INFORMATION TO USERS

This material was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in "sectioning" the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again – beginning below the first row and continuing on until complete.

4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from "photographs" if essential to the understanding of the dissertation. Silver prints of "photographs" may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.

5. PLEASE NOTE: Some pages may have indistinct print. Filmed as received.

Xerox University Microfilms

300 North Zeeb Road
Ann Arbor, Michigan 48106

76-24,371

SCHAPIRO, Philip Joseph, 1927-
SHAPE CLASSIFICATION OF METRIC
COMPACTA.

The University of Oklahoma, Ph.D., 1976
Mathematics

Xerox University Microfilms, Ann Arbor, Michigan 48106

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

SHAPE CLASSIFICATION OF METRIC COMPACTA

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

PHILIP JOSEPH SCHAPIRO

Norman, Oklahoma

1976

SHAPE CLASSIFICATION OF METRIC COMPACTA

APPROVED BY

Leonard R. Rubin

Harold V. Huneke

David C. Kay

W. T. Reid

[illegible]

DISSERTATION COMMITTEE

TABLE OF CONTENTS

SHAPE CLASSIFICATION OF METRIC COMPACTA

CHAPTER 1

INTRODUCTION

Borsuk began in [4] an attempt to classify the plane shapes by classifying the shapes of plane continua. He employed $\check{H}_1(-,Z)$, the first Čech homology group functor with integer coefficients, to demonstrate that the rank of the group provides an order preserving bijection between the classes of plane continua and the ordinals $[0,\omega]$. He then attacked the metric compacta by utilizing a functor Λ from the category of compact subsets of the Hilbert cube and fundamental sequences to the category of 0-dimensional compact metric spaces and perfect maps. For each compactum $X, \Lambda(X)$ is the u.s.c. decomposition of the space into components, and for each fundamental sequence $f, \Lambda(f)$ is a perfect map induced by the sequence. For a shape equivalence, $\Lambda(f)$ is a homeomorphism which maps each component to one of the same shape. However, two simple plane compacta for which there exists a homeomorphism of component spaces with this property were shown to have different shapes. Borsuk concluded by describing the problem of classifying the plane compacta as "an open, and a rather difficult problem".

We have developed classifying objects for the plane compacta by adding to Borsuk's space of components and assignment of ordinals designating shape a means of expressing topologically the unique way in which the components converge to one another. Our definitions are phrased in a generality sufficient to apply to all metric compacta. We begin by examining the kth Čech homology group of a metric compactum for a positive integer k and a compact Hausdorff abelian topological group G as coefficient group. Within $\check{H}_k(C,G)$ there is a compact subspace composed of the class of all bounding cycles and the homology classes of non-bounding cycles produced by single components. This one point union of groups, denoted by $\overline{S}_k(C,G)$, can be linked in a natural way by a relation to the base space $\Lambda(C)$ of components of C, and this provides a new type of object, $S_k(C,G)$. The topology of the base space describes the convergence of the components of C, whereas the topology of the total space $\overline{S}_k(C,G)$ describes the convergence of the classes of cycles carried by individual components. Morphisms between these objects for different compacta are defined so as to create a range category for a functor $S_k(-,G)$.

When G has a multiplicative structure which makes it a field, then it is possible for each of the groups $\overline{L}_\alpha$ within $\overline{S}_k(C,G)$ to be a compact vector space over G, and one may sometimes choose a Schauder basis for each $\overline{L}_\alpha$. In particular, a topologically meaningful set of bases can always be chosen for $\overline{S}_{N-1}(C,Z_2)$ when C is a compactum in euclidean N space by considering a defining sequence M for C. In this case a subspace, denoted by $\mathrm{Str}_{N-1}(M)$, which is the union of sets L_α of augmented bases in $\overline{L}_\alpha$ is selected. It is linked by a mapping to the space of components. This creates an object $\mathrm{Str}_{N-1}(M)$ which is a subobject of $S_{N-1}(C,Z_2)$ in a weak sense, for it bears no algebraic structure. The function Str_{M-1} is not functorial, and it is necessary to show that it is a shape invariant. This invariant yields for the compactum topological information which is veiled by the group structure carried by the values of $S_{M-1}(-,Z_2)$. For example, the augmented bases L_α, each composed of classes of non-bounding cycles, correspond to

1) sets of components of $S^M - C_\alpha$,

2) subsets of C_α which bound these components.

We alter these collections slightly and then form the union of each. Topologies are defined for each union, and therefrom objects isomorphic to $Str_{M-1}(M)$ are created. These isomorphisms express a new kind of duality for a compactum in S^M, yielding more information than can be obtained from pure homology and cohomology groups.

The work is arranged as follows. In the first chapter the new invariants are defined and their basic properties established. They are used to classify the plane compacta and the toroidally defined subcompacta of R^3. In the second chapter duality relations are introduced. In the third chapter we formulate a condition necessary and sufficient for two u.s.c. decompositions of the 2-sphere generated by compacta to be equivalent under a homeomorphism of pairs. A corollary demonstrates that the two compacta must have the same shape, in sharp contrast to the situation in the 3-sphere.

Expense has prevented us from including the second and third chapters as part of this copy. They will be submitted elsewhere for publication.

CHAPTER II

CLASSIFICATION BY INVARIANTS

1. Preliminary discussion and necessary notation. We shall apply the version of shape theory for metric compacta expressed in terms of ANR sequences and mappings of systems developed in [29]. For each positive integer n, M_n will be an ANR (compact metric) space and $i_{n+1} : M_{n+1} \longrightarrow M_n$ a continuous bonding map. The sequence is said to be associated with the space C if C = inv lim (M,i). There is a function c from the positive integers to themselves such that c(n) is the number of components of M_n. We denote the components of M_n by $M_{n,j}$ for $1 \leq j \leq c(n)$. There is a set A of sequences α of positive integers $\alpha(n)$ subject to the conditions: 1) $1 \leq \alpha(n) \leq c(n)$ and 2) $i_{n+1}\left[M_{n+1,\alpha(n+1)}\right] \subseteq M_{n,\alpha(n)}$, and such that each component C_α of C is the inverse limit of the sequence (M^α, i^α), where $M_n^\alpha = M_{n,\alpha(n)}$ and $i_{n+1}^\alpha = i_{n+1}\big|M_{n+1}^\alpha$. Let $V_{q,p}$ be equal to $i_{q+1} \cdot i_{q+2} \cdots i_p : M_p \longrightarrow M_q$ and $V_{q,p}^\alpha$ be $V_{q,p}\big|M_p^\alpha$.

We use the notation f <u>htp</u> g to indicate that the continuous map f is homotopic to g. A mapping (f,ψ) of system (M,i) to $(\underline{M},\underline{i})$ is a pair of sequences with ψ an increasing map of the positive integers into themselves and f_n a continuous map of $M_{\psi(n)}$ to $\underline{M}_n$ with the property that for each n we have $\underline{i}_n \cdot f_n$ <u>htp</u> $f_{n-1} \cdot V_{\psi(n-1),\psi(n)}$. The concept of homotopic mappings of systems is as in [29].

We desire to compute the Cech homology groups $\check{H}_k(C,G)$, where G is a compact Hausdorff abelian topological group (hereafter denoted by the symbol CTAG). The continuity theorem [16] for compact T_2 spaces and a CTAG as coefficient group asserts that there is a natural equivalence between $\check{H}_k(C,G)$ and inv lim $(\check{H}_k \cdot M, \check{H}_k \cdot i; G)$. For an ANR (compact metric) all homology theories which satisfy the axioms in [16] agree in value. We therefore apply to the sequence (M,i) that formulation of a homology theory which at the moment is most convenient.

<u>Hypothesis 1.1.</u> Let G be a fixed CTAG until section 6.

<u>2. The functor S_0</u>

The domain of S_0 will be the set of ANR sequences and mappings of systems. The range will be the category of 0-dimensional metric compacta and continuous mappings. We apply the compact Hausdorff abelian topological group functor $H_0(-,Z_2)$ to the sequence (M,i). For each ANR M_n, $H_0(M_n)$ is a free abelian 2-group generated by a set $\{m_{n,j} \mid 1 \leq j \leq c(n)\}$, where $m_{n,j}$ corresponds to the component $M_{n,j}$ of M_n. Let I_{n+1} denote the continuous homomorphism $H_0(i_{n+1})$. The mapping I_{n+1} takes each generator $m_{n+1,j}$ to the generator $m_{n,i}$ corresponding to the component $M_{n,i}$ in which $i_{n+1}[M_{n+1,j}]$ lies. We shall define $S_0(M,i)$ to be a geometrically significant subset of inv lim $(H_0 \cdot M, I)$, which is a subspace of $\prod_{n=1}^{\infty} H_0(M_n)$ provided with the Tychonov topology. The set $S_0(M)$ is defined as the image of the u.s.c. decomposition space $\Lambda(C)$ under a mapping T_M defined as follows. For each component C_α of C there is a unique sequence (M^α, i^α) such that C_α equals inv lim (M^α, i^α). To each M_n^α there corresponds a generator m_n^α of $H_0(M_n)$. Let $T_M(C\ )$ be the sequence m^α so defined. Because $I_{n+1}(m_{n+1}^\alpha)$ equals m_n^α, the sequence is a thread of the inverse limit of $(H_0 \cdot M, I)$. Define the topology on $S_0(M)$ to be the topology induced on the set from the product $\prod_{n=1}^{\infty} H_0(M_n)$. This topology is thus the topology of coordinatewise convergence. For a subset B of an inverse limit of a sequence of discrete spaces, this topology has the special property that for a thread m^α, the set of open sets $\{p \mid p \in B \text{ and } p_k = m_k^\alpha\}$ for $k \geq 1$ is a local basis at m^α. For a mapping (f, ψ) from system (M, i) to $(\underline{M}, \underline{i})$, the value of $S_0(f)$ is defined to be the restriction of inv lim $(H_0 . f, \psi)$ to $S_0(M)$.

We now show that S_0 is a functor. We must first demonstrate that $S_0(f)$ maps $S_0(M)$ into $S_0(\underline{M})$. Consider that for each n the diagram

(2.1)

$$\begin{tikzcd} M_{\psi(n)} \arrow[d, "f_n"] & M_{\psi(n+1)} \arrow[l, "V_{\psi(n),\psi(n+1)}"'] \arrow[d, "f_{n+1}"] \\ \underline{M}_n & \underline{M}_{n+1} \arrow[l, "\underline{i}_{n+1}"] \end{tikzcd}$$

commutes up to homotopy. For a sequence M^α generating a component C_α of C, f_n must map $M^\alpha_{\psi(n)}$ for each n to an ANR component $\underline{M}_{n,\beta(n)}$ so that the diagram

(2.2)

$$
\begin{tikzcd}
M^{\alpha}_{\psi(n)} \arrow[d, "f_n^{\alpha}"] & & M_{\psi(n+1)}^{\alpha} \arrow[ll, "V^{\alpha}_{\psi(n),\psi(n+1)}"'] \arrow[d, "f_{n+1}^{\alpha}"] \\
\underline{M}_{n,\beta(n)} & & \underline{M}_{n+1,\beta(n+1)} \arrow[ll, "\underline{i}_{n+1}^{\beta}"]
\end{tikzcd}
$$

commutes up to homotopy. Here $f_n^{\alpha} = f_n | M_{\psi(n)}^{\alpha}$. Thus the sequence $\underline{M}^{\beta}$ is an ANR sequence for some component $\underline{C}_{\beta} = \text{inv lim } (\underline{M}^{\beta}, \underline{i}^{\beta})$ of the compactum $\underline{C}$ generated by $\underline{M}$. We may observe that

$$H_0(f_n \cdot V_{\psi(n),\psi(n+1)})(m_{n+1}^{\alpha}) = \underline{I}_{n+1} \cdot H_0(f_{n+1})(m_{n+1}^{\alpha}).$$

The right side of the equation equals $\underline{I}_{n+1}(\underline{m}_{n+1}^{\beta})$, and the left side equals $H_0(f_n)(m_n^{\alpha}) = \underline{m}_n^{\beta}$. Because $I_{n+1}(\underline{m}_{n+1}^{\beta}) = \underline{m}_n^{\beta}$, the thread is not only a thread of the inverse limit but is also the image of a component $\underline{C}_{\beta}$ of $\underline{C}$ under the mapping $T_{\underline{M}}$.

<u>Lemma 2.3.</u> For (M,i) the mapping $S_0(1,1)$ is the identity mapping on $S_0(M)$. For $f : (M,i) \longrightarrow (\underline{M},\underline{i})$ and $g: (\underline{M},\underline{i}) \longrightarrow (\overline{M},\overline{i})$, $S_0(g \cdot f) = S_0(g) \cdot S_0(f)$.

This follows from the fact that S_0 is the restriction of the functor $\check{H}_0$, and our demonstration that the image of $S_0(M)$ under $S_0(f)$ lies within $S_0(\underline{M})$.

<u>Lemma 2.4.</u> The mapping $S_0(f)$ is continuous.

Because the functor $\check{H}_0$ converts a mapping of systems to a continuous mapping, the result follows directly.

<u>Lemma 2.5.</u> If f and g are homotopic mappings of systems, then $S_0(f) = S_0(g)$.

The mappings $\check{H}_0(f)$ and $\check{H}_0(g)$ are equal, so that their restrictions to $S_0(M)$ are equal.

<u>Lemma 2.6.</u> The mapping inv lim $(H_0 \cdot f, \ )$ is continuous.

It is clear that for a compactum C the mapping T_M provides a bijection of the decomposition space $\Lambda(C)$ onto $S_0(M)$ for each ANR sequence for C. We see that all shape equivalent systems lead to isomorphic values of S_0.

<u>Lemma 2.7.</u> The mapping T_M is a homeomorphism.

Consider a component C_α of C and a basic open neighborhood B of $m^\alpha = T_M(C_\alpha)$ in $S_0(M)$ of the form $\{p \mid p \in S_0(M) \text{ and } p_l = m_l^\alpha\}$ for some $l \geq 1$. The open set $D = \{K \mid K \in \Lambda(C) \text{ and } K^* \text{ lies in } M_l^\alpha\}$ of $\Lambda(C)$ is mapped by T_M into B. (Here K^* is the image of K under the lth projection from C.) Thus T_M is continuous at each component, and because $S_0(M)$ is a T_2 space and $\Lambda(C)$ is compact, it is a homeomorphism.

We may proceed further and show that Λ is naturally equivalent to a functor derived from S_0. For each metric compactum choose an ANR sequence $(M(C),i(C))$. Then the functions $T_{M(C)}$ from $\Lambda(C)$ to $S_0(M(C))$ define a natural equivalence from the functor Λ to the functor S_0^C. Here $S_0^C(C) = S_0(M(C),i(C))$. We shall drop the superscript when there is no danger of confusion.

There is a natural equivalence N from the functor inv lim $\cdot H_0$ to $H_0 \cdot$inv lim on the category under consideration [16]. The second functor is intrinsic in nature. Form for each sequence (M,i) the restriction $N_S(M)$ of $N(M)$ to $S_0(M)$.

<u>Lemma 2.8.</u> The function N_S is a natural transformation from

S_o to a functor H_S whose values are well defined subsets of the intrinsically defined 0th Cech homology group functor values.

Thus, if M and $\underline{M}$ are equivalent systems for C under f, then the diagram

$$\begin{tikzcd} S_0(\underline{M}) \arrow[rr, "N_S(\underline{M})"] & & \check{H}_0(C) \\ S_0(M) \arrow[u, "S_0(f)"] \arrow[urr, "N_S(M)"'] & & \end{tikzcd}$$

commutes.

3. The functor $\overline{S}_k(-,G)$.

Let $k \geq 1$. We shall define the functor $\overline{S}_k(-,G)$ with domain the set of ANR sequences and mappings of systems. Its range will be the category of metric compacta and continuous mappings, To the sequence (M,i) apply the functor $H_k(-,G)$. Let $L = \text{inv lim } (H_k{\cdot}M, H_k{\cdot}i)$ in the category of compact T_2 abelian topological groups. The space L bears the topology of coordinatewise convergence. For each component C_α of C there is a unique sequence M^α of ANR components and j^α of embeddings such that the diagram

(3.1)
$$\begin{tikzcd} M_n^\alpha \arrow[r, "j_n^\alpha"'] & M_n \\ M_{n+1}^\alpha \arrow[u, "i_{n+1}^\alpha"] \arrow[r, "j_{n+1}^\alpha"] & M_{n+1} \arrow[u, "i_{n+1}"'] \end{tikzcd}$$

commutes. The sequence (j^α, i) constitutes a mapping from the system (M^α, i^α) to (M,i). Apply the functor H_k to the diagram (3.1). Let $J_n^\alpha = H_k(j_n^\alpha)$, $I_{n+1}^\alpha = H_k(i_{n+1}^\alpha)$, and $I_{n+1} = H_k(i_{n+1})$. The resulting diagram

(3.2)
$$\begin{tikzcd}
H_k(M_n^\alpha) \arrow[r, "J_n^\alpha"] & H_k(M_n) \\
H_k(M_{n+1}^\alpha) \arrow[u, "I_{n+1}^\alpha"] \arrow[r, "J_{n+1}^\alpha"] & H_k(M_{n+1}) \arrow[u, "I_{n+1}"']
\end{tikzcd}$$

is also commutative. The sequence $(J^\alpha,1)$ constitutes a mapping of the sequence $(H_k \cdot M^\alpha, I^\alpha)$ to $(H_k \cdot M, I)$. Now M_n^α is an open and closed subset of M_n, so that J_n^α is an embedding into the group $H_k(M_n)$. The group $H_k(M_n^\alpha)$ is compact and $H_k(M_n)$ is Hausdorff, so that $H_k(M_n^\alpha)$ is a closed subset of $H_k(M_n)$. From this we may deduce that $J_\infty^\alpha = \text{inv lim } (J^\alpha,1)$ is a closed topological group embedding of $L^\alpha = \text{inv lim } (H_k \cdot M^\alpha, I^\alpha)$ as a closed subgroup $\overline{L}_\alpha$ of L. Each thread t of L_α is mapped to $J_\infty^\alpha(t)$ of L. We shall identify $J_\infty^\alpha(t)$ with t when convenient.

Suppose $C_\alpha \neq C_\beta$, $t^\alpha \in L^\alpha - \{0\}$, and $t^\beta \in L_\beta - \{0\}$. Then $J_\infty^\alpha(t^\alpha) \neq J_\infty^\beta(t^\beta)$. For there exists an integer p such that $M_p^\alpha \neq M_p^\beta$ and neither of the pth coordinates of $J_\infty^\alpha(t^\alpha)$ and $J_\infty^\beta(t^\beta)$ are zero. However, $H_k(M_p)$ contains $H_k(M_p^\alpha) \oplus H_k(M_p^\beta)$ as a direct summand. Therefore the pth coordinates of the two elements in L must differ. When $C_\alpha \neq C_\beta$, then $\overline{L}_\alpha \cap \overline{L}_\beta = \{0\}$.

We define $\overline{S}k(M,i;G)$, which we shall often denote by $\overline{S}_k(M)$, to be the union $\bigcup \{L_\alpha \mid \alpha \in A\}$, provided with the topology induced from L.

Let $j_{n,i} : M_{n,i} \longrightarrow M_n$ be the inclusion mapping and $J_{n,i} = H_k(j_{n,i})$. Suppose that $x \in L_\alpha$ and $y = J_\infty^\alpha(x)$. For each n, $H_k(M_n) \cong \prod_{j=1}^{c(n)} H_k(M_{n,j})$ with the product topology. The nth coordinate y_n

of y is a c(n) tuple and only the $\alpha(n)$th coordinate $y_{n,\alpha(n)}$ is different from zero. Clearly $y_{n,\alpha(n)} = x_n$. The set of sets $\{z \mid z \in \overline{S}_k(M)$ and for each i, $1 \leq i \leq c(n)$, $z_n \in \prod_{i=1}^{c(n)} D_i\}$, where $n \geq 1$ and D_i is an open neighborhood of $y_{n,i}$, is a set of basic open neighborhoods of y in $\overline{S}_k(M)$. This local base becomes simpler when each $H_k(m_{n,j})$ is a discrete group. Then we find that the set of sets $\{z \mid z \in \overline{S}_k(M)$ and $z_n = y_n\}$ for $n \geq 1$ is an open local base at y.

