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A COMPARISON OF THE POSTULATIONAL STRUCTURE OF THE SYNTHETIC, TRANSFORMATION AND VECTOR APPROACHES TO PLANE GEOMETRY

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CHAPTER I

INTRODUCTION

Purpose of Study

The nature of the high school geometry course in the United States remains unstable and subject to change. Evidence of this fact is found in the appearance of articles written in the late sixties such as "The Dilemma in Geometry", by Allendoerfer;[1] "What Shall We Teach in High School Geometry?", by Adler;[2] and "What Should High School Geometry Be?", by Buck.[3] Each of these articles recognizes the inadequacy of the curriculum groups of the late fifties and early sixties, such as the School Mathematics Study Group (S. M. S. G.) and the University of Illinois Committee on School Mathematics (U. I. C. S. M.), in solving the problem of the geometry curriculum. Although there was a great amount of time spent on geometry by these curriculum projects, they concentrated

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[1]Carl B. Allendoerfer, "The Dilemma in Geometry," Mathematics Teacher, LXII, No. 3 (1969), p. 169.

[2]Irving Adler, "What Shall We Teach in High School Geometry?", Mathematics Teacher, LXI, No. 3 (1968), p. 227.

[3]Charles Buck, "What Should High School Geometry Be?", Mathematics Teacher, LXI, No. 5 (1968), p. 470.

basically on merely revising Euclid's geometry. Thus, for the most part, the geometry program in the United States has continued to be essentially Euclidean geometry.

Meanwhile, the sixties have seen the development of several different approaches to geometry. One approach makes use of transformations to develop geometry, and has found advocates both in the United States and abroad. For example, two new programs in England, the Nuffield Mathematics Teaching Project and the Leicestershire Experiment, make use of this approach. Elliott states in his report, "All the high school projects mentioned treat geometry mainly as a study in transformations...."[4] In the United States, Schuster indicates that there is concern for introducing transformations into the geometry programs:

> During the past several years, there has developed something of a stir concerning the introduction of geometric transformations into the high school program. This stir is due partly to the influence of certain European programs and partly to our own growing up--the result of a decade or more of curriculum reform.[5]

The K-13 Geometry Project at the University of Ontario is writing a great deal of material using transformations. The director, H. S. M. Coxeter, said at the joint meeting of the Mathematics Association of America (M. A. A.)--National Council of Teachers of Mathematics (N. C. T. M.) on geometry held in Houston during January, 1967: "We follow the British and Russians in recommending the introduction of geometric transformations as early as possible, not only as a tricky way to prove theorems

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[4]Andrew Elliott, Geometry in the Secondary School, a compendium of papers presented in Houston, Texas, January 29, 1967 (Washington, D. C.: National Council of Teachers of Mathematics, 1968), p. 15.

[5]Seymour Schuster, Geometry in the Secondary School, p. 29.

but as a means of inculcating a feeling for space."[6]

The vector approach is another which has received attention. This approach was popularized by Dieudonne and Choquet of France when they presented it at an Organization for European Economic Co-operation seminar held at Royaumont, France, in 1959. In 1963, the U.I.C.S.M. in the United States began to develop such a program. This approach relies heavily on algebra and the notions of vector space. The integration of algebra and geometry is one of the main objectives of some of the new programs. Herbert Vaughn, one of the principal writers of the U.I.C.S.M. materials, describes their development as follows: "Three dimensional Euclidean geometry is developed as a theory of an inner product space T--the set of translations--acting on a set E of points--the points of Euclidean space."[7]

Critics are almost unanimous in stating that plane geometry as it is now taught is dominated too much by the geometry of Euclid both in sequence and presentation. They agree that there is merit in new approaches such as transformations and vectors. In the report of the Cambridge Conference on School Mathematics it was recognized that "there are many different routes to follow in teaching geometry and that each has its advantage."[8]

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[6] H.S.M. Coxeter, Geometry in the Secondary School, p. 11.

[7] Herbert Vaughan, Geometry in the Secondary School, p. 24.

[8] Cambridge Conference on School Mathematics. Report of the Conference. Goals for School Mathematics (Boston: Houghton Mifflin Company, 1963), p. 47.

There is no doubt that the geometry curriculum is being subjected to close scrutiny, and that there will be many different recommendations concerning the best way to develop the subject. As Adler says, "...thus, in spite of all the world wide activity directed toward revising the geometry curriculum, teachers are still faced with the question, 'What shall we teach in high school geometry?'"[9]

Whereas experimentation in the geometry curriculum is derived in part from a desire to bring geometry and algebra closer together, there is another important aspect. There has been a continuing concern about logical defects in the postulational system, which in turn has led to several different suggested postulational approaches for eliminating those defects. These diverse approaches to Euclidean geometry (which for two thousand years had only one set of postulates) can lead to confusion for the person who has studied only the geometry of Euclid. Meserve says,

> We especially need to allow and encourage secondary school teachers to comprehend what we are trying to accomplish by our independent approaches to geometry. Only with this background can teachers enter into the true spirit of the teaching of geometry.[10]

Presently, secondary teachers fall roughly into two categories, those who had their pre-service training prior to the institution of the Committee on the Undergraduate Program in Mathematics (C.U.P.M.) recommendations and those whose training was at schools that tried to meet those recommendations. As a general rule, neither group is

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[9]Irving Adler, "What Shall We Teach in High School Geometry?", Mathematics Teacher, LXI, No. 3 (1968), p. 227.

[10]Bruce Meserve, Geometry in the Secondary School, p. 2.

educated to teach the new geometry curriculums that are being suggested. For those teachers in the first category, the college geometry course was probably advanced plane geometry which concentrated on constructions with compass and straight edge similar to that found in College Geometry by Nathan Altschiller-Court.[11] The teachers in the second group have a more modern background, but normally it would not include work in plane geometry from either the transformation or vector approach. A typical curriculum for this group would include a course in foundations of geometry and one in non-Euclidean geometry. Often the latter is replaced by work in projective geometry or topology. The foundations course often consists of a "clean" treatment of Euclidean geometry based on either Hilbert's or Birkhoff's axioms. The textbook by Moise, Elementary Geometry from an Advanced Standpoint,[12] is a widely used text in this type of course. Hence, for the most part, the preparation of secondary teachers in the area of geometry is primarily concentrated on Euclidean geometry from the synthetic or the metric approach.

One indication of the possible change in the geometry curriculum is the type of textbook written for teacher preparation. Moise's book, previously mentioned, has been quite popular. This popularity is probably attributable to the fact that his book parallels the S.M.S.G. geometry course. There have been several new books published

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[11]Nathan Altschiller-Court, College Geometry (Richmond: Johnson Publishing Company, 1925).

[12]Edwin E. Moise, Elementary Geometry from an Advanced Standpoint (Reading, Mass.: Addison-Wesley Publishing Company, Inc., 1963).

for teacher education courses which develop geometry through some technique other than the synthetic. Hall and Szabo's book, Plane Geometry, an Approach Through Isometries,[13] is an example of this type of book. A book by Choquet, Geometry in a Modern Setting,[14] develops plane geometry through the ideas of a vector space. Rainich and Dowdy also use the vector approach in their book, Geometry for Teachers.[15] It is evident that these authors believe that the geometry curriculum will change in such a way as to include modern algebraic methods. If this is so, the secondary mathematics teachers in the United States will be faced with the problem of reorienting their thinking about high school geometry.

The purpose of the present study is to show the relationship of the development of plane geometry by each of the three methods--synthetic, transformation, and vector. The study is intended for those interested teachers who know that the geometry program is likely to change, and who wish to know how these changes are related to the geometry that they have been teaching.

STATEMENT OF PROBLEM

This study compares three different methods of developing high school geometry. Three approaches to the subject will

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[13]Dick Wick Hall and Steven Szabo, Plane Geometry, An Approach Through Isometries (Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1971).

[14]Gustave Choquet, Geometry in a Modern Setting (Paris: Hermann, 1969; Boston: Houghton Mifflin Company, 1969).

[15]G. R. Rainich and S. M. Dowdy, Geometry for Teachers (New York: John Wiley and Sons, Inc., 1968).

be analyzed: synthetic, vector and transformation. The analysis will be directed toward answering the question, "How does each approach develop the basic topics of incidence, order, parallelism, congruence and similarity?"

DESIGN OF STUDY

The investigation is an analysis and comparison of three different ways of developing Euclidean plane geometry. Each approach will be analyzed with regard to the necessary axiomatic structure needed to develop the usual topics of plane geometry. These include incidence, order, distance, congruence, parallelism, and similarity.

The comparison between approaches is based upon the relationship between the postulates, definitions and theorems which result from them. For example, in the synthetic approach the development of the congruence of triangles hinges on the "Side-Angle-Side" postulate, whereas in the transformation approach this postulate is a theorem. The proof of this theorem is based on a postulate pertaining to a particular type of transformation called reflections. The procedure used in the study to discuss the basic question will be to point out such differences as well as the similarities which exist.

A program exemplifying each of the different approaches will be analyzed. This is done so that the development of each approach will not have to be given in the study. It should be understood that the study is not attempting to compare sequence and content of the specific programs, but rather describe the relationship of the different treatments of geometry that they represent.

The text by Moise and Downs, Geometry,[16] will be used for the synthetic approach. This book parallels the treatment done by S.M.S.G.'s Geometry.[17] These two works use the metric approach similar to that suggested by Birkhoff in 1932.[18] These materials are most representative of the type of program being used in the high school geometry courses today in the United States.

The material selected for the vector approach is the two volume text, A Vector Approach to Euclidean Geometry,[19] written by Vaughn and Szabo. This text is the culmination of the efforts of U.I.C.S.M. since 1963 to develop a vector geometry. Although this material is not widely used, it has been tested in classrooms around the country and a number of teachers have studied it in summer institutes. The material is now in the process of being published commercially.

A high school text by Coxford and Usiskin, Geometry: A Transformation Approach,[20] will be used as a guide for the transformation treatment. This book was just recently published and represents one of the few high school texts organized with transformations as a central theme.

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[16] Edwin E. Moise and Floyd L. Downs Jr., Geometry (Reading, Mass.: Addison-Wesley Publishing Company, 1967).

[17] School Mathematics Study Group, Mathematics for High School, Geometry (New Haven, Mass.: Yale University Press, 1961).

[18] G. D. Birkhoff, "A Set of Postulates for Plane Geometry," Annals of Mathematics, 33 (1932), p. 329.

[19] Herbert Vaughn and Steven Szabo, A Vector Approach to Euclidean Geometry (New York: Macmillan Company, 1971).

[20] A. F. Coxford and A. P. Usiskin, Geometry: A Transformation Approach, (River Forest, Ill.: Laidlaw Brothers, 1971).

Chapter I contains a discussion of purpose of the study, the design of the study, and a background and review of literature.

Chapter II is devoted to a comparison of the primitive terms, the incidence and the order postulates in each program.

Chapter III contains the analysis and comparison of the topic of congruence, especially the congruence of triangles.

Chapter IV presents a similar discussion of the general topics of parallelism and similarity.

Chapter V is devoted to a comparison of the objectives of each program to generally accepted objectives for geometry. Also included is a discussion of methods of proof and sequence of topics.

Chapter VI is the conclusion and recommendations for further study.

Background and Review of Literature

The question of what constitutes plane geometry is not a recent one. It is a generally accepted fact that geometry began in Egypt as long ago as 3000 B.C. It was basically empirical, with results verified by experimentation. Greek geometry started with the work of Thales of Miletus (640-546 B.C.), who apparently was the first to use the deductive method in geometry. However, Euclid (about 300 B.C.), established the postulational method in geometry; in his highly significant work, the *Elements*, he attempted to demonstrate all the geometry knownto the Greeks by starting with five postulates and five common notions. Blumenthal states:

> This epoch-making work, written between 330 and 320 B.C., has probably had more influence on the molding of our present civilization than has any other creation of the Greek intellect. Though far from attaining the perfection it aspired to, the *Elements* commanded the admiration of mankind for more than two thousand years and established a standard for rigorous demonstration that was not surpassed until modern times.[21]

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[21]Leonard Blumenthal, *A Modern View of Geometry* (San Francisco: W. H. Freeman and Company, 1961).

Although Euclid's geometry was a remarkable achievement, it was not without error. He did not, for instance, recognize the necessity of undefined terms; instead, he attempted to define everything. Neither did he realize, nor at least make it known, that he tacitly assumed certain relations such as "betweenness." These issues were to be the cause of much controversy in the field of geometry during the nineteenth and twentieth centuries.

During his lifetime critics began to investigate Euclid's fifth postulate, which is called the "parallel postulate." This postulate is more complicated than the four previous ones, and hence many mathematicians believed that it could be deduced from the others. Girolamo Saccheri (1667-1733), an Italian mathematician, made one of the more thorough investigations of the problem. Saccheri was fascinated by indirect proof, and his attempts to use this type of argument eventually led mathematicians to suspect that the fifth postulate was in fact independent of the other four. Karl Friedrich Gauss (1777-1855), Jonas Bolyai (1802-1860), and Nicolai Lobachevsky (1793-1856) were the first three mathematicians to work on the problem of independence. It is generally believed that Gauss was the first to realize that a postulate contradicting the fifth postulate could be substituted for it, and that another geometry as consistent as Euclidean geometry could be developed. Although Gauss was the first to recognize this, Lobachevsky's work was the first to be published. The geometries developed using such a postulate were called non-Euclidean.

The work of these three men had far-reaching effects in the field of mathematics. Eves states, "Shortly after the first quarter of the nineteenth century a geometrical event took place which proved to be of tremendous significance, not only for geometry, but for the whole of mathematics."[22] The existence of this new geometry raised serious questions about "truths" in mathematics, since it was logically consistent yet based on a postulate which negated one of Euclid's. Prior to this time, Euclid's postulates were believed to be "true" because they were considered to be a representation of physical space (and supposedly every person knew them intuitively). Eves adds, "The discovery of a non-Euclidean geometry compelled mathematicians to adopt a new viewpoint of their subject."[23] Mathematicians now began to realize that the postulates for a mathematical system could be chosen arbitrarily and without any need for physical significance. The work done in non-Euclidean geometry created an interest among mathematicians in devising a set of postulates from which Euclidean geometry could be developed without inconsistencies. Mortiz Pasch (1843-1930) was the first man to complete such a work. Although his work was the first, the most widely accepted is that of David Hilbert (1862-1943). Hilbert's treatment of plane Euclidean geometry, which he published in a work entitled Grundlagen der Geometrie[24] (Foundations of Geometry) rests on fifteen postulates that involve five primitive (undefined) terms. The postulates

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[22] Howard Eves, A Survey of Geometry (Boston: Allyn and Bacon, Inc., 1963), p. 371.

[23] *Ibid.*, p. 371.

[24] David Hilbert, The Foundations of Geometry (La Salle, Ill.: Open Court Publishing Co., 1902).

deal with the topics of connection, order, congruence, parallelism and continuity, and define implicitly the primitive terms "point", "line", "on", "between", and "congruent." Hilbert's work was significant not just because of his logical treatment of plane geometry, but also because, as Eves says, "By implanting the postulational method in nearly all of mathematics since 1900, Hilbert's Grundlagen der Geometrie represents a definite landmark in the history of mathematical thought."[25]

Following Hilbert's work, several different postulate sets were derived by men such as Oswald Veblen, Gilbert Robinson, and Henry Forder. In 1904 G. B. Halstead wrote a high school textbook, Rational Geometry,[26] in which he developed plane geometry by using Hilbert's postulates. The public school was not ready for such a thorough treatment of plane Euclidean geometry, as may be seen by the report of the Committee of Fifteen on the Geometry Syllabus, a joint committee of the National Education Association and the American Federation of Teachers of Mathematics and Natural Science. The Thirty-Second Yearbook of the N.C.T.M. reports, "The committee did not deem it desirable to postulate the existence of points, lines, and angles, to assume Pasch's postulate, or to postulate continuity. It was recommended, however, that teachers 'mention incidently that tacit assumptions are always made.'"[27]

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[25]Eves, p. 387.

[26]G. B. Halstead, Rational Geometry (New York: John Wiley and Sons, Inc., 1904).

[27]National Council of Teachers of Mathematics. A History of Mathematics Education in the United States and Canada. Thirty-Second Yearbook. (Washington, D.C.: National Council of Teachers of Mathematics, 1970), p. 191.

In 1940 George D. Birkhoff and a Harvard colleague, Ralph Beatley, wrote a high school text based on a postulate set which Birkhoff had reported earlier in 1930. His postulate set was based upon the assumption of a one-to-one correspondence between the real numbers and the points on a line, thus permitting the measurement of segments. Through measurement, the concepts of betweenness and congruence were treated. He also postulated measurement of angles in the same manner. Birkhoff and Beatley's Basic Geometry[28] did not become popular as a high school text, but was a significant book. As Tuller states, "This was too radical a departure from Euclid to gain favor at the time. We mention it here because of the bridge it affords from Hilbert's treatment to that of the S.M.S.G., and also because it points up the fact that the same geometry can be developed from such very different postulate systems."[29] Hunte's investigation into the changes of the postulational structure used in high school geometry texts revealed that the Hilbert and Birkhoff schemes are a basis for postulational structure for most of the texts written during the sixties.[30]

The nineteenth century was a period of concern for geometry. Projective geometry had been revived and many prominent mathematicians, such as Alfred Cayley (1821-1895), believed that projective geometry was

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[28] G. D. Birkhoff and R. Beatley, Basic Geometry (Chicago: Scott, Foresman and Company, 1940).

[29] Annita Tuller, A Modern Introduction to Geometries (Princeton, New Jersey: D. Van Nostrand Company, Inc., 1967), p. 23.

[30] Beryl Eleanor Hunte, "Demonstrative Geometry During the Twentieth Century: An Account of the Various Sequences Used in the Subject Matter of Demonstrative Geometry from 1900 to Present Time." (Unpublished Ph. D. dissertation, New York University, 1965.)

all geometry. In a series of lectures given in 1872, Felix Klein attempted to use transformations on a set to classify all geometries. Klein based his classifications on the fact that transformations, which are one-to-one onto functions, form a group, and that the sub-groups of this group give different geometries. Klein states, "It was just this which constituted the leading thought of my Erlanger Programm: Given any group of transformations in space which includes the principal group as a sub-group, then the invariant theory of this group gives a definite kind of geometry, and every possible geometry can be obtained in this way."[31] Although his basic work deals with different types of geometries, he also wrote on the development of plane Euclidean geometry through the use of transformations he called the "subgroup of proper movements."[32] His development required postulates of connection, order and continuity. Klein's work has been a strong influence in the European curriculums which are oriented toward the transformation concept.

In 1960 S.M.S.G. published their first work on geometry. This work used Birkhoff's notion of the one-to-one correspondence between the reals and the points on a line. Their connection axioms and plane separation axiom are similar to those of Hilbert. As Tuller says,

> As we look over this set of postulates we note that it combines the ideas of Hilbert and Birkhoff in a form simple enough for use on the secondary school level and gives a basis for the early introduction of analytic methods.[33]

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[31] Felix Klein, Elementary Mathematics from an Advanced Standpoint: Geometry, trans. by E. R. Hedrick and C. A. Noble (New York: Dover Publications, Inc., 1939), p. 133.

[32] Ibid., p. 161.

[33] Tuller, p. 24.

This comment appears in the Teacher's Commentary of S.M.S.G. Geometry: "The treatment of geometry in this book is very different, especially the first few chapters, from the treatment that nearly everybody is used to."[34] Their work has been used as a guide by a number of authors for commercial publication.[35]

In 1963 U.I.C.S.M. began work on a course developed through the ideas of vector space. This program has now been completed and is in the process of being published commercially by Macmillan Company. Johnson, in his doctoral dissertation, showed that by assuming the properties of a Euclidean vector space with inner product that the twenty-two assumptions found in S.M.S.G. Geometry could be proved as theorems.[36] In a report on the use of vectors in the geometry curriculum in the German Gymnasium, Athen points out that in Germany vectors have been used in developing geometry in the middle and upper parts of the Gymnasium (ages thirteen to nineteen) since 1955. In his article, he describes the vector approach taken in the German schools. The theory is built on an intuitive introduction to translations and from this an axiomized vector space is developed.[37]

In 1969 Choquet published a textbook for teacher preparation

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[34]School Mathematics Study Group. Mathematics for High School, Geometry (New Haven, Mass.: Yale University Press, 1961), p. ix.

[35]See, for instance, Richard D. Anderson, Jack W. Garon and Joseph Gremillion, School Mathematics Geometry (Boston: Houghton Mifflin Company, 1966).

