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# QUASI-ORTHONORMAL FRAMES ON SEMI-RIEMANNIAN n-MANIFOLDS

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QUASI-ORTHONORMAL FRAMES ON SEMI-

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## QUASI-ORTHONORMAL FRAMES ON SEMI-RIEMANNIAN n-MANIFOLDS

# CHAPTER I

## INTRODUCTION

A semi-Riemannian manifold is a differentiable (i.e., of class $C^\infty$) n-manifold M on which a real valued, bilinear, symmetric and non-singular function $\langle\ ,\ \rangle$ on ordered pairs of tangent vectors at each point has been singled out. The functional $\langle\ ,\ \rangle$ is called the semi-Riemannian metric of M. The unique connection D associated with $\langle\ ,\ \rangle$ is called the semi-Riemannian connection. Let U be a (connected) neighborhood of a point of M. The investigation in this dissertation is concerned mainly with the geometry on $U \subset M$ and thus is local in nature.

The study of the local differential geometry of a semi-Riemannian manifold M by means of Ricci's 'invariant theory' is well-known [2, pp. 96-142]. But for problems involving null vector fields, the theory cannot be applied directly. A quasi-orthonormal frame is a frame consisting partly of null and partly of non-null vector fields. In 1945, Y. C. Wong developed an invariant theory with respect to a quasi-orthonormal frame and applied it successfully to determine pairs of semi-Riemannian 3-manifolds with corresponding geodesics. Other authors made use of quasi-orthonormal frames in their works; for example, Ruse did so in his study of partly null parallel planes in a semi-Riemannian manifold [5; 6]. In essence this paper is confined to the investigation of quasi-orthonormal frames and is similar to what has been done with orthonormal frames on a semi-Riemannian n-manifold.

In 1953, T. K. Pan defined and studied variation and subvariation related to an orthonormal frame on a Riemannian n-manifold. In 1966, R. N. Sen and B. Gupta defined and investigated sets of coefficients of rotation induced by a mapping between Riemannian n-manifolds. The results of Pan will be generalized in Chapter II and those of Sen and Gupta in Chapter III. The terms quasi-variation (of a first kind or of a second kind), quasi-subvariation, induced quasi-variation etc are introduced for corresponding concepts related to quasi-orthonormal frames on semi-Riemannian manifolds. The last chapter deals with applications. It is a study of congruences by means of the invariants mentioned above.

As commonly used, theorems are more significant assertions than propositions. Vector fields on $U \subset M$ are denoted by the capital letters X and Y with or without indices on them; while functions on U and scalars by other capital letters. Indices are represented by small letters. Displayed equations are numbered serially by chapters. If equation (3) of Chapter II is mentioned in this same chapter, it is cited as (3) and if referred to in a different chapter specified as (II.3). Similar conventions apply to theorems and propositions. For other notations, we generally follow Hicks [3].

## CHAPTER II

## QUASI-ORTHONORMAL FRAMES ON A SEMI-RIEMANNIAN MANIFOLD

1. Definitions and some relations. Let M be a $C^\infty$ n-manifold with a semi-Riemannian metric $\langle\ ,\ \rangle$ and U a neighborhood of a point p in M. Vector fields X and Y on U are said to be orthogonal if $\langle X,Y\rangle = 0$. A vector field on U which is orthogonal to itself is called a null vector field; otherwise, it is called a non-null vector field. In relativity theory, X is called time-like, light-like or space-like, respectively, if $\langle X,X\rangle > 0$, $\langle X,X\rangle = 0$ or $\langle X,X\rangle < 0$. When $\langle X,X\rangle = \pm 1$, X is called a unit vector field.

Let $Y^1, Y^2, \ldots, Y^n$ be an orthonormal frame on U; i.e., the $Y^a$'s are linearly independent vector fields such that $\langle Y^a,Y^a\rangle = \pm 1$, where a is not summed, and $\langle Y^a,Y^b\rangle = 0$, when $a \neq b$. The number I of the $Y^a$'s such that $\langle Y^a,Y^a\rangle = -1$, called the index of $\langle\ ,\ \rangle$ on U, is a constant which does not depend on the choice of the frame [1, p. 208]. Let S be the smaller of the numbers I and n - I. Consider a fixed integer R, where $1 \leq R \leq S$. Then we can renumber the $Y^a$'s such that each $Y^a$ is time-like for a = 1, 2,..., R and each $Y^a$ is space-like for a = R + 1, R + 2,..., 2R. Define n vector fields $X^1, X^2, \ldots, X^n$ on U by

$$X^{2i-1} = \frac{1}{\sqrt{2}}(Y^i + Y^{i+R}), \quad i = 1, 2, \ldots, R,$$

(1) $$X^{2i} = \frac{1}{\sqrt{2}}(Y^i - Y^{i+R}), \quad i = 1, 2, \ldots, R,$$

$$X^r = Y^r, \quad r = 2R + 1, 2R + 2, \ldots, n.$$

For simplicity, we avoid writing the ranges of variables used as indices repeatedly by fixing their ranges henceforth as follows:

$$a, b, c, d = 1, 2, \ldots, n,$$
$$h, k, m = 1, 2, \ldots, 2R,$$
$$r, s, t = 2R + 1, 2R + 2, \ldots, n.$$

Also, for a given h we associate $h_o$ defined by

$$h_o = \begin{cases} h + 1, & \text{if } h \text{ is odd}, \\ h - 1, & \text{if } h \text{ is even}, \end{cases}$$

and for a given b we have $b_o$ defined by

$$b_o = \begin{cases} h_o, & \text{if } b = h, \\ r, & \text{if } b = r. \end{cases}$$

Similar interpretations go to $k_o$, $m_o$, $a_o$, $c_o$ and $d_o$.

It is easy to see that the $X^a$'s defined by (1) are linearly independent vector fields on U satisfying

(2) $$\langle X^h, X^r \rangle = 0 = \langle X^h, X^k \rangle, \text{ when } h \neq k_o,$$

(3) $$\langle X^h, X^{h_o}\rangle = 1,$$

(4) $$\langle X^r, X^s\rangle = E_r\delta^r_s, \text{ (not summed on } r),$$

where $E_r = \langle X^r, X^r\rangle = \pm 1$, and $\delta^r_s$ is the Kronecker delta. Such a sequence $(X^a) = (X^1, X^2, \ldots, X^n)$ of n linearly independent vector fields on U is said to form a <u>quasi-orthonormal frame of nullity</u> R <u>on</u> U. Note that the term nullity as used here has a special meaning.

