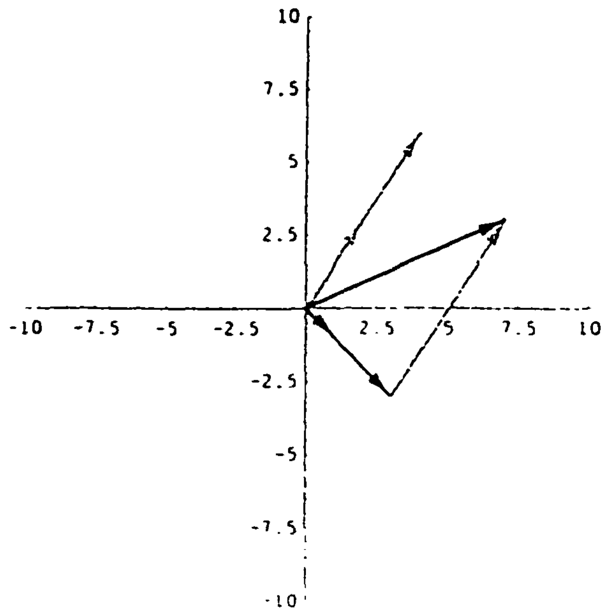
a) Find [v]$_B$ (coordinate vector of v with respect to the basis B) and [v]$_S$.
b) Describe the vector v (Find its component values)
c) Find the transition matrix P from the basis B to the basis S.
d) Show that P.[v]$_B$=[v]$_S$
e) Are vector spaces W and V isomorphic? (Explain) If your answer is yes, define an isomorphism between the vector spaces W and V.
f) Write a linear transformation T: W-> V
g) Find T(v). (Indicate the position of T(v) on the graph given above)
h) Find [T(v)]$_S$ by using the graph of V given above
i) Evaluate [T(v)]$_S$ algebraically (show your work). For this, you will need to find matrix of T with respect to the bases B and S,. and you will need to use [v]$_B$ (coordinate vector of v with respect to the basis B.)

Example 1

Linear combination of the vectors u and v, au+bv

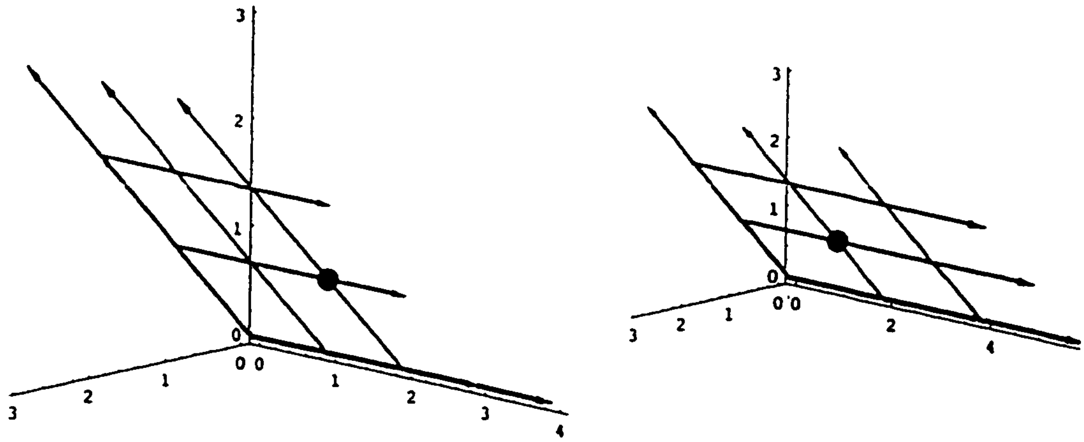
In[44]:=

```
a = 2; b = 3;
u1 = Vector[{2, 3}, Color → Hue[.5],
    TailWidth → .01, HeadScale → .05];
u2 = Vector[{1, -1}, Color → Hue[.7], TailWidth → .01,
   HeadScale → .05]; s = MyArrow[{1, -1},
   {b, -b}, HeadScale → .05, Color → Hue[.7]];
t = MyArrow[{b, -b}, {2 a + b, 3 a - b},
   HeadScale → .05, Color → Hue[.5]];
ad = Vector[{2 a + b, 3 a - b}, Color → Hue[1],
   TailWidth → .01, HeadScale → .05];
ext = MyArrow[{2, 3}, {2 a, 3 a},
   HeadScale → .05, Color → Hue[.5]];
Show[u1, u2, s, t, ad, ext,
 PlotRange → {{-10, 10}, {-10, 10}},
 Axes → True, AspectRatio → Automatic]
```



