

0.1 Leavitt Path Algebras and their (irreducible and indecomposable) representations.

Most of the results in this section are not new, but some of the proofs are. Standard terminology is in italics. Less standard terminology is bolded.

Let Γ be a *directed graph* consisting of V , a set of *vertices*, E , a set of *arrows*, and $s, t : E \longrightarrow V$ (the *source and target maps* respectively). We will only work in the finite case, which means $V \sqcup E$ is a finite set. We will call a vertex v a *sink* when $s^{-1}(v) = \emptyset$. A *path* in Γ is either a finite sequence of arrows $e_1 e_2 \cdots e_n$ where $t e_i = s e_{i+1}$ for $i \in \{1, 2, \dots, n-1\}$ or a vertex. The length of the path is n , and the length of a path that is a vertex is 0. We extend the domain of s and t to paths where $s v = t v = v$ for $v \in V$, and for $p = p_1 \cdots p_n$ where $p_i \in E$, $s p = s p_1$, $t p = t p_n$. In the case where $s p = t p$, we will call p a *closed path*. When the set $\{s p_i\}_{i=1}^n$ has n distinct vertices, then we call p a *cycle*. In the case where there is a path between any two vertices, we will call the graph *strongly connected*.

From this, we define the *path algebra* $\mathbb{F}\Gamma$ in two different ways, and then show that these separate constructions define the same module. We will construct $\mathbb{F}\Gamma_1$ as the algebra over some field $\mathbb{F}$ with a basis consisting of the set of paths in Γ (denoted by $Path(\Gamma)$), and multiplication of paths p, q given by:

$$pq = \begin{cases} p & t p = q \\ q & p = s q \\ p_1 \cdots p_n q_1 \cdots q_k & p = p_1 \cdots p_n, q = q_1 \cdots q_k, t p = s q \\ 0 & \text{otherwise} \end{cases}$$

We abuse notation (and will continue to abuse notation) by having elements of the algebra and elements of the sets V, E , have the same notation.

Here is the other construction of this object: for $\mathbb{F}\Gamma_2$, take the non unital algebra over a field $\mathbb{F}$ generated by $V \sqcup E$ with the following relations:

$$(V) \quad uv = \delta_{u,v} v \text{ for all } u, v \in V$$

$$(E) \quad s e e = e \text{ and } e t e = e \text{ for all } e \in E$$

Here, δ is the Kronecker delta. We also call the nontrivial monomials $v \in V$ and $e_1 \cdots e_n$ where $e_i \in E$, where $te_i = se_{i+1}$ for $i \in \{1, 2, \dots, n-1\}$ paths. We again extend s and t to paths.

$\mathbb{F}\Gamma_1$ and $\mathbb{F}\Gamma_2$ do not have a unit when V is infinite. In the case where V is finite, then $1 = \sum_V v$. We can put a partial order on V . For $v, w \in V$, $v \geq w$ means there is a $p \in \text{Path}(\Gamma)$ such that $sp = v, tp = w$. Equivalence classes happen among strongly connected subsets of vertices.

$\mathbb{F}\Gamma_1$ is isomorphic to $\mathbb{F}\Gamma_2$. We can define a map from $\mathbb{F}\Gamma_2$ to $\mathbb{F}\Gamma_1$ as identity on the generators. The relations (V) and (E) are satisfied in the target. The map is surjective as all basis elements in $\mathbb{F}\Gamma_1$ are images of paths in $\mathbb{F}\Gamma_2$. The map is also injective: for any linear combination of paths in $\mathbb{F}\Gamma_2$, the image in $\mathbb{F}\Gamma_1$ is zero iff all of the coefficients are zero. Thus, the map is an isomorphism, and we will just refer to $\mathbb{F}\Gamma$ from now on.

We extend construction of the path algebra on the *doubled graph*. Here we double the edge set by adding new edges $E^* := \{e^* | e \in E\}$ where e^* goes in the opposite direction as e of the original digraph.

$$\begin{aligned} V_{double} &= V & E_{double} &= E \sqcup E^* \\ s|_E &= s & t|_E &= t \\ se^* &= te & te^* &= se \quad \text{for all } e \in E^* \end{aligned}$$

We again extend s and t to paths.

From the path algebra on the doubled graph, we create $L(\Gamma)$, the *Leavitt Path Algebra* (LPA), where we impose the the following additional relations:

$$(CK1) \quad e^* f = \delta_{e,f} te \text{ for all } e^* \in E^*, f \in E,$$

$$(CK2) \quad \sum_{se=v} ee^* = \forall (\text{nonsink}) v \in V.$$

Here, CK stands for Cuntz-Krieger and δ is again the Kronecker delta.

