

## 0.1 Generalized twisted rational Chen modules

We will now construct a module on which  $L(\Gamma)$  acts. This will be a previously unknown generalization of rational Chen modules. For these new modules, we will consider the case where  $\Gamma$  contains a directed cycle. Let  $C = c_0 \cdot c_1 \cdots c_n$  be a primitive closed path.

We define  $\mathbb{F}P_C$  as a vector space over  $\mathbb{F}$  with basis  $P_C$ , where  $P_C$  is the set of paths in  $Path(\Gamma)$  that end at  $w = sC$ , but are not equal to a path  $qC$  for any path  $q$ .

Then  $M_C := \mathbb{F}[x, x^{-1}] \otimes_{\mathbb{F}} \mathbb{F}P_C$  as a vector space.

We will define the action of  $L(\Gamma)$  on  $M_C$  using pure tensors of  $M_C$  acted on by  $V \sqcup E \sqcup E^*$  on the right and have the action extend linearly and multiplicatively. We denote a generic pure tensor (ignoring coefficients) in  $M_C$  by  $x^m \otimes a_1 a_2 \cdots a_k$  where  $a_1 a_2 \cdots a_k$  is in  $P_C$  ( $a_i \in E$  or  $a_1 \cdots a_k = w$ ).

$$(x^m \otimes a_1 a_2 \cdots a_k) \alpha = \begin{cases} x^m \otimes a_1 a_2 \cdots a_k & \alpha = s a_1 \\ x^m \otimes a_2 \cdots a_k & \alpha = a_1 \\ x^m \otimes \alpha^* a_1 \cdots a_k & \alpha \in E^*, s \alpha = s a_1 \\ x^{m-1} \otimes c_1 c_2 \cdots c_n & \alpha = c_0, a_1 \cdots a_k = w \\ x^{m+1} \otimes w & \alpha = c_0^*, a_i = c_i \text{ for } 1 \leq i \leq n, k = n \\ 0 & \text{otherwise} \end{cases}$$

This defines an action of  $L(\Gamma)$  on  $M_C$  satisfying the path algebra relations as well as CK1 and CK2. This is clear for relations (V), (E) and (CK1). This is also true for relation (CK2) because given a pure tensor in  $M_C$ , at most one  $e \in s^{-1}(v)$  will yield a nonzero result after its action. For this  $e$ , acting by the factor  $e e^*$  is the same as acting by  $v$ .

We also have a left action of  $\mathbb{F}[x, x^{-1}]$  on  $M_C$  via multiplication by Laurent polynomials.

We have the categories  $\mathcal{M}_{\mathbb{F}[x, x^{-1}]}$  of left  $\mathbb{F}[x, x^{-1}]$  modules and  $\mathcal{M}_{L(\Gamma)}$  of right  $L(\Gamma)$  modules. Using  $M_C$ , we define two functors. Firstly, there is a functor

$$\mathcal{F}_C : \mathcal{M}_{\mathbb{F}[x, x^{-1}]} \longrightarrow \mathcal{M}_{L(\Gamma)}$$

where  $\mathcal{F}_C(X) = X \otimes_{\mathbb{F}[x, x^{-1}]} M_C$  for all objects  $X$  and  $\mathcal{F}_C(f) = f \otimes id_{M_C}$  for morphisms  $f$ . We also have, in the opposite direction,

$$\mathcal{G}_C : \mathcal{M}_{L(\Gamma)} \longrightarrow \mathcal{M}_{\mathbb{F}[x, x^{-1}]}$$

where  $\mathcal{G}_C(X) = Hom^{L(\Gamma)}(M_C, X)$  and for  $f : X \longrightarrow Y$ , we have the map  $\mathcal{G}_C(f)$ , where  $Hom^{L(\Gamma)}(M_C, X) \ni x \mapsto f \circ x \in Hom^{L(\Gamma)}(M_C, Y)$ .

Recall that  $\mathbb{F}[x, x^{-1}]$  is a PID, as it is a localization of the PID  $\mathbb{F}[x]$ . Thus, every finitely generated module  $X$  over  $\mathbb{F}[x, x^{-1}]$  is equal to the sum of  $\mathbb{F}[x] / (f(x)^k)$  where  $f(x)$  is some irreducible nonconstant polynomial, or the zero polynomial. As both  $\mathcal{G}_C$  and  $\mathcal{F}_C$  respect direct sums, we will focus on the case where  $X$  is finitely generated.

We will call  $\mathcal{F}_C(X) = M_{C,f}$  when  $X = \mathbb{F}[x] / (f(x))$ . Note that:

$$M_{f,C} \cong \left( \mathbb{F}[x, x^{-1}] / (f(x)) \right) \otimes_{\mathbb{F}} \mathbb{F}P_C$$

**Theorem 1.**  $\mathcal{G}_C \circ \mathcal{F}_C$  is naturally isomorphic to the identity functor.

We will need a lemma:

**Lemma 2.** Given  $\varphi \in Hom^{L(\Gamma)}(M_C, M_{C,f})$ ,  $\varphi$  is determined uniquely and

completely by an element of  $\mathbb{F}[x] / (f(x))$ .

*Proof.* We want to know the set

$$\left\{ \sum_i h_i(x) \otimes p_i \parallel \sum_i h_i(x) \otimes p_i C^n C^{*n} = \sum_i h_i(x) \otimes p_i \text{ for all } n \in \mathbb{N} \right\}$$

because this is the set of possible images for  $1 \otimes v$  under an  $L(\Gamma)$  morphism ( $M_C$  is a cyclic  $L(\Gamma)$  module generated by  $1 \otimes v$ ). Given a particular element, let  $N$  be a number such that  $C^N$  is longer than  $p_i$  for all  $i$ . When we consider  $\sum_i h_i(x) \otimes p_i C^N C^{*N}$ , the only terms that will have a nonzero result are those where  $p_i$  is an initial segment of  $C^N$ . For such  $p_i \in P_C$ , this means that  $(h_i(x) \otimes p_i) C^N = (h_i(x) \otimes v) q_i$  where  $q_i = c_{k_i} \cdots c_n C^{N_i}$ . If  $p_i$  has positive length,  $k_i \neq 0$ , and thus  $q_i$  is not a path starting with a power of  $C$ . (Suppose  $c_k c_{k+1} \cdots c_n C^N = C^N c_0 \cdots c_{n-k}$ . As these are elements of  $Path(\Gamma)$ , we have  $c_k \cdots c_n c_{n+k} = c_0 \cdots c_n$ , where  $c_{i+n+1} := c_i$ . We have  $c_i \cong c_j \pmod{n+1}$  and this contradicts the fact that  $C$  is primitive.) Thus  $(h_i(x) \otimes v) q_i = 0$  for all  $i$  with  $p_i$  of positive length. As,  $\sum_i h_i(x) \otimes p_i C^N C^{*N} = \sum_i h_i(x) \otimes p_i$ , this means  $\sum_i h_i(x) \otimes p_i = h(x) \otimes v$ .  $\square$

*Proof of theorem.* Consider the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ Hom^{L(\Gamma)}(M_C, X \otimes_{\mathbb{F}} M_C) & \xrightarrow{\mathcal{G}_C \circ \mathcal{F}_C(f)} & Hom^{L(\Gamma)}(M_C, Y \otimes_{\mathbb{F}} M_C) \end{array}$$

Note that  $x \mapsto f(x) \mapsto \varphi_{f(x)}$  and  $x \mapsto \varphi_x \mapsto f \otimes id_{M_C} \circ \varphi_x$ . Note that, by the lemma, both maps are completely determined by a polynomial, and that polynomial is the first factor in the image of  $1 \otimes v$ . Let us call  $\varphi_x(1 \otimes v) = h(x) \otimes v$ . Then  $\varphi_{f(x)}(1 \otimes v) = f(\varphi_x)(1 \otimes v) = f \circ h(x) \otimes v$  and  $f \otimes id_{M_C} \circ \varphi_x(1 \otimes v) = f \circ h(x) \otimes v$ . Thus the diagram commutes when we restrict to the category of finitely generated  $\mathbb{F}[x, x^{-1}]$  modules.

