

We will construct modules that are not Chen modules over $L(1, 2)$ (a construction that can easily be modified for $L(1, n)$ where $n \geq 2$). To remind the reader, the underlying graph Γ is:



0.1 Papillon

We will consider a quiver representation that satisfies the isomorphism condition, and demonstrate why this is not a Chen module.

Let the vector space at the single vertex be

$$X = \{ (a_i)_{i \in \mathbb{N}} \mid (a_i)_{i \in \mathbb{N}} \text{ is } 2^k\text{-periodic for some } k \in \mathbb{N} \}$$

The edges, e_0 and e_1 respectively, will act in the following manner:

$$(a_0, a_1, a_2, \dots) e_0 = (a_0, a_2, a_4, \dots)$$

$$(a_0, a_1, a_2, \dots) e_1 = (a_1, a_3, a_5, \dots)$$

Notice that this module satisfies the isomorphism condition. Therefore, it is a Leavitt module, and:

$$(a_0, a_1, a_2, \dots) e_0^* = (a_0, 0, a_1, 0, \dots)$$

$$(a_0, a_1, a_2, \dots) e_1^* = (0, a_0, 0, a_1, \dots)$$

We will call this module Papillon. You may recall a very similar module from the introduction to this thesis. The action is exactly the same ($e_i \longleftrightarrow x_i$, $e_i^* \longleftrightarrow y_i$) but the space X is markedly different from the space of finite sequences.

For any $m \in M$, where M is an $L(\Gamma)$ module, we define:

$$A_m := \{ p \in \text{Path}(\Gamma) \mid p = p_1 p_2 \dots p_n, \ m(sp) p_0 \dots p_{n-1} \neq 0, \ mp = 0, \text{ where } p_i \in E \}$$

Lemma 1. *For Chen modules, A_m has infinitely many elements when $m \neq 0$, unless we have a rational Chen Module defined by a cycle with no exit.*

Proof. For any nonzero $m \in M$, M a Chen Module, m is a linear combination of infinite tail equivalent paths (this means that for any set $\{\alpha_1, \dots, \alpha_n\}$ of tail equivalent infinite paths, there is a k_{α_i} for each α_i such that all paths in the set are the same at and beyond the edge of index k_{α_i} . For any $m = \sum \alpha$, without loss of generality, let k_{α_1} be the largest of the k_{α_i} . There exists a path p of length k_{α_1} such that $mp \neq 0$ (by the repeated application of (CK2) exactly k_{α_1} times). We can see that A_{mp} has infinitely many elements - it consists of all paths $\alpha_{k_{\alpha_1}+1}\alpha_{k_{\alpha_1}+2} \cdots \alpha_{k_{\alpha_1}+N}e$ where $N \in \mathbb{N}$ and $e \in E$, $e \neq \alpha_{k_{\alpha_1}+N+1}$ (this is possible as the infinite path is not on a cycle without exits). As $A_{mp} \subseteq A_m$, this shows that A_m also has infinitely many elements. $\square$

This is not a property shared by the Papillon over $L(1, 2)$ described above - given any nonzero $x \in X$, any edge that does not annihilate x will divide the length of its period by 2. This means that $|A_x| \leq k$ where 2^k is the smallest period of x . It is also clear that Papillon is not a rational Chen module on a cycle with no exit, given that our Γ has exits for any rational path. Papillon is the same as the module generated by the indicator function for P_∞ on Γ . The correspondence is clear when you have the constant sequence correspond to $\mathbb{1}_{P_\infty}$.

0.2 Modules with the action of Leavitt defined by polynomials

We want to obtain new representations for $L(\Gamma)$ acting on:

1. $\mathbb{Z}$ -indexed sequences
2. Laurent polynomials ($\mathbb{F}[x^{-1}, x]$)
3. Polynomials
4. $x^{-1}\mathbb{F}[x^{-1}]$

We will relate the actions of (1) and (2) with the correspondence

$$\underline{f} = \begin{cases} f_k & -m \leq k \leq l \\ 0 & \text{else} \end{cases} \leftrightarrow f(z) = \sum_{k=-m}^l f_k x^k$$

. We can consider (3) and (4) as a restriction of (2).

