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Leavitt Path Algebras

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New Simple Representations of
Leavitt Path Algebras

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To my parents

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Abstract

This thesis is about representations of Leavitt Path Algebras (LPAs). Specifically, we first generalize a previously known construction of twisted Chen modules over the Leavitt Path Algebra of a directed graph. We then give some thought to its extension to other modules, and present new classes of simple Leavitt Path Algebra modules previously unknown. These are modules generated by indicator functions of closed sets of P_∞ , the set of all infinite paths of a directed graph.

Chapter 1

Introduction

This thesis is mostly about irreducible representations of Leavitt Path Algebras (LPAs). Specifically, we first generalize a previously known construction of twisted Chen modules over the Leavitt Path Algebra of a directed graph. We then give some thought to its extension to other modules, and present new classes of simple Leavitt Path Algebra modules previously unknown.

As we study modules over a particular algebra, it is useful to note that these modules are equivalent to representations of the algebra. A *representation* of an $\mathbb{F}$ -algebra A is a algebra homomorphism:

$$\rho : A \longrightarrow \text{End}_{\mathbb{F}}(\Omega)$$

where Ω is a vector space over $\mathbb{F}$, and $\text{End}_{\mathbb{F}}(\Omega)$ refers to the algebra of linear endomorphisms on Ω . We will make the choice that our endomorphisms will act on the right, so that the expression xf makes sense for $x \in \Omega$, $f \in \text{End}_{\mathbb{F}}(\Omega)$. As an algebra homomorphism, ρ respects addition, multiplication (which is composition for endomorphisms), and scalar multiplication:

$$\rho(a + b) = \rho(a) + \rho(b), \quad \rho(ab) = \rho(a)\rho(b), \quad \rho(\lambda a) = \lambda\rho(a)$$

for $a, b \in A$ and $\lambda \in \mathbb{F}$. A module over an $\mathbb{F}$ -algebra A is a vector space Ω and a pairing

$\Omega \times A \longrightarrow \Omega$ denoted $(x, a) \longrightarrow xa$. This pairing respects the addition and scalar multiplication of both A and Ω , as well as the multiplication of A so:

$$(x, a + b) \longrightarrow xa + xb, \quad (x, ab) \longrightarrow (xa)b, \quad (x, \lambda a) \longrightarrow \lambda(xa)$$

$$(x + y, a) \longrightarrow xa + ya, \quad (\lambda x, a) \longrightarrow \lambda(xa)$$

for all $a, b \in A$, $x, y \in \Omega$, and $\lambda \in \mathbb{F}$. Using the relation $(x)\rho(a) = xa$, one can see how modules over the algebra completely define a representation ρ and vice versa. As a common abuse of notation, we often omit ρ when speaking about a representation.

The theory of Leavitt Path Algebras is the confluence of three threads of mathematics, involving ring theory, operator algebras, and quiver representations. The main thread of our story is within ring theory, and began with investigations of W. Leavitt into the extent of the failure of the Invariant Basis Number (IBN) property of a unital ring (with $1 \neq 0$) around 1960 [30]. Any module over such a ring R that has a basis is a free (right) module over R , and any module over a ring R that has a finite basis is finitely generated free (right) module over R that is isomorphic to R^n for some n . A basis of a free module is a subset of the module such that all elements of the module can be written uniquely as R -linear combinations of elements of the basis. A ring has the IBN property when any two finite bases have the same number of elements, that is: if $R^n \cong R^m$ as R modules, then $n = m$. For example, one of the first results taught in linear algebra is that all fields have the IBN property. Other examples include (unital) skew fields, commutative rings, Noetherian rings, finite dimensional algebras, and any ring with a quotient that has the IBN property.

When a ring does not have the IBN property, then $R^n \cong R^m$ tells you nothing about n and m . At the time, it was already well known that there are rings R such that the free module of rank one is isomorphic to the free module of rank two. From this, it follows that $R^n \cong R^m$ for any $m, n \in \mathbb{N}$. For instance, $R = \text{End}(\mathbb{F}\mathbb{N})$, where $\mathbb{F}\mathbb{N}$ is the formal vector space with basis $\mathbb{N} = \{0, 1, 2, \dots\}$ over the field $\mathbb{F}$ (equivalently $\mathbb{F}$ -sequences of finite support) has

the property $R \cong R^2$. Leavitt wanted to find a ring that met a finer condition - a ring R such that $R^m \cong R^n$ as R -modules (for $m < n$), but $R^k \not\cong R^n$ for $0 < k < n$, $k \neq m$. We say a ring that meets this condition is non-IBN of type (m, n) . Leavitt's early investigations focused on finding rings of type $(1, n)$

To realize the module isomorphism between R and R^n is a straightforward task. It is a fact that $\text{Hom}(R^m, R^n)$ is isomorphic as a vector space over $\mathbb{F}$ to $M_{m \times n}(R)$ where matrices act via left multiplication by elements. Viewing the module homomorphisms from R to R^n and vice versa as matrices, we get:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = I_{R^n} \text{ and } \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = I_R$$

This ensures that if R has elements x_i, y_i , $1 \leq i, j \leq n$ such that $\sum x_i y_i = 1$ and $y_i x_j = \delta_{ij}$, then $R \cong R^n$. Leavitt then defined $R = L_{\mathbb{F}}(n)$ as the algebra over the field $\mathbb{F}$ in noncommuting variables x_i, y_i for $1 \leq i \leq n$, quotiented by the ideal generated by:

$$\left\{ \sum x_i y_i - 1 \right\}_{1 \leq i \leq n} \cup \left\{ y_i x_j - \delta_{ij} \right\}_{1 \leq i, j \leq n}$$

where δ_{ij} is the Kronecker delta.

A concrete realization of $L_{\mathbb{F}}(2)$ is the operator algebra generated by upsampling and downsampling operators of signal processing [36] acting on (finite) sequences, where the x_i is downsampling and the y_i is upsampling. The action is defined as follows:

$$(a_0, a_1, a_2, \dots)x_1 = (a_0, a_2, a_4, \dots)$$

$$(a_0, a_1, a_2, \dots)x_2 = (a_1, a_3, a_5, \dots)$$

$$(a_0, a_1, a_2, \dots)y_1 = (a_0, 0, a_1, 0, a_2, 0, \dots)$$

$$(a_0, a_1, a_2, \dots)y_2 = (0, a_0, 0, a_1, 0, a_2, \dots)$$

The vector space of finitely supported sequences is isomorphic to the vector space $\mathbb{FN}$, and thus we can write the upsampling and downsampling operators as infinite matrices (acting on row vectors on the right):

$$x_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} x_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \left[\begin{array}{c|c} 0 & x_1 \\ 0 & \\ \vdots & \end{array} \right]$$

$$y_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} y_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \left[\begin{array}{c|c} 0 & y_1 \\ 0 & \\ \vdots & \end{array} \right]$$

Thus, we have a subalgebra of $End(\mathbb{FN})$ that also fails to have the IBN property.

Similarly, we can realize $L_{\mathbb{F}}(n)$ as the following downsampling and upsampling operators

(which also have realizations as matrices):

$$\begin{aligned}
(a_0, a_1, a_2, a_3, \dots, a_n, \dots)x_1 &= (a_0, a_n, a_{2n}, \dots) \\
(a_0, a_1, a_2, a_3, \dots, a_n, \dots)x_2 &= (a_1, a_{n+1}, a_{2n+1}, \dots) \\
&\vdots \\
(a_0, a_1, a_2, a_3, \dots, a_n, \dots)x_n &= (a_{n-1}, a_{2n-1}, a_{3n-1}, \dots) \\
\\
(a_0, a_1, a_2, a_3, \dots, a_n, \dots)y_1 &= (a_0, \underbrace{0, 0, \dots, 0}_{n-1}, a_1, \underbrace{0, 0, \dots, 0}_{n-1}, a_{2n}, \dots) \\
(a_0, a_1, a_2, a_3, \dots, a_n, \dots)y_2 &= (0, a_0, \underbrace{0, 0, \dots, 0}_{n-1}, a_1, \underbrace{0, 0, \dots, 0}_{n-1}, a_{2n}, \dots) \\
&\vdots \\
(a_0, a_1, a_2, a_3, \dots, a_n, \dots)y_n &= (\underbrace{0, 0, \dots, 0}_{n-1}, a_0, \underbrace{0, 0, \dots, 0}_{n-1}, a_1, \underbrace{0, 0, \dots, 0}_{n-1}, a_{2n}, \dots)
\end{aligned}$$

In order to show that $L_{\mathbb{F}}(n) \not\cong L_{\mathbb{F}}(n)^k$ for $1 < k < n$, Leavitt had to (essentially) compute the nonstable K-theory of $L_{\mathbb{F}}(n)$. The nonstable K-theory of a ring R , denoted $V(R)$, is the commutative monoid of isomorphism classes of finitely generated projective (right) modules over R under the operation of direct sum. A familiar, related idea is $K_0(R)$, the Grothendieck group of this commutative monoid. The *Grothendieck group of a commutative monoid* A is the group with underlying set $A \times A$ quotiented by the equivalence relation:

$$(a, b) \sim (c, d) \Leftrightarrow \exists e \in A \text{ such that } (a + d + e = b + c + e).$$

Addition is given by $[a, b] + [c, d] = [a + c, b + d]$ where $[a, b]$ denotes the equivalence class

containing (a, b) . The neutral element is $[0, 0]$, and the additive inverse of $[a, b]$ is $[b, a]$. There is a canonical monoid homomorphism from $V(R)$ to $K_0(R)$, given by $a \mapsto [a, 0]$. In general, this homomorphism is not injective. For example - consider an (additive) cyclic monoid A generated by a with the relation $5a = 7a$. We see that A has seven elements $\{0, a, \dots, 6a\}$ but the Grothendieck group $K_0(A)$ has only two elements $\{0, [a, 0]\}$, since $2[a, 0] = 0$. Leavitt needed to work with $V(R)$ rather than $K_0(R)$, because although $K_0(R)$ can detect if a ring has the IBN property or not, it cannot detect the type (m, n) of a ring. (R has the IBN property if and only if $[R] \in K_0(R)$ has infinite order.)

Related to Leavitt's work, P. M. Cohn and G. Bergman made important advances in ring theory in the 1970s. In 1973, Cohn gave a general way to invert homomorphisms between finitely generated projective modules, via inverting certain matrices over a ring R (a process called Cohn localization or universal localization) [16]. The Leavitt algebra $L_{\mathbb{F}}(n)$ is a Cohn localization of the polynomial algebra in n noncommuting variables. In 1974, Bergman described a construction that takes any monoid where $a + b = 0$ implies $a = 0$ (called a conical monoids) and creates a ring with that monoid as its nonstable K-theory. In the case of finite cyclic monoids, Bergman's construction does give the Leavitt algebras [31].

The second thread began in the 1940s, when I. M. Gelfand and M. Naimark investigated representations of what would later be called C^* -algebras, which are generalized Weyl-von Neumann operator algebras [23]. These were part of von Neumann's attempt to axiomatize quantum mechanics. Soon afterwards, Gelfand, Naimark and Segal gave an abstract characterization of closed $*$ -subalgebras of bounded linear operators on some Hilbert space [34].

Definition 1.1. A C^* -algebra is a Banach algebra A (a complete normed algebra over $\mathbb{C}$ satisfying $\|xy\| \leq \|x\| \cdot \|y\|$) with an anti-automorphism denoted by $(\)^*$ such that, for all $x, y \in A$ and all $\lambda \in \mathbb{C}$:

$$x^{**} = ((x)^*)^* = x, \quad (x + y)^* = x^* + y^*, \quad (xy)^* = y^* x^*, \quad (\lambda x)^* = \bar{\lambda} x^*$$

$$\text{and} \quad \|x^*x\| = \|x^*\| \|x\|.$$

An example of a finite dimensional C^* -algebra is the algebra of $n \times n$ matrices with entries in $\mathbb{C}$, where $*$ is conjugate transpose, and $\|\cdot\|$ is the operator norm. A commutative example is $C(X)$, complex valued continuous functions on a compact Hausdorff topological space X , where (for $f \in C(X)$), $f^* := \bar{f}$ is complex conjugation, and $\|f\| := \sup_{x \in X} |f(x)|$.

The Gelfand-Naimark theorem states that all (unital) commutative C^* -algebras are isometrically isomorphic to a $C(X)$ for some compact Hausdorff X . Moreover, this gives a (co-)functorial correspondence between compact Hausdorff spaces and commutative C^* -algebras called Gelfand duality. The Gelfand-Naimark-Segal theorem states that all C^* -algebras are isometrically isomorphic to a closed subalgebra of bounded linear operators on some Hilbert space. Unlike the commutative case, neither the subalgebra nor the Hilbert space are uniquely determined. This has lead to the definition of a "noncommutative (or quantum) space" as the "space" which corresponds to a noncommutative C^* -algebra in the noncommutative geometry of Alain Connes [17].

In the late 1970s, J. Cuntz defined a C^* -algebras later called Cuntz algebras and denoted $\mathcal{O}_n$ [18]. These were the first explicit examples of separable simple infinite C^* -algebras, although Dixmier had proven their existence earlier. Every simple infinite C^* -algebra contains $\mathcal{O}_n$ as a quotient. It was much later observed that $\mathcal{O}_n$ is the (universal) completion of the Leavitt algebra $L_{\mathbb{C}}(n)$. Let us think back to the example of $L_{\mathbb{C}}(n)$ acting faithfully on $\mathbb{CN}$ via downsampling and upsampling operators (where we have specified the field to be complex numbers). Although $\mathbb{CN}$ is not a Hilbert space, it is dense in the space of square summable sequences l_2 . The action of $L_{\mathbb{C}}(n)$ extends to this Hilbert space. When we complete $L_{\mathbb{C}}(n)$ with respect to the operator norm, we have a concrete realization of $\mathcal{O}_n$.

The Cuntz algebras were a significant breakthrough in understanding and classifying C^* -algebras, and were generalized to Cuntz-Kreiger algebras. Later, these algebras were further generalized to graph C^* -algebras by M. Enomoto and Y. Watatani in 1980 [20]. The Cuntz algebra $\mathcal{O}_n$ corresponds to the graph C^* -algebra on the n -petaled rose (one

vertex, n loops). Graph C^* -algebras were popularized by I. Raeburn and his coauthors in the late 1990s and early 2000s. In particular, quantum spheres can be realized as graph C^* algebras.