Suppose that (f,ψ) is a mapping from the system (M,i) for C to the system $(\underline{M},\underline{i})$ for $\underline{C}$. We shall define the mapping $\overline{S}(f)$ from $\overline{S}_k(M)$ to $\overline{S}_k(\underline{M})$. We have seen that for each n there is a homotopy commutative square

(3.3)

$$\begin{tikzcd} M_{\psi(n)} \arrow[d, "f_n"] & M_{\psi(n+1)} \arrow[l, "V_{\psi(n),\psi(n+1)}"'] \arrow[d, "f_{n+1}"] \\ \underline{M}_n & \underline{M}_{n+1} \arrow[l, "\underline{i}_{n+1}"] \end{tikzcd}$$

For each component C_α of C there is a component $\underline{C}_\beta$ of $\underline{C}$ such that the truly commutative squares

(3.4)

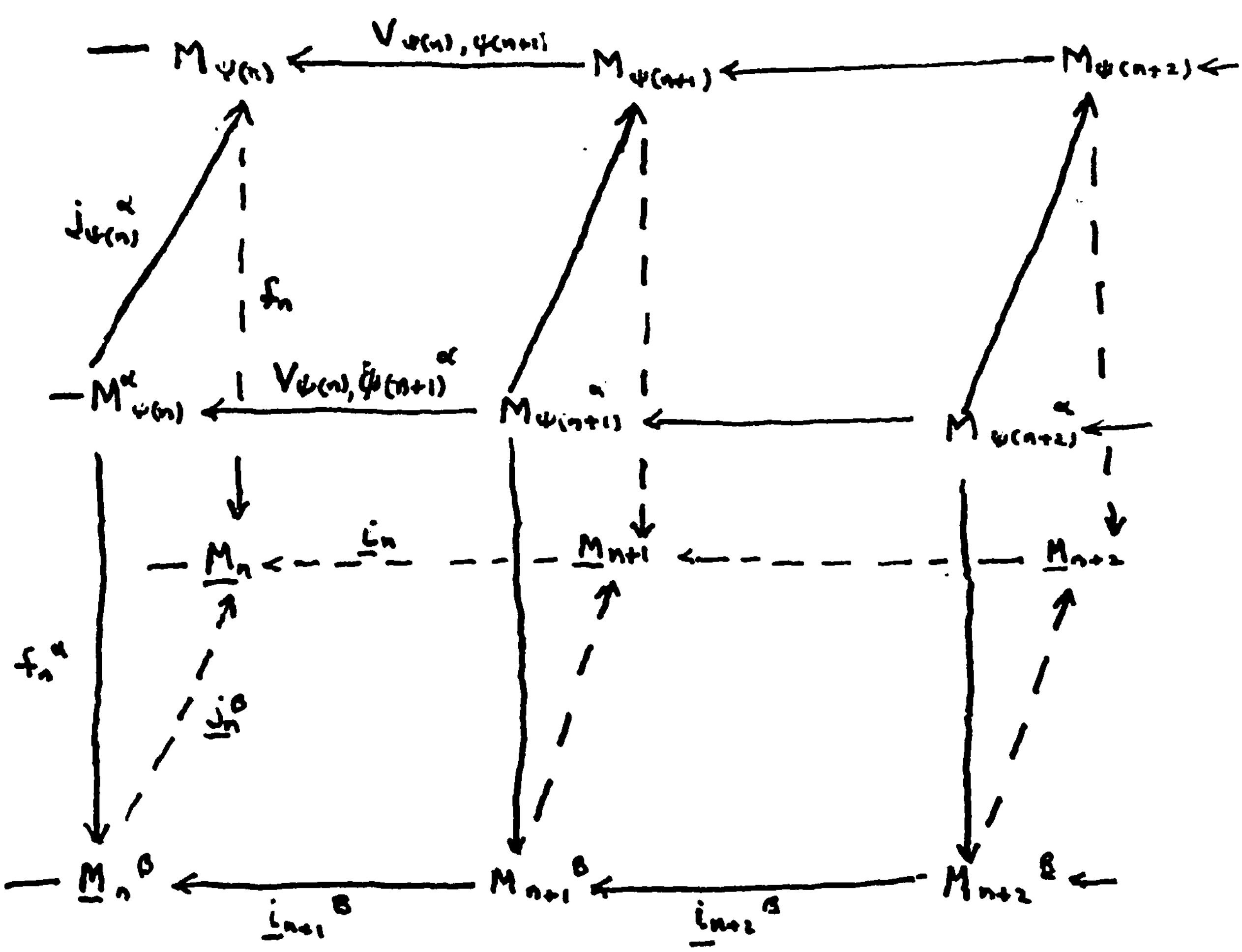
$$\begin{tikzcd} M_{\psi(n)}^{\alpha} \arrow[r, "J_{\psi(n)}^{\alpha}"] \arrow[d, "f_n^{\alpha}"] & M_{\psi(n)} \arrow[d, "f_n"] \\ \underline{M}_n^{\beta} \arrow[r, "\underline{j}_n^{\beta}"] & \underline{M}_n \end{tikzcd}$$

form a morphism of the ladder with homotopy commutative squares

(3.5)
$$\begin{tikzcd}
M_{\psi(n)}^{\alpha} \arrow[d, "f_n^{\alpha}"] & M_{\psi(n+1)} \arrow[l, "V_{\psi(n),\psi(n+1)}^{\alpha}"'] \arrow[d, "f_{n+1}^{\alpha}"] \\
\underline{M}_n^{\beta} & \underline{M}_{n+1}^{\beta} \arrow[l, "\underline{i}_{n+1}^{\beta}"']
\end{tikzcd}$$

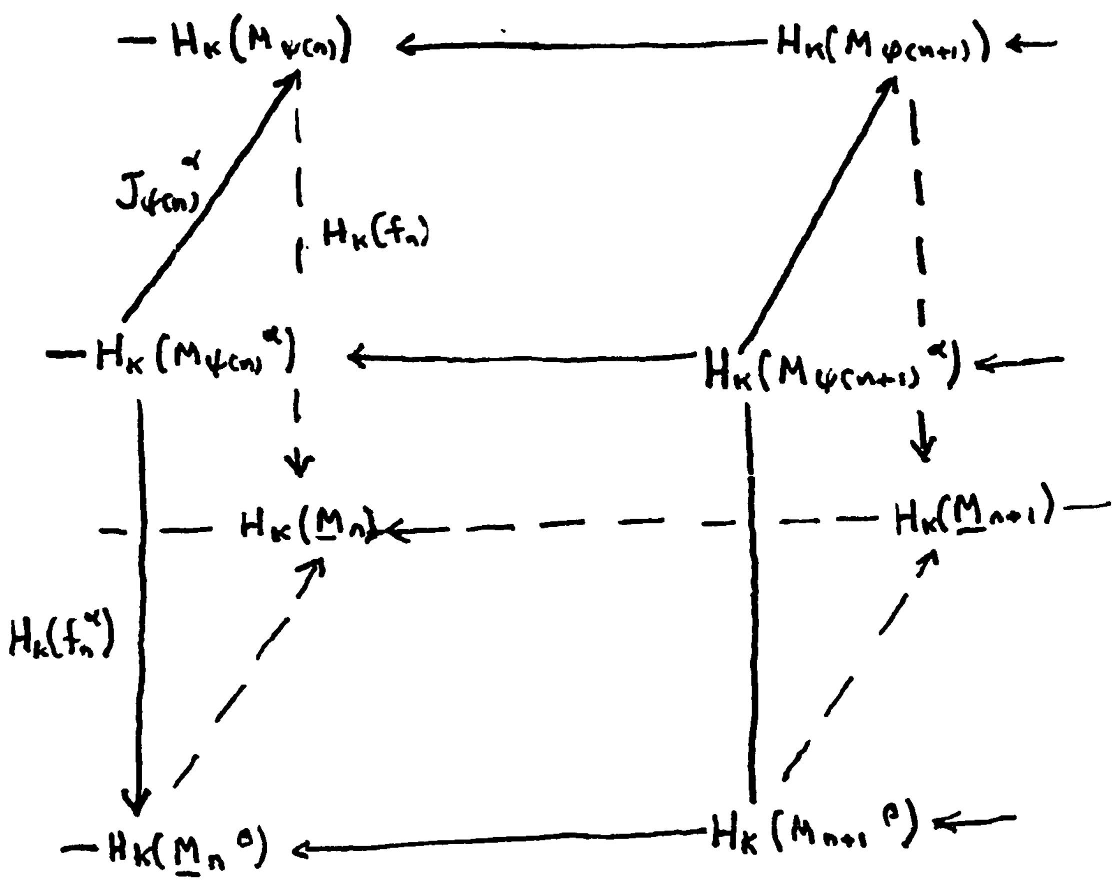
into the ladder with squares (3.3). A portion of a projective view of the resulting three dimensional diagram is

(3.6)



We may apply the kth homology group functor H_k to the diagram (3.6). When we do so the mappings in the horizontal planes are converted to closed embeddings, and the three dimensional diagram is converted to the commutative diagram of topological groups and morphisms below.

(3.7)



From (3.7) we can conclude that inv lim $(H_k\cdot f,\Psi)$ is a continuous homomorphism from inv lim $(H_k\cdot M, H_k\cdot i)$ to inv lim $(H_k\cdot \underline{M}, H_k\cdot \underline{i})$ and inv lim $(H_k\cdot f^{\alpha},\Psi)$ is a continuous homomorphism from inv lim $(H_k\cdot M^{\alpha}$, $H_k\cdot i^{\alpha})$ to inv lim $(H_k\cdot \underline{M}^{\beta}, H_k\cdot i^{\beta})$. Moreover, the mapping inv lim $(H_k\cdot j^{\alpha},1)$ and the mapping inv lim $(H_k\cdot j^{\beta},1)$ are closed group embeddings making the diagram

(3.8)
$$\begin{tikzcd}
\operatorname{inv\,lim}(H_k\cdot M^\alpha, H_k\cdot i^\alpha) \arrow[d, "\varprojlim(H_k\cdot f^\alpha,\psi)"'] & & \varprojlim(H_k\cdot M, I) \arrow[ll, "\varprojlim(J^\alpha,1)"'] \arrow[d, "\varprojlim(H_k\cdot f,\psi)"] \\
\operatorname{inv\,lim}(H_k\cdot \underline{M}^\beta, H_k\cdot i^\beta) & & \varprojlim(H_k\cdot \underline{M}, \underline{I}) \arrow[ll, "\varprojlim(\underline{J}^\beta,1)"']
\end{tikzcd}$$

commute. We rewrite the diagram for brevity

(3.9)
$$\begin{tikzcd}
\bar{L}_\alpha \arrow[d, "F_{\beta,\alpha}"'] & L \arrow[l, "in\,\alpha"'] \arrow[d, "F"] \\
\bar{\underline{L}}_\beta & \underline{L} \arrow[l, "\underline{in}\,\beta"']
\end{tikzcd}$$

We may follow elements belonging to $\bar{S}_k(M)$ under the mapping F. The thread t is taken by F to the thread s with $s_n = H_k(f_n)(t_{\varphi(n)})$. The diagram (3.8) shows that when $t \in \bar{L}_\alpha$, the image of L_α, then F converts it to a thread in $\underline{L}_\beta$. We therefore can define $\bar{S}_k(f)$ to be the mapping which acts in this way on each thread in $\bar{S}_k(M)$. We observe that $\bar{S}_k(f)$ can be identified with the restriction of F to $\bar{S}_k(M)$.

<u>Lemma 3.10.</u> If g is a mapping from the system $(\underline{M},\underline{i})$ to the system $(\bar{M},\bar{I})$ for $\bar{C}$, then $\bar{S}_k(g\cdot f) = \bar{S}_k(g)\cdot \bar{S}_k(f)$.

This follows from the fact that the values of $\bar{S}_k$ for the mappings can be identified with the restriction of the mapping obtained by applying the functors H_k and inv lim.

It is clear that the following are true.

Lemma 3.11 . If (1,1) is the identity mapping for (M,i), then $\overline{S}_k(1,1)$ is the identity mapping for $\overline{S}_k(M)$.

Lemma 3.12. If f and g are homotopic mappings of systems from (M,i) to ($\underline{M}$,$\underline{i}$), then $\overline{S}_k(f) = \overline{S}_k(g)$.

The proof follows by the use of well known techniques when the mappings $\overline{S}_k(f)$ and $\overline{S}_k(g)$ are identified with restrictions.

Lemma 3.13. We may therefore conclude that $\overline{S}_k$ is a covariant functor from the category of ANR sequences and mappings thereof to the category of 0-dimensional metric compacta and perfect maps. The functor factors through the homotopy category of maps of systems.

Lemma 3.14. The space $\overline{S}_k(M)$ is compact.

The proof of this assertion employs lemma (3.14.1).

Lemma 3.14.1. Suppose that $s \in L$ and N is a positive integer such that $s_N \neq 0$ and for each $n \geq N$ there exists a component A_n of M_n with the property that $s_n \in J_{n,j}[H_k(A_n)]$, where $A_n = M_{n,j}$. Then $s \in \overline{S}_k(M)$.

Let M_n^* be A_n for $n \geq N$. By finite induction downward let M_n^* for $n < N$ be the unique component of M_n in which $i_{n+1}[M_{n+1}^*]$ lies. For each n, $H_k(M_n)$ is the finite direct product of $H_k(M_{n,j})$ for all components $M_{n,j}$ of M_n. The mapping I_{n+1} maps each subspace $H_k(M_{n+1,i})$ of $H_k(M_{n+1})$ into the subspace $H_k(M_{n,j})$ of $H_k(M_n)$ for the component $M_{n,j}$ in which $i_{n+1}[M_{n+1,i}]$ lies. But $I_{n+1}(s_{n+1}) = s_n$ because s is a thread of the inverse limit, so that $A_{n+1} \subseteq A_n$ for $n \geq N$. Consequently $s \in \overline{L}_\alpha$.

To prove lemma (3.14) we show that $\overline{S}_k(M)$ is a closed subspace of L. Suppose that t is a sequence of threads of $S = \overline{S}_k(M)$ converging to s in L. If $s = 0$, then it belongs to S. If $s \neq 0$, there is a least integer N such that $s_N \neq 0$. Then $s_p \neq 0$ for $p \geq N$. Now s_p is a c(p) tuple in $\prod_{j=1}^{c(p)} H_k(M_{p,j})$. Take a basic open neighborhood $H = \prod_{j=1}^{c(p)} D_j$ of s_p. If the jth coordinate of s_p is different from zero, choose D_j so that $0 \notin D_j$. Because t converges to s, there is an integer I(p) such that when $i \geq I(p)$, t^i_p belongs to H. But $t^{I(p)}_p$, the pth coordinate of $t^{I(p)}$, belongs to $J_{p,j}[H_k(M_{p,j})]$ for some component $M_{p,j}$ of M_p, that is, only the jth slot is different from zero. This forces all but the jth slot of s_p to be equal to zero. Thus, s_p belongs to $J_{p,j}[H_k(M_{p,j})]$. Let $A_p = M_{p,j}$. Now we can apply lemma 3.14.1.

<u>4. The relation ρ_k and mapping σ_k.</u> The classical means for relating H_k and H_0 is the notion of a pair of spaces and a sequence of boundary operators. Here we shall relate $\overline{S}_k$ and S_0 through a continuous mapping σ_k and a relation ρ_k which is an extension of it. Each non-zero thread t of $\overline{S}_k(M)$ belongs to some subgroup $\overline{L}_\alpha$ for some component C_α of C. To each C_α corresponds a unique ANR sequence M^α, and to this sequence corresponds a unique thread m^α of $S_0(M)$. We therefore define σ_k so that $\sigma_k(t) = m$. The relation ρ_k is defined as $\sigma \cup [\{0\} \times S_0(M)]$, making each thread of $S_0(M)$ correspond to the common zero of all the groups $\overline{L}_\alpha$. (When k is fixed throughout a discussion, we shall drop it as a subscript.)

The mapping can be described in terms of mappings of individual coordinates of the threads t and m. For each integer n for which

$H_k(M_n)$ is not zero we define a mapping v_n from the subset $\bigcup_{i=1}^{c(n)} J_{n,i}[H_k(M_{n,i})] - \{0\}$ in $\prod_{i=1}^{c(n)} H_k(M_{n,i})$ to $H_0(M_n)$. By abuse of language we shall speak when convenient of mapping $\bigcup H_k(M_{n,i}) - \{0\}$. The mapping is defined on each of the disjoint subsets $J_{n,i}[H_k(M_{n,i}) - \{0\}]$ so that its entire image is $m_{n,i}$, the generator corresponding to $M_{n,i}$ in $H_0(M_n)$. Precisely, v_n takes each non-zero $c(n)$ tuple x to the element $m_{n,i}$ for which the ith slot is not zero. We define a map ν on $\overline{S}_k(M) - 0$ as follows. For each non-zero thread there is a least integer N for which $t_N \neq 0$. Let $\nu(t)$ be the thread whose nth coordinate for $n \geq N$ is $v_n(t)$. The mapping is well defined and can be shown to be identical to σ. We use ν to show the following lemma.

Lemma 4.1. The mapping σ is a continuous mapping from $S = \overline{S}_k(M) - \{0\}$ to $S_0(M)$.

proof: Suppose that x belongs to S and $m^{\alpha} = \sigma(x)$. Let $B_n = \{t \mid t \in S_0(M) \text{ and } t_n = m_n^{\alpha}\}$ for some $n \geq 1$ be a basic open neighborhood of m^{α}. Because $x \neq 0$, there is a least integer p such that $x_p \neq 0$. Take $q = p+n$. Then x_q is a $c(q)$ tuple with the $\alpha(q)$ th coordinate not zero and all of the remaining coordinates equal to zero. There is an open neighborhood $Q = \prod_{i=1}^{c(q)} D_i$, where $D_i = H_k(M_{q,i})$ for $i \neq \alpha(q)$ and $D_{\alpha(q)}$ is a neighborhood of $x_{q,i}$ that does not contain zero. Then $v_q[D] = \{m_q^{\alpha}\}$, so that $\nu[\{y \mid y \in \overline{S}_k(M) \text{ and } y_q \in Q\}] \subseteq B_q \subseteq B_n$.

5. The functor $S_k(G)$. We define the functor $S_k(G)$ to have domain the set of ANR sequences and mappings of systems. The value of $S_k(G)$ will be the triple $\rho_k : \overline{S}_k(M) \longrightarrow S_0(M)$. For a mapping f

from system (M,i) to $(\underline{M},\underline{i})$, the value of $S_k(f)$ will be the pair $(\overline{S}_k(f),S_0(f))$. The diagram

(5.1)
$$\begin{tikzcd}
\overline{S}_k(M) \arrow[r, "\overline{S}_k(f)"] \arrow[d, "e_k"] & \overline{S}_k(\underline{M}) \arrow[d, "\underline{e}_k"] \\
S_0(M) \arrow[r] & S_0(\underline{M})
\end{tikzcd}$$

commutes in the sense that for $t \in \overline{S}_k(M)$ such that $\overline{S}_k(f)(t) = 0$, we have $\overline{S}_k(f)(t)$ belonging to $\underline{e}_k^{-1}[S_0(f)(e(t))]$.

We therefore define a morphism of objects in the range category to be a pair of continuous mappings making a diagram commute in this sense. Composition of the morphisms (f_k,f_0) and (g_k,g_0) is defined to be $(g_k \cdot f_k, g_0 \cdot f_0)$. With this definition one may verify that

Lemma 5.2. The set of triples derived from metric compacta forms a category under composition of mappings.

We deduce the following theorem from what has been shown above.

Theorem 5.3. $S_k(G)$ is a functor on the category of ANR sequences and mappings of systems.

Corollary 5.4 If the system $(\underline{M},\underline{i})$ shape dominates (M,i), then $S_k(\underline{M})$ dominates $S_k(M)$ in the range category, that is, there exist morphisms $F = (f_k,f_0)$ and $G = (g_k,g_0)$, where $F : S_k(M) \longrightarrow S_k(\underline{M})$ and $G : S_k(\underline{M}) \longrightarrow S_k(M)$, such that $G \cdot F$ is an identity in the range category.

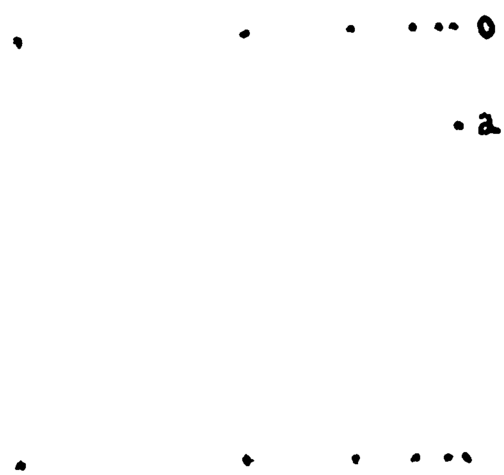
Corollary 5.5. Shape equivalent ANR systems define isomorphic invariants under S_k.

We may form a new functor $S_k^{\,C}(G)$ whose domain will be the category of metric compacta and shape mappings. Select for each compactum a particular ANR sequence M and let $S_k^{\,C}(C)$ be $S_k(M)$. Foe each shape mapping select a particular mapping of systems in the class and apply the functor S_k to it. The mappings can be shown to be well defined and to define a functor. (We shall drop the superscript when there is no danger of confusion.)

<u>6. Application of the functor $S_k(Z_2)$.</u> When Z_2 is used as a coefficient group, the total space as well as the base space is homeomorphic to a closed subset of the Cantor set. This makes the value of $S_k(Z_2)$ conveniently calculable and conceptualizable.