We want to extend $*$ as a linear antiautomorphism on the path algebra of the doubled graph, where

$$(e)^* := e^* \quad (e^*)^* := e \quad (v)^* := v \quad (fg)^* = g^* f^*$$

for $e \in E$, $v \in V$, and $f, g \in E \sqcup E^*$. The relations hold under this automorphism, as:

$$(CK1) \quad (e^* f)^* = f^* (e^*)^* = f^* e = \delta_{f,et} f = (\delta_{e,ft} e)^* \text{ for all } e, f \in E,$$

$$(CK2) \quad \left(\sum_{se=v} e e^* \right)^* = \sum_{se=v} e e^* = v = v^*, \quad \forall \text{ nonsink } v \in V.$$

For $p = p_1 \cdots p_n$ where $p_i \in E \sqcup E^*$, we can see that $p^* = p_n^* \cdots p_1^*$. We see that $L(\Gamma)$ is spanned by $\{pq^* \mid p, q \in \text{Path}(\Gamma), tp = tq\} \subseteq L(\Gamma)$ by CK1. Also, note that $p^* q = 0$ for all $p, q \in \text{Path}(\Gamma)$ unless there exists $r \in \text{Path}(\Gamma)$ such that $p = rq$ or $q = pr$. We will call $e^* \in E^*$ a *dual arrow*, and p^* such that $p \in \text{Path}(\Gamma)$ a *dual path*.

Whenever one defines an algebra in terms of generators and relations, the question of whether the algebra is trivial or some of the generators map to 0 should be settled. We already know that the path algebra is not trivial, as $\mathbb{F}\Gamma$ was defined with the paths of Γ as a basis. However, $L(\Gamma)$ has more generators and more relations. We can use representation theory to make sure all the generators of $L(\Gamma)$ are nonzero.

For our argument, it is useful to note that the category of quiver representations is equivalent to the category of path algebra modules [?]. This is straightforward - to form a quiver representation from a module M , assign the vector space Mv to the vertex v and the linear map assigned to e will be the map that e induces from Mse to Mte . To form a module from a quiver representation, let M be the direct sum of all the vector spaces assigned to each vertex, and let v act as projection and then inclusion: $M \xrightarrow{pr} Mv \hookrightarrow M$, and e acts via: $M \xrightarrow{pr} Mse \xrightarrow{e} Mte \hookrightarrow M$.

The category of representations of $L(\Gamma)$ are equivalent to the full subcategory of modules M over $\mathbb{F}\Gamma$ such that the **Isomorphism Condition** holds [?]:

$$(IC) \quad (\cdot e)_{se=v} : Mv \xrightarrow{\cong} \bigoplus_{se=v} Mte \quad \text{for all nonsinks } v \in V$$

By CK1, e restricted to Mse yields a surjection onto Mte , and e^* yields a left inverse to e , hence e^* restricted to Mte is injective.

Example 1. To each vertex assign the vector space $\mathbb{F}\mathbb{N}$. Then, for each nonsink vertex, consider an isomorphism $\mathbb{F}\mathbb{N} \longrightarrow \bigoplus_{se=v} \mathbb{F}\mathbb{N}$. We can find such an isomorphism because vector space isomorphism classes are uniquely determined by their dimension, and both $\bigoplus_{se=v} \mathbb{F}\mathbb{N}$ and $\mathbb{F}\mathbb{N}$ are of countably infinite dimension. Then to each arrow e , assign the composition of the isomorphism from $\mathbb{F}\mathbb{N}$ to $\bigoplus_{se=v} \mathbb{F}\mathbb{N}$ with the projection to the summand corresponding to e .

Note that by our construction, this quiver representation satisfies the Isomorphism Condition, making it an $L(\Gamma)$ module. Each dual arrow e^* on $Mte \cong \mathbb{F}\mathbb{N}$ is given by the inclusion of this summand into $\oplus_{se=v} \mathbb{F}\mathbb{N}$ composed with the inverse on the isomorphism above.

The generators $V \sqcup E \sqcup E^*$ cannot be 0 in $L(\Gamma)$ as they do not act as 0 on the module. We also infer that the spanning set $\{pq^* \mid tp = tq\}$ consists of nontrivial elements of $L(\Gamma)$.

There is a $\mathbb{Z}$ -grading on $L(\Gamma)$ and $\mathbb{F}\Gamma$ as a consequence of assigning all $v \in V$ grade 0, all $e \in E$ grade 1, and all $e^* \in E^*$ grade -1 . This is compatible with the relations (V), (E), and (CK1), (CK2), as all these relations are homogeneous. As a consequence, homogeneous elements of degree n are described as $\sum_i p_i q_i^*$ where the length of p_i minus the length of q_i is n for all i . Hence, $L(\Gamma)$ is a $\mathbb{Z}$ -graded $*$ -algebra, where $*$ reverses the grading (with multiplication by -1).

We can use the $\mathbb{Z}$ -grading of these algebras to prove that $\mathbb{F}\Gamma$ embeds into $L(\Gamma)$ [?]. First, look at the map that sends $\mathbb{F}\Gamma$ to $L(\Gamma)$. This is a $\mathbb{Z}$ -graded morphism, and the kernel must be a graded ideal. Look at a homogeneous element in the ideal, say $\sum_{i=1}^n \lambda_i p_i$, where p_i have the same length for $1 \leq i \leq n$. Taking the image of this element in $L(\Gamma)$, act on it on the left with p_i^* . One obtains $\lambda_i t p_i$, which is only trivial if $\lambda_i = 0$ for $1 \leq i \leq n$, as $t p_i$ is a non trivial element of $L(\Gamma)$. Thus, the kernel is trivial, and the map is an injection.