Leavitt path algebras (examined in detail in the next section) were defined in 2004 by P. Ara, M. A. Moreno, and E. Pardo [10] and (independently) by G. Abrams and G. Aranda-Pino [4] as algebraic analogues of graph C^* -algebras. Soon after their inception, Mark Tomforde observed that Leavitt path algebras were dense $*$ -subalgebras of graph C^* -algebras [37]. This generalizes that the completion of $L_{\mathbb{C}}(n)$ is $\mathcal{O}_n$, as was previously mentioned.

The last thread began in 1972, when quiver representations associated to a directed graph were defined and investigated by P. Gabriel [21]. A *quiver representation* is a functor from a directed graph (thought of here as a small category whose objects are vertices and whose morphisms are directed paths) to the category of vector spaces. Equivalently, a quiver representation assigns to each vertex a vector space, and to each arrow a map from the vector space assigned to the source to the vector space assigned to the target. A quiver representation is equivalent to a module over $\mathbb{F}\Gamma$, the path algebra. $\mathbb{F}\Gamma$ is the algebra of formal linear combinations of directed paths, where the product of paths is defined by concatenation when applicable (and zero otherwise).

Gabriel classified all finite dimensional quiver representations of finite representation type and tame representation type, showing almost all quivers have wild representation type. He proved that all *irreducible quivers* (that is, the underlying graph is connected) will be of *finite type* (admitting only a finite number of indecomposable representations) if and only if the underlying graph without orientations is a simply-laced Dynkin diagram. In the next year, J. Bernstein, I. M. Gelfand, and V. A. Ponomarev made Gabriel's work more accessible and introduced new techniques such as Coxeter functors, which opened up the area to further advancements and generalizations by V. Kac and others [13].

In this thesis, we will focus on infinite dimensional representations of Leavitt path algebras

of exponential growth, as finite dimensional representations of Leavitt path algebra have been completely classified [28]. To this purpose, A. Koç and M. Özaydın observed in 2015 that representations of Leavitt path algebras are equivalent to a full subcategory of quiver representations that satisfy an isomorphism condition (which will be our viewpoint also). This fact is the representation-theoretic consequence of the Leavitt path algebra of Γ being a Cohn localization of $\mathbb{F}\Gamma$.

According to K. Rangaswamy, a leading expert in the theory of Levitt Path Algebras, "The module theory over Leavitt path algebras is still at an infant stage [33]." For instance, up until now, the only known simple representations of Levitt Path Algebras were mild generalizations of modules called Chen modules [14]. The main contribution of this thesis is to expand upon that knowledge, by construct new families of simple representations of Leavitt path algebras by utilizing topological Markov chains.

Chapter 2

Preliminaries

2.1 Leavitt Path Algebras and their (irreducible and indecomposable) representations.

Most of the results in this section are not new, but some of the proofs are. Standard terminology is in *italics*. Less standard terminology is **bolded**.

Let Γ be a *directed graph* consisting of V , a set of *vertices*, E , a set of *arrows*, and $s, t : E \longrightarrow V$ (the *source and target maps* respectively). We will only work in the finite case, which means $V \sqcup E$ is a finite set. We will call a vertex v a *sink* when $s^{-1}(v) = \emptyset$. A *path* in Γ is either a finite sequence of arrows $e_1 e_2 \cdots e_n$ where $t e_i = s e_{i+1}$ for $i \in \{1, 2, \dots, n-1\}$ or a vertex. The length of the path is n , and the length of a path that is a vertex is 0. We extend the domain of s and t to paths where $s v = t v = v$ for $v \in V$, and for $p = p_1 \cdots p_n$ where $p_i \in E$, $s p = s p_1$, $t p = t p_n$. In the case where $s p = t p$, we will call p a *closed path*. When the set $\{s p_i\}_{i=1}^n$ has n distinct vertices, then we call p a *cycle*. In the case where there is a path between any two vertices, we will call the graph *strongly connected*.

From this, we define the *path algebra* $\mathbb{F}\Gamma$ in two different ways, and then show that these

separate constructions define the same module. We will construct $\mathbb{F}\Gamma_1$ as the algebra over some field $\mathbb{F}$ with a basis consisting of the set of paths in Γ (denoted by $Path(\Gamma)$), and multiplication of paths p, q given by:

$$pq = \begin{cases} p & tp = q \\ q & p = sq \\ p_1 \cdots p_n q_1 \cdots q_k & p = p_1 \cdots p_n, q = q_1 \cdots q_k, tp = sq \\ 0 & otherwise \end{cases}$$

We abuse notation (and will continue to abuse notation) by having elements of the algebra and elements of the sets V, E , have the same notation.

Here is the other construction of this object: for $\mathbb{F}\Gamma_2$, take the non unital algebra over a field $\mathbb{F}$ generated by $V \sqcup E$ with the following relations:

$$(V) \quad uv = \delta_{u,v}v \text{ for all } u, v \in V$$

$$(E) \quad se e = e \text{ and } e te = e \text{ for all } e \in E$$

Here, δ is the Kronecker delta. We also call the nontrivial monomials $v \in V$ and $e_1 \cdots e_n$ where $e_i \in E$, where $te_i = se_{i+1}$ for $i \in \{1, 2, \dots, n-1\}$ paths. We again extend s and t to paths.

$\mathbb{F}\Gamma_1$ and $\mathbb{F}\Gamma_2$ do not have a unit when V is infinite. In the case where V is finite, then $1 = \sum_V v$. We can put a partial order on V . For $v, w \in V$, $v \geq w$ means there is a $p \in Path(\Gamma)$ such that $sp = v, tp = w$. Equivalence classes happen among strongly connected subsets of vertices.

$\mathbb{F}\Gamma_1$ is isomorphic to $\mathbb{F}\Gamma_2$. We can define a map from $\mathbb{F}\Gamma_2$ to $\mathbb{F}\Gamma_1$ as identity on the generators. The relations (V) and (E) are satisfied in the target. The map is surjective as all basis elements in $\mathbb{F}\Gamma_1$ are images of paths in $\mathbb{F}\Gamma_2$. The map is also injective: for any linear

combination of paths in $\mathbb{F}\Gamma_2$, the image in $\mathbb{F}\Gamma_1$ is zero iff all of the coefficients are zero. Thus, the map is an isomorphism, and we will just refer to $\mathbb{F}\Gamma$ from now on.

We extend construction of the path algebra on the *doubled graph*. Here we double the edge set by adding new edges $E^* := \{e^* | e \in E\}$ where e^* goes in the opposite direction as e of the original digraph.

$$V_{double} = V \quad E_{double} = E \sqcup E^*$$

$$s|_E = s \quad t|_E = t$$

$$s e^* = t e \quad t e^* = s e \quad \text{for all } e \in E^*$$

We again extend s and t to paths.

From the path algebra on the doubled graph, we create $L(\Gamma)$, the *Leavitt Path Algebra* (LPA), where we impose the the following additional relations:

$$(CK1) \quad e^* f = \delta_{e,f} t e \text{ for all } e^* \in E^*, f \in E,$$

$$(CK2) \quad \sum_{s e = v} e e^* = \forall (\text{nonsink}) v \in V.$$

Here, CK stands for Cuntz-Krieger and δ is again the Kronecker delta.

We want to extend $*$ as a linear antiautomorphism on the path algebra of the doubled graph, where

$$(e)^* := e^* \quad (e^*)^* := e \quad (v)^* := v \quad (fg)^* = g^* f^*$$

for $e \in E$, $v \in V$, and $f, g \in E \sqcup E^*$. The relations hold under this automorphism, as:

$$(CK1) \quad (e^* f)^* = f^* (e^*)^* = f^* e = \delta_{f,e} t f = (\delta_{e,f} t e)^* \text{ for all } e, f \in E,$$

$$(CK2) \quad \left(\sum_{s e = v} e e^* \right)^* = \sum_{s e = v} e e^* = v = v^*, \quad \forall \text{ nonsink } v \in V.$$

For $p = p_1 \cdots p_n$ where $p_i \in E \sqcup E^*$, we can see that $p^* = p_n^* \cdots p_1^*$. We see that $L(\Gamma)$ is spanned by $\{pq^* \mid p, q \in \text{Path}(\Gamma), tp = tq\} \subseteq L(\Gamma)$ by CK1. Also, note that $p^*q = 0$ for all $p, q \in \text{Path}(\Gamma)$ unless there exists $r \in \text{Path}(\Gamma)$ such that $p = rq$ or $q = pr$. We will call $e^* \in E^*$ a *dual arrow*, and p^* such that $p \in \text{Path}(\Gamma)$ a *dual path*.

Whenever one defines an algebra in terms of generators and relations, the question of whether the algebra is trivial or some of the generators map to 0 should be settled. We already know that the path algebra is not trivial, as $\mathbb{F}\Gamma$ was defined with the paths of Γ as a basis. However, $L(\Gamma)$ has more generators and more relations. We can use representation theory to make sure all the generators of $L(\Gamma)$ are nonzero.

For our argument, it is useful to note that the category of quiver representations is equivalent to the category of path algebra modules [19]. This is straightforward - to form a quiver representation from a module M , assign the vector space Mv to the vertex v and the linear map assigned to e will be the map that e induces from Mse to Mte . To form a module from a quiver representation, let M be the direct sum of all the vector spaces assigned to each vertex, and let v act as projection and then inclusion: $M \xrightarrow{pr} Mv \hookrightarrow M$, and e acts via: $M \xrightarrow{pr} Mse \xrightarrow{e} Mte \hookrightarrow M$.

The category of representations of $L(\Gamma)$ are equivalent to the full subcategory of modules M over $\mathbb{F}\Gamma$ such that the **Isomorphism Condition** holds [28]:

$$(IC) \quad (\cdot e)_{se=v} : Mv \xrightarrow{\cong} \bigoplus_{se=v} Mte \quad \text{for all nonsinks } v \in V$$

By CK1, e restricted to Mse yields a surjection onto Mte , and e^* yields a left inverse to e , hence e^* restricted to Mte is injective.

Example 1. To each vertex assign the vector space $\mathbb{F}\mathbb{N}$. Then, for each nonsink vertex, consider an isomorphism $\mathbb{F}\mathbb{N} \longrightarrow \bigoplus_{se=v} \mathbb{F}\mathbb{N}$. We can find such an isomorphism because vector space isomorphism classes are uniquely determined by their dimension, and both $\bigoplus_{se=v} \mathbb{F}\mathbb{N}$ and $\mathbb{F}\mathbb{N}$ are of countably infinite dimension. Then to each arrow e , assign the composition of

the isomorphism from $\mathbb{F}\mathbb{N}$ to $\bigoplus_{se=v}\mathbb{F}\mathbb{N}$ with the projection to the summand corresponding to e .

Note that by our construction, this quiver representation satisfies the Isomorphism Condition, making it an $L(\Gamma)$ module. Each dual arrow e^* on $Mte \cong \mathbb{F}\mathbb{N}$ is given by the inclusion of this summand into $\bigoplus_{se=v}\mathbb{F}\mathbb{N}$ composed with the inverse on the isomorphism above.

The generators $V \sqcup E \sqcup E^*$ cannot be 0 in $L(\Gamma)$ as they do not act as 0 on the module. We also infer that the spanning set $\{pq^* \mid tp = tq\}$ consists of nontrivial elements of $L(\Gamma)$.

There is a $\mathbb{Z}$ -grading on $L(\Gamma)$ and $\mathbb{F}\Gamma$ as a consequence of assigning all $v \in V$ grade 0, all $e \in E$ grade 1, and all $e^* \in E^*$ grade -1 . This is compatible with the relations (V), (E), and (CK1), (CK2), as all these relations are homogeneous. As a consequence, homogeneous elements of degree n are described as $\sum_i p_i q_i^*$ where the length of p_i minus the length of q_i is n for all i . Hence, $L(\Gamma)$ is a $\mathbb{Z}$ -graded $*$ -algebra, where $*$ reverses the grading (with multiplication by -1).

We can use the $\mathbb{Z}$ -grading of these algebras to prove that $\mathbb{F}\Gamma$ embeds into $L(\Gamma)$ [25]. First, look at the map that sends $\mathbb{F}\Gamma$ to $L(\Gamma)$. This is a $\mathbb{Z}$ -graded morphism, and the kernel must be a graded ideal. Look at a homogeneous element in the ideal, say $\sum_{i=1}^n \lambda_i p_i$, where p_i have the same length for $1 \leq i \leq n$. Taking the image of this element in $L(\Gamma)$, act on it on the left with p_i^* . One obtains $\lambda_i t p_i$, which is only trivial if $\lambda_i = 0$ for $1 \leq i \leq n$, as $t p_i$ is a non trivial element of $L(\Gamma)$. Thus, the kernel is trivial, and the map is an injection.

We will need some more definitions to continue. Note that in order for a ring R to be *Artinian*, that is having the *Descending Chain Condition* on ideals, all descending sequences of ideals of R , say $I_1 \supseteq I_2 \supseteq \dots$ are eventually constant. A ring R is *Noetherian* when it has the *Ascending Chain Condition* on ideals: all ascending sequences of ideals $I_1 \subseteq I_2 \subseteq \dots$ are eventually constant. A ring has *Unbounded Generating Number* when for each positive integer m , any set of generators for the free right R -module R^m has cardinality greater than or equal to m . A cycle has an exit when for any p that is a cycle, there is a path q such that

$sq = sp$, but $qr \neq p$ for any path r .