From (2), (3) and (4) we know readily that the 2R vector fields $X^h$ are null, the n - 2R vector fields $X^r$ are unit and the n vector fields $X^a$ are mutually orthogonal except for the R pairs $X^h$ and $X^{h_o}$. These equations may be represented more compactly by

(5) $$\langle X^a, X_b\rangle = \delta^a_b,$$

where $X_a$ for $1 \leq a \leq 2R$ and $2R < a \leq n$ are defined by

$$X_h = X^{h_o} \text{ and } X_r = E_r X^r.$$

Since a given orthonormal frame on a semi-Riemannian manifold always gives rise to some quasi-orthonormal frames of nullity R >0 as shown above, there exist quasi-orthonormal frames on U. Unless otherwise stated, every frame is assumed to be quasi-orthonormal of a fixed nullity R > 0. Note that the $Y^a$'s also satisfy (5) with R = 0 and so we may consider an orthonormal frame on a semi-Riemannian manifold as a quasi-orthonormal frame of zero nullity.

Let $(X^a)$ be a quasi-orthonormal frame on U. Then

$$D_{X_c} X^a = \sum_b \langle D_{X_c} X^a, X_b \rangle X^b,$$

where D is the unique semi-Riemannian connection associated with the semi-Riemannian metric $\langle\ ,\ \rangle$. The invariants

$$V^a_{bc} = \langle D_{X_c} X^a, X_b \rangle \tag{6}$$

are called <u>coefficients of rotation of</u> $(X^a)$. When the frame is orthonormal, the invariants $V^a_{bc}$ are the familiar Ricci coefficients of rotation [2, p. 97]. Now,

$$\begin{aligned} 0 = X_c \langle X^a, X_b \rangle &= \langle D_{X_c} X^a, X_b \rangle + \langle X^a, D_{X_c} X_b \rangle \\ &= V^a_{bc} + E_{a_o} E_{b_o} V^{b_o}_{a_o c}, \text{ (not summed on } a_o \text{ and } b_o) \end{aligned}$$

where we define $E_{a_o}$ (similarly for $E_{b_o}$ and $E_{c_o}$) by

$$E_{a_o} = \begin{cases} 1, & \text{if } a = h, \\ E_r, & \text{if } a = r. \end{cases}$$

Hence the coefficients of rotation satisfy [8, p. 162]

$$V^a_{bc} + E_{a_o} E_{b_o} V^{b_o}_{a_o c} = 0, \text{ (not summed on } a_o \text{ and } b_o). \tag{7}$$

In particular, we have

$$V^h_{h_o a} = 0 = V^r_{ra}. \tag{8}$$

2. <u>Quasi-variations and quasi-subvariations</u>. Let p be a point of U and $(X^a)$ a quasi-orthonormal frame on U.

Let C be the integral curve thru p of a fixed but arbitrarily chosen null or non-null vector field $Y = A_c X^c$ on U. Let $X = Y/H$, where

$$H = (A_1^2 + A_2^2 + \dots + A_n^2)^{\frac{1}{2}}$$

When $X^a$ is displaced locally and $X^b$ is displaced parallelly along C, the initial rate at which $\langle X^a, X^b \rangle$ changes is measured by

$$P_{ab} = \langle D_X X^a, X^b \rangle = \langle D_{A_c X^c/H} X^a, X^b \rangle = E_{b_o} L/H,$$

where

$$L = \sum_h A_h V^a_{b_o h_o} + \sum_r A_r V^a_{b_o r}.$$

Now,

$$\frac{\partial P_{ab}}{\partial A_s} = E_{b_o}\left[\frac{H^2 E_s V^a_{b_o s} - L A_s}{H^3}\right],$$

$$\frac{\partial P_{ab}}{\partial A_k} = E_{b_o}\left[\frac{H^2 V^a_{b_o k_o} - L A_k}{H^3}\right].$$

Hence the vanishing of the partial derivatives of $P_{ab}$ with respect to the A's implies

(9) $$A_k L - H^2 V^a_{b_o k_o} = 0, \quad A_s L - E_s H^2 V^a_{b_o s} = 0.$$

The extreme value of $P_{ab}$ considered as a function of the A's, which we denote by $V_{ab}$, is reached when the A's satisfy (9). We find

$$\sum_c (V^a_{b_o c})^2 = \sum_h (LA_h/H^2)(V^a_{b_o h_o}) + \sum_r (E_r LA_r/H^2)(V^a_{b_o r})$$

$$= (L/H^2)\left(\sum_h A_h V^a_{b_o h_o} + \sum_r E_r A_r V^a_{b_o r}\right)$$

$$= L^2/H^2 = (E_{b_o} L/H)^2 = (V_{ab})^2,$$

where L and H have the fixed values given by (9). Consequently

(10) $$V_{ab} = \left[\sum_c (V^a_{b_o c})^2\right]^{\frac{1}{2}}.$$

The above derivation of $V_{ab}$ was made on the assumption that $P_{ab}$ is a function of the A's. When $P_{ab}$ does not depend on the A's, it is easily shown that $V^a_{b_o c} = 0$ for every c and hence $P_{ab} \equiv 0$. Then $V_{ab} = 0$ and (10) also holds.

If the above procedure were applied to a Riemannian manifold with an orthonormal frame $(X^a)$, then the invariant $V_{ab}$ reduces to the variation of $X^a$ with respect to $X^b$ first introduced by T. K. Pan [4]. So, we may call $V_{ab}$ as given by (10) the <u>quasi-variation of</u> $X^a$ <u>with respect to</u> $X^b$.

Note that $V_{ab}$ is derived by using either a null or a non-null Y. Now suppose Y is non-null. Let $X = Y/|\langle Y,Y\rangle|^{\frac{1}{2}}$ and let C be the integral curve of X through $p \in U$. When $X^a$ is displaced locally and $X^b$ is displaced parallelly along C, the rate at which the scalar function $\langle X^a,X^b\rangle$ changes is measured by

$$B_{ab} = \langle D_X X^a, X^b\rangle = E_{b_o} T/|K|^{\frac{1}{2}},$$

where

$$K = \langle Y,Y\rangle = \sum_h A_h A_{h_o} + \sum_r E_r A_r^2,$$

$$T = \sum_h A_h V^a_{b_o h_o} + \sum_r E_r A_r V^a_{b_o r}.$$

The vanishing of the partial derivatives of $B_{ab}$ with respect to the A's implies

$$E_K T A_c - |K| V^a_{b_o c} = 0, \tag{11}$$

where

$$E_K = \begin{cases} 1, & \text{if } K > 0, \\ -1, & \text{if } K < 0. \end{cases}$$

(14) $$SV_{ab} = \left[ \sum_{c \neq b} (V^a_{b_o c})^2 \right]^{\frac{1}{2}}.$$

(15) $$SW_{ab} = \left[ \left| \sum_h V^a_{b_o h} V^a_{b_o h_o} + \sum_r E_r (V^a_{b_o r})^2 - E_{b_o} V^a_{b_o b} V^a_{b_o b_o} \right| \right]^{\frac{1}{2}},$$

which are respectively called the <u>quasi-subvariation of</u> $X^a$ <u>with respect to</u> $X^b$ <u>of the first kind</u> and <u>of the second kind</u>.