The functor  $\mathcal{G}_C$  preserves inclusion (if  $X$  is a  $L(\Gamma)$  submodule of  $Y$ , then  $Hom^{L(\Gamma)}(M_C, X)$  is a submodule of  $Hom^{L(\Gamma)}(M_C, Y)$ ). Thus, if  $\mathbb{F}[x, x^{-1}] / (f(x))$  is simple (which happens when  $f$  is irreducible), then so is  $M_{f,C}$ .  $\square$

Note that in the case that  $C$  is an exclusive cycle (a cycle with no exits), we have  $M_C = \mathbb{F}[x, x^{-1}] \otimes_{\mathbb{F}[x, x^{-1}]} (sC)L(\Gamma)$ . This is because  $CC^* = C^*C = sC$ , so  $C$  and  $C^*$  are inverses of each other as they act on the left of  $(sC)L(\Gamma)$  in this case.

Here is a general way to understand if a module you observe is a Chen module:

Let us define  $\alpha = \alpha_0\alpha_1\alpha_2\cdots$  as a regular infinite path in  $L(\Gamma)$  ( $\alpha_i \in E$ ,  $t\alpha_i = s\alpha_{i+1} \ \forall i \in \mathbb{N}$ ). We will use the notion that an element  $m$  of a module  $M$  survives along  $\alpha$  if  $m\alpha_n$  is nonzero for all  $n \in \mathbb{N}$ .

We will define the set  $P_\infty^m := \{\alpha \text{ an infinite path in } L(\Gamma) \mid m \text{ survives along } \alpha\}$  and often put the tail equivalence (denoted  $P_\infty^m / \sim$ ). We also define

$$P_\infty^M := \bigcup_{m \in M} P_\infty^m.$$

**Lemma 3.** *For  $M$  a simple  $L(\Gamma)$  module,  $P_\infty^m / \sim = P_\infty^M / \sim$ .*

*Proof.* Clearly  $P_\infty^m \subseteq P_\infty^{m'}$ . Consider  $\alpha \in E_\infty^{m'}$ . As  $M$  is simple, there is a  $\lambda \in L(\Gamma)$  such that  $m\lambda = m'$ . Thus:

$$m'\alpha_n = m \sum_{i=1}^k p_i q_i^* \alpha_n = m \sum_{i=1}^K p_i \alpha'_n$$

where  $\alpha'_n = q_i^* \alpha_n$ ,  $n \geq \max\{l(q_i)\}_{i=1}^k$  and  $mp_i \alpha'_n \neq 0$  for  $1 \leq i \leq K$ . Thus we have  $K$  paths that are tail equivalent to  $\alpha$  in  $P_\infty^m$ . Thus  $E_\infty^{m'} / \sim = P_\infty^m / \sim$ , which yields that  $P_\infty^m / \sim = P_\infty^m / \sim$   $\square$

**Lemma 4.** For  $M$  simple,  $P_\infty^m / \sim = \{[\alpha]\} \Rightarrow |P_\infty^m| < \infty$ .

*Proof.* Take a nonzero element  $m \in M$ . Look at  $P_\infty^m$ . If, by way of contradiction,  $|P_\infty^m| = \infty$ , then there must be at least two nonidentical paths that  $m$  survives along, say  $p_0$  and  $p'_0$ . Since  $M$  is simple,  $|E_\infty^{mp_0}|$  and  $|E_\infty^{mp'_0}|$  are both infinite. One can continue in this way, forming two infinite paths that are not the same at each step, and therefore cannot be tail equivalent. This contradicts  $P_\infty^m / \sim = \{[\alpha]\}$ .  $\square$

**Corollary 5.** For  $M$  simple,  $P_\infty^m / \sim = \{[\alpha]\} \Rightarrow$  there is a  $p \in \text{Path}(\Gamma)$  such that  $E_\infty^{mp} = \{\alpha'\}$  where  $\alpha'$  is a truncation of  $\alpha$

*Proof.* By the above,  $P_\infty^m$  is finite. Look at the element of  $P_\infty^m$  that takes the longest to become tail equivalent to  $\alpha$ , say  $\beta$ . If  $\beta_k \in \text{Path}(\Gamma)$  is the initial part of this infinitely long path that does not conform to  $\alpha$ , then  $E_\infty^{m\beta_k} = \{\alpha'\}$   $\square$

**Lemma 6.** For  $M$  simple,  $P_\infty^m / \sim = \{[\alpha]\} \iff M$  is a Chen Module.

*Proof.*  $\Leftarrow$  is clear by definition. For  $\Rightarrow$ , take any nonzero element of  $M$ . There is a path  $p \in \text{Path}(\Gamma)$  such that  $mp$  survives only along a truncation of  $\alpha$  (which, since we are working with equivalence classes, we can take this path and call it  $\alpha'$ ). Consider the module homomorphism from the Chen module to  $M$  that sends  $\alpha$  to  $mp$ . This will be onto as  $M$  is simple, and one-to-one as Chen modules are simple.  $\square$

**Lemma 7.** *Given a simple module  $M$  such that there is an  $0 \neq m \in M$  where  $m f(C^*) = 0$ , ( $f(x)$  is a polynomial in  $\mathbb{F}[x]$ ,  $f(0) = 1$ ),  $P_\infty^m = \{C^\infty\}$ .*

*Proof.* With the condition that  $m f(C^*) = 0$ , we can show that  $mp = 0$  where  $p \in \text{Path}(\Gamma)$  is anything that deviates from  $C^\infty$ . For  $f(x) = \sum_{i=0}^n a_i x^i$  we can write  $-a_n m C^{*n} - a_{n-1} m C^{*(n-1)} - \dots - a_1 m C^* = m$ . We consider  $mp = (-a_n m C^{*n} - a_{n-1} m C^{*(n-1)} - \dots - a_1 m C^*)p$ . This may set some summands equal to zero, as it is assumed that  $p$  deviates from  $C^\infty$ . For the summands it does not set equal to zero, we consider the identity  $-a_n m C^{*n} - a_{n-1} m C^{*(n-1)} - \dots - a_1 m C^* = m$ , we get  $-a_n m C^{*(n+k)} - a_{n-1} m C^{*(n-1+k)} - \dots - a_1 m C^{*(k+1)} = m C^{*k}$ , which allows us to rewrite  $m C^{*k}$  in terms of summands with strictly larger powers of  $C^*$ . We can inflate these powers of  $C^{*k}$  to be longer than any  $p$  that deviates from  $C^\infty$ . Hence,  $mp = 0$ .

