

This thesis is mostly about irreducible representations of Leavitt Path Algebras (LPAs). Specifically, we first generalize a previously known construction of twisted Chen modules over the Leavitt Path Algebra of a directed graph. We then give some thought to its extension to other modules, and present new classes of simple Leavitt Path Algebra modules previously unknown.

As we study modules over a particular algebra, it is useful to note that these modules are equivalent to representations of the algebra. A *representation* of an $\mathbb{F}$ -algebra A is a algebra homomorphism:

$$\rho : A \longrightarrow \text{End}_{\mathbb{F}}(\Omega)$$

where Ω is a vector space over $\mathbb{F}$, and $\text{End}_{\mathbb{F}}(\Omega)$ refers to the algebra of linear endomorphisms on Ω . We will make the choice that our endomorphisms will act on the right, so that the expression xf makes sense for $x \in \Omega$, $f \in \text{End}_{\mathbb{F}}(\Omega)$. As an algebra homomorphism, ρ respects addition, multiplication (which is composition for endomorphisms), and scalar multiplication:

$$\rho(a + b) = \rho(a) + \rho(b), \quad \rho(ab) = \rho(a)\rho(b), \quad \rho(\lambda a) = \lambda\rho(a)$$

for $a, b \in A$ and $\lambda \in \mathbb{F}$. A module over an $\mathbb{F}$ -algebra A is a vector space Ω and a pairing $\Omega \times A \longrightarrow \Omega$ denoted $(x, a) \longrightarrow xa$. This pairing respects the addition and scalar multiplication of both A and Ω , as well as the multiplication of A so:

$$\begin{aligned} (x, a + b) &\longrightarrow xa + xb, & (x, ab) &\longrightarrow (xa)b, & (x, \lambda a) &\longrightarrow \lambda(xa) \\ (x + y, a) &\longrightarrow xa + ya, & (\lambda x, a) &\longrightarrow \lambda(xa) \end{aligned}$$

for all $a, b \in A$, $x, y \in \Omega$, and $\lambda \in \mathbb{F}$. Using the relation $(x)\rho(a) = xa$, one can see how modules over the algebra completely define a representation ρ and vice versa. As a common abuse of notation, we often omit ρ when speaking about a representation.

The theory of Leavitt Path Algebras is the confluence of three threads of mathematics, involving ring theory, operator algebras, and quiver representations. The main thread of our story is within ring theory, and began with investigations of W. Leavitt into the extent of the failure of the Invariant Basis Number (IBN) property of a unital ring (with $1 \neq 0$) around 1960 [?]. Any module over such a ring R that has a basis is a free (right) module over R , and any module over a ring R that has a finite basis is finitely generated free (right) module over R that is isomorphic to R^n for some n . A basis of a free module is a subset of the module such that all elements of the module

can be written uniquely as R -linear combinations of elements of the basis. A ring has the IBN property when any two finite bases have the same number of elements, that is: if $R^n \cong R^m$ as R modules, then $n = m$. For example, one of the first results taught in linear algebra is that all fields have the IBN property. Other examples include (unital) skew fields, commutative rings, Noetherian rings, finite dimensional algebras, and any ring with a quotient that has the IBN property.

When a ring does not have the IBN property, then $R^n \cong R^m$ tells you nothing about n and m . At the time, it was already well known that there are rings R such that the free module of rank one is isomorphic to the free module of rank two. From this, it follows that $R^n \cong R^m$ for any $m, n \in \mathbb{N}$. For instance, $R = \text{End}(\mathbb{F}\mathbb{N})$, where $\mathbb{F}\mathbb{N}$ is the formal vector space with basis $\mathbb{N} = \{0, 1, 2, \dots\}$ over the field $\mathbb{F}$ (equivalently $\mathbb{F}$ -sequences of finite support) has the property $R \cong R^2$. Leavitt wanted to find a ring that met a finer condition - a ring R such that $R^m \cong R^n$ as R -modules (for $m < n$), but $R^k \not\cong R^n$ for $0 < k < n$, $k \neq m$. We say a ring that meets this condition is non-IBN of type (m, n) . Leavitt's early investigations focused on finding rings of type $(1, n)$