We will need some more definitions to continue. Note that in order for a ring R to be *Artinian*, that is having the *Descending Chain Condition* on ideals, all descending sequences of ideals of R , say $I_1 \supseteq I_2 \supseteq \dots$ are eventually constant. A ring R is *Noetherian* when it has the *Ascending Chain Condition* on ideals: all ascending sequences of ideals $I_1 \subseteq I_2 \subseteq \dots$ are eventually constant. A ring has *Unbounded Generating Number* when for each positive integer m , any set of generators for the free right R -module R^m has cardinality greater than or equal to m . A cycle has an exit when for any p that is a cycle, there is a path q such that $sq = sp$, but $qr \neq p$ for any path r .

Many investigations of LPAs focus on the relationship between Γ and $L(\Gamma)$. These results are summarized here and quoted [?, Introduction]:

- (i) $L(\Gamma)$ has DCC (Descending Chain Condition) on right (or left) ideals [?, Theorem 2.6] if and only if Γ is acyclic (that is, Γ has no directed cycles) if and only if $L(\Gamma)$ is von Neumann regular [?, Theorem 1] if and only if $L(\Gamma)$ is finite

dimensional if and only if $L(\Gamma)$ is isomorphic to a direct sum of matrix algebras (over the ground field $\mathbb{F}$) [?, Corollaries 3.6 and 3.7].

- (ii) $L(\Gamma)$ has ACC (Ascending Chain Condition) on right (or left) ideals [?, Theorem 3.8] if and only if the cycles of Γ have no exits if and only if $L(\Gamma)$ is locally finite dimensional (i.e., a graded algebra with each homogeneous summand being finite dimensional) if and only if $L(\Gamma)$ is a principal ideal ring [?, Proposition 17] in which case $L(\Gamma)$ is isomorphic to a direct sum of matrix algebras over $\mathbb{F}$ and/or matrix algebras over $\mathbb{F}[x, x^{-1}]$ (the Laurent polynomial algebra) [?, Theorems 3.8 and 3.10].
- (iii) $L(\Gamma)$ has finite GK (Gelfand-Kirillov) dimension, equivalently $L(\Gamma)$ has polynomial growth if and only if the cycles in Γ are mutually disjoint [?, Theorem 5] if and only if all simple $L(\Gamma)$ -modules are finitely presented [?, Theorem 4.5]. In fact (i) and (ii) are special cases of (iii): Γ is acyclic if and only if the GK dimension of $L(\Gamma)$ is 0. The digraph Γ has a cycle but the cycles of Γ have no exits if and only if the GK dimension of $L(\Gamma)$ is 1. The first instance of $L(\Gamma)$ with GK dimension > 1 is given by the Toeplitz digraph

$$\Gamma : \begin{array}{c} \bullet \\ \bullet \end{array} \longrightarrow \begin{array}{c} \bullet \\ \bullet \end{array} \quad ([?], [?], [?, Example 5.6]).$$

$\vdots$

- (iv) $L(\Gamma)$ has a nonzero finite dimensional quotient if and only if Γ has a sink or a cycle such that there is no path from any other cycle to it [?, Theorem 6.5] if and only if $L(\Gamma)$ has UGN (Unbounded Generating Number) [?, Theorem 3.16] if and only if $L(\Gamma) \oplus L(\Gamma)$ is not a quotient of $L(\Gamma)$ [?, Corollary 6.7]. If $L(\Gamma)$ has finite Gelfand-Kirillov dimension then $L(\Gamma)$ has a nonzero finite dimensional quotient and if $L(\Gamma)$ has a nonzero finite dimensional quotient then $L(\Gamma)$ has IBN. Neither of these implications is reversible. [?, Corollary 6.9]

We have a Galois connection between 2^V (subsets of V) and ideals of

$L(\Gamma)$. A *Galois Connection* is a pair of order reversing functions $F : A \longrightarrow B$, $G : B \longrightarrow A$ between two posets A and B , such that for $a \in A$, $b \in B$, $a \leq GF(a)$ and $b \leq FG(b)$. The connection is as follows:

$$\begin{array}{c} X \xrightarrow{F} (X) \\ I \cap V \xleftarrow{G} I \end{array}$$

where (X) denotes the ideal generated by X . The partial order we use on the poset 2^V is inclusion, and the partial order we use on the set of ideals is reverse inclusion. This Galois Connection establishes a bijection between particular subsets of vertices and graded ideals. These subsets of vertices are *hereditary* and *saturated*: a subset of vertices is hereditary if for $v \in H$, if there is a $p \in \text{Path}(\Gamma)$ such that $sp = v$, then $tp \in H$, and a subset of vertices is saturated if $te \in H$ for all $e \in E$ where $se = v$, then $v \in H$. In general, this bijective correspondence is called a *Galois Correspondence* between $\{a \in A \mid GF(a) = a\}$ and $\{b \in B \mid FG(b) = b\}$.