It follows easily from (7) that the quasi-variation of either kind of a vector field of a quasi-orthonormal frame with respect to another vector field of the frame is equal to the quasi-variation of the same kind of the latter field with respect to the former. Moreover, from (8), the quasi-variation of either kind of a vector field of the frame with respect to itself vanishes.

# CHAPTER III

## INDUCED INVARIANTS

1. Induced coefficients of rotation. Let $M = (M, \langle\ ,\ \rangle)$ and $\overline{M} = (\overline{M}, \langle\langle\ ,\ \rangle\rangle)$ be two semi-Riemannian n-manifolds with semi-Riemannian connections D and $\overline{D}$, respectively. Let $F\colon U \subset M \longrightarrow F(U) \subset \overline{M}$ be a diffeomorphism. Then a connection is induced by F in each of the neighborhoods U and F(U), the one on U being called the induced $\overline{D}$ and that on F(U) the induced D. They are defined by

$$\overline{D}_X Y = F_*^{-1}(\overline{D}_{F_* X} F_* Y),$$

$$D_{\overline{X}}\overline{Y} = F_*(D_{F_*^{-1}\overline{X}} F_*^{-1}\overline{Y}),$$

for all vector fields X,Y on U and $\overline{X},\overline{Y}$ on F(U).

If D = induced $\overline{D}$, then $\overline{D}$ = induced D and F is connection preserving. In the following, we exclude this trivial case in which U and F(U) have essentially the same semi-Riemannian structure.

Let $(X^a)$ and $(\overline{X}^a)$ be quasi-orthonormal frames on U and F(U), respectively. Form the following four sets of coefficients of rotation:

(1a) $$\overline{N}^a_{bc} \equiv \langle\langle \overline{D}_{\overline{X}_c} \overline{X}^a, \overline{X}_b \rangle\rangle,$$

(1b) $$N^a_{bc} \equiv \langle \overline{D}_{X_c} X^a, X_b \rangle,$$

(1c) $$V^a_{bc} \equiv \langle D_{X_c} X^a, X_b \rangle,$$

(1d) $$\overline{V}^a_{bc} \equiv \langle\langle D_{\overline{X}_c} \overline{X}^a, \overline{X}_b \rangle\rangle.$$

We call the N's the <u>induced coefficients of rotation of the vector fields of</u> $(X^a)$ <u>on</u> U and the $\overline{V}$'s the <u>induced coefficients of rotation of</u> $(\overline{X}^a)$ <u>on</u> F(U).

<u>Theorem</u> 1. Among the four sets of coefficients of rotation in (1) associated with two diffeomorphic semi-Riemannian manifolds, there exist the following relations:

$$V^a_{bc} = \frac{1}{2}\Big[(N^a_{bc} - N^a_{cb}) + E_{a_o}E_{c_o}(N^{c_o}_{ba_o} - N^{c_o}_{a_ob})$$
$$+ E_{a_o}E_{b_o}(N^{b_o}_{ca_o} - N^{b_o}_{a_oc})\Big],$$

$$\overline{N}^a_{bc} = \frac{1}{2}\Big[(\overline{V}^a_{bc} - \overline{V}^a_{cb}) + E_{a_o}E_{c_o}(\overline{V}^{c_o}_{ba_o} - \overline{V}^{c_o}_{a_ob})$$
$$+ E_{a_o}E_{b_o}(\overline{V}^{b_o}_{ca_o} - \overline{V}^{b_o}_{a_oc})\Big].$$

<u>Proof</u>: For all vector fields X and Y on U, we define B in terms of D and the induced $\overline{D}$ by

(2) $$B(X,Y) = \overline{D}_X Y - D_X Y.$$

Then B is a vector-valued 2-covariant tensor field on U [3, p. 64]. Writing $B(X_c, X^a) = B^a_{bc} X^b$ and using (2), (1b) and (1c), we obtain

$$B^a_{bc} = \overline{\Gamma}^a_{bc} - \Gamma^a_{bc} = N^a_{bc} - V^a_{bc}, \tag{3}$$

where $\overline{\Gamma}^a_{bc}$ and $\Gamma^a_{bc}$ are the connection coefficients of the induced $\overline{D}$ and D, respectively. In their paper [7, p. 210], Sen and Gupta employed an identity which in our notations is of the form

$$B^a_{bc} = -\frac{1}{2}\left\{\overline{D}_{X_c}\langle X^a, X_b\rangle + \overline{D}_{X_b}\langle X^a, X_c\rangle - \overline{D}_{X^a}\langle X_c, X_b\rangle\right\}. \tag{4}$$

By use of the following formula [1, p. 239]

$$\begin{aligned} \overline{D}_{X^a}\langle X^b, X^c\rangle &= X^a\langle X^b, X^c\rangle - \langle \overline{D}_{X^a}X^b, X^c\rangle - \langle X^b, \overline{D}_{X^a}X^c\rangle \\ &= -\langle \overline{D}_{X^a}X^b, X^c\rangle - \langle X^b, \overline{D}_{X^a}X^c\rangle, \end{aligned} \tag{5}$$

we may reduce (4) to

$$\begin{aligned} B^a_{bc} = \frac{1}{2}\Big[&\langle \overline{D}_{X_c}X^a, X_b\rangle + \langle X^a, \overline{D}_{X_c}X_b\rangle + \langle \overline{D}_{X_b}X^a, X_c\rangle \\ &+ \langle X^a, \overline{D}_{X_b}X_c\rangle - \langle \overline{D}_{X^a}X_c, X_b\rangle - \langle X_c, \overline{D}_{X^a}X_b\rangle\Big]. \end{aligned} \tag{6}$$

Combination of (3) and (6) yields the first relation of the theorem.

The second relation can be shown similarly.

<u>Corollary</u> 1. $V^a_{bc} - V^a_{cb} = N^a_{bc} - N^a_{cb}$ and $\overline{N}^a_{bc} - \overline{N}^a_{cb} =$

$\overline{V}^a_{bc} - \overline{V}^a_{bc}$.