Many investigations of LPAs focus on the relationship between Γ and $L(\Gamma)$. These results are summarized here and quoted [28, Introduction]:

- (i) $L(\Gamma)$ has DCC (Descending Chain Condition) on right (or left) ideals [6, Theorem 2.6] if and only if Γ is acyclic (that is, Γ has no directed cycles) if and only if $L(\Gamma)$ is von Neumann regular [7, Theorem 1] if and only if $L(\Gamma)$ is finite dimensional if and only if $L(\Gamma)$ is isomorphic to a direct sum of matrix algebras (over the ground field $\mathbb{F}$) [1, Corollaries 3.6 and 3.7].
- (ii) $L(\Gamma)$ has ACC (Ascending Chain Condition) on right (or left) ideals [6, Theorem 3.8] if and only if the cycles of Γ have no exits if and only if $L(\Gamma)$ is locally finite dimensional (i.e., a graded algebra with each homogeneous summand being finite dimensional) if and only if $L(\Gamma)$ is a principal ideal ring [5, Proposition 17] in which case $L(\Gamma)$ is isomorphic to a direct sum of matrix algebras over $\mathbb{F}$ and/or matrix algebras over $\mathbb{F}[x, x^{-1}]$ (the Laurent polynomial algebra) [1, Theorems 3.8 and 3.10].
- (iii) $L(\Gamma)$ has finite GK (Gelfand-Kirillov) dimension, equivalently $L(\Gamma)$ has polynomial growth if and only if the cycles in Γ are mutually disjoint [8, Theorem 5] if and only if all simple $L(\Gamma)$ -modules are finitely presented [12, Theorem 4.5]. In fact (i) and (ii) are special cases of (iii): Γ is acyclic if and only if the GK dimension of $L(\Gamma)$ is 0. The digraph Γ has a cycle but the cycles of Γ have no exits if and only if the GK dimension of $L(\Gamma)$ is 1. The first instance of $L(\Gamma)$ with GK dimension > 1 is given by the Toeplitz digraph

$$\Gamma : \begin{array}{c} \circ \\ \bullet \end{array} \longrightarrow \bullet \quad ([26], [9], [28, \text{Example 5.6}]).$$

⋮

- (iv) $L(\Gamma)$ has a nonzero finite dimensional quotient if and only if Γ has a sink or a cycle such that there is no path from any other cycle to it [28, Theorem 6.5] if and only if $L(\Gamma)$ has UGN (Unbounded Generating Number) [2, Theorem 3.16] if and only if $L(\Gamma) \oplus L(\Gamma)$ is not a quotient of $L(\Gamma)$ [28, Corollary 6.7]. If $L(\Gamma)$ has finite Gelfand-Kirillov dimension then $L(\Gamma)$ has a nonzero finite dimensional quotient and if $L(\Gamma)$ has a nonzero finite dimensional quotient then $L(\Gamma)$ has IBN. Neither of these implications is reversible. [28, Corollary 6.9]

We have a Galois connection between 2^V (subsets of V) and ideals of $L(\Gamma)$. A *Galois Connection* is a pair of order reversing functions $F : A \longrightarrow B$, $G : B \longrightarrow A$ between two posets A and B , such that for $a \in A$, $b \in B$, $a \leq GF(a)$ and $b \leq FG(b)$. The connection is as follows:

$$\begin{array}{ccc} X & \xrightarrow{F} & (X) \\ I \cap V & \xleftarrow{G} & I \end{array}$$

where (X) denotes the ideal generated by X . The partial order we use on the poset 2^V is inclusion, and the partial order we use on the set of ideals is reverse inclusion. This Galois Connection establishes a bijection between particular subsets of vertices and graded ideals. These subsets of vertices are *hereditary* and *saturated*: a subset of vertices is hereditary if for $v \in H$, if there is a $p \in \text{Path}(\Gamma)$ such that $sp = v$, then $tp \in H$, and a subset of vertices is saturated if $te \in H$ for all $e \in E$ where $se = v$, then $v \in H$. In general, this bijective correspondence is called a *Galois Correspondence* between $\{a \in A \mid GF(a) = a\}$ and $\{b \in B \mid FG(b) = b\}$.

To support the claim of a Galois Correspondence, we will need a lemma:

Lemma 1. *For any nontrivial graded ideal I of $L(\Gamma)$, $I \cap V \neq \emptyset$.*

Proof. Let $0 \neq x \in I$, where $x = \sum_i \lambda_i p_i q_i^*$ is a homogeneous element. As $1 = \sum_V v$, there is a $v \in V$ such that $xv \neq 0$. Since $v = \sum_{se=v} ee^*$, when v is not a sink and e^* is always injective, this means there is an e_1 such that $xe_1 \neq 0$ when v is not a sink. One can continue to find $e_j \in E$ such that $xve_1 e_2 \cdots e_n \neq 0$ and $e_1 \cdots e_n$ is longer than any q_i^* that appears in the sum $\sum_i \lambda_i p_i q_i^*$ as long as te_j is not a sink for $j \in 1, 2, \dots, n-1$. If there is a sink that appears as te_j , there is no q_i^* that can be longer than it, as no starred path can precede a sink. Thus, by CK1, we have $xve_1 e_2 \cdots e_n = \sum_l \lambda_k p'_k \neq 0$. By similar argument, we can find a $w \in V$ and a sequence of $f^* \in E^*$ such that $f_n^* f_{n-1}^* \cdots f_1^* w x v e_1 e_2 \cdots e_n \neq 0$. This is a nonzero homogeneous element of grade 0. By CK1, this must be a sum of paths, and the only paths that are of grade 0 are vertices. In particular, up to scaling by an element of $\mathbb{F}$, $f_n^* f_{n-1}^* \cdots f_1^* w x v e_1 e_2 \cdots e_n = t f_n$. $\square$

Proof of claim of Galois Correspondence. Suppose H is a hereditary saturated subset of vertices. It is clear that $H \subseteq (H) \cap V$. Since H generates (H) , for any vertex $v \in (H) \cap V \setminus H$, we have an expression: $\sum_i \lambda_i p_i q_i^* h_i p'_i q'_i{}^* = v$ where $h_i \in H$ and p_i, q_i, p'_i, q'_i are paths in $Path(\Gamma)$. We see that, by using the relation of CK1, $\sum_i \lambda_i p_i q_i^* h_i p'_i q'_i{}^*$ simplifies to $\sum_j \lambda_j a_j b_j^*$ where a_j, b_j are paths in $Path(\Gamma)$ and $ta_j \in H$ since H is hereditary (remember: $e^*(se)e = te$). For all paths p such that $sp = v$ that are either of length equal to the longest b_j^* or where tp is a sink, we can see that $tp \in H$. This is because $p^*vp = tp$ and $p^* \left(\sum_j \lambda_j a_j b_j^* \right) p$ is a nonzero element with grade zero, where each summand has no generators from E^* . This is a vertex that is a descendant of an element of H , and therefore in H itself. By invoking that H is a saturated subset of vertices, for all p such that tp immediately precedes a sink or the length of p is one less than the length of the longest b_j , we have tp is also in H . We can repeat this process until $v \in H$, which means $H = (H) \cap V$.

Suppose that I is a graded ideal. It is clear that $(I \cap V) \subset I$. Observe $I \cap V$ is a hereditary saturated set of vertices (if $v \in I$, $p^*vp = tp \in I$, and if $te \in I$ for all e such that $se = v$, then $\sum_{se=v} e te e^* = se \in I$). Consider the graded module $L(\Gamma) / I \longrightarrow \mathbb{L}(\Gamma) / (I \cap V)$. The

graded kernel of this morphism contains no ideals as it contains no vertices by our lemma, so it must be an isomorphism. $\square$

For H a hereditary saturated subset of vertices, we have an isomorphism:

$$L(\Gamma) / (H) \cong L(\Gamma \setminus H)$$

where $\Gamma \setminus H$ is the full subgraph of Γ on the set of vertices $V \setminus H$.

For the map from $L(\Gamma)$ to $L(\Gamma \setminus H)$:

$$v \mapsto v \text{ for } v \in V \setminus H$$

$$v \mapsto 0 \text{ for } v \in H$$

$$e \mapsto e \text{ for } e \text{ such that } \{te, se\} \subseteq V \setminus H$$

$$e \mapsto 0 \text{ for } e \text{ such that } \{te, se\} \not\subseteq V \setminus H$$

We see that the homomorphism from left to right is graded. Via the Galois correspondence above, the graded kernel contains H and no other vertices, and is therefore exactly (H) .

This correspondence between hereditary saturated subsets of vertices and graded ideals of $L(\Gamma)$, extends to the non-graded case. The condition that the only hereditary saturated subsets of V are $\emptyset$ and V , and that any cycle has an exit are together equivalent to the condition that $L(\Gamma)$ is a simple algebra [3].

Most of the literature on representation theory is on representations of finite dimensional algebras. However, in order for $L(\Gamma)$ to be finite dimensional, Γ must have no directed cycles. This is equivalent to $L(\Gamma)$ having DCC. This is an incredibly restrictive condition on Γ . In this case, $L(\Gamma)$ is a finite sum of matrix algebras, indexed by the sinks of Γ . The summand corresponding to a sink is an algebra of square matrices, where each $n \times n$ matrix has n equal to the number of directed paths that end at that sink [1]. The representation theory of these algebras is well understood.

The representation theory of $L(\Gamma)$ for Γ with no directed cycles is well understood as a consequence of Morita equivalence of rings. Two rings R and S are *Morita equivalent* if there is a functor from the category of R modules to the category of S modules and another functor going the other way, such that their composition is naturally isomorphic to the identity. Using this functorial viewpoint, one finds that modules over $R \oplus S$ (where R and S are rings with unity) are the same as modules over R and modules over S as $M \mapsto M(1, 0)$ and $M \mapsto M(0, 1)$ maps the category of $R \oplus S$ modules to the category of R modules and S modules (respectively), and the direct sum of an R module and an S module has a canonical $R \times S$ module structure (R acts trivially on the second summand, S acts trivially on the first). Thus we can break up a representation of $\oplus_{sinks} M_n(\mathbb{F})$ into representations of $M_n(\mathbb{F})$. We can then call upon the fact that, for any ring R , the representation theory of R is the same as that of $M_n(R)$ [15]. Once we have reduced the representation of $M_n(\mathbb{F})$ to the representation theory of $\mathbb{F}$, we understand that these modules are just vector spaces completely characterized by their dimension.

When an algebra is not finite dimensional, mathematicians often focus on finite dimensional representations of the algebra. However, finite dimensional representations of Leavitt Path Algebras are now completely understood [27]. In general, given a module M , the **support** V_M of the module M is the set of vertices v such that $Mv \neq 0$. When one looks at a module M over $L(\Gamma)$, consider the **support subgraph** Γ_M (that is, the full subgraph of Γ on the support of M). The cycles of the support subgraph have no exits [11].

In parallel to how representation theory in general evolved, we may wish to extend our understanding from modules over matrix algebras to modules over $\mathbb{F}[x]$, or a Principal Ideal Domain (PID). With this in mind, one can recall that finitely generated modules over a PID are well understood. For any module M over a PID, say R , there is a unique sequence of nested ideals $I_n \supseteq I_{n-1} \supseteq \cdots \supseteq I_1$ such that $M \cong \bigoplus_{i=1}^n R/I_i$. In general, the subcategory of finitely generated modules is too unruly to work with unless the ring is at least Noetherian, as submodules of finitely generated modules over a Noetherian ring are also finitely generated.

Leavitt Path Algebras are often not Noetherian. A Leavitt Path Algebra is Noetherian precisely when the cycles of Γ have no exit [28]. This condition is rather restrictive on Γ .

Instead, we restrict our attention to the category of modules that are of *finite type*, which are both Artinian and Noetherian. This is the same as the category of modules with a finite length composition series: a module M is said to have a *composition series of length n* when there exists a sequence of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that M_i/M_{i-1} is a simple module for $i \in \{1, 2, \dots, n\}$ (where a *simple module* is a nonzero module with no submodules other than 0 and itself). This module category will often exclude the free module $L(\Gamma)$ (of rank one), but will still contain a breadth of interesting examples. We will also have two tools to make use of here - the *Jordan-Hölder theorem* and the *Krull-Remak-Schmidt theorem*. Their statements are as follows (J-H found in [32], K-R-S found in [29]):

Theorem (Jordan-Hölder). *Let M be a Λ -module of finite length, and $F := (0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_l = M)$ and $G := (0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_m = M)$ two composition series for M . Then, for each simple Λ -module S , we have that the number of times S appears as a quotient F_i/F_{i-1} for $i \in \{1, 2, \dots, l\}$ is the same as the number of times S appears as a quotient G_i/G_{i-1} for $i \in \{1, 2, \dots, m\}$, and hence $l = m$.*

Theorem (Krull-Remak-Schmidt). *Let $M \neq 0$ be a module which is both Noetherian and Artinian. Then M is a finite direct sum of its indecomposable modules. Up to permutation, the indecomposable summands are uniquely determined up to isomorphism.*

Indecomposable modules are modules that cannot be written as the direct sum of two submodules. While it is clear that a simple module is indecomposable, it is not true that an indecomposable module is simple. When our module category is semisimple, then indecomposable also implies simple, but the finite length modules of $L(\Gamma)$ is not a semisimple category. While this category is not semisimple, we can still try to understand simple modules and how they might be extended to indecomposable modules.

As a brief example of why the category of finite length modules over $L(\Gamma)$ is not semisimple, consider the graph consisting of one vertex v and a single arrow e that forms a loop. It follows

from (CK1) and (CK2) that for the arrow e , the dual arrow e^* is its inverse. This algebra is isomorphic to $\mathbb{F}[x^{-1}, x]$, the Laurent polynomial algebra, via the map $1 \mapsto v$, $x \mapsto e$. It is well known that for $f(x) \in \mathbb{F}[x] \subseteq \mathbb{F}[x^{-1}, x]$, an irreducible polynomial where $f(0) = 1$, then $\mathbb{F}[x^{-1}, x]/(f(x)^k)$ for $k \geq 2$ is indecomposable, but not simple.

Currently, the known simple modules over the Leavitt Path Algebra where Γ is finite ($|V \sqcup E| < \infty$) are either the projective simple modules $vL(\Gamma)$, where v is a sink ($s^{-1}v = \emptyset$) or modules defined by X. Chen in 2013 [14] (and later generalized). Chen modules are generated by an *infinite path* $\cdots \alpha_2^* \alpha_1^* \alpha_0^* = \alpha$, where $\alpha_i \in E$ for all $i \in \mathbb{N}$, and $t\alpha_i = s\alpha_{i+1}$. The action is defined as follows :

$$(\cdots \alpha_2^* \alpha_1^* \alpha_0^*)p = \begin{cases} \cdots \alpha_2^* \alpha_1^* \alpha_0^* & p = s\alpha_0 \\ \cdots \alpha_3^* \alpha_2^* \alpha_1^* & p = \alpha_0 \\ \cdots \alpha_2^* \alpha_1^* \alpha_0^* e^* & p = e, \text{ } te = s\alpha_0, \text{ } e \in E \\ 0 & \text{else} \end{cases}$$

This defines the action with generators $V \sqcup E \sqcup E^*$, which extends to all of $L(\Gamma)$. The action of any pq^* preserves tail equivalence, where *tail equivalence* for two infinite paths $\alpha = \cdots \alpha_2^* \alpha_1^* \alpha_0^*$ and $\beta = \cdots \beta_2^* \beta_1^* \beta_0^*$ means that there is a k_α and k_β such that $\alpha_{k_\alpha+n} = \beta_{k_\beta+n}$ for all $n \in \mathbb{N}$. Thus, Chen modules M have elements that are formal linear combinations of tail equivalent infinite paths.