To support the claim of a Galois Correspondence, we will need a lemma:

Lemma 1. *For any nontrivial graded ideal I of $L(\Gamma)$, $I \cap V \neq \emptyset$.*

Proof. Let $0 \neq x \in I$, where $x = \sum_i \lambda_i p_i q_i^*$ is a homogeneous element. As $1 = \sum_V v$, there is a $v \in V$ such that $xv \neq 0$. Since $v = \sum_{se=v} ee^*$, when v is not a sink and e^* is always injective, this means there is an e_1 such that $xe_0 \neq 0$ when v is not a sink. One can continue to find $e_j \in E$ such that $xve_1e_2 \cdots e_n \neq 0$ and $e_1 \cdots e_n$ is longer than any q_i^* that appears in the sum $\sum_i \lambda_i p_i q_i^*$ as long as te_j is not a sink for $j \in 1, 2, \dots, n-1$. If there is a sink that appears as te_j , there is no q_i^* that can be longer than it, as no starred path can precede a sink. Thus, by CK1, we have $xve_1e_2 \cdots e_n = \sum_l \lambda_k p'_k \neq 0$. By similar argument, we can find a $w \in V$ and a sequence of $f^* \in E^*$ such that $f_n^* f_{n-1}^* \cdots f_1^* wxve_1e_2 \cdots e_n \neq 0$. This is a nonzero homogeneous element of grade 0. By CK1, this must be a sum of paths, and the only paths that are of grade 0 are vertices. In particular, up to scaling by an element of $\mathbb{F}$, $f_n^* f_{n-1}^* \cdots f_1^* wxve_1e_2 \cdots e_n = tf_n$. $\square$

Proof of claim of Galois Correspondence. Suppose H is a hereditary saturated subset of vertices. It is clear that $H \subseteq (H) \cap V$. Since H generates (H) , for any vertex $v \in (H) \cap V \setminus H$, we have an expression: $\sum_i \lambda_i p_i q_i^* h_i p'_i q_i'^* = v$ where $h_i \in H$ and p_i, q_i, p'_i, q'_i are paths in $\text{Path}(\Gamma)$. We see that, by using

the relation of CK1, $\sum_i \lambda_i p_i q_i^* h_i p_i' q_i'^*$ simplifies to $\sum_j \lambda_j a_j b_j^*$ where a_j, b_j are paths in $Path(\Gamma)$ and $ta_j \in H$ since H is hereditary (remember: $e^*(se)e = te$). For all paths p such that $sp = v$ that are either of length equal to the longest b_j^* or where tp is a sink, we can see that $tp \in H$. This is because $p^*vp = tp$ and $p^* \left(\sum_j \lambda_j a_j b_j^* \right) p$ is a nonzero element with grade zero, where each summand has no generators from E^* . This is a vertex that is a descendant of an element of H , and therefore in H itself. By invoking that H is a saturated subset of vertices, for all p such that tp immediately precedes a sink or the length of p is one less than the length of the longest b_j , we have tp is also in H . We can repeat this process until $v \in H$, which means $H = (H) \cap V$.

Suppose that I is a graded ideal. It is clear that $(I \cap V) \subset I$. Observe $I \cap V$ is a hereditary saturated set of vertices (if $v \in I$, $p^*vp = tp \in I$, and if $te \in I$ for all e such that $se = v$, then $\sum_{se=v} e te e^* = se \in I$). Consider the graded module $L(\Gamma) / I \rightarrow L(\Gamma) / (I \cap V)$. The graded kernel of this morphism contains no ideals as it contains no vertices by our lemma, so it must be an isomorphism. $\square$

For H a hereditary saturated subset of vertices, we have an isomorphism:

$$L(\Gamma) / (H) \cong L(\Gamma \setminus H)$$

where $\Gamma \setminus H$ is the full subgraph of Γ on the set of vertices $V \setminus H$.

For the map from $L(\Gamma)$ to $L(\Gamma \setminus H)$:

$$\begin{aligned} v &\mapsto v \text{ for } v \in V \setminus H \\ v &\mapsto 0 \text{ for } v \in H \\ e &\mapsto e \text{ for } e \text{ such that } \{te, se\} \subseteq V \setminus H \\ e &\mapsto 0 \text{ for } e \text{ such that } \{te, se\} \not\subseteq V \setminus H \end{aligned}$$

We see that the homomorphism from left to right is graded. Via the Galois correspondence above, the graded kernel contains H and no other vertices, and is therefore exactly (H) .

This correspondence between hereditary saturated subsets of vertices and graded ideals of $L(\Gamma)$, extends to the non-graded case. The condition that the only hereditary saturated subsets of V are $\emptyset$ and V , and that any cycle has an exit are together equivalent to the condition that $L(\Gamma)$ is a simple algebra [?].

Most of the literature on representation theory is on representations of finite dimensional algebras. However, in order for $L(\Gamma)$ to be finite dimensional, Γ must have no directed cycles. This is equivalent to $L(\Gamma)$ having DCC. This is an incredibly restrictive condition on Γ . In this case, $L(\Gamma)$ is a finite sum of matrix algebras, indexed by the sinks of Γ . The summand corresponding to a sink is an algebra of square matrices, where each $n \times n$ matrix has n equal to the number of directed paths that end at that sink [?]. The representation theory of these algebras is well understood.