Proof: This follows from Theorem 1 by direct computations.

Let p be a point of U. From (5), we have

$$-(N^a_{bc})_p = -\langle \overline{D}_{X_c} X^a, X_b \rangle_p$$

which is the initial rate of change of $\langle X^a, X_b \rangle$ when $X^a$ is displaced locally and $X_b$ is displaced parallelly with respect to the induced $\overline{D}$ along the integral curve of $X_c$ thru p. Thus $N^a_{bc}$ as well as $\overline{V}^a_{bc}$ possess suitable geometric meanings.

2. Induced quasi-variations. The four sets of coefficients of rotation associated with diffeomorphic semi-Riemannian manifolds may be used to define eight sets of quasi-variations. Besides $V_{ab}$ and $W_{ab}$ on U and their analogues $\overline{N}_{ab}$ and $\overline{Z}_{ab}$ on F(U), we have the following:

$$N_{ab} = \left[\sum_c (N^a_{b_o c})^2\right]^{\frac{1}{2}},$$

$$\overline{V}_{ab} = \left[\sum_c (\overline{V}^a_{b_o c})^2\right]^{\frac{1}{2}}$$

(7)

$$\overline{W}_{ab} = \left[\left|\sum_h \overline{V}^a_{b_o h} \overline{V}^a_{b_o h_o} + \sum_r E_r (\overline{V}^a_{b_o r})^2\right|\right]^{\frac{1}{2}},$$

$$Z_{ab} = \left[\left|\sum_h N^a_{b_o h} N^a_{b_o h_o} + \sum_r E_r (N^a_{b_o r})^2\right|\right]^{\frac{1}{2}}.$$

which are called the induced quasi-variations. There are geometric interpretations of these induced quasi-variations.

Consider $Z_{ab}$, for example. Let p be a point of U and $(X^a)$ a frame on U. When $X^a$ is displaced locally and $X_b$ is displaced parallelly with respect to the induced $\bar{D}$ along a non-null curve C thru p whose unit tangent vector field is X, the initial rate at which $<X^a, X_b>$ changes is measured by $-<\bar{D}_X X^a, X_b>$. Thus $(Z_{ab})_p$ is the absolute critical value of $<\bar{D}_X X^a, X_b>$ as C ranges over all the non-null curves thru p.

3. Semi-Riemannian manifolds in conformal correspondence. Let $F: U \longrightarrow F(U)$ be a weakly conformal mapping, i.e.,

$$<<F_* X, F_* Y>> \circ F = e^{2w} <X,Y>, \tag{8}$$

for all vector fields X,Y on U, where w is some function on U and e is the base of the natural logarithm.

In classical differential geometry one usually uses mapping by 'equal coordinates' which is equivalent to taking F as the identity mapping. Then (8) simplifies to

$$<<X,Y>> = e^{2w} <X,Y> ,$$

which is assumed throughout this section.

Let $(X^a)$ be a quasi-orthonormal frame on U; then $(\bar{X}^a) = (e^{-w} X^a)$ is a quasi-orthonormal frame on $F(U) = U$. Consequently, we have that

$$\begin{aligned} N^a_{bc} &= e^w \{\bar{N}^a_{bc} + \delta^a_b \bar{X}_c(w)\}, \\ \bar{N}^a_{bc} &= e^{-w} \{N^a_{bc} - \delta^a_b X_c(w)\}, \end{aligned} \tag{9}$$

$$\overline{V}^a_{bc} = e^{-w}\{V^a_{bc} - \delta^a_b X_c(w)\},$$

(10)

$$V^a_{bc} = e^{w}\{\overline{V}^a_{bc} + \delta^a_b \overline{X}_c(w)\},$$

$$V^a_{bc} = e^{w}\{\overline{N}^a_{bc} - \delta^a_c \overline{X}_b(w) + E_{a_o}E_{b_o}\delta^{b_o}_c \overline{X}_{a_o}(w)\},$$

(11)

$$\overline{N}^a_{bc} = e^{-w}\{V^a_{bc} + \delta^a_c X_b(w) - E_{a_o}E_{b_o}\delta^{b_o}_c X_{a_o}(w)\},$$

$$\overline{V}^a_{bc} = e^{-w}\{N^a_{bc} + E_{a_o}E_{b_o}\delta^{b_o}_c X_{a_o}(w) - 2\delta^a_b X_c(w) - \delta^a_c X_b(w)\},$$

(12)

$$N^a_{bc} = e^{w}\{\overline{V}^a_{bc} + 2\delta^a_b \overline{X}_c(w) + \delta^a_c \overline{X}_b(w) - E_{a_o}E_{b_o}\delta^{b_o}_c \overline{X}_{a_o}(w)\},$$

where (11) is derived by means of Theorem 1 and equation (9) as follows:

$$2V^a_{bc} = (N^a_{bc} - N^a_{cb}) + E_{a_o}E_{c_o}(N^{c_o}_{ba_o} - N^{c_o}_{a_o b}) + E_{a_o}E_{b_o}(N^{b_o}_{ca_o} - N^{b_o}_{a_o c})$$

$$= e^{w}\{(\overline{N}^a_{bc} + \delta^a_b \overline{X}_c(w) - \overline{N}^a_{cb} - \delta^a_c \overline{X}_b) + E_{a_o}E_{c_o}(\overline{N}^{c_o}_{ba_o} +$$

$$\delta^{c_o}_b \overline{X}_{a_o}(w) - \overline{N}^{c_o}_{a_o b} - \delta^{c_o}_{a_o}\overline{X}_b(w)) + E_{a_o}E_{b_o}(\overline{N}^{b_o}_{ca_o} +$$

$$\delta^{b_o}_c \overline{X}_{a_o}(w) - \overline{N}^{b_o}_{a_o c} - \delta^{b_o}_{a_o}\overline{X}_c(w))\}$$

$$= e^{w}\{(\overline{N}^a_{bc} - E_{a_o}E_{b_o}\overline{N}^{b_o}_{a\ c}) + E_{a_o}(E_{c_o}\overline{N}^{c_o}_{ba_o} + E_{b_o}\overline{N}^{b_o}_{ca_o}) -$$

$$(\overline{N}^a_{cb} + E_{a_o}E_{c_o}\overline{N}^{c_o}_{a_o b}) - (\delta^a_c + E_{a_o}E_{c_o}\delta^{c_o}_{a_o})\overline{X}_b(w) +$$

$$+ (\delta^a_b - E_{a_o}E_{b_o}\delta^{b_o}_{a_o})\overline{X}_c(w) + (E_{a_o}E_{b_o}\delta^{b_o}_c + E_{a_o}E_{c_o}\delta^{c_o}_{a_o})\overline{X}_{a_o}(w)\}$$

$$= 2e^w\{\overline{N}^a_{bc} - \delta^a_c\overline{X}_b(w) + E_{a_o}E_{b_o}\delta^{b_o}_c\overline{X}_{a_o}(w)\}$$