[36]Alonzo Franklin Johnson, "S.M.S.G. Geometry as a Real Vector Space." (Unpublished Ed. D. dissertation, Oklahoma State University, 1967.)

[37]Herman Athen, "The Teaching of Vectors in the German Gymnasium," Mathematics Teacher, LIX, No. 5 (1966), p. 382.

based on the vector approach. He sums up his view of the presentation needed for high school geometry:

> The situation can be summed up as follows: We have a "royal" road based on the concepts of "vector space and inner product"; but pupils cannot be cannon-balled along this road without preparation, especially at an age when they are not very familiar with algebraic operations.[38]

He bases his axiomatization on the additive structure of the line, parallels and symmetry.

In 1964 the Wesleyan Coordinate Geometry Group was formed to work on a program which would develop Euclidean geometry as a coordinatized affine plane, with the real number system used as a set of coordinates on a line. In the words of G. P. Johnson, one of the authors, their development "bears resemblance to the Cartesian product development, as does any coordinate geometry."[39] From this type of treatment they build up the affine geometry, incidence and parallelism, quite easily, but "these axioms are such that Euclidean structure is by no means immediately apparent in them."[40]

A great amount of curriculum work on geometry has been done in Europe. Meeting through a committee of the Organization for European Economic Co-operation in 1961, a number of European mathematicians made proposals stressing either the transformation or the vector treatment of geometry. In his article, Richard Gast indicates that at the present time transformations are included in the curriculums throughout

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[38] Choquet, p. 14.

[39] G.P. Johnson, Geometry in the Secondary School, p. 20.

[40] *Ibid.*

most of Europe.[41]

A number of proposals have been made to unify the synthetic and transformational approaches. Adler suggests two solutions:[42] to give isometries a central role, such as has been done in Denmark, using a set of axioms, such as those in Guggenheimer's book;[43]; or to include a unit on isometries with the modified Hilbert axioms used in S.M.S.G. Earl M. L. Beard, in his doctoral dissertation, presents "a postulate set and sequence of theorems for plane geometry based on isometries with some topics not common to the curriculum that uses transformations."[44]

The use of transformations as an integral part of the postulational structure in the high school geometry course has led to research inte the changes in attitudes and retention on the part of the student using transformation geometry. Usiskin, co-author of the text used as a guide for the transformation approach in this study, found that the transformation approach aided in student understanding of congruence and similarity.[45] In a follow-up study, Kort, using the same text, (Coxford-Usiskin), found an increase in the students' retention of

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[41] Richard H. Gast, "The High School Geometry Controversy; Is Transformation Geometry the Answer?" Mathematics Teacher, LXIV, No. 1, (1971), p. 37.

[42] Irving Adler, Mathematics Teacher (1968), p. 237.

[43] Heinrich W. Guggenheimer, Plane Geometry and its Groups (San Francisco: Holden-Day, Inc., 1967).

[44] Earl M. L. Beard, "An Axiom System for High School Geometry Based on Isometries." (Unpublished Ph. D. dissertation, University of Wisconsin, 1968.)

[45] Zalman Philip Usiskin, "The Effects of Teaching Euclidean Geometry Via Transformations on Student Achievement and Attitudes in Tenth-Grade Geometry." (Unpublished Ph. D. dissertation, University of Michigan, 1969.)

concepts relating to congruence and similarity by those using the transformation approach.[46]

Some mathematicians have argued for geometry to remain separate from algebra. Charles Buck calls for a geometry curriculum similar to the traditional high school geometry course. He says it should be "a course in synthetic geometry with intuition and spirals in the more difficult points."[47] Allendoerfer suggests a curriculum for geometry from the elementary school to twelfth grade. He believes geometry can be applied to algebra, science, etc., but states, "We must strive to teach our geometry courses with a truly geometry flavor, and not merely as an exercise in algebra or logic."[48]

There are those who call for the geometry program to include all three approaches. The Secondary School Mathematics Curriculum Improvement Study has written a mathematics program for grades seven through twelve, which attempts to unify secondary mathematics through basic concepts.[49] In their program they have presented geometry from all three approaches beginning with an introduction to transformations in the seventh grade.

A review of the literature indicates that there is experimentation with the geometry curriculum. Many of the proposals are radically dif-

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[46]Anthone Paul Kort, "Transformation vs. Non-Transformation in Tenth-Grade Geometry: Effects on Retention of Geometry and on Transfer in Eleventh-Grade Mathematics." (Unpublished Ph. D. dissertation, Northwestern University, 1971.)

[47]Charles Buck, p. 473.

[48]Carl B. Allendoerfer, Mathematics Teacher (1969), p. 169.

[49]Secondary School Mathematics Curriculum Improvement Study, Unified Modern Mathematics (New York: Columbia University, 1968).

ferent from the existing curriculum, and it is uncertain which of these may find their way into the high school curriculum.

CHAPTER II

PRIMITIVE TERMS, INCIDENCE, ORDER AND DISTANCE

Primitive Terms

A primitive term in a mathematical system is one which is not defined but rather is understood by the properties ascribed to it by the accepted postulates. The primitive terms are then used to define other terms in the system. The role of the primitive term is one of the primary differences between more recent approaches to geometry and the approach of Euclid. Euclid attempted to define every term, the result of which was that some are meaningless. For example, "A point is that which has no part."[1] Blumenthal says,

> Though Euclid clearly recognized the necessity for unproved propositions in his scheme, it is doubtful that he realized the necessity for primitive (undefined) notions. And yet it is obvious that the attempt to define everything must result in a vicious circle or in an infinite regression.[2]

The primitive (undefined) terms of the programs being reviewed are somewhat different, and this difference contributes considerably to the method of development of each program. As a starting point of this study, a brief description of the primitive terms is given.

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[1] Tuller, p. 179.

[2] Blumenthal, p. 3.

Each of the three programs depends on an understanding of sets ("set" itself being an undefined term), including the operations of union and intersection and the relations of membership and inclusion. They all begin with the notion that geometry is the study of a certain set whose elements are called points. Vaughan-Szabo state that "space is a set - we shall call it '$\mathcal{E}$' - and its members are customarily called points."[3] Coxford-Usiskin say, "In geometry, space is the set of all points."[4] Moise-Downs write that "...points, lines and planes will be regarded as sets of points. (You may regard this statement, if you like, as our first postulate.)"[5]

The idea of considering lines, planes and space as sets is quite common in recent texts, and enables the authors to use the accepted notions and symbolism of sets. For instance, a line is a set of points and a point is considered to be an element of the line.

Other than "point" the undefined terms that are used in the synthetic approach are "line", "plane" and "space." The properties of these sets are given in the first ten postulates of Moise-Downs. Those same terms are the undefined terms in Coxford-Usiskin and are described in the first and third postulates. However, Vaughan-Szabo develop their program with "point" and "translation" as the primitive terms. Thus, the first significant difference among the programs is in the primitive terms that are used as basic building blocks. This means of

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[3]Vaughan-Szabo, p. 39.

[4]Coxford-Usiskin, p. 2.

[5]Moise-Downs, p. 16.

course that if Vaughan-Szabo discuss the usual theorems pertaining to lines and planes then they will have to define lines and planes by use of points and translations or of other terms which have themselves been defined by translations.

In a discussion of the undefined terms it should be mentioned that all three programs under investigation use the real numbers without an axiomatic development of them. Thus, they all assume that the student has studied the real numbers and their properties to the level normally reached in an Algebra I course.

Incidence

As was mentioned in the previous section both the synthetic and the transformation approaches use "point", "line" and "plane" as primitive terms, and the properties which those sets possess are found in those postulates that are commonly referred to as the incidence postulates. These postulates are quite similar in Moise-Downs and Coxford-Usiskin. There are some slight differences, and one such difference is discussed to illustrate the effect of varying postulates. Postulate One (b) of Coxford-Usiskin states that "two distinct lines intersect in at most one point." Postulate One (c) says that "every pair of distinct points lies in at least one line."[6] By contrast Moise-Downs state in Postulate Four that "for every two distinct points there is exactly one line that contains them."[7] At first glance, it appears that Moise-Downs have been too restrictive by stating "exactly one line" rather than

[6]Coxford-Usiskin, p. 18.

[7]Moise-Downs, p. 57.

"at least one line." However, this restriction allows them to prove the statement, "Two distinct lines intersect in at most one point," which is Postulate One (b) in Coxford-Usiskin. On the other hand, Coxford-Usiskin can, using their postulates, prove "that through any two distinct points there is exactly one line," which is of course the fourth postulate of Moise-Downs. This example is included in this study to point out that although the two approaches seem to be the same, slight variations can occur which cause postulates in one to become theorems in the other.

It is not the purpose of this study to discuss small variations in the postulates such as the previous example but rather to discuss those differences that truly make the three treatments different, for example, how the vector approach to incidence compares to the transformation and synthetic approaches.

Before presenting a discussion of the manner in which Vaughan-Szabo develop incidence, a description of those topics that are introduced prior to, and are necessary for, the development of incidence is included. The vector approach postulates the existence of a set of mappings, $\mathfrak{T}$, from the set of points $\mathcal{E}$ to $\mathcal{E}$. These mappings are called translations. A translation is a mapping that maps each point of the plane onto a point in the plane and which has certain distance and direction preserving properties. Intuitively, a translation is a mapping that moves each point the same distance in the same direction. Of course, the notions of distance and direction must be defined; so the fact that translations possess these properties must either be postulated or implied by the postulates. Postulate One has two parts; the first says that given any two points A and B, the two points determine a translation ($B - A \in \mathfrak{T}$).

The second part of Postulate One states that a point and a translation determine another point $(A + \vec{a} \in \mathcal{E})$. The second postulate tells how the translations and points guaranteed in the first postulate relate to each other. Again, the second postulate has two parts. Part (a) states that the image of point A under the translation determined by points A and B is point B; i.e., $A + (B - A) = B$. The second part says that any given translation $\vec{a}$ is determined by any point and the image of that point; i.e., $\vec{a} = (A + \vec{a}) - A$. The essence of these two postulates is that if you pick any two points there is only one translation which will map one onto the other.

The authors have a rather unique notation associated with their concept of translations. Instead of using the traditional f(X) or (X)f for the image of a point X under a mapping f, they choose to use $X + \vec{f}$ as the image of point X under the translation $\vec{f}$. The authors seem to confuse the issue of notation even more by again using "+" as the symbol for the operation of composition of mappings. For example, they write $\vec{a} + \vec{b}$ rather than $\vec{a} \circ \vec{b}$. The symbol "-" is used in much the same way. $A - \vec{a}$ means the image of A under the inverse of $\vec{a}$, and $\vec{a} - \vec{b}$ is $\vec{a}$ followed by the inverse of $\vec{b}$. Evidently, the purpose for this type of notation is to strengthen the concept that many properties which are true for real number addition are also true in their two forms of "addition." As a matter of fact, the third postulate lends support to this idea.

The third postulate states that $(B - A) + (C - B) = C - A$; that is, the translation which takes A to B followed by the translation which takes B to C is equivalent to the translation which takes A to C. This postulate gives a convenient and familiar arithmetic for translations.

The next step in the general development of the vector approach is to point out that the translations, using the operation of composition, form a commutative group. This is stated in Postulate Four: "$\mathcal{T}$ is a commutative group with respect to composition."[8] This postulate is the basis for using $-\vec{a}$ as the inverse of the translation $\vec{a}$ and $\vec{0}$ as the identity translation.

The real numbers are then introduced, and their properties with respect to points and translations are specified in five postulates. The postulates can be summarized, "The group $\mathcal{T}$ of translations admits the real numbers as operators."[9] With the addition of these postulates the authors have made $\mathcal{T}$, with respect to composition of mappings, a vector space over the real numbers.

Because of the fact that Vaughan-Szabo use translations as one of their basic tools, the theorems in the first few chapters generally appear to be algebraic in nature. For example, the list of theorems at the end of Chapter Three includes such theorems as $-\vec{a} = A - (A + \vec{a})$ and $\vec{a} - (B - C) = C - (B - \vec{a})$. Contrast these with a theorem such as "Vertical angles have the same measure" which is found in the third chapter of Moise-Downs and it points out one of the basic reasons why Vaughan-Szabo believe that their program has merit. The fact that through the use of vector space concepts they can relate algebra and geometry is significant to them. As a matter of fact, they often use

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[8] Vaughan-Szabo, p. 165.

[9] *Ibid.*, p. 231.

theorems that are valid in the field of real numbers to motivate theorems pertaining to translations and points. For example, the cancellation property for multiplication, "If $c \neq 0$, and $ac = bc$, then $a = b$", leads to a similar property with translations, "If $c \neq 0$, and $\vec{a}c = \vec{b}c$, then $\vec{a} = \vec{b}$". The proofs for many of these theorems are patterned after the proofs of the analogous real number theorems. From this observation, it should be evident that the type of proof used in the vector approach is considerably different from that found in most high school geometry texts.

A concept which is associated with a vector space is that of linear dependence of translations (vectors). Intuitively, two vectors in the same plane can be thought of as linearly dependent if they have the same or opposite directions. Actually, Vaughan-Szabo postulate the properties of linear independence for space, but the concept is easily adapted to the plane. Linear dependence is the concept that the authors use to tie together points and translations so that they can be used to define lines and planes.

Since lines and planes are not primitive terms in the vector approach, their development is of some importance in this study. To begin with, Vaughan-Szabo define collinear points in terms of the vectors determined by those points. If three points, A, B, C are such that the translations (B - A, C - A), determined by those points are linearly dependent, then the points are collinear. The concept of collinearity is thus defined in terms of translations and linear dependence. Once the idea of collinearity has been introduced, the authors can define a line by specifying that it be a set of points that contains at least two points, and furthermore that for every pair of distinct points in the set, a point is in the set if and only if the three points are collinear.

Using their definition of a line, how do they prove the incidence property--"Two points determine a line"--that was found as a postulate in Moise-Downs? To develop this theorem is certainly desirable but somewhat tedious. They begin by showing that a point C is on a line l if and only if there exists a real number x such that $C = A + (B - A)x$. The key to this proof is a theorem proved in the section on linear dependence which says that $C - A$ is a multiple of $B - A$ if $(C - A, B - A)$ are linearly dependent. Thus, they have a condition for a point being on the line containing points A and B. The next step is to define the set $\overleftrightarrow{AB}$ as the set of all X such that there exists a real number x so that $X = A + (B - A)x$. They now need to prove that $\overleftrightarrow{AB}$ actually satisfies the definition of a line. The proof of this is quite lengthy and again uses those theorems relating linear dependence and multiples of translations. The general outline of the proof is as follows:

> The fact that $\overleftrightarrow{AB}$ is a subset of $\mathcal{E}$ which contains two points comes from the definition of $\overleftrightarrow{AB}$ and letting $x = 0$ and $x = 1$ to get points A and B respectively. Next we need to show that for every distinct pair of points, $P, Q \in \overleftrightarrow{AB}$, then for every point C, $C \in \overleftrightarrow{AB}$ if and only if $(Q - P, C - P)$ is linearly dependent. If $Q - P$ and $C - P$ are linearly dependent, then $C - P$ is a multiple of $Q - P$, say $C - P = (Q - P)m$. $Q - P$ is a multiple of $B - A$, $Q - P = (B - A)n$, hence $C - P = (B - A)mn$. Since

$P \in \overleftrightarrow{AB}$, then $P - A = (B - A)r$. Now, rewriting $C - A$ as $(P - A) + (C - P)$ we see that $C - A = (B - A)(mn + r)$. Thus, $C - A$ is a multiple of $B - A$ and $C \in \overleftrightarrow{BA}$.

Now assume $C \in \overleftrightarrow{AB}$. This implies there exists a real number c such that $C = A + (B - A)c$. Since $P \in \overleftrightarrow{AB}$ we have $P = A + (B - A)p$ and thus $C - P = (B - A)(c - p)$. Similarly if $Q = A + (B - A)q$, we have $Q - P = (B - A)(q - p)$. Combining these two statements it says

$$(Q - P)(p - c) + (C - P)(q - p) = 0.$$

Since $q \neq p$, $(Q - P, C - P)$ are linearly dependent.

Thus, they have proved that $\overleftrightarrow{AB}$ is a line and the only line that contains points A and B. This theorem is significant in the fact that it can be used to prove other incidence theorems. Also, as we will see in Chapter Four of this study, it is a basis for the theorems required for the parallel theorems.

The definition and subsequent development of a plane and its properties follows closely that of a line. The concept of points being coplanar is first defined in terms of the linear dependence of the translations determined by the points. Then the definition of a plane is given in terms of coplanar points. The theorem, "Three non-collinear points determine a plane," is a key one, and the proof follows closely its counterpart, "Two points determine a line." Both theorems are postulates in the other two treatments.

Once Vaughan-Szabo have developed as theorems those incidence properties that are postulates in the other approaches, then they can prove other incidence theorems in the same manner as the theorems are proved in the synthetic or transformation approach. This type of proof represents a departure from the type of proof that is used in most of the theorems in the vector approach. For example, the theorem, "A line and a point not on that line determine a plane," is a standard incidence theorem in synthetic geometry. The proof simply uses the fact that a line contains two distinct points and since the given point is not on the line, then we have three noncollinear points. Since the existence of three noncollinear points is known, the postulate that three noncollinear points determine a plane can be applied. The proof in Vaughan-Szabo is exactly the same; however, the fact that three noncollinear points determine a plane is not a postulate but rather a theorem.

There are some incidence theorems that can be proved in the vector approach which could not be proved in the synthetic or transformation approaches. The theorem, "The medians of a triangle are concurrent in a point which divides each in the ratio 2 to 1," is such an example. The proof of this theorem is presented to contrast it with proofs using synthetic methods.

> Suppose A, B, C are noncollinear and E, F are the midpoints of $\overline{BC}$ and $\overline{AB}$ respectively. The medians, $\overline{AE}$ and $\overline{CF}$, will have a point in common if and only if there exists real numbers x, y such that $A + (E - A)x = C + (F - C)y$. This is true because the left side of the equation represents those points on $\overleftrightarrow{AE}$ for any real number x. Similarly, the right side is $\overleftrightarrow{FC}$ for any real number y. Using the algebra of translations, we find that $E - A = (C - A) + (E - C) = C - A + 1/2(B - C)$. Furthermore, $F - C =$

$(A - C) + (F - A) = (A - C) + 1/2(B - A) = (A - C) - 1/2(A - C) + 1/2(B - C) = 1/2(A - C) + 1/2(B - C)$. Finally, we see that $A + (E - A)x = C + (F - C)y$ if and only if $(1 - x - 1/2y)(A - C) + (1/2x - 1/2y)(B - C) = \vec{0}$. The previous statement is a linear combination of $(A - C)$ and $(B - C)$ which is equal to $\vec{0}$. But $A - C$ and $B - C$ are linearly independent because A, B, C are noncollinear. Thus, $1 - x - 1/2y = 0$ and $1/2\, x - 1/2\, y = 0$. From these two equations we can solve for the values of x and y and we see that $x = y = 2/3$.

The proof of this theorem in the synthetic or transformation approach can be done in several ways; however, each requires that other topics be developed previously. For example, one proof often used is based on the properties of a parallelogram formed using some of the points given in the hypothesis. But the properties that are used relating to the parallelogram are based on theorems whose proofs require congruence and parallel postulates. Often a proof of the medians theorem is based on Ceva's theorem.[10] Here again, proof of Ceva's theorem requires properties of similar triangles. Other proofs are based on coordinate systems or parallel lines. In any case, the proof of the medians theorem can not usually be done with only the incidence theorems in synthetic or transformation geometry.

Order

"Order is one of the most basic and pervasive of mathematical ideas."[11]

[10] For statement and proof of theorem see Miller, Leslie H., College Geometry (New York: Appleton-Century-Croft, 1957), p. 40.

[11] Walter Prenowitz and Meyer Jordan, Basic Concepts of Geometry (Waltham, Mass.: Blaisdell Publishing Company, 1965), p. 184.

Euclid did not even recognize the necessity for a discussion of order, and consequently, there are proofs based on his postulates which seemingly are valid, yet are not.[12]

"There are two well-known theories of order called the theory of precedence and the theory of betweenness."[13] Precedence is a binary relation; whereas betweenness is a ternary relation. The first widely recognized discussion of order in geometry was that of Hilbert, who developed the betweenness relation without using the real numbers.[14] Today, most authors approach order in geometry through an association with real numbers, a scheme that was devised by Birkhoff.[15]

With respect to order, again the vector approach differs from the other two. Coxford-Usiskin and Moise-Downs both use a definition of betweenness based on postulating a one-to-one correspondence between the points on a line and the real numbers. The real number associate of a point is called its coordinate. The postulate that establishes this one-to-one correspondence is commonly referred to as the "ruler" postulate. The betweenness relation can then be defined by letting C be between A and B if the coordinate of C is between those of A and B.