The representation theory of $L(\Gamma)$ for Γ with no directed cycles is well understood as a consequence of Morita equivalence of rings. Two rings R and S are *Morita equivalent* if there is a functor from the category of R modules to the category of S modules and another functor going the other way, such that their composition is naturally isomorphic to the identity. Using this functorial viewpoint, one finds that modules over $R \oplus S$ (where R and S are rings with unity) are the same as modules over R and modules over S as $M \mapsto M(1, 0)$ and $M \mapsto M(0, 1)$ maps the category of $R \oplus S$ modules to the category of R modules and S modules (respectively), and the direct sum of an R module and an S module has a canonical $R \times S$ module structure (R acts trivially on the second summand, S acts trivially on the first). Thus we can break up a representation of $\bigoplus_{\text{sinks}} M_n(\mathbb{F})$ into representations of $M_n(\mathbb{F})$. We can then call upon the fact that, for any ring R , the representation theory of R is the same as that of $M_n(R)$ [?]. Once we have reduced the representation of $M_n(\mathbb{F})$ to the representation theory of $\mathbb{F}$, we understand that these modules are just vector spaces completely characterized by their dimension.

When an algebra is not finite dimensional, mathematicians often focus on finite dimensional representations of the algebra. However, finite dimensional representations of Leavitt Path Algebras are now completely understood [?]. In general, given a module M , the **support** V_M of the module M is the set of vertices v such that $Mv \neq 0$. When one looks at a module M over $L(\Gamma)$, consider the **support subgraph** Γ_M (that is, the full subgraph of Γ on the support of M). The cycles of the support subgraph have no exits [?].

In parallel to how representation theory in general evolved, we may wish to extend our understanding from modules over matrix algebras to modules over $\mathbb{F}[x]$, or a Principal Ideal Domain (PID). With this in mind, one can recall that finitely generated modules over a PID are well understood. For any module M over a PID, say R , there is a unique sequence of nested ideals $I_n \supseteq I_{n-1} \supseteq \cdots \supseteq I_1$ such that $M \cong \bigoplus_{i=1}^n R/I_i$. In general, the subcategory of finitely generated modules is too unruly to work with unless the ring

is at least Noetherian, as submodules of finitely generated modules over a Noetherian ring are also finitely generated. Leavitt Path Algebras are often not Noetherian. A Leavitt Path Algebra is Noetherian precisely when the cycles of Γ have no exit [?]. This condition is rather restrictive on Γ .

Instead, we restrict our attention to the category of modules that are of *finite type*, which are both Artinian and Noetherian. This is the same as the category of modules with a finite length composition series: a module M is said to have a *composition series of length n* when there exists a sequence of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that M_i/M_{i-1} is a simple module for $i \in \{1, 2, \dots, n\}$ (where a *simple module* is a nonzero module with no submodules other than 0 and itself). This module category will often exclude the free module $L(\Gamma)$ (of rank one), but will still contain a breadth of interesting examples. We will also have two tools to make use of here - the *Jordan-Hölder theorem* and the *Krull-Remak-Schmidt theorem*. Their statements are as follows (J-H found in [?], K-R-S found in [?]):

Theorem (Jordan-Hölder). *Let M be a Λ -module of finite length, and $F := (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_l = M)$ and $G := (0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_m = M)$ two composition series for M . Then, for each simple Λ -module S , we have that the number of times S appears as a quotient F_i/F_{i-1} for $i \in \{1, 2, \dots, l\}$ is the same as the number of times S appears as a quotient G_i/G_{i-1} for $i \in \{1, 2, \dots, m\}$, and hence $l = m$.*

Theorem (Krull-Remak-Schmidt). *Let $M \neq 0$ be a module which is both Noetherian and Artinian. Then M is a finite direct sum of its indecomposable modules. Up to permutation, the indecomposable summands are uniquely determined up to isomorphism.*

Indecomposable modules are modules that cannot be written as the direct sum of two submodules. While it is clear that a simple module is indecomposable, it is not true that an indecomposable module is simple. When our module category is semisimple, then indecomposable also implies simple, but the finite length modules of $L(\Gamma)$ is not a semisimple category. While this category is not semisimple, we can still try to understand simple modules and how they might be extended to indecomposable modules.

As a brief example of why the category of finite length modules over $L(\Gamma)$ is not semisimple, consider the graph consisting of one vertex v and a single arrow e that forms a loop. It follows from (CK1) and (CK2) that for the arrow e , the dual arrow e^* is its inverse. This algebra is isomorphic to $\mathbb{F}[x^{-1}, x]$,

the Laurent polynomial algebra, via the map $1 \mapsto v$, $x \mapsto e$. It is well known that for $f(x) \in \mathbb{F}[x] \subseteq \mathbb{F}[x^{-1}, x]$, an irreducible polynomial where $f(0) = 1$, then $\mathbb{F}[x^{-1}, x]/(f(x)^k)$ for $k \geq 2$ is indecomposable, but not simple.