Using a similar argument, consider  $C^k$  acting on  $m$  for some  $k \in \mathbb{N}$ . When we act on  $m$  where it has been rewritten with powers strictly larger than  $k$ . Then we get  $m C^k = m g(C^*)$  for some polynomial  $g$ . This cannot equal zero, as  $m g(C^*) C^{*k} = m$ .  $\square$

**Corollary 8.** *Given a simple module  $M$  such that there is a  $f(x) \in \mathbb{F}[x]$  where  $f(0) = 1$  and a  $0 \neq m \in M$  such that  $mf(C^*) = 0$ , then  $M$  is a (generalized, twisted) rational Chen module.*

## 0.2 Extension of Generalized Twisted Chen Modules

**Lemma 9.** *Given a commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & E & \xrightarrow{g} & B & \longrightarrow & 0 \\ & & \downarrow id & & \downarrow \varphi & & \downarrow id & & \\ 0 & \longrightarrow & A & \xrightarrow{f'} & E' & \xrightarrow{g'} & B & \longrightarrow & 0 \end{array}$$

where the rows are short exact sequences,  $\varphi$  is an isomorphism.

*Proof.* Suppose  $\varphi(e) = 0$  for  $e \in E$ . Then  $g'\varphi(e) = 0$ , which by commutativity implies that  $g(e) = 0$ . By exactness,  $e \in \text{Im}(f)$ . Since  $f$  is injective,  $f^{-1}(e) = a$  and by commutativity,  $\varphi f(a) = f'(a) = 0$ . Since  $f'$  is injective, this means  $a = 0$ , so  $f(a) = e = 0$ .

As the diagram is commutative,  $\varphi$  is an injection.

Let  $e' \in E$ . Since  $g', g$  are both surjective, there is an  $e \in E$  such  $g(e) = g'(e')$ . By commutativity,  $g'\varphi(e) = g(e)$ , so  $g'\varphi(e) = g'(e')$ . This means  $\varphi(e) + x = e'$  where  $x$  is some element of  $\text{Ker}(g')$ . Since  $\text{Ker}(g') = \text{Im}(f')$  by exactness, then  $x = f(a)$  for some  $a \in A$ . By commutativity,  $\varphi f(a) = f'(a)$ .

Thus,  $\varphi(e + f(a)) = e'$ .

Therefore,  $\varphi$  is also a surjection, and thus an isomorphism.

□

Notice that we only needed a surjective map from  $A$  to  $A$ , rather than an isomorphism.

Given a projective module  $P$  (the first step of a projective resolution of  $B$  and a surjection  $\epsilon : P \longrightarrow B$ , then given any module  $E$  and a surjective map  $\pi : E \longrightarrow B$ , we have an  $\tilde{\epsilon}$  such that the following diagram commutes.

$$\begin{array}{ccc} & P & \\ & \downarrow \epsilon & \\ E & \xrightarrow{\pi} & B \end{array}$$

$\tilde{\epsilon}$  (dashed arrow from  $P$  to  $E$ )

We will use this setup in two future diagrams.

**Lemma 10.** *Given a short exact sequence:*

$$0 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} B \longrightarrow 0$$

*there is an extension of  $B$  by  $A$  that is isomorphic to  $E$ .*

*Proof.* First, consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{a \mapsto (a,0)} & A \oplus P & \xrightarrow{(a,\alpha) \mapsto \alpha} & P & \longrightarrow & 0 \\
& & \downarrow id & & \downarrow \begin{bmatrix} \iota \\ \tilde{\epsilon} \end{bmatrix} & & \downarrow \epsilon & & \\
0 & \longrightarrow & A & \xrightarrow{\iota} & E & \xrightarrow{\pi} & B & \longrightarrow & 0
\end{array}$$

(A dashed arrow labeled  $\tilde{\epsilon}$  points from  $A \oplus P$  to  $E$ .)

Clearly the top sequence is short exact. We will check that it is commutative on the squares.

For the left square:  $a \mapsto (a, 0) \mapsto \iota(a) + \epsilon(0) = \iota(a)$  and  $a \mapsto a \mapsto \iota(a)$

For the right square:  $(a, \alpha) \mapsto \alpha \mapsto \epsilon(\alpha)$  and  $(a, \alpha) \mapsto \iota(a) + \tilde{\epsilon}(\alpha) \mapsto \pi\iota(a) + \pi\tilde{\epsilon}(\alpha) = 0 + \epsilon(\alpha) = \epsilon(\alpha)$

As a consequence of this diagram, we get that  $\begin{bmatrix} \iota \\ \tilde{\epsilon} \end{bmatrix}$  is a surjective map. We will call the kernel of this surjective map  $M$ . Note that  $M = \{(a, \alpha) \in A \oplus P \mid \iota(a) = -\tilde{\epsilon}(\alpha)\}$ . Thus, we get the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{a \mapsto [a,0]} & A \oplus P / M & \xrightarrow{[a,\alpha] \mapsto \epsilon(\alpha)} & B & \longrightarrow & 0 \\
& & \downarrow id & & \downarrow \begin{bmatrix} \iota \\ \tilde{\epsilon} \end{bmatrix} & & \downarrow id & & \\
0 & \longrightarrow & A & \xrightarrow{\iota} & E & \xrightarrow{\pi} & B & \longrightarrow & 0
\end{array}$$

We note that the top right map is well defined. This is because when  $[a, \alpha] = [a + a', \alpha + \alpha']$ , then  $\iota(a') = -\tilde{\epsilon}(\alpha')$ . This implies  $\pi\iota(a') = \pi\tilde{\epsilon}(\alpha')$ , so  $0 = \epsilon(\alpha')$  (as  $\pi\iota = 0$  and  $\pi\tilde{\epsilon} = \epsilon$ ). Thus,  $\epsilon(\alpha + \alpha') = \epsilon(\alpha)$ . The top sequence is again short exact.

So, given a fixed projective module  $P$  which projects onto  $B$  via  $\epsilon$ , we have  $A \oplus P / M \cong E$  where  $M$  is calculated using  $\tilde{\epsilon}$  (which uses the information of  $\pi$ ) and  $\iota$ .

□

Note that  $M = \{(a, \alpha) \in A \oplus P \mid \iota(a) = -\tilde{\epsilon}(\alpha)\} = \{(a, \alpha) \in A \oplus P \mid a = -\iota^{-1}(\tilde{\epsilon}(\alpha))\}$  (this makes sense as  $\iota$  is injective). Here,  $\alpha$  is an element of the  $\ker(\epsilon)$ , as  $0 = \pi\iota(a) = \pi\tilde{\epsilon}(\alpha) = \epsilon(\alpha)$ . Let us then consider  $M$  as the graph of a homomorphism from  $K := \ker(\epsilon)$  to  $A$  (where the domain is the second summand and the range is the first summand). We will be more specific in our notation then:

$$M_\varphi = \{(a, k) \in A \oplus K \mid a = -\varphi(k)\}$$

for  $\varphi \in \text{Hom}(K, A)$ . Thus we have a correspondence between  $\text{Hom}(K, A)$  and equivalence classes in  $\text{Ext}(B, A)$ . Both of these sets have algebraic structure -  $\text{Hom}(K, A)$  is an additive group, as is  $\text{Ext}(B, A)$  under Baer sum. We will show that this correspondence respects the additive structures of both. We will use the notation  $+_{\text{Baer}}$  to denote the Baer sum and  $E \times_B E'$  to denote the pullback with respect to  $B$ . Recall that the Baer sum is a quotient of the pullback with respect to the pushforward (using two SESs with the same start and end).