To realize the module isomorphism between R and R^n is a straightforward task. It is a fact that $\text{Hom}(R^m, R^n)$ is isomorphic as a vector space over $\mathbb{F}$ to $M_{m \times n}(R)$ where matrices act via left multiplication by elements. Viewing the module homomorphisms from R to R^n and vice versa as matrices, we get:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = I_{R^n} \text{ and } \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = I_R$$

This ensures that if R has elements x_i, y_i , $1 \leq i, j \leq n$ such that $\sum x_i y_i = 1$ and $y_i x_j = \delta_{ij}$, then $R \cong R^n$. Leavitt then defined $R = L_{\mathbb{F}}(n)$ as the algebra over the field $\mathbb{F}$ in noncommuting variables x_i, y_i for $1 \leq i \leq n$, quotiented by the ideal generated by:

$$\left\{ \sum x_i y_i - 1 \right\}_{1 \leq i \leq n} \cup \left\{ y_i x_j - \delta_{ij} \right\}_{1 \leq i, j \leq n}$$

where δ_{ij} is the Kronecker delta.

A concrete realization of $L_{\mathbb{F}}(2)$ is the operator algebra generated by upsampling and downsampling operators of signal processing [?] acting on

(finite) sequences, where the x_i is downsampling and the y_i is upsampling. The action is defined as follows:

$$\begin{aligned}(a_0, a_1, a_2, \dots)x_1 &= (a_0, a_2, a_4, \dots) \\ (a_0, a_1, a_2, \dots)x_2 &= (a_1, a_3, a_5, \dots) \\ (a_0, a_1, a_2, \dots)y_1 &= (a_0, 0, a_1, 0, a_2, 0, \dots) \\ (a_0, a_1, a_2, \dots)y_2 &= (0, a_0, 0, a_1, 0, a_2, \dots)\end{aligned}$$

The vector space of finitely supported sequences is isomorphic to the vector space $\mathbb{FN}$, and thus we can write the upsampling and downsampling operators as infinite matrices (acting on row vectors on the right):

$$x_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} x_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \left[\begin{array}{c|c} 0 & x_1 \\ 0 & \\ \vdots & \end{array} \right]$$

$$y_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} y_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \left[\begin{array}{c|c} 0 & y_1 \\ 0 & \\ \vdots & \end{array} \right]$$

Thus, we have a subalgebra of $End(\mathbb{FN})$ that also fails to have the IBN property.

Similarly, we can realize $L_{\mathbb{F}}(n)$ as the following downsampling and upsampling operators (which also have realizations as matrices):

$$(a_0, a_1, a_2, a_3, \dots, a_n, \dots)x_1 = (a_0, a_n, a_{2n}, \dots)$$

$$\begin{aligned}
(a_0, a_1, a_2, a_3, \dots, a_n, \dots)x_2 &= (a_1, a_{n+1}, a_{2n+1}, \dots) \\
&\vdots \\
(a_0, a_1, a_2, a_3, \dots, a_n, \dots)x_n &= (a_{n-1}, a_{2n-1}, a_{3n-1}, \dots) \\
\\
(a_0, a_1, a_2, a_3, \dots, a_n, \dots)y_1 &= (a_0, \underbrace{0, 0, \dots, 0}_{n-1}, a_1, \underbrace{0, 0, \dots, 0}_{n-1}, a_{2n}, \dots) \\
(a_0, a_1, a_2, a_3, \dots, a_n, \dots)y_2 &= (0, a_0, \underbrace{0, 0, \dots, 0}_{n-1}, a_1, \underbrace{0, 0, \dots, 0}_{n-1}, a_{2n}, \dots) \\
&\vdots \\
(a_0, a_1, a_2, a_3, \dots, a_n, \dots)y_n &= (\underbrace{0, 0, \dots, 0}_{n-1}, a_0, \underbrace{0, 0, \dots, 0}_{n-1}, a_1, \underbrace{0, 0, \dots, 0}_{n-1}, a_{2n}, \dots)
\end{aligned}$$