Currently, the known simple modules over the Leavitt Path Algebra where Γ is finite ($|V \sqcup E| < \infty$) are either the projective simple modules $vL(\Gamma)$, where v is a sink ($s^{-1}v = \emptyset$) or modules defined by X. Chen in 2013 [?] (and later generalized). Chen modules are generated by an *infinite path* $\cdots \alpha_2^* \alpha_1^* \alpha_0^* = \boldsymbol{\alpha}$, where $\alpha_i \in E$ for all $i \in \mathbb{N}$, and $t\alpha_i = s\alpha_{i+1}$. The action is defined as follows :

$$(\cdots \alpha_2^* \alpha_1^* \alpha_0^*)p = \begin{cases} \cdots \alpha_2^* \alpha_1^* \alpha_0^* & p = s \alpha_0 \\ \cdots \alpha_3^* \alpha_2^* \alpha_1^* & p = \alpha_0 \\ \cdots \alpha_2^* \alpha_1^* \alpha_0^* e^* & p = e, \text{ } t e = s \alpha_0, \text{ } e \in E \\ 0 & \text{else} \end{cases}$$

This defines the action with generators $V \sqcup E \sqcup E^*$, which extends to all of $L(\Gamma)$. The action of any pq^* preserves tail equivalence, where *tail equivalence* for two infinite paths $\boldsymbol{\alpha} = \cdots \alpha_2^* \alpha_1^* \alpha_0^*$ and $\boldsymbol{\beta} = \cdots \beta_2^* \beta_1^* \beta_0^*$ means that there is a $k_{\boldsymbol{\alpha}}$ and $k_{\boldsymbol{\beta}}$ such that $\alpha_{k_{\boldsymbol{\alpha}}+n} = \beta_{k_{\boldsymbol{\beta}}+n}$ for all $n \in \mathbb{N}$. Thus, Chen modules M have elements that are formal linear combinations of tail equivalent infinite paths.

The infinite path $\boldsymbol{\alpha}$ that generates a Chen module may be eventually periodic or not. An infinite path is eventually periodic if there exists $n, m \in \mathbb{N}$ where $m \geq 1$ such that $\alpha_{n+k} = \alpha_{n+m+k}$ for all $k \in \mathbb{N}$. We call the modules generated by eventually periodic infinite paths *rational* Chen modules (and we call the other Chen modules *irrational*). In the case of rational Chen modules generated by $(\boldsymbol{C}^*)^\infty = \cdots \boldsymbol{C}^* \boldsymbol{C}^* \boldsymbol{C}^*$ where $\boldsymbol{C}$ is some primitive closed path (*primitive* means that there is no other path $\boldsymbol{D}$ such that $\boldsymbol{D}^k = \boldsymbol{C}$ for $k \geq 2$), generalizations were defined by X. Chen [?], P. Ara, and K. Rangaswamy [?]. The former twists the action of $e \in E$ by the gauge action (arrows act as a nonzero scalar multiple of their previous actions), and the latter is a further twisting that can only be done when the field is not algebraically closed and the $\boldsymbol{C}$ is an *exclusive* cycle (meaning there is no other cycle with a common vertex). In 3.1 below, we will extend these constructions via irreducible polynomials.

0.2 Graph C^* -algebras and their representations on Hilbert spaces

A $*$ -algebra A over $\mathbb{F}$ (where $\mathbb{F}$ has an involution $c \longrightarrow \bar{c}$) is an algebra with a map $*$ which is a linear anti-automorphism and an involution:

$$\begin{aligned}(x + y)^* &= x^* + y^* \\ (xy)^* &= y^* x^* \\ (cx)^* &= \bar{c} x^* \\ (x^*)^* &= x\end{aligned}$$

for all $x, y \in A$, $c \in \mathbb{F}$.

A Cuntz-Kreiger Γ -family is a family of mutually orthogonal projections $\{P_v \mid v \in V\}$ (projections are elements such that $P_v^2 = P_v = P_v^*$, and mutually orthogonal means that $P_u P_v = \Delta_{u,v}^{n-1} P_u$) and partial isometries $\{S_e \mid e \in E\}$ (partial isometries are elements such that $S_e^* S_e$ is a projection). that generate a $*$ -algebra such that:

$$(CK1) \quad S_e^* S_e = P_{s_e} \text{ for all } e \in E$$

$$(CK2) \quad \sum_{t \in v} S_e S_e^* = P_v \text{ for all } v \in V \text{ such that } t^{-1}(v) \neq \emptyset.$$

where Γ is a graph as before. One noticeable difference is that composition is right to left. For paths $p = p_1 p_2 \cdots p_n$ in Γ with $p_i \in E$, we define $S_p := S_{p_1} S_{p_2} \cdots S_{p_n}$. When $p = v$, a path of length 0, we define $S_p = P_v$.

Now we can define a graph C^* -algebra, denoted $C^*(\Gamma)$. It is a universal algebra such that for each C^* -algebra with the same Γ -family, there exists a $*$ -homomorphism from $C^*(\Gamma)$ to it such that $\{P_v \mid v \in V\}$ and $\{S_e \mid e \in E\}$ are both preserved pointwise. This universal algebra will not retain the same $\mathbb{Z}$ -grading that the $*$ -algebra generated by the Cuntz-Kreiger Γ -family has (when one uses the analogous $\mathbb{Z}$ -grading placed on $L(\Gamma)$).