We start with a polynomial $m_0(z) = \sum_{i=0}^n a_i x^i$, with the following conditions:

1. $a_0 \neq 0$
2. $\lambda = \sum_{i=0}^n a_i^2 \neq 0$
3. $\sum_{i-j=k} a_i a_j = 0$ for k even, $k \neq 0$

As a consequence of the last condition, the degree n is odd. We will also define $m_1(z) = z^n m_0(-z^{-1})$. For a Laurent polynomial $f(z)$, the action is as follows:

$$f(z) v = f(z)$$

$$f(z) e_i^* = \lambda^{-1} m_i(z) f(z^2) \text{ for } i = 0, 1$$

$$f(z) e_i = \frac{1}{2} (m_i(z^{-1/2}) f(z^{1/2}) + m_i(-z^{-1/2}) f(-z^{1/2})) \text{ for } i = 0, 1$$

On $\mathbb{Z}$ -indexed sequences, the action is:

$$\underline{f} v = \underline{f}$$

$$\underline{f} e_0^* = \begin{cases} \lambda^{-1} \sum_{k=0}^{(n-1)/2} f_{i-k} a_{2k} & i \text{ is even} \\ \lambda^{-1} \sum_{k=0}^{(n-1)/2} f_{i-k} a_{2k+1} & i \text{ is odd} \end{cases}$$

$$\underline{f} e_1^* = \begin{cases} \lambda^{-1} \sum_{k=0}^{(n-1)/2} -f_{i-k} a_{n-2k} & i \text{ is even} \\ \lambda^{-1} \sum_{k=0}^{(n-1)/2} f_{i-k} a_{n-(2k+1)} & i \text{ is odd} \end{cases}$$

$$\underline{f} e_0 = (\dots, \sum_{k=0}^n f_{2i-k} a_k, \dots)$$

$$\underline{f}e_1 = (\dots, \sum_{k=0}^n f_{2i-k}a_{n-k}, \dots)$$

The conditions on m_0 ensures that the relations of the Leavitt path algebra hold.

Even though the action of e_i may seem problematic, you always get a Laurent polynomial or $\mathbb{Z}$ -indexed sequence due to cancellation of terms. Another way to describe the action is: e_i^* upsamples (double all exponents) and multiplies by m_i ; e_i multiplies by $\bar{m}_i$ (where $\bar{m}_i(x) = m_i(x^{-1})$) and then downsamples (halve all exponents and get rid of non integer powers). To verify (CK1) and (CK2), it is handy to have the identities:

$$m_i(x)\bar{m}_j(x) + m_i(-x)\bar{m}_j(-x) = 2\lambda\delta_{ij}$$

$$\bar{m}_0(x)m_0(x) + \bar{m}_1(x)m_1(x) = 2\lambda$$

$$\bar{m}_0(x)m_0(-x) + \bar{m}_1(x)m_1(-x) = 0$$

All of these identities are a consequence of having $\begin{bmatrix} m_0(x) & m_0(-x) \\ m_1(x) & m_1(-x) \end{bmatrix}$ be almost a "unitary" matrix, so that :

$$\begin{bmatrix} m_0(x) & m_0(-x) \\ m_1(x) & m_1(-x) \end{bmatrix} \begin{bmatrix} \bar{m}_0(x) & \bar{m}_1(x) \\ \bar{m}_0(-x) & \bar{m}_1(-x) \end{bmatrix} = \begin{bmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{bmatrix}$$

$$\begin{bmatrix} \bar{m}_0(x) & \bar{m}_1(x) \\ \bar{m}_0(-x) & \bar{m}_1(-x) \end{bmatrix} \begin{bmatrix} m_0(x) & m_0(-x) \\ m_1(x) & m_1(-x) \end{bmatrix} = \begin{bmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{bmatrix}$$

To check (CK1) for Laurent polynomials:

$$\begin{aligned} f(z)e_i^*e_j &= (\lambda^{-1}m_i(z)f(z^2))e_j \\ &= \lambda^{-1}(m_i(z^{1/2})f(z)m_j(z^{-1/2}) + m_i(-z^{1/2})f(z)m_j(-z^{-1/2})) \\ &= \lambda^{-1}\frac{1}{2}(m_i(z^{1/2})\bar{m}_j(z^{1/2}) + m_i(-z^{1/2})\bar{m}_j(z^{1/2}))f(z) \\ &= \lambda^{-1}\frac{1}{2}2\lambda\delta_{ij}f(z) \\ &= \delta_{ij}f(z) \end{aligned}$$

To check (CK2):

$$\begin{aligned}
f(z)(e_0 e_0^* + e_1 e_1^*) &= \frac{1}{2} (m_0(z^{-1/2})f(z^{1/2}) + m_0(-z^{-1/2})f(-z^{1/2})) e_0^* \\
&\quad + \frac{1}{2} (m_1(z^{-1/2})f(z^{1/2}) + m_1(-z^{-1/2})f(-z^{1/2})) e_1^* \\
&= \frac{1}{2} \lambda^{-1} (m_0(z^{-1})f(z)m_0(z) + m_0(-z^{-1})f(-z)m_0(z)) \\
&\quad + \frac{1}{2} \lambda^{-1} (m_1(z^{-1})f(z)m_1(z) + m_1(-z^{-1})f(-z)m_1(z)) \\
&= \frac{1}{2} \lambda^{-1} f(z) (\bar{m}_0(z)m_0(z) + \bar{m}_1(z)m_1(z)) \\
&\quad + \frac{1}{2} \lambda^{-1} f(-z) (\bar{m}_0(-z)m_0(z) + \bar{m}_1(-z)m_1(z)) \\
&= \frac{1}{2} \lambda^{-1} f(z)(2\lambda) + \frac{1}{2} \lambda^{-1} f(-z)(0) = f(z)
\end{aligned}$$

The action on $\mathbb{Z}$ -indexed sequences is analogous.