In order to show that $L_{\mathbb{F}}(n) \not\cong L_{\mathbb{F}}(n)^k$ for $1 < k < n$, Leavitt had to (essentially) compute the nonstable K-theory of $L_{\mathbb{F}}(n)$. The nonstable K-theory of a ring R , denoted $V(R)$, is the commutative monoid of isomorphism classes of finitely generated projective (right) modules over R under the operation of direct sum. A familiar, related idea is $K_0(R)$, the Grothendieck group of this commutative monoid. The *Grothendieck group of a commutative monoid* A is the group with underlying set $A \times A$ quotiented by the equivalence relation:

$$(a, b) \sim (c, d) \Leftrightarrow \exists e \in A \text{ such that } (a + d + e = b + c + e).$$

Addition is given by $[a, b] + [c, d] = [a + c, b + d]$ where $[a, b]$ denotes the equivalence class containing (a, b) . The neutral element is $[0, 0]$, and the additive inverse of $[a, b]$ is $[b, a]$. There is a canonical monoid homomorphism from $V(R)$ to $K_0(R)$, given by $a \mapsto [a, 0]$. In general, this homomorphism is not injective. For example - consider an (additive) cyclic monoid A generated by a with the relation $5a = 7a$. We see that A has seven elements $\{0, a, \dots, 6a\}$ but the Grothendieck group $K_0(A)$ has only two elements $\{0, [a, 0]\}$, since $2[a, 0] = 0$. Leavitt needed to work with $V(R)$ rather than $K_0(R)$, because although $K_0(R)$ can detect if a ring has the IBN property or not, it cannot detect the type (m, n) of a ring. (R has the IBN property if and only if $[R] \in K_0(R)$ has infinite order.)

Related to Leavitt's work, P. M. Cohn and G. Bergman made important advances in ring theory in the 1970s. In 1973, Cohn gave a general way to invert homomorphisms between finitely generated projective modules, via inverting certain matrices over a ring R (a process called Cohn localization or universal localization) [?]. The Leavitt algebra $L_{\mathbb{R}}(n)$ is a Cohn localization of the polynomial algebra in n noncommuting variables. In 1974, Bergman described a construction that takes any monoid where $a + b = 0$ implies $a = 0$ (called a conical monoids) and creates a ring with that monoid as its nonstable K-theory. In the case of finite cyclic monoids, Bergman's construction does give the Leavitt algebras [?].

The second thread began in the 1940s, when I. M. Gelfand and M. Naimark investigated representations of what would later be called C^* -algebras, which are generalized Weyl-von Neumann operator algebras [?]. These were part of von Neumann's attempt to axiomatize quantum mechanics. Soon afterwards, Gelfand, Naimark and Segal gave an abstract characterization of closed $*$ -subalgebras of bounded linear operators on some Hilbert space [?].

Definition 0.1. A C^* -algebra is a Banach algebra A (a complete normed algebra over $\mathbb{C}$ satisfying $\|xy\| \leq \|x\| \cdot \|y\|$) with an anti-automorphism denoted by $()^*$ such that, for all $x, y \in A$ and all $\lambda \in \mathbb{C}$:

$$x^{**} = ((x)^*)^* = x, \quad (x + y)^* = x^* + y^*, \quad (xy)^* = y^* x^*, \quad (\lambda x)^* = \bar{\lambda} x^*$$

and $\|x^* x\| = \|x^*\| \|x\|$.

An example of a finite dimensional C^* -algebra is the algebra of $n \times n$ matrices with entries in $\mathbb{C}$, where $*$ is conjugate transpose, and $\|\cdot\|$ is the operator norm. A commutative example is $C(X)$, complex valued continuous functions on a compact Hausdorff topological space X , where (for $f \in C(X)$), $f^* := \bar{f}$ is complex conjugation, and $\|f\| := \sup_{x \in X} |f(x)|$.

The Gelfand-Naimark theorem states that all (unital) commutative C^* -algebras are isometrically isomorphic to a $C(X)$ for some compact Hausdorff X . Moreover, this gives a (co-)functorial correspondence between compact Hausdorff spaces and commutative C^* -algebras called Gelfand duality. The Gelfand-Naimark-Segal theorem states that all C^* -algebras are isometrically isomorphic to a closed subalgebra of bounded linear operators on some Hilbert space. Unlike the commutative case, neither the subalgebra nor the Hilbert space are uniquely determined. This has lead to the definition of a "noncommutative (or quantum) space" as the "space" which corresponds

to a noncommutative C^* -algebra in the noncommutative geometry of Alain Connes [?].