The infinite path α that generates a Chen module may be eventually periodic or not. An infinite path is eventually periodic if there exists $n, m \in \mathbb{N}$ where $m \geq 1$ such that $\alpha_{n+k} = \alpha_{n+m+k}$ for all $k \in \mathbb{N}$. We call the modules generated by eventually periodic infinite paths *rational* Chen modules (and we call the other Chen modules *irrational*). In the case of rational Chen modules generated by $(C^*)^\infty = \cdots C^* C^* C^*$ where C is some primitive closed path (*primitive* means that there is no other path D such that $D^k = C$ for $k \geq 2$), generalizations were defined by X. Chen [14], P. Ara, and K. Rangaswamy [12]. The former

twists the action of $e \in E$ by the gauge action (arrows act as a nonzero scalar multiple of their previous actions), and the latter is a further twisting that can only be done when the field is not algebraically closed and the C is an *exclusive* cycle (meaning there is no other cycle with a common vertex). In 3.1 below, we will extend these constructions via irreducible polynomials.

2.2 Graph C*-algebras and their representations on Hilbert spaces

A **-algebra* A over $\mathbb{F}$ (where $\mathbb{F}$ has an involution $c \longrightarrow \bar{c}$) is an algebra with a map $*$ which is an linear anti-automorphism and an involution:

$$(x + y)^* = x^* + y^*$$

$$(xy)^* = y^* x^*$$

$$(cx)^* = \bar{c}x^*$$

$$(x^*)^* = x$$

for all $x, y \in A$, $c \in \mathbb{F}$.

A Cuntz-Kreiger Γ -*family* is a family of mutually orthogonal projections $\{P_v \mid v \in V\}$ (*projections* are elements such that $P_v^2 = P_v = P_v^*$, and mutually orthogonal means that $P_u P_v = \Delta_{u,v}^{n-1} P_u$) and partial isometries $\{S_e \mid e \in E\}$ (*partial isometries* are elements such that $S_e^* S_e$ is a projection). that generate a *-algebra such that:

$$(CK1) \quad S_e^* S_e = P_{s_e} \text{ for all } e \in E$$

$$(CK2) \quad \sum_{te=v} S_e S_e^* = P_v \text{ for all } v \in V \text{ such that } t^{-1}(v) \neq \emptyset.$$

where Γ is a graph as before. One noticeable difference is that composition is right to left. For paths $p = p_1 p_2 \cdots p_n$ in Γ with $p_i \in E$, we define $S_p := S_{p_1} S_{p_2} \cdots S_{p_n}$. When $p = v$, a path of length 0, we define $S_p = P_v$.

Now we can define a graph C^* -algebra, denoted $C^*(\Gamma)$. It is a universal algebra such that for each C^* -algebra with the same Γ -family, there exists a $*$ -homomorphism from $C^*(\Gamma)$ to it such that $\{P_v \mid v \in V\}$ and $\{S_e \mid e \in E\}$ are both preserved pointwise. This universal algebra will not retain the same $\mathbb{Z}$ -grading that the $*$ -algebra generated by the Cuntz-Krieger Γ -family has (when one uses the analogous $\mathbb{Z}$ -grading placed on $L(\Gamma)$).

From the work of Gelfand-Naimark [22] and Segal [35], we know that $C^*(\Gamma)$ has a realization as a closed $*$ -subalgebra of bounded linear operator on some Hilbert space $\mathcal{H}$ [37], in fact:

$$C^*(\Gamma) = \overline{\text{span}}\{S_p S_q^* \mid p, q \in \text{Path}(\Gamma), sp = sq\}$$

As an example of a graph C^* algebra consider the Toeplitz digraph as before:

$$\Gamma : \begin{array}{ccc} & e & \\ & \curvearrowright & \\ \bullet & \xrightarrow{f} & \bullet \\ v & & w \end{array}$$

Let $\mathcal{H}$ be the Hilbert space of square summable sequences over $\mathbb{C}$ indexed by $\mathbb{N}$. Consider the action on $\mathcal{H}$ defined by:

$$\begin{aligned} (a_0, a_1, a_2, a_3, \dots) P_v &= (0, a_1, a_2, a_3, \dots) & (a_0, a_1, a_2, a_3, \dots) P_w &= (a_0, 0, 0, 0, \dots) \\ (a_0, a_1, a_2, a_3, \dots) S_e &= (0, a_2, a_3, a_4, \dots) & (a_0, a_1, a_2, a_3, \dots) S_f &= (a_1, 0, 0, 0, \dots) \\ (a_0, a_1, a_2, a_3, \dots) S_e^* &= (0, 0, a_1, a_2, \dots) & (a_0, a_1, a_2, a_3, \dots) S_f^* &= (0, a_0, 0, 0, \dots) \end{aligned}$$

When one completes this $*$ -algebra with respect to the operator norm, then you obtain $C^*(\Gamma)$.

2.3 Symbolic Dynamics, Topological Markov Chains and Perron-Frobenius Theory

Let $\mathcal{A}$ be a (finite) alphabet. A *full $\mathcal{A}$ -shift space* is the space $\mathcal{A}^{\mathbb{N}}$, with a *shift map* $\sigma : \mathcal{A}^{\mathbb{N}} \rightarrow \mathcal{A}^{\mathbb{N}}$ such that for all $(x_i)_{i \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$, we have $(\sigma x)_i = x_{i+1}$ for $i \geq 1$. We will call a subset of $\mathcal{A}^{\mathbb{N}}$ a *shift space* if it is invariant under σ . Here $\mathcal{A}^{\mathbb{N}}$ is a topological space with the product topology where $\mathcal{A}$ has the discrete topology, and σ is a continuous self map.

Words in the alphabet $\mathcal{A}$ are finite strings of elements of $\mathcal{A}$. We will be interested in *Shift of Finite Type* (SFT) spaces, which are shift spaces S defined by a finite list of *forbidden* words - that is, words that do not appear as a substring of any element of X . These space can also be described by a finite set of *allowed* words, where, given a set of allowed words, we can think of constructing elements of X letter by letter, checking against the list to see if each letter appended allows the end of the word to appear on the allowed list. An SFT is called *irreducible* if, for any two allowed words w_1, w_3 there is an allowed word w_2 such that the word $w_1 w_2 w_3$ is allowed.

We have the discrete topology on $\mathcal{A}$, and the product topology on $\mathcal{A}^{\mathbb{N}}$. Projection onto an appropriate coordinate separates any two distinct points in $\mathcal{A}^{\mathbb{N}}$, showing that $\mathcal{A}^{\mathbb{N}}$ (and hence any subset $\mathcal{A}^{\mathbb{N}}$) is totally disconnected. By Tychonoff's theorem, $\mathcal{A}^{\mathbb{N}}$ is compact if and only if $\mathcal{A}$ is finite. Moreover, $\mathcal{A}^{\mathbb{N}}$ is metrizable. For example, we can use the metric:

$$d(a_1 a_2 a_3 \cdots, b_1 b_2 b_3 \cdots) = \sum_{\{i \mid a_i \neq b_i\}} \frac{1}{2^i}.$$

The shift σ on $\mathcal{A}^{\mathbb{N}}$ is continuous. A subset of $\mathcal{A}^{\mathbb{N}}$ defined by a collection of forbidden words is closed and shift invariant.

We will look at $P_{\infty} := P_{\infty}(\Gamma)$, the space of infinite paths on the digraph Γ . Here, $\mathcal{A} = E$,

the set of arrows of the digraph Γ , and P_∞ is obtained by declaring the set $\{ef \mid te \neq sf\}$ as forbidden. P_∞ is a *Topological Markov Chain*, or a *1-step SFT*. This is because as we build the space, we need only examine the previous letter, or arrow, in order to determine if the next letter/arrow is an allowed successor.

The topology placed on P_∞ is the restriction of the product topology on $E^\mathbb{N}$. Notice that P_∞ is closed as it is defined by forbidden words. As a closed subset of a compact set, P_∞ is compact. It is also metrizable and totally disconnected.

Example 2.

$$P_\infty(\mathcal{Q}) = \{*\} \quad ; \quad P_\infty(\mathcal{G} \cdot \mathcal{Q}) = \text{Cantor set}.$$

Some combinatorial properties of the digraph Γ , equivalent to important algebraic properties of $L(\Gamma)$ (quoted earlier in section 2.1), are also detected by the cardinality of P_∞ :

1. $P_\infty = \emptyset$ if and only if Γ is acyclic.

When Γ has no sinks:

2. P_∞ is a finite set if and only if the cycles of Γ have no exits.
3. P_∞ is countable if and only if the cycles of Γ are pairwise disjoint.

$P_\infty(\Gamma)$ is also separable. This is because $Path(\Gamma)$ is countable, and hence so is the subset of (finite) paths that can be extended to some infinite path. Picking an infinite extension of such a finite path, we get a countable dense subset of P_∞ since $\{pP_\infty \mid p \in Path(\Gamma)\}$ is a basis for the topology of P_∞ (here pP_∞ is the set of all infinite paths with initial segment $p \in Path(\Gamma)$).

We can also endow P_∞ with a Borel probability measure via the *Perron-Frobenius Theorem* [24].

Given a real nonnegative $n \times n$ matrix A , we can define a digraph Γ_A where $V = \{1, 2, \dots, n\}$, and there is an arrow from i to j if $a_{i,j} > 0$. However, Γ_A has at most one arrow from i to j . Now A is *irreducible* (that is, there is a positive integer $k = k(i, j)$ such that

$(A^k)_{i,j} > 0$) if and only if Γ_A is strongly connected. If A is *stochastic* (that is, the sum of the entries of each row is 1), we get a probability distribution for a random walk on Γ .

Conversely, to define a random walk on a directed graph Γ , we assign a positive probability $p(e)$ to each arrow e , such that $\sum_{se=v} p(e) = 1$ for each nonsink $v \in V$. If $V = \{1, 2, \dots, n\}$ and Γ has no sinks, then we define the stochastic matrix $A = (a_{ij})$, where a_{ij} is the sum of $p(e)$ for e from i to j . Now $(A^k)_{i,j}$ is the probability of going from i to j in k steps, hence Γ is strongly connected if and only if A is irreducible.

Theorem (Perron-Frobenius). *Let A be an $n \times n$ real matrix A that is irreducible and stochastic. Then the largest eigenvalue of A is 1, which has multiplicity one. It has a unique left eigenvector $\mathbf{z}$ with each entry z_i positive and $\sum_{i=1}^n z_i = 1$.*

Proof. Consider the map A on $\Delta^{n-1} = \{(x_i)_{i=1}^n \mid x_i \geq 0, \sum_i x_i = 1\} \subseteq \mathbb{R}^n$ (where A acts on the right). This defines a self map on Δ^{n-1} , since for $\mathbf{x} \in \Delta^{n-1}$, we have $\sum_{i=1}^n \sum_{j=1}^n x_i a_{ij} = \sum_{i=1}^n x_i \sum_{j=1}^n a_{ij} = \sum_{i=1}^n x_i = 1$. By Brouwer fixed-point theorem, there exists a vector $\mathbf{z}$ such that $\mathbf{z}A = \mathbf{z}$. Thus A has an left eigenvector of eigenvalue 1.

This vector $\mathbf{z} = [z_1 \ z_2 \ \dots \ z_n]$ has at least one entry, say z_i , that is strictly positive since it is an element of Δ^{n-1} . Since $\mathbf{z}A = \mathbf{z}$, we have the formula $z_j = \sum_{i=1}^n z_i a_{ij}$. We also know that $\mathbf{z}A^k = \mathbf{z}$ for all $k \geq 1$. Recall that for $k = k(i, j)$, we have $(A^k)_{i,j} > 0$. As a consequence, z_j must be positive when z_i is positive. Thus, $\mathbf{z}$ is an element of the interior of Δ^{n-1} .

Suppose, by way of contradiction, there are two distinct eigenvectors of eigenvalue 1 for A , and call the second eigenvector $\mathbf{y}$. Then, if the sum of the coordinates of $\mathbf{y}$ is not 0, then consider $\frac{1}{\sum_i y_i} \mathbf{y}$ as an eigenvector of eigenvalue 1 whose entries sum to 1. If the sum of the coordinates of $\mathbf{y}$ is 0, then consider $\mathbf{y} + \mathbf{z}$ as an eigenvector of eigenvalue 1 whose entries sum to 1. Given an eigenvector of eigenvalue 1 whose entries sum to 1 that is not equal to a multiple of $\mathbf{z}$, say $\mathbf{u}$, consider

$$(\lambda \mathbf{z} - (1 - \lambda) \mathbf{u})A = \lambda \mathbf{z} - (1 - \lambda) \mathbf{u}$$

for all values of λ . This line must intersect the boundary of Δ^{n-1} for some values of λ . Earlier, we concluded that eigenvectors of A of eigenvalue 1 and whose coordinates sum to 1 cannot lie on the boundary of Δ^{n-1} , so this is a contradiction.

Now, let λ be the largest eigenvalue of A , for the right eigenvector of $\mathbf{y}$, as left eigenvalues are equal to right eigenvalues. We will assume, without loss of generality, that the largest entry of $\mathbf{y}$, say y_i is equal to 1 (this can be done taking a constant multiple of $\mathbf{y}$). Then we have:

$$\lambda = \lambda y_i = \sum_{j=1}^n a_{ij} y_j \leq \sum_{j=1}^n |a_{ij} y_j| \leq \sum_{j=1}^n a_{ij} \leq 1.$$

As we have already found a eigenvector of eigenvalue 1, this bound is sharp. □

Chapter 3

New Irreducible Representations

3.1 Generalized twisted rational Chen modules

We will now construct a module on which $L(\Gamma)$ acts. This will be a previously unknown generalization of rational Chen modules. For these new modules, we will consider the case where Γ contains a directed cycle. Let $C = c_0 \cdot c_1 \cdots c_n$ be a primitive closed path.

We define $\mathbb{F}P_C$ as a vector space over $\mathbb{F}$ with basis P_C , where P_C is the set of paths in $Path(\Gamma)$ that end at $w = sC$, but are not equal to a path qC for any path q .

Then $M_C := \mathbb{F}[x, x^{-1}] \otimes_{\mathbb{F}} \mathbb{F}P_C$ as a vector space.