Out[45]= - Graphics -

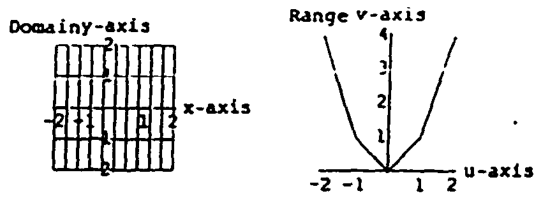
- **Coordinate vector of v=(1,2,1) with respect to the basis S={(1,0,1), (0,1,0) } of the vector space SPAN{(1,0,1), (0,1,0)} and with respect to the basis L={(1,0,1), (0,2,0)} of the same vector space.**



Samples of maps that are Not Linear Transformations

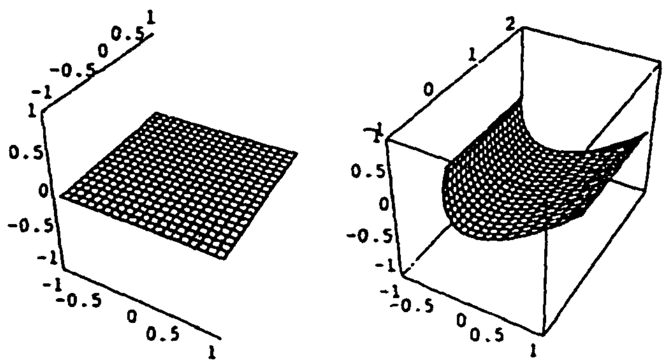
In R^2

```
In[96]:= p = CoordinatePlot[
        {y, y^2}, {x, -2, 2, .4}, {y, -2, 2, 1}]
```



In R^3

```
In[97]:= g= ParametricPlot3D[{x,y,0}, {x, -1,1}, {y,-1,1}, Boxed->False];
        g1= ParametricPlot3D[{y,x+y^2,0}, {x,-1,1}, {y,-1,1} ]; Show[GraphicsArray[{{g,
        g1}}]]
```



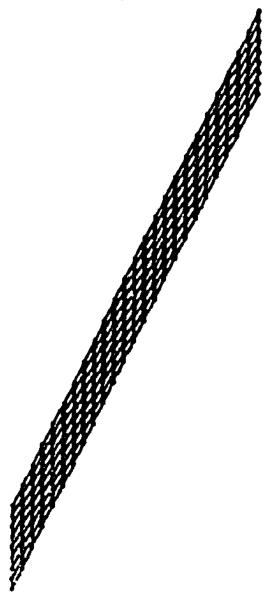
■ Example 10

Is the set {(1, 2), (0, 1), (2, 4)} a spanning set for R^2?

■ Answer:

Yes it is spanning the vector space R^2. Run the following cell;

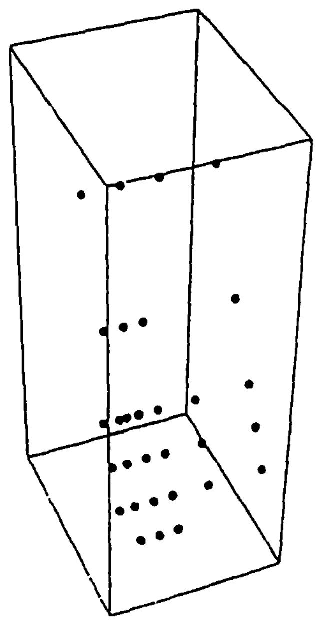
```
In[89]:= i = {1, 2}; j = {0, 1}; k = {2, 4};
        picture = BasisPicture[{i, j, k},
            4, HeadScale → .1, TailWidth → .004];
        Show[picture, Axes → None];
```

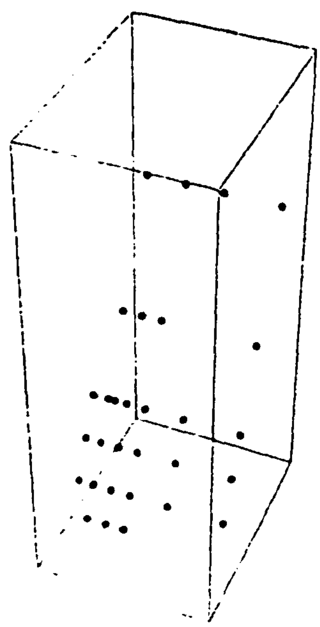


Example 3

Set of Linear combinations of the vectors
u=(1,0,0) and v=(0,1,2).

```
In[2]:= Do[Show[p1, p2, p3, p4, p5, p6, p7, p8, p9, p10, p11,
        p12, p13, p14, p15, p16, p17, p18, p19, p20, p21,
        p22, p23, p24, p25, p26, p27, p28, p29, p30, p31,
        ViewPoint -> {n, -2.400, 2.000}], {n, -1, 4, 1}]
```





APPENDIX D

Background Questionnaire and Pretest Fall 1999

Math 3333-001/002

As part of a study on testing effectiveness of different instructional approaches to teaching Linear Algebra courses, the following questionnaire has been requested . Please respond to each item as correctly as possible to the extent of your knowledge.