In the late 1970s, J. Cuntz defined a C^* -algebras later called Cuntz algebras and denoted $\mathcal{O}_n$ [?]. These were the first explicit examples of separable simple infinite C^* -algebras, although Dixmier had proven their existence earlier. Every simple infinite C^* -algebra contains $\mathcal{O}_n$ as a quotient. It was much later observed that $\mathcal{O}_n$ is the (universal) completion of the Leavitt algebra $L_{\mathbb{C}}(n)$. Let us think back to the example of $L_{\mathbb{C}}(n)$ acting faithfully on $\mathbb{C}N$ via downsampling and upsampling operators (where we have specified the field to be complex numbers). Although $\mathbb{C}N$ is not a Hilbert space, it is dense in the space of square summable sequences l_2 . The action of $L_{\mathbb{C}}(n)$ extends to this Hilbert space. When we complete $L_{\mathbb{C}}(n)$ with respect to the operator norm, we have a concrete realization of $\mathcal{O}_n$.

The Cuntz algebras were a significant breakthrough in understanding and classifying C^* -algebras, and were generalized to Cuntz-Kreiger algebras. Later, these algebras were further generalized to graph C^* -algebras by M. Enomoto and Y. Watatani in 1980 [?]. The Cuntz algebra $\mathcal{O}_n$ corresponds to the graph C^* -algebra on the n -petaled rose (one vertex, n loops). Graph C^* -algebras were popularized by I. Raeburn and his coauthors in the late 1990s and early 2000s. In particular, quantum spheres can be realized as graph C^* algebras.

Leavitt path algebras (examined in detail in the next section) were defined in 2004 by P. Ara, M. A. Moreno, and E. Pardo [?] and (independently) by G. Abrams and G. Aranda-Pino [?] as algebraic analogues of graph C^* -algebras. Soon after their inception, Mark Tomforde observed that Leavitt path algebras were dense $*$ -subalgebras of graph C^* -algebras [?]. This generalizes that the completion of $L_{\mathbb{C}}(n)$ is $\mathcal{O}_n$, as was previously mentioned.

The last thread began in 1972, when quiver representations associated to a directed graph were defined and investigated by P. Gabriel [?]. A *quiver representation* is a functor from a directed graph (thought of here as a small category whose objects are vertices and whose morphisms are directed paths) to the category of vector spaces. Equivalently, a quiver representation assigns to each vertex a vector space, and to each arrow a map from the vector space assigned to the source to the vector space assigned to the target. A quiver representation is equivalent to a module over $\mathbb{F}\Gamma$, the path algebra. $\mathbb{F}\Gamma$ is the algebra of formal linear combinations of directed paths, where the product of paths is defined by concatenation when applicable (and zero otherwise).

Gabriel classified all finite dimensional quiver representations of finite

representation type and tame representation type, showing almost all quivers have wild representation type. He proved that all *irreducible quivers* (that is, the underlying graph is connected) will be of *finite type* (admitting only a finite number of indecomposable representations) if and only if the underlying graph without orientations is a simply-laced Dynkin diagram. In the next year, J. Bernstein, I. M. Gelfand, and V. A. Ponomarev made Gabriel's work more accessible and introduced new techniques such as Coxeter functors, which opened up the area to further advancements and generalizations by V. Kac and others [?].

In this thesis, we will focus on infinite dimensional representations of Leavitt path algebras of exponential growth, as finite dimensional representations of Leavitt path algebra have been completely classified [?]. To this purpose, A. Koç and M. Özaydın observed in 2015 that representations of Leavitt path algebras are equivalent to a full subcategory of quiver representations that satisfy an isomorphism condition (which will be our viewpoint also). This fact is the representation-theoretic consequence of the Leavitt path algebra of Γ being a Cohn localization of $\mathbb{F}\Gamma$.

According to K. Rangaswamy, a leading expert in the theory of Levitt Path Algebras, "The module theory over Leavitt path algebras is still at an infant stage [?]." For instance, up until now, the only known simple representations of Levitt Path Algebras were mild generalizations of modules called Chen modules [?]. The main contribution of this thesis is to expand upon that knowledge, by construct new families of simple representations of Leavitt path algebras by utilizing topological Markov chains.