Similarly, we have that

(13) $$N^s_{sc} = X_c(w),\ \overline{V}^s_{sc} = -\overline{X}_c(w),$$

(14) $$N^a_{bc} - N^a_{cb} = e^w(\overline{N}^a_{bc} - \overline{N}^a_{cb}),\ V^a_{bc} - V^a_{cb} = e^w(\overline{V}^a_{bc} - \overline{V}^a_{cb}),\ \text{when } b,c \neq a,$$

(15) $$N_{ab} = e^w\overline{N}_{ab},\ Z_{ab} = e^w\overline{Z}_{ab},\ V_{ab} = e^w\overline{V}_{ab},\ W_{ab} = e^w\overline{W}_{ab},\ N_{ab}\overline{V}_{ab} = V_{ab}\overline{N}_{ab},\ Z_{ab}\overline{W}_{ab} = \overline{Z}_{ab}W_{ab},\ \text{when } b \neq a_o,$$

(16) $Z_{ss} = 0 = \overline{Z}_{ss}$, $W_{ss} = 0 = \overline{W}_{ss}$, if the gradient of $w$ is in the direction of one of the null vector fields of the frame,

(17) $V_{ab} = e^w\overline{N}_{ab}$, $W_{ab} = e^w\overline{Z}_{ab}$, if the gradient of $w$ is orthogonal to both $\overline{X}_{a_o}$ and $\overline{X}_{b_o}$,

(18) $N_{ab} = e^w\overline{V}_{ab}$, $Z_{ab} = e^w\overline{W}_{ab}$, if $a \neq b$ and the gradient of $w$ is orthogonal to both $X_{a_o}, X_{b_o}$,

(19) $$N_{ss} = e^w\overline{V}_{ss},\ Z_{ss} = e^w\overline{W}_{ss},$$

where (13) is derived by use of (9) and (10) and the rest by (9) to (13).

CHAPTER IV

# CONGRUENCES OF MEMBERS OF A QUASI-ORTHONORMAL FRAME

1. Some conventions. A vector field X on $U \subset M$ determines a family of integral curves of X as defined in [1, p. 121], which is sometimes called the congruence of X. A curve of a congruence is thus a parametrized curve. It is proved there that a translation is the only allowable reparametrization of such a curve.

Since a vector field and a congruence determine each other, the two terms will be used interchangeably sometimes without distinction for the sake of simplifying statements.

A congruence of a vector field X is usually written as a congruence X. Unless otherwise noted, every congruence mentioned in this chapter is one determined by a member of a quasi-orthonormal frame $(X^a)$ on U.

2. Geodesic and normal congruences. A congruence $X^a$ is parallel along a congruence $X^b$ if the vector field $X^a$ is parallel along every curve of the congruence $X^b$; i.e., $D_{X^b}X^a = 0$. A congruence $X^a$ is a geodesic congruence if it is parallel along itself.

Since $D_{X^b} X^a = E_{b_o} V^a_{cb_o} X^c$, we have:

Proposition 1. A member $X^a$ is parallel along another member $X^b$ if and only if $V^a_{cb_o} = 0$ for every c. A member $X^a$ is geodesic if and only if $V^a_{ca_o} = 0$ for every c.

The coefficient of rotation $V^a_{bb} = \langle D_{X_b} X^a, X_b \rangle$ is called the tendency of $X^a$ in the direction $X_b$.

Proposition 1 and equation (II.7) lead directly to:

Proposition 2. A member $X^a$ is geodesic if and only if the tendency of $X^c$ in the direction $X_{a_o}$ vanishes for every c.

A null or non-null congruence X on U is called a normal congruence if X is a scalar multiple of a gradient, i.e., if there exist functions H and G on U such that X = G(grad H).

The following lemma is due to Wong [8, p. 163].

Lemma 1. A member $X^a$ is normal if and only if $V^a_{bc} = V^a_{cb}$, for $b,c \neq a$.

Proposition 3. Let a null member $X^h$ be normal. Then $X^h$ is geodesic if and only if the tendency $V^{h_o}_{h_o h_o}$ vanishes.

Proof: If $X^h$ is geodesic, then $V^h_{ah_o} = 0$ for every a. Hence $0 = V^h_{hh_o} = - V^{h_o}_{h_o h_o}$.

Conversely, suppose $V^{h_o}_{h_o h_o} = 0$. Then $V^{h}_{hh_o} = 0$. By Lemma 1 and equation (II.8) we have $V^{h}_{ah_o} = V^{h}_{h_o a} = 0$ for $a \neq h$. Hence the member $X^h$ is geodesic.

From (II.7) a given coefficient of rotation may have a representation with a different set of indices. We write " $V^a_{bc} = 0$, for $b \neq c \neq a \neq b$" if $V^a_{bc}$ vanishes when (a) the indices a, b and c are pairwise distinct and (b) the indices of its other representations from (II.7) are also pairwise distinct.

<u>Theorem</u> 1. Each of the members of $(X^a)$ is normal if and only if (a) $V^a_{bc} = 0$, for $b \neq c \neq a \neq b$, (b) $V^h_{ah} = V^{h_o}_{ah_o}$, for every a and h such that $h, h_o \neq a$ and (c) $V^a_{hh} = 0$, for every a and h such that $a \neq h$.

<u>Proof</u>: Suppose the congruence $X^a$ is normal for every a. By (II.7) and Lemma 1, we have

(a) $$V^a_{bc} = V^a_{cb} = - E_{a_o} E_{c_o} V^{c_o}_{a_o b} = - E_{a_o} E_{c_o} V^{c_o}_{b a_o}$$

$$= E_{a_o} E_{b_o} V^{b_o}_{c a_o} = E_{a_o} E_{b_o} V^{b_o}_{a_o c} = - V^a_{bc},$$

when the sets $\{a, b, c\}$, $\{a_o, c_o, b\}$ and $\{a_o, b_o, c\}$ have pairwise distinct elements, or

$$V^a_{bc} = 0, \text{ for } b \neq c \neq a \neq b;$$

(b) $V^h_{ah} = - E_{a_o} V^{a_o}_{h_o h} = - E_{a_o} V^{a_o}_{hh_o} = V^{h_o}_{ah_o}$, when $h, h_o \neq a$;

(c) $V^a_{hh} = - E_{a_o} V^{h_o}_{a_o h} = - E_{a_o} V^{h_o}_{ha_o} = 0$, when $a \neq h$.