From the work of Gelfand-Naimark [?] and Segal [?], we know that $C^*(\Gamma)$ has a realization as a closed $*$ -subalgebra of bounded linear operator on some Hilbert space $\mathcal{H}$ [?], in fact:

$$C^*(\Gamma) = \overline{\text{span}}\{S_p S_q^* \mid p, q, \in \text{Path}(\Gamma), s p = s q\}$$

As an example of a graph C^* algebra consider the Toeplitz digraph as before:

$$\Gamma : \begin{array}{ccc} & e & \\ & \curvearrowright & \\ \bullet & \xrightarrow{f} & \bullet \\ v & & w \end{array}$$

Let $\mathcal{H}$ be the Hilbert space of square summable sequences over $\mathbb{C}$ indexed by $\mathbb{N}$. Consider the action on $\mathcal{H}$ defined by:

$$\begin{aligned} (a_0, a_1, a_2, a_3, \dots)P_v &= (0, a_1, a_2, a_3, \dots) & (a_0, a_1, a_2, a_3, \dots)P_w &= (a_0, 0, 0, 0, \dots) \\ (a_0, a_1, a_2, a_3, \dots)S_e &= (0, a_2, a_3, a_4, \dots) & (a_0, a_1, a_2, a_3, \dots)S_f &= (a_1, 0, 0, 0, \dots) \\ (a_0, a_1, a_2, a_3, \dots)S_e^* &= (0, 0, a_1, a_2, \dots) & (a_0, a_1, a_2, a_3, \dots)S_f^* &= (0, a_0, 0, 0, \dots) \end{aligned}$$

When one completes this $*$ -algebra with respect to the operator norm, then you obtain $C^*(\Gamma)$.

0.3 Symbolic Dynamics, Topological Markov Chains and Perron-Frobenius Theory

Let $\mathcal{A}$ be a (finite) alphabet. A *full $\mathcal{A}$ -shift space* is the space $\mathcal{A}^{\mathbb{N}}$, with a *shift map* $\sigma : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{A}^{\mathbb{N}}$ such that for all $(x_i)_{i \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$, we have $(\sigma x)_i = x_{i+1}$ for $i \geq 1$. We will call a subset of $\mathcal{A}^{\mathbb{N}}$ a *shift space* if it is invariant under σ . Here $\mathcal{A}^{\mathbb{N}}$ is a topological space with the product topology where $\mathcal{A}$ has the discrete topology, and σ is a continuous self map.

Words in the alphabet $\mathcal{A}$ are finite strings of elements of $\mathcal{A}$. We will be interested in *Shift of Finite Type* (SFT) spaces, which are shift spaces S defined by a finite list of *forbidden* words - that is, words that do not appear as a substring of any element of X . These space can also be described by a finite set of *allowed* words, where, given a set of allowed words, we can think of constructing elements of X letter by letter, checking against the list to see if each letter appended allows the end of the word to appear on the allowed list. An SFT is called *irreducible* if, for any two allowed words w_1, w_3 there is an allowed word w_2 such that the word $w_1 w_2 w_3$ is allowed.

We have the discrete topology on $\mathcal{A}$, and the product topology on $\mathcal{A}^{\mathbb{N}}$. Projection onto an appropriate coordinate separates any two distinct points in $\mathcal{A}^{\mathbb{N}}$, showing that $\mathcal{A}^{\mathbb{N}}$ (and hence any subset $\mathcal{A}^{\mathbb{N}}$) is totally disconnected. By Tychonoff's theorem, $\mathcal{A}^{\mathbb{N}}$ is compact if and only if $\mathcal{A}$ is finite. Moreover, $\mathcal{A}^{\mathbb{N}}$ is metrizable. For example, we can use the metric:

$$d(a_1a_2a_3\cdots, b_1b_2b_3\cdots) = \sum_{\{i \mid a_i \neq b_i\}} \frac{1}{2^i}.$$

The shift σ on $\mathcal{A}^{\mathbb{N}}$ is continuous. A subset of $\mathcal{A}^{\mathbb{N}}$ defined by a collection of forbidden words is closed and shift invariant.

We will look at $P_{\infty} := P_{\infty}(\Gamma)$, the space of infinite paths on the digraph Γ . Here, $\mathcal{A} = E$, the set of arrows of the digraph Γ , and P_{∞} is obtained by declaring the set $\{ef \mid te \neq sf\}$ as forbidden. P_{∞} is a *Topological Markov Chain*, or a *1-step SFT*. This is because as we build the space, we need only examine the previous letter, or arrow, in order to determine if the next letter/arrow is an allowed successor.

The topology placed on P_{∞} is the restriction of the product topology on $E^{\mathbb{N}}$. Notice that P_{∞} is closed as it is defined by forbidden words. As a closed subset of a compact set, P_{∞} is compact. It is also metrizable and totally disconnected.

Example 2.

$$P_{\infty}(\begin{smallmatrix} \circ \\ \vdots \end{smallmatrix}) = \{*\} \quad ; \quad P_{\infty}(\begin{smallmatrix} \curvearrowright & \bullet & \curvearrowleft \end{smallmatrix}) = \text{Cantor set}.$$

Some combinatorial properties of the digraph Γ , equivalent to important algebraic properties of $L(\Gamma)$ (quoted earlier in section 2.1), are also detected by the cardinality of P_{∞} :

1. $P_{\infty} = \emptyset$ if and only if Γ is acyclic.