Vaughan-Szabo use a precedence relation for order. This relation is defined by using the "sense" of a vector $\vec{a}$, $[\vec{a}]+$, which is defined as all non-zero vectors, $\vec{x}$, for which there exists a positive real

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[12]For example, see Prenewitz and Jordan, p. 4.

[13]Prenowitz and Jordan, p. 184.

[14]Hilbert, The Foundations of Geometry.

[15]Birkhoff, Annals of Mathematics (1932), pp. 329-32.

number x such that $\vec{x} = \vec{a}x$. Using [a]+, they define "a half line." This definition leads to a theorem relating the reals and points on a half line as follows; $\overrightarrow{PQ}$ is the set of all X for which there exists a real number greater than zero such that $X = P + (Q - P)x$. This essentially describes the condition necessary for being on one side of a point.

Although the question of order must be considered in proofs, the topic yields very few interesting geometric theorems, with the exception of one rather interesting group, the separation theorems. This is that group of theorems that must have seemed so obvious to Euclid that he neglected to discuss them. Two examples of this kind of theorem are "If a line intersects one side of a triangle, not at a vertex, then it must intersect one of the other sides of the triangle" and the "Crossbar Theorem," which says, "If a ray is in the interior of an angle of a triangle, then it must intersect the opposite side of the triangle." It is this type of theorem that is needed to remove some of the logical defects in Euclid's geometry.[16] In the synthetic and transformation approaches, these theorems are necessary to the proofs of several theorems. For instance, the so-called "Crossbar" theorem is used in some proofs of the theorem that the base angles of an isosceles triangle are congruent.

The proofs of the separation theorems found in the synthetic and transformation programs are founded on a postulate called the "plane separation" postulate. Intuitively, it says that a line in a plane divides the plane into three convex sets; the set of points on either

[16]For an example of a proof which neglects the separation properties see Prenowitz and Jordan, p. 4.

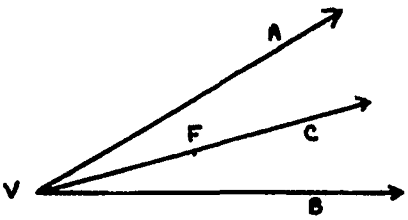
side of the line and the line itself. The term "convex" is used to describe the property that these sets must possess. A set is a convex set if for every pair of points, P and Q, in the set the segment $\overline{PQ}$ is included in the set. Both Coxford-Usiskin and Moise-Downs call these sets "half-planes." Both texts include the Plane Separation postulate which states that for any line in a plane, the points in the plane, not on the line, form two convex sets and if P is in one set and Q in the other, then $\overline{PQ}$ intersects the line.

In the vector approach the separation theorems can be proved from the properties of translations. The one property that seems essential is the necessary and sufficient condition that a point be an element of a ray: $\overrightarrow{AB} = \{X : \exists \; x \geq o, X = A + (B - A)x\}$.

There is such a significant difference in the manner of proof of separation theorems in the vector approach as compared to the synthetic and transformation approaches that the proofs of two "common" theorems are included to illustrate the difference. The proofs will be given using synthetic methods first. The first theorem states that "If a point C lies in the interior of $\angle$AVB, then all of $\overrightarrow{VC}$, except V, is in the interior of $\angle$AVB." By definition, point D is in the interior of $\angle$ABC if D is on the same side of $\overleftrightarrow{BA}$ as C and on the same side of $\overleftrightarrow{BC}$ as A.

Given: C is in the interior of $\angle$AVB.

Prove: $\overrightarrow{VC}$ - V is in the interior of $\angle$AVB.



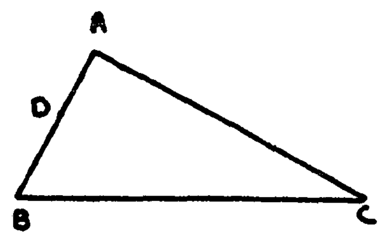
For ray $\overrightarrow{VC}$ to be in the interior of $\angle$AVB, we need to show that

$\overrightarrow{VC}$ - V lies on the same side of $\overleftrightarrow{VB}$ as A and on the same side of $\overleftrightarrow{VA}$ as B. So assume that F is some point of $\overrightarrow{VC}$ and that F and C are on opposite sides of $\overleftrightarrow{VB}$. By the separation axiom, $\overline{FC}$ intersects $\overleftrightarrow{VB}$. Since $\overline{FC} \subset \overleftrightarrow{VC}$ and $\overleftrightarrow{VC}$ intersects $\overleftrightarrow{VB}$ at only V, then V must be between F and C. But this contradicts the definition that $F \in \overrightarrow{VC}$, since the vertex of the ray, V, cannot be between any of the points of the ray. Thus, our assumption is wrong and any point of $\overrightarrow{VC}$ is on the same side of $\overleftrightarrow{VB}$ as C. The proof can be completed by using the same type of argument to show that $\overrightarrow{VC}$ is on the same side of $\overleftrightarrow{VA}$ as B.

The second theorem says that "Every side of a triangle lies, except for its end points, in the interior of the opposite angle." This theorem is one that is used in the proof of the SSS congruence theorem.

Given: $\triangle ABC$

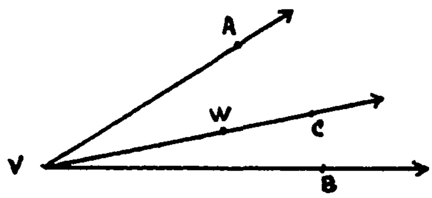
Prove: Any point of $\overline{AB}$, except A and B, is in the interior of $\angle ACB$.



Let D be any point of $\overline{AB}$ except A and B. By definition D is between A and B and thus C is not on $\overline{AD}$. Since $\overline{AD} \subset \overleftrightarrow{AD}$ and $\overleftrightarrow{AD} \cap \overleftrightarrow{CB} = B$, then $\overline{AD}$ cannot intersect $\overleftrightarrow{CB}$. Thus, by the separation axiom, A and D are on the same side of $\overleftrightarrow{CB}$. The same type of argument can be used to show that B and D are on the same side of $\overleftrightarrow{AC}$. Hence, D is in the interior of $\angle ACB$.

Notice that the key to both of these proofs is the plane separation axiom.

Now consider the proofs of these same two theorems from the vector standpoint.

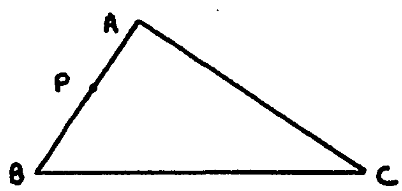


Given: C is in the interior of $\angle AVB$

Prove: $\overrightarrow{VC} - V$ is in the interior of $\angle AVB$

To begin with, a point is in the interior of $\angle AVB$ if and only if there exist points X and Y on $\overrightarrow{VA}$ and $\overrightarrow{VB}$ respectively such that $C - V = (X - V) + (Y - V)$. A statement equivalent to this is verified in the following statements. If X is on $\overrightarrow{VA}$, then there exists $x > o$ such that $X = V + (A - V)x$ and Y on $\overrightarrow{VB}$ implies there exists $y > o$ such that $Y = V + (B - V)y$. Hence, C in the interior of $\angle AVB$ if and only if there exists $x, y > o$ such that $C - V = (X - V) + (Y - V) = (A - V)x + (B - V)y$.

Now consider W a point on $\overrightarrow{VC}$. By definition, there exists $w > o$ such that $W = V + (C - V)w$. Hence, $(W - V)1/w = C - V$. Using the fact that C is in the interior of $\angle AVB$ we know there exists $x, y > o$ such that $C - V = (A - V)x + (B - V)y$, which upon substitution yields $(W - V)1/w = (A - V)x + (B - V)y$ or equivalently $(W - V) = (A - V)wx + (B - V)wy$. But this equation is a sufficient condition for W to be in the interior of $\angle AVB$. The proof of the second theorem is similar in style.



Given: $\triangle ABC$

Prove: Any point on segment $\overline{AB}$, except A and B, is in the interior of $\angle ACB$.

Let P by any point of $\overline{AB}$, except A or B. By the definition of segment $\overline{AB}$, P is on $\overline{AB}$ if and only if there exists x such that $0 < x < 1$ and $P = A + (B - A)x$ But

$$\begin{aligned} P - C &= (A - C) + ((B - C) - (A - C))x \\ &= A - C + (B - C)x - (A - C)x \\ &= (A - C)(1 - x) + (B - C)x \end{aligned}$$

But $P - C = (A - C)(1 - x) + (B - C)x$ is a sufficient condition for P to be interior to $\angle ACB$.

As can be seen, the proofs of these two theorems are algebraic in nature. Several steps in the proofs require that equations be manipulated to determine certain conditions.

In conclusion, we see that there is a considerable difference between the two approaches taken to separation. First, the synthetic and transformation approaches use a postulate, the plane separation postulate, which describes a general condition for points being in a particular half-plane. Therefore, the proofs in this approach hinge on the relation "in the same half-plane." On the other hand, the vector approach uses some theorems that require the finding of real numbers to determine if points are in particular half-planes. The proofs in this instance are algebraic in form.

Distance

Vaughan-Szabo develop the notion of distance in a different way from Coxford-Usiskin and Moise-Downs. The latter books use the intuitive idea that distance between points can be "measured by a ruler." In order to make this intuitive idea part of their axiomatic structure both books postulate the existence of a "ruler." On the other hand, Vaughan-Szabo base their development of distance on the norm of a vector. The notion of the norm of a vector is intuitively the distance between any two points which determine that vector. However, they desire to state their postulates in terms of points and translations (vectors); thus norm of a vector is defined in terms of an operation on vectors called "dot multiplication."

A closer look at both methods of developing the concept of distance reveals that each approach uses the real numbers. Both Coxford-Usiskin and Moise-Downs postulate that to every pair of points there corresponds a non-negative real number. Vaughan-Szabo postulates that $\vec{a} \cdot \vec{a}$, the dot product of $\vec{a}$ and $\vec{a}$, is a real number.

Coxford-Usiskin and Moise-Downs postulate that the points on any line can be placed in a one-to-one correspondence with the real numbers in such a way that if you have any two points, A and B, then A can be made to correspond to zero and B with a positive real number. The one-to-one correspondence is the basis for what we term a "coordinate system" or "coordinatizing a line." The latter part is referred to in Moise-Downs as "Ruler Placement" and enables us to establish a coordinate system anywhere on the line. The distance between any two points is then defined as the absolute value of the difference of their coordinates.

In the vector treatment the dot product of two vectors is a real number. From this they define the norm of a vector, $\|\vec{a}\|$, to be $\sqrt{\vec{a} \cdot \vec{a}}$. The motivation behind defining the norm in terms of the dot product does not seem to be clearly evident. In their discussion they develop dot product by looking at the projection of one vector on another. This in turn leads to an intuitive discussion of orthogonal components. From some of the properties of orthogonal components such as comp $_c$ $(\vec{a} + \vec{b})$ = comp $_c \vec{a}$ + comp $_c \vec{b}$, they suggest that orthogonal "componenting" of vectors is similar to division. For instance, the property mentioned above corresponds to $(a + b) \div c = (a \div c) + (b \div c)$. If this be the case, then using the real numbers as a guide there should be an operation with vectors similar to multiplication of reals. With the property of reals $a \cdot b = (a \div b) \cdot b^2$ in mind they would like to have the relation $\vec{a} \cdot \vec{b}$ = comp $_b \vec{a} \cdot \|\vec{b}\|^2$. To use this definition they would need to postulate some properties of the norm. Instead, they chose to postulate the properties of the dot product, which are given in five postulates. Once the norm of a vector is defined, then the distance between two points is defined as the norm of the translation determined by those two points, that is $d(A, B) = \|B - A\|$.

The concept of distance is introduced much earlier in Coxford-Usiskin and Moise-Downs than it is in Vaughan-Szabo. Coxford-Usiskin and Moise-Downs include this in their first postulates because much of the later work depends on distance. As has been mentioned, both books define betweenness of points in terms of distance. Another important use of the concept of distance is in defining congruent segments. In the vector material, distance is brought up rather late in the course. Their first volume contains a development of the affine properties. In the second volume they discuss perpendicularity and distance.

The idea of distance defined in terms of the norm of a vector leads to some different types of proofs for some familiar theorems pertaining to lengths of sides of polygons. For example, consider the proof of the Pythagorean theorem. The proof of this theorem is usually delayed in the synthetic approach until similarity is discussed. However, in the vector approach it can be proved using a proof based on properties of the dot product.

"If $\triangle ABC$ is a right triangle, then the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides. $AB^2 = AC^2 + BC^2$"

In $\triangle ABC$ let $A - B = \vec{c}$, $B - C = \vec{a}$ and $C - A = \vec{b}$. Since these three points determine a triangle, then $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

$\triangle ABC$ is a right triangle; so call $\angle C$ the right angle. From this fact we have $\overleftrightarrow{AC}$ perpendicular to $\overleftrightarrow{CB}$; consequently, $\vec{a} \cdot \vec{b} = 0$. Since the length of the segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ is the same as those of $\|\vec{c}\|$, $\|\vec{b}\|$ and $\|\vec{a}\|$ respectively, then norms can be used to prove our theorem. From the definition of norms we have $\|\vec{a}\|^2 = \vec{a} \cdot \vec{a}$.

$\vec{a} + \vec{b} + \vec{c} = \vec{0}$ implies

$\vec{c} = -(\vec{a} + \vec{b})$ so

$\|\vec{c}\| = \|-(\vec{a} + \vec{b})\| = \|\vec{a} + \vec{b}\|$.

$\vec{c} \cdot \vec{c} = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + 2\vec{a} \cdot \vec{b}$

Using the fact that $\vec{a} \cdot \vec{b} = 0$, we have

$\vec{c} \cdot \vec{c} = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b}$ or $\|\vec{c}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2$

or $AB^2 = BC^2 + AC^2$

Summary

It can be seen from the discussion in Chapter Two that the synthetic and transformation approaches are essentially the same with respect to incidence, order and distance. It is also evident that the vector approach differs greatly from the other two with respect to these topics. It has been pointed out that the difference is not only in content but also in the general manner in which proofs are presented. Certainly the sequence of topics is different between the vector and the other two, although for the most part the vector approach does include those theorems which are normally found in Euclidean plane geometry.

CHAPTER III

CONGRUENCE

The Role of Congruence

Congruence, particularly congruence of triangles, is one of the central ideas of the geometry of Euclid. The proofs of many of his theorems relied on corresponding parts of congruent triangles. The importance of congruence to the geometry curriculum of today has not diminished. In the present day curriculum a considerable amount of time is spent studying congruent triangles, and they are used in the proofs of theorems throughout the year's study of geometry. In all probability, the topic most associated with geometry by high school students would be congruent triangles.

Until recently, congruence had been treated from the standpoint of superposition. If two sets of points would "fit," one on top of the other, then they were congruent. Schuster comments,

> Virtually all the high school textbooks up to 1955 discussed congruence in terms of superposition, as did Euclid. Superposition is unsatisfactory as a logical basis for the notion of congruence. First of all, it is a physical notion not a mathematical one.[1]

Intuitively, the notion of superimposing one triangle onto another is

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[1]Schuster, Geometry in the Secondary School, p. 30.

fine, but in the traditional treatments of geometry there were no provisions in the postulational system to move one triangle on another. Golos points out:

> It has been used and misused ever since and with much more frequency than Euclid would have favored. Euclid himself does not use this technique very often but, evidently, saw no way to prove the fourth theorem (SAS) without it. 2

During the last decade, objections to the lack of mathematical foundation for congruence as presented in the high school curriculum have brought about a rethinking of how it could be presented. The School Mathematics Study Group, whose work has influenced many textbook writers, worked on this problem in the late fifties and early sixties. Generally, they adopted a scheme which had been proposed earlier by Birkhoff, and have used this scheme in their book Geometry.[3] Birkhoff's plan was to base congruence on the metric properties of segments and angles, since congruence intuitively implies the same size and shape. By using measurement as a key concept they could remove the difficulties inherent in the notion of superposition. For the most part, the geometry books now available use some type of development similar to that of SMSG.

Within the last decade there has been renewed interest in the United States in the idea of superposition, but on a postulational basis. The idea is rooted in the work on transformations done by Klein. This approach to geometry has been relatively popular in some European

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[2]Golos, Ellery B., Foundations of Euclidean and Non-Euclidean Geometry (New York: Holt, Rinehart and Winston, Inc., 1968), p. 56.

[3]School Mathematics Study Group.

curriculums and has come to be known as transformation geometry. The foundation of this approach is distance-preserving mappings, called isometries. In this type of development, sets of points are congruent if there is some distance-preserving mapping which maps one set to the other.

Congruence is a relatively minor topic in the vector approach as presented by Vaughan-Szabo. Most theorems are proved without using congruent sets. Choquet, one of the main proponents of the vector approach, does not mention congruence in his book, Geometry in a Modern Setting.[4] He even suggests in the preface that the use of triangles has hindered the development of vector techniques in geometry. However, Vaughan-Szabo do include a section on congruence; but for the most part, the use of congruence is not used in the overall development. Their discussion is basically transformational in nature, and can be derived quite naturally from translations.

The work in Chapter Two of this study indicated that the synthetic and transformation approaches were quite similar with regard to incidence, order, etc., and that both differed considerably from the vector approach. Chapter Three points out that the synthetic and transformation approaches differ in their postulational basis for congruence and that the vector approach does not stress the importance of congruence as do the other two approaches.

The Concept of Congruence

Moise-Downs approaches congruence through the use of measurement.

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[4]Choquet, Geometry in a Modern Setting, 1969.

This is opposite to the traditional idea where measurement was defined in terms of congruence. The first time the term is used in their book is in the section on angular measure, where they say, "Two angles with the same measure are called congruent."[5] Segments are congruent if they have the same measure or length. The use of measurement to determine congruence of angles and segments seems to be quite natural. Intuitively, it seems logical that two segments would have to be of the same length if one is to be a "replica" of the other. For this reason, this approach to congruence is often called the "metric approach."

Mathematically, Moise-Downs have used the same concept of congruence as have the other two texts except on a more limited scale. In the beginning they have restricted the notion of congruence to segments, angles and triangles. Later they include circles and arcs of circles. The concepts are similar in that Moise-Downs uses distance-preserving mappings to define congruent triangles. However, the mappings they use are not mappings of all the points in a plane but rather mappings of the vertices of one triangle to the vertices of another. Moise-Downs refers to this kind of mapping as a one-to-one correspondence of vertices.

A one-to-one correspondence of vertices of a triangle is called a congruence if it satisfies two conditions. First, the distances between the vertices must be the same as the distances between their images; and second, the measures of the three angles formed by the vertices must be the same as the measures of the angles formed by their images.

As mentioned before, the transformation approach is an attempt to

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[5]Moise-Downs, p. 105.

put the idea of superposition on a mathematical rather than physical basis. The technique for doing this is to use mappings of points in a plane. More specifically, it uses those mappings, called isometries, which have the property that they preserve distance. The term "distance-preserving" means that the distance between any two points is the same as the distance between the images of those two points. Coxford-Usiskin introduce reflection in a line as the first example of a distance-preserving mapping. To reflect the plane in a line, each point is mapped to a point which may be thought of as "in the same position" on the other side of the line. Precisely, this can be defined in terms of perpendicular lines and distances. Thus, if A is not on the line, then A' is the image of A if the line is the perpendicular bisector of AA'. If A is an element of the line, then the image of A is itself.

It can ultimately be shown that any distance-preserving mapping is the composition of a number of reflections; hence, only the properties of the reflection need to be studied. Some of the properties of reflections that are postulated are preservation of distance, angle measure, collinearity, betweenness and orientation of vertices of polygons. Building on these properties, Coxford-Usiskin define sets to be congruent if there is an isometry which maps one set to the other.