Conversely, suppose (a), (b) and (c) hold. Let the member $X^h$ be fixed and null. Then

(i) $V^h_{bc} = 0 = V^h_{cb}$, when the sets $\{h, b, c\}$ and $\{h_o, b_o, c\}$ have pairwise distinct elements;

(ii) $V^h_{kk_o} = - V^{k_o}_{h_ok_o} = V^k_{h_ok} = V^h_{k_ok}$, when $k, k_o \neq h$;

(iii) $V^h_{bh_o} = - E_{b_o} V^{b_o}_{h_oh_o} = 0 = V^h_{h_ob}$, when $b \neq h$.

Hence $V^h_{bc} = V^h_{cb}$, when $b, c \neq h$, i.e., the member $X^h$ is normal. Next, let the member $X^s$ be fixed and non-null. Then

(iv) $V^s_{bc} = 0 = V^s_{cb}$, when the sets $\{s, b, c\}$ and $\{s, b_o, c\}$ have pairwise distinct elements;

(v) $$V^s_{hh_o} = - E_s V^{h_o}_{sh_o} = - E_s V^h_{sh} = V^s_{h_oh}.$$

From (iv) and (v) we conclude that the member $X^s$ is normal.

When the congruence of each member of $(X^a)$ is normal, we call $(X^a)$ a <u>special</u> <u>frame</u>.

<u>Proposition</u> 4. The tendency of a non-null member of a special frame in the direction of another non-null member vanishes if and only if the first is parallel along the latter.

Proof: Let $X^r$ and $X^s$ be two non-null members of a

3. Associate curvature. Let $D_{X^b}X^a$ be the associated vector field of $X^a$ with respect to $X^b$. The invariant $||X^{ab}||$ defined by

(1) $$||X^{ab}|| = |\langle D_{X^b}X^a, D_{X^b}X^a\rangle|^{\frac{1}{2}}$$

which in terms of the coefficients of rotation is given by

$$||X^{ab}|| = \left[\,|\sum_h V^a_{hb_o} V^a_{h_o b_o} + \sum_r E_r (V^a_{rb_o})^2|\,\right]^{\frac{1}{2}}$$

is called the associate curvature of the member $X^a$ with respect to the member $X^b$.

In a Riemannian manifold, the associate curvature of one congruence with respect to another is a measure of the deviation of the first congruence from parallelism along the second in that its vanishing is the condition for parallelism. It is obvious that this is no more so in a semi-Riemannian manifold.

By Theorem 1, equation (1) becomes

(2a) $$||X^{hh}|| = 0,$$

(2b) $$||X^{hh_o}|| = \left[\,|\sum_k V^h_{kh_o} V^h_{k_o h_o} + \sum_r E_r (V^h_{rh_o})^2|\,\right]^{\frac{1}{2}} = ||X^{h_o h}||$$

(2c) $$||X^{hk}|| = 0, \text{ for } h, h_o \neq k,$$

(2d) $$||X^{hs}|| = |V^h_{ss}|, \text{ (not summed on } s)$$

(2e) $$||X^{sh}|| = 0,$$

(2f) $$||X^{sr}|| = |V^s_{rr}|, \text{ (not summed on } r).$$

<u>Theorem</u> 2. The associate curvature of any member of a special frame with respect to any one of the null members vanishes, except possibly for the R distinct associate curvatures $||X^{hh_o}|| = ||X^{h_oh}||$.

<u>Proof</u>: This follows directly from (2a), (2b), (2c) and (2e).

From (2d) and (2f), we have the following theorem.

<u>Theorem</u> 3. The associate curvature of a member of a special frame with respect to any non-null member is numerically equal to the tendency of the former member in the direction of the latter.

<u>Theorem</u> 4. The associate curvature of a member of a special frame with respect to any non-null member vanishes if and only if the latter is geodesic.

<u>Proof</u>: Let $X^a$ and $X^s$ be two members of a special frame such that the non-null $X^s$ is geodesic. By Proposition 2, $V^a_{ss} = 0$ for every a; hence, by (2d) and (2f), we have $||X^{as}|| = 0$ for every a.

Conversely, suppose $||X^{as}|| = 0$ for every a. Then $V^a_{ss} = 0$ for every a. Hence $X^s$ is geodesic.

4. <u>Quasi-variations</u>. Applying Theorem 1 to (II.13), we have

(3a) $$W_{hm} = 0, \text{ for } h \neq m_o,$$

(3b) $$W_{hh_o} = \left[\left|\sum_k V^h_{hk} V^h_{hk_o} + \sum_r E_r (V^h_{hr})^2\right|\right]^{\frac{1}{2}},$$

(3c) $$W_{sh} = |V^s_{h_o s}| = |V^h_{ss}|, \text{ (not summed on s)}$$

(3d) $$W_{sr} = \left[\left|E_r(V^s_{rr}) + E_s(V^s_{rs})^2\right|\right]^{\frac{1}{2}}$$
$$= \left[\left|E_r(V^s_{rr}) + E_s(V^r_{ss})^2\right|\right]^{\frac{1}{2}},$$
(not summed on r and s).

When the quasi-variation (quasi-subvariation) of a member of a frame $(X^a)$ with respect to another member is numerically equal to a coefficient of rotation, we call the former member a quasi-principal (respectively, quasi-subprincipal) congruence with respect to the latter.

Theorem 5. The quasi-variation of each null member of a special frame with respect to another null member vanishes, except possibly for the R quasi-variations $W_{hh_o} = W_{h_o h}$.

Proof: This is evident from (3a) and (3b).

Theorem 6. Every null (non-null) member of a special frame is quasi-principal with respect to every non-null (respectively, null) member. Moreover, the quasi-variation of a null (non-null) member of a special frame with respect to a non-null (respectively, null) member vanishes if and only if the tendency of the null member in the direction of the non-null one vanishes.

<u>Proof</u>: The first part follows from (3c) and the remark following (II.15). The second part follows from (3c) and the fact that the tendency of a null $X^h$ in the direction of a non-null $X^s$ is numerically equal to the tendency of $X^h$ in the direction $X_s$.

<u>Corollary</u>. The quasi-subvariation of a null (non-null) member of a special frame with respect to every non-null (respectively, null) member vanishes.