When Γ has no sinks:

2. P_{∞} is a finite set if and only if the cycles of Γ have no exits.
3. P_{∞} is countable if and only if the cycles of Γ are pairwise disjoint.

$P_{\infty}(\Gamma)$ is also separable. This is because $Path(\Gamma)$ is countable, and hence so is the subset of (finite) paths that can be extended to some infinite path. Picking an infinite extension of such a finite path, we get a countable dense

subset of P_∞ since $\{pP_\infty \mid p \in \text{Path}(\Gamma)\}$ is a basis for the topology of P_∞ (here pP_∞ is the set of all infinite paths with initial segment $p \in \text{Path}(\Gamma)$).

We can also endow P_∞ with a Borel probability measure via the *Perron-Frobenius Theorem* [?].

Given a real nonnegative $n \times n$ matrix A , we can define a digraph Γ_A where $V = \{1, 2, \dots, n\}$, and there is an arrow from i to j if $a_{i,j} > 0$. However, Γ_A has at most one arrow from i to j . Now A is *irreducible* (that is, there is a positive integer $k = k(i, j)$ such that $(A^k)_{i,j} > 0$) if and only if Γ_A is strongly connected. If A is *stochastic* (that is, the sum of the entries of each row is 1), we get a probability distribution for a random walk on Γ .

Conversely, to define a random walk on a directed graph Γ , we assign a positive probability $p(e)$ to each arrow e , such that $\sum_{se=v} p(e) = 1$ for each nonsink $v \in V$. If $V = \{1, 2, \dots, n\}$ and Γ has no sinks, then we define the stochastic matrix $A = (a_{ij})$, where a_{ij} is the sum of $p(e)$ for e from i to j . Now $(A^k)_{i,j}$ is the probability of going from i to j in k steps, hence Γ is strongly connected if and only if A is irreducible.

Theorem (Perron-Frobenius). *Let A be an $n \times n$ real matrix A that is irreducible and stochastic. Then the largest eigenvalue of A is 1, which has multiplicity one. It has a unique left eigenvector $\mathbf{z}$ with each entry z_i positive and $\sum_{i=1}^n z_i = 1$.*

Proof. Consider the map A on $\Delta^{n-1} = \{(x_i)_{i=1}^n \mid x_i \geq 0, \sum_i x_i = 1\} \subseteq \mathbb{R}^n$ (where A acts on the right). This defines a self map on Δ^{n-1} , since for $\mathbf{x} \in \Delta^{n-1}$, we have $\sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} = \sum_{i=1}^n x_i \sum_{j=1}^n a_{ij} = \sum_{i=1}^n x_i = 1$. By Brouwer fixed-point theorem, there exists a vector $\mathbf{z}$ such that $\mathbf{z}A = \mathbf{z}$. Thus A has an left eigenvector of eigenvalue 1.

This vector $\mathbf{z} = [z_1 \ z_2 \ \dots \ z_n]$ has at least one entry, say z_i , that is strictly positive since it is an element of Δ^{n-1} . Since $\mathbf{z}A = \mathbf{z}$, we have the formula $z_j = \sum_{i=1}^n z_i a_{ij}$. We also know that $\mathbf{z}A^k = \mathbf{z}$ for all $k \geq 1$. Recall that for $k = k(i, j)$, we have $(A^k)_{i,j} > 0$. As a consequence, z_j must be positive when z_i is positive. Thus, $\mathbf{z}$ is an element of the interior of Δ^{n-1} .

Suppose, by way of contradiction, there are two distinct eigenvectors of eigenvalue 1 for A , and call the second eigenvector $\mathbf{y}$. Then, if the sum of the

coordinates of $\mathbf{y}$ is not 0, then consider $\frac{1}{\sum_i y_i} \mathbf{y}$ as an eigenvector of eigenvalue 1 whose entries sum to 1. If the sum of the coordinates of $\mathbf{y}$ is 0, then consider $\mathbf{y} + \mathbf{z}$ as an eigenvector of eigenvalue 1 whose entries sum to 1. Given an eigenvector of eigenvalue 1 whose entries sum to 1 that is not equal to a multiple of $\mathbf{z}$, say $\mathbf{u}$, consider

$$(\lambda \mathbf{z} - (1 - \lambda) \mathbf{u})A = \lambda \mathbf{z} - (1 - \lambda) \mathbf{u}$$

for all values of λ . This line must intersect the boundary of Δ^{n-1} for some values of λ . Earlier, we concluded that eigenvectors of A of eigenvalue 1 and whose coordinates sum to 1 cannot lie on the boundary of Δ^{n-1} , so this is a contradiction.

Now, let λ be the largest eigenvalue of A , for the right eigenvector of $\mathbf{y}$, as left eigenvalues are equal to right eigenvalues. We will assume, without loss of generality, that the largest entry of $\mathbf{y}$, say y_i is equal to 1 (this can be done taking a constant multiple of $\mathbf{y}$). Then we have:

$$\lambda = \lambda y_i = \sum_{j=1}^n a_{ij} y_j \leq \sum_{j=1}^n |a_{ij} y_j| \leq \sum_{j=1}^n a_{ij} \leq 1.$$

As we have already found a eigenvector of eigenvalue 1, this bound is sharp. $\square$