One way to view a mapping of the points of a plane is to think of the mapping as "moving" the point to its image. Thus, if one set of points is to be congruent to another, then there should be some way to map the plane onto itself so that the image of the first set would coincide with the second set of points. Intuitively, you could think of this as having the two sets of points on separate planes parallel to each other, and then sliding one plane around until the first set "fits" on top of the other. It is in this manner that the transformation approach is related to superposition. Schuster says, "The notion of an isometry is a mathematical abstraction of the idea of superposition, but as a mathematical abstraction it has many advantages."[6]

In general, the use of such mappings does not require that you actually specify an isometry which will transform one set into another, but rather just prove the existence of one. Usually, problems that may be solved by using congruence are stated in such a way that the existence of an isometry is guaranteed by the hypothesis. Quite often there will be only enough given to say, "Let t be the isometry that will map $\overline{AB}$ to $\overline{CD}$, whatever that isometry may be." For instance, in proving the SAS congruence theorem the hypothesis of the theorem assumes enough information to prove the existence of an isometry which will map the one triangle to the other regardless of how the two are situated. On the other hand, in proving "the base angles of an isoscàles triangle are congruent" a specific reflection is required--the reflection about the median to the base.

The concept of congruence in the vector approach is the same as that in the transformation approach. The primary difference lies not in the concept, but in the postulational structure required and the importance placed on congruence in the overall development.

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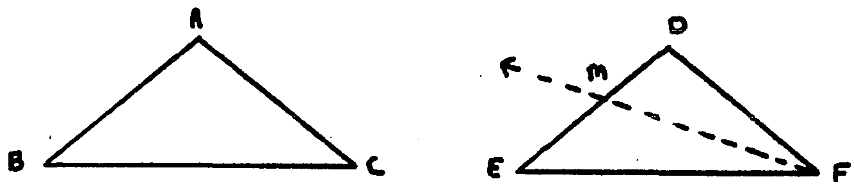
[6]Schuster, Geometry in the Secondary School, p. 33.

Synthetic Development

It was previously mentioned that the idea of congruence in the synthetic approach depends heavily on measurement. Both congruent angles and congruent segments are defined as those whose measures are the same. Thus, for the most part, the postulational basis for congruence rests on the postulates permitting measurement of segments and angles. Postulate One sets up a positive number to be associated with every pair of distinct points. The number is called the distance between the points. The second postulate says that all the points on a line can be put into a one-to-one correspondence with the real numbers; thus, to find the number associated with any two points, one must find the absolute value of the difference of the numbers corresponding to them. The third postulate is regarded by Moise-Downs as the Ruler Placement Postulate and states that for any two given points, one point can be associated with zero and the other with a positive number. This postulate allows the coordinate system to be placed anywhere on the line. Since there is an infinite number of ways to associate the real numbers with points on a line, the third postulate states that for any two given points, one can be associated with zero and the other with a positive number.

Postulates Eleven, Twelve, Thirteen and Fourteen constitute a group of postulates that are sometimes referred to as the "protractor" postulates. This group of postulates establishes properties used in the measurement of angles. Postulate Eleven states every angle is associated with a real number between 0 and 180. Postulate Twelve is referred to as the Angle Construction Postulate and Postulate Thirteen prevides for, in certain instances, the addition of the measures of two angles. Postulate Thirteen is named the Angle Addition Postulate. The last postulate in this group establishes the measure of a "straight" angle to be 180.

Congruent triangles are two triangles for which there exists a congruence. By a congruence, Moise-Downs mean a one-to-one correspondence of the vertices of the two triangles in which the corresponding segments and angles are congruent. At this point, they introduce, as a postulate, the minimum number of parts that must be congruent in two triangles in order that the two triangles be congruent. This is, of course, the "side-angle-side" postulate (SAS). Moise-Downs also state "angle-side-angle" (ASA) and "side-side-side" (SSS) as postulates; however, they later show that these statements can be proved as theorems. For completeness, we will assume that they are not postulates and discuss their proof.

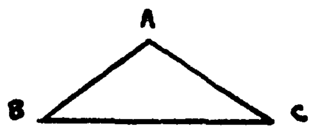


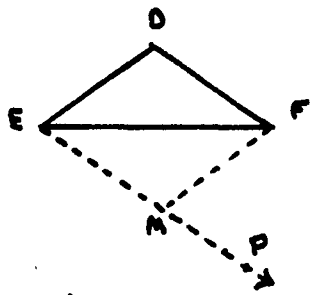
Given: $\overline{BC} \cong \overline{EF}$; $\angle ACB \cong \angle DFE$; $\angle ABC \cong \angle DEF$

Prove: $\triangle ABC \cong \triangle DEF$

By the ruler postulate select the point, M, on $\overrightarrow{ED}$ such that $\overline{BA} \cong \overline{EM}$. Thus, $\triangle ABC \cong \triangle MEF$ by SAS. Since the triangles are congruent, $\angle ACB \cong \angle MFE$, which in turn implies that $\angle DFE \cong \angle MFE$. By the Angle Construction Postulate there is only one point on $\overline{DE}$ such that F is the vertex of the angle, $\overrightarrow{FE}$ is one side and it has measure equal to $m\angle ACB$. But, points M and D both satisfy these conditions; hence, $M = D$ and $\triangle ABC \cong \triangle DEF$.

The proof of this theorem rests on three postulates. First is the Ruler Postulate which allows the selection of point M. Second is the SAS congruence postulate and the third is the Angle Construction Postulate. This theorem illustrates the extensive use of measurement in the treatment of Moise-Downs. The SSS congruence theorem uses these same postulates as well as "the base angles of an isosceles triangle are congruent." The later theorem is an easy result of SAS.





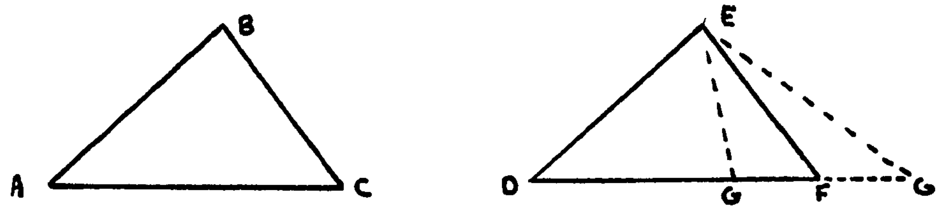
Given: $\overline{AB} \cong \overline{DE}$; $\overline{BC} \cong \overline{EF}$; $\overline{AC} \cong \overline{DF}$

Prove: $\triangle ABC \cong \triangle DEF$

Using the Angle Construction Postulate with $\overrightarrow{EF}$, select a point P such that $\angle ABC \cong \angle FEP$ and P is in the half-plane determined by $\overleftrightarrow{EF}$ and not containing point D. Using the Ruler Postulate, select the point M on $\overrightarrow{EP}$ such that $\overline{EM} \cong \overline{AB}$. Thus, $\triangle MEF \cong \triangle ABC$ by SAS. Consequently, $\overline{ME} \cong \overline{AB} \cong \overline{DE}$; $\overline{MF} \cong \overline{AC} \cong \overline{DF}$; so $\triangle DFM$ and $\triangle DEM$ are isosceles triangles. By the base angles theorem, $\angle DMF \cong \angle MDF$ and $\angle DME \cong \angle MDE$, and hence by the Angle Addition Postulate $\angle EDF \cong \angle EMF$. This yields $\triangle DEF \cong \triangle MEF$ by SAS, from which we conclude $\triangle ABC \cong \triangle MEF \cong \triangle DEF$.

In the proofs of both the ASA and SSS congruence theorems, the authors have assumed that the relation of congruence is an equivalence relation. For example, in the proof of SSS just presented, they use the fact that if $\overline{ME} \cong \overline{AB}$ and $\overline{AB} \cong \overline{DE}$, then $\overline{ME} \cong \overline{DE}$, thus assuming that "$\cong$" has the transitive property. In their development of congruence, Moise-Downs does not mention that the congruence relation is an equivalence relation.

Once the SAS congruence postulate is available, theorems pertaining to geometric inequalities can be proved. The Exterior Angle theorem is a key theorem in this group. This theorem states that an exterior angle of a triangle is greater than either of the remote interior angles. The Exterior Angle theorem provides a basis for the proof of SAA which traditionally was delayed until the parallel postulate was available.



Given: $\triangle ABC$ and $\triangle DEF$; $\overline{AB} \cong \overline{DE}$, $\angle BAC \cong \angle EDF$, $\angle BCA \cong \angle EFD$.

Prove: $\triangle ABC \cong \triangle DEF$.

Let G be a point of $\overrightarrow{DF}$, such that $\overline{DG} \cong \overline{AC}$. By SAS we have $\triangle ABC \cong \triangle DEG$. Therefore, $\angle G \cong \angle C \cong \angle F$. But we must have 1) F between D and G, 2) G between D and F, or 3) G = F. If F is between D and G, then $\angle F$ is an exterior angle of $\triangle EFG$, and thus $\angle F > \angle G$, which is false. If G is between D and F, then $\angle G$ is an exterior angle of $\triangle EFG$, and $\angle G > \angle F$, which is false. Thus, by elimination, G = F, and $\triangle ABC \cong \triangle DEF$.

From the preceding proofs, we see that the postulational basis for congruence in the synthetic approach is the measurement of properties vested in the ruler and protractor postulates and the SAS triangle congruence postulate.

Transformation Development

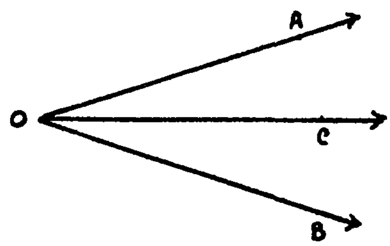
In the transformation approach, two sets are congruent if there is an isometry which maps one set to another. The reflection is the building block for all isometries. This observation is not mentioned by the authors until later in the course; but they imply that reflections are important by the fact that they postulate their properties. Since they have defined sets as being congruent if one is the image of the other under an isometry, the reflection postulate serves the role of establishing the existence and characteristics of one such type of mapping. The Reflection Postulate, which has five parts, provides the basis for all the congruence theorems. The first part is that every point of a plane has an image under reflection. The second part is that lines are reflected into lines. The fact that reflections preserve betweenness and distance are the third and fourth parts, respectively. The last part of the Reflection Postulate states that angle measure is preserved by reflections. A closer look at this postulate reveals that it says "if you move sets around in the right way the properties of size and shape are preserved."

Another postulate pertaining to reflections is needed. This postulate states that reflections reverse the orientation of the vertices of any polygon from clockwise to counterclockwise or vice versa.

A group of theorems that are quite obvious, yet are important to the further development of congruence theorems, relate to the reflection image of sets of collinear points. For example, if A and B determine a line, then the image of $\overleftrightarrow{AB}$ is the line determined by r(A) and r(B). Similar theorems are true for $\overline{AB}$, $\overrightarrow{AB}$ and $\overrightarrow{BA}$. An extension of this idea is the theorem which says that if $r(P_1) = P_1'$, $r(P_2) = P_2'$, etc., then if $P_1P_2 \ldots P_n$ is a polygon with vertices $P_1P_2 \ldots P_n$, then $r(P_1P_2 \ldots P_n) = P_1'P_2' \ldots P_n'$.

Before discussing the concept of congruence, Coxford-Usiskin introduce the notion of the composition of reflections. The composition of two mappings is another mapping and is simply the first mapping followed by the second. They then define the composition of r_m and r_l, $r_m \circ r_l$, to be a translation if line m is parallel to line l, and to be a rotation if m is not parallel to l. Since reflections preserve a number of properties, and since translations and rotations are the composition of reflections, then they preserve these same properties. The authors then define those transformations that are reflections or the composition of reflections as isometries. Later in the text they introduce the glide reflection as the composition of a reflection and a translation. Thus, they have four basic types of isometries--reflections, translations, rotations, glide reflections. At this point they do not show that every distance-preserving mapping has to be one of these four types; but in the last chapter they do present this notion. Hence, they are consistent in defining isometries as reflections or the composition of reflections. After defining reflections, postulating their properties and defining isometries, they then define two sets as being congruent if there exists an isometry which maps one set to the other. The reflexive, symmetric and transitive properties of congruence can be proved by using the properties of reflections, the properties of composition of mappings and the fact that isometries have inverses.

A concept that plays a very important role in the development of transformation geometry and is not used to any extent in the synthetic development is symmetry. Intuitively, a set is reflection-symmetric if there is a line which divides it into two halves that are identical (for example, the diameter of a circle). A set is reflection-symmetric if there is a reflecting line about which the set and its image coincide. A theorem which is used extensively states that the angle bisector is a line of symmetry of an angle.

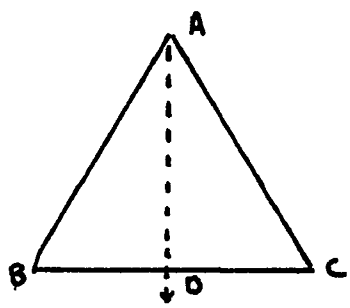


Given: $\angle AOB$ with $\overrightarrow{OC}$ the bisector of $\angle AOB$.

Prove: $r(\overrightarrow{OA}) = \overrightarrow{OB}$, where r is the reflection about $\overrightarrow{OC}$.

Let $r(A) = A'$. We know $r(O) = O$ and $r(C) = C$; hence, $r(\angle AOC) = \angle A'OC$. Since $m\angle AOC = m\angle BOC$, we have $m\angle A'OC = m\angle BOC$. According to the angle measurement postulate, there is only one ray such that $\angle BOC$ has a given measure; thus, $\overrightarrow{OA'} = \overrightarrow{OB}$ and $r(\overrightarrow{OA}) = \overrightarrow{OB}$. We can similarly show that $r(\overrightarrow{OB}) = \overrightarrow{OA}$.

As an illustration of how this concept can be used, consider the proof of the theorem, "The base angles of an isosceles triangle are congruent."



Given: $\triangle ABC$ isosceles with $AB = AC$.

Prove: $\angle ABC \cong \angle ACB$.

Let D be the point of intersection of $\overline{BC}$ and the angle bisector of $\angle BAC$. Call r the reflection about the line $\overleftrightarrow{AD}$. Now let $r(B) = B'$. We know that $r(\overline{AB}) = \overline{AC}$, $r(A) = A$ and that $AB = AB'$. Since $AB' = AB = AC$, then by the Ruler Postulate $B' = C$. This gives $r(\triangle ABC) = \triangle ACB$ and hence $\triangle ABC \cong \triangle ACB$. But if the two triangles are congruent then $\angle ABC \cong \angle ACB$.

This proof is similar to that used by Moise-Downs except that Coxford-Usiskin have made symmetry an integral part of their proof.

Symmetry is used also with quadrilaterals that have the reflection-symmetric property. A quadrilateral which has two distinct pairs of adjacent sides congruent is called a kite. Kites have a line of symmetry which is one of the diagonals. This fact can be used to prove several geometric facts relating to quadrilaterals, particularly rhombuses, as well as to prove the SSS congruence theorem.

The proofs of the familiar triangle congruence theorems given by Coxford-Usiskin illustrate two different styles. In the proof of the SSS theorem they prove the existence of an isometry which maps one triangle to the other. In the proof of the SAS theorem they show how to construct the isometry and use the properties of those specific mappings to prove the two triangles congruent. The proof of SSS is reminiscent of the technique used by Euclid in that they simply say that a triangle can be moved without showing how to do it. The authors, of course, have postulational foundation for the existence of such mappings, whereas Euclid did not. The proofs will be presented to illustrate two different styles of proof used by Coxford-Usiskin.

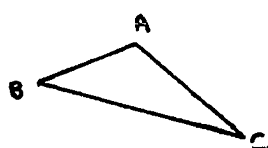
Given: $\overline{AB} \cong \overline{DE}$; $\overline{BC} \cong \overline{EF}$ and $\overline{AC} \cong \overline{DF}$

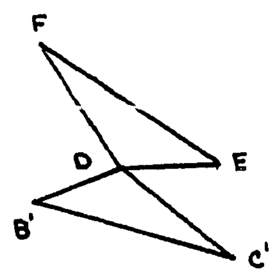
Prove: $\triangle ABC \cong \triangle DEF$

Since $\overline{BC} \cong \overline{EF}$, there exists an isometry t, such that t($\overline{BC}$) = $\overline{EF}$, with t(B) = E and t(C) = F. Let t(A) = A'. If A' is in the same half-plane formed by $\overleftrightarrow{EF}$ with D, then let r be a reflection about $\overleftrightarrow{EF}$. Hence, r o t is an isometry. We then have $\overline{AB} \cong \overline{A'E} \cong \overline{DE}$ and $\overline{AC} \cong \overline{A'E} \cong \overline{DE}$. Thus, DEA'F is a kite with line $\overleftrightarrow{EF}$ as a line of symmetry. Consequently, $\triangle DEF \cong \triangle A'EF$. This gives $\triangle ABC \cong \triangle A'EF \cong \triangle DEF$.

A key to the proof is simply the existence of an isometry which maps $\overline{BC}$ to $\overline{EF}$; yet the specific isometry is not used.

One of the strong points of transformation geometry is that there is a postulational foundation for "moving" one triangle to fit on another. So that this notion might be presented, the proof of the SAS theorem is given. The proof is given for the case where the triangles have the same orientation.





Given: $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, $\angle BAC \cong \angle EDF$

Prove: $\triangle ABC \cong \triangle DEF$

Let T be the translation that maps A to D. Call T(B) = B' and T(C) = C'; thus T($\triangle ABC$) = $\triangle DB'C'$. From the previous statement we have $\angle BAC \cong \angle B'DC'$ and hence $\angle B'DC' \cong \angle EDF$. Also, $\overline{AB} \cong \overline{DB'}$, $\overline{AC} \cong \overline{DC'}$ implies that $\overline{DB'} \cong \overline{DE}$, $\overline{DC'} \cong \overline{DF}$.

Let R be the rotation through an angle of measure equal to

$m\angle B'DE$ with center D. We have that $R(\overrightarrow{DB'}) = \overrightarrow{DE}$. Since R preserves distance and $\overline{DB'} \cong \overline{DE}$, we find that $R(D) = D$ and $R(B') = E$. Using $\angle B'DC \cong \angle EDF$ and $m\angle B'DC' + m\angle C'DE = m\angle B'DE$, we substitute and find $m\angle EDF + m\angle C'DE = m\angle B'DE$ or

$$m\angle C'DF = m\angle B'DE.$$

From this we conclude that $R(\overrightarrow{DC'}) = \overrightarrow{DF}$. Because of the distance-preserving nature of R and $\overline{DC'} \cong \overline{DF}$, we have that $R(C') = F$. Hence, $\triangle DEF \cong \triangle DB'C' \cong \triangle ABC$.

A similar type of proof can be given for ASA. Once the proofs of the congruence theorems have been given, these theorems are used in solving problems in the same manner as the synthetic approach.

From the preceding discussion, it is evident that Coxford-Usiskin make use of the measurement of segments and angles as did Moise-Downs. It should also be clear that Coxford-Usiskin have used a postulate relating to reflections as a replacement for the SAS congruence postulate.

Vector

In their book, Vaughan-Szabo do not emphasize congruence as much as Moise-Downs or Coxford-Usiskin. They present the concept in thirty-three pages of their 1170 page book. One reason for de-emphasizing congruence may be that Vaughan-Szabo have methods other than congruent triangles by which they can prove geometric properties. It appears that they have included congruence because it is such an important concept in other approaches to geometry.

The approach to congruence that Vaughan-Szabo have taken is via isometries. This approach is quite natural, considering that a translation is a special type of isometry. Although congruence is not a major theme in their book, the development should be of interest to the reader of this study because of its contrast to the way in which Coxford-Usiskin develop isometries. In the Coxford-Usiskin development, reflections are used as the basic building blocks of all isometries. Their basic structure is to postulate the necessary properties of reflections. In the Vaughan-Szabo development, translations are used to develop isometries. From the postulates pertaining to translations, they are able to prove those properties of reflections postulated in Coxford-Usiskin.

To begin their discussion of congruence, Vaughan-Szabo define an isometry as a one-to-one mapping f of the plane onto the plane such that $d(f(X), f(Y)) = d(X,Y)$ for all points X and Y. Two sets of points are congruent if there exists an isometry which maps one set to the other. Their first example of an isometry is a translation. This fact, that a translation is an isometry, is easily proved by observing that

$$d(P + \vec{a}, Q + \vec{a}) = \|(Q + a) - (P + a)\| = \|Q - P\| = d(P,Q).$$

There is no difficulty in showing that the composition of any number of translations is an isometry or even the composition of any number of isometries is an isometry. A reflection in a plane is defined to be a mapping f such that

$f(X) = X + (M - X)2$ where M is the foot of the perpendicular from X to the plane.