We now have a way of generating representations of $L(1, 2)$. Let's look at the representation that is defined by $m_0 = 1 + x$ (and thus $m_1 = 1 - x$). Regardless of whether you look at this in the space of Laurent Polynomials or bi-infinite 2^k periodic sequences, if you want to produce simple representations, you find that you need to restrict to one of three spaces: infinite sequences (indexed by natural numbers), $\mathbb{F}[x]$, or $x^{-1}\mathbb{F}[x^{-1}]$. For the action on infinite sequences, it is helpful to look at the basis of rows of the infinite Hadamard matrix constructed recursively:

$$H_0 = [1] \text{ and } H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$$

Using the action on $\underline{f} = (\dots, f_0, f_1, f_2, \dots, f_{2^k-1} \dots)$, which simplifies as:

$$\underline{f} e_0^* = \frac{1}{2} (\dots, f_0, f_0, f_1, f_1, f_2, f_2, \dots, f_{2^k-1}, f_{2^k-1}, \dots)$$

$$\underline{f} e_1^* = \frac{1}{2} (\dots, f_0, -f_0, f_1, -f_1, f_2, -f_2, \dots, f_{2^k-1}, -f_{2^k-1}, \dots)$$

$$\underline{f} e_0 = (\dots, f_0 + f_1, f_2 + f_3, f_4 + f_5 \dots, f_{2^k-2} + f_{2^k-1}, f_0 + f_1, \dots)$$

$$\underline{f}e_1 = (\dots, f_0 - f_1, f_2 - f_3, f_4 - f_5 \dots, f_{2^k-2} - f_{2^k-1}, f_0 - f_1, \dots)$$

We get that that rows go to scalar multiples of rows under the action of e_0^* and e_1^* . This allows us to see that we can obtain all 2^k periodic sequences (as rows of the Hadamard transform form a basis of 2^k periodic sequences). As all sequences can be reduced to the constant sequence (e_0, e_1 divide the period of any sequence by 2, and they cannot both annihilate a given element), and the constant sequence survives (up to constant multiple) only along the infinite path $e_0 e_0 e_0 \dots$, this is a Chen Module (one can see from a future lemma that the feature of an element only surviving along one path uniquely defines a Chen Module).

On polynomials, the action is markedly different. This module is simple because all nontrivial polynomials can be reduced by repeated action of e_0, e_1 to a nontrivial constant polynomial. By repeated action of $e_0^* + e_1^*, e_0^* - e_1^*$, we are able to take x^n and obtain x^{2n} or x^{2n+1} . We are able to obtain all monomials of power greater than or equal to 0 this way. This module is the same as Papillon - 1 corresponds to $(1, 1, 1, 1, \dots)$ and monomials correspond to the rows of the infinite Hadamard transform.

Lastly, the action of $L(1, 2)$ on $x^{-1}\mathbb{F}[x^{-1}]$ is a nontrivial twist of the Papillon. We will show that all twists by the gauge action are nontrivial.

Lemma 2. *For all $(a, b) \in \mathbb{F}^\times \times \mathbb{F}^\times$, Papillon twisted by the gauge action of (a, b) is a distinct simple module up to isomorphism.*

Proof. We will check that the only thing that stabilizes Papillon is $(1, 1)$. We will consider the twisted action of $L(1, 2)$ on Papillon, where $\tilde{e}_0 = ae_0$ and $\tilde{e}_1 = be_1$, for $(a, b) \in \mathbb{F}^\times \times \mathbb{F}^\times$. Consider that $(1, 1, 1, 1, \dots)$ is fixed under the action of e_0 and e_1 . Scalar multiples of this sequence are the only ones fixed under the action of e_0 and e_1 . However, there is no nontrivial constant sequence fixed by $\tilde{e}_0$ and $\tilde{e}_1$, with the exception of when $(a, b) = (1, 1)$. Thus, the action is faithful. $\square$

The action of $L(1, 2)$ on $x^{-1}\mathbb{F}[x^{-1}]$ is a nontrivial twist of the Papillon, by the scalars $(1, -1)$.