We will define the action of $L(\Gamma)$ on M_C using pure tensors of M_C acted on by $V \sqcup E \sqcup E^*$ on the right and have the action extend linearly and multiplicatively. We denote a generic pure tensor (ignoring coefficients) in M_C by $x^m \otimes a_1 a_2 \cdots a_k$ where $a_1 a_2 \cdots a_k$ is in P_C ($a_i \in E$ or $a_1 \cdots a_k = w$).

$$(x^m \otimes a_1 a_2 \cdots a_k) \alpha = \begin{cases} x^m \otimes a_1 a_2 \cdots a_k & \alpha = s a_1 \\ x^m \otimes a_2 \cdots a_k & \alpha = a_1 \\ x^m \otimes \alpha^* a_1 \cdots a_k & \alpha \in E^*, s \alpha = s a_1 \\ x^{m-1} \otimes c_1 c_2 \cdots c_n & \alpha = c_0, a_1 \cdots a_k = w \\ x^{m+1} \otimes w & \alpha = c_0^*, a_i = c_i \text{ for } 1 \leq i \leq n, k = n \\ 0 & \text{otherwise} \end{cases}$$

This defines an action of $L(\Gamma)$ on M_C satisfying the path algebra relations as well as CK1 and CK2. This is clear for relations (V), (E) and (CK1). This is also true for relation (CK2) because given a pure tensor in M_C , at most one $e \in s^{-1}(v)$ will yield a nonzero result after its action. For this e , acting by the factor $e e^*$ is the same as acting by v .

We also have a left action of $\mathbb{F}[x, x^{-1}]$ on M_C via multiplication by Laurent polynomials.

We have the categories $\mathcal{M}_{\mathbb{F}[x, x^{-1}]}$ of left $\mathbb{F}[x, x^{-1}]$ modules and $\mathcal{M}_{L(\Gamma)}$ of right $L(\Gamma)$ modules. Using M_C , we define two functors. Firstly, there is a functor

$$\mathcal{F}_C : \mathcal{M}_{\mathbb{F}[x, x^{-1}]} \longrightarrow \mathcal{M}_{L(\Gamma)}$$

where $\mathcal{F}_C(X) = X \otimes_{\mathbb{F}[x, x^{-1}]} M_C$ for all objects X and $\mathcal{F}_C(f) = f \otimes id_{M_C}$ for morphisms f .

We also have, in the opposite direction,

$$\mathcal{G}_C : \mathcal{M}_{L(\Gamma)} \longrightarrow \mathcal{M}_{\mathbb{F}[x, x^{-1}]}$$

where $\mathcal{G}_C(X) = Hom^{L(\Gamma)}(M_C, X)$ and for $f : X \longrightarrow Y$, we have the map $\mathcal{G}_C(f)$, where

$$\text{Hom}^{L(\Gamma)}(M_C, X) \ni x \mapsto f \circ x \in \text{Hom}^{L(\Gamma)}(M_C, Y).$$

Recall that $\mathbb{F}[x, x^{-1}]$ is a PID, as it is a localization of the PID $\mathbb{F}[x]$. Thus, every finitely generated module X over $\mathbb{F}[x, x^{-1}]$ is equal to the sum of $\mathbb{F}[x] / (f(x)^k)$ where $f(x)$ is some irreducible nonconstant polynomial, or the zero polynomial. As both $\mathcal{G}_C$ and $\mathcal{F}_C$ respect direct sums, we will focus on the case where X is finitely generated.

We will call $\mathcal{F}_C(X) = M_{C,f}$ when $X = \mathbb{F}[x] / (f(x))$. Note that:

$$M_{f,C} \cong \left(\mathbb{F}[x, x^{-1}] / (f(x)) \right) \otimes_{\mathbb{F}} \mathbb{F}P_C$$

Theorem 2. $\mathcal{G}_C \circ \mathcal{F}_C$ is naturally isomorphic to the identity functor.

We will need a lemma:

Lemma 3. Given $\varphi \in \text{Hom}^{L(\Gamma)}(M_C, M_{C,f})$, φ is determined uniquely and completely by an element of $\mathbb{F}[x] / (f(x))$.

Proof. We want to know the set

$$\left\{ \sum_i h_i(x) \otimes p_i \parallel \sum_i h_i(x) \otimes p_i C^n C^{*n} = \sum_i h_i(x) \otimes p_i \text{ for all } n \in \mathbb{N} \right\}$$

because this is the set of possible images for $1 \otimes v$ under an $L(\Gamma)$ morphism (M_C is a cyclic $L(\Gamma)$ module generated by $1 \otimes v$). Given a particular element, let N be a number such that C^N is longer than p_i for all i . When we consider $\sum_i h_i(x) \otimes p_i C^N C^{*N}$, the only terms that will have a nonzero result are those where p_i is an initial segment of C^N . For such $p_i \in P_C$, this means that $(h_i(x) \otimes p_i) C^N = (h_i(x) \otimes v) q_i$ where $q_i = c_{k_i} \cdots c_n C^{N_i}$. If p_i has positive length, $k_i \neq 0$, and thus q_i is not a path starting with a power of C . (Suppose $c_k c_{k+1} \cdots c_n C^N = C^N c_0 \cdots c_{n-k}$. As

these are elements of $Path(\Gamma)$, we have $c_k \cdots c_n c_{n+k} = c_0 \cdots c_n$, where $c_{i+n+1} := c_i$. We have $c_i \cong c_j \pmod{n+1}$ and this contradicts the fact that C is primitive.) Thus $(h_i(x) \otimes v)q_i = 0$ for all i with p_i of positive length. As, $\sum_i h_i(x) \otimes p_i C^N C^{*N} = \sum_i h_i(x) \otimes p_i$, this means $\sum_i h_i(x) \otimes p_i = h(x) \otimes v$. $\square$

Proof of theorem. Consider the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ Hom^{L(\Gamma)}(M_C, X \otimes_{\mathbb{F}} M_C) & \xrightarrow{\mathcal{G}_C \circ \mathcal{F}_C(f)} & Hom^{L(\Gamma)}(M_C, Y \otimes_{\mathbb{F}} M_C) \end{array}$$

Note that $x \mapsto f(x) \mapsto \varphi_{f(x)}$ and $x \mapsto \varphi_x \mapsto f \otimes id_{M_C} \circ \varphi_x$. Note that, by the lemma, both maps are completely determined by a polynomial, and that polynomial is the first factor in the image of $1 \otimes v$. Let us call $\varphi_x(1 \otimes v) = h(x) \otimes v$. Then $\varphi_{f(x)}(1 \otimes v) = f(\varphi_x)(1 \otimes v) = f \circ h(x) \otimes v$ and $f \otimes id_{M_C} \circ \varphi_x(1 \otimes v) = f \circ h(x) \otimes v$. Thus the diagram commutes when we restrict to the category of finitely generated $\mathbb{F}[x, x^{-1}]$ modules.

The functor $\mathcal{G}_C$ preserves inclusion (if X is a $L(\Gamma)$ submodule of Y , then $Hom^{L(\Gamma)}(M_C, X)$ is a submodule of $Hom^{L(\Gamma)}(M_C, Y)$). Thus, if $\mathbb{F}[x, x^{-1}] / (f(x))$ is simple (which happens when f is irreducible), then so is $M_{f,C}$. $\square$

Note that in the case that C is an exclusive cycle (a cycle with no exits), we have $M_C = \mathbb{F}[x, x^{-1}] \otimes_{\mathbb{F}[x, x^{-1}]} (sC)L(\Gamma)$. This is because $CC^* = C^*C = sC$, so C and C^* are inverses of each other as they act on the left of $(sC)L(\Gamma)$ in this case.

Here is a general way to understand if a module you observe is a Chen module:

Let us define $\alpha = \alpha_0 \alpha_1 \alpha_2 \cdots$ as a regular infinite path in $L(\Gamma)$ ($\alpha_i \in E$, $t \alpha_i = s \alpha_{i+1} \forall i \in \mathbb{N}$). We will use the notion that an element m of a module M survives along α if $m \alpha_n$ is nonzero for all $n \in \mathbb{N}$.

We will define the set $P_{\infty}^m := \{\alpha \text{ an infinite path in } L(\Gamma) \mid m \text{ survives along } \alpha\}$ and often

put the tail equivalence (denoted $P_\infty^m / \sim$). We also define $P_\infty^M := \bigcup_{m \in M} P_\infty^m$.

Lemma 4. *For M a simple $L(\Gamma)$ module, $P_\infty^m / \sim = P_\infty^M / \sim$.*

Proof. Clearly $P_\infty^m \subseteq P_\infty^M$. Consider $\alpha \in E_\infty^{m'}$. As M is simple, there is $a \in L(\Gamma)$ such that $m\lambda = m'$. Thus:

$$m'\alpha_n = m \sum_{i=1}^k p_i q_i^* \alpha_n = m \sum_{i=1}^K p_i \alpha'_n$$

where $\alpha'_n = q_i^* \alpha_n$, $n \geq \max\{l(q_i)\}_{i=1}^k$ and $mp_i \alpha'_n \neq 0$ for $1 \leq i \leq K$. Thus we have K paths that are tail equivalent to α in P_∞^m . Thus $E_\infty^{m'} / \sim = P_\infty^m / \sim$, which yields that $P_\infty^m / \sim = P_\infty^M / \sim$. $\square$

Lemma 5. *For M simple, $P_\infty^m / \sim = \{[\alpha]\} \Rightarrow |P_\infty^m| < \infty$.*

Proof. Take a nonzero element $m \in M$. Look at P_∞^m . If, by way of contradiction, $|P_\infty^m| = \infty$, then there must be at least two nonidentical paths that m survives along, say p_0 and p'_0 . Since M is simple, $|E_\infty^{mp_0}|$ and $|E_\infty^{mp'_0}|$ are both infinite. One can continue in this way, forming two infinite paths that are not the same at each step, and therefore cannot be tail equivalent. This contradicts $P_\infty^m / \sim = \{[\alpha]\}$. $\square$

Corollary 6. *For M simple, $P_\infty^m / \sim = \{[\alpha]\} \Rightarrow$ there is a $p \in \text{Path}(\Gamma)$ such that $E_\infty^{mp} = \{\alpha'\}$ where α' is a truncation of α*

Proof. By the above, P_∞^m is finite. Look at the element of P_∞^m that takes the longest to become tail equivalent to α , say β . If $\beta_k \in \text{Path}(\Gamma)$ is the initial part of this infinitely long path that does not conform to α , then $E_\infty^{m\beta_k} = \{\alpha'\}$. $\square$

Lemma 7. *For M simple, $P_\infty^m / \sim = \{[\alpha]\} \iff M$ is a Chen Module.*

Proof. $\Leftarrow$ is clear by definition. For $\Rightarrow$, take any nonzero element of M . There is a path $p \in \text{Path}(\Gamma)$ such that mp survives only along a truncation of α (which, since we are working with equivalence classes, we can take this path and call it α'). Consider the module homomorphism from the Chen module to M that sends α to mp . This will be onto as M is simple, and one-to-one as Chen modules are simple. $\square$

Lemma 8. *Given a simple module M such that there is an $0 \neq m \in M$ where $mf(C^*) = 0$, ($f(x)$ is a polynomial in $\mathbb{F}[x]$, $f(0) = 1$), $P_\infty^m = \{C^\infty\}$.*

Proof. With the condition that $mf(C^*) = 0$, we can show that $mp = 0$ where $p \in \text{Path}(\Gamma)$ is anything that deviates from C^∞ . For $f(x) = \sum_{i=0}^n a_i x^i$ we can write $-a_n m C^{*n} - a_{n-1} m C^{*(n-1)} - \dots - a_1 m C^* = m$. We consider $mp = (-a_n m C^{*n} - a_{n-1} m C^{*(n-1)} - \dots - a_1 m C^*)p$. This may set some summands equal to zero, as it is assumed that p deviates from C^∞ . For the summands it does not set equal to zero, we consider the identity $-a_n m C^{*n} - a_{n-1} m C^{*(n-1)} - \dots - a_1 m C^* = m$, we get $-a_n m C^{*(n+k)} - a_{n-1} m C^{*(n-1+k)} - \dots - a_1 m C^{*(k+1)} = m C^{*k}$, which allows us to rewrite $m C^{*k}$ in terms of summands with strictly larger powers of C^* . We can inflate these powers of C^{*k} to be longer than any p that deviates from C^∞ . Hence, $mp = 0$.

Using a similar argument, consider C^k acting on m for some $k \in \mathbb{N}$. When we act on m where it has been rewritten with powers strictly larger than k . Then we get $m C^k = mg(C^*)$ for some polynomial g . This cannot equal zero, as $mg(C^*)C^{*k} = m$. $\square$

Corollary 9. *Given a simple module M such that there is a $f(x) \in \mathbb{F}[x]$ where $f(0) = 1$ and a $0 \neq m \in M$ such that $mf(C^*) = 0$, then M is a (generalized, twisted) rational Chen module.*

3.2 Extension of Generalized Twisted Chen Modules

Lemma 10. *Given a commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & E & \xrightarrow{g} & B & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \varphi & & \downarrow \text{id} & & \\ 0 & \longrightarrow & A & \xrightarrow{f'} & E' & \xrightarrow{g'} & B & \longrightarrow & 0 \end{array}$$

where the rows are short exact sequences, φ is an isomorphism.

Proof. Suppose $\varphi(e) = 0$ for $e \in E$. Then $g'\varphi(e) = 0$, which by commutativity implies that $g(e) = 0$. By exactness, $e \in \text{Im}(f)$. Since f is injective, $f^{-1}(e) = a$ and by commutativity, $\varphi f(a) = f'(a) = 0$. Since f' is injective, this means $a = 0$, so $f(a) = e = 0$.

As the diagram is commutative, φ is an injection.

Let $e' \in E$. Since g', g are both surjective, there is an $e \in E$ such $g(e) = g'(e')$. By commutativity, $g'\varphi(e) = g(e)$, so $g'\varphi(e) = g'(e')$. This means $\varphi(e) + x = e'$ where x is some element of $\text{Ker}(g')$. Since $\text{Ker}(g') = \text{Im}(f')$ by exactness, then $x = f(a)$ for some $a \in A$. By commutativity, $\varphi f(a) = f'(a)$. Thus, $\varphi(e + f(a)) = e'$.

Therefore, φ is also a surjection, and thus an isomorphism.

□

Notice that we only needed a surjective map from A to A , rather than an isomorphism.

Given a projective module P (the first step of a projective resolution of B and a surjection $\epsilon : P \longrightarrow B$, then given any module E and a surjective map $\pi : E \longrightarrow B$, we have an $\tilde{\epsilon}$ such that the following diagram commutes.

$$\begin{array}{ccc} & P & \\ \tilde{\epsilon} \swarrow & \downarrow \epsilon & \\ E & \xrightarrow{\pi} & B \end{array}$$

We will use this setup in two future diagrams.

Lemma 11. *Given a short exact sequence:*

$$0 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} B \longrightarrow 0$$

there is an extension of B by A that is isomorphic to E .