The concept of a reflection in Vaughan-Szabo is defined as a reflection in a plane but can be easily adapted to a reflection in a line. This definition is motivated by the physical example of reflecting in mirrors. The authors have done enough preliminary work to prove that reflections in planes are isometries of the plane. Since only reflections in a line were dealt with in the transformation approach, only that special case will be presented here. Hence, if the definition of reflection is modified, it becomes $f(X) = M + (M - X)$, where M is the foot of the perpendicular from X to the reflecting line.

From the definition we see

$$f(P) = M + (M - P)$$

$$f(P) = 2\,M - P$$

$$2\,M = f(P) + P$$

$$M = f(P)1/2 + P\,1/2$$

$$M = P + f(P)\,1/2 - P\,1/2$$

$$M = P + (f(P) - P)\,1/2$$

from which we conclude that M is the midpoint of the segment $\overline{P\,f(P)}$. Also, application of the definition tells us that $f(M) = M$.

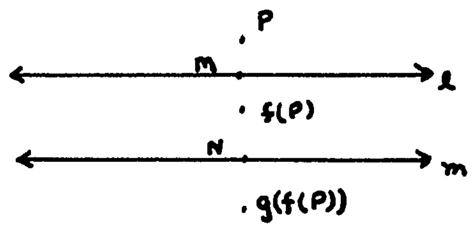
The proof that reflections in a line are isometries follows from several properties of dot multiplication and orthogonal translations.

An interesting theorem which was discussed in Coxford-Usiskin and is noteworthy because it pertains to the relationship between reflections and translations is the one which states that "The composition of two reflections about parallel lines is a translation."

Given: $m \parallel l$

f is a reflection in l

g is a reflection in m



Prove: $g \circ f$ is a translation

$f(P) = M + (M - P)$

$g(f(P)) = M + (N - (M + (M - P))$

$g(f(P)) = N + (N - M) + (P - M)$

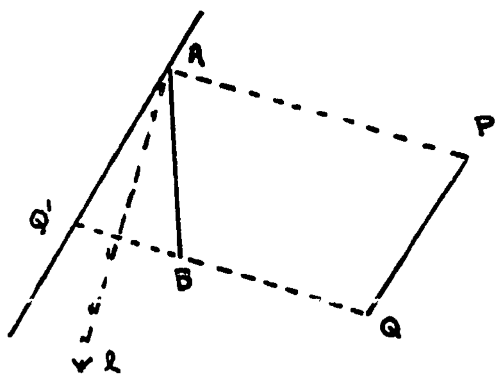
$g(f(P)) = P + (N - M)2$

Hence, $g \circ f$ is $(N - M)2$

The authors use the properties of norms to prove that isometries preserve lines, rays, segments, planes, parallelism and perpendicularity. Once these properties have been established, the next stage is to show that if segments have the same length, then there exists an isometry that maps one to the other. This is followed by establishing a theorem saying that if two triangles are such that their corresponding sides are the same length, then there exists an isometry which maps corresonding vertices to each other.

Given: $AB = PQ$

Prove: There exists an isometry f such that $f(P) = A$ and $f(Q) = B$.



Let $\vec{a} = A - P$ and $Q' = Q + \vec{a}$

Now if $Q' = B$, then a is the desired isometry.

If $Q' \neq B$, let l be the perpendicular bisector of $Q'B$. Define g to be a reflection in line l; hence, $g(Q') = B$.

Since $A \in l$, we have $g(A) = A$.

Let $f = g \circ \vec{a}$. We see that

$f(P) = g(\vec{a}(P)) = g(A) = A$ and

$$f(Q) = g(\vec{a}(Q)) = g(Q') = B.$$

The proof of the triangle theorem is simply an extension of this proof.

Given: A, B, C and D, E, F noncollinear with

$AB = DE$, $BC = EF$ and $AC = DF$

Prove: There exists an isometry f such that $f(D) = A$, $f(E) = B$ and $f(F) = C$.

By the previous theorem there exists an isometry, g, such that $g(D) = A$ and $g(E) = B$. Call $g(F) = F'$. If $F' = C$, then g is the isometry desired. If $F' \neq C$, then let h be the reflection in the perpendicular bisector of $F'C$. It is easily shown that $h \circ g$ is the desired isometry.

This theorem is the proof for the SSS congruence theorem. The verification for SAS is a direction application of SSS and the definition of the cosine of an angle.

Summary

Because of the emphasis that has been placed on congruence throughout the study of geometry, the desire to make it as meaningful as possible seems quite logical. It appears that Moise-Downs and Coxford-Usiskin are representative of two different approaches to the concept of congruence. The former rely heavily on metric properties to form their concept of congruence. The latter base their development on a way of "moving" points so that they coincide with other points.

In the synthetic approach, the "side-angle-side" postulate is the foundation for congruence. From this postulate other triangle congruence theorems can be proved. Generally, congruence in Moise-Downs is discussed only for triangles. However, congruent triangles are used in proving theorems related to parallelism and similarity.

Isometries, distance preserving mappings, are used to develop congruence in the transformation approach. Isometries are defined as the composition of reflections and Coxford-Usiskin show that there are four types of isometries: reflections, rotations, translations, glide reflections. Since each of these is the composition of reflections, the properties of reflections are postulated. Those properties that are postulated are preservation of distance, collinearity, betweenness and angle measure. Using these as the basic assumptions about congruence, Coxford-Usiskin prove the triangle congruence theorems.

Congruence in the vector approach is not as important as in the other two treatments. The topic is presented near the end of the course in Vaughan-Szabo. The development of congruence in the vector approach is similar to that in the transformation approach except that in the vector approach the properties of translations are postulated. From translations Vaughan-Szabo prove that reflections and rotations are isometries. The properties of isometries are then used to prove the triangle congruence theorems.

CHAPTER IV

PARALLELISM AND SIMILARITY

Parallelism

The geometry which has traditionally been taught in the secondary curriculum in the United States has been called Euclidean plane geometry because of the assumption pertaining to parallel lines first made by Euclid. Euclid's Fifth Postulate is the equivalent of the statement that through any point not on a line there is a unique line through the given point parallel to the given line. As mentioned before, mathematicians tried for many years to prove this postulate until, almost simultaneously, three different men showed that geometries exist in which this postulate is not true. These geometries became known as non-Euclidean. However, Euclid's geometry remains as the model of space that is most consistent with our intuition. Moise-Downs and Coxford-Usiskin assume the Parallel Postulate as part of their basic structure. Although Vaughan-Szabo do not assume a parallel postulate, they have made enough assumptions to be able to prove it. Thus, all three of the approaches being reviewed are treatments of Euclidean geometry.

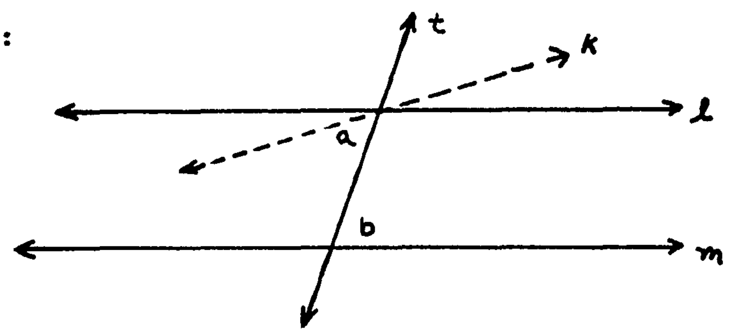
It is evident from the previous paragraph that once again the synthetic and transformation approaches are similar in structure, whereas the vector approach is considerably different. There are, however, several proofs in which the synthetic and transformation approaches differ. The synthetic approach relies on the Angle Construction Postulate and congruent triangles for the foundation theorems, whereas the transformation approach uses properties of reflections and symmetry. In order to contrast these three approaches, each general development of parallelism is given.

Synthetic

By definition, lines in the same plane are parallel if their intersection is empty. This definition is used by both Moise-Downs and Coxford-Usiskin.

Prior to introducing the parallel postulate, Moise-Downs proves some sufficient conditions for lines to be parallel. These conditions are the converse to some theorems which can be proved as a result of the parallel postulate. One of these is "If the alternate interior angles are congruent, then the lines are parallel." The proof of this theorem is based on the Exterior Angle Theorem, which can be proved by congruent triangles and geometric inequalities. Thus, in their development they present parallelism following the chapters on congruence and inequalities.

The converse of this theorem is a result obtainable from the Parallel Postulate which states, "Through a given external point there is only one parallel to a given line." The proof of the Alternate Interior Angles Theorem is as follows:



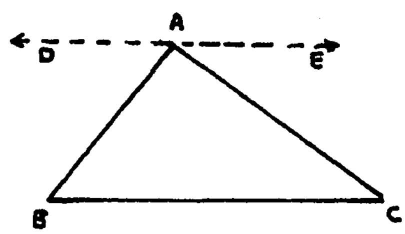
Given: 1 and m parallel with t a transversal

Prove: $\angle a \cong \angle b$

Assume $\angle a$ is not congruent to $\angle b$. By the Angle Construction Postulate there is a ray, consequently a line, k, through point P, which forms $\angle c$ such that $\angle c \cong \angle b$. Since $\angle c$ and $\angle b$ are alternate interior angles, then $k \parallel m$. However, this is a contradiction, because k and 1 both pass through P and are parallel to m. Thus, $\angle a \cong \angle b$.

The proof of the Corresponding Angles Theorems is easily done by using vertical angles and the Alternate Interior Angle Theorem. Both of these theorems are used to verify properties of parallelograms.

Another theorem which is valid as a result of the Parallel Postulate is the one pertaining to the sum of the measure of the three angles of a triangle.



Given: $\triangle ABC$

Prove: $m\angle A + m\angle B + m\angle C = 180$

First construct the line through A which is parallel to $\overleftrightarrow{BC}$ (which is guaranteed by the Parallel Postulate). We know $m\angle DAB + m\angle BAC + m\angle EAC = 180$. By the Alternate Interior Angle Theorem, $m\angle C = m\angle EAC$ and $m\angle DAB = m\angle B$. Thus by substituting

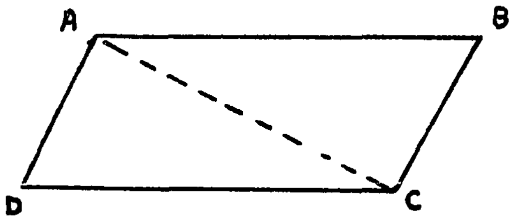
$$m\angle B + m\angle A + m\angle C = 180.$$

In order to illustrate the contrast between the three approaches, a proof of the same theorem from the three texts is presented. Each proof is found in the section which pertains to that particular approach.

There are numerous properties of parallelograms that can be verified with similar proofs. Chapter Nine on parallelograms and their properties follows the work on congruence and parallel lines. The theorem stating that opposite sides of a parallelogram are congruent will be presented.

Given: ABCD is a parallelogram

Prove: AD = BC and AB = DC



Construct the diagonal $\overline{AC}$. Considering $\overleftrightarrow{AC}$ as a transversal, we find that $\angle BAC \cong \angle DCA$ and $\angle DAC \cong \angle BCA$ because they are alternate interior angles formed by parallel lines and a transversal. Hence, $\triangle ADC \cong \triangle CBA$ by ASA. From the congruent triangles, we know that $\overline{AD} \cong \overline{BC}$ and $\overline{AB} \cong \overline{DC}$.

The treatment of parallelism as presented by Moise-Downs is basically the same as that found in most secondary texts used in the United States today.

Transformation

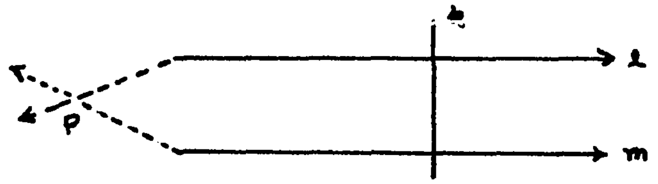
The general plan of development in Coxford-Usiskin is much the same as that found in Moise-Downs. Although the plan is basically the same, the sequence of topics they use is slightly different. Coxford-Usiskin discuss parallel lines and the Parallel Postulate prior to proving the congruence theorems. The proofs of the triangle congruence theorems are not dependent on the Parallel Postulate; hence, it appears that the authors have chosen this order because they wanted to introduce at this time the notion of indirect proof. Some of the important theorems of this section on parallel lines can be proved by the indirect method. Based on the way Coxford-Usiskin prove the properties of parallelograms, it is necessary for them to delay the chapter on parallelograms until after they have done the triangle congruence theorems.

Coxford-Usiskin begin their work on parallel lines by verifying some sufficient conditions for lines to be parallel. The next step is to use these conditions with the Parallel Postulate to prove the Alternate Interior Angle Theorem. Once this theorem is available, the theorems pertaining to parallelograms and the "sum of the angles of a triangle" can be proved. The main difference between the synthetic and transformation approaches lies in the way they prove the sufficient conditions and the Alternate Interior Angle Theorem. The proofs of the key theorems leading to the proof of the Alternate Interior Angle Theorem are given in the study. Again, we will see the use of isometries in the proof of a geometric theorem.

A theorem which describes a sufficient condition for parallel lines is "Two lines perpendicular to the same line are parallel." The proof of this theorem found in Coxford-Usiskin is as follows.

Given: $l \perp k$; $m \perp k$

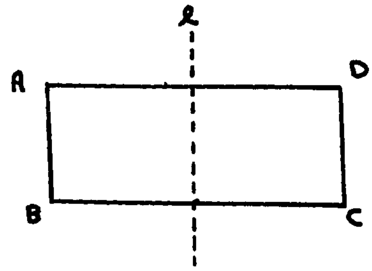
Prove: $l \parallel m$



Assume l is not parallel to m; thus, they intersect at a point, say P. We know P is not a point of k, so $r_k(P) \neq P$, where r_k is a reflection about k. Since $P \in l \cap m$, then $r_k(P)$ belongs to $r_k(l) \cap r_k(m)$. Because l and m are perpendicular to k, $r_k(l) = l$ and $r_k(m) = m$. Thus, we have $l \cap m = P$ and $l \cap m = r_k(P)$ with

$P \neq r_k(P)$. This is a contradiction of the fact that two distinct lines intersect in only one point, hence $l \parallel m$.

The Parallel Postulate in Coxford-Usiskin states that "Through a point not on a line there is exactly one line parallel to the given line." The first theorem the authors prove is: "If a line is perpendicular to one of two parallel lines, it is perpendicular to the other." The proof follows from the Parallel Postulate using an indirect argument. With this theorem available it is possible to prove a theorem relating to the symmetry of a rectangle which is needed in the proof of the Alternate Interior Angle Theorem.

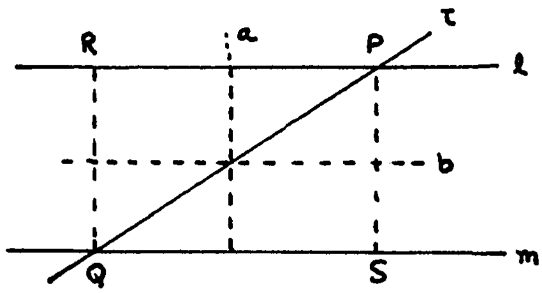


Given: ABCD is a rectangle; l is the perpendicular bisector of $\overline{BC}$

Prove: l is a line of symmetry for ABCD

Since l is the perpendicular bisector of $\overline{BC}$, $r_l(B) = C$ and $r_l(C) = B$. By the theorem just mentioned, $\overleftrightarrow{AB} \parallel l$, so $r_l(\overleftrightarrow{AB}) \parallel l$ and $\overleftrightarrow{DC} \parallel l$. Because $r_l(B) = C$, we know that $r_l(\overleftrightarrow{AB})$ contains C. By the Parallel Postulate there is exactly one line through C parallel to l, so $r_l(\overleftrightarrow{AB}) = \overleftrightarrow{CD}$. We know that $r_l(A) \in \overleftrightarrow{CD}$. Making use of the fact that $\overleftrightarrow{AD} \perp l$, we find that $r_l(\overleftrightarrow{AD}) = \overleftrightarrow{AD}$; hence, $r_l(A) \in \overleftrightarrow{AD}$. $r_l(A) \in \overleftrightarrow{CD}$, $r_l(A) \in \overleftrightarrow{AD}$ and $\overleftrightarrow{CD} \cap \overleftrightarrow{AD} = D$ implies that $r_l(A) = D$. If $r_l(A) = D$, then $r_l(D) = A$ and thus $r_l(ABCD) = DCBA$, making l a line of symmetry.

A key theorem in all the work with parallel lines and parallelograms is the Alternate Interior Angle Theorem. The postulational basis for this theorem is vested in the Parallel Postulate, but does require other theorems to prove it. The proof is as follows:



Given: $l \parallel m$; t a transversal intersecting l at P and Q.

Prove: $\angle P \cong \angle Q$

Construct the perpendicular to m through P and call the point of intersection S. Likewise, locate point R. Since $l \parallel m$ and $\overleftrightarrow{PS} \perp m$, then $\overleftrightarrow{PS} \perp l$. Similarly, $\overleftrightarrow{RQ} \perp m$. By definition this makes PSQR a rectangle. The perpendicular bisector of the sides of PSQR are lines of symmetry; call them a and b. Hence, $r_a(PSQR) = RQSP$ and $r_b(PSQR) = SPRQ$. Now $r_b(r_a(P)) = r_b(R) = Q$; thus there exists an isometry which maps P to Q and by definition $\angle P \cong \angle Q$.

The remaining development of the topic of parallelism in Coxford-Usiskin is done as it is in Moise-Downs. For instance, the proofs done in the last section relating to the sides of a parallelogram and the sum of the angles of a triangle are exactly the same.

Vector

Unlike the other two treatments, the vector approach requires no special postulate concerning parallel lines. As a matter of fact, the traditional Parallel Postulate can be proved as a theorem. Since this is drastically different from the other two approaches and the discussion reveals more insight into the significance of the concept of linear dependence, the development will be presented in some detail. The general theory is based on the direction of translations. The definition of parallel lines in the vector approach is also slightly different from that found in Moise-Downs and Coxford-Usiskin. Vaughan-Szabo define lines to be parallel if they have the same direction. The condition that distinct parallel lines have an empty intersection is a theorem in Vaughan-Szabo.

From previous work with lines in the vector approach, we saw that two points, C and D, are on line $\overleftrightarrow{AB}$ if and only if D - C is a scalar multiple of B - A; that is, if and only if $D - C \in [B - A]$. Since any two points determine a line, the direction of that line should be determined by those points. It seems logical to determine the direction of the line by the set of all scalar multiples, [B - A]. Thus, the direction of l, [l], is defined as the class of all translations, $\vec{x}$, where $\vec{x} = Z - Y$ for any points $Y, Z \in l$. We see that if $l = \overleftrightarrow{AB}$, then $[B - A] = \{Z - Y : Z, Y \in l\}$. But this is simply saying that the translation Z - Y is a scalar multiple of B - A. Thus, it seems justifiable to use [B - A] as the notation for both the set of scalar multiples of B - A and the direction of $\overleftrightarrow{AB}$. Notice that this concept is related to points on a specific line.

Now if $\vec{a}$ is any translation with $\vec{a} \neq \vec{0}$, then for any point A, $A + \vec{a}$ and A are two points that determine a line. The direction of this line is $[A + \vec{a} - A]$. The direction of $\vec{a}$, $[\vec{a}]$, is defined to mean $[A + \vec{a} - A]$ for any point A. This definition enables one to refer to the general direction of a translation.

The logical extension of the direction concept is to define a line through a given point in a given direction. Hence, the authors define $A\vec{a} = \{X : X - A \in [\vec{a}]\}$. By definition, $X - A \in [\vec{a}]$ if $X = A + \vec{a}$. This leads to the useful conclusion that $\overleftrightarrow{A[\vec{a}]} = \overleftrightarrow{A(A + \vec{a})} = \overleftrightarrow{AB}$ if $B = A + \vec{a}$. The proof of this theorem is given below.

Since $B = A + \vec{a}$, then $B - A = \vec{a}$. Now $X \in \overleftrightarrow{AB}$ iff $X = A + (B - A)x$

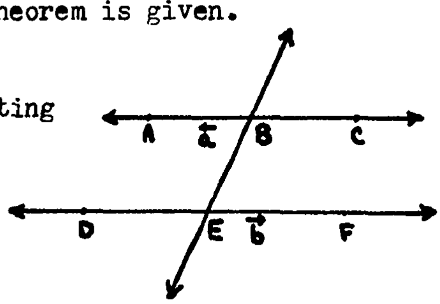
iff $X - A = (B - A)x$ iff $X - A \in [B - A] = [\vec{a}]$.