**Theorem 0.1.** *There is a correspondence between  $\text{Hom}(K, A)$  as an additive group and  $\text{Ext}(B, A)$ , an additive group under Baer sum*

*Proof.*

$$\begin{array}{ccc}
(A \oplus P) \times_B (A \oplus P) & \xrightarrow{\quad\quad\quad} & A \oplus P \\
\downarrow & & \downarrow \\
(A \oplus P / M_{\varphi_1}) \times_B (A \oplus P / M_{\varphi_2}) & & A \oplus P / M_{\varphi_1 + \varphi_2} \\
\downarrow & & \\
(A \oplus P / M_{\varphi_1}) +_{Baer} (A \oplus P / M_{\varphi_2}) & & 
\end{array}$$

The first pullback uses the map  $(a, \alpha) \mapsto \epsilon(\alpha)$  on both summands. The second pullback uses the map  $[a, \alpha] \mapsto \epsilon(\alpha)$  on both summands. The final Baer sum is with respect to the short exact sequence seen on the top of the last diagram.

The downward maps are all quotients, and the horizontal map does the following:

$$((a_1, \alpha_1), (a_2, \alpha_2)) \mapsto (a_1 + a_2 - \varphi_2(\alpha_1 - \alpha_2), \alpha_1 + \alpha_2)$$

Altogether, this demonstrates an isomorphism between

$$(A \oplus P / M_{\varphi_1}) +_{Baer} (A \oplus P / M_{\varphi_2})$$

and  $A \oplus P / M_{\varphi_1 + \varphi_2}$ .

Note that  $0 \in Hom(K, A)$  corresponds to  $A \oplus P / M_0 \cong A \oplus B$ . This makes

the correspondence between  $Hom(K, A)$  and  $Ext(B, A)$  a homomorphism. We wish to know the kernel of this isomorphism, i.e. for what  $\varphi \in Hom(K, A)$  is  $A \oplus P / M_\varphi \cong A \oplus B$ . Suppose we have a isomorphism  $\theta$  from  $A \oplus P / M_\varphi$  to  $A \oplus B$  such that the following diagram commutes:

$$\begin{array}{ccccccc}
& & P & \hookrightarrow & A \oplus P & & \\
& & & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{a \mapsto (a,0)} & A \oplus P / M_\varphi & \xrightarrow{[a,\alpha] \mapsto \epsilon(\alpha)} & B \longrightarrow 0 \\
& & \downarrow id & & \downarrow \theta & & \downarrow id \\
0 & \longrightarrow & A & \hookrightarrow & A \oplus B & \xrightarrow{proj_2} & B \longrightarrow 0 \\
& & & & \downarrow proj_1 & & \\
& & & & A & & 
\end{array}$$

Then the map from the top left  $P$  to the bottom center  $A$  is  $p \mapsto proj_2(\theta[0, p])$ , which we will call  $\psi$  is a homomorphism whose restriction to  $K$  is  $\varphi$ . Thus,  $\varphi$  extends to  $P$ . Suppose conversely that  $\varphi : K \longrightarrow A$  extends to  $\psi : P \longrightarrow A$ . Then,

$$\theta[a, \alpha] = (a + \psi(\alpha), \epsilon(\alpha))$$

makes the above diagram commute.

So  $Ext(B, A) \cong coker\{Hom(P, A) \longrightarrow Hom(K, A)\}$

Now, consider the following short exact sequences of modules:

$$A_{n-1} \hookrightarrow A_n \xrightarrow{\pi} B_n \quad A_n \hookrightarrow A_{n+1} \longrightarrow B_{n+1}$$

$$K \xrightarrow{\kappa} P \xrightarrow{\epsilon} B_{n+1}$$

As before,  $P$  is a projective module.

Consider the sequence:

$$0 \rightarrow \text{Hom}(B_{n+1}, A_{n-1}) \rightarrow \text{Hom}(B_{n+1}, A_n) \rightarrow \text{Hom}(B_{n+1}, B_n) \rightarrow \text{Ext}(B_{n+1}, A_{n-1})$$

$$\rightarrow \text{Ext}(B_{n+1}, A_n) \rightarrow \text{Ext}(B_{n+1}, B_n) \rightarrow \text{Ext}^2(B_{n+1}, A_{n-1}) \rightarrow \cdots$$

We wish to check that the sequence is exact.

$0 \rightarrow \text{Hom}(B_{n+1}, A_{n-1}) \rightarrow \text{Hom}(B_{n+1}, A_n) \rightarrow \text{Hom}(B_{n+1}, B_n)$  is exact, as this is the result of the functor  $\text{Hom}(B_{n+1}, \_)$  acting on the first SES. We also know  $\text{Ext}(B_{n+1}, A_{n-1}) \rightarrow \text{Ext}(B_{n+1}, A_n) \rightarrow \text{Ext}(B_{n+1}, B_n) \rightarrow \text{Ext}^2(B_{n+1}, A_{n-1}) \rightarrow \cdots$  is exact, as it is a result of the functor  $\text{Ext}^*(B_{n+1}, \_)$  acting on the first SES. We wish to see if the map  $f : \text{Hom}(B_{n+1}, B_n) \rightarrow \text{Ext}(B_{n+1})$  such that  $\theta \xrightarrow{f} A_{n-1} \oplus P / M_{\tilde{\theta}}$  is a map in the sequence above that will combine these sequences to be one long exact sequence, where  $\tilde{\theta}\epsilon$  exists

(as  $P$  is projective) and makes the below diagram commute, and  $\bar{\theta} = \tilde{\theta}\epsilon \circ \iota$ .

$$\begin{array}{ccccc}
K & \xrightarrow{\iota} & P & \xrightarrow{\epsilon} \twoheadrightarrow & B_{n+1} \\
\downarrow \bar{\theta} & & \downarrow \tilde{\theta}\epsilon & & \downarrow \theta \\
A_{n-1} & \hookrightarrow & A_n & \xrightarrow{\pi} \twoheadrightarrow & B_n
\end{array}$$

We will first check if  $\ker(f) = \text{img}(\text{Hom}(B_{n+1}, A_n) \rightarrow \text{Hom}(B_{n+1}, B_n))$ . We know from the previously established isomorphism ( $\text{Ext}(B_{n+1}, A_{n-1}) \cong \text{coker}\{\text{Hom}(P, A_{n-1}) \rightarrow \text{Hom}(K, A_{n-1})\}$ ) that if  $\bar{\theta}$  is the restriction of the map  $\hat{\theta} : P \rightarrow A_{n-1}$ , then it is in  $\ker(f)$ .

$$\begin{array}{ccccc}
K & \xrightarrow{\iota} & P & \xrightarrow{\epsilon} \twoheadrightarrow & B_{n+1} \\
\downarrow \bar{\theta} & \swarrow \hat{\theta} & \downarrow \tilde{\theta}\epsilon & & \downarrow \theta \\
A_{n-1} & \hookrightarrow & A_n & \xrightarrow{\pi} \twoheadrightarrow & B_n
\end{array}$$

Consider  $(\tilde{\theta}\epsilon - \hat{\theta})\epsilon^{-1} = \varphi \in \text{Hom}(B_{n+1}, A_n)$ . Notice that  $\varphi$  is well defined, as for  $k \in \ker(\epsilon)$ ,  $(\tilde{\theta}\epsilon - \hat{\theta})(k) = (\tilde{\theta}\epsilon - \hat{\theta})(\iota^{-1}k) = 0$  by the commutativity of the lower left hand triangle in the diagram (this triangle is commutative because of the commutivity of the top triangle and the injectivity of  $\iota$ ).