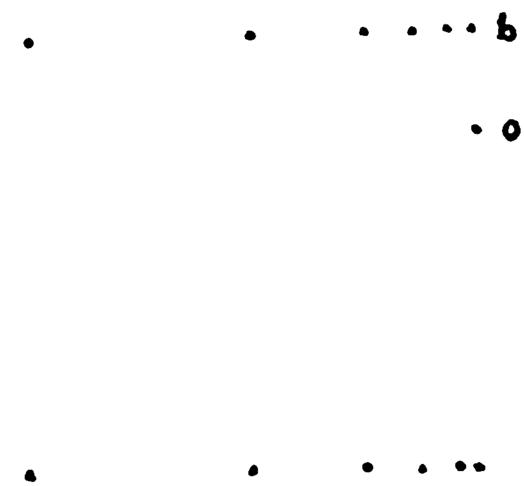
Borsuk has presented in [4] a pair of plane compacta and has demonstrated that one does not shape dominate the other. We generalize his example and produce in each euclidean space of dimension $N+1 \geq 3$ analogous compacta K_N and $K_{\underline{N}}$ comprised of N-dimensional submanifolds. Consider a polyhedral N-dimensional, closed, connected submanifold D of R^{N+1}. There is a polyhedral embedding h of $D \times [0,1]$ in R^{N+1} such that $h[D \times \{0\}] = D$. Let $K_{\underline{N}}$ be $D \cup \bigcup_{n=1}^{\infty} h\, D \times \{1/n\}$. In a cell in $D \times (1/(n+1),\ 1/n)$ embed a polyhedral copy D_n of D for each $n \geq 1$. Let K_N be $D \cup \bigcup_{n=1}^{\infty} D_n$.

For a closed, connected, orientable N-manifold D, $H_N(D,Z_2) = Z_2$. The values of $S_N(K_N,\ Z_2)$ and $S_N(K_{\underline{N}},\ Z_2)$ can be computed in a straightforward manner by the use of regular neighborhoods. Each K_N has the same value of $S_N(K_N,\ Z_2)$, and this value is pictured in figure 6.1 as embedded in the plane. The common value for $S_N(K_{\underline{N}},\ Z_2)$ is depicted in figure 6.2.



$S_N(K_N)$

Figure 6.1



$S_N(K_{\underline{N}})$

Figure 6.2

Neither of the compacta K_N and $K_{\underline{N}}$ dominates the other. If K_N were to shape dominate $K_{\underline{N}}$, there would be morphisms $F = (f_N, f_0)$ and $G = (g_N, g_0)$ with $F : S_N(K_N) \longrightarrow S_N(K_{\underline{N}})$ and $G : S_N(K_{\underline{N}}) \longrightarrow S_N(K_N)$ such that $F \circ G$ is the identity on $S_N(K_{\underline{N}})$. The mapping g_N would be a fiber preserving closed embedding of $\overline{S}_N(K_{\underline{N}})$ in $\overline{S}_N(K_N)$. Then $g(0)$ would be equal to 0, and $g(b)$ would be equal to a. But b is a limit point of $\overline{S}_N(K_{\underline{N}})$, whereas its image a is an isolated point of the containing space. It cannot be a limit point of the image of g_N.

If $K_{\underline{N}}$ were to dominate K_N, then $G \circ F$ would be the identity. In this case f_N would have to be a fiber preserving closed embedding. Then $f_N(0)$ would be 0, and $f_N(a)$ equal to b. Now 0 is a limit point of the domain, but its image, 0, is an isolated point of the containing space.

In general the fiber structure of the values of the invariant will be more complicated than those in these simple examples.

<u>7. Applications of the functor $S_1(T_0)$</u> Let T_0 be the group of complex numbers of unit norm under multiplication provided with the norm topology.

Let K be the class of compacta generated by ANR sequences (M,i) such that there exists a shape equivalent sequence (T,P) with the following properties.

7.1) Each component $T_{n,j}$ of T_n is a wedge of $g(n,j)$ copies $U_{n,j,i}$ of T_0 with 1 as the common point.

7.2) Each bonding mapping P_{n+1} when restricted to a component $T_{n+1,k}$ (and denoted by $P_{n+1,j}$) maps $T_{n+1,k}$ into a component $T_{n,j}$ in the following special fashion. Each circle $U_{n+1,k,i}$ of $T_{n+1,k}$ is mapped into a circle $U_{n,j,l}$ of $T_{n,j}$ so that for each x in $U_{n+1,k,i}$ $P_{n+1}(x) = x^{d(n+1,k,i)}$, $d(n+1,k,i)$ being a non-negative integer. We have made identifications here which will later be treated more precisely.

There are two interesting subclasses of K arrived at by specializing either of the conditions (7.1) or (7.2).

7.3) The set of compacta with sequences (T,P) for which each $g(n,j) = 1$ is characterized by the property that each shape class contains a compactum with a toroidal defining sequencein S^3. Thus, suppose compactum C has toroidal defining sequence (M,i) in S^3. For each n choose $T_n = \bigcup_{j=1}^{c(n)} \{j\} \times T_0$. Suppose that $i_{n+1}[M_{n+1,k}] \subseteq M_{n,j}$. Let $i_{n+1,k} = i_{n+1}| M_{n+1,k}$, and $d(n+1,k)$ be the degree of of $i_{n+1,k}$. Define $P_{n+1,k}$ so that for (k,x) in $T_{n+1,k}$, $P_{n+1,k}((k,x)) = (j, x^{d(n+1,k)})$. Let $P_{n+1} = \bigcup_{k=1}^{c(n+1)} P_{n+1,k}$. Then the sequence (T,P) is shape equivalent to (M,i). On the other hand suppose that

(T,P) is a sequence with properties (7.1-7.3).Then it can be seen that we may construct a toroidal defining sequence (M,i) in S^3 which is shape equivalent to (T,P) and generates a homeomorphic compactum.

7.4) The set of compacta for which there exists an equivalent sequence (T,P) for which the value of each d(n,j,i) equals 1 or 0 will be investigated in the next section.

We shall now classify the toroidally defined compacta by means of $S_1(T_0)$. Suppose that C is a compactum asociated with ANR sequence (M,i) , (T,P) is an equivalent sequence obeying (7.1-7.3), and $C^* =$ inv lim (T,P). Each component C^*_α of C^* is generated by the sequence (T^α,P^α), and is therefore a point, a circle, or a solenoid. Thus C^* is a 1-dimensional representative with the maximum possible local connectedness, a term that can be made more precise if necessary. Each $T_n^\alpha = T_{n,\alpha(n)} = \{\alpha(n)\} \times T_0$. Each thread x of C^*_α is a sequence of pairs $(\alpha(n),t(n))$, where $t_n \in T_0$. We denote the two projections from $Z \times T_0$ by $proj_1$ and $proj_2$. Then $proj_1(x_n) = \alpha(n)$, and $proj_2(x_n) = t(n)$. The space C^* bears the topology of coordinatewise convergence. The set of sets $N_{n,e}(x) = \{y \mid$ y belongs to C^*, $proj_1(y_n) = proj_1(x_n)$, and $|proj_2(y_n) - proj_2(x_n)| < e\}$ for all $e > 0$ and all positive integers n forms a basic set of open neighborhoods of x in C^*.

Let E be the set of all identity elements of C^*, that is, $E = \bigcup\{1_\alpha \mid \alpha \in A\} = \{\{(n,(\alpha(n),1)) \mid n \geq 1\} \mid \alpha \in A\}$.(The definition of A is given in section 1.)

The value of $\overline{S}_1(T,P;T_0)$ is either the group (1) or the one point union of circles and solenoids bearing a topology which is indicative of shape. This value is a subspace of L = inv lim $(H_1 \cdot T,$

$H_1 \cdot P$). The value of $H_1(T_n)$ is $\prod_{j=1}^{c(n)} \{j\} \times T_0$. The group $\overline{L}_\alpha$ is composed of threads x each of which is a sequence of elements x_n which belong to $J_{n,\alpha(n)}[T_0]$. In other words, x_n is a c(n) tuple with only the $\alpha(n)$th coordinate different from 1.The set of sets $U_{n,e}(x) = \{ y \mid y \in \overline{S}_1(T) \text{ and } |x_{n,i} - y_{n,i}| < e \text{ for } 1 \leq i \leq c(n) \}$ for n a positive integer and e a positive number is a local base of open neighborhoods of x in $\overline{S}_1(T)$. The mapping σ takes each non-zero x in $\overline{L}_\alpha$ to the thread m^α of $S_0(T)$. There is a canonical topological group isomorphism h_α from C^*_α to $\overline{L}_\alpha$ which maps the thread x to $h_\alpha(x)$ with $(h_\alpha(x))_n = J_{n,\alpha(n)}(\text{proj}_2(x))$. In words, the nth coordinate of x is $(\alpha(n), \text{proj}_2(x_n))$ and it is merely formally changed by h to the c(n) tuple with $\text{proj}_2(x_n)$ in the $\alpha(n)$th slot and 1 elsewhere. The union H_T of all the h_α for $\alpha \in A$ is a closed continuous surjection H_T of C^* onto $\overline{S}_1(T)$. On each C^*_α, H_T is an embedding. Thus the opening statement of the paragraph can be verified. Let h_T be H_T restricted to $C^* - E$. Then h_T is a homeomorphism of $C^* - E$ onto $\overline{S}_1(T) - \{0\}$.

There is a canonical mapping from C^* to $\Lambda(C^*)$ which takes each point x to the component C^* to which it belongs. We can define the mapping K_T from C^* to $S_0(T)$ by setting $(K_T(x))_n = m_{n,\text{proj}_1(x_n)}$. In words, K_T takes the thread x and picks out for each n the first coordinate of x_n, and this coordinate identifies the component of M_n in which the nth coordinate of x lies. The generator corresponding to this component is chosen by K_T as the nth coordinate of a thread. The mapping K_T is a closed continuous surjection. Let k_T be K_T restricted to E. This mapping is a homeomorphism onto $S_0(T)$. We also observe that $K_T|(C^* - E) = \sigma \cdot h_T|(C^* - E)$.

Hypothesis 7.5. Assume that compacta C and $\underline{C}$ are associated with ANR sequences which are equivalent to sequences (T,P) and ($\underline{T}$,$\underline{P}$) obeying (7.1-7.3). Let $C^* = \text{inv lim}\ (T,P)$ and $\underline{C}^* = \text{inv lim}\ (\underline{T},\underline{P})$.

Theorem 7.6. The compacta C and $\underline{C}$ have the same shape iff $S_1(T)$ is isomorphic to $S_1(\underline{T})$.

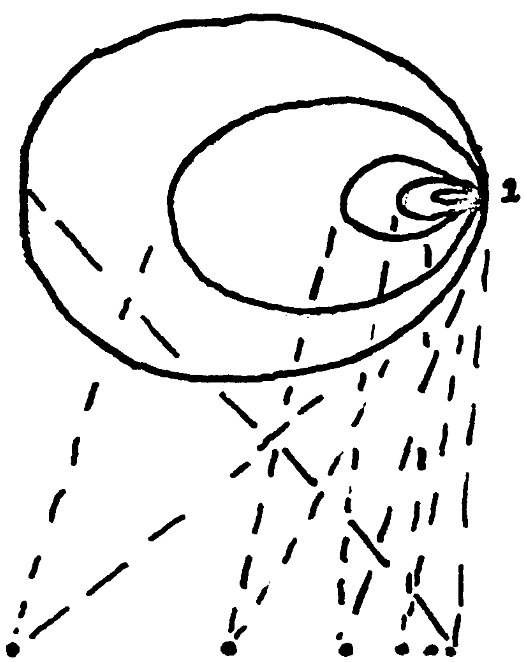
Sufficiency has already been demonstrated; we derive the necessary condition as a corollary to the stronger theorem which follows.

Theorem 7.7. If $S_1(T)$ is isomorphic to $S_1(\underline{T})$, then C^* is homeomorphic to $\underline{C}^*$.

Proof: We define the mapping g from C^* to $\underline{C}^*$ on two disjoint subsets. Let $g(x) = h_{\underline{T}}^{-1}.f_1.h_T(x)$ for $x \in C^* - E$; let $g(x) = k_{\underline{T}}^{-1}.f_0.k_T(x)$ for $x \in E$. The mapping g is a bijection of C^* onto $\underline{C}^*$. On the open set $C^* - E$, g is continuous. We show that g is continuous at each identity element 1_α of E. Let $\underline{1}_\alpha = g(1_\alpha)$. Denote $N_{n,e}(\underline{1}_\alpha)$ by G. The mapping g is a homeomorphism from E to $\underline{E}$, so that there exists an integer q and real positive number d with the property that each identity in $N_{q,d}(1_\alpha)$ is mapped into G. The mapping f_1 is also a homeomorphism, so that there exists an open neighborhood $U_{p,b}(1)$ with the property that $f_1[U_{p,b}(1)] \subseteq U_{n,e}(1)$. Take $r = p+q$ and a less than both b and d. Consider the basic open set $D = N_{r,a}(1_\alpha)$. Clearly $g[D \cap E] \subseteq G$. To show that $g[D \cap (C^*-E)] \subseteq G$, follow in detail the action of g on an element x of the subset. The coordinate x_r of x is of the form $(\alpha(r),v)$, where $v \in T_0$ and $0 < |v-1| < a$. The mapping h_T takes x to a thread z of $\overline{S}_1(T) - \{1\}$ which belongs to $U_{p,b}(1) - \{1\}$. The mapping f_1 takes $U_{p,b}(1) - \{1\}$ to threads in $U_{n,e}(\underline{1}) - \{\underline{1}\}$. The restriction of $h_{\underline{T}}^{-1}$ takes these elements into $N_{n,e}(\underline{1}_\alpha)$. Therefore $g[D] \subseteq N_{n,e}(\underline{1}_\alpha)$. So g is continuous at 1_α. But C^* is compact and $\underline{C}^*$ is Hausdorff, so that we may conclude that g is a homeomorphism.

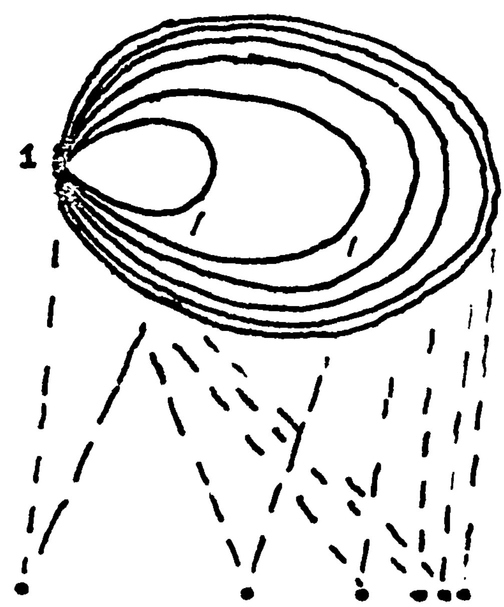
<u>Corollary 7.8.</u> The shape classes of toroidally defined compacta in S^3 correspond to the homeomorphism classes of the compacta associated with ANR sequences obeying (7.1-7.3).

It is clear upon inspection that the compacta K_1 and $K_{\underline{1}}$ satisfy both of the conditions appearing in (7.8). They are not homeomorphic, and therefore they do not have the same shape. We may also distinguish between their shapes by applying the functor $S_1(T_0)$. Pictures of $S_1(K_1,T_0)$ and $S_1(K_{\underline{1}},T_0)$ are presented in figures (7.9) and (7.10) respectively. It is clear that $\overline{S}_1(K_1)$ and $\overline{S}_1(K_{\underline{1}})$ are not homeomorphic so that the invariants are not isomorphic.



$S_1(K_1,T_0)$

Figure 7.9



$S_1(K_{\underline{1}},T_0)$

Figure 7.10

8.0 Plane compacta. Consider a compactum C in the plane. Complete the plane to S^2 by adding the point at infinity, $\infty = \{R^2\}$. There exists a sequence M of compact, polyhedral 2-manifolds with boundary such that $M_{n+1} \subseteq \mathring{M}_n \subseteq R^2$, $\bigcap_{n=1}^{\infty} M_n = C$, and each component of M_n is a disk with holes or a disk. Let i denote the sequence of embeddings $i_n : M_n \rightarrow M_{n-1}$ of each manifold in the preceding. Then in a natural way the space C is the inverse limit of the sequence (M,i). Each component manifold $M_{n,j}$ of M_n is a disk or disk with g(n,j) holes. Each hole is an open subdisk of R^2, denoted by $(d_{n,j})_i$, bounded by a polyhedral 1-sphere, denoted by $(b_{n,j})_i$, for $1 \leq i \leq g(n,j)$. Let $(d_{n,j})_0$ denote the remaining disk of $S^2 - M_{m,j}$, and $(b_{n,j})_0$ denote the remaining bounding curve of $M_{n,j}$, the outer boundary. When dealing with a sequence M^{α} generating a component C_{α} of C, we shall drop from the symbol $(b_{n,j})_i$ the index j, which must equal $\alpha(n)$, and let $(b_n^{\alpha})_i$ denote the ith boundary of M_n^{α}.

We shall construct a sequence W of sets W_n of triples $(B_{n,j}, T_{n+1,k}, B_{n+1,k})$ from which one can reconstruct a defining sequence $(\overline{M},\overline{i})$ in the plane which is shape equivalent to (M,i). For each manifold component $M_{n,j}$ we assign a set $B_{n,j}$. If $M_{n,j}$ is a disk then $B_{n,j}$ consists of the pair (n,j); if $M_{n,j}$ is a disk with g(n,j) holes then $B_{n,j}$ will be the ordered set $\{(b_{n,j})_i \mid 1 \leq i \leq g(n,j)\}$. For each $n \geq 1$, the mapping i_{n+1} takes each component $M_{n+1,k}$ into the component $M_{n,j}$ of which it is a subset in a way that can be described by a matrix T associated with $B_{n,j}$ and $B_{n+1,k}$. If $M_{n,j}$ or $M_{n+1,k}$ is a disk, let T be the empty matrix. If $M_{n,j}$ is not a disk, the matrices for the set $\mathcal{M}$ of components of M_{n+1} which are not disks are calculated by an inductive procedure. Order the elements of $\mathfrak{M}$ so that $M_{n+1,h}$ $M_{n+1,k}$ if $M_{n+1,h}$ lies in one of the disks $(d_{n+1,k})_i$ complementary to $M_{n+1,k}$ in R^2. This relation induces a partial ordering on the members of $\mathfrak{M}$. Select a maximal member $M_{n+1,k}$ of $\mathfrak{M}$. Form a matrix with $g(n+1,k)$ rows and $g(n,j)$ columns.row by row. Each row is to designate the circles of $M_{n,j}$ linked by a particular circle of $M_{n+1.k}$. A "1" appears in the ith row and pth column of T if the circle $(b_{n+1,k})_i$ links the circle $(b_{n,j})_p$; otherwise a "0" is to appear there. The triple $(B_{n,j}, T, B_{n+1,k})$ is added to the set of those already determined, the component $M_{n+1,k}$ is removed from the set $\mathfrak{M}$ and the process is repeated. It is characteristic of a set of nested planar manifolds that each matrix T has no column with more than a single "1" in it, for two circles of an embedded component $M_{n+1,k}$ cannot link the same circle of $M_{n,j}$ in which it is contained.

Although the sequence W does not contain much of the inessential information carried by the sequence M, it is not the only such sequence derived from compacta of a given shape class.The details of the pattern (M,i) are not characteristic of shape, rather the properties of the pattern in the limit as n becomes infinite seem to be significant. We seek a way of expressing this limiting information and obliterating details.

Apply the homology functor $H_1(Z_2)$ to the sequence (M,i). The sequence W allows one to construct the sequence $(H_1 \cdot M, H_1 \cdot i)$, for it provides the generators of $H_1(M_n)$ and mappings of generators which generate each $H_1(i_{n+1})$. On the other hand, the sequence $(H_1 \cdot M, H_1 \cdot i)$ allows one to reconstruct inductively a sequence $(\overline{M},\overline{i})$ which is shape equivalent to (M,i). However, the sequence $(H_1 \cdot M, H_1 \cdot i)$ is not the only sequence deriveable from a shape class. A sequence of this kind still contains unnecessary information, and cannot serve as an invariant.

The CTAG inv $\lim(H_1 \cdot M, H_1 \cdot i)$ contains too little information. The set of plane shapes is uncountable [4] , but the set of 2-groups which can serve as values for homology groups of plane compacta is countable, as one may deduce from the theorems of [16]. We therefore derive from the sequence W an invariant expressive only of shape. This invariant is related algebraically to the invariant $S_k(Z_2)$ which we have already defined.

One of the steps in this search is to find a simple defining sequence for a compactum which will simplify proofs. It is helpful, though not necessary, to choose a defining sequence M so as to satisfy the conditions formulated by Spiez in [36]. We may choose (M,i) so that we may define sequences of functions $\phi_n : \{M_{n,i}\}_{i=1}^{c(n)} \longrightarrow \{M_{n+1,j}\}_{j=1}^{c(n+1)}$ and $a_n : \{M_{n,i}\}_{i=1}^{c(n)} \longrightarrow [0,\omega]$ with the following properties.