Suppose now we define $\overleftrightarrow{A[l]} = \{X : X - A \in [l]\}$. This set is a line which contains A and whose direction is the same as l. In this definition, we see that if $A \in m$ and $[m] = [\vec{a}]$, then $m = \overleftrightarrow{A[\vec{a}]}$. Restated this says that if A is on l, then $l = \overleftrightarrow{A[l]}$.

Recalling that two lines, m and l, are parallel if and only if $[m] = [l]$ we can easily prove one of the basic assumptions of Euclidean geometry: "Parallel lines have an empty intersection."

Suppose m and l are distinct parallel lines, with direction of $[\vec{a}]$. Since $m \neq l$, then there exists a point $X \in l$ such that $X \notin m$. Furthermore, let Y be any point of m other than X. Assume that $m \cap l = Z$. From the preceding paragraphs we find that $\overleftrightarrow{XZ} = \overleftrightarrow{X[\vec{a}]} = l$ and $\overleftrightarrow{YZ} = \overleftrightarrow{Y[\vec{a}]} = m$. This implies that $Z = X + \vec{a}r$ and $Z = Y + \vec{a}s$. Hence, $X + \vec{a}r = Y + \vec{a}s$ or $X = Y + \vec{a}(s - r)$. Thus, $X \in m$; which is a contradiction.

A theorem that was significant in the other approaches with regard to parallelism was the Alternate Interior Angle Theorem. This theorem is not essential to the further development in the vector approach, but is included in the vector approach because it represents a standard theorem of plane geometry. In order to contrast the three styles of proof, the verification of this theorem is given.



Given: $l \parallel m$; t a transversal intersecting l and m at points B and E.

Prove: $\angle ABE \cong \angle FEB$

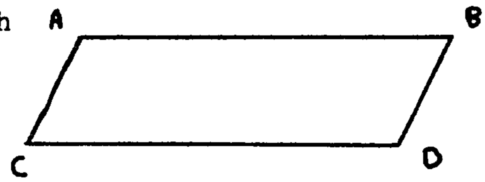
Let $\vec{a}$ and $\vec{b}$ be unit vectors in the direction of A - B and F - E respectively. Since l and m are parallel and $\vec{a}$ and $\vec{b}$ are unit vectors, then $\vec{a} = -\vec{b}$. Now, Cos $\angle ABE = \vec{a} \cdot \vec{c} / \|\vec{c}\|$, where $\vec{c} = B - E$ and Cos $\angle FEB = -\vec{b} \cdot \vec{c} / \|\vec{c}\| = \vec{a} \cdot \vec{c} / \|\vec{c}\| =$ Cos $\angle ABE$.

But if two angles have the same cosine, then they are congruent. Notice that although the incidence properties of parallel lines can be proved quite early, properties relating to angles must be delayed until dot multiplication and angle measure have been introduced.

Another theorem that was presented previously is: "Opposite sides of a parallelogram are congruent." Again, the proof of this theorem from the vector standpoint illustrates the different techniques used to prove the same results. This proof reflects the importance of several key concepts such as direction of translations, the addition postulate for translations, linear dependence and length of translations.

Given: ABCD is a parallelogram with $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

Prove: AC = BD and AB = CD



Since $\overleftrightarrow{AC} \parallel \overleftrightarrow{BD}$, $[D - B] = [C - A]$ and $D - B = (C - A)x$. Likewise, $D - C = (B - A)y$. By Postulate Three, $C - B = A - B + C - A$ and $C - B = D - B + C - D$. Upon substitution in the last equation we have $C - B = (C - A)x + (A - B)y$. Thus, $A - B + C - A = (A - B)y + (C - A)x$ or $(A - B)(1 - y) + (C - A)(1 - x) = \vec{0}$. Since $(A - B)$ and $(C - A)$ are linearly independent, then $1 - y = 0$ and $1 - x = 0$. Hence, we conclude that $x = y = 1$ and that $D - C = B - A$ and $D - B = C - A$. If the translations are the same, then they have the same length.

Similarity

Intuitively, sets of points are similar if they have the same shape. It is obvious that similarity should be related somewhat to congruence since congruent sets have both the same shape and size. Of course, as it develops, congruence is a special case of similarity. Since the two concepts are closely related, it is reasonable to expect that the different books being investigated would treat similarity in much the same manner as they do congruence. This is precisely what they do. Moise-Downs use congruence of angles and measurement of segments to define similarity, while both Coxford-Usiskin and Vaughan-Szabo use special types of mappings.

Synthetic

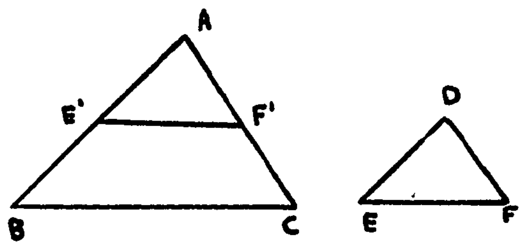
Moise-Downs define a similarity of triangles to be a one-to-one correspondence of the vertices of the triangle such that the corresponding angles are congruent and the corresponding sides are proportional. Moise-Downs have limited the discussion of similarity to triangles.

The general sequence of theorems developed by each book is much the same. Each begins by proving a theorem that pertains to the ratio of the segments formed when a segment parallel to one side of a triangle is constructed. This theorem is used to prove some sufficient conditions for similarity of triangles such as the SSS and AAA similarity theorems for triangles. Each book also discusses similarity of right triangles and trigonometric ratios.

The key theorem in the proof of the SSS and AAA similarity theorems is called the Basic Proportionality Theorem in Moise-Downs, whereas Coxford-Usiskin refer to it as the Side-Splitting Theorem. The theorem states that "If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off segments which are proportional to these sides." The proof as presented in Moise-Downs uses properties of the areas of triangles, particularly the relationship of the altitudes and bases of triangles with the same area. This means that Moise-Downs must develop the topic of area prior to that of similarity. The introduction of area earlier in the course is a departure from the traditional sequence of topics (Coxford-Usiskin and Vaughan-Szabo both do area much later). In the development of area, Moise-Downs must postulate several properties of area measure. The use of area to prove the Basic Proportionality Theorem is not essential, but Moise-Downs use it as a means of integrating algebraic methods into the geometry course at an earlier time. In his text, Elementary Geometry From an Advanced Standpoint,[1] Moise presents a proof of this theorem using the properties of real numbers. As will be seen later, Coxford-Usiskin are able to obtain this result from the properties of a special type of mapping.

The basic similarity theorems are the culmination of the discussion of similar triangles. The proof of the AAA similarity theorem as done in Moise-Downs is presented below.

Given: $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$



[1]Moise, p. 138.

Prove: $AB / DE = BC / EF = AC / DF$

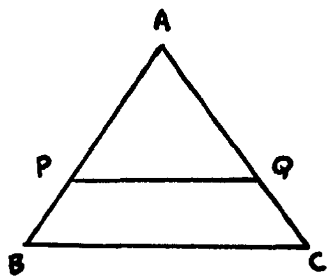
If $\triangle ABC$ is congruent to $\triangle DEF$, then all the ratios are equal to 1. If not, either $AB > DE$ or $AB < DE$. Without loss of generality, assume $AB > DE$. Hence, there exists a point E' on AB and a point F' on AC such that $DE = AE'$ and $DF = AF'$. Thus, $\triangle AE'F \cong \triangle DEF$. By properties of congruence $\angle E' \cong \angle B$ and $\angle F' \cong \angle C$ and thus $\overleftrightarrow{E'F} \parallel \overleftrightarrow{BC}$. By the Basic Proportionality Theorem we have the desired result.

Transformation

The basic tool used with similarity by Coxford-Usiskin is a size transformation. A mapping, S, is a size transformation with magnitude k and center C if for any point A, S(A) is on $\overrightarrow{CA}$ where $CS(A) = kCA$, $k > 0$. Like their development of reflections, the authors must postulate the properties of this type mapping. The Size Transformation Postulate says that size transformations of magnitude k preserve collinearity, betweenness, angles, angle measure and that the ratio of the image and preimage is k. It is easily verified that a line and its image are parallel under size transformations. It can also be shown that if a triangle is mapped by a size transformation the image will be a triangle in which the ratio of the sides of the image to the preimage is k. The proof of the Side-Splitting Theorem makes use of the notion of a size transformation.

Given: $\overline{PQ} \parallel \overline{BC}$

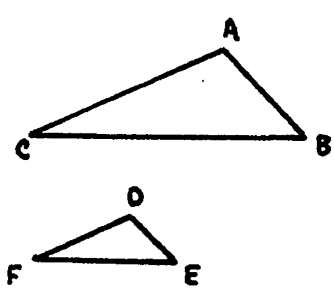
Prove: There exists a size transformation such that $S(\triangle APQ) = \triangle ABC$



Let S be the size transformation with center A and magnitude AB / AP. By definition, S(A) = A and S(P) = B. We know S(Q) is on $\overrightarrow{AQ}$. Also, S($\overline{PQ}$) is parallel to $\overline{PQ}$ and contains B since S(P) = B. By the Parallel Postulate we know S($\overline{PQ}$) is on $\overleftrightarrow{BC}$; hence, S(Q) is on $\overleftrightarrow{BC}$. The only point of intersection of $\overrightarrow{AQ}$ and $\overleftrightarrow{BC}$ is C, so S(Q) = C. Thus, we have S($\triangle$ APQ) = $\triangle$ ABC.

We see from this proof that properties of parallel lines are necessary to the development of similarity in Coxford-Usiskin as well as in Moise-Downs.

Since size transformations require a single point as the center, sets would have to be in a certain position in order to use them exclusively for similarity. Thus, the notion that Coxford-Usiskin uses is to transform one of the sets into the necessary position by using an isometry. This is the motivation behind the definition of a similarity transformation. A transformation is a similarity transformation if it is the composite of isometries and a size transformation. By definition, two sets of points are similar if there exist a similarity transformation that maps one set to another. All the properties preserved by both types of mappings would be preserved by the composition of them. Hence, similarity transformations preserve collinearity, betweenness, angles, angle measure and proportionality of segments. The proof of the AAA similarity for triangles illustrates the use of similarity transformations.



Given: $\triangle$ ABC and $\triangle$ DEF; $\angle A \cong \angle D$

$\angle B \cong \angle E$, $\angle C \cong \angle F$

Prove: $\triangle ABC \sim \triangle DEF$

Apply a size transformation, S, to $\triangle$DEF with center D and magnitude AB / DE. S($\triangle$ DEF) = D'E'F'. Since angle measure is preserved, $\angle$ D' $\cong$ $\angle$ A and $\angle$ E' $\cong$ $\angle$ B. We see that D'E' = DE • AB / DE = AB; hence, $\triangle$ ABC $\cong$ $\triangle$ D'E'F' by the ASA congruence theorem. By definition, there exists an isometry, t, which maps $\triangle$ D'E'F' to $\triangle$ ABC and so t o s is the desired similarity transformation.

Vector

Vaughan-Szabo present similarity in a way almost identical to that of Coxford-Usiskin. The only differences are in the terminology and the fact that Vaughan-Szabo do not have to postulate the properties of size transformations which they call uniform stretching--or shrinkings. A uniform stretching or shrinking about a point O is defined to be a mapping, g, such that g(X) = O + (X - O)m, m > 0. The authors define a similitude to be a mapping f such that d(f(X), f(Y)) = d(X, Y)m. It is easily proved that a similitude is the composition of an isometry and a uniform stretching or shrinking.

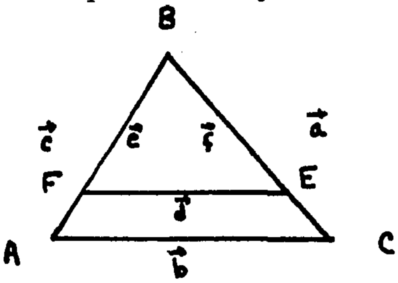
Suppose f is a similitude, then $\| f(X) - f(Y) \| = \| X - Y \| \, m$.

The mapping g has an inverse, $g^{-1} = O + (X - O)\, 1/m$.

The desired isometry is $f \circ g^{-1}$ since $\| f \circ g^{-1}(X) - f \circ g^{-1}(Y) \| = \| X - Y \|$.

Hence, the similitude is the same kind of mapping as a similarity transformation. Similarity, like congruence, in the Vaughan-Szabo development must be done after the concepts of distance and angle measure, which are done late in the program.

Actually, the vector approach does not need similarity to develop the concepts of proportional segments. The ratio of translations is a natural extension of the concept of linear dependence. If two translations, $\vec{a}$ and $\vec{b}$, have the same direction, then $\vec{a} = \vec{b}x$ and thus the ratio of these two translations can be defined such that $\vec{a} : \vec{b} = x$. The ratio of translations can then be used to discuss the ratio of the segments determined by these two translations. As an illustration of how proportionality theorems can be proved independent of similarity, consider the theorem referred to as the Basic Proportionality Theorem in Moise-Downs.



Given: $\triangle$ ABC with $\overleftrightarrow{FE} \parallel \overleftrightarrow{AC}$

Prove: FB / AB = EB / BC = FE / AC

Since $\overleftrightarrow{FE} \parallel \overleftrightarrow{AC}$, we know $\vec{d} = \vec{b}x$. F is a point of $\overline{AB}$; thus $\vec{e} = \vec{c}y$ and similarly $\vec{f} = \vec{a}z$. Now, $-\vec{d} = \vec{e} + \vec{f}$ so by substitution $-\vec{b}x = \vec{c}y + \vec{a}z$ or $(\vec{c} + \vec{a})x = \vec{c}y + \vec{a}z$. From this $\vec{c}(x - y) + \vec{a}(x - z) = \vec{0}$. Since $\vec{a}, \vec{b}, \vec{c}$ are noncollinear, they are linearly independent and thus $x - y = 0$ and $x - z = 0$ or $x = y = z$.

Summary

The three approaches presented are called Euclidean geometries because they have either a postulate or a theorem which says that "for a given line and a given point not on the line there is only one line through the point parallel to the given line." Moise-Downs and Coxford-Usiskin both have this as a postulate, whereas Vaughan-Szabo can prove it as a theorem. An important theorem in the use of parallel lines is the Alternate Interior Angle Theorem. Moise-Downs use geometric inequalities and the Exterior Angle Theorem to derive sufficient conditions for parallel lines, whereas Coxford-Usiskin use perpendicular lines. In the proof of the Alternate Interior Angle Theorem Moise-Downs use an indirect proof and Coxford-Usiskin use the symmetric properties of a rectangle. After proving this theorem, the two approaches are quite similar.

The topic of parallelism in Vaughan-Szabo is based on the direction of translations. The Parallel Postulate of the other two texts can be proved as a theorem. Some theorems in the synthetic approach that depend on parallel lines for their proof can be proved much more simply by vector methods.

The approach to similarity in Moise-Downs relates only to triangles. No special postulates are made in Moise-Downs concerning similar triangles. However, they do use the properties of area to prove the significant theorems about similar triangles.

Similarity in the transformation approach is developed through mappings. These mappings are called similarity transformations and are the composition of isometries and size transformations. Size transformations are assumed to preserve collinearity, angles, angle measure and betweenness. The proofs of similarity theorems in the transformation approach are much like the proofs of congruence theorems. The same general development of similarity is used in the vector approach, except the names are different. Size transformations are called stretchings or shrinkings and similarity transformations are named similitudes.

CHAPTER V

OBJECTIVES, METHODS OF PROOF, SEQUENCE OF TOPICS

In the three preceding chapters, this study has investigated the ways in which three books, representing the synthetic, transformation, and vector approaches to geometry, develop the topics of incidence, order, distance, congruence, parallelism, and similarity. In the course of this investigation, the ways in which these different books use their underlying assumptions to develop each of these topics has been demonstrated. In addition to the differences in postulational structure, the investigation has alluded to differences in other variables--*objectives*, *methods of proof*, *sequence of topics*. Certainly, these variables are so closely related to the postulational structure that any differences seen among them in the three programs can be attributed to, or have contributed to, the differences in postulational structures. For this reason, a discussion relating to each of the variables--*objectives*, *sequence of topics*, *methods of proof*--will be presented.

Objectives

Any discussion of the relative merits of the three programs that have been reviewed must consider the goals that they are attempting to achieve. The objectives of the high school geometry curriculum are difficult to list, because there is not a commonly accepted view of what geometry is. Adler states, "The lack of agreement on the organization of tenth-grade geometry reflects lack of agreement about its goals."[1] Some mathematicians view geometry as the study of geometric figures, others as a method of proof. Klein thinks of geometry as a study of the invariants of transformation groups.[2] The problem is compounded by the fact that there is no one "geometry"; for example, there are affine, projective, Euclidean, vector, and differential geometries. The objectives that one outlines for a geometry course are surely dependent upon the kind of geometry that he wishes to study and his concept of the nature of geometry.

The Commission on Mathematics of the College Entrance Examination Board states in its report three main objectives for the high school geometry curriculum:

1) The acquisition of information about geometric figures in the plane and in space.
2) The development of an understanding of the deductive method as a way of thinking and a reasonable skill in applying this method to mathematical situations.
3) The provision of opportunities for original and creative thinking by students.[3]

These objectives are quite compatible with those of the textbooks used in the United States today. There is, however, concern among mathematics

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[1]Irving Adler, "Criteria of Success in the Seventies," Mathematics Teacher, LXV, No. 1 (1972), p. 34.

[2]Klein, p. 133.

[3]College Entrance Examination Board, Report of the Commission on Mathematics. Program for College Preparatory Mathematics (New York: College Entrance Examination Board, 1959), p. 23.

educators about the goals of the high school geometry curriculum. In the last several years a number of suggestions have been made pertaining to the place of geometry in the mathematics curriculum. In a recent article, Fehr states the following three objectives: "intellectual formation," "information about space," and "skill in geometric problem solving."[4] Fehr's use of the term "intellectual formation" can be interpreted as an understanding of what geometry is and the methods used to study it. Allendoerfer quotes a similar set of goals for high school geometry:

1) An understanding of the basic facts about geometric figures.
2) An understanding of the basic facts about geometric transformations such as reflections, rotations, and translations.
3) An appreciation of the deductive method.
4) An opportunity for imaginative thinking.
5) An integration of geometric ideas with other parts of mathematics.[5]

Adler mentions five significant goals:

1) Exploration of relationships among geometric facts previously learned.
2) Introduction to the role of transformations of space in the study of geometry.
3) Mastery of a variety of techniques.
4) Development of critical thinking.
5) Development of an understanding of the nature of a mathema-

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[4]Howard Fehr, "The Present Year-Long Course in Euclidean Geometry Must Go," Mathematics Teacher, LXV, No. 2 (1972), p. 102.

[5]Allendoerfer, Mathematics Teacher (1969), p. 165.

tical model.[6]

There appears to be some agreement among the more recent suggestions for the goals of a contemporary high school geometry curriculum. First, most authorities would agree that a geometry course should lead to the acquisition of geometric "facts." There are, of course, differences in opinions about what geometric facts should be acquired. Generally speaking, most authorities are referring to the affine and metric properties of sets of points in space, such as a line and a point not on the line determine a plane or the sum of the angles of a triangle, and to the relationships that may exist among different sets, such as congruence of triangles.

Secondly, most authorities would agree that a mastery of the deductive method of thought is a worthwhile objective for the high school geometry course. For many years, Euclidean geometry was the only example of the axiomatic method in the secondary school. During the twentieth century there has been an interest on the part of mathematicians to axiomatize every field of mathematics. Inductive reasoning is still viewed as an essential tool for discovering and understanding generalizations, but ultimately the desire is to prove them deductively. Some believe that geometry is the secondary subject best suited for a thorough understanding of the deductive method because of the students' intuitive experience with physical space. Others think that geometry should not be the first experience with deductive thinking but should certainly continue with the use of that method. Whether or not geometry is the first experience with deduction, most mathematics educators would agree that, since the deductive method is

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[6]Adler, Mathematics Teacher (1968), p. 228.

an essential part of modern mathematics, geometry should be developed by this method. To develop geometry axiomatically does not seem to be the question, but rather which set of axioms to use. Adler sums up the situation, "While there is agreement that the core of tenth-grade geometry should be its development as an axiomatic system, there is no agreement on which system of axioms is appropriate for this level."[7]

A third objective that most authorities seem to agree on is the fact that geometry should reinforce, as much as possible, the mathematical concepts studied in other mathematics courses. Geometry is not an isolated subject and therefore algebraic concepts, such as the real numbers and functions, should be integrated into the geometry curriculum. In line with this objective is the idea of many mathematicians that geometry should reflect the methods of investigation used in contemporary mathematics. A statement by Fehr expresses this point of view:

> With this conception, instruction in geometry must be brought more and more into relation with algebra and its structure, and thus it must be developed so as to permit and exhibit the use of algebraic structures and techniques. This is the spirit of the times. In this respect, a very important objective should be to develop geometry so that it leads to a basic understanding of vector space and linear algebra.[8]

In the following paragraphs, each approach will be examined with respect to the three broad objectives for the geometry curriculum outlined above.