Thus,  $\ker(f) \subseteq \text{img}(\text{Hom}(B_{n+1}, A_n) \rightarrow \text{Hom}(B_{n+1}, B_n))$

Given  $\theta \in \text{img}(\text{Hom}(B_{n+1}, A_n) \rightarrow \text{Hom}(B_{n+1}, B_n))$ , we have the follow-

ing diagram:

$$\begin{array}{ccccc}
 K & \xrightarrow{\iota} & P & \xrightarrow{\epsilon} & B_{n+1} \\
 & \searrow \bar{\theta} & & \searrow \tilde{\theta}\epsilon & \downarrow \theta \\
 & & A_{n-1} & \hookrightarrow & A_n \xrightarrow{\pi} B_n
 \end{array}$$

where  $\tilde{\theta}\epsilon$  is the lift of  $\pi\theta\epsilon$ . Consider  $\tilde{\theta}\epsilon - \theta\epsilon$ . We know  $\pi\tilde{\theta}\epsilon = \pi\theta\epsilon$ , so  $\tilde{\theta}\epsilon - \theta\epsilon(p) \in \ker(\pi) = A_{n-1}$  for all  $p \in P$ . Also,  $(\tilde{\theta}\epsilon - \theta\epsilon)(\iota k) = \tilde{\theta}\epsilon(\iota k) = \bar{\theta}(k)$ . Therefore,  $\theta \in \ker(f)$ .

Now, let us check that  $\text{img}(f) = \ker(\text{Ext}(B_{n+1}, A_{n-1}) \rightarrow \text{Ext}(B_{n+1}, A_n))$ .

Using our previously established isomorphism, in order to determine if  $\text{img}(f) \subseteq \ker(\text{Ext}(B_{n+1}, A_{n-1}) \rightarrow \text{Ext}(B_{n+1}, A_n))$ , we need to know when, given a  $\theta \in \text{Hom}(B_{n+1}, B_n)$  (and the resulting  $\bar{\theta}$  that determines a homomorphism from  $K$  to  $A_{n-1}$ ), can we extend  $\bar{\theta}$  to a map  $\hat{\theta}$  from  $P$  to  $A_n$ . This is a direct consequence of this previous diagram:

$$\begin{array}{ccccc}
 K & \xrightarrow{\iota} & P & \xrightarrow{\epsilon} & B_{n+1} \\
 \downarrow \bar{\theta} & & \downarrow \hat{\theta} & & \downarrow \theta \\
 A_{n-1} & \hookrightarrow & A_n & \xrightarrow{\pi} & B_n
 \end{array}$$

We also need to establish that if we have  $\bar{\theta}$ , a homomorphism from  $K$  to  $A_{n-1}$  that extends to  $\hat{\theta}$  from  $P$  to  $A_n$ , whether it is the image of some

$$\theta \in \text{Hom}(B_{n+1}, B_n).$$

$$\begin{array}{ccccc} K & \xrightarrow{\iota} & P & \xrightarrow{\epsilon} \twoheadrightarrow & B_{n+1} \\ \downarrow \bar{\theta} & & \downarrow \hat{\theta} & & \downarrow \theta \\ A_{n-1} & \hookrightarrow & A_n & \xrightarrow{\pi} \twoheadrightarrow & B_n \end{array}$$

where  $\theta = \pi \bar{\theta} \epsilon^{-1}$ . This is well defined as, although  $\epsilon$  has kernel  $\iota(K)$ , we have  $\pi \hat{\theta} \iota(K) = \pi \bar{\theta}(K) = 0$

Any  $\varphi \in \text{Hom}^{L(\Gamma)}(M_C, A)$  is determined uniquely by an element  $a$  in  $\{a \in A \mid a(wI_C) = 0\}$  where

$$wI_C := \text{span}\{pq^* \in L(\Gamma) \mid p, q \in \text{Path}(\Gamma), sp = w, \exists N \in \mathbb{N} \text{ s.t. } (C^*)^N p = 0\}$$

It is clear why this condition is necessary. It is sufficient because given any  $L(\Gamma)$  module  $A$ , any  $a \in \{a \in A \mid a(wI_C) = 0\}$  defines a morphism by determining its output on  $1 \otimes v$ .

Note that  $\{a \in Av \mid aC^m C^{*n} = a \text{ for all } n \in \mathbb{N}\} \supseteq \{a \in A \mid a(wI_C) = 0\}$ . Since  $\varphi(1 \otimes v) = \varphi(1 \otimes v)C^m C^{*n} = \varphi(1 \otimes v)v$ , it is clear why we have this containment. We have the reverse containment because if, by way of contradiction, there is  $a \in \{a \in Av \mid aC^m C^{*n} = a \text{ for all } n \in \mathbb{N}\}$  where  $awI_C \neq 0$ , then for all  $n \in \mathbb{N}$ , we have  $aC^m C^{*n} wI_C \neq 0$ , which is a contradiction of the definition of  $wI_C$ .

□

### 0.3 Reducing a Digraph to Strongly Connected $\Gamma$ with Multiple Cycles

We can do the reduction algorithm on  $\Gamma$  that eliminates loopless nonsinks [?]. The output of this reduction algorithm is not unique, but it is a Morita equivalence. Thus we have changed the algebra but preserved the module category - in particular, our simple modules under this functor are still simple.

We will fix a simple module  $M$  over some  $L(\Gamma)$ .

Consider  $V_M$ , the support of  $M$ . Since  $M$  is a simple module, we can also denote this set as  $V_{\rightsquigarrow w}$  for any  $w$  in the unique minimal equivalence class in  $V_M$  (the partial order is given by  $v \leq w$  when there is a path from  $w$  to  $v$ ). Here is the reasoning: for any vertex  $v$  not in  $V_M$ , all vertices less than  $v$  are also not in  $V_M$ , and if  $te \in V \setminus V_M$  for all  $e$  such that  $se = v$ , then  $v \in V \setminus V_M$ . Thus,  $V \setminus V_M$  is a hereditary saturated set. We also see that for any  $w$ , a minimal element of  $V_M$ , all other elements of its equivalence class are in  $V_M$ . If not, then the fact that  $V \setminus V_M$  is hereditary means that  $w$  would not be in  $V_M$ , a contradiction. We have that this minimal equivalence class is unique, as if  $m_1 \in Mv_1$ ,  $m_2 \in Mv_2$  are two nonzero elements for  $v_1, v_2$  in different

minimal equivalence classes. By simplicity, there exists  $\sum_i p_i q_i^* \in L(\Gamma)$  such that  $m_1 \sum_i p_i q_i^* \in L(\Gamma) = m_2$ . Thus,  $tp_i \in V_M$ , and  $tp_i$  is less than  $v_1$  and  $v_2$ , which is a contradiction. Also, it is clear that any vertices greater than  $w$  are in  $V_M$ , as for any path  $p$  where  $tp = w$ , we have  $p^*$  as an injective map from  $Mw$  to  $Msp$ .