Other modules can be constructed by considering the action defined $m_0 = 1 + x^{2n+1}$, $m_1 = 1 - x^{2n+1}$ on Laurent polynomials ($n \in \mathbb{N}$). On monomials, the action simplifies as follows:

$$x^k e_0 = \begin{cases} x^{k/2} & k \text{ is even} \\ x^{(k-2n-1)/2} & k \text{ is odd} \end{cases}$$

$$x^k e_1 = \begin{cases} x^{k/2} & k \text{ is even} \\ -x^{(k-2n-1)/2} & k \text{ is odd} \end{cases}$$

$$x^k e_0^* = x^{2k} + x^{2k+2n+1}$$

$$x^k e_1^* = x^{2k} - x^{2k+2n+1}$$

Given this action we have this decomposition of $\mathbb{F}[x, x^{-1}]$ into these simple modules:

$$\mathbb{F}[x, x^{-1}] = \mathbb{F}[x^{2n+1}] \oplus x^{-2n-1} \mathbb{F}[x^{-2n-1}] \bigoplus_O \left(\bigoplus_{k \in \tilde{O}} \mathbb{F} x^k \right)$$

Where O is a nontrivial orbit of $\langle 2 \rangle$ (the multiplicative group generated by 2 in $\mathbb{Z}/(2n+1)\mathbb{Z}$), and $\tilde{O}$ is the preimage of O under the map from $\mathbb{Z}$ to $\mathbb{Z}/(2n+1)\mathbb{Z}$. This is because the action of e_0, e_0^*, e_1, e_1^* on a x^k yields terms whose degree is either $2k, 2k+2n+1, k/2$, or $(k-2n-1)/2$. All of the algebraic operations involved (multiplying or dividing by 2, adding or subtracting $2n+1$) leave invariant the orbits of $\langle 2 \rangle$ in $\mathbb{Z}/(2n+1)\mathbb{Z}$.

As far as why each piece is simple - notice that any Laurent polynomial under the repeated action of e_0 and e_1 becomes a polynomial of reduced degree and increased order until the order is greater than or equal to $-2n-1$ and the degree is less than or equal to 0. It just so happens that 1 and x^{-2n-1} are fixed under the action of e_0, e_1 . Repeated application of e_0, e_1 (such that the action does not yield the 0 polynomial) to a polynomial of order strictly greater than $-2n-1$ and degree less than 0 keeps multiplying the powers of each term by 1/2, and adjusting to where each term is strictly between $-2n-1$ and 0 - thus giving a basis of polynomials of terms with powers in the same orbit of $\langle 2 \rangle$. As 2 and 2^{-1} are units in $\mathbb{Z}/(2n+1)\mathbb{Z}$, all elements of the orbit are achieved through repeated application of e_0, e_1 .

While $\mathbb{F}[x^{2n+1}]$ and $x^{-2n-1} \mathbb{F}[x^{-2n-1}]$ are recognizable as Papillon and the Papillon twisted by $(1, -1)$ (this via the vector isomorphism $x^k \mapsto x^{k(2n+1)}$), the modules indexed by the orbits of $\langle 2 \rangle$ in $\mathbb{Z}/(2n+1)\mathbb{Z}$ are different. It is

clear that these modules are not Chen, as monomials are all nonzero under the action of e_0 and e_1 forever. At the same time, these modules are not Papillon- this is clear when you consider the O always have even and odd elements. Look at the action of e_0, e_1 on odd powered monomials. If there was a polynomial such that it was fixed (up to scalar multiple) under that action of e_0, e_1 , it would need to have all powers between $-2n - 1$ and 0 . The polynomial must have all even or odd degree (as e_0 and e_1 differ only by a scalar on even or odd power terms - but not on polynomials with mixed even and odd power terms). This cannot happen, as all orbits of $\langle 2 \rangle$ have even and odd numbers:

Lemma 3. *All orbits of $\langle 2 \rangle$ in $\mathbb{Z}/(2n + 1)\mathbb{Z}$ have even and odd numbers*

Proof. Given any even element of an orbit, if you multiply by 2 enough, you get something larger than $2n + 1$. When you mod by $2n + 1$, you will get something odd. If you start with something odd instead, multiplication by 2 will either yield an even number (if your starting number was less than or equal to n) or an odd number less than n (if your starting number was greater than or equal to $n + 1$). When in the latter case, multiply by 2 again, giving an even number. $\square$

There is no element fixed up to scalar by e_0, e_1 . Thus, these modules are not the same as Papillon (where the constant sequence is fixed). They cannot be any module generated by an indicator function on P_∞ , as we have just shown that this module is not the one generated by $\mathbb{1}_{P_\infty}$.

Thus, we have constructed a module which is neither a Chen module, nor a module generated by an indicator function on P_∞ .