Proof. First, consider the following diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{a \mapsto (a,0)} & A \oplus P & \xrightarrow{(a,\alpha) \mapsto \alpha} & P \longrightarrow 0 \\
& & \downarrow id & & \downarrow \begin{bmatrix} \iota \\ \tilde{\epsilon} \end{bmatrix} & \nearrow \tilde{\epsilon} & \downarrow \epsilon \\
0 & \longrightarrow & A & \xrightarrow{\iota} & E & \xrightarrow{\pi} & B \longrightarrow 0
\end{array}$$

Clearly the top sequence is short exact. We will check that it is commutative on the squares.

For the left square: $a \mapsto (a, 0) \mapsto \iota(a) + \epsilon(0) = \iota(a)$ and $a \mapsto a \mapsto \iota(a)$

For the right square: $(a, \alpha) \mapsto \alpha \mapsto \epsilon(\alpha)$ and $(a, \alpha) \mapsto \iota(a) + \tilde{\epsilon}(\alpha) \mapsto \pi\iota(a) + \pi\tilde{\epsilon}(\alpha) = 0 + \epsilon(\alpha) = \epsilon(\alpha)$

As a consequence of this diagram, we get that $\begin{bmatrix} \iota \\ \tilde{\epsilon} \end{bmatrix}$ is a surjective map. We will call the kernel of this surjective map M . Note that $M = \{(a, \alpha) \in A \oplus P \mid \iota(a) = -\tilde{\epsilon}(\alpha)\}$. Thus, we get the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{a \mapsto [a,0]} & A \oplus P / M & \xrightarrow{[a,\alpha] \mapsto \epsilon(\alpha)} & B \longrightarrow 0 \\
& & \downarrow id & & \downarrow \begin{bmatrix} \iota \\ \tilde{\epsilon} \end{bmatrix} & & \downarrow id \\
0 & \longrightarrow & A & \xrightarrow{\iota} & E & \xrightarrow{\pi} & B \longrightarrow 0
\end{array}$$

We note that the top right map is well defined. This is because when $[a, \alpha] = [a + a', \alpha + \alpha']$, then $\iota(a') = -\tilde{\epsilon}(\alpha')$. This implies $\pi\iota(a') = \pi\tilde{\epsilon}(\alpha')$, so $0 = \epsilon(\alpha')$ (as $\pi\iota = 0$ and $\pi\tilde{\epsilon} = \epsilon$). Thus, $\epsilon(\alpha + \alpha') = \epsilon(\alpha)$. The top sequence is again short exact.

So, given a fixed projective module P which projects onto B via ϵ , we have $A \oplus P / M \cong E$ where M is calculated using $\tilde{\epsilon}$ (which uses the information of π) and ι .

□

Note that $M = \{(a, \alpha) \in A \oplus P \mid \iota(a) = -\tilde{\epsilon}(\alpha)\} = \{(a, \alpha) \in A \oplus P \mid a = -\iota^{-1}(\tilde{\epsilon}(\alpha))\}$ (this makes sense as ι is injective). Here, α is an element of the $\ker(\epsilon)$, as $0 = \pi\iota(a) = \pi\tilde{\epsilon}(\alpha) = \epsilon(\alpha)$. Let us then consider M as the graph of a homomorphism from $K := \ker(\epsilon)$ to A (where

the domain is the second summand and the range is the first summand). We will be more specific in our notation then:

$$M_\varphi = \{(a, k) \in A \oplus K \mid a = -\varphi(k)\}$$

for $\varphi \in \text{Hom}(K, A)$. Thus we have a correspondence between $\text{Hom}(K, A)$ and equivalence classes in $\text{Ext}(B, A)$. Both of these sets have algebraic structure - $\text{Hom}(K, A)$ is an additive group, as is $\text{Ext}(B, A)$ under Baer sum. We will show that this correspondence respects the additive structures of both. We will use the notation $+_{\text{Baer}}$ to denote the Baer sum and $E \times_B E'$ to denote the pullback with respect to B . Recall that the Baer sum is a quotient of the pullback with respect to the pushforward (using two SESs with the same start and end).

Theorem 3.1. *There is a correspondence between $\text{Hom}(K, A)$ as an additive group and $\text{Ext}(B, A)$, an additive group under Baer sum*

Proof.

$$\begin{array}{ccc}
(A \oplus P) \times_B (A \oplus P) & \xrightarrow{\quad\quad\quad} & A \oplus P \\
\downarrow & & \downarrow \\
(A \oplus P / M_{\varphi_1}) \times_B (A \oplus P / M_{\varphi_2}) & & A \oplus P / M_{\varphi_1 + \varphi_2} \\
\downarrow & & \\
(A \oplus P / M_{\varphi_1}) +_{\text{Baer}} (A \oplus P / M_{\varphi_2}) & &
\end{array}$$

The first pullback uses the map $(a, \alpha) \mapsto \epsilon(\alpha)$ on both summands. The second pullback uses the map $[a, \alpha] \mapsto \epsilon(\alpha)$ on both summands. The final Baer sum is with respect to the short exact sequence seen on the top of the last diagram.

The downward maps are all quotients, and the horizontal map does the following:

$$((a_1, \alpha_1), (a_2, \alpha_2)) \mapsto (a_1 + a_2 - \varphi_2(\alpha_1 - \alpha_2), \alpha_1 + \alpha_2)$$

Altogether, this demonstrates an isomorphism between

$$(A \oplus P / M_{\varphi_1}) +_{Baer} (A \oplus P / M_{\varphi_2})$$

and $A \oplus P / M_{\varphi_1 + \varphi_2}$.

Note that $0 \in \text{Hom}(K, A)$ corresponds to $A \oplus P / M_0 \cong A \oplus B$. This makes the correspondence between $\text{Hom}(K, A)$ and $\text{Ext}(B, A)$ a homomorphism. We wish to know the kernel of this isomorphism, i.e. for what $\varphi \in \text{Hom}(K, A)$ is $A \oplus P / M_\varphi \cong A \oplus B$. Suppose we have a isomorphism θ from $A \oplus P / M_\varphi$ to $A \oplus B$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
& & P & \hookrightarrow & A \oplus P & & \\
& & & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{a \mapsto (a,0)} & A \oplus P / M_\varphi & \xrightarrow{[a,\alpha] \mapsto \epsilon(\alpha)} & B \longrightarrow 0 \\
& & \downarrow id & & \downarrow \theta & & \downarrow id \\
0 & \longrightarrow & A & \hookrightarrow & A \oplus B & \xrightarrow{proj_2} & B \longrightarrow 0 \\
& & & & \downarrow proj_1 & & \\
& & & & A & &
\end{array}$$

Then the map from the top left P to the bottom center A is $p \mapsto proj_2(\theta[0, p])$, which we will call ψ is a homomorphism whose restriction to K is φ . Thus, φ extends to P . Suppose conversely that $\varphi : K \longrightarrow A$ extends to $\psi : P \longrightarrow A$. Then,

$$\theta[a, \alpha] = (a + \psi(\alpha), \epsilon(\alpha))$$

makes the above diagram commute.

So $\text{Ext}(B, A) \cong \text{coker}\{\text{Hom}(P, A) \longrightarrow \text{Hom}(K, A)\}$

Now, consider the following short exact sequences of modules:

$$A_{n-1} \hookrightarrow A_n \xrightarrow{\pi} B_n \quad A_n \hookrightarrow A_{n+1} \twoheadrightarrow B_{n+1}$$

$$K \xrightarrow{\kappa} P \xrightarrow{\epsilon} B_{n+1}$$

As before, P is a projective module.

Consider the sequence:

$$\begin{aligned} 0 \rightarrow \operatorname{Hom}(B_{n+1}, A_{n-1}) &\rightarrow \operatorname{Hom}(B_{n+1}, A_n) \rightarrow \operatorname{Hom}(B_{n+1}, B_n) \rightarrow \operatorname{Ext}(B_{n+1}, A_{n-1}) \\ &\rightarrow \operatorname{Ext}(B_{n+1}, A_n) \rightarrow \operatorname{Ext}(B_{n+1}, B_n) \rightarrow \operatorname{Ext}^2(B_{n+1}, A_{n-1}) \rightarrow \cdots \end{aligned}$$

We wish to check that the sequence is exact.

$0 \rightarrow \operatorname{Hom}(B_{n+1}, A_{n-1}) \rightarrow \operatorname{Hom}(B_{n+1}, A_n) \rightarrow \operatorname{Hom}(B_{n+1}, B_n)$ is exact, as this is the result of the functor $\operatorname{Hom}(B_{n+1}, _)$ acting on the first SES. We also know $\operatorname{Ext}(B_{n+1}, A_{n-1}) \rightarrow \operatorname{Ext}(B_{n+1}, A_n) \rightarrow \operatorname{Ext}(B_{n+1}, B_n) \rightarrow \operatorname{Ext}^2(B_{n+1}, A_{n-1}) \rightarrow \cdots$ is exact, as it is a result of the functor $\operatorname{Ext}^*(B_{n+1}, _)$ acting on the first SES. We wish to see if the map $f : \operatorname{Hom}(B_{n+1}, B_n) \rightarrow \operatorname{Ext}(B_{n+1})$ such that $\theta \xrightarrow{f} A_{n-1} \oplus P / M_{\bar{\theta}}$ is a map in the sequence above that will combine these sequences to be one long exact sequence, where $\tilde{\theta}\epsilon$ exists (as P is projective) and makes the below diagram commute, and $\bar{\theta} = \tilde{\theta}\epsilon \circ \iota$.

$$\begin{array}{ccccc} K & \xrightarrow{\iota} & P & \xrightarrow{\epsilon} & B_{n+1} \\ \downarrow \bar{\theta} & & \downarrow \tilde{\theta}\epsilon & & \downarrow \theta \\ A_{n-1} & \hookrightarrow & A_n & \xrightarrow{\pi} & B_n \end{array}$$

We will first check if $\ker(f) = \operatorname{img}(\operatorname{Hom}(B_{n+1}, A_n) \rightarrow \operatorname{Hom}(B_{n+1}, B_n))$. We know from the previously established isomorphism $(\operatorname{Ext}(B_{n+1}, A_{n-1}) \cong \operatorname{coker}\{\operatorname{Hom}(P, A_{n-1}) \rightarrow \operatorname{Hom}(K, A_{n-1})\})$ that if $\bar{\theta}$ is the restriction of the map $\hat{\theta} : P \rightarrow A_{n-1}$, then it is in $\ker(f)$.

$$\begin{array}{ccccc}
K & \xrightarrow{\iota} & P & \xrightarrow{\epsilon} & B_{n+1} \\
\downarrow \bar{\theta} & \swarrow \hat{\theta} & \downarrow \tilde{\theta}\epsilon & & \downarrow \theta \\
A_{n-1} & \hookrightarrow & A_n & \xrightarrow{\pi} & B_n
\end{array}$$

Consider $(\tilde{\theta}\epsilon - \hat{\theta})\epsilon^{-1} = \varphi \in \text{Hom}(B_{n+1}, A_n)$. Notice that φ is well defined, as for $k \in \ker(\epsilon)$, $(\tilde{\theta}\epsilon - \hat{\theta})(k) = (\tilde{\theta}\epsilon - \hat{\theta})(\iota^{-1}k) = 0$ by the commutativity of the lower left hand triangle in the diagram (this triangle is commutative because of the commutativity of the top triangle and the injectivity of ι).

Thus, $\ker(f) \subseteq \text{img}(\text{Hom}(B_{n+1}, A_n) \rightarrow \text{Hom}(B_{n+1}, B_n))$

Given $\theta \in \text{img}(\text{Hom}(B_{n+1}, A_n) \rightarrow \text{Hom}(B_{n+1}, B_n))$, we have the following diagram:

$$\begin{array}{ccccc}
K & \xrightarrow{\iota} & P & \xrightarrow{\epsilon} & B_{n+1} \\
& \searrow \bar{\theta} & \swarrow \tilde{\theta}\epsilon & & \downarrow \theta \\
& & A_{n-1} & \hookrightarrow & A_n \xrightarrow{\pi} B_n
\end{array}$$

where $\tilde{\theta}\epsilon$ is the lift of $\pi\theta\epsilon$. Consider $\tilde{\theta}\epsilon - \theta\epsilon$. We know $\pi\tilde{\theta}\epsilon = \pi\theta\epsilon$, so $\tilde{\theta}\epsilon - \theta\epsilon(p) \in \ker(\pi) = A_{n-1}$ for all $p \in P$. Also, $(\tilde{\theta}\epsilon - \theta\epsilon)(\iota k) = \tilde{\theta}\epsilon(\iota k) = \bar{\theta}(k)$ Therefore, $\theta \in \ker(f)$.

Now, let us check that $\text{img}(f) = \ker(\text{Ext}(B_{n+1}, A_{n-1}) \rightarrow \text{Ext}(B_{n+1}, A_n))$.

Using our previously established isomorphism, in order to determine if $\text{img}(f) \subseteq \ker(\text{Ext}(B_{n+1}, A_{n-1}) \rightarrow \text{Ext}(B_{n+1}, A_n))$, we need to know when, given a $\theta \in \text{Hom}(B_{n+1}, B_n)$ (and the resulting $\bar{\theta}$ that determines a homomorphism from K to A_{n-1}), can we extend $\bar{\theta}$ to a map $\hat{\theta}$ from P to A_n . This is a direct consequence of this previous diagram:

$$\begin{array}{ccccc}
K & \xrightarrow{\iota} & P & \xrightarrow{\epsilon} & B_{n+1} \\
\downarrow \bar{\theta} & & \downarrow \hat{\theta} & & \downarrow \theta \\
A_{n-1} & \hookrightarrow & A_n & \xrightarrow{\pi} & B_n
\end{array}$$

We also need to establish that if we have $\bar{\theta}$, a homomorphism from K to A_{n-1} that extends to $\hat{\theta}$ from P to A_n , whether it is the image of some $\theta \in \text{Hom}(B_{n+1}, B_n)$.

$$\begin{array}{ccccc}
K & \xrightarrow{\iota} & P & \xrightarrow{\epsilon} & B_{n+1} \\
\downarrow \bar{\theta} & & \downarrow \hat{\theta} & & \downarrow \theta \\
A_{n-1} & \hookrightarrow & A_n & \xrightarrow{\pi} & B_n
\end{array}$$

where $\theta = \pi \bar{\theta} \epsilon^{-1}$. This is well defined as, although ϵ has kernel $\iota(K)$, we have $\pi \hat{\theta} \iota(K) = \pi \bar{\theta}(K) = 0$

Any $\varphi \in \text{Hom}^{L(\Gamma)}(M_C, A)$ is determined uniquely by an element a in $\{a \in A \mid a(wI_C) = 0\}$ where

$$wI_C := \text{span}\{pq^* \in L(\Gamma) \mid p, q \in \text{Path}(\Gamma), sp = w, \exists N \in \mathbb{N} \text{ s.t. } (C^*)^N p = 0\}$$

It is clear why this condition is necessary. It is sufficient because given any $L(\Gamma)$ module A , any $a \in \{a \in A \mid a(wI_C) = 0\}$ defines a morphism by determining its output on $1 \otimes v$.