Synthetic

Of the three approaches being reviewed in this study, the approach used by Moise-Downs is closest to the geometry of Euclid and the high

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[7]Adler, Mathematics Teacher (1972), p. 34.

[8]Fehr, Mathematics Teacher (1972), p. 152.

school geometry curriculum as it existed prior to 1960. In the preface, Moise-Downs state:

> An examination of the Table of Contents for this book will indicate that we have closely followed the recommendations of the Commission of Mathematics of the College Entrance Examination Board and have been strongly influenced by the text entitled Geometry, written by the School Mathematics Study Group (SMSG).[9]

The Commission, in Program for College Preparatory Mathematics,[10] suggests a sequence of topics which are suitable for their objectives. The sequence is quite "standard" and reflects the Commission's view that synthetic geometry is the geometry best suited for the high school curriculum. Generally, the Moise-Downs text follows this sequence of topics. However, there are exceptions, such as the inclusion of a chapter on geometric inequalities. In addition, the Commission mentions several defects in the geometry of Euclid that should be remedied:

1) Inadequate axiomatic structure for basis for congruence.
2) Lack of algebra in geometry.
3) Failure to recognize the need for order axioms.[11]

The S. M. S. G. text, Geometry, attempts to correct these three points. This book is patterned after a scheme first suggested by Birkhoff, which uses the properties of the reals. As has been shown, Moise-Downs have used this same general outline.

In attempting to put congruence on a more logical basis, Moise-

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[9]Moise-Downs, p. v.

[10]College Entrance Examination Board, p. 38.

[11]*Ibid.*, p. 23.

Downs have appealed to the use of measurement. In the development of Euclid, segments were the same length if they were congruent. Moise-Downs have reversed this idea and thus define segments as congruent if they have the same length. It has been suggested that this approach to geometry offers several advantages. Birkhoff and Beatley state,

> In the first place, it is severely logical. In this respect it offers as satisfactory training as that of Euclid and is much more direct. It lends itself admirably to explicit consideration of the place of undefined terms, definitions, and assumptions in any chain of logical reasoning.... In the second place, it takes advantage of the knowledge of number and of linear and angular mensuration which the student possesses.... In the third place, it leads very naturally to the elementary facts of analytic geometry.[12]

By using the reals to measure segments, Moise-Downs have gone a long way toward eliminating the other defects mentioned by the Commission. The lack of any type of order relation in the geometry of Euclid is partly eliminated by defining a relation called "betweenness" on the points on a line.

Some algebraic methods are integrated into the course by Moise-Downs; for example, the reals are used for measurement of angles and segments, and the area postulates are introduced much earlier in the sequence.

The content of the geometry course presented by Moise-Downs is basically Euclidean in nature. The course contains few, if any, geometric concepts from the modern era, such as transformations. Seemingly, the most worthwhile goal of a synthetic course such as that of Moise-Downs is the understanding of the deductive method. Fehr states,

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[12] George D. Birkhoff and Ralph Beatley, "A New Approach to Elementary Geometry," Mathematics Teacher LXI, No. 8 (1968), p. 770.

"The survival of Euclid's geometry rests on the assumption that it is the only subject available at the secondary school level to introduce students to an axiomatic development of mathematics."[13] Many mathematicians, including Fehr, would argue that algebra or some other type of geometry could be used to demonstrate the axiomatic method. One of the objections to the use of synthetic geometry as a good illustration of the deductive method is the fact that we require students to verify geometric properties that they already accept. Also, intuitive geometry is becoming increasingly popular in the elementary and junior high schools, and consequently high school students are more familiar with the geometric facts studied in a traditional geometry course. A logical question to consider is, "Why not develop an understanding of the deductive method through material other than that for which the students already have a previous intuitive understanding?"

The synthetic approach to geometry provides very little integration of algebra and geometry. Moise-Downs have included a section on coordinate geometry, but the topic is isolated from the main development. The function concept is one of the basic concepts in all mathematics. The synthetic approach makes little use of this important and unifying notion. There is no mention, for example, of relating algebraic structures to sets of points, of groups of transformations or of vector spaces. Eccles states, quite emphatically, "A student's algebraic skills stagnate at best. Furthermore, the momentum for expanding his grasp of algebraic ideas comes to a halt during a year devoted to the usual

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[13] Fehr, Mathematics Teacher (1972), p. 151.

geometry course."[14] Because of the reaction against a compartmentalized mathematics curriculum, i.e., non-integrated geometry courses, mathematics educators have begun to suggest new geometry curriculums. The transformation and vector approaches analyzed in this study are two such proposals.

Transformation

The Coxford-Usiskin text states in its preface that "contemporary geometric thinking is poorly represented in school mathematics."[15] The authors feel that the transformation approach, as well as the vector and coordinate approaches, is in line with contemporary mathematics. Transformations are used by Coxford-Usiskin for several reasons.

1) They simplify the mathematical development.
2) They require no more algebra than traditional courses.
3) They can be understood by students of widely varying abilities.
4) They give a unifying concept to the geometry course.
5) They provide assistance for future work in algebra and analysis.[16]

The authors state that the following objectives were used to guide their choice of content:

1) That material be at the level of the "average" student in geometry.
2) That the standard content of Euclidean geometry be thoroughly

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[14] Frank Eccles, "Transformations in High School Geometry," Mathematics Teacher, LXV, No. 2 (1972), p. 165.

[15] Coxford-Usiskin, p. v.

[16] Ibid., p. vi.

represented.

3) That the text require no special teacher training.

4) That transformations be used to simplify and unify mathematics.[17]

In the beginning, Coxford-Usiskin discuss the properties of sets of points in a way similar to that of Moise-Downs. Coxford-Usiskin comment, "This early content is similar to that found in SMSG-based materials (see, for example, Moise and Downs, 1964)."[18] The topics of congruence and similarity, which are commonly part of the geometry course, are presented, but using transformations. In general, Coxford-Usiskin present the standard content, but from a different approach. There are, of course, topics that are outgrowths of the transformations that are not normally found in "standard" synthetic texts; for example, groups.

The transformation approach as presented by Coxford-Usiskin is an axiomatic treatment. They do delay the formal proof of theorems and the inclusion of student problems requiring formal proofs until Chapter Six. This allows the student to develop an intuitive feeling for the use of transformations before requiring him to present formal proofs.

The transformation approach is one attempt to inject some contemporary mathematics into the geometry curriculum. Foremost in this attempt is the emphasis on the fundamental concept of function. In a standard mathematics curriculum, students encounter mainly algebraic functions. The transformations as presented by Coxford-Usiskin provide the opportunity

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[17] Ibid.

[18] Zalman Usiskin and Arthur Coxford, "A Transformation Approach to Tenth-Grade Geometry," Mathematics Teacher, LXV, No. 1 (1972), p. 21.

to see functions used in geometry. Properties of functions (e. g. one-to-one, inverses) are brought into the presentation. Composition of functions (transformation) can be used to introduce an elementary study of groups. Coxford-Usiskin present the concept of a group in a later chapter and show that the set of all isometries form a group under composition. If the plane is coordinatized, the matrices associated with transformations can be derived. Coxford-Usiskin do this in Chapter Twenty. The matrix representation of transformations of the plane are used in advanced studies in geometry and analysis. The transformation approach appears to be a step in the direction of relating high school geometry to other areas of mathematics and yet one which retains the flavor of the synthetic course.

Vector

The objectives of the course designed by Vaughan-Szabo are stated by Szabo:

1) To teach three-space geometry.
2) To demonstrate for the students how one uses plausible intuitions to create a formal mathematical system.
3) To relate the study of algebra and geometry in the secondary mathematics curriculum.[19]

The authors have chosen a vector approach to Euclidean geometry to accomplish these objectives. To them the concept of a vector space is one of the central ideas in mathematics today. This theory is supported by the fact that the vector space concept has applications in the areas of alge-

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[19]Steven Szabo, "An Approach to Euclidean Geometry through Vectors," Mathematics Teacher, LIX, No. 3 (1966), p. 218.

bra, geometry and analysis. The particular vector space they use is a set of mappings--translations--from three-space ($\mathcal{E}$) to three-space ($\mathcal{E}$).

One of the overall differences between the approach taken by Vaughan-Szabo and that of the other two investigated in this study is the fact that it endeavors to discuss more than just the standard topics of high school geometry. As Szabo states, "From the properties of these mappings on $\mathcal{E}$, one can deduce all of the standard, and much that is not now standard, high school geometry."[20] The extra material is what usually is called linear algebra and as the authors say in the preface to their book, "This extra material furnishes a foundation which ties the rest together."[21]

The program of Vaughan-Szabo appears to rely more on inductive reasoning than do the other two approaches. However, this is not to say that they do not stress deductive thinking, because ultimately they do develop a formal system. The vector approach gives the students more opportunity to form generalizations through inductive methods than do the synthetic or the transformation approaches, because the students generally have little previous understanding of vectors or vector space concepts. The formation of these generalizations provides a good opportunity for the authors to demonstrate the role of postulates in a formal system. For example, in the beginning the authors provide exercises that should lead the students to see that by specifying two points a translation is determined; thus, the authors have their first postulate, $A - B \in \mathcal{T}$. In most cases, the theorems are suggested by a set of developmental exercises that

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[20]Ibid., p. 218.

[21]Vaughan-Szabo, p. v.

precede the statement of the theorem. The emphasis is on both discovering the theorem and proving it deductively.

In the case of theorems relating to geometric "facts" of which the students already have an intuitive understanding, the students might tend to feel that they "walked around the block to get next door." The artificial nature of the vector approach is a criticism of some authorities. In the Cambridge Report, those who disagreed with the strong emphasis on vectors called this procedure "flank-attacks." They stated,

> ...the descriptions that are offered to him cannot be justified merely by the fact that after a lengthy development they turn out to be appropriate, adequate, and accurate.... For this reason, we believe that the logical simplicity of a vectorial approach to the geometry of space does not justify its use in a first treatment.[22]

A noticeable feature of the Vaughan-Szabo text is the emphasis on quantifiers. The authors include a chapter on the use of the universal and existential quantifiers. Most high school texts do not formally discuss this topic. The fact that Vaughan-Szabo do discuss quantifiers is mentioned, not to argue a case for or against them, but rather to illustrate that they do stress an understanding of deductive proof.

The vector approach to geometry is an attempt to answer the criticism that our present high school geometry curriculum is not related to contemporary mathematics. Besides reinforcing a student's intuitive notion of geometric properties through formal proof, the vector approach has the fundamental objective of introducing the concepts of a vector space. There are several reasons why this seems to be a worthwhile objective.

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[22]Cambridge Conference on School Mathematics, p. 79.

First, it is a means of integrating algebra and geometry. The structure of a vector space is directly related to that of the field of real numbers; precisely, the reals are a one-dimensional vector space. Many of the properties of reals are also properties of a vector space; for example, the associative property for "addition." Many of the proofs use vector algebraic relations which are amenable to algebraic treatment. For example, in the proof of the medians theorem that was previously presented, we have equations such as $(A - C) - 1/2\ (A - C) + 1/2\ (B - C) = 1/2\ (A - C) + 1/2\ (B - C)$ that can be manipulated by properties that are algebraic in nature.

Second, the students are learning about a structure that can be useful in the study of other fields of mathematics. The concept of a vector space is used extensively in the solution of systems of linear equations. In the analysis of N-dimensional spaces, vectors, linear transformations and their associated matrices are used to define concepts such as derivatives. The importance to the future study of mathematics of the ideas discussed in the vector approach seems unquestioned.

Methods of Proof

One of the objectives of each of the programs investigated is an understanding of the deductive method as a way of thinking and the techniques necessary to use this method in mathematics; that is, the authors are interested in a study of the axiomatic method--drawing conclusions from assumed statements. The procedure used to determine the truth of those statements not assumed is called "proof." A mathematical proof of a statement is not unique. First, the proof is dependent upon the original assumptions. Primarily, this study has shown how different assumptions can be used to arrive at the same conclusions. Secondly, the proof of a statement is guided by the concepts that the student is dealing with (*e. g.* lines, triangles, translations) and the amount of verified knowledge that he has about them. For example, in doing a proof using vectors, he probably would not look for congruent triangles, since the concept is not introduced until late in the course. Proof is also dictated by certain "rules" of logic. Mathematicians have traditionally used several different methods of proof: direct proof, indirect proof, proof by contraposition, proof of existence, proof by enumeration.

Certainly, an integral part of mathematics is the conjecturing of theorems that are to be proved. Although this process is not, strictly speaking, a part of deductive thinking, it plays an important role in mathematics. Often the proof of a theorem is stressed at the expense of permitting the student to discover the theorem in the first place.

This section will discuss the different methods of proofs used by each approach, how they provide for inductive reasoning, and how the style of proof is affected by the different concepts inherent in each of the different approaches.

Synthetic

The most frequently used method of proof is a direct one in which the student arranges a chain of syllogisms from the given hypothesis to the desired conclusion. Most of the proofs done by the authors of Moise-Downs, as well as those done by the student, use the direct method. The authors present proofs starting in the section on betweenness; however, the students are not asked to present formal proofs until they have studied congruent triangles. Prior to this time, the authors give the student a hypothesis and ask what can be concluded. Often they also ask, "What postulate or theorem supports your answer?" Moise-Downs present a special section on proof in which they state that the students should look for facts that are consequences of the hypothesis. The section on congruent triangles provides a good opportunity for students to do original proofs using the direct method.

In Chapter Six the authors discuss the indirect method of proof. This method of proof is used to prove a number of theorems concerning lines and planes: for example, the theorem which states that "Given a line and a point not on the line, there is exactly one plane containing both of them." This type of theorem is also used to present existence proofs. The students are made aware that the part of the theorem that states "exactly one" means that they are obligated to prove the existence of at least one. The authors also use indirect proofs for some of the theorems pertaining to parallel lines, for instance, the Alternate Interior Angle Theorem. In some instances an indirect proof takes a somewhat different form. When there are a few possibilities for the conclusion, often it is more convenient to show that all but one of the possibilities are false. Such a case is the theorem "In any triangle $\triangle ABC$, if $\angle C > \angle B$, then $AB > AC$." There appears to be no discussion of proof by the contrapositive or no occasion to use an enumeration proof in Moise-Downs.

Moise-Downs have attempted to include some discovery type exercises in their treatment. For example, in the chapter on geometric inequalities we find this as the first exercise: "In each of these triangles, $m\angle A >$

m $\angle B$. What conjecture can you make about the sides opposite $\angle A$ and $\angle B$?" There are many instances in geometry where analogies can be found because the subject pertains to sets in one, two, and three dimensions. For instance, in the topic of separation a point "separates" a line; a line "separates" a plane; a plane "separates" a space. Another example of an analogous situation is the perpendicular bisector of a segment and the perpendicular bisecting plane of a segment. Geometry also presents the opportunity for generalizations. Properties of parallelograms and the sum of the interior angles of a polygon are instances where students may form generalizations.

Transformation

The transformation approach, like the synthetic, primarily uses the direct method of proof. Coxford-Usiskin introduce proof in Chapter Six and from that time on the student is asked to prove a number of statements.

Many of the first propositions to be proved pertain to properties of transformations. The indirect method of proof is demonstrated and used for theorems similar to those in Moise-Downs, i. e., two lines perpendicular to the same line are parallel. Coxford-Usiskin present the idea of the contrapositive, but only do one proof using it. There is no discussion of existence proofs or enumeration proofs.

The primary difference between Moise-Downs and Coxford-Usiskin with regard to proof is not in methods of proof but in the fact that Coxford-Usiskin concentrate on using transformations and their properties in their proofs. By investigating the properties of sets of points under transformations, the authors present the students with the chance to discover theorems. The topics presented in the transformation approach are quite similar to those in the synthetic with the exception of transformations. Hence, similar situations are presented for inductive reasoning. The authors do capitalize upon the fact that isometries are a special case of similarities by allowing the student to speculate and prove properties of similarities.

Vector

Vaughan-Szabo are much more thorough than are either Moise-Downs or Coxford-Usiskin in their discussion of methods of proof. This does not seem to be something that is necessary for a presentation of geometry from the vector viewpoint, but rather something that the authors think is necessary for an understanding of the deductive method. Vaughan-Szabo seem to have developed their text with the view in mind of requiring the students to prove most of the significant theorems, whereas the other two books present the proofs for most of the essential theorems. It appears that this is a pedagogical difference between the texts rather than a mathematical difference between the three approaches.

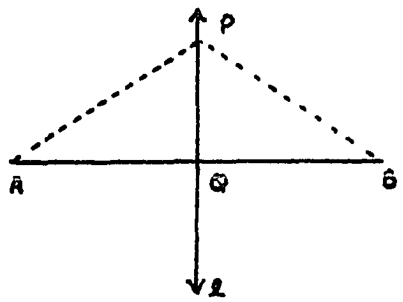
In the vector approach, as in the other two approaches, most of the proofs use the direct method. However, there are more theorems in the vector approach that could be proved by using the contrapositive; for example, if $\vec{a}$, $\vec{b}$ are linearly independent, then $\vec{a} \neq 0$ and $\vec{b} \neq 0$. There are more biconditional theorems in the vector approach, such as in the section on linear dependence. In some instances, these theorems can be proved one way by use of the contrapositive. Vaughan-Szabo have made no use of proof by contradiction. Although most of the proofs in Vaughan-Szabo make use of the vector techniques, there are theorems where synthetic and transformation techniques can be applied. For example, after the properties of reflections have been developed they can be used to prove theorems pertaining to congruence. Also, synthetic methods are illustrated in problems using the triangle congruence theorems.

Illustrations

The three approaches to geometry do vary somewhat with regard to the different techniques of proof. However, the main difference in proof pertains to the kinds of relationships one might look for in structuring the proof. For example, consider the proof of the same theorem from the point of view of each approach. The theorem is "Any point on the perpendicular bisector of a segment is equidistant from the ends of the segment." It will be observed that congruent triangles are used in the synthetic proof.

Given: $\overline{AB}$, with l the perpendicular bisector of $\overline{AB}$ and P a point of l.

Prove: AP = BP



Let Q be the point of intersection of l and AB. By hypothesis AQ = BQ and $\angle AQP \cong \angle BQP$. Thus, $\triangle AQP \cong \triangle BQP$ by SAS. From the congruent triangles, AP = BP.

The argument of this theorem in the transformation approach can be done before congruent triangles are discussed, because it uses symmetry. Using the same hypothesis, conclusion and diagram, we have the following proof:

Since l is the perpendicular bisector of $\overline{AB}$, by definition

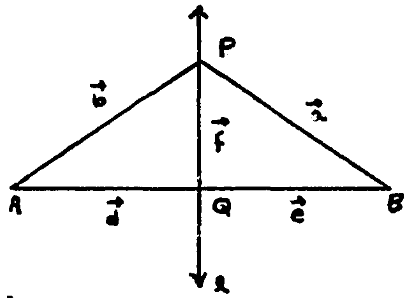
r_1 (A) = B and r_1 (B) = A. P is a point of l, so r_1 (P) = P.

Because reflections preserve segments AP = r_1 (AP) = BP.

The proof used in the vector approach demonstrates the use of algebraic relationships to prove the theorem. Consider the same diagram, but name the translations as follows:

Let $P - A = \vec{b}$, $B - P = \vec{a}$, $Q - B = \vec{e}$

$A - Q = \vec{d}$ and $P - Q = \vec{f}$.



By hypothesis, $\vec{d} = \vec{e}$, $\vec{f} \cdot \vec{d} = 0$ and $\vec{f} \cdot \vec{e} = 0$. We see that $\vec{b} = -\vec{d} + \vec{f}$ and $\vec{a} = -\vec{f} - \vec{e}$; hence, $\|\vec{b}\| = \|\vec{f} - \vec{d}\|$ and $\|\vec{a}\| = \|-\vec{f} - \vec{e}\| = \|-\vec{f} - \vec{d}\|$. But $\|\vec{f} - \vec{d}\| = \vec{f} \circ \vec{f} - 2\vec{f} \cdot \vec{d} + \vec{d} \cdot \vec{d}$ and $\|-\vec{f} - \vec{d}\| = \vec{f} \cdot \vec{f} + 2\vec{f} \cdot \vec{d} + \vec{d} \cdot \vec{d}$. Since $\vec{f} \cdot \vec{d} = 0$, we have $\|\vec{a}\| = \|\vec{b}\|$.