We can use the isomorphism mentioned earlier to get:

$$L(\Gamma) / (V \setminus V_M) \cong L(\Gamma_{\rightsquigarrow w})$$

where  $\Gamma_{\rightsquigarrow w}$  is the full subgraph of  $\Gamma$  on  $V_{\rightsquigarrow w}$ . Thus, we have changed the algebra again, but to one that is isomorphic.

We will denote  $\bar{w} := \sum_{v \in [w]} v$ , where  $[w]$  is the set of vertices in the same equivalence class as  $w$ . We will also denote  $\Gamma_{[w]}$  as the full subgraph on  $[w]$ .

**Lemma 11.** *The functor*

$$- \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)$$

*gives an equivalence of categories between  $L(\Gamma_{[w]}) \cong L(\Gamma_{[w]})$  modules  $M$  and  $L(\Gamma)$  modules  $N$  generated by  $N\bar{w}$ . The adjoint functor*

$$Hom^{\bar{w}L(\Gamma)\bar{w}}(\bar{w}L(\Gamma), -) = -\bar{w}$$

*is the functor between  $L(\Gamma)$  modules  $N$  generated by  $N\bar{w}$  and  $L(\Gamma_{[w]})$  modules  $M$ .*

*Proof.*

$$M \xrightarrow{-\otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)} M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma) \xrightarrow{-\bar{w}} (M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)) \bar{w}$$

We need to show  $(M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)) \bar{w} \cong M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)\bar{w}$ . We can show this by viewing  $M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)$  as

$$(M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)\bar{w}) \oplus (M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)(1 - \bar{w}))$$

Notice that projection onto the first summand gives a section of  $-\bar{w}$ . Thus,  $(M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)) \bar{w} \cong M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)\bar{w}$ . Then, we have  $M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)\bar{w} \cong M$ .

For the reverse direction, consider  $N$  an  $L(\Gamma)$  module such that  $N\bar{w}L(\Gamma) = N$ .

$$N \xrightarrow{-\bar{w}} N\bar{w} \xrightarrow{-\otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)} N\bar{w} \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)$$

By the evaluation map  $n\bar{w} \otimes x \mapsto n\bar{w}x$  and the assumption that  $N\bar{w}L(\Gamma) = N$ , the composition of these maps are the identity.  $\square$

Notice that if you have a  $L(\Gamma)$  module  $N$ , the image of  $N$  under the composition of these functors is the submodule of  $N$  that is isomorphic to  $N\bar{w}L(\Gamma)$ . This can be seen by viewing  $N$  as  $N\bar{w}L(\Gamma) \oplus N(1 - \bar{w})L(\Gamma)$ . We see that  $N\bar{w}L(\Gamma)\bar{w} \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)$  is isomorphic to  $N\bar{w}L(\Gamma)$  via the evaluation map. We can also see  $N(1 - \bar{w})L(\Gamma)\bar{w} \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)$  is trivial as the only

elements of  $(1 - \bar{w})L(\Gamma)$  are linear combinations of dual paths that start at vertices not in  $[w]$ . However, all paths of  $\bar{w}L(\Gamma)\bar{w}$  start and end at elements of  $[w]$ . By CK1, this is isomorphic to the zero module.

Since  $M = M\bar{w}L(\Gamma)$  when  $M$  is simple, we may restrict attention to  $L(\Gamma_{[w]})$ .

After the previous reductions, we are now considering a simple  $L(\Gamma)$  module, where  $\Gamma$  is strongly connected and has full support  $V = V_M$ .

There are three possibilities for  $w$ :

- $w$  is a sink
- $w$  is on a single cycle.
- $\Gamma_{[w]}$  contains multiple cycles

For the first case,  $L(\Gamma) = \mathbb{F}$ , and  $M$  is any one dimensional vector space over  $\mathbb{F}$ . In the module over  $L(\Gamma)$  before we did our reductions on  $\Gamma$ , we have that  $M = wL(\Gamma)$ , a projective simple module (in fact all projective simple modules over  $L(\Gamma)$  are of this form). The dimension of  $wL(\Gamma)$  may or may not be finite. It is finite dimensional iff  $w$  there are no cycles leading to  $w$  [?].

In the second case,  $L(\Gamma) = \mathbb{F}[x, x^{-1}]$ . Since this is a PID (and if  $M$  is simple, it means that  $M$  is finitely generated), we know all modules over it. These modules are the Chen modules. All modules (in the case of disjoint cycles) are Chen modules [?]. These modules can be further twisted and generalized as was done in the previous section.

Our new simple  $L(\Gamma)$  modules appear in the case that  $[w]$  contains multiple cycles.

## 0.4 Spaces of step functions and infinite product paths

Our new modules are submodules of *step functions* (linear combinations of indicator functions) on the space of infinite paths in  $\Gamma$ , denoted by

$$P_\infty(\Gamma) = P_\infty = \{e_1 e_2 e_3 \cdots \mid t e_i = s e_{i+1} \ \forall i \in \mathbb{N}_{>0}, e_i \in E\}.$$

The topology placed on this is the restriction of the product topology on  $E^\mathbb{N}$  (where  $E$  has the discrete topology). Basic open sets are determined by a path  $p = p_1 p_2 \cdots p_n \in \text{Path}(\Gamma)$  (where  $p_i \in E$ ):

$$pP_\infty := \{e_1 e_2 e_3 \cdots \in P_\infty \mid e_i = p_i \text{ for } 1 \leq i \leq n\}.$$

Recall from the preliminaries that  $P_\infty$  is a closed, compact space that is Borel measurable.

We would like to have a probability measure on this space. This can be done by making a weight function from  $E \sqcup V$  to  $\mathbb{R}$  that assigns to each edge

in  $E$  a weight that is strictly positive, with the condition that  $\sum_{se=v} w(e) = 1$ .  
 With these weights in mind, we can create the matrix:

$$W = \left( \sum_{\substack{se=v \\ te=u}} w(e) \right)_{v,u \in V}$$

Thus we have that the sum of each row is 1.

We can use Perron-Frobenius Theory to find the left eigenvector with positive coordinates and eigenvalue 1. We make the choice of eigenvector unique by demanding that  $\sum_{v \in V} a_v = 1$ . We define  $w(v) := a_v$

This weight function extends to a measure  $\mu$  on basic open sets (which are also closed). For  $p = p_1 p_2 \cdots p_n \in \text{Path}(\Gamma)$  (where  $p_i \in E$ ), we define  $\mu(p P_\infty) := w(sp_1)w(p_1) \cdots w(p_n) := w(sp)w(p)$ .