8.1) M_1 is a disk. Each component $M_{n,i}$ of M_n contains as a subset a "main" component $\phi_n(M_{n,i})$ and possibly a "subsidiary" component $M_{n+1,j}$ of M_{n+1}. If $M_{n,i}$ has $k \geq 0$ holes, then $\phi(M_{n,i})$ has k (or k+1) holes and is embedded in $M_{n,i}$ with no (one) null homologous boundary curve. The subsidiary component is either a disk or an annulus. If an annulus, it lies in an annular component of $M_{n,i} - \phi_n(M_{n,i})$ and is not contractible therein.

8.2) The value of $a_n(M_{n,i})$ is the shape number (that is, the number of disk components of $R^2 - \bigcap_{j\geq 1} \phi_{n+j}(\ldots(\phi_n(M_{n,i})))$).

8.3) If $M_{n,i}$ has k holes and $k < a_n(M_{n,i})$, then $\phi_n(M_{n,i})$ has k+1 holes.

When the conditions of Spiez are imposed, the sequence W derived from the defining sequence (M,i) becomes quite simple. If $M_{n,j}$ is not a disk, but has h holes, then the matrix T for the main component $M_{n+1,k}$ will be the h X h identity matrix in the case that $M_{n+1,k}$ has h holes, and the identity matrix augmented by a row of zeros otherwise. The matrix for a subsidiary one will be a 1 x h matrix with a "1" in only 1 column or with a "1" in each of the h columns if the component is not a disk.

We shall call a defining sequence obeying the conditions 8.1 - 8.3 a Spiez sequence. Each Spiez sequence for a plane compactum is an ANR sequence satisfying conditions 7.1, 7.2, and 7.4.

Lemma 8.4. The set of components of a plane compactum with shape number greater than one is countable.

Proof: We may label the at most two components in each $M_{n,i}$ so that $M_{n+1,1}$ is the main component and $M_{n+1,0}$ is the possibly existing subsidiary component. Then there is a bijection from the set of components C_α of C to a subset of the set of sequences taking the values zero and one only. If a component C_α has shape number greater than one, then there must be an integer p such that M_n^α is a main component from n = p on. The sequence corresponding to C_α will have value equal to one from some point on. But the set of all sequences of zeros and ones which are eventually equal to one is at most countable.

Remark 8.5. The set of all circles in the plane with center at (-1,0) and passing through a point of the Cantor set is an example of a plane compactum with an uncountable set of components of non-trivial shape.

Let us examine the relation between the sequence M^{α} which generates a component C_{α} of C and the shape number of C_{α} for a Spiez sequence. There are three possibilities for (M^{α}, i^{α}):

8.6) The sequence i^{α} may contain a cofinal sequence of null-homotopic embeddings.

8.7) The sequence i^{α} may contain no null-homotopic embeddings from some point on, and the sequence M^{α} may contain a subsequence of annuli $M^{\alpha}_{n(i)}$.

8.8) The sequence M^{α} may be composed of main components with two or more holes from some point on.

When (8.6) obtains, the C_{α} is of trivial shape with shape number equal to zero. Suppose that (8.7) obtains. There is a least integer k for which i^{α}_n is an essential embedding for $n \geq k$. Then M^{α}_k must have at least one hole. It cannot have two holes, for then i^{α}_k would be essential. Thus M^{α}_k is an annulus. Choose a sequence $(\underline{M}, \phi)$ as follows. Let $\underline{M}$ be the sequence of all annuli $M^{\alpha}_{n(j)}$ with $n(j) \geq k$ in the sequence M^{α}. We have $\underline{M}_1 = M^{\alpha}_k$, $\underline{M}_j = M^{\alpha}_{n(j)}$, and $\underline{M}_{j+1} = M^{\alpha}_{n(j+1)}$. Let $\phi_{j+1} = V^{\alpha}_{n(j),n(j+1)} = i^{\alpha}_{n(j)+1} \cdots i^{\alpha}_{n(j+1)}$. There are two subcases:

8.7.1) From some integer on the ϕ_j are all essential embeddings.

8.7.2) There is a subsequence of ϕ the members of which are null-homotopic embeddings.

When (8.7.2) prevails, the component C_α has trivial shape. Let us consider the case (8.7.1) more closely. Look at the segments of (M^α, i^α) of the form:

$$(8.9)\qquad M^\alpha_{n(j)} \xleftarrow{i^\alpha_{n(j)+1}} M^\alpha_{n(j)+1} \leftarrow \cdots \longleftarrow M^\alpha_{n(j+1)} \quad .$$

The composition of the mappings in each segment is ϕ_j. The first begins with $n(1)=k$ and the composition is ϕ_2. In each segment the genus of M_n^α rises in single steps from a value of one for $M_{n(j)}^\alpha$ until $M_{n(j+1)-1}^\alpha$ when it again drops to one for the annulus $M_{n(j+1)}^\alpha$. The matrix T in the triple $\{B_{1,\alpha(1)}, T, B_{1+1,\alpha(1+1)}\}$ representing the action of the mapping i_{1+1}^α on the boundary circles of M_{1+1}^α is an identity matrix augmented by a row of zeros except for the last mapping in the segment, $i^\alpha_{n(j+1)}$. We can see that ϕ_j for $j \geq 3$ will be essential if the circle $(b_{n(j)})_1$ is deformable in the preceding manifold component $M_{n(j)-1}^\alpha$ to either the circle $(b_{n(j)-1})_1$ or the union $\cup\{(b_{n(j)-1})_i \mid 1 \leq i \leq g(n(j)-1, \alpha(n(j)-1)\}$. (Alternatively, one may say $(b_{n(j)-1})_0$ in the second case.) It will be null-homotopic if $(b_{n(j)})$ is deformable to $(b_{n(j)-1})_i$ with $i > 1$. Thus the statement (8.7.2) is equivalent to the following statement. There is a subsequence of $\underline{H}$ of annuli whose bounding circles $(b_{h(i)})_1$ are deformable in the preceding manifold to $(b_{h(i)-1})_l$ with $l > 1$.The statement (8.7.1) is equivalent to the following statement. From some integer on all the components which are annuli have the property that $(b_n)_1$ is deformable in M_{n-1}^α to either $(b_{n-1})_1$ or $(b_{n-1})_0$. Here C_α has the shape of an annulus.

Let us consider the consequences of (8.7.1) more closely in order to determine the value N of the index counting the stage at which "a hole in C first appears". There are two possible cases according to the behavior of the sequence ϕ.

8.10) All the ϕ_j are essential. The define N to be k. We have seen that M_N^{α} is an annulus, and it is inessentially embedded in M_{N-1}^{α}. The condition(8.1) requires M_{N-1}^{α} to be a disk. Furthermore, all $V_{N,n}$ for $n > N$ are essential. This is because there is an inclusion map u such that $V_{N,n} \cdot u = \phi_2 \cdot \phi_3 \cdots \phi_j$ for sufficiently high values of j.

8.11) At least one inclusion ϕ_j is inessential.Let q be last integer for which ϕ_q is inessential. Now ϕ_q is the composition of the mappings in the segment

$$8.12) \qquad M_Q^{\alpha} \xleftarrow{i_{Q+1}^{\alpha}} M_{Q+1}^{\alpha} \leftarrow \cdots \xleftarrow{i_R^{\alpha}} M_R^{\alpha} ,$$

where R = n(q) and Q = n(q-1). The length of this segment must be greater than one, that is $R-Q > 1$. For if $R = Q+1$, then the annulus M_R^{α} would be embedded in M_Q^{α} so as to be contractible. This is, however forbidden by (8.1). Moreover, the value of Q cannot be less than k. The manifold M_R^{α} is an annulus with its bounding circle $(b_R)_1$ deformable in M_{R-1}^{α} to a circle $(b_{R-1})_l$ with $l > 1$. The component M_{R-1} has R-Q holes, and the disk $(d_R)_1$ contains $(d_{R-1})_l$ as a subdisk. We compute that $(b_R)_1$ is contractible in M_{Q+l-2}^{α} but not in M_{Q+l-1}^{α}. The circle $(b_{Q+l-}$ is contractible in M_{Q+l-2}^{α}. The manifold M_{Q+l-2}^{α} has only l-1 holes in it, and the boundaries of these holes are the circles $(b_{Q+l-2})_i$ for $1 \leq i \leq l-1$. In this case choose $N = Q+l-1$. One can then show that all the maps $V_{N,n}$ are essential.

In the case (8.9) the component C_{α} has shape number,sh, in [2,4]. We proceed with an analysis which has features similar to the one above. We shall define an integer for each bounded component of $R^2 - C_\alpha$ which will count the manifold at which "the hole first appears". We do this by first defining a pair of integers N and P with $N \leq P$ which will be very useful in the computations of future sections. Let $\underline{M}$ denote the necessarily finite subsequence of M^α comprised of all annuli $M^\alpha_{n(j)}$, for $1 \leq j \leq a$. Let ϕ_{j+1} be as before the inclusion map taking $\underline{M}_{j+1} = M^\alpha_{n(j+1)}$ into $\underline{M}_j$. There are four cases to consider.

8.13) There is only one annulus $\underline{M}_1 = M^\alpha_{n(1)}$ in the sequence $\underline{M}$. Then let both P and N be equal to the index n(1) of this annulus.

If there is more than one annulus in the sequence M^α, let P be the index of the last annulus. Now we shall proceed as in the paragraphs above.

8.14) If all the inclusions ϕ_j are essential, let $N = n(1)$, the index of the first annulus.

If they are not, let q be the last integer for which ϕ_q is inessential. Again ϕ_q is the composition of the inclusion maps in the segment of the form (8.12), where $R = n(q)$ and $Q = n(q-1)$. There are two distinct cases which can arise.

8.15) The manifold M^α_{R-1} is a disk. Then let $N = R$.

8.16) None of the M^α_n with $Q < n < R$ is a disk. Then let $N = Q+i-1$ for the appropriate $i > 1$ as in (8.12) above.

The number of holes in M^α_n rises in single steps from the value of one for M^α_P as n increases for all four cases. If sh is finite it reaches the constant value sh.

Let us relate the sequence M^α for a component C_α of shape number sh greater than zero to the bounded complementary domains B_i^α, $1 \leq i \leq sh$, in the plane. For sh equal to one, there is the least integer N for which all the inclusions $V_{N,n}^\alpha$ are essential. There is the cofinal subsequence of M^α composed of the $M_{n(j)}^\alpha$ with $n(j) \geq N$ which we have defined above. Each of the bounded components of R^2 - the annulus $M_{n(j)}^\alpha$ is a disk $(d_{n(j)})_1$. For $n(j) < n < n(j+1)$, where $M_{n(j)}^\alpha$ and $M_{n(j+1)}^\alpha$ are successive annuli, there are two or more disk components of $R^2 - M_n^\alpha$. If there are integers n with $N < n < n(1)$, then $R^2 - M_n^\alpha$ has two or more disk components. We define a monotone non-decreasing sequence G^α of open sets G_n^α whose union will be B_1^α. For $n < N$ let $G_n = \phi$, because there is no bounded component of $R^2 - M_n^\alpha$ which is a subset of B_1^α. If there are integers n with $N < n < n(1)$, then for $N \leq n < n(1)$ let $G_n^\alpha = (d_n)_i$. The integer i is, as we have seen, greater than one, and the symbol denotes the disk which lies in $d_{n(1)}$. If there are not, let $G_N^\alpha = (d_N)_1$. For each of the integers $n(j) \geq N$, let $G_{n(j)}^\alpha = (d_{n(j)})_1$, For $n(j) < n < n(j+1)$, let $G_n = (d_n)_1$ or $\bigcup \{ (d_n)_i \mid 1 \leq i \leq g(n, \alpha(n)) \}$. Clearly $(d_{n(j)})_1 \subseteq G_n \subseteq (d_{n(j+1)})_1$ for these values of n. For all n we have $G_n \subseteq G_{n+1}$, and $B_1^\alpha = \bigcup_{n=1} G_n^\alpha$. Observe that the integer N identifies the manifold M_N at which the hole B_1^α first appears, and that for $n < N$ we have $B_1^\alpha \not\subseteq M_n$ and not contained in its complement.

For $sh > 1$ we may for B_1^α define a non-decreasing sequence G_1^α such that $B_1^\alpha = \bigcup_{n=1}^\infty G_{1,n}^\alpha$. For $n \geq P$, let $G_{1,n}^\alpha = (d_n)_1$. When $n < N$ let $G_{1,n}^\alpha = \phi$. For the integers n with $N \leq n < P$ there are two possibilities. For cases (8.14-15) let $G_{1,n(j)}^\alpha = (d_{n(j)})_1$ for

$N \leq n(j) < P$. If there exist values of n with $n(j) < n < n(j+1)$, then let $G_{1,n}^{\alpha} = \bigcup\{(d_n)_i \mid 1 \leq i \leq g(n,\alpha(n))\}$. For case (8.16) the same recipe is followed with the additional stipulation that for $Q+i-1 \leq n < P$, $G_{1,n}^{\alpha} = (d_n)_i$, where i is the appropriate integer greater than one. Each of the bounded components of $R^2 - \mathcal{C}_\alpha$, B_i^{α} with $i > 1$, is the non-decreasing union of the disks $\{(d_n)_i \mid P+i-1 \leq n\}$. Therefore, let $G_{i,n}^{\alpha} = \emptyset$ for $n < P+i-1$; let $G_{i,n}^{\alpha} = (d_n)_i$ otherwise. For $i > 1$, M_{P+i-1} is the first manifold whose complement contains a subdomain of B_i^{α}. For $n < P+i-1$, $B_i^{\alpha} \subsetneq M_n^{\alpha}$. Finally we define a sequence G_0^{α} with $G_{0,n}^{\alpha} = \bigcup_{i=1}^{sh} G_{i,n}^{\alpha}$. Then $G_{0,n}^{\alpha} = G_{1,n}^{\alpha}$ for $n \leq P$, and $G_{0,n}^{\alpha} = \bigcup\{(d_n)_i \mid 1 \leq i \leq g(n,\alpha(n))\}$ for $n > P$. The set B_0^{α} is defined to be the union $\bigcup_{n=1}^{\infty} G_{0,n}^{\alpha}$.

9. <u>U-generating spaces for topological group unions.</u> All groups in this section are abelian. Let (M,i) be an ANR sequence and G a CTAG. Because the space M_n is an ANR(Compact Metric), each group $H_k(M_n,G)$ is the topological group quotient of rather special groups. When the group G carries a multiplicative structure which makes it a field, it is finite because it is compact. The finite fields are each isomorphic to one of the residue class fields Z/p^nZ, where either $p = 2$ and $n = 1$, or p is an odd prime and n is a positive integer. Then for $G = Z/p^nZ$ each $H_k(M_n,G)$ becomes a topological G module, for the multiplication is perforce continuous. The topological group $L = \text{inv lim } (H_k \cdot M, H_k \cdot i)$ can be given the structure of a topological G module by defining the product of a thread t and an element g of G as the thread s with $s_n = gt_n$. This makes each subgroup L_α a topological G module.

Each group $H_k(M_n,G)$ is [24] topologically isomorphic to the discrete group $\prod_{i=1}^{n} Z/q_iZ$, where n is a positive integer and the q_i are positive integers. It is not necessary that this group be able to carry the structure of a free G module. However, when $G = Z/pZ$, each element will have order p. Each CTAG which has this property is topologically isomorphic to a product group $\prod_{i \in I} \{i\} \times Z/pZ$ with the Tychonov topology. Such a group becomes in a natural way a Z/pZ vector space.

Let us assume for the remainder of this section that $G = Z/pZ$. Then there is a mapping q from the positive integers to the non-negative integers such that each group $H_k(M_n,G)$ is the discrete topological G module $\prod_{i=1}^{q(n)} \{i\} \times Z/pZ$. The group L and each group $\overline{L}$ is either the zero group or a compact Z/pZ vector space $\prod_{i=1}^{k} \{i\} \times Z/pZ$, where $1 \leq k \leq \omega$.

In a CTAG one may define [8] the notions of a summable family and a convergent series. In a totally disconnected CTAG these concepts coincide [8]. Indeed, it can be shown that a series $\sum_{n=1}^{\infty} x_n$ converges iff $\lim_{n\to\infty} x_n = 0$.

We extend to the class of compact, T_2, Z/pZ vector spaces the classical definitions for real Banach spaces.

Definition 9.1. The sequence t of elements t_i of the Z/pZ vector space $D = \prod_{i=1}^{\infty} \{i\} \times Z/pZ$ with the Tychonov topology is said to be a basis iff for each element x of D there exists a unique function Ψ from the positive integers to Z/pZ such that $x = \sum_{i=1}^{\infty} \Psi_i t_i$.

Definition 9.2. The basis t is said to be a Schauder basis iff the coordinate functionals $Co_n : D \to Z/pZ$ such that $Co_n(x) = \Psi_n$

are continuous.

Definition 9.3. The basis t is said to be unconditional iff the series expansion for each element is summable (alternatively, unconditionally or commutatively convergent).

Remark 9.4. For the vector space D every basis is an unconditional basis.

Let us derive several properties possessed by every basis for the vector space D. Let T be the set of basis elements t_i.

Lemma 9.5. Each basis t is ω-linearly independent, that is, if $\sum_{i=1}^{\infty} \psi_i t_i = 0$ then all the ψ_i equal 0.

Lemma 9.6. Let H be the Z/pZ topological vector space $\prod_{j \in J} \{i\} \times Z/pZ$ with the Tychonov topology and f a mapping of T into H.

9.6.1) If F is a continuous Z/pZ linear mapping from D into H such that $F|T = f$, then for each element x in D we have $F(x) = \sum_{n=1}^{\infty} Co_n(x)\, f(t_n)$.

9.6.2) The mapping F defined in (9.6.1) is a continuous Z/pZ linear mapping.

Lemma 9.7. Every basis for D is a Schauder basis.

This follows from (9.6) by extension of a bijection onto the canonical basis for D.

We require some results concerning a change of basis.

Remark 9.8. Let ψ be a sequence of mappings ψ_j from the positive integers into Z/pZ. Define the sequence $\underline{t}$ by letting $\underline{t}_j = \sum_{i=1}^{\infty} \psi_{j,i} t_i$. Then if $\underline{t}$ is a basis for D, for each positive integer i there must be a value of j for which $\psi_{j,i} \neq 0$.

For the next two lemmas let $p = 2$.

Lemma 9.9. Define $t_0 = \sum_{n=1}^{\infty} t_n$. Then $\sum_{n=0}^{\infty} t_n = 0$. Equivalently, for $k \geq 0$ we have $t_k = \sum_{n \neq k} t_n$.

Lemma 9.10. For each $k \geq 0$, the sequence of t_i with $i \neq k$ is a basis for D.

Suppose now that M is a positive integer and $D = \prod_{i=1}^{M} \{i\} \times Z/pZ$. Then the well known analogues of the preceding theorems can be obtained by replacing the infinity appearing in them by M.

Let d be either a positive integer or be replaced by infinity in the following definition. Let H be a CTAG topologically isomorphic to $\prod_{j=1}^{d} \{j\} \times Z_2$ carrying the product topology. Let $S = \bigcup \{ K_a \mid a \in A \}$ be a subspace of H with each K_a a closed subgroup of S. Let $K_a \cap K_b = 0$ if $a \neq b$.

Definition 9.11. The subspace B of S is said to be a U-generating space for S iff for each group $\underline{H}$ isomorphic to $\prod_{i=1}^{d} \{i\} \times Z_2$ and each subspace $\underline{S} = \bigcup \{ \underline{K}_b \mid b \in \underline{A} \}$, where the closed subgroups $\underline{K}_b$ and $\underline{K}_a$ intersect in 0, and each continuous, addition preserving mapping f from B into $\underline{S}$ with the property that $f[K_a \cap B] \subseteq \underline{K}_b$ for some $b \in \underline{A}$, there is a unique extension F of f to S which is continuous and a homomorphism on each K_a.

Remark 9.12. If B is a U-generating space for S, $B \cap K_a$ contains a basis for K_a.

10. The functions $\overline{Str}_1(Z_2)$ and $\overline{St}_1(Z_2)$. We define function $\overline{Str}_1(Z_2)$ whose domain is the class of Spiez sequences, a subset of the class of ANR sequences. The value of $\overline{Str}_1(M,Z_2)$ is a subspace of $\overline{S}_1(M,Z_2)$. It is defined in a uniform way over the domain but is not functorial. We define the function $\overline{St}_1(Z_2)$ so that its value $\overline{St}_1(M,Z_2)$

is the pointed space $(\overline{Str}_1(M,Z_2) \cup \{0\}, 0)$, again bearing the topology induced from L. Here 0 denotes the zero of the inverse limit group L.