<u>Sequence of Topics</u>

The sequence of topics presented in a course is dictated by two considerations: mathematical and pedagogical. In some cases, the authors of a text may group a number of topics together because they all are part of some larger structure. In other situations they may sequence a group of topics because certain theorems can be used to prove theorems in the following topics. Each of these would be considered to be a mathematical motive. On the other hand, the authors' sequence may be determined by the fact that they think a particular topic is easier to comprehend, or it illustrates techniques that are useful later. This kind of consideration is pedagogical. Analysis of the texts reveals some apparent mathematical and pedagogical reasons for the sequence of topics chosen in each of the three presentations.

Synthetic

Moise-Downs present points, lines, planes, angles, and triangles in the first chapters. These are the fundamental "objects" that are to be discussed in the synthetic approach. Congruence is one of the most important relationships that exist between certain sets and thus is presented in the next chapter. Also, properties of congruent triangles are used in many of the proofs of theorems in later sections. Next, the authors introduce geometric inequalities. An important theorem in this chapter, the Exterior Angle Theorem, is used in the proof of the Alternate Interior Angle Theorem in a later chapter. The work done up through the chapter on inequalities is a part of what is sometimes referred to as Absolute Geometry, i. e., geometry which is independent of some type of Parallel Postulate. Hence, it appears that Moise-Downs want to emphasize that part of geometry that is common to both Euclidean and non-Euclidean geometries. This chapter is followed by the topic of parallel lines and the properties that exist if one assumes Euclid's parallel postulate. Proofs of many of the properties of parallelograms found in this section use congruent triangles; hence, this is another reason for introducing parallel lines after congruent triangles. A chapter on area follows the parallel chapter. In more traditional synthetic texts, one probably would find similar triangles as the next chapter. Moise-Downs have chosen to introduce area because they believe that area provides simpler proofs of some of the similar triangle theorems. Of the topics we have investigated, similarity is the last one presented in Moise-Downs. Parallel lines are used in the proofs of many theorems, for example, the SSS similarity theorem.

Transformation

As with Moise-Downs, the first three chapters of Coxford-Usiskin relate to points, lines, planes, and angles. Since the authors are going to use isometries to develop the topic of congruence, they introduce reflections and composition of reflections next. At this time, Coxford-Usiskin develop the concept of congruence but do not introduce the familiar triangle congruence theorems (SAS, ASA, SSS). Instead, they bring in the concept of line symmetry. The symmetry of quadrilaterals, such as kites, is used to prove the Alternate Interior Angle Theorem and the Triangle Congruence Theorems. According to Coxford, they "used quadrilaterals because we wanted to stress the importance of polygons other than triangles."[23] Following the chapter on parallels, the authors return to the topic of triangle congruence theorems and their applications in proving theorems about parallelograms. Coxford indicated that they felt the use of congruent triangles in geometric proofs was a technique that should be emphasized, but that they wanted to delay introducing these theorems until the students were familiar with proofs using transformations and symmetry.

Finally, Coxford-Usiskin develop the concept of similarities. Using their approach to similar triangles, i. e., similarity transformations, the authors could have presented this topic in the begin-

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[23]Arthur F. Coxford, private interview held during meeting of the National Council of Teachers of Mathematics, Las Vegas, Nevada, February, 1972.

ning. Since an isometry is a special kind of similarity transformation, they could have proceeded from the general to the specific. However, many of the theorems use parallel lines; thus, they have chosen to present similarity following parallelism.

Vector

It is difficult to compare the sequence in Vaughan-Szabo with the other two. Having selected vectors as the basic tool with which to work, it is essential that in the beginning the properties of a vector space be presented. Hence, the extent of the algebraic material not found in the other two approaches is quite large. Their beginning chapters deal with translations and the properties of a vector space. These first chapters include scalar multiplication and linear dependence. As far as the geometric content is concerned, they have first presented the affine properties and then the metric properties. Chapter Seven ties the vector concepts to traditional geometry by developing the notions of lines and planes and their incidence properties. The next concept introduced is parallelism. Up until this time, there has been no discussion of distance properties. Hence, in the next chapters the authors introduce distance and angle measure through dot multiplication. Once they have distance properties, Vaughan-Szabo can then develop the ideas of isometries. Again, similarity is the last of the topics to be presented.

CHAPTER VI

CONCLUSIONS AND SUGGESTIONS FOR FURTHER STUDY

Conclusions

Recent statements by well-known mathematics educators have indicated that there is concern about the content of the tenth-grade geometry curriculum. There have been a number of suggestions relating to possible ways of developing high school geometry. This study has compared two programs which reflect some of those suggestions with a synthetic program which typifies those currently being used in high schools throughout the United States.

One of the primary concerns of those opposed to the present geometry curriculum is the fact that geometry as it now exists in the mathematics curriculum is an isolated subject. Coxford-Usiskin and Vaughan-Szabo have attempted to unify geometry and algebra by making the function concept an integral part of their basic structure. Coxford-Usiskin have been much more moderate in their attempt than have Vaughan-Szabo.

Coxford-Usiskin use functions (transformations) to provide a better understanding of the geometric concepts of congruence, similarity, and symmetry. In the Euclidean development of congruence, geometric figures are congruent if one can be superimposed on the other. Coxford-Usiskin have attempted to present a mathematical development of this physical notion. This development is based on mappings of the set of points in the plane onto itself. In particular, the mappings that are used are distance-preserving mappings, called isometries. Sets of points are considered to be congruent if one is the image of the other with respect to an isometry. The axiomatic foundation for congruence is presented in the postulates related to the properties of a particular type of isometry called reflections. The other types of isometries (rotations, translations, glide reflections) are shown to be the composition of reflections. By using isometries, the authors are able to prove the traditional triangle congruence theorems.

Since congruence is a special case of similarity, Coxford-Usiskin have treated similarity much as they did congruence. For the presentation of similarity, the authors needed a mapping that would enlarge or shrink a set of points. They have called this type of mapping a size transformation. Consequently, postulates pertaining to the properties of size transformations are stated.

Vaughan-Szabo have stressed the concept of functions (translations) in their development of geometry. However, they have used them to develop a much larger structure, a vector space. The primitive terms in the vector approach are points and translations. Using these basic objects and postulates for a vector space, they are able to prove the basic assumptions of Euclidean geometry. To be able to prove the basic assumptions of Euclidean geometry it was necessary that Vaughan-Szabo present a detailed study of the algebra of vectors which is not normally a part of the geometry curriculum. In some instances, such as the incidence properties of points and lines, the vector approach presents a rather tedious and artificial development of what might be considered an intuitively obvious property.

In many ways the transformation and synthetic approaches are quite similar. Each reflects the idea that the basic postulational structure of the high school geometry course should be based on assumptions related to points, lines, planes, etc. Essentially both have the same set of incidence, order, and measurement postulates. Furthermore, both approaches view parallelism as a basic concept and build this topic on the assumption of the Parallel Postulate. Congruence is one of the key concepts in both the synthetic and transformation treatments. Congruent triangles are used in almost every topic. The most significant difference between these two approaches is the manner in which they develop congruence.

On the other hand, the vector approach as presented by Vaughan-Szabo appears to be a radical departure from the synthetic course. In the beginning, lines and planes are not considered as undefined terms but rather are defined in terms of points and vectors. Parallelism is not developed from a Parallel Postulate but rather from the notion of vectors with the same direction. Distance, which is presented in the synthetic and transformation approaches by coordinatizing a line, is developed in the vector approach by dot multiplication of vectors. Also, angle measure is defined in terms of dot multiplication. Congruence, which is quite important in the other two approaches, is used very little in the vector approach. Although the vector approach deals with the traditional geometry theorems, it appears to be concentrating on the study of linear algebra.

One of the objectives of the geometry curriculum is to develop an understanding of deductive proof. There is diversity among the three programs regarding the techniques for constructing a proof. Proofs in the synthetic and transformation programs are somewhat similar in the sense that the consequences of the hypothesis of a theorem seem more easily recognized, because the objects that they relate to appear to be models of physical objects. However, in the transformation approach, isometries and their properties are often used in proofs, whereas they are not used in the synthetic approach. On the other hand, proofs in the vector approach are more algebraic. Consequently, in the vector approach geometric properties must be translated into algebraic statements, thus tending to compound the problem of devising a proof. Admittedly, there are theorems whose proofs using synthetic or transformation methods are quite difficult and yet have an easy solution using vectors.

This analysis of the three programs emphasizes the fact that there is no unique postulational foundation for high school geometry. Each approach presents concepts and techniques for problem-solving that are a worthwhile part of a student's mathematical education. If this be the case, we should strive for a presentation of geometry which reflects each of these approaches to geometry.

Suggestions for Further Study

A review of the literature reveals that in the past decade mathematics educators in the United States have become increasingly concerned about the high school geometry curriculum. The result has been the proposal of various ways of reconstructing the high school geometry course.

In most instances the suggestions have been centered around unifying algebra and geometry. Some, such as Coxford-Usiskin and Vaughan-Szabo, have written texts exemplifying their proposals.

There is little doubt that this is a period when there is much examination and speculation concerning the geometry curriculum. It is possible that the future high school course will incorporate some of the new innovations. This study has explored the similarities and the differences between two suggested programs and the existing curriculum. By examining the postulational structure of each program, this analysis has endeavored to isolate some of the areas for possible research.

Ultimately, the relative merits of the different suggestions should be examined by experimental research. This investigation suggests some areas in which experimentation might provide insight into the problem of what the geometry curriculum should be.

1. One of the primary objectives of both the transformation and vector approaches is a better understanding of the function concept. Therefore, a statistical study could be conducted testing whether students in subsequent mathematics courses who have been taught by transformation or vector programs have a better knowledge of functions than do those students who have been taught by the synthetic approach.

2. A statistical experiment could be designed using students who have completed a transformation course and those who have completed a synthetic course in order to test the hypothesis that the transformation approach provides a deeper insight into the concept of congruence.

3. One of the main goals of the vector approach is to provide a continuation of the study of algebraic concepts. Separate groups that have studied the vector, transformation, or synthetic course could be tested following the completion of a subsequent algebra course to test the assumption that the vector approach enhances the later study of algebra.

4. The acquisition of geometric facts is a stated objective of all geometry curriculums. Often this objective is obscured by the overemphasis of the development of a formal system.

An experiment could be conducted following a year's study among groups that have studied geometry from one of the three approaches analyzed or a group in which there was no axiomatic development in order to test the hypothesis that a structured presentation detracts from learning basic geometric facts.

5. Analysis of the three programs revealed the use of different schemes for constructing proofs. The introduction and development of deductive proof is considered to be a major function of the geometry curriculum. A study could be designed to test the ability of students to devise proofs in later mathematics courses after studying geometry from one of the three approaches.

6. Each of the programs analyzed in this study presents an axiomatic development of geometry. One of the primary factors determining success in these courses is that a student be at the formal operational level, in other words, that he be capable of abstract thinking. Further research could be conducted to determine the percentage of students in the tenth grade that are in the formal operational stage.

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APPENDIX

POSTULATES

Synthetic--Moise-Downs

Postulate 1. The Distance Postulate. To every pair of different points there corresponds a unique positive number.

Postulate 2. The Ruler Postulate. The points of a line can be placed in correspondence with the real numbers in such a way that

(1) to every point of the line there corresponds exactly one real number;

(2) to every real number there corresponds exactly one point of the line; and

(3) the distance between any two points is the absolute value of the difference of the corresponding numbers.

Postulate 3. The Ruler Placement Postulate. Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

Postulate 4. The Line Postulate. For every two points there is exactly one line that contains both points.

Postulate 5

(a) Every plane contains at least three noncollinear points.

(b) Space contains at least four noncoplanar points.

Postulate 6. If two points of a line lie in a plane, then the line lies in the same plane.

Postulate 7. The Plane Postulate. Any three points lie in a least one plane, and any three noncollinear points lie in exactly one plane.

Postulate 8. If two different planes intersect, then their intersection is a line.

Postulate 9. The Plane Separation Postulate. Given a line and a plane containing it. The points of the plane that do not lie on the line form two sets such that

(1) each of the sets is convex, and

(2) if P is in one of the sets and Q is in the other, then the segment $\overline{PQ}$ intersects the line.

Postulate 10. The Space Separation Postulate. The points of space that do not lie in a given plane form two sets, such that

(1) each of the sets is convex, and

(2) if P is in one of the sets and Q is in the other, then the segment $\overline{PQ}$ intersects the plane.

Postulate 11. The Angle Measurement Postulate. To every angle $\angle BAC$ there corresponds a real number between 0 and 180.

Postulate 12. The Angle Construction Postulate. Let $\overrightarrow{AB}$ be a ray on the edge of the half-plane H. For every number r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$, with P in H, such that $m\angle PAB = r$.

Postulate 13. The Angle Addition Postulate. If D is in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.

Postulate 14. The Supplement Postulate. If two angles form a linear pair, then they are supplementary.

Postulate 15. The SAS Postulate. Every SAS correspondence is a congruence.

Postulate 16. The ASA Postulate. Every ASA correspondence is a congruence.

Postulate 17. The SSS Postulate. Every SSS correspondence is a congruence.

Postulate 18. The Parallel Postulate. Through a given external point there is only one parallel to a given line.

Postulate 19. The Area Postulate. To every polygonal region there corresponds a unique positive real number.

Postulate 20. The Congruence Postulate. If two triangles are congruent, then the triangular regions determined by them have the same area.

Postulate 21. The Area Addition Postulate. Suppose that the region R is the union of two regions R_1 and R_2. Suppose that R_1 and R_2 intersect in at most a finite number of segments and points. Then $aR = aR_1 + aR_2$.

Postulate 22. The Unit Postulate. The area of a square region is the square of the length of its edge.

Postulate 23. The Unit Postulate. The volume of a rectangular parallelepiped is the product of the altitude and the area of the base.

Postulate 24. Cavalieri's Principle. Given two solids and a plane. Suppose that every plane parallel to the given plane, intersecting one of the two solids, also intersects the other, and gives cross sections with the same area. Then the two solids have the same volume.

Transformation--Coxford-Usiskin

1. Point-Line-Plane.
   a. A line is a set of points and contains at least two distinct points.
   b. Two distinct lines intersect in a most one point.
   c. Every pair of distinct points lies in at least one line.
   d. A plane is a set of points and contains at least three points which are not collinear.
   e. For any three non-collinear points, there is exactly one plane which contains them.
   f. If two distinct points are in a plane, the line determined by these points is a subset of the plane.
   g. If two distinct planes intersect, their intersection is a line.
   h. Space contains at least four points not all in the same place.

2. Ruler. The points of a line can be placed in correspondence with the real numbers so that
   a. to every point of the line there corresponds exactly one real number called its coordinate;
   b. to every real number there corresponds exactly one point called the graph of the number on the line;
   c. to each pair of points there corresponds a unique real number called the distance between the points; and
   d. for any two distinct points A and B of a line, the line can be coordinatized in such a way that A corresponds to zero and B corresponds to a positive number.

3. Plane Separation. Every line in a plane separates that plane into two convex sets.

4. Protractor. To each angle, say $\angle APB$, there corresponds a unique real number greater than or equal to 0 and less than or equal to 180, called the measure of the angle and denoted by $m\angle APB$, so that each of the following statements holds.
   a. If both sides of an angle are the same ray, then the measure of the angle is 0.
   b. If the sides of an angle are opposite rays, then the measure of the angle is 180.
   c. If $\overrightarrow{PC}$ (except for point P) is in the interior of $\angle APB$, then $m\angle APC + m\angle CPB = m\angle APB$.
   d. In each half-plane of a line $\overleftrightarrow{PA}$ there is exactly one ray whose union with $\overrightarrow{PA}$ is an angle having a given measure k between 0 and 180.

5. Reflection.
   a. Given a line of reflection, every point has exactly one image.

b. The reflection image of a line l over any reflecting line m is a line, call it l'.
c. The reflection image of a point P is between the images of points A and B if and only if P is between A and B.
d. Let m be a reflecting line. If A' is the image of point A over m and B' is the image of point B over m, then AB = A'B'.
e. The reflection image of an angle is an angle of the same measure.

6. Orientation. Let $A_1, A_2, A_3, \cdots, A_n$ be the n vertices of a convex polygon. Then,
   a. the path from A_1 to A_2 to A_3 to $\cdots$ to A_n is either clockwise oriented or counterclockwise oriented, but not both;
   b. if path $A_1A_2A_3\cdots A_n$ is clockwise oriented, its reflection image, path $A'_1A'_2A'_3\cdots A'_n$, is counterclockwise oriented, and vice versa.

7. Parallel. Through a point not on a line there is exactly one parallel to the given line.

8. Size Transformation. Each size transformation preserves
   a. collinearity,
   b. betweenness, and
   c. angles and angle measure.
   d. Under a size transformation of magnitude k, the distance between images is k times the distance between their preimages.

9. Area.
   a. To each closed region R, there corresponds a positive real number.
   b. If R is a square region with side of length s, then $\alpha R = s^2$.
   c. If R is the union of two regions S and T with no common interior points, then $\alpha R = \alpha S + \alpha T$.

10. Volume.
   a. To each closed surface S, there corresponds a positive real number.
   b. Let S be a right prism with altitude h and base area B. Then, $\mathcal{V} S = B \cdot h$.
   c. If S is the union of two solids Q and R which have no common interior points, then $\mathcal{V} S = \mathcal{V} Q + \mathcal{V} R$.
   d. Cavalieri's Principle. Let S and T be solids and X be a plane. If every plane which intersects either S or T, and is parallel to X, intersects both S and T in figures having the same area, then $\mathcal{V} S = \mathcal{V} T$.

Vector--Vaughan-Szabo

Postuate 1. (a) $B - A \in \Upsilon$ (b) $A + \vec{a} \in \mathcal{E}$

<u>Postulate</u> <u>2</u>. (a) $A + (B - A) = B$ (b) $\vec{a} = (A + \vec{a}) - A$

<u>Postulate</u> <u>3</u>. <u>The</u> <u>Bypass</u> <u>Postulate</u>. $(B - A) + (C - B) = C - A$

<u>Postulate</u> $\underline{4}_{\underline{0}}$. (a) $\vec{a} + \vec{b} \in \mathcal{T}$ (b) $\vec{0} \in \mathcal{T}$ (c) $-\vec{a} \in \mathcal{T}$

<u>Postulate</u> $\underline{4}_{\underline{1}}$. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

<u>Postulate</u> $\underline{4}_{\underline{2}}$. $\vec{a} + \vec{0} = \vec{a}$

<u>Postulate</u> $\underline{4}_{\underline{3}}$. $\vec{a} + \vec{-a} = \vec{0}$

<u>Postulate</u> $\underline{4}_{\underline{4}}$. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

<u>Postulate</u> $\underline{4}_{\underline{5}}$. $\vec{a} \cdot 1 = \vec{a}$

<u>Postulate</u> $\underline{4}_{\underline{6}}$. $\vec{a} \cdot (b + c) = \vec{a} \cdot b + \vec{a} \cdot c$

<u>Postulate</u> $\underline{4}_{\underline{7}}$. $(\vec{a} + \vec{b}) \cdot c = \vec{a} \cdot c + \vec{b} \cdot c$

<u>Postulate</u> $\underline{4}_{\underline{8}}$. $(\vec{a} \cdot b) \cdot c = \vec{a} \cdot (bc)$

<u>Postulate</u> $\underline{4}_{\underline{9}}$. There are three linearly independent members of $\mathcal{T}$.

<u>Postulate</u> $\underline{4}_{\underline{10}}$. There are not four linearly independent members of $\mathcal{T}$.

<u>Postulate</u> $\underline{4}_{\underline{11}}$. $\vec{a} \cdot \vec{a} > 0 \quad [a \neq 0]$

<u>Postulate</u> $\underline{4}_{\underline{12}}$. $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

<u>Postulate</u> $\underline{4}_{\underline{13}}$. $(\vec{a}a) \cdot \vec{b} = (\vec{a} \cdot \vec{b})a$

<u>Postulate</u> $\underline{4}_{\underline{14}}$. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$