We then extend this measure to all closed sets  $S$  of  $P_\infty$  via:

$$\mu(S) = \lim_{n \rightarrow \infty} \mu \left( \bigcap_n \{p \cdot P_\infty \mid p \in Pr_n(S)\} \right) = \lim_{n \rightarrow \infty} \sum_{p \in Pr_n(S)} \mu(p \cdot P_\infty)$$

where  $Pr_n(e_1 e_2 \cdots) = e_1 e_2 \cdots e_n$ . Since the sequence is monotone decreasing and bounded below by 0, it converges.

We wish to establish which linear combinations of step functions are trivial. To that end consider:

$$\mathbb{F} (2^X \setminus \{\emptyset\}) \longrightarrow \mathbb{F}^X$$

$$\sum_{i=1}^n \lambda_i X_i \longmapsto \sum_i \lambda_i \mathbb{1}_{X_i}$$

where  $2^X$  is the power set of  $X$ . Notice that multiplication of indicator functions corresponds to intersection of the corresponding sets. We can see that  $A + B - A \cup B$  is in their kernel when  $A$  and  $B$  are disjoint sets. Consider an arbitrary element of the kernel. Using the previous relation, we can rewrite any  $\sum_i \lambda_i X_i$  as an equivalent sum using disjoint sets, such as

$$\sum_{\substack{a_i \in \{0,1\} \\ \bigcap_{i=1}^n X_i^{a_i} \neq \emptyset}} (a_1 \lambda_1 + \cdots a_n \lambda_n) \bigcap_{i=1}^n X_i^{a_i}$$

where  $X_i^1 := X_i$  and  $X_i^0 := X \setminus X_i$ . These sets are all disjoint, and the sum is mapped to the trivial function if and only if each coefficient is 0.

The action of  $L(\Gamma)$  on  $\mathbb{F}^{P_\infty}$  is given by:

$$\begin{array}{lll} \text{for } v \in V & (\mathbb{1}_S)v = \mathbb{1}_{vS} & \text{where } vS = \{x \in S \mid sx = v\} \\ \text{for } e \in E & (\mathbb{1}_S)e = \mathbb{1}_{e^*S} & \text{where } e^*S = \{x \mid ex \in S\} \\ \text{for } e^* \in E^* & (\mathbb{1}_S)e^* = \mathbb{1}_{eS} & \text{where } eS = \{ex \mid x \in S, sx = te\} \end{array}$$

When  $\mathbb{F} = \mathbb{C}$ , we can adjust the action of  $L(\Gamma)$  on  $\mathbb{F}^{P_\infty}$  to ensure that

both  $v$  and  $ee^*$  act as partial isometries:

$$\begin{aligned}
\text{for } v \in V \quad & (\mathbb{1}_S)v = \mathbb{1}_{vS} \quad & \text{where } vS = \{x \in S \mid sx = v\} \\
\text{for } e \in E \quad & (\mathbb{1}_S)e = \frac{w(te)}{w(se)w(e)} \mathbb{1}_{e^*S} \quad & \text{where } e^*S = \{x \mid ex \in S\} \\
\text{for } e^* \in E^* \quad & (\mathbb{1}_S)e^* = \frac{w(se)w(e)}{w(te)} \mathbb{1}_{eS} \quad & \text{where } eS = \{ex \mid x \in S, sx = te\}
\end{aligned}$$

All relations of the Leavitt Path Algebra hold:

$$\begin{aligned}
\text{for } v, u \in V \quad & (\mathbb{1}_S)vu = (\mathbb{1}_{vS})u = \mathbb{1}_{uvS} = \mathbb{1}_{\delta_{u,v}uS} = (\mathbb{1}_S)\delta_{u,v}v \\
\text{for } e \in E \quad & (\mathbb{1}_S)see = (\mathbb{1}_{seS})e = \mathbb{1}_{e^*seS} = \mathbb{1}_{e^*S} = (\mathbb{1}_S)e \\
\text{for } e \in E \quad & (\mathbb{1}_S)ete = (\mathbb{1}_{e^*S})te = \mathbb{1}_{te e^*S} = \mathbb{1}_{e^*S} = (\mathbb{1}_S)e \\
\text{for } e, f \in E \quad & (\mathbb{1}_S)e^*f = (\mathbb{1}_{eS})f = \mathbb{1}_{f^*eS} = \mathbb{1}_{\delta_{f,e}tfS} = (\mathbb{1}_S)\delta_{e,f}te \\
\text{for } v \text{ (nonsink)} \in V \quad & \sum_{se=v} (\mathbb{1}_S)ee^* = \sum_{se=v} (\mathbb{1}_{e^*S})e^* = \sum_{se=v} \mathbb{1}_{ee^*S} = \mathbb{1}_{vS} = (\mathbb{1}_S)v
\end{aligned}$$

The last relation is true as  $ee^*S = \{x \in S \mid x = ex'\}$ . These sets are disjoint for each  $e \in E$ , thus the indicator function of the sum corresponds to the indicator function of their union. Thus,  $\bigcup_{se=v} ee^*S = \{x \in S \mid x = ex'\} = vS$ . This action extends to step functions, which are functions that are linear combination of indicator functions of closed sets  $S$  on  $P_\infty$ . We will consider the indicator functions of closed sets, as they are measurable.

However, when we have indicator functions of closed sets, we also have

indicator functions of *locally closed sets*, which are intersections of open sets and closed sets. We also have indicator functions of *constructible sets*, which are unions of locally closed sets. This is detailed in the following lemma.

Before we detail the proof of the lemma, the following facts are useful:

1. In general,  $\mathbb{1}_{A \cap B} = \mathbb{1}_A \cdot \mathbb{1}_B$ .
2. The intersection of two locally closed sets is a locally closed set - note when  $U_i$  is open and  $F_i$  is closed for  $i \in \{0, 1\}$ , we have:

$$(U_0 \cap F_0) \cap (U_1 \cap F_1) = (U_0 \cap U_1) \cap (F_0 \cap F_1).$$

**Lemma 12.** *The following are equivalent:*

1.  $f = \sum_i \lambda_i \mathbb{1}_{S_i}$ , where  $S_i$  are nonempty, constructible sets.
2.  $f = \sum_i \lambda_i \mathbb{1}_{S_i}$ , where  $S_i$  are nonempty, locally closed sets.
3.  $f = \sum_i \lambda_i \mathbb{1}_{S_i}$ , where  $S_i$  are nonempty, closed sets.
4.  $f = \sum_i \lambda_i \mathbb{1}_{S_i}$ , where  $S_i$  are nonempty, disjoint, constructible sets.
5.  $f = \sum_i \lambda_i \mathbb{1}_{S_i}$ , where the image of  $f$  is finite and each  $S_i$  is nonempty and constructible.
6.  $f = \sum_{f^{-1}(\lambda_i)} \lambda_i \mathbb{1}_{f^{-1}(\lambda_i)}$ , where the collection of sets  $\{f^{-1}(\lambda_i)\}_{\lambda_i \in \mathbb{F}}$  is locally finite and each  $S_i$  is nonempty and constructible.