From this point until the contrary is indicated the coefficient group will be Z_2, and the symbol "Z_2" will be omitted unless necessary. We assume also that (M,i) is a Spiez sequence for a plane compactum C. For the sequence (M^{*},i^{*}) generating C_{*}, let M^{*} and P^{*} be the integers defined in section 8. We shall select for each component C_{*} a canonical set $\mathcal{L}_{*}$ of threads of the group $\overline{L}_{*}$. If C_{*} has trivial shape, let $\mathcal{L}_{*}$ be the empty set, for neither of the functions will consider such components. If C_{*} has the shape of an annulus, we select the only non-zero thread of $\overline{L}_{*}$, for as we shall see the precise component of $R^2 - C_{*}$ in which convergence to C_{*} takes place is of no significance. If C_{*} has shape number greater than one, we shall select one thread for each component of $R^2 - C_{*}$, for as we shall later illustrate, the designation of the components in which convergence occurs is necessary to specify a plane shape.

Each group $H_1(M_n)$ is isomorphic to the discrete product group $\prod_{j=1}^{c(n)} H_1(M_{n,j})$. Each group $H_1(M_{n,j})$ is the zero group or is isomorphic to the product group $(Z_2)^{g(n,j)}$. The set $\{(b_{n,j})_i \mid 1 \leq i \leq g(n,j)\}$ can be identified with a basis for $H_1(M_{n,j})$ considered as a Z_2 vector space. We shall also identify the elements of this set with a subset of the basis for $H_1(M_n)$ to avoid clumsy and unnecessary notation.

For a component C_{*} with sh = 1, let $\mathcal{L}_{*}$ consist of the thread t to be defined using section 8. (Where there is no danger of confusion, the symbol "α" will be omitted. For $n < N$, let $t_n = 0$. If there are integers n with $N < n < n(1)$, then let $t_n = (b_n)_i$ for the appropriate integer i greater than one for $N \leq n < n(1)$. For each of the integers $n(j) \geq N$ let $t_{n(j)} = (b_{n(j)})_1$. For $n(j) < n < n(j+1)$, let $t_n = \Sigma_{i=1}^{g(n,\alpha(n))} (b_n)_i$ or $(b_n)_1$. From the discussion in section 8 one can show that $I_{n+1}(t_{n+1}) = H_1(i_{n+1})(t_{n+1}) = t_n$.

When sh > 1, select threads t_i for i 1 as follows. For $n < P+i-1$ let $t_{i,n} = 0$. For $n \geq P+i-1$, let $t_{i,n} = (b_n)_i$. Clearly $I_{n+1}(t_{i,n+1}) = t_{i,n}$. To define the sequence t_1, let $t_{1,n} = 0$ for $n < N$ and $t_{1,n} = (b_n)_1$ for $n \geq P$. For the integers $N \leq n < P$ there are two possibilities. For cases (8.14-15) let $t_{1,n(j)} = (b_{n(j)})_1$. If there exist values of n with $n(j) < n < n(j+1)$, let $t_{1,n} = \Sigma_{i=1}^{g(n,\alpha(n))}(b_n)_i$ or $(b_n)_1$. In case (8.16) one defines for $Q+i-1 \leq n < R$ the value of $t_{1,n}$ to be $(b_n)_i$ for the appropriate value of i greater than one. It is straightforward to show that $I_{n+1}(t_{1,n+1}) = t_{1,n}$ for all values of n. A sequence t_0 is defined by letting $t_{0,n} = t_{1,n}$ for $n \leq P$ and $\Sigma_{i=1}^{g(n)} t_{i,n}$ for $n > P$. That $I_{n+1}(t_{0,n+1}) = t_{0,n}$ for $n < P$ follows from the proof for t_1. For $n \geq P$, the result follows because $I_{n+1}((b_{n+1})_i)$ equals $(b_n)_i$ when $1 \leq i < g(n,\alpha(n))$, and $I_{n+1}(b_{n+1})_{g(n)}$ equals zero. (Hereafter we shall omit the symbol $\alpha(n)$ from the value of g.) Thus all the sequences t are threads of $\overline{L}_\alpha$. The symbols t_i^α will hereafter denote these canonically defined elements of $\overline{L}_\alpha$.

The space $\overline{Str}_1(M)$ will be the set $\{t_\alpha \mid C_\alpha$ is a component of $C\}$ carrying the topology induced from L.

<u>11. The functions Str_1 and St_1.</u> Define σt to be the restriction of $\sigma : \overline{S}_1(M) \to S_0(M)$ to $\overline{Str}_1(M)$. Let $\mathbf{e}t$ be the restriction of $\mathbf{e}$

to $\overline{St}_1(M)$. The mapping σt is a continuous mapping into, but not necessarily onto, $S_0(M)$.

We define the function Str_1 with domain the set of all Spiez sequences M to have the triple $\sigma t : \overline{Str}_1(M) \longrightarrow S_0(M)$ as its value at M. The value of the function St_1 at M is defined to be the triple $\sigma t : \overline{St}_1 \longrightarrow S_0(M)$.

If f is a mapping from one Spiez sequence (M,i) to another $(\underline{M},\underline{i})$, $\overline{S}_1(f)$ need not map $\overline{Str}_1(M)$ into $\overline{Str}_1(\underline{M})$ even if f is derived from a homotopy equivalence of polyhedra.

<u>12. Schauder bases and $\overline{Str}_1$.</u> Suppose that s is thread of $S_1(M)$. Let q be an increasing sequence of positive integers.

<u>Lemma 12.1</u>. The set of sets $\{ x \mid x \in \overline{S}_1(M) \text{ and } x_{q(j)} = s_{q(j)} \}$ for all j is a local base of open and closed sets for s in $\overline{S}_1(M)$.

<u>Lemma 12.2</u>. Suppose that y is a sequence of threads in $\overline{S}_1(M)$. The sequence y converges to s iff for each value of j there is an integer I such that when $i \geq I$ we have $y_{i,q(j)} = s_{q(j)}$.

Suppose that A is a subset of $\overline{S}_1(M)$ and $s \in A$.

<u>Lemma 12.3</u>. The preceding lemmas remain true if A is substituted for $\overline{S}_1(M)$.

<u>Lemma 12.4</u>. Suppose that component C_α has shape number, sh, greater than one, but finite. Then the set $\{t_i \mid 1 \leq i \leq sh \}$ is a Schauder basis for $\overline{L}_\alpha$.

Proof; Each element $\Sigma_{i=1}^{sh} r_i t_i$, where $r_i \in Z_2$, belongs to $\overline{L}_\alpha$ because $\overline{L}_\alpha$ is a Z_2 vector space. Suppose that x is an element of $\overline{L}_\alpha$. We have seen that for $n \geq P+sh-1$, M_n^α has sh holes. The Z_2 vector space $H_1(M_n^\alpha)$ has as a basis the set $\{b_{n,i} \mid 1 \leq i \leq sh \}$. Furthermore,

$t_{i,n} = b_{n,i}$ for positive i. Thus for these values of n there exist unique $r_{n,i}$ in Z_2 such that $x_n = \Sigma_{i=1}^{sh} r_{n,i}\, t_{i,n}$. We shall show that $r_{n+1,i} = r_{n,i}$. Now

$$\Sigma_{i=1}^{sh} r_{n,i}\, t_{i,n} = x_n$$

$$= I_{n+1}(x_{n+1}), \text{ (because x is a thread)},$$

$$= I_{n+1}(\Sigma_{i=1}^{sh} r_{n+1,i}\, t_{i,n+1})$$

$$= \Sigma_{i=1}^{sh} r_{n+1,i}\, I_{n+1}(t_{i,n+1}), \text{(because}$$

I_{n+1} is a Z_2 linear mapping),

$$= \Sigma_{i=1}^{sh} r_{n+1,i}\, t_{i,n}, \text{ because } n \geq P+sh-1.$$

Then we can conclude that $r_{n+1,i} = r_{n,i}$, because the $t_{i,n}$ form a basis. Now let $R_i = r_{P+sh-1,i}$. Then the threads x and $\Sigma_{i=1}^{sh} R_i\, t_i$ have the same nth coordinates for $n \geq P+sh-1$. They have the same coordinates for $n < P+sh-1$ because they are threads. Consequently they are equal.

Corollary 12.5. Suppose that sh is finite and greater than one. Let $0 \leq k \leq sh$. The set $\{ t_i \mid i \neq k \}$ is an unconditional Schauder basis for $\overline{L}_\alpha$.

This follows from the finite dimensional analogue of (9.10) and the other lemmas of that section.

Lemma 12.6. Suppose that C_α has infinite shape number. Then the set $\{t_i \mid 1 \leq i\}$ is a basis for $\overline{L}_\alpha$.

The series $\Sigma_{i=1} r_i\, t_i$ belongs to $\overline{L}_\alpha$ for r_i in Z_2. To prove this let $s(n) = \Sigma_{i=1}^{n} r_i\, t_i$. The set $R_q = \{(x,y) \mid x \text{ and } y \in \overline{L}_\alpha \text{ and } x_q - y_q = 0\}$ for $q \geq P$ is a base for the only uniformity on $\overline{L}_\alpha$. Now $t_{i,q} = 0$ for $q < P+i-1$, and this is true when $q < i$. Therefore when m and n are greater than q, $(s(n), s(m))$ belongs to R_q, for $s(n)-s(m) = \Sigma_{i=m}^{n} r_i\, t_i$. Thus the sequence s is Cauchy and in a compact group converges.

Let us now prove the converse. Suppose that x belongs to $\overline{L}_\alpha$. Let $n \geq P$. Then $\{ b_{n,i} \mid 1 \leq i \leq n-P+1 \}$ is a basis for $H_1(M_n^\alpha)$, and $b_{n,i} = t_{i,n}$. There is a unique set of elements $r_{n,i}$ of Z_2 such that $x_n = \Sigma_{i=1}^{n-P+1} r_{n,i}\, t_{i,n}$. We can deduce that

$$\begin{aligned} \Sigma_{i=1}^{n-P+1} r_{n,i}\, t_{i,n} &= x_n \\ &= I_{n+1}(x_{n+1}) \\ &= I_{n+1}(\Sigma_{i=1}^{2+n-P} r_{n+1,i}\, t_{i,n+1}) \\ &= \Sigma_{i=1}^{2+n-P} r_{n+1,i}\, I_{n+1}(t_{i,n+1}) . \end{aligned}$$

Now, for $1 \leq i \leq n-P+1$ we have $I_{n+1}(t_{i,n+1}) = t_{i,n}$ and $I_{n+1}(t_{n-P+2,n+1})$ equal to zero. Then the right hand side becomes

$$= \Sigma_{i=1}^{n-P+1} r_{n+1,i}\, t_{i,n}.$$

We then can conclude that $r_{n+1,i} = r_{n,i}$ for $1 \leq i \leq n-P+1$. Let $R_i = r_{i+P-1,i}$ for $i \geq 1$. Then $x_n = \Sigma_{i=1}^{k} R_i t_i$ for $k \geq n-P+1$. Therefore by lemma 12.2, the series converges to x.

<u>Corollary 12.7</u>. Let $k \geq 0$, and C_α have infinite shape number. Then the set $\{ t_i \mid i \neq k\}$ is an unconditional Schauder basis for $\overline{L}_\alpha$.

This follows from (9.4), (9.7), and (9.10).

<u>Remark 12.8</u>. The mapping h with $h(0) = 0$ and $h(1/n) = t_{n-1}$ for $n \geq 1$ is a homeomorphism of the subspace $\{0\} \cup \{ 1/n \mid n \geq 1\}$ of the real line onto the subspace $\rho t^{-1}\{C_\alpha\}$ of $\overline{L}_\alpha$.

Hereafter let L_α denote $\mathcal{L}_\alpha \cup \{0\} = \rho t^{-1}\{C_\alpha\} = \{t_i \mid 0 \leq i\} \cup \{0\}$.

We establish only that h is continuous at zero, the remainder of the proof being straightforward. The set of sets $\{ t_i \mid t_{i,p+j} = 0 \}$ is a local base at zero when j takes all non-negative values. For

$j \geq 0$ the neighborhood $\{0\} \quad \{1/n \mid n \geq j+3\}$ is mapped into the set $\{0\} \cup \{t_i \mid i \geq j+2\}$. But $t_{i,P+j} = 0$ for $i \geq j+2$.

<u>13. Basic topological properties of $\overline{Str}_1(M)$ and $\overline{St}_1(M)$.</u> Suppose that the shape number of C_α is equal to one.

<u>Lemma 13.1.</u> There is an increasing sequence of positive integers $q(j)$ such that $M_{q(j)}$ is an annulus and $V_{q(j),q(j+1)}(t_{1,q(j+1)}) = t_{1,q(j)}$. The sets $u_j = \{ y \mid y \in \overline{St}_1(M)$ and $y_{q(j)} = t_{1,q(j)} \}$ for $j \geq 1$ is a local base of open and closed sets in $\overline{St}_1(M)$ at t_1.

Suppose that C_α has shape number greater than one and t_i is a thread of $\mathcal{L}_\alpha$.

<u>Lemma 13.2.</u> For $n \geq P$, each M_n^α is a disk with two or more holes. The sets $u_j = \{ y \mid y \in \overline{St}_1(M)$ and $y_j = t_{i,j}\}$ for $j \geq P+i-1$ is a local base of open and closed sets in $\overline{St}_1(M)$ at t_i. Zero does not belong to any of these sets.

<u>Lemma 13.3.</u> No point of $\overline{L}_\alpha - L_\alpha$ can be a limit point of $\overline{S}_1(M) - \overline{L}_\alpha$.

We treat two cases separately.

13.3.1) C_α has finite shape number sh greater than one. Suppose that s is a limit point of $\overline{S}_1(M) - \overline{L}_\alpha$, s belongs to $\overline{L}_\alpha - L_\alpha$, and x is a sequence of threads of $\overline{S}_1(M) - \overline{L}_\alpha$ converging to s. Now $s = \Sigma_{i=1}^{sh} r_i t_i$. We shall show that either all the r_i are equal to one or only one of them is. For the integer $q = P+sh$, there is an integer J such that when $j \geq J$ we have $x_{j,q}$, the qth coordinate of x_j, equal to s_q. Now x_j does not belong to $\overline{L}_\alpha$. Therefore there is an integer $d > q$ such that $x_{J,d} \neq s_d$. Let d be the least such integer. Then $x_{J,d-1} = s_{d-1}$, and $d-1 \geq q$. Now M_{d-1}^α is a main component with sh ≥ 2 holes. Suppose that x_J

belongs to $\overline{L}_\beta$, where $C_\beta = \sigma t(x_J)$. Then $M_d\beta$ is a subsidiary component which can only be a disk or annulus. It cannot be a disk, for this assumption would require that $x_{J,d}$ be zero. Then $x_{J,d-1}$ would have to be zero, which contradicts the fact that $x_{J,d-1} = s_{d-1}$, which is not zero. Therefore M_d^β is an annulus. By condition (8.1) $(b_d^\beta)_1$ must be deformable in $M_{d-1}^\beta = M_{d-1}^\alpha$ to one of the circles $(b_{d-1}^\alpha)_i$ or to the union $\Sigma_{i=1}^{sh}(b_{d-1}^\alpha)_i$. Then we have $x_{J,d-1} = t^\alpha_{i,d-1}$ or $\Sigma_{i=1}^{sh}\ t^\alpha_{i,d-1}$. Since $x_{J,d-1} = s_{d-1}$, it follows that $s_{d-1} = t_{i,d-1}$ or $\Sigma_{i=1}^{sh}\ t_{i,d-1}$. Because I_{n+1} takes $t_{i,n+1}$ to $t_{i,n}$ when $n \geq P+sh-1$, $s_n = t_{i,n}$ or $\Sigma_{i=1}^{sh}\ t_{i,n}$ for the values of $n \geq d-1$. Then $s = \Sigma_{i=1}^{sh}\ t_i$ or t_i for some $i \geq 1$.

13.3.2) The shape number of C_α is infinite. Then $s = \Sigma_{i=1}^{\infty}\ r_i t_i$. We shall show that either only one of the r_i is not zero or all are non-zero. Let k be the first positive integer for which $r_k \neq 0$. There is an integer J such that when $j \geq J$, $x_{k+P-1} = s_{k+P-1}$. Let q be the greatest integer for which $x_{J,q} = s_q$. Now $H_1(M_n^\alpha)$ is a Z_2 vector space with basis $\{ (b_n^\alpha)_i \mid 1 \leq i \leq n-P+1 \}$ for $n \geq P$. The manifold M_{q+1}^β, where $C^\beta = \sigma t(x_J)$, must be an annulus by the argument given above. Then s_q equals $t_{i,q}$ or $\Sigma_{i=1}^{q-P+1}\ t_{i,q}$ by the argument given above. Let $n(1) = q$. We now treat the two different possibilities by induction.[1]

Having defined an increasing sequence of integers $n(h)$ for $1 \leq h \leq H$ such that $s_{n(h)} = t_{i,n(h)}$ for these values, let us find an integer $n(H+1) > n(H)$ such that the equality holds for H+1. There is an integer J such that when $j \geq J$, $x_{j,n(H)+1} = s_{n(H)+1}$. Let q be as before the greatest integer for which $x_{J,q} = s_q$. Again one may deduce that s_q is equal to $t_{d,q}$ for some $d \geq 1$ or to $\Sigma_{i=1}^{q-P+1}\ t_{i,q}$.

[1]See section 1, ADDENDA, p. 70

If the first possibility obtains, the fact that for each e we have $V_{n(H),q}(t_{e,q})$ equal to zero or to $t_{e,n(H)}$ implies that d = i. It is not possible for the second possibility to occur, for $V_{n(H),q}(\Sigma_{i=1}^{q-P+1} t_{i,q}) = \Sigma_{i=1}^{n(H)-P+1} t_{i,n(H)}$. Let n(H+1) = q. We may therefore conclude that there is an increasing sequence of integers n(h) such that $s_{n(h)} = t_{i,n(h)}$. Consequently $s = t_i$.

Having defined a sequence of integers n(h) with $1 \leq h \leq H$ such that $s_{n(h)} = \Sigma_{i=1}^{n(h)-P+1} t_{i,n(h)}$ we can find a still larger integer n(H+1) for which $s_{n(H+1)} = \Sigma_{i=1}^{n(H+1)-P+1} t_{i,n(H+1)}$. This enables us to deduce that $s = \Sigma_{i=1}^{\infty} t_i$.

<u>Lemma 13.4</u>. The space $\overline{St}_1(M)$ is closed in $\overline{S}_1(M)$.

If a sequence in $\overline{St}_1(M)$ converges to zero, it converges to an element of this space, for zero belongs to it. If a sequence x of elements of this space converges to a non-zero element of $\overline{S}_1(M)$, then there is an integer J such that when $j \geq J$, $x_j \neq 0$. Application of the preceding lemma completes the proof.

<u>Corollary 13.5</u>. The space $\overline{St}_1(M)$ is compact.

<u>Lemma 13.6</u>. The space $\overline{St}_1(M)$ is the one point compactification of the space $\overline{Str}_1(M)$, where in case $\overline{Str}_1(M)$ is compact, this compactification is defined to be the result of adding an additional isolated point to the space.

This follows from the uniqueness of the one point compactification of a locally compact, non-compact space.

<u>Corollary 13.7</u>. For any component C_α of C, L_α is a discrete space in the relative topology. If $\overline{L}_\alpha$ is infinite, it is perfect, but the points of $\overline{L}_\alpha - L_\alpha$ are limit points of $\overline{L}_\alpha$ and not of $\overline{S}_1(M) - \overline{L}_\alpha$.

Corollary 13.8. If C_α has infinite shape number, there are no group structure relations between the elements of $\mathfrak{X}_\alpha$. The relation $t_0 = \Sigma_{i=1}^{\infty} t_i$ is a topological group relation and requires the notion of convergence.

Lemma 13.9. Two Spiez systems (M,i) and $(\underline{M},\underline{i})$ have isomorphic values of St_1 iff they have isomorphic values of Str_1.

This is true because any homeomorphism of locally compact Hausdorff spaces has an extension which is a homeomorphism of the one point compactifications which maps one added point to the other.

The next two lemmas are supplements to (12.2)

Lemma 13.10. For x in L, the sets $\{ y \mid y \in L \text{ and } y_n = 0\}$ are open and closed Z_2 linear subspaces, and the sets $\{ y \mid y \in L \text{ and } y_n = x_n \}$ are open and closed flats for $n \geq 1$. Each point x of a subspace A of L has a local base of open and closed sets formed by intersecting a set of these flats with A.

Lemma 13.11. The nth projection operator from L onto $H_1(M_n)$ is a continuous homomorphism for all n. Therefore if $x = \Sigma_{i=1}^{sh(C_\alpha)} r_i t_i^\alpha$, then $x_n = \Sigma_{i=1}^{sh} r_i t_{i,n}^\alpha$.

Lemma 13.12. Suppose that t_1^β is the only thread of $\mathfrak{X}_\beta$. There exists an increasing sequence of integers n(j) such that the pairs of disjoint open and closed neighborhoods

$u_{n(j)} = \{y \mid y \in \overline{St_1}(M) \text{ and } y_{n(j)} = t_{1,n(j)}^\beta\}$ and

$w_{n(j)} = \{y \mid y \in \overline{St_1}(M) \text{ and } y_{n(j)} = 0 \}$

have the following properties.