Note that $\{a \in Av \mid aC^n C^{*n} = a \text{ for all } n \in \mathbb{N}\} \supseteq \{a \in A \mid a(wI_C) = 0\}$. Since $\varphi(1 \otimes v) = \varphi(1 \otimes v)C^n C^{*n} = \varphi(1 \otimes v)v$, it is clear why we have this containment. We have the reverse containment because if, by way of contradiction, there is $a \in \{a \in Av \mid aC^n C^{*n} = a \text{ for all } n \in \mathbb{N}\}$ where $awI_C \neq 0$, then for all $n \in \mathbb{N}$, we have $aC^n C^{*n} wI_C \neq 0$, which is a contradiction of the definition of wI_C .

□

3.3 Reducing a Digraph to Strongly Connected Γ with Multiple Cycles

We can do the reduction algorithm on Γ that eliminates loopless nonsinks [28]. The output of this reduction algorithm is not unique, but it is a Morita equivalence. Thus we have changed the algebra but preserved the module category - in particular, our simple modules under this functor are still simple.

We will fix a simple module M over some $L(\Gamma)$.

Consider V_M , the support of M . Since M is a simple module, we can also denote this set as $V_{\rightsquigarrow w}$ for any w in the unique minimal equivalence class in V_M (the partial order is given by $v \leq w$ when there is a path from w to v). Here is the reasoning: for any vertex v not in V_M , all vertices less than v are also not in V_M , and if $te \in V \setminus V_M$ for all e such that $se = v$, then $v \in V \setminus V_M$. Thus, $V \setminus V_M$ is a hereditary saturated set. We also see that for any w , a minimal element of V_M , all other elements of its equivalence class are in V_M . If not, then the fact that $V \setminus V_M$ is hereditary means that w would not be in V_M , a contradiction. We have that this minimal equivalence class is unique, as if $m_1 \in Mv_1$, $m_2 \in Mv_2$ are two nonzero elements for v_1, v_2 in different minimal equivalence classes. By simplicity, there exists $\sum_i p_i q_i^* \in L(\Gamma)$ such that $m_1 \sum_i p_i q_i^* \in L(\Gamma) = m_2$. Thus, $tp_i \in V_M$, and tp_i is less than v_1 and v_2 , which is a contradiction. Also, it is clear that any vertices greater than w are in V_M , as for any path p where $tp = w$, we have p^* as an injective map from Mw to Msp .

We can use the isomorphism mentioned earlier to get:

$$L(\Gamma) / (V \setminus V_M) \cong L(\Gamma_{\rightsquigarrow w})$$

where $\Gamma_{\sim w}$ is the full subgraph of Γ on $V_{\sim w}$. Thus, we have changed the algebra again, but to one that is isomorphic.

We will denote $\bar{w} := \sum_{v \in [w]} v$, where $[w]$ is the set of vertices in the same equivalence class as w . We will also denote $\Gamma_{[w]}$ as the full subgraph on $[w]$.

Lemma 12. *The functor*

$$- \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)$$

gives an equivalence of categories between $L(\Gamma_{[w]}) \cong L(\Gamma_{[w]})$ modules M and $L(\Gamma)$ modules N generated by $N\bar{w}$. The adjoint functor

$$\text{Hom}^{\bar{w}L(\Gamma)\bar{w}}(\bar{w}L(\Gamma), -) = -\bar{w}$$

is the functor between $L(\Gamma)$ modules N generated by $N\bar{w}$ and $L(\Gamma_{[w]})$ modules M .

Proof.

$$M \xrightarrow{- \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)} M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma) \xrightarrow{-\bar{w}} (M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)) \bar{w}$$

We need to show $(M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)) \bar{w} \cong M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)\bar{w}$. We can show this by viewing $M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)$ as

$$(M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)\bar{w}) \oplus (M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)(1 - \bar{w}))$$

Notice that projection onto the first summand gives a section of $-\bar{w}$. Thus, $(M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)) \bar{w} \cong M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)\bar{w}$. Then, we have $M \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)\bar{w} \cong M$.

For the reverse direction, consider N an $L(\Gamma)$ module such that $N\bar{w}L(\Gamma) = N$.

$$N \xrightarrow{-\bar{w}} N\bar{w} \xrightarrow{- \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)} N\bar{w} \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)$$

By the evaluation map $n\bar{w} \otimes x \mapsto n\bar{w}x$ and the assumption that $N\bar{w}L(\Gamma) = N$, the composition of these maps are the identity. $\square$

Notice that if you have a $L(\Gamma)$ module N , the image of N under the composition of these functors is the submodule of N that is isomorphic to $N\bar{w}L(\Gamma)$. This can be seen by viewing N as $N\bar{w}L(\Gamma) \oplus N(1 - \bar{w})L(\Gamma)$. We see that $N\bar{w}L(\Gamma)\bar{w} \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)$ is isomorphic to $N\bar{w}L(\Gamma)$ via the evaluation map. We can also see $N(1 - \bar{w})L(\Gamma)\bar{w} \otimes_{\bar{w}L(\Gamma)\bar{w}} \bar{w}L(\Gamma)$ is trivial as the only elements of $(1 - \bar{w})L(\Gamma)$ are linear combinations of dual paths that start at vertices not in $[w]$. However, all paths of $\bar{w}L(\Gamma)\bar{w}$ start and end at elements of $[w]$. By CK1, this is isomorphic to the zero module.

Since $M = M\bar{w}L(\Gamma)$ when M is simple, we may restrict attention to $L(\Gamma_{[w]})$.

After the previous reductions, we are now considering a simple $L(\Gamma)$ module, where Γ is strongly connected and has full support $V = V_M$.

There are three possibilities for w :

- w is a sink
- w is on a single cycle.
- $\Gamma_{[w]}$ contains multiple cycles

For the first case, $L(\Gamma) = \mathbb{F}$, and M is any one dimensional vector space over $\mathbb{F}$. In the module over $L(\Gamma)$ before we did our reductions on Γ , we have that $M = wL(\Gamma)$, a projective simple module (in fact all projective simple modules over $L(\Gamma)$ are of this form). The dimension of $wL(\Gamma)$ may or may not be finite. It is finite dimensional iff w there are no cycles leading to w [27].

In the second case, $L(\Gamma) = \mathbb{F}[x, x^{-1}]$. Since this is a PID (and if M is simple, it means that M is finitely generated), we know all modules over it. These modules are the Chen modules. All modules (in the case of disjoint cycles) are Chen modules [12]. These modules can be further twisted and generalized as was done in the previous section.

Our new simple $L(\Gamma)$ modules appear in the case that $[w]$ contains multiple cycles.

3.4 Spaces of step functions and infinite product paths

Our new modules are submodules of *step functions* (linear combinations of indicator functions) on the space of infinite paths in Γ , denoted by

$$P_\infty(\Gamma) = P_\infty = \{e_1 e_2 e_3 \cdots \mid te_i = se_{i+1} \forall i \in \mathbb{N}_{>0}, e_i \in E\}.$$

The topology placed on this is the restriction of the product topology on $E^\mathbb{N}$ (where E has the discrete topology). Basic open sets are determined by a path $p = p_1 p_2 \cdots p_n \in \text{Path}(\Gamma)$ (where $p_i \in E$):

$$pP_\infty := \{e_1 e_2 e_3 \cdots \in P_\infty \mid e_i = p_i \text{ for } 1 \leq i \leq n\}.$$

Recall from the preliminaries that P_∞ is a closed, compact space that is Borel measurable.

We would like to have a probability measure on this space. This can be done by making a weight function from $E \sqcup V$ to $\mathbb{R}$ that assigns to each edge in E a weight that is strictly positive, with the condition that $\sum_{se=v} w(e) = 1$. With these weights in mind, we can create the matrix:

$$W = \left(\sum_{\substack{se=v \\ te=u}} w(e) \right)_{v,u \in V}$$

Thus we have that the sum of each row is 1.

We can use Perron-Frobenius Theory to find the left eigenvector with positive coordinates and eigenvalue 1. We make the choice of eigenvector unique by demanding that $\sum_{v \in V} a_v = 1$.

We define $w(v) := a_v$

This weight function extends to a measure μ on basic open sets (which are also closed). For $p = p_1 p_2 \cdots p_n \in \text{Path}(\Gamma)$ (where $p_i \in E$), we define $\mu(pP_\infty) := w(sp_1)w(p_1) \cdots w(p_n) := w(sp)w(p)$.

We then extend this measure to all closed sets S of P_∞ via:

$$\mu(S) = \lim_{n \rightarrow \infty} \mu \left(\bigcap_n \{p \cdot P_\infty \mid p \in Pr_n(S)\} \right) = \lim_{n \rightarrow \infty} \sum_{p \in Pr_n(S)} \mu(p \cdot P_\infty)$$

where $Pr_n(e_1 e_2 \cdots) = e_1 e_2 \cdots e_n$. Since the sequence is monotone decreasing and bounded below by 0, it converges.

We wish to establish which linear combinations of step functions are trivial. To that end consider:

$$\begin{aligned} \mathbb{F} (2^X \setminus \{\emptyset\}) &\longrightarrow \mathbb{F}^X \\ \sum_{i=1}^n \lambda_i X_i &\longmapsto \sum_i^n \lambda_i \mathbb{1}_{X_i} \end{aligned}$$

where 2^X is the power set of X . Notice that multiplication of indicator functions corresponds to intersection of the corresponding sets. We can see that $A + B - A \cup B$ is in their kernel when A and B are disjoint sets. Consider an arbitrary element of the kernel. Using the previous relation, we can rewrite any $\sum_i^n \lambda_i X_i$ as an equivalent sum using disjoint sets, such as

$$\sum_{\substack{a_i \in \{0,1\} \\ \bigcap_{i=1}^n X_i^{a_i} \neq \emptyset}} (a_1 \lambda_1 + \cdots a_n \lambda_n) \bigcap_{i=1}^n X_i^{a_i}$$

where $X_i^1 := X_i$ and $X_i^0 := X \setminus X_i$. These sets are all disjoint, and the sum is mapped to the trivial function if and only if each coefficient is 0.

The action of $L(\Gamma)$ on $\mathbb{F}^{P_\infty}$ is given by:

for $v \in V$	$(\mathbb{1}_S)v = \mathbb{1}_{vS}$	where $vS = \{x \in S \mid sx = v\}$
for $e \in E$	$(\mathbb{1}_S)e = \mathbb{1}_{e^*S}$	where $e^*S = \{x \mid ex \in S\}$
for $e^* \in E^*$	$(\mathbb{1}_S)e^* = \mathbb{1}_{eS}$	where $eS = \{ex \mid x \in S, sx = te\}$

When $\mathbb{F} = \mathbb{C}$, we can adjust the action of $L(\Gamma)$ on $\mathbb{F}^{P_\infty}$ to ensure that both v and ee^* act

as partial isometries:

$$\begin{array}{lll}
\text{for } v \in V & (\mathbb{1}_S)v = \mathbb{1}_{vS} & \text{where } vS = \{x \in S \mid sx = v\} \\
\text{for } e \in E & (\mathbb{1}_S)e = \frac{w(te)}{w(se)w(e)} \mathbb{1}_{e^*S} & \text{where } e^*S = \{x \mid ex \in S\} \\
\text{for } e^* \in E^* & (\mathbb{1}_S)e^* = \frac{w(se)w(e)}{w(te)} \mathbb{1}_{eS} & \text{where } eS = \{ex \mid x \in S, sx = te\}
\end{array}$$

All relations of the Leavitt Path Algebra hold:

$$\begin{array}{ll}
\text{for } v, u \in V & (\mathbb{1}_S)vu = (\mathbb{1}_{vS})u = \mathbb{1}_{uvS} = \mathbb{1}_{\delta_{u,v}uS} = (\mathbb{1}_S)\delta_{u,v}v \\
\text{for } e \in E & (\mathbb{1}_S)se e = (\mathbb{1}_{seS})e = \mathbb{1}_{e^*seS} = \mathbb{1}_{e^*S} = (\mathbb{1}_S)e \\
\text{for } e \in E & (\mathbb{1}_S)ete = (\mathbb{1}_{e^*S})te = \mathbb{1}_{te e^*S} = \mathbb{1}_{e^*S} = (\mathbb{1}_S)e \\
\text{for } e, f \in E & (\mathbb{1}_S)e^*f = (\mathbb{1}_{eS})f = \mathbb{1}_{f^*eS} = \mathbb{1}_{\delta_{f,e}tfS} = (\mathbb{1}_S)\delta_{e,f}te \\
\text{for } v \text{ (nonsink)} \in V & \sum_{se=v} (\mathbb{1}_S)ee^* = \sum_{se=v} (\mathbb{1}_{e^*S})e^* = \sum_{se=v} \mathbb{1}_{ee^*S} = \mathbb{1}_{vS} = (\mathbb{1}_S)v
\end{array}$$

The last relation is true as $ee^*S = \{x \in S \mid x = ex'\}$. These sets are disjoint for each $e \in E$, thus the indicator function of the sum corresponds to the indicator function of their union. Thus, $\bigcup_{se=v} ee^*S = \{x \in S \mid x = ex'\} = vS$.

This action extends to step functions, which are functions that are linear combination of indicator functions of closed sets S on P_∞ . We will consider the indicator functions of closed sets, as they are measurable.

However, when we have indicator functions of closed sets, we also have indicator functions of *locally closed sets*, which are intersections of open sets and closed sets. We also have indicator functions of *constructible sets*, which are unions of locally closed sets. This is detailed in the following lemma.