*Proof.*

(1)  $\implies$  (2) We have  $f = \sum_i \lambda_i \mathbb{1}_{S_i}$ , where  $S_i$  are constructible sets. Consider that each constructible set  $S_i$  is the union of finitely many locally closed sets  $A_k$ . Note that

$$\bigcup_{k=1}^n A_k = \bigcup_{\substack{a_k \in \{0,1\} \\ \bigcap_{k=1}^n A_k^{a_k} \neq \emptyset}} \left( \bigcap_{k=1}^n A_k^{a_k} \right)$$

where  $A_k^1 = A_k$ , and  $A_k^0 = P_\infty \setminus A_k$ .

The compliment of a locally closed set is a linear combination of indicator functions of locally closed sets. For  $U$  an open set,  $F$  a closed set:

$$\mathbb{1}_{P_\infty \setminus (U \cap F)} = \mathbb{1}_{P_\infty \setminus U} + \mathbb{1}_{P_\infty \setminus F} - \mathbb{1}_{(P_\infty \setminus U) \cap (P_\infty \setminus F)}$$

Since the intersection of locally closed sets is a locally closed set, and intersection of sets corresponds to multiplication of their indicator functions, we have successfully rewritten  $f$  as a linear combination of indicator functions of locally closed sets.

(2)  $\implies$  (3) We have  $f = \sum_i \lambda_i \mathbb{1}_{S_i}$ , where  $S_i$  are locally closed sets. Note that for  $U$  an open set,  $F$  a closed set:

$$\mathbb{1}_{U \cap F} = \mathbb{1}_F - \mathbb{1}_{F \cap (P_\infty \setminus U)}.$$

We have thus successfully rewritten  $f$  as a linear combination of indicator

functions of closed sets.

(3)  $\implies$  (4) We have  $f = \sum_i \lambda_i \mathbb{1}_{S_i}$ , where  $S_i$  are closed sets. Consider the fact that we can partition the union of the  $S_i$  into disjoint sets locally closed sets:

$$\bigcup_{\substack{a_i \in \{0,1\} \\ \bigcap_{i=1}^n S_i^{a_i} \neq \emptyset}} \left( \bigcap_{i=1}^n S_i^{a_i} \right)$$

where  $S_i^1 = S_i$ , and  $S_i^0 = P_\infty \setminus S_i$ . Then, we can rewrite  $f$  as:

$$\sum_{\substack{a_i \in \{0,1\} \\ \bigcap_{i=1}^n S_i^{a_i} \neq \emptyset}} (a_1 \lambda_1 + \cdots a_n \lambda_n) \mathbb{1}_{\bigcap_{i=1}^n S_i^{a_i}}$$

(4)  $\implies$  (5) is trivial.

(5)  $\implies$  (6) Consider  $f = \sum_i \lambda_i \mathbb{1}_{S_i}$ , where the image of  $f$  is finite and each  $S_i$  is constructible. We can, without loss of generality, rewrite  $f$  as a linear combination of indicator functions of disjoint sets (using (1)  $\implies$  (2)  $\implies$  (3)  $\implies$  (4)). Then, let  $X_\lambda = \bigcup \{S_i \mid \lambda_i = \lambda\}$ . These  $X_\lambda$  are constructible, and there are finitely many of them (as Image of  $f$  is finite). We can then rewrite  $f$  as:

$$f = \sum_{\lambda} \lambda \mathbb{1}_{X_\lambda}$$

(6)  $\implies$  (1) is trivial.

□

**Lemma 13.** *If  $f = \sum_i \lambda_i \mathbb{1}_{S_i} \neq 0$  is linear combination of indicator functions with  $S_i$  distinct nonempty closed sets, then there exists a path  $p$  such that  $fp = \lambda \mathbb{1}_S$  with  $S$  closed,  $S \neq \emptyset$ .*

*Proof.* We can rewrite  $f$  as  $\sum_i \mu_i \mathbb{1}_{X_i}$  a linear combination of indicator functions of disjoint nonempty constructible sets by our lemma. As the sets are disjoint, we can find an  $n \in \mathbb{N}$  and a  $p \in Pr_n(X_1)$  such that  $p \notin Pr_n(X_i)$ ,  $2 \leq i \leq n$ . When we act on  $f$  with  $p$ , then  $fp$  only has output 0 and  $\mu_1$ . Consider how when  $p$  acts on  $\sum_i \lambda_i \mathbb{1}_{S_i}$  with  $S_i$  distinct, if  $fp$  must be the indicator of a disjoint union of closed sets in order to have one output other than 0. As a finite union of closed sets is closed,  $fp = \lambda \mathbb{1}_S$  with  $S$  closed,  $S \neq \emptyset$ .  $\square$

Thus, all simple modules in the space of step functions are generated by a single step function.

Then the question remain of which step functions generate simple modules.  $S$  is *defined by a set of forbidden words* if infinite paths in  $S$  have no paths in the set of forbidden paths appears as a sub-segment. You may similarly think of  $S$  as being *defined by allowed paths* (where allowed paths have no forbidden path as a sub-segment). For  $S$  defined by a set of forbidden words, we say that  $S$  is *irreducible* that for any two allowed paths  $x$  and  $z$ , there is an allowed path  $y$  such that  $xyz$  is a valid path.

**Lemma 14.** *If  $(\mathbb{1}_S)v \neq 0$  for irreducible  $S$ , then for any allowed path  $p$  such that  $tp = v$ ,  $(\mathbb{1}_S)p = \lambda(\mathbb{1}_S)v$ .*

*Proof.* To understand this, consider  $S$  as a set defined by allowed words. The

set of infinite paths that start with  $p$  and those that start with  $tp$  are in  $1 - 1$  correspondence.  $\square$

**Lemma 15.** *If  $(\mathbb{1}_S)v \neq 0$  for irreducible  $S$ , then there is an element  $x$  of  $\mathbb{F}\Gamma$  such that  $(\mathbb{1}_S)vx = \mathbb{1}_S$ .*

*Proof.* For  $u \in V$  there is an allowed word  $p_u$  such that  $sp_u = v$ ,  $tp_u = u$  (this is due to the irreducible condition). Thus,  $\sum_{u \in V} \frac{w(tp)}{w(u)w(p)} p_u$  is an element of  $\mathbb{F}\Gamma$  that sends  $(\mathbb{1}_S)v$  to  $\mathbb{1}_S$ .  $\square$

**Theorem 16.** *All irreducible nonempty  $S \subset P_\infty$  define indicator functions  $\mathbb{1}_S$  whose associated cyclic modules are simple.*

*Proof.* Consider  $\mathbb{1}_S \sum_i p_i q_i^*$ , where  $\sum_i p_i q_i^* \in L(\Gamma)$  such that  $\mathbb{1}_S \sum_i p_i q_i^* \neq 0$ . We can find a path  $r$  longer than any of the  $q_i$  such that  $\mathbb{1}_S \sum_i p_i q_i^* r \neq 0$ . By this, we obtain  $\lambda \mathbb{1}_S tr \neq 0$ . By our corollary, there is an element  $x \in \mathbb{F}\Gamma$  (scaled appropriately) that will send  $\lambda \mathbb{1}_S tr$  to  $\mathbb{1}_S$ . Thus, we can go from any generic element of the cycloc module back to the generator  $\mathbb{1}_S$  via the action of  $L(\Gamma)$ . Thus, the module is simple.  $\square$