13.12.1) The set of $u_{n(j)}$ for $j \geq 1$ is a local base at t_1^β.

13.12.2) The set of $w_{n(j)}$ is a local base at 0.

13.12.3) For each component C_α of C with shape number greater than one, either $L_\alpha \cap u_{n(j)} = \emptyset$ or $u_{n(j)}$ contains precisely the threads $t_0{}^\alpha$ and $t_1{}^\alpha$ and the remaining elements of $\mathcal{I}_\alpha$ belong to $w_{n(j)}$.

Proof: Consider n(j) for $j \geq 1$. Denote it by N. Suppose that a thread $t_i{}^\alpha$ of L_α for C_α with shape number sh greater than one belongs to u_N. Denote $t_1{}^\beta$ by x. Then $t_{i,N}{}^\alpha = x_N$. For $n > P^\alpha$, $M_n{}^\alpha$ is a main component with two or more holes. We show first that it is not possible for P^α to be less than N. From the equality $t_{i,N}{}^\alpha = x_N$ we can conclude that $M_N{}^\alpha = M_N{}^\beta$. But $M_N{}^\beta$ is an annulus, whereas when $P^\alpha < N$, M_N^α has two or more holes. Because $P^\alpha \geq N$, we can deduce that $t_{j,N}{}^\alpha = 0$ for $j > 1$ and so $t_j{}^\alpha$ belongs to w_N. Moreover, $t_{0,n}{}^\alpha = t_{1,n}{}^\alpha$ for $n \leq P^\alpha$. Consequently, since one of $t_1{}^\alpha$ and $t_0{}^\alpha$ belongs to u_N, so does the other.

We remark that for $n(j) < n < n(j+1)$ the conclusion need not be true. It is possible that $t_1{}^\alpha$ not belong to u_n even though $t_0{}^\alpha$ does.

Lemma 13.13. Suppose that $t_k{}^\beta$ is a thread of $\mathcal{I}_\beta$ for a component C_β of shape number sh greater than one. Then the pairs of disjoint open and closed neighborhoods

$$u_n = \{\, y \mid y \in \overline{St_1}(M) \text{ and } y_n = t_{k,n}{}^\beta \,\} \text{ and}$$
$$w_n = \{\, y \mid y \in \overline{St_1}(M) \text{ and } y_n = 0 \,\}$$

for $n \geq P$ when $k = 0$ and for $n \geq P + k-1$ otherwise have the properties (13.13.1-3) listed in section 2, ADDENDA, on page 70.

Denote $t_k{}^\beta$ by x. Assume that $n \geq P$. Suppose that $t_i{}^\alpha$ belongs to u_n. Then $t_{i,n}{}^\alpha = x_n$. For the component C_α there is an integer P^α such that $M_P^\alpha\alpha$ is the last annulus in the sequence. For $n = P^\beta$ the next part of the argument is the same as the one above. For $n \geq P^\beta$, the assumption that $P^\alpha < n$ requires that $M_n{}^\alpha$, which equals $M_n{}^\beta$, be a a main component with two or more holes in both the sequences M^α and M^β. This is forbidden by (8.1-3). Therefore $P^\alpha > n$. But when $n \leq P^\alpha$, we have $t_{i,n}{}^\alpha = 0$ for $i > 1$, so that $t_i{}^\alpha$ belongs to w_n. Moreover, $t_{1,n}{}^\alpha = t_{0,n}{}^\alpha$, so that both $t_1{}^\alpha$ and $t_0{}^\alpha$ belong to u_n.

Theorem 13.14. Suppose that (M,i) and $(\underline{M},\underline{i})$ are Spiez sequences and that (h_1,h_0) is an isomorphism from $\overline{St}_1(M)$ to $\overline{St}_1(\underline{M})$. Then there is a unique extension of h_1 to $\overline{h} : \overline{S}_1(M) \to \overline{S}_1(\underline{M})$ such that $(\overline{h},h_0)$ is an isomorphism from $S_1(M)$ to $S_1(\underline{M})$.

Suppose that $\overline{h}$ is an extension of h_1 with the properties described above. Then for each C_α, $\overline{h}|\overline{L}_\alpha$ is a topological Z_2 vector space isomorphism which is an extension of $h_1|L_\alpha$. Therefore by (9.6), for each x in $\overline{L}_\alpha$ $\overline{h}(x) = \Sigma_{i=1}^{sh(C_\alpha)} Co_i(x)\, h_1(t_i{}^\alpha)$. Moreover, if we define $\overline{h}$ in this manner it will be a topological Z_2 vector space isomorphism on each fiber $\overline{L}_\alpha$. It will have as its inverse $\overline{h}^{-1}$ the mapping which is defined from $h_1{}^{-1}$ in the same manner.

We must now show that $\overline{h}$ is a continuous mapping. The lemma (13.3) shows that we need to demonstrate that $\overline{h}$ is continuous at the points of L_β which are limit points of $\overline{S}_1(M) - \overline{L}_\beta$ We show first that $\overline{h}$ is continuous at a point x in $\mathcal{L}_\beta$. Let $y = \overline{h}(x)$ which equals $h_1(x)$. Suppose that H is an open neighborhood of y in $\overline{S}_1(\underline{M})$. There is a pair of disjoint open and closed neighborhoods $\underline{u}_k = \{ d \mid d \in \overline{S}_1(\underline{M})$ and $d_k = y_k\}$ and $\underline{w}_k = \{ d \mid d \in \overline{S}_1(\underline{M})$ and $d_k = 0 \}$ with $\underline{u}_k \subseteq H$. There is a pair of disjoint neighborhoods $\underline{u}_n$ and $\underline{w}_n$ of y and of 0 in $\overline{St}_1(\underline{M})$ with the properties give in (13.12) and (13.13) and refining $\underline{u}_k$ and $\underline{w}_k$. There is a disjoint pair of neighborhoods G_x of x and G_0 of 0 in $\overline{St}_1(M)$ such that $h_1[G_x] \subseteq \underline{u}_n$ and $h_1[G_0] \subseteq \underline{w}_n$. We may find a pair of disjoint neighborhoods u_q and w_q of x and of 0 in $\overline{St}_1(M)$ possessing the properties (13.12) or (13.13) and refining G_x and G_0. Consider the neighborhood $v_q = \{ d \mid d \in \overline{S}_1(M)$ and $d_q = x_q \}$. Suppose that z belongs to v_q. Then z belongs to the fiber $\overline{L}_\alpha$ over component C_α, where $C_\alpha = \sigma(z)$. If $\overline{L}_\alpha$ has only one non-zero element, then $\overline{h}(z) = h_1(z)$, which belongs to H. If C_α has shape number, sh, greater than one, then $z = \Sigma_{i=1}^{sh} r_i\ t_i^{\alpha}$. Now the values of the qth coordinates $t_{i,q}^{\alpha}$ of t_i^{α} are zero when $i > 1$. Therefore from (13.11), $z_q = t_{1,q}^{\alpha}$. Now $\overline{h}(z) = \Sigma_{i=1}^{sh} r_i\ h_1(t_i^{\alpha})$. Because $t_1^{\alpha} \in u_q$, $h_1(t_1^{\alpha}) \in \underline{u}_n$; because $t_i^{\alpha} \in w_q$ for $i > 1$, $h_1(t_i^{\alpha}) \in \underline{w}_n$. By (13.11) $(\overline{h}(z))_n = \ \ r_i\ (h_1(t_i^{\alpha}))_n$. For $i > 1$, $(h_1(t_i^{\alpha}))_n = 0$ because these elements belong to $\underline{w}_n$. On the other hand $(h_1(t_1^{\alpha}))_n = v_n$. Thus, $(h(z))_n = v_n$, so that h(z) belongs to $\underline{u}_n$, a subset of H. If C_α has shape number equal to one, then $\overline{h}(t_1\ )$ belongs to H if t_1^{α} belongs to v_q because $\overline{h} = h_1$ on such a fiber. Now on $\overline{L}_\beta$, $\overline{h}$ is continuous, so that there exists an open neighborhood D of x in $\overline{S}_1(M)$ such that $\overline{h}[D] \subseteq \overline{L}_\beta$ H. Then $\overline{h}[D \cap v_q] \subseteq H$. Finally we can conclude that $\overline{h}$ is continuous at all points of $\overline{S}_1(M) - \{0\}$.

The inverse pair (h_1^{-1}, h_0^{-1}) is also an isomorphism, and the same reasoning can be applied to it. Produce from h_1^{-1} a mapping by the process above. It will be equal to $\overline{h}^{-1}$, as we have remarked. It is continuous on $\overline{S}_1(\underline{M}) - \{0\}$, for we may carry out a proof similar to the one above. Then $\overline{h} - \{(0,0)\}$ is a homeomorphism. Consequently, $\overline{h}$, its extension to the one point compactifications of $\overline{S}_1(M) - \{0\}$ and $\overline{S}_1(\underline{M}) - \{0\}$ is a homeomorphism.

14. The invariance of the functions Str_1 and St_1.

Theorem 14.1. Suppose that (M,i) and ($\underline{M}$,$\underline{i}$) are Spiez sequences for compacta C and $\underline{C}$ respectively. Suppose that in the range category

(h_1, h_0) is an isomorphism from $S_1(M)$ to $S_1(\underline{M})$. Then there is a homeomorphism h making the diagram

$$
\begin{tikzcd}
\overline{Str}_1(M) \arrow[r, "h"] \arrow[d, "\sigma t"] & Str_1(\underline{M}) \arrow[d, "\underline{\sigma t}"] \\
S_0(M) \arrow[r, "h_0"] & S_0(\underline{M})
\end{tikzcd}
$$

commute. Let $he = h \cup \{(0,0)\}$. Then (he, h_0) is an isomorphism from $St_1(M)$ to $St_1(\underline{M})$.

Let (g_1, g_0) be the isomorphism inverse to (h_1, h_0), and let $\overline{h}$ be the restriction of h_1 to $\overline{St}_1(M)$. The sets $\mathcal{L}_\alpha = \{ t_i \mid 0 \leq i\} \subseteq L_\alpha \subseteq \overline{L}_\alpha$ and $\underline{\mathcal{L}}_\alpha = \{T_i \mid 0 \leq i\} \subseteq \underline{L}_\alpha \subseteq \overline{\underline{L}}_\alpha$ have their usual meaning.

The mapping $\overline{h}$ takes zero to zero. It maps $\mathcal{L}_\alpha$ into $\overline{\underline{L}}_\alpha$, and not necessarily into $\underline{\mathcal{L}}_\alpha$. It must be a bijection from the limit points of $\overline{Str}_1(M)$ onto the limit points of $\overline{Str}_1(\underline{M})$ in the image $h_1[\overline{Str}_1(M)]$. Each of the limit points x of $\overline{Str}_1(M)$ in $\mathcal{L}_\alpha$ is a limit point of $\overline{Str}_1(M) - \overline{L}_\alpha$ by (13.3) and (13.7). Let $y = \overline{h}(x)$. The thread y must lie in $\overline{\underline{L}}_\alpha$, and it must be a limit point of $\overline{S}_1(\underline{M}) - \overline{\underline{L}}_\alpha$, for $\overline{L}_\alpha$ is mapped onto $\overline{\underline{L}}_\alpha$ and $\overline{S}_1(M) - \overline{L}_\alpha$ onto $\overline{S}_1(\underline{M}) - \overline{\underline{L}}_\alpha$. However, only the points of $\underline{\mathcal{L}}_\alpha$ can be limit points of $\overline{S}_1(\underline{M}) - \overline{\underline{L}}_\alpha$, by (13.3). Thus any point of $\mathcal{L}_\alpha$ which is a limit point is mapped into $\underline{\mathcal{L}}_\alpha$ by $\overline{h}$. It is only the isolated points of $\overline{Str}_1(M)$ which are mapped into $\overline{\underline{L}}_\alpha$ but not necessarily into $\underline{\mathcal{L}}_\alpha$. We must define the mapping h on the isolated points of $\overline{Str}_1(M)$ so that it is one to one and onto , fiber preserving and continuous. We observe that when C_α has shape number equal to one then there is only one way to map $\mathfrak{L}_\alpha$ to $\underline{\mathfrak{L}}_\alpha$, so that we only have to redefine $\overline{h}$ on the fibers over components with shape number greater than one. It is desireable that h possess the property $h(t_q^{\alpha}) = \Sigma_{i=0,\neq q}^{sh} h(t_i^{\alpha})$ possessed by $\overline{h}$. If we define h to be a bijection on each fiber, it will have this property because of (9.9). This implies that if all but one thread of $\mathfrak{L}_\alpha$ is a limit point, the mapping h can be taken equal to $\overline{h}$ on $\mathfrak{L}_\alpha$.

The set $\mathfrak{L}_\alpha$ is the disjoint union of A_α, the set of limit points of $\overline{Str_1}(M)$ in $\mathfrak{L}_\alpha$, and B_α, the set of isolated points of $\overline{Str_1}(M)$ in $\mathfrak{L}_\alpha$. A similar notation will be used for $Str_1(\underline{M})$. The mapping $\overline{h}$ takes A_α onto $\underline{A}_\alpha \subseteq \underline{\mathfrak{L}}_\alpha$, and B_α onto $C \subseteq \underline{\overline{L}}_\alpha$. The set $\mathfrak{L}_\alpha$ contains one element more than necessary for a Schauder basis for $\overline{L}_\alpha$. We shall form a basis V_α for $\overline{L}_\alpha$ in the case that B_α contains two or more elements by dropping a t_i belonging to B_α. In the case that B_α is finite drop the t_i with the highest value of the index i. If B_α is infinite drop a t_i with $i > 1$. Form $\underline{V}_\alpha$ for $\underline{\overline{L}}_\alpha$ in the same way. To be definite, drop the first t_i with $i > 1$. Then let $h(x) = \overline{h}(x)$ for x in A_α, and let h be order preserving with respect to the index i labelling the threads t_i in $V_\alpha \cap B_\alpha$ and T_i in $\underline{V}_\alpha \cap \underline{B}_\alpha$. Then h is a bijection of V_α onto $\underline{V}_\alpha$. If an element t_k has been dropped from B_α to form V_α, let $h(t_k) = \Sigma_{i=0,\neq k}^{sh} h(t_i)$.

We shall first prove that h is continuous. Suppose that x in $\mathfrak{L}_\alpha$ is a limit point of $\overline{Str_1}(M)$ and let $y = h(x)$, which equals $\overline{h}(x)$. Let $\underline{G}$ be an open neighborhood of y. By (13.12) and (13.13) there is an open subneighborhood $\underline{u}(y) \subseteq \underline{G}$ of y in $\overline{Str_1}(\underline{M})$ such that for each component $\underline{C}_\gamma$ of shape number greater than one, either $\underline{L}_\gamma \cap \underline{u}(y) = \emptyset$ or $\underline{L}_\gamma \cap \underline{u}(y) = \{T_0^\gamma, T_1^\gamma\}$. Because $\overline{h}$ is continuous, there is an open neighborhood G of x such that $\overline{h}[G] \subseteq \underline{u}(y)$. Again, by (13.12) and (13.13) there is an open closed neighborhood $u(x) \subseteq G$ of x in $\overline{Str_1}(M)$ and $u(x)$ has the property that for C_γ of shape number greater than one either $L_\gamma \cap u(x) = \emptyset$ or $L_\gamma \cap u(x) = \{t_0^\gamma, t_1^\gamma\}$. Suppose now that s is a thread in $u(x)$. Then $x = t_i^\gamma$ for some component $C_\gamma = \sigma t(x)$. If $\mathfrak{L}_\gamma = \{s\}$, then $h(s) = \overline{h}(s)$. Because $s \in G$, $\overline{h}(s) \in \underline{G}$. If $\mathfrak{L}_\gamma$ contains more than one element, then i is either zero or one, and both t_1^γ and t_0^γ belong to $u(x)$. Now $\overline{h}$ maps $u(x)$ into $\underline{u}(y)$, and it must, as a fiber preserving mapping , take t_0^γ and t_1^γ to some T_i^γ and T_j^γ of $\underline{\mathfrak{L}}_\gamma$. But only T_0^γ and T_1^γ can belong to $\underline{u}(y)$. Therefore $\overline{h}\ \{t_0^\gamma, t_1^\gamma\} = \{T_0^\gamma, T_1^\gamma\}$.

Let $E = \{t_0^\gamma, t_1^\gamma\}$ and $\underline{E} = \{T_0^\gamma, T_1^\gamma\}$. There are three cases to consider:

1) both of the members of E are limit points of $\overline{Str_1}(M)$,

2) neither of them is, and

3) precisely one of them is.

Because $\overline{h}[E] = \underline{E}$, and $\overline{h}$ is a bijection of limit points, a similar statement can be made for $\underline{E}$. If case j obtains for E, it also does for $\underline{E}$.

In the first case, $h|E$ has been defined so that it is equal to $\overline{h}|E$. Thus $h[E] = \underline{E}$.

In the second case, the recipe for h changes according to the nature of B_γ, the set of points of $\mathfrak{L}_\gamma$ isolated in $\overline{Str_1}(M)$. If B_γ contains three or more elements, then both the elements of E belong to the basis V_γ. Then there are three or more elements in $\underline{B}_\gamma$, and both the members of $\underline{E}$ belong to $\underline{V}_\gamma$. Because h was defined to be order preserving with respect to the index i on $V_\gamma \cap B_\gamma$, $h[E] = \underline{E}$. If B_γ contains precisely the two elements of E, $\underline{B}_\gamma$ contains only the two elements of $\underline{E}$. Then $h(t_0^\gamma) = T_0^\gamma$ by the recipe for h. The mapping h takes t_1^γ =

$\Sigma_{i=0,\neq 1}^{sh} t_i^{\gamma}$ to $\Sigma_{i=0,\neq 1}^{sh} T_i^{\gamma}$. Because this series equals T_1^{γ} by (9.9), $h[F] = \underline{E}$. We have already accounted for the situation in which B_γ contains fewer than two elements.

In the third case, when exactly one element of E is a limit point, its image is also. Then by the recipe for h, the isolated point in E is carried to the isolated point in $\underline{E}$ because h is defined to be order preserving with respect to i on B_γ.

We have in all three cases shown that $h[L_\gamma \cap u(x)] \subseteq \underline{G}$. This, together with the remarks about components of annular shape allows us to conclude that h is continuous at x. We have shown that h is continuous at each limit point of its domain. It is clearly continuous everywhere.

The definition for h is perfectly symmetric with respect to the systems M and $\underline{M}$. Therefore the same proof can be used to show that h^{-1} is continuous, for if one applies the same treatment to $\overline{h}^{-1}$ one will obtain h^{-1}. Thus h is a homeomorphism of $\overline{Str_1}(M)$ onto $\overline{Str_1}(\underline{M})$. But any homeomorphism of locally compact Hausdorff spaces can be extended to one homeomorphism of their one point compactifications, and $h \cup \{(0,0)\}$ is such an extension. Therefore it is a homeomorphism of $\overline{St_1}(M)$ onto $\overline{St_1}(\underline{M})$, and it preserves distinguished points.

<u>Corollary 14.2</u>. If one of the invariants S_1, St_1, or Str_1 yields isomorphic values for M and $\underline{M}$, then all of them do.

<u>Corollary 14.3</u>. Suppose that f is an equivalence from one Spiez sequence (M,i) to another, $(\underline{M},\underline{i})$. Then there is a homeomorphism h making the diagram

$$\begin{tikzcd}
\overline{Str}_1(M) \arrow[r, "h"] \arrow[d, "\rho t"] & \overline{Str}_1(\underline{M}) \arrow[d, "\underline{\rho t}"] \\
S_0(M) \arrow[r, "S_0(f)"] & S_0(\underline{M})
\end{tikzcd}$$

commute. The mapping $h \cup \{(0,0)\}$ preserves distinguished points and makes the diagram

$$\begin{tikzcd}
\overline{St}_1(M) \arrow[r, "he"] \arrow[d, "\rho t"] & St_1(\underline{M}) \arrow[d, "\underline{\rho t}"] \\
S_0(M) \arrow[r, "S_0(f)"] & S_0(\underline{M})
\end{tikzcd}$$

commute.

We may therefore define functions $St_1^{\,c}$ and $Str_1^{\,c}$ whose domains will be the set of plane compacta by selecting a Spiez sequence M for each compactum and calculating the values of $St_1(M)$ and $Str_1(M)$. Use of a different Spiez sequence for a compactum will lead to an isomorphic value. The superscript will be omitted where possible.

We may define similar functions on the shape classes of plane compacta by selecting a compactum in the plane from each shape class and using the procedure described above.

15. <u>A uniqueness theorem for the functions.</u>

<u>Theorem 15.1.</u> Suppose that (M,i) and $(\underline{M},\underline{i})$ are Spiez sequences for compacta C and $\underline{C}$ and that $(\overline{h},\overline{g})$ is an isomorphism from $St_1(M)$ to $St_1(\underline{M})$. Then there is a mapping (f, ϕ) from system M to $\underline{M}$ which induces $(\overline{h},\overline{g})$ and is an equivalence.