1. To the best of your knowledge, please indicate Mathematics courses you have taken in high school and the year you have taken them.

Elementary Algebra in 19......
______Algebra in 19.......
______Geometry in 19......
______Algebra II in 19......
______Trigonometry in 19.....
______Precalculus in 19.......
______Advanced Math in 19.....
______Calculus in 19.....
______Others (Please list)

1. To the best of your knowledge , please indicate Mathematics courses you have taken in college and the year you have taken them.

______Elementary Algebra in 19......
______Intermediate Algebra in 19.......
______Mathematics for critical thinking in 19......
______Introduction to elementary functions in 19......
______Elementary Functions in 19.....
______Calculus I for business , life and social sciences in 19.....
______Calculus and analytic geometry I in 19....
______Calculus II for business, life and social sciences in 19.....
______Arithmetic for elementary teachers in 19.....
______Calculus and analytic geometry II in 19.....
______Calculus and analytic geometry III in 19....
______Calculus and analytic geometry IV in 19 ......
______Engineering Mathematics I in 19.....
______Others (Please list)

2. Are you taking Linear Algebra for the first time?

Yes No

If you are repeating Linear Algebra this semester, indicate the reasons:
______Failed the course the first time
______Dropped the course due to a failing grade
______Dropped the course for other reasons
______Did not fail the course, but I am repeating it for other reasons

3. Please indicate your intended major:
_____Engineering (not computer science)
_____Computer science
_____Geoscience
_____Mathematics
_____Physics
_____Chemistry
_____Life science
_____Education
_____Fine or Applied arts
_____Commerce or business related majors
_____Humanities, Liberal arts, or Social sciences
(English, History, Psychology, Sociology, ect.)
_____Undecided
_____Others (Please list)

4. Please indicate the enrollment status that best describes you:
______Full- time student
______Part-time student (more than one course)
______Single Course taker

5. Please indicate your race:
____White
____Hispanic
____African American
____American Indian
_____None of the above

6. Please indicate the student status that best describes you (You can chose more than one)
________Freshman _______Sophomore _________Junior
________Senior _______Graduate student
________Just came back to school after taking some time off
________Have a job _______Others (Please list)

7. Would you be willing to participate in a tutorial interview session addressing effectiveness of linear algebra instructions?

Yes No

If your answer is yes, please write either an e-mail address or a phone number where you can be reached.

8. How much experience have you had with computer algebra systems (e.g. Mathematica, Derive, MathCad , etc.)
_____Not at all
_____Less than once a month
_____At least once a month
______More than once a month

9. To the best of your knowledge, answer each of the following questions. Please show your work and Justify your answers. *Note that these questions are for a research study, not to evaluate you for this class.*

Q1.

. Given the following vectors v=(1,2,3,4) and w=(0,-1,4,5). Find v+w.

Q2. Given f'(3)=4 (Derivative of f at x=3), g'(2)=5 and g(2)=3. By using the definition given below, find h'(2) (derivative of h at x=2) where h(x)=f(g(x)).

Definition: Derivative of h(x)=f(g(x)) at a given point x=a is defined as h'(a)=f'(g(a)) g'(a) where f and g are any differentiable functions.

Q3. Solve the following equation for A.

$$A+\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}=\begin{bmatrix}2 & 0\\ 1 & 1\end{bmatrix}.$$

Q4. Given a function f(x)= x^2+2x+1. Find the values of a that satisfy the equation f(a)=4.

Q5. Solve the following equation for a.

a (c-2) + (k+3) = a +10

10. Read each statement and then circle the response that matches your feelings. Use the following rating scale:

Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree
1	2	3	4	5

Mathematics is my favorite subject	1 2 3 4 5
A mathematical topic is of little importance if it has no real world Applications	1 2 3 4 5
Use of software, such as *Mathematica*, *MathCad*, or *Derive*, enhances learning of college algebra	1 2 3 4 5
It is necessary to use symbols to define most mathematical concepts	1 2 3 4 5
The best way to learn mathematics is to study visual representations of given concepts	1 2 3 4 5
Geometrical demonstrations enhance learning of mathematical concepts	1 2 3 4 5
Computation is an important Mathematics skill	1 2 3 4 5
In mathematics courses, hard work can make up for having less ability than other students	1 2 3 4 5
I like to use technology (calculators, computers etc.) in my classes.	1 2 3 4 5
Algebra is my favorite subject	1 2 3 4 5
Mathematics is useful	1 2 3 4 5
I really need a mathematics textbook with clear explanations to do well in a mathematics course	1 2 3 4 5
The best way to learn mathematics is to find good examples of kinds of problems you have to solve and try to follow its pattern.	1 2 3 4 5
A student's mathematics program should emphasize theory as well as applications	1 2 3 4 5

Post-Questionnaire
Sec 002

Please complete the following questionnaire. This will be used for a study on comparing effectiveness of two different teaching styles for undergraduate linear algebra classes. This information will not be used for your grade.