<u>Theorem</u> 7. Let $X^s$ and $X^r$ be two fixed non-null members of a special frame. If $X^s$ and $X^r$ are both time-like or both space-like, then the quasi-variation of $X^s$ with respect to $X^r$ vanishes if and only if each is parallel along the other. If one of $X^s$ and $X^r$ is time-like and the other is space-like, then the quasi-variation of $X^s$ with respect to $X^r$ vanishes if and only if the tendency of $X^s$ in the direction $X_r$ is numerically equal to the tendency of $X^r$ in the direction $X_s$.

<u>Proof</u>: Suppose $X^r$ and $X^s$ are both time-like or both space-like. Then $E_s = E_r$ and hence by (3d), $W_{sr} = 0$ if and only if $V^s_{rr} = 0 = V^r_{ss}$. The conclusion now follows from Proposition 4.

Now suppose one of $X^s$ and $X^r$ is time-like and the other is space-like. Then $E_s = - E_r$. Hence, $W_{sr} = 0$ if and only if $(V^s_{rr})^2 = (V^r_{ss})^2$, i.e., if and only if $|V^s_{rr}| = |V^r_{ss}|$.

Another geometric interpretation of (3d) is given by the following theorem.

Theorem 8. A non-null member of a special frame is quasi-principal with respect to another non-null member if one is parallel along the other. Moreover if the quasi-variation of one non-null member with respect to another non-null member is numerically equal to the tendency of one in the direction of the other, then one is parallel along the other.

Corollary. Every non-null member of a special frame is quasi-principal with respect to another non-null member. Moreover a non-null member is quasi-principal with respect to another non-null member if the tendency of either one in the direction of the other vanishes.

From (II.13) and (II.15), we obtain

$$(4) \qquad W^2_{ab} - SW^2_{ab} = \begin{cases} |A + V^a_{h_o h} V^a_{h_o h_o}| - |A|, \text{ if } b = h, \\ \\ |B + E_s (V^a_{ss})^2| - |B|, \text{ if } b = s, \end{cases}$$

where $|A| = SW^2_{ah}$ and $|B| = SW^2_{as}$.

The following theorem is a consequence of (4).

Theorem 9. The difference between the squares of the quasi-variation and the quasi-subvariation of any one of n - 1 members of a frame with respect to the remaining member vanishes if each of the tendencies of the former members in the direction of the latter vanishes.

Using (4) and Proposition 2, we also have the following theorem.

Theorem 10. The quasi-variation and the quasi-subvariation of each of n - 1 members of a frame with respect to a remaining non-null member are equal if and only if the latter is geodesic.

Suppose $X^a$ is a fixed normal member of a frame. Then by Lemma 1 and equation (II.8), when a takes on the values h and s, we have that

(5a) $$W^2_{hk} = \left|\sum_m V^h_{mk_o} V^h_{m_o k_o} + \sum_r E_r (V^h_{rk_o})^2 + 2V^h_{k_o h} V^h_{k_o h_o}\right| , \text{ for } h \neq k_o,$$

(5b) $$W^2_{ht} = \left|\sum_m V^h_{mt} V^h_{m_o t} + \sum_r E_r (V^h_{rt})^2 + 2V^h_{th} V^h_{th_o}\right|,$$

(5c) $$W^2_{st} = \left|\sum_m V^s_{mt} V^s_{m\,t} + \sum_r E_r (V^s_{rt})^2 + E_s (V^s_{ts})^2\right|, \text{ for } s \neq t.$$

Theorem 11. Let $X^a$ be a fixed normal and $X^b$ any other member of a frame. If the tendency of $X^b$ in the direction $X_{a_o}$ vanishes, then the quasi-variation of $X^a$ with respect to $X^b$ is equal to the associate curvature of $X^a$ with respect to $X^b$.

Proof: The conclusion follows from (1), (5a), (5b) and (5c) since

$$V^h_{k_oh_o} = - V^k_{h_oh_o},\ V^h_{th_o} = - E_tV^t_{h_oh_o} \text{ and } V^s_{ts} = - E_sE_tV^t_{ss}.$$

Theorem 12. Let $X^a$ be a normal member of a frame which is parallel along another member $X^b$. If the tendency of $X^b$ in the direction $X_{a_o}$ vanishes, then the quasi-variation of $X^a$ with respect to $X^b$ also vanishes.

Proof: Since $X^a$ is parallel along $X^b$, the associate curvature of $X^a$ with respect to $X^b$ vanishes. The conclusion now follows from Theorem 11.

Theorem 13. If a given non-null geodesic member of a frame is parallel along each of the remaining members, then the quasi-variation of the given member with respect to every member vanishes.

Proof: Let $X^n$ be the given non-null geodesic member. Since $X^n$ is parallel along a member $X^b$ if and only if $V^n_{cb_o} = 0$ for every c and for $b \neq n$, we have $W_{nb} = 0$ for $b \neq n$.

Theorem 14. If each of the non-null members of a frame is geodesic and parallel along every other member, then the quasi-variation of each non-null member with respect to another member vanishes.

Proof. This is an immediate consequence of Proposition 1.

5. Induced coefficients of rotation and quasi-variations. We consider a coordinate neighborhood of U of M with the two connections D and induced $\overline{D}$ as was discussed in Chapter III. The frame $(X^a)$ is understood to be a quasi-orthonormal frame.

Proposition 5. A member $X^a$ of a frame is parallel with respect to D along another member $X^c$ if and only if

$$N^a_{bc_o} + E_{c_o}E_{a_o}N^c_{ba_o} + E_{b_o}E_{a_o}N^{b_o}_{c_oa_o} =$$

$$N^a_{c_ob} + E_{c_o}E_{a_o}N^c_{a_ob} + E_{b_o}E_{a_o}N^{b_o}_{a_oc_o},$$

for every b.

Proof: Use Proposition 1 and Theorem III.1 to get the desired result.

Proposition 6. A member $X^a$ of a frame is geodesic with respect to D if and only if $N^a_{ba_o} = N^a_{a_ob}$ for every b.

Proof: A necessary and sufficient condition for $X^a$ to be geodesic is $V^a_{ba_o} = 0$ for every b. By Corollary III.1, this condition is equivalent to that stated above.

Proposition 7. A member $X^a$ is normal if and only if $N^a_{bc} = N^a_{cb}$ for $b,c \neq a$.

Proof: This follows from Lemma 1 and Corollary III.1.