In the proof which follows we shall make identifications which are necessary to avoid the complicated presentation which otherwise would result.

The first step in the proof is to define for each Spiez sequence (M,i) an equivalent sequence (G,w) similar to those described in (7.1). For this purpose we use the sequence of sets of triples described in section 8. For each $n \geq 1$ define a set of components $G_{n,j}$ for $1 \leq j \leq c(n)$ as follows.

15.2.1) If $M_{n,j}$ is a disk, let $G_{n,j}$ be the trivial group, indexed, that is, $G_{n,j} = \{j\} \times (1)$.

15.2.2) If $M_{n,j}$ is a disk with $g(n,j)$ holes, each bounded by a circle $(b_{n,j})_i$ for $1 \leq i \leq g(n,j)$, then let $G_{n,j}$ be the indexed wedge $\{j\} \times \; V_{i=1}^{g(n,j)} \{i\} \times T_0$ of $g(n,j)$ groups $\{i\} \times T_0$. The topology on the wedge is the quotient topology after the unities of the groups have been identified. We let (j,i,y) denote any non-unity element of the wedge and $(j,-,1)$ or more simply $1_{n,j}$ denote the common unity.

15.2.3) Let $G_n = \bigcup_{j=1}^{c(n)} G_{n,j}$.

Suppose that $W_{p,k}$ is a wedge $\{k\} \times V_{i=1}^{P} \{i\} \times T_0$ of P copies of T_0 and $W_{q,j}$ is a wedge $\{j\} \times V_{i=1}^{Q} \{i\} \times T_0$ of Q copies. We define three kinds of mappings $d_{-,i}$. Let T stand for $\{k\} \times \{i\} \times T_0$.

15.2.4) Define $d_{0,i} : T \to W_{q,j}$ so that for each x in T_0

$$d_{0,i}((k,i,x)) = (j,-,1).$$

15.2.5) Define $d_{m,i}$ for $1 \leq m \leq Q$ so that

$$d_{m,i}((k,i,x)) = (j,m,x).$$

15.2.6) Define $d_{+,i}$ so that for all $e^{i\theta}$ in T_0, where $0 \leq \theta < 2\pi$,

$$d_{+,i}((k,i,e^{i\theta})) = (j,1 + [\theta/2\pi \cdot Q], e^{iQ\theta}).,$$

the brackets are used to denote the greatest integer function.

Definition 15.2.7. A mapping $f : W_{p,k} \to W_{q,j}$ is said to be of type Z_2 if $f = \bigcup_{i=1}^{P} d_{e,i}$, where e can take the values 0, +, or m, with $1 \leq m \leq Q$.

For each n we define mappings $w_{n+1,k} : G_{n+1,k} \to G_{n,j}$ when $M_{n+1,k} \subseteq M_{n,j}$ as follows.

15.2.8) If the matrix T of the triple described in section 8 is the empty matrix, let $w_{n+1,k} = \bigcup_{i=1}^{g(n+1,k)} d_{0,i}$. If $G_{n+1,k}$ is a point, then $w_{n+1,k}(1_{n+1,k}) = 1_{n,j}$.

15.2.9) If the matrix is the identity matrix let $w_{n+1,k} = \bigcup_{i=1}^{g(n+1,k)} d_{i,i}$.

15.2.10) If the matrix T is the identity matrix augmented by a row of zeros, let $w_{n+1,k} = d_{0,g(n+1,k)} \cup \bigcup_{i=1}^{g(n+1,k)-1} d_{i,i}$.

15.2.11) If the matrix T is a row of ones, let $w_{n+1,k} = d_{+,1}$.

Define $w_{n+1} = \bigcup_{k=1}^{c(n+1)} w_{n+1,k}$. Let $W_{p,q}$ be the composite $w_{p+1} \cdots w_q : G_q \to G_p$.

Let $C^* = \text{inv lim } (G,w)$. Each component C^*_α of C^* is naturally the inv lim (G^α, w^α).

Lemma 15.3. There is a mapping of systems (F,1) from (M,i) to (G,w) which is an equivalence. This equivalence can be chosen so that $H_1(F_n)((b_{n,j})_i) = \{j\} \times \{i\} \times T_0$, where the set on the right is identified with a homology generator.

This follows directly from the description of a Spiez sequence.

Choose such a mapping (F,1). Denote the image of $\overline{St}_1(M)$ under $\overline{S}_1(F)$ in $\overline{S}_1(G)$ by St'(G). Let ρt be ρ restricted to St'(G). Denote the triple $\rho t : St'(G) \to S_0(G)$ by $St_1(G)$. The elements of the space

$St'(G)$ are threads simply related to the threads of $\overline{St}_1(M)$. Denote the generators of $H_1(G_{n,j})$ corresponding to the loops $\{j\} \times \{i\} \times T_0$ by $(b_{n,j})_i$. Then the thread $\overline{S}_1(F)(t_i^{\alpha})$ will have the same coordinates in both spaces. We shall denote this image thread also by t_i^{α} and shall carry over much of the notation used with $\overline{St}_1(M)$ to $St'(G)$. It must be added, parenthetically, that the problem of defining the subspace $\overline{St}_1(M)$ without embedding the compactum in a euclidean space is still under consideration. Let $\beta = \overline{S}_1(F)|\overline{St}_1(M)$.

Now choose a second mapping of systems $(\underline{F},1) : (\underline{M},\underline{i}) \to (\underline{G},\underline{w})$. Make the corresponding identifications and definitions.

Lemma 15.4. There exists an isomorphism (h,g) making the diagram

$$\begin{tikzcd}
St_1(M) \arrow[r, "{(\beta,\, S_0(F))}"] \arrow[d, "{(\bar{h},\, \bar{g})}"] & St_1(G) \arrow[d, "{(h,\, g)}"] \\
St_1(\underline{M}) \arrow[r, "{(\underline{\beta},\, S_0(\underline{F}))}"] & St_1(\underline{G})
\end{tikzcd}$$

commute.

We shall define an equivalence of systems $(f,\phi) : (G,w) \to (\underline{G},\underline{w})$ which induces (h,g). Theorem (15.1) will then immediately follow.

To initiate the second step of the proof express $S_0(G)$ as an inverse limit of finite subsets of $H_0(G_n)$ with discrete topology rather than a subspace of the inverse limit of these groups. Let $E_n = \{m_{n,j} \mid 1 \leq j \leq c(n)\}$. Let $A_n = \{\, \mathrm{proj}_n^{-1}\{m_{n,j}\} \mid 1 \leq j \leq c(n)\}$, where proj_n is the nth projection of $S_0(G)$ into $H_0(G_n)$. The set E_n is a finite open partition of $\mathrm{proj}_n\left[S_0(G)\right]$. The set A_n is a finite open partition of $S_0(G)$. The set $A = \bigcup_{n=1}^{\infty} A_n$ is a base for the topology on $S_0(G)$. Similar statements can be made for $\underline{G}$, defining the corresponding sets $\underline{E}_n$, $\underline{A}_n$, and $\underline{A}$.

We now define by induction an increasing function ψ from the positive integers to themselves and a sequence of functions r_n. For each positive n the function r_n will map the set $E_{\psi(n)}$ into $\underline{E}_n$ in such a way that (r, ψ) constitutes a mapping from the sequence $(E, H_0 \cdot w)$ to $(\underline{E}, H_0 . \underline{w})$. (Here the component maps of the mapping of systems are continuous maps making the appropriate squares commute.) Because G_1 and $\underline{G}_1$ are points, let $\psi(1) = 1$ and $r(m_{1,1}) = \underline{m}_{1,1}$. Suppose that we have defined ψ on $[1,n)$ and also for each k with $1 \leq k < n$, the mapping r_k such that $H_0(\underline{w}_k) \cdot r_k = r_{k-1} \cdot H_0(W_{\psi(k-1),\psi(k)})$. Suppose that m is a thread of $S_0(G)$. Then there is a unique set $\underline{d}$ in $\underline{A}_n$ such that $g(m) \in \underline{d}$. Because g is continuous, there is a set d in A such that $g[d] \subseteq \underline{d}$. There is thus a cover B of $S_0(G)$ by these sets d. Because $S_0(G)$ is compact, B has a finite subcover C. Choose the largest index q for which A_q contains a member of C. Let $\psi(n)$ be $q+\psi(n-1)$. Then $A_{\psi(n)}$ is a finite open partition refining B. The mapping g is surjective so that the set of $\underline{d}$ originally selected is all of $\underline{A}_n$. Thus the set of images of the members of $A_{\psi(n)}$ refines $\underline{A}_n$. To each element $m_{\psi(n),i}$ of $E_{\psi(n)}$ corresponds a unique d of $A_{\psi(n)}$; $g[d]$ lies in a unique element $\underline{d}$ of $\underline{A}_n$; and to this corresponds a single element $\underline{m}_{n,j}$ of $\underline{E}_n$. Let $r_n(m_{\psi(n),i}) = \underline{m}_{n,j}$.

Now $A_{\psi(n)}$ refines $A_{\psi(n-1)}$, and $\underline{A}_n$ refines $\underline{A}_{n-1}$. The equality $H_0(\underline{w}_n) \cdot r_n = r_{n-1} \cdot H_0(W_{\psi(n-1),\psi(n)})$ can be deduced from this.

The pair (r,ψ) will induce g, for it has been defined so that for each m in $S_0(G)$, we have $\underline{\text{proj}}_n(g(m)) = r_n(m_{\psi(n)})$.

The third step in the proof is to express $St'(G)$ as an inverse limit of finite sets with discrete topology. For this purpose define the sets $C_{n,j}$ for each $G_{n,j}$ of non-trivial shape in the following way.

15.5.1) If $G_{n,j}$ is a copy of T_0, let $C_{n,j}$ be $\{(b_{n,j})_1\}$.

15.5.2) Otherwise, let $C_{n,j} = \{(b_{n,j})_i \mid 0 \leq i \leq g(n,j)\}$.

Let $C_n = \{0\} \cup \bigcup_{j=1}^{c(n)} C_{n,j}$, the union over all non-trivial components of G_n. Let $C = \bigcup_{n=1}^{\infty} C_n$. Let $D_n = \{\text{proj}_n^{-1}\{d\} \mid d \in C_n\}$ and $D = \bigcup_{n=1}^{\infty} D_n$, where proj_n is the nth projection of $St'(G)$ into $H_1(G_n)$.

The set C_n constitutes a finite open partition of $\text{proj}_n[St'(G)]$ in $H_1(G_n)$. The set D_n is a finite open partition of $St'(G)$. The set D is a base for the topology on $St'(G)$. One can show that $St'(G)$ is the inverse limit of the sequence $(C_n, H_1(w_{n+1}|C_{n+1}))_{n=1}^{\infty}$. Similar statements can be made for the system $\underline{G}$, giving rise to $\underline{C}_{n,j}$, $\underline{C}_n$, $\underline{D}_n$, and $\underline{D}$.

We now define by induction an increasing mapping ϕ from the positive integers to themselves, a sequence f of continuous mappings f_n from $G_{\phi(n)}$ to $\underline{G}_n$, and a sequence p of mappings p_n from $C_{\phi(n)}$ to $\underline{C}_n$. Begin the induction by letting $\phi(1) = 1$, $f(1) = 1$, and p_1 a map from C_1 to $\underline{C}_1$ such that $p_1(0) = 0$. Suppose that we have defined ϕ on $(0,n)$; and for each k with $1 \leq k < n$ have defined p_k from $C_{\phi(k)}$ to $\underline{C}_k$ so that $p_{\phi(k-1)} \cdot H_1(W_{\phi(k-1),\phi(k)}) = H_1(\underline{w}_k) \cdot p_k$, and f_k so that $f_{k-1} \cdot W_{\phi(k-1),\phi(k)} \ \underline{\text{htp}}\ \underline{w}_k \cdot f_k$, and p_k is the restriction of $H_1(f_k)$.

The set $\underline{D}_n$ is a finite open partition of $St'(G)$, so that for each thread x of $St'(G)$ the thread $h(x)$ belongs to a unique set $\underline{d}$

belonging to $\underline{D}_n$. There is a neighborhood d of x belonging to D and which is mapped by h into $\underline{d}$. The set B of such neighborhoods d covers St'(G). Because St'(G) is compact there is a finite subcover. Choose an integer $\phi(n)$ larger than $\phi(n-1)$, $\psi(n)$, and all the indices j of the sets D_j to which the members of the subcover belong. Then $D_{\phi(n)}$ refines B. For each element e of $C_{\phi(n)}$ there is a unique d in $D_{\phi(n)}$ corresponding to it, a unique $\underline{d}$ in $\underline{D}_n$ in which $h[d]$ lies, and a unique element in $\underline{C}_n$ corresponding to $\underline{d}$. Let $p_n(e)$ be this member of $\underline{C}_n$.

Lemma 15.6. Let $\alpha_n = r_n \cdot H_0(W_{\psi(n),\phi(n)})$. Then (α,ϕ) is a mapping from $(E,H_0 \cdot w)$ to $(\underline{E}, H_0 \cdot \underline{w})$, and inv lim $(\alpha,\phi) = g$.

This follows from the choice of $\phi(n) > \psi(n)$.

The mapping p_n has the following properties. First of all, we see that $H_1(\underline{w}_n) \cdot p_n = p_{n-1} \cdot H_1(W_{\phi(n-1),\phi(n)})$ on $C_{\phi(n)}$. One can deduce this result from the fact that $D_{\phi(n)}$ refines $D_{\phi(n-1)}$ and $\underline{D}_n$ refines $\underline{D}_{n-1}$. Second, p_n has been defined in such a way that the diagram

(15.7.1)

$$\begin{tikzcd} St'(G) \arrow[r, "h"] \arrow[d, "proj_{\phi(n)}"'] & St'(\underline{G}) \arrow[d, "\underline{proj}_n"] \\ C_{\phi(n)} \arrow[r, "p_n"] & \underline{C}_n \end{tikzcd}$$

commutes. Third, p_n maps zero to zero. Fourth, one can show that p_n is fiber preserving over α_n in the extended sense. Consider the diagram (15.7.2), where $\underline{u}_n$ is the relation $\underline{v}_n \cup \{0\} \times \underline{A}_n$, and $\underline{v}_n$ is defined in section 4.

(15.7.2)

$$\begin{tikzcd} \underline{C}_n \arrow[d] & St'(\underline{G}) \arrow[l, "\underline{proj}_n"'] \arrow[d, "\underline{et}"] \\ \underline{E}_n & S_o(\underline{G}) \arrow[l, "\underline{proj}_n"'] \end{tikzcd}$$

The diagram (15.7.3), where $u_{\phi(n)}$ is a similar extension of $v_{\phi(n)}$, commutes in the extended sense.

(15.7.3)

$$\begin{tikzcd} C_{\phi(n)} \arrow[d, "u_{\phi(n)}"] & St'(G) \arrow[l, "proj_{\phi(n)}"] \arrow[d, "et"] \\ E_{\phi(n)} & S_o(G) \arrow[l, "proj_{\phi(n)}"'] \end{tikzcd}$$

The diagram

(15.7.4)

$$\begin{tikzcd} St'(G) \arrow[r, "h"] \arrow[d, "et"] & St'(\underline{G}) \arrow[d, "\underline{et}"] \\ S_o(G) \arrow[r] & S_o(\underline{G}) \end{tikzcd}$$

commutes in the extended sense. The diagram

(15.7.5)

$$\begin{tikzcd} S_o(G) \arrow[r, "g"'] \arrow[d, "proj_{\phi(n)}"] & S_o(\underline{G}) \arrow[d, "\underline{proj}_n"] \\ E_{\phi(n)} \arrow[r, "\alpha_n"] & \underline{E}_n \end{tikzcd}$$

commutes by lemma (15.6). We shall prove that the diagram

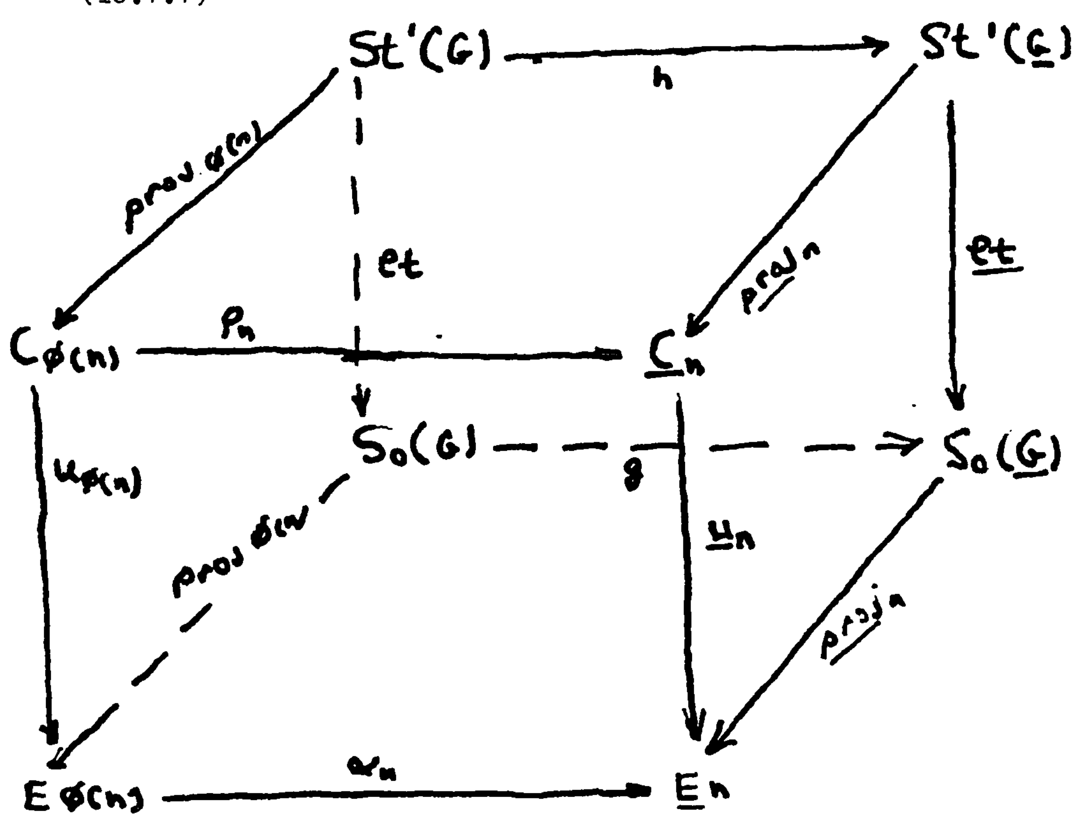
(15.7.6)

$$\begin{tikzcd} C_{\phi(n)} \arrow[r, "\rho_n"] \arrow[d, "u_{\phi(n)}"] & \underline{C}_n \arrow[d, "\underline{u}_n"] \\ A_{\phi(n)} \arrow[r, "\alpha_n"] & \underline{A}_n \end{tikzcd}$$

commutes for any non-zero element c of $C_{\phi(n)}$. This claim implies that all the non-zero elements of $C_{\phi(n),j}$ for component $G_{\phi(n),j}$ are mapped to the set $\underline{C}_{n,k}$ for a single component $\underline{G}_{n,k}$ of G_n.

We prove the claim by observing that the diagram (15.7.6) is the front face of the cube (15.7.7) whose other faces are given in diagrams (15.7.1-5). The proof involves chasing diagrams.

(15.7.7)



Suppose that c belongs to $C_{\phi(n)}$ and is not zero. Then there is an element y of St'(G) such that $proj_{\phi(n)}(y) = c$. From the top face we deduce that $p_n(c) = p_n(proj_{\phi(n)}(y)) = \underline{proj}_n(h(y))$. If $P_n(c)$ is zero the result follows. Suppose it is not zero. Then $\underline{u}_n(p_n(c)) = \underline{u}_n(\underline{proj}_n(h(y)))$. Now h is a zero preserving homeomorphism and y is not zero, so that $h(y) \neq 0$. We may then use the right side face to derive the equality $\underline{u}_n(\underline{proj}_n(h(y))) = \underline{proj}_n(\underline{\rho t}(h(y)))$ in the strict sense. Employing the back face we deduce that $\underline{\rho t}(h(y)) = g(\rho t(y))$ in the strict sense. From this one can see that $\underline{u}_n(p_n(c)) = \underline{proj}_n(g(\rho t(y)))$. Using the bottom face, one can verify that this equals $\alpha_n(proj_{\phi(n)}(\rho t(y)))$. The left side face can be used to convert this to $\alpha_n(u_{\phi(n)}(proj_{\phi(n)}(y)))$. This finally allows us to conclude that $\underline{u}_n(\underline{proj}_n(h(y))) = \alpha_n(u_{\phi(n)}(c))$.