1. Please indicate the courses you have taken and the year you have taken them.
   ____ Discrete Mathematics in 19.....
   ____ Foundation of Analysis in 19....

2. Have long have you been a student at OU?

   ____

3. Are you an international student ?
   YES NO
   If your answer is yes, when did you come to U.S.A?

   ____

4. Indicate how difficult learning each of the following has been for you:

	Very Difficult	Somewhat Difficult	Not Difficult at all
Vector Spaces			
Matrices			
System of linear Equations			
Linear Transformations			

5. Indicate how helpful *Mathematica* related activities have been for you in learning the following :

	Not Helpful	Somewhat Helpful	Very Helpful
Vector Space concepts			
Matrices			
System of linear equations			
Transformations			
Writing proofs			
Understanding Abstract Material			
Visualizing basic concepts			
Numerical Calculations			
Applications			
Understanding Definitions			

6. Read each statement and then circle the response that matches your feelings. Use the following choices:

1. Strongly Disagree   2. Somewhat Disagree   3. Disagree

4. Agree   5. Somewhat Agree   6. Strongly Agree

Since taking this mathematics course, I believe I have a much clearer understanding of mathematical concepts that will be important in my career.	1 2 3 4 5 6
The technology we used made the course enjoyable.	1 2 3 4 5 6
Symbolic manipulations made learning of mathematical concepts easier.	1 2 3 4 5 6
I really did not need technology to learn the material covered in this class.	1 2 3 4 5 6
A student's mathematics program should emphasize theory as well as applications.	1 2 3 4 5 6
It is important that I know how to use modern technological tools.	1 2 3 4 5 6
A mathematical topic is of little importance if it has no applications.	1 2 3 4 5 6
Visualization is an important mathematics skill.	1 2 3 4 5 6
Geometrical demonstrations enhanced learning of abstract materials.	1 2 3 4 5 6
Basic algebraic operations on Matrices are important mathematics skills.	1 2 3 4 5 6
Computer assisted instructions, such as MATHEMATICA, MathCad, DERIVE, can enhance learning of the material covered in this class.	1 2 3 4 5 6
Proving mathematical statements enhanced my problem solving skill.	1 2 3 4 5 6
Computation is an important mathematics skill for this class.	1 2 3 4 5 6
Matrix Theory will be useful in my choice of career.	1 2 3 4 5 6

1. Strongly Disagree 2. Somewhat Disagree 3. Disagree

4. Agree 5. Somewhat Agree 6. Strongly Agree

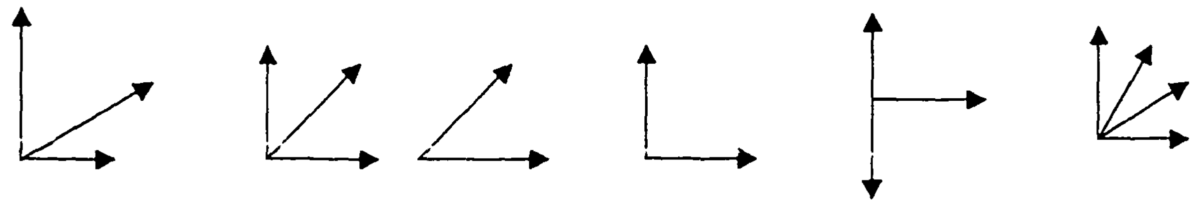
The technology we used is appropriate for this course.	1 2 3 4 5 6
Theory of Vector Spaces will be useful In my choice of career.	1 2 3 4 5 6
Algebra is my favorite subject	1 2 3 4 5 6
I like Mathematics	1 2 3 4 5 6
I have enjoyed the class	1 2 3 4 5 6
For this class, I have used outside help at least once a week	1 2 3 4 5 6

7. Indicate how helpful each of the following has been for you in learning Linear algebra material.

Instructional Device	Not Helpful	Somewhat Helpful	Very Helpful
Mathematica Notebooks (Examples and demonstrations given on Mathematica lessons) and Mathematica related activities			
Calculators (graphing or other)			
E-mail (individual, discussion groups, etc..)			
Internet(Class Web site, other web sites)			
Lectures			
Proving mathematical statements			
Office hours			
Homework from the textbook			
Other assignments (e.g. projects)			
The Textbook (other than homework)			
working alone			
working with a partner			
Studying for tests			
Examples given in class			
Applications			

8. On each graph below, shown a set of vectors originated from (0,0). Circle the ones that are ***linearly independent***.

A. B. C. D. **E.** F.



9. If you feel strongly about any issues related to Linear Algebra that we have neglected in this questionnaire, please write your comments below.