Theorem 15. Each member of a quasi-orthonormal frame is normal if and only if (a) $N^a_{bc} = N^a_{cb}$, for $b \neq c \neq a \neq b$, (b) $N^a_{hh_o} = N^a_{h_oh}$, for every a and h such that $h, h_o \neq a$ and (c) $N^{h_o}_{ha} = N^{h_o}_{ah}$, for every a and h such that $a \neq h$.

Proof: Theorem 1 and Corollary III.1 lead to this theorem.

A congruence $X^a$ is said to be $\bar{D}$-parallel along a congruence $X^b$ if $\bar{D}_{X^b}X^a = 0$. A congruence $X^a$ is a $\bar{D}$-geodesic congruence if it is $\bar{D}$-parallel along itself.

Since $D_{X^b}X^a = E_{b_o}N^a_{cb_o}X^c$, the following proposition ensues.

Proposition 8. A member $X^a$ is $\bar{D}$-parallel along another member $X^b$ if and only if $N^a_{cb_o} = 0$ for every c. A member $X^a$ is $\bar{D}$-geodesic if and only if $N^a_{ca_o} = 0$ for every c.

The $\bar{D}$-associate curvature of $X^a$ with respect to $X^b$ is defined as

$$||X^{(ab)'}|| = |<\bar{D}_{X^b}X^a, \bar{D}_{X^b}X^a>|^{\frac{1}{2}}$$

which is equivalent to the following in terms of the induced coefficients of rotation:

$$(6) \qquad ||X^{(ab)'}|| = \left[\left|\sum_h N^a_{hb_o} N^a_{h_ob_o} + \sum_r E_r (N^a_{rb_o})^2\right|\right]^{\frac{1}{2}}.$$

The following proposition is a direct consequence of Proposition 8 and equation (6).

Proposition 9. If $X^a$ is $\bar{D}$-parallel along $X^b$, then the $\bar{D}$-associate curvature of $X^a$ with respect to $X^b$ vanishes. If $X^a$ is $\bar{D}$-geodesic, then its $\bar{D}$-associate curvature with respect to itself vanishes.

Proposition 10. The following statements are equivalent:

(a) $X^a$ is both D-geodesic and $\bar{D}$-geodesic.

(b) $N^a_{a_0 b} = 0$ for every b.

(c) $N^a_{ba_0} = V^a_{ba_0}$ for every b.

Proof: Suppose (a) holds. Then $N^a_{ba_0} = N^a_{a_0 b}$ and $N^a_{ba_0} = 0$, for every b, and conversely, these conditions imply (a). It is now easy to see the equivalence of (a) and (b).

Using Corollary III.1 with the fact that $V^a_{a_0 b} = 0$, we have $N^a_{ba_0} - N^a_{a_0 b} = V^a_{ba_0} - V^a_{a_0 b} = V^a_{ba_0}$. Hence (b) is equivalent to (c).

We may call the induced quasi-variation $Z_{ab}$ as the $\bar{D}$-quasi-variation of $X^a$ with respect to $X^b$.

Theorem 16. If each member of a quasi-orthonormal frame is both D-geodesic and $\bar{D}$-geodesic, then both the $\bar{D}$-quasi-variation and the $\bar{D}$-associate curvature of any member with respect to itself vanish.

Proof: This follows from equations (III.7) and (6) and Proposition 10.

Theorem 17. If each member of a quasi-orthonormal frame is D-parallel along every member, then the $\bar{D}$-quasi-variation $Z_{ab}$ vanishes for every a and b.

Proof: By Proposition 8, we have $N^a_{bc} = 0$ for every a, b, and c. Hence $Z_{ab} = 0$ for every a and b.

Theorem 18. Let $X^s$ be a non-null normal member of a quasi-orthonormal frame. If the tendency of another member $X^b$ in the direction $X^s$ vanishes, then the $\bar{D}$-quasi-variation $Z_{sb}$ is equal to the $\bar{D}$-associate curvature $||X^{(sb)'}||$.

Proof: The normality of $X^s$ implies that $N^s_{ab} = N^s_{ab}$ for $a,b \neq s$. Hence, for $b \neq s$, we have

$$Z_{sb} = \left[ \left| \sum_h N^s_{b_o h} N^s_{b_o h_o} + \sum_r E_r (N^s_{b_o r})^2 \right| \right]^{\frac{1}{2}}$$

$$= \left[ \left| \sum_h N^s_{h b_o} N^s_{h_o b_o} + \sum_{r \neq s} E_r (N^s_{r b_o})^2 + E_s (N^s_{b_o s})^2 \right| \right]^{\frac{1}{2}}.$$

The conclusion now follows from

$$N^s_{b_o s} - N^s_{s b_o} = V^s_{b_o s} - V^s_{s b_o} = V^s_{b_o s} = - E_{b_o} E_s V^b_{ss} = 0,$$

which is true because $V^b_{ss} = 0$.

Theorem 19. The $\bar{D}$-quasi-variation of a non-null $X^s$ with respect to itself is equal to the $\bar{D}$-associate curvature of $X^s$ with respect to itself if the tendency of each $X^a$ in the direction $X^s$ vanishes for every $a \neq s$.

<u>Proof</u>: For $a \neq s$, we have that

$$N^s_{sa} - N^s_{as} = V^s_{sa} - V^s_{as} = - V^s_{as} = E_{a_o} E_s V^{a_o}_{ss} = 0$$

since $V^a_{ss} = 0$ for every a. Hence $N^s_{sa} = N^s_{as}$, for every a, and $Z_{ss} = ||X^{(ss)'}||$.

REFERENCES


1. Bishop, R. L., and Goldberg, S. I. Tensor Analysis on Manifolds. The Macmillan Company, New York, 1968.

2. Eisenhart, L. P. Riemannian Geometry. Princeton University Press, New Jersey, 1926, Second Printing, 1949.

3. Hicks, N. J. Notes on Differential Geometry. D. van Nostrand Company, New Jersey, 1965.

4. Pan, T. K. "Variation of congruences of curves of an orthogonal ennuple in a Riemannian space." Canadian Jour. of Math., 5(1953), 519 - 523.

5. Ruse, H. S. "On parallel fields of planes in a Riemannian space." Quart. Jour. Math. Oxford Ser., 20(1949) 218 - 234

6. ----------. "Parallel planes in a Riemannian $V_n$." Proc. Roy. Soc. Edinburgh Sect A 68(1950), 78 - 92.

7. Sen, R. N., and Gupta, B. "On orthogonal ennuples in a pair of Riemannian spaces." Proc. Nat. Inst. Sci. India Part A 32(1966), 210 - 222.

8. Wong, Y. C. "Quasi-orthogonal ennuple of congruences in a Riemannian space." Annals of Math. 46(1945), 158 -173.