Fifth, if $G_{\phi(n),j}$ is not a circle, the mapping p_n is additive in the sense that $P_n(\Sigma_{i=1}^{g(\phi(n),j)} (b_{\phi(n),j})_i)=\Sigma\ p_n((b_{\phi(n),j})_i)$.The mapping h is a bijection on each fiber, so that by (9.9) h must be additive on each fiber. But we have shown that $p_n \cdot proj_{\phi(n)} = \underline{proj}_n . h$. The projection $\underline{proj}_n$ is additive on each fiber and surjective, so that we can conclude that p_n must be additive. For, consider that $\Sigma_{i=1}^{g(\phi(n),j)}(b_{\phi(n),j})_i = proj_{\phi(n)}\ t_0$ so that with $N = \phi(n)$

$$p_n(proj_N(t_0))= \underline{proj}_n(h(t_0)) = \underline{proj}_n(\Sigma_{i=1}^{sh} h(t_i)) =$$
$$= \Sigma\ \underline{proj}_n(h(t_i)) = \Sigma\ p_n(proj_N(t_i))$$
$$= \Sigma_{i=1}^{g(N,j)} p_n(b_{N,j})_i.$$

We now define f_n as the union of mappings $f_{n,j} : G_{N,j} \to \underline{G}_n$. To $G_{N,j}$ corresponds a thread $m_{N,j}$ of $S_0(G)$, and to $\underline{m}_{n,k} = \alpha_n(m_{N,j})$ corresponds a component $G_{n,k}$.

15.8.1) Suppose that $G_{N,j}$ is a point $1_{N,j}$. Then let $f_{n,j}(1_{N,j}) = 1_{n,k}$.

15.8.2) Suppose that $G_{N,j}$ is a wedge of $g(N,j)$ copies of T_0. Then the set $C_N \supseteq \{ (b_{N,j})_i \mid 1 \leq i \leq g(N,j)\}$. For each such i

a) if $p_n((b_{N,j})_i) = 0$, let $f_{n,j,i} = d_{0,i}$,

b) if it equals $(b_{n,k})_q$, let $f_{n,j,i} = d_{q,i}$

c) if it equals $\Sigma_{m=1} (b_{n,k})_m$, let $f_{n,j,i} = d_{+,i}$.

Finally, let $f_{n,j} = \bigcup_{i=1}^{g(N,j)} f_{n,j,i}$.

<u>Lemma 15.9.</u> Consider the diagram

(15.9.1)

$$\begin{tikzcd}
& W_{Q+1} \arrow[dl, "f_{Q+1}"'] & \cdots \arrow[l, "f_{Q+2}"'] & \\
W_Q & & & W_R \arrow[ul] \arrow[dl] \\
& \underline{W}_{Q+1} \arrow[ul] & \cdots \arrow[l, "g_{Q+2}"'] &
\end{tikzcd}$$

where each of the W_i is a point or a wedge of a finite number of copies of T_0 and each f_i and g_i is a mapping of type Z_2. Let F be the composition of the maps in the upper row and G , of the lower. If $H_1(F) = H_1(G)$, then F is homotopic to G.

The lemma can be proved by showing F and G induce homomorphisms $F^{\#}$ and $G^{\#}$ which belong to the same conjugacy class. Then by theorem 8.11 of [35], they are homotopic, for the domain is a path connected CW complex and the range is a space of type $(\pi,1)$. We can see that $F^{\#}$ and $G^{\#}$ are actually equal here, by observing that for mappings of type Z_2 the effect of the f_i on generators of singular integral homology can be identified with its effect on generators of fundamental groups. There is but one difference, the product of free generators of the fundamental group in the appropriate order replacing the sum of basis elements.

Lemma 15.10. Suppose that in (15.9.1) each W_i is a finite disjoint union of wedges of copies of T_0 or points, and that on each component each f_i and g_i is of type Z_2. If $H_0(F) = H_0(G)$ and $H_1(F) = H_1(G)$ then F is homotopic to G.

The equality of the values of H_0 allows us to find for each component of W_R a diagram (15.9.1) meeting the conditions of (15.9) One then applies (15.9) and shows that F and G are homotopic on all components.

Lemma 15.11. For each n, $H_0(f_n) = \alpha_n$ on $E_{\phi(n)}$ and $H_1(f_n) = p_n$ on $C_{\phi(n)}$.

This follows from the definition of (f,ϕ) which has been formulated so as to produce this result.

Lemma 15.12. The sequence (f,ϕ) is a mapping of systems.

To prove this result we show that $\underline{w}_n \cdot f_n$ htp $f_{n-1} \cdot W_{\phi(n-1),\phi(n)}$. Because $H_0(\underline{w}_n) \cdot \alpha_n = \alpha_{n-1} \cdot H_0(W)$ on the generators of $H_0(G_{\phi(n)})$, we can derive the chain of equalities $H_0(\underline{w}_n \cdot f_n) = H_0(\underline{w}_n) \cdot H_0(f_n) = H_0(f_{n-1}) \cdot H_0(W)$. Similarly, because p_n is fiber preserving over α_n, we can from the equality $H_1(\underline{w}_n) \cdot p_n = p_{n-1} \cdot H_1(W)$ on the generators of $H_1(G_{\phi(n)})$ deduce that $H_1(\underline{w}_n \cdot f_n) = H_1(f_{n-1} \cdot W)$. Therefore by (15.10) the two maps are homotopic.

Lemma 15.13. The mapping of systems (f,ϕ) induces (h,g).

This follows from two facts. First (f,ϕ) induces (α,ϕ), the inverse limit of which is g. Second (f,ϕ) induces (p,ϕ) the inverse limit of which is h.

Theorem 15.14. The mapping (f,ϕ) is an equivalence.

The morphism (h^{-1}, g^{-1}) is an isomorphism. Use the procedure described above to derive a mapping of systems $(\underline{f},\underline{\phi}) : (\underline{G},\underline{w}) \to (G,w)$. In the process one defines $(\underline{\alpha},\underline{\phi})$ and $(\underline{p},\underline{\phi})$ which are induced through the application of H_0 and H_1 to the sequence $(\underline{f},\underline{\phi})$.

Now inv lim $(\alpha,\phi) = g$, and inv lim $(\underline{\alpha},\underline{\phi}) = \underline{g}$. Because the inverse limit is functorial, inv lim$((\alpha,\phi) \cdot (\underline{\alpha},\underline{\phi}))$ = inv lim (α,ϕ) . inv lim $(\underline{\alpha},\underline{\phi}) = g\ g^{-1} = 1|S_0(\underline{G})$. Similarly, inv lim $((\underline{\alpha},\underline{\phi}) \cdot (\alpha,\phi))$ = inv lim $(\underline{\alpha},\underline{\phi})$. inv lim$(\alpha,\phi) = 1|S_0(G)$.

Let $\underline{\beta} = H_0(\underline{W}_{n,\phi\underline{\phi}(n)})$. From the preceding discussions one may conclude that the diagram

(15.14.1)
$$
\begin{tikzcd}
\underline{E}_{\phi\phi(n)} \arrow[ddd, "\underline{\beta}"] & S_0(\underline{G}) \arrow[l, "\underline{proj}_{\phi\phi(n)}"'] \arrow[rr, "\underline{proj}_{\phi\phi(n)}"] \arrow[ddd, "1"] \arrow[dr, "g^{-1}"] & & \underline{E}_{\phi\phi(n)} \arrow[d, "\underline{f}_{\phi(n)}"] \\
& & S_0(G) \arrow[r, "proj"'] \arrow[ddl, "g"] & E_{\phi(n)} \arrow[dd, "\alpha_n"] \\
& & & \\
\underline{E}_n & S_0(\underline{G}) \arrow[l, "\underline{proj}_n"'] \arrow[rr, "\underline{proj}_n"] & & \underline{E}_n
\end{tikzcd}
$$

commutes. This yields the equalities

$\underline{proj}_n = \underline{\beta} \cdot \underline{proj}_{\phi\underline{\phi}(n)}$, and $\underline{proj}_n = \alpha_n \cdot \underline{\alpha}_{\phi(n)} \cdot \underline{proj}_{\phi\underline{\phi}(n)}$. Because $\underline{proj}_{\phi\phi(n)}$ is a surjection one may conclude that $\alpha_n \cdot \underline{\alpha}_{\phi(n)} = H_0(\underline{W}_{n,\phi\underline{\phi}(n)})$ on $E_{\underline{\phi}\phi(n)}$. Therefore $H_0(f_n\ \underline{f}_{\phi(n)}) = H_0(\underline{W}_{n,\phi\underline{\phi}(n)})$.

In a similar fashion one may show that $H_0(\underline{f}_n\ f_{\underline{\phi}(n)}) = H_0(W_{n,\phi\underline{\phi}(n)})$.

One may in the same way show that inv lim $((p,\phi) \cdot (\underline{p},\underline{\phi}))$ = inv lim (p,ϕ) . inv lim $(\underline{p},\underline{\phi}) = h\ h^{-1} = 1|St'(\underline{G})$ and inv lim $((\underline{p},\underline{\phi}) \cdot (p,\phi))$ = inv lim $(\underline{p},\underline{\phi})$. inv lim $(p,\phi) = h^{-1}\ h = 1|St'(G)$. Let $\underline{\gamma} = H_1(\underline{W}_{n,\underline{\phi}o/(n)})\ C_{\underline{\phi}\phi(n)}$. Then, from the discussion above one may conclude that the diagram

(15.14.2)

$$\begin{tikzcd}
\underline{C}_{\phi\phi(n)} \arrow[dd, "\underline{Y}"'] & St'(\underline{G}) \arrow[l, "\underline{proj}_{\phi\phi(n)}"'] \arrow[rr, "\underline{proj}_{\phi\phi(n)}"'] \arrow[dd, "1"'] \arrow[dr, "h^{-1}"] & & \underline{C}_{\phi\phi(n)} \arrow[d, "\underline{p}_{\phi(n)}"] \\
& & st'(G) \arrow[r, "proj"] \arrow[dl, "h"] & C_{\phi(n)} \arrow[d, no head] \\
\underline{C}_n & St'(\underline{G}) \arrow[l, "\underline{proj}_n"'] \arrow[rr, "\underline{proj}_n"] & & \underline{C}_n
\end{tikzcd}$$

commutes. This yields the result

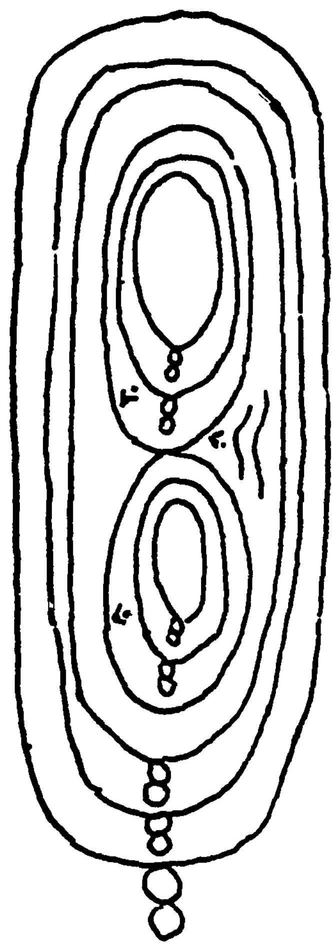
$$\underline{proj}_n = \underline{Y} \cdot \underline{proj}_{\phi\phi(n)}, \text{ and } \underline{proj}_n = p_n \, \underline{p}_{\phi(n)} \cdot \underline{proj}_{\phi\phi(n)}.$$

Because $\underline{proj}_{\phi\phi(n)}$ is a surjection, one can conclude that $p_n \, p_{\phi(n)} = H_1(\underline{W}_{n,\phi\phi(n)})$ on $C_{\phi\phi(n)}$. Therefore $H_1(f_n \, f_{\phi(n)}) = H_1(\underline{W}_{n,\phi\phi(n)})$. Similarly, $H_1(\underline{f}_n \, f_{\underline{\phi}(n)}) = H_1(W_{n,\phi\underline{\phi}(n)})$. By applying (15.10) one can conclude that $f_n \, \underline{f}_{\phi(n)} \; \underline{htp} \; \underline{W}_{n,\phi\phi(n)}$ and $\underline{f}_n \, f_{\underline{\phi}(n)} \; \underline{htp} \; W_{n,\phi\underline{\phi}(n)}$. This implies that $(f.\phi)$ and $(\underline{f},\underline{\phi})$ are inverse shape equivalences.

<u>Corollary 15.15</u>. The isomorphism classes of the invariant S_1 characterize the shape classes of plane compacta.

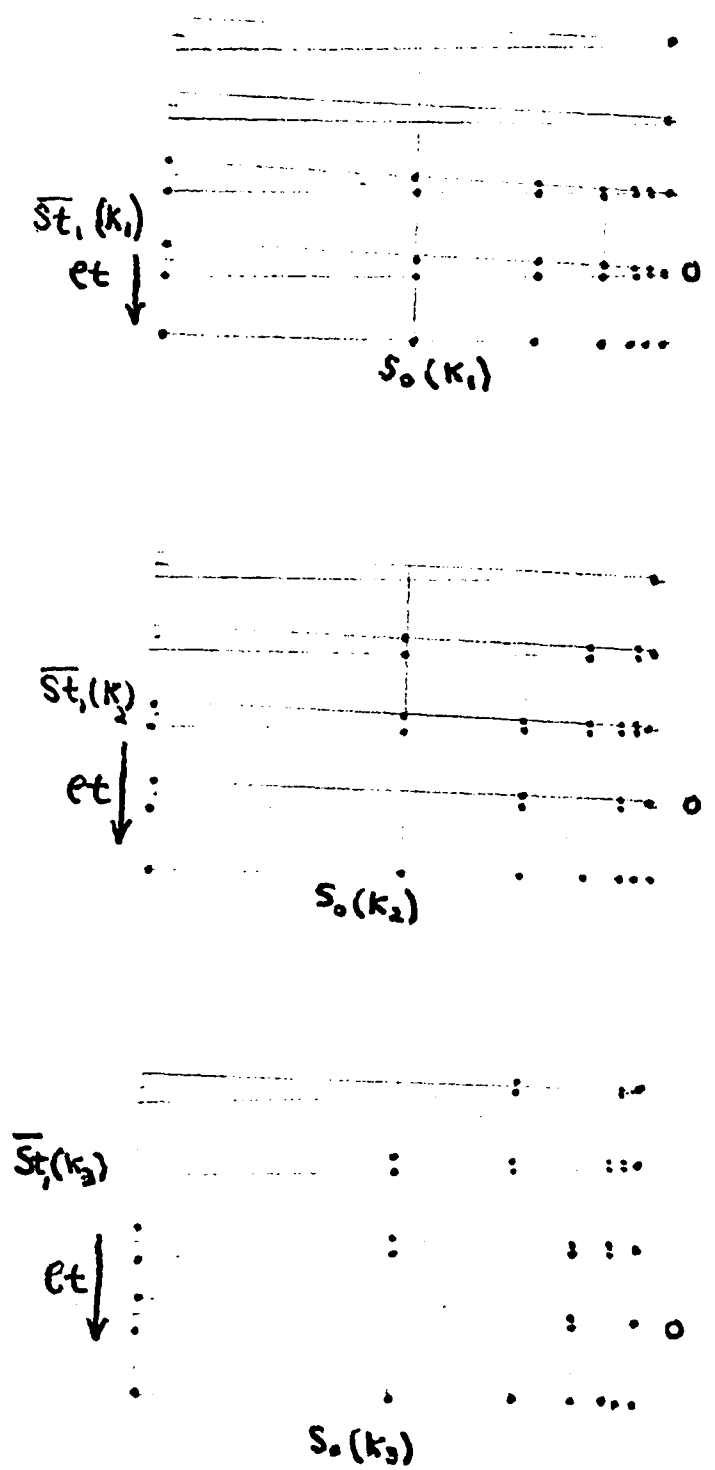
<u>16. Illustrative examples</u>. We present three compacta K_1, K_2, and K_3 which will also be useful in later chapters. For brevity and clarity we exhibit K_3 in figure (16.1) and give instructions for constructing K_1 and K_2 from it. The compactum K_2 is obtained by omitting those components of K_3 in one of the disks complementary to the limiting figure eight; the compactum K_2, by omitting components from two of the complementary disks. In figure (16.1) are presented the values of $St_1(K_i)$ for $1 \leq i \leq 3$. It is apparent that these values are not isomorphic, so that the compacta do not have the same shape. Nevertheless, their complements in the plane are homeomorphic.

Values of St_2



k_3

Figure 10.1



ADDENDA

1. <u>The second induction step in (13.3.2).</u> If $n(1) \neq P$, proceed with the induction in the second paragraph of (13.3.2). Suppose that $n(1) = P$. Then there is an integer J such that when $j \geq J$ we have $x_{j,P+1} = s_{P+1}$. Let q be as before the greatest integer for which $x_{j,q} = s_q$. Again we may deduce that $s_q = t_{d,q}$ for some d 1 or $s_q = \Sigma_{i=1}^{q-P+1} t_{i,q}$. If the first possibility obtains, then the fact that $H_1(V_{P,q})(t_{i,q}) = 0$ for $i \geq 1$ implies that $d = 1$. In either case let $n(2) = q$ and proceed with the induction in the second paragraph of (13.3.2).

2. <u>Properties 13.13.1-3.</u>

13.13.1) The set of u_n is a local base at t_k^{β}.

13.13.2) The set of w_n is a local base at 0.

13.13.3) For each component C_α of C with shape number greater than one, either $L_\alpha \cap u_n = \emptyset$ or u_n contains precisely the elements t_0^{α} and t_1^{α} and the remaining elements of L_α belong to w_n.

BIBLIOGRAPHY

1. Alexandrov, P.S., "Topological Duality Theorems I", A.M.S. Translations Ser. 2, Vol. 30, 1-102.

2. ————————, "Topological Duality Theorems II", A.M.S. Translations Ser. 2, Vol. 30, 103-235

3. Armentrout, S., Lininger, L., and Meyer, D.V., Pac. J. Math. 24 205 (1958).

4. Borsuk, K., "Theory of Shape", Lecture Notes Series, No. 28, Aarhus University, Feb. 1971.

5 —————, Fund. Math. 85 197-201 (1974)

6. —————, Fund. Math. 79 141 (1973)

7. —————, "General Topology and Its Relations to Modern Algebra and Analysis III", Academic Press, 1972.

8. Bourbaki, N., "General Topology", Addisen-Wesley, 1966.

9. Chapman, T.A., Fund. Math. 76 183 (1972)

10. ——————, Fund. Math. 76 261 (1972)

11. Chogosvili, G.S., "Generalized Products and Limits and their Applications in Homology Theory", Russ. Math. Surveys, Vol. 21, No 4, p. 13 (1966).

12. Curtis, M.L., and Fort, M.K., Jr., Proc. A.M.S. 8 577 (1957)

13. Edwards, D.A., "Etale Homotopy Theory and Shape", Lecture Notes in Math., 428, Springer Verlag, Berlin, 1974.

14. Edwards, D.A., and Geoghegan, R., "The Stability Problem in Shape and a Whitehead Theorem in Pro-Homotopy", (Preprint)

15. ————————————————, "Shapes of Complexes, Ends of Manifolds, Homotopy Limits, and the Wall Obstruction", (Pre-

print).

16. Eilenberg, S., and Steenrod, N., "Foundations of Algebraic Topology", Princeton Univ. Press, 1952.

17. Fox, R.H. Fund. Math. 74 48 (1972)

18. Freudenthal, H., Comp. Math. 4 145 (1937)

19. Geohegan, R., and Summerhil, R.R., Trans. A.M.S. 179 281 (1973)

20. Godlewski, S., and Novak, , Bull. Acad. Pôl. Sci., 20 387 (1972)

21. Handel, D. and Segal, J., Gen. Top. and its Appl. 3 111 (1973)

22. ______________________, Q.J.Math. 24 37 (1973)

23. Henderson, D.W., Bull. Acad. Pol. Sci. XIX 747 (1971)

24. Hewitt, E. and Ross, K.A., "Abstract Harmonic Analysis", Academic Press, 1963.

25. Kelley, J.L., Proc. N.A.S. Wash. 26 192 (1940)

26. ___________, "General Topology", Van Nostrand, 1955.

27. Kuratowski, K., "Topology", Vols. 1,2, Academic Press, 1966 and 1968.

28. Mardesic, S., Bull. Acad. Pol. Sci., XIX 623 (1971)

29. Mardesic, S., and Segal, J., Fund. Math. 72

30. Mardesic, S., Ill. J. Math 4 278 (1960) 41 (1971)

31. Mazurkiewicz, S., and Sierpinski, W., Fund. Math. 1 17 (1920)

32. Michael, E.A., Trans. A.M.S. 71 152 (1951)

33. Moore, R.L., Monat. für Math. u. Phys. 36 81 (1929)

34. Moszynska, M., Fund. Math. 80 221 (1973)

35. Spanier, E., "Algebraic Topology", McGraw-Hill, 1966.

36. Spiez, S. Fund. Math. 78 145 (1973)

37. Trybulec, A., Bull. Acad. Pol. Sci. XXI 727 (1973)

38. Whyburn, G.T., "Analytic Topology", A.M.S. Colloq. Publ., Vol. 27, 1942.

39. Wilder, R.L., "Topology of Manifolds", A.M.S. Colloq. Publ., Vol. 32, 1949.