Before we detail the proof of the lemma, the following facts are useful:

1. In general, $\mathbb{1}_{A \cap B} = \mathbb{1}_A \cdot \mathbb{1}_B$.
2. The intersection of two locally closed sets is a locally closed set - note when U_i is open and F_i is closed for $i \in \{0, 1\}$, we have:

$$(U_0 \cap F_0) \cap (U_1 \cap F_1) = (U_0 \cap U_1) \cap (F_0 \cap F_1).$$

Lemma 13. *The following are equivalent:*

1. $f = \sum_i \lambda_i \mathbb{1}_{S_i}$, where S_i are nonempty, constructible sets.
2. $f = \sum_i \lambda_i \mathbb{1}_{S_i}$, where S_i are nonempty, locally closed sets.
3. $f = \sum_i \lambda_i \mathbb{1}_{S_i}$, where S_i are nonempty, closed sets.
4. $f = \sum_i \lambda_i \mathbb{1}_{S_i}$, where S_i are nonempty, disjoint, constructible sets.
5. $f = \sum_i \lambda_i \mathbb{1}_{S_i}$, where the image of f is finite and each S_i is nonempty and constructible.
6. $f = \sum_{f^{-1}(\lambda_i)} \lambda_i \mathbb{1}_{f^{-1}(\lambda_i)}$, where the collection of sets $\{f^{-1}(\lambda_i)\}_{\lambda_i \in \mathbb{F}}$ is locally finite and each S_i is nonempty and constructible.

Proof.

(1) $\implies$ (2) We have $f = \sum_i \lambda_i \mathbb{1}_{S_i}$, where S_i are constructible sets. Consider that each constructible set S_i is the union of finitely many locally closed sets A_k . Note that

$$\bigcup_{k=1}^n A_k = \bigcup_{\substack{a_k \in \{0,1\} \\ \bigcap_{k=1}^n A_k^{a_k} \neq \emptyset}} \left(\bigcap_{k=1}^n A_k^{a_k} \right)$$

where $A_k^1 = A_k$, and $A_k^0 = P_\infty \setminus A_k$.

The compliment of a locally closed set is a linear combination of indicator functions of locally closed sets. For U an open set, F a closed set:

$$\mathbb{1}_{P_\infty \setminus (U \cap F)} = \mathbb{1}_{P_\infty \setminus U} + \mathbb{1}_{P_\infty \setminus F} - \mathbb{1}_{(P_\infty \setminus U) \cap (P_\infty \setminus F)}$$

Since the intersection of locally closed sets is a locally closed set, and intersection of sets corresponds to multiplication of their indicator functions, we have successfully rewritten f as a linear combination of indicator functions of locally closed sets.

(2) $\implies$ (3) We have $f = \sum_i \lambda_i \mathbb{1}_{S_i}$, where S_i are locally closed sets. Note that for U an open set, F a closed set:

$$\mathbb{1}_{U \cap F} = \mathbb{1}_F - \mathbb{1}_{F \cap (P_\infty \setminus U)}.$$

We have thus successfully rewritten f as a linear combination of indicator functions of closed sets.

(3) $\implies$ (4) We have $f = \sum_i \lambda_i \mathbb{1}_{S_i}$, where S_i are closed sets. Consider the fact that we can partition the union of the S_i into disjoint sets locally closed sets:

$$\bigcup_{\substack{a_i \in \{0,1\} \\ \bigcap_{i=1}^n S_i^{a_i} \neq \emptyset}} \left(\bigcap_{i=1}^n S_i^{a_i} \right)$$

where $S_i^1 = S_i$, and $S_i^0 = P_\infty \setminus S_i$. Then, we can rewrite f as:

$$\sum_{\substack{a_i \in \{0,1\} \\ \bigcap_{i=1}^n S_i^{a_i} \neq \emptyset}} (a_1 \lambda_1 + \cdots a_n \lambda_n) \mathbb{1}_{\bigcap_{i=1}^n S_i^{a_i}}$$

(4) $\implies$ (5) is trivial.

(5) $\implies$ (6) Consider $f = \sum_i \lambda_i \mathbb{1}_{S_i}$, where the image of f is finite and each S_i is constructible. We can, without loss of generality, rewrite f as a linear combination of indicator functions of disjoint sets (using (1) $\implies$ (2) $\implies$ (3) $\implies$ (4)). Then, let $X_\lambda = \bigcup \{S_i \mid \lambda_i = \lambda\}$. These X_λ are constructible, and there are finitely many of them

(as Image of f is finite). We can then rewrite f as:

$$f = \sum_{\lambda} \lambda \mathbb{1}_{X_{\lambda}}$$

(6) $\implies$ (1) is trivial. □

Lemma 14. *If $f = \sum_i \lambda_i \mathbb{1}_{S_i} \neq 0$ is linear combination of indicator functions with S_i distinct nonempty closed sets, then there exists a path p such that $fp = \lambda \mathbb{1}_S$ with S closed, $S \neq \emptyset$.*

Proof. We can rewrite f as $\sum_i \mu_i \mathbb{1}_{X_i}$ a linear combination of indicator functions of disjoint nonempty constructible sets by our lemma. As the sets are disjoint, we can find an $n \in \mathbb{N}$ and a $p \in Pr_n(X_1)$ such that $p \notin Pr_n(X_i)$, $2 \leq i \leq n$. When we act on f with p , then fp only has output 0 and μ_1 . Consider how when p acts on $\sum_i \lambda_i \mathbb{1}_{S_i}$ with S_i distinct, if fp must be the indicator of a disjoint union of closed sets in order to have one output other than 0. As a finite union of closed sets is closed, $fp = \lambda \mathbb{1}_S$ with S closed, $S \neq \emptyset$. □

Thus, all simple modules in the space of step functions are generated by a single step function.

Then the question remain of which step functions generate simple modules.

S is *defined by a set of forbidden words* if infinite paths in S have no paths in the set of forbidden paths appears as a sub-segment. You may similarly think of S as being *defined by allowed paths* (where allowed paths have no forbidden path as a sub-segment). For S defined by a set of forbidden words, we say that S is *irreducible* that for any two allowed paths x and z , there is an allowed path y such that xyz is a valid path.

Lemma 15. *If $(\mathbb{1}_S)v \neq 0$ for irreducible S , then for any allowed path p such that $tp = v$, $(\mathbb{1}_S)p = \lambda(\mathbb{1}_S)v$.*

Proof. To understand this, consider S as a set defined by allowed words. The set of infinite paths that start with p and those that start with tp are in 1 – 1 correspondence. □

Lemma 16. *If $(\mathbb{1}_S)v \neq 0$ for irreducible S , then there is an element x of $\mathbb{F}\Gamma$ such that $(\mathbb{1}_S)vx = \mathbb{1}_S$.*

Proof. For $u \in V$ there is an allowed word p_u such that $sp_u = v$, $tp_u = u$ (this is due to the irreducible condition). Thus, $\sum_{u \in V} \frac{w(tp)}{w(u)w(p)} p_u$ is an element of $\mathbb{F}\Gamma$ that sends $(\mathbb{1}_S)v$ to $\mathbb{1}_S$. $\square$

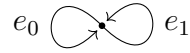
Theorem 17. *All irreducible nonempty $S \subset P_\infty$ define indicator functions $\mathbb{1}_S$ whose associated cyclic modules are simple.*

Proof. Consider $\mathbb{1}_S \sum_i p_i q_i^*$, where $\sum_i p_i q_i^* \in L(\Gamma)$ such that $\mathbb{1}_S \sum_i p_i q_i^* \neq 0$. We can find a path r longer than any of the q_i such that $\mathbb{1}_S \sum_i p_i q_i^* r \neq 0$. By this, we obtain $\lambda \mathbb{1}_S tr \neq 0$. By our corollary, there is an element $x \in \mathbb{F}\Gamma$ (scaled appropriately) that will send $\lambda \mathbb{1}_S tp$ to $\mathbb{1}_S$. Thus, we can go from any generic element of the cycloc module back to the generator $\mathbb{1}_S$ via the action of $L(\Gamma)$. Thus, the module is simple. $\square$

Chapter 4

Other Simple Modules over $L(1, 2)$

We will construct modules that are not Chen modules over $L(1, 2)$ (a construction that can easily be modified for $L(1, n)$ where $n \geq 2$). To remind the reader, the underlying graph Γ is:



4.1 Papillon

We will consider a quiver representation that satisfies the isomorphism condition, and demonstrate why this is not a Chen module.

Let the vector space at the single vertex be

$$X = \{ (a_i)_{i \in \mathbb{N}} \mid (a_i)_{i \in \mathbb{N}} \text{ is } 2^k\text{-periodic for some } k \in \mathbb{N} \}$$

The edges, e_0 and e_1 respectively, will act in the following manner:

$$(a_0, a_1, a_2, \dots) e_0 = (a_0, a_2, a_4, \dots)$$

$$(a_0, a_1, a_2, \dots) e_1 = (a_1, a_3, a_5, \dots)$$

Notice that this module satisfies the isomorphism condition. Therefore, it is a Leavitt module, and:

$$(a_0, a_1, a_2, \dots)e_0^* = (a_0, 0, a_1, 0, \dots)$$

$$(a_0, a_1, a_2, \dots)e_1^* = (0, a_0, 0, a_1, \dots)$$

We will call this module Papillon. You may recall a very similar module from the introduction to this thesis. The action is exactly the same ($e_i \longleftrightarrow x_i$, $e_i^* \longleftrightarrow y_i$) but the space X is markedly different from the space of finite sequences.

For any $m \in M$, where M is an $L(\Gamma)$ module, we define:

$$A_m := \{p \in \text{Path}(\Gamma) \mid p = p_1 p_2 \dots p_n, \ m(sp)p_0 \dots p_{n-1} \neq 0, \ mp = 0, \ \text{where } p_i \in E\}$$

Lemma 18. *For Chen modules, A_m has infinitely many elements when $m \neq 0$, unless we have a rational Chen Module defined by a cycle with no exit.*

Proof. For any nonzero $m \in M$, M a Chen Module, m is a linear combination of infinite tail equivalent paths (this means that for any set $\{\alpha_1, \dots, \alpha_n\}$ of tail equivalent infinite paths, there is a k_{α_i} for each α_i such that all paths in the set are the same at and beyond the edge of index k_{α_i} . For any $m = \sum \alpha$, without loss of generality, let k_{α_1} be the largest of the k_{α_i} . There exists a path p of length k_{α_1} such that $mp \neq 0$ (by the repeated application of (CK2) exactly k_{α_1} times). We can see that A_{mp} has infinitely many elements - it consists of all paths $\alpha_{k_{\alpha_1}+1} \alpha_{k_{\alpha_1}+2} \dots \alpha_{k_{\alpha_1}+N} e$ where $N \in \mathbb{N}$ and $e \in E$, $e \neq \alpha_{k_{\alpha_1}+N+1}$ (this is possible as the infinite path is not on a cycle without exits). As $A_{mp} \subseteq A_m$, this shows that A_m also has infinitely many elements. $\square$

This is not a property shared by the Papillon over $L(1, 2)$ described above - given any nonzero $x \in X$, any edge that does not annihilate x will divide the length of its period by 2. This means that $|A_x| \leq k$ where 2^k is the smallest period of x . It is also clear that Papillon

is not a rational Chen module on a cycle with no exit, given that our Γ has exits for any rational path. Papillion is the same as the module generated by the indicator function for P_∞ on Γ . The correspondence is clear when you have the constant sequence correspond to $\mathbb{1}_{P_\infty}$.

4.2 Modules with the action of Leavitt defined by polynomials

We want to obtain new representations for $L(\Gamma)$ acting on:

1. $\mathbb{Z}$ -indexed sequences
2. Laurent polynomials ($\mathbb{F}[x^{-1}, x]$)
3. Polynomials
4. $x^{-1}\mathbb{F}[x^{-1}]$

We will relate the actions of (1) and (2) with the correspondence

$$\underline{f} = \begin{cases} f_k & -m \leq k \leq l \\ 0 & \text{else} \end{cases} \leftrightarrow f(z) = \sum_{k=-m}^l f_k x^k$$

. We can consider (3) and (4) as a restriction of (2).

We start with a polynomial $m_0(z) = \sum_{i=0}^n a_i x^i$, with the following conditions:

1. $a_0 \neq 0$
2. $\lambda = \sum_{i=0}^n a_i^2 \neq 0$
3. $\sum_{i-j=k} a_i a_j = 0$ for k even, $k \neq 0$

As a consequence of the last condition, the degree n is odd. We will also define $m_1(z) = z^n m_0(-z^{-1})$. For a Laurent polynomial $f(z)$, the action is as follows:

$$f(z) v = f(z)$$

$$f(z) e_i^* = \lambda^{-1} m_i(z) f(z^2) \text{ for } i = 0, 1$$

$$f(z) e_i = \frac{1}{2} \left(m_i(z^{-1/2}) f(z^{1/2}) + m_i(-z^{-1/2}) f(-z^{1/2}) \right) \text{ for } i = 0, 1$$

On $\mathbb{Z}$ -indexed sequences, the action is:

$$\underline{f} v = \underline{f}$$

$$\underline{f} e_0^* = \begin{cases} \lambda^{-1} \sum_{k=0}^{(n-1)/2} f_{i-k} a_{2k} & i \text{ is even} \\ \lambda^{-1} \sum_{k=0}^{(n-1)/2} f_{i-k} a_{2k+1} & i \text{ is odd} \end{cases}$$

$$\underline{f} e_1^* = \begin{cases} \lambda^{-1} \sum_{k=0}^{(n-1)/2} -f_{i-k} a_{n-2k} & i \text{ is even} \\ \lambda^{-1} \sum_{k=0}^{(n-1)/2} f_{i-k} a_{n-(2k+1)} & i \text{ is odd} \end{cases}$$

$$\underline{f} e_0 = (\dots, \sum_{k=0}^n f_{2i-k} a_k, \dots)$$

$$\underline{f} e_1 = (\dots, \sum_{k=0}^n f_{2i-k} a_{n-k}, \dots)$$

The conditions on m_0 ensures that the relations of the Leavitt path algebra hold.

Even though the action of e_i may seem problematic, you always get a Laurent polynomial or $\mathbb{Z}$ -indexed sequence due to cancellation of terms. Another way to describe the action

is: e_i^* upsamples (double all exponents) and multiplies by m_i ; e_i multiplies by $\bar{m}_i$ (where $\bar{m}_i(x) = m_i(x^{-1})$) and then downsamples (halve all exponents and get rid of non integer powers). To verify (CK1) and (CK2), it is handy to have the identities:

$$m_i(x)\bar{m}_j(x) + m_i(-x)\bar{m}_j(-x) = 2\lambda\delta_{ij}$$

$$\bar{m}_0(x)m_0(x) + \bar{m}_1(x)m_1(x) = 2\lambda$$

$$\bar{m}_0(x)m_0(-x) + \bar{m}_1(x)m_1(-x) = 0$$

All of these identities are a consequence of having $\begin{bmatrix} m_0(x) & m_0(-x) \\ m_1(x) & m_1(-x) \end{bmatrix}$ be almost a "unitary" matrix, so that :

$$\begin{bmatrix} m_0(x) & m_0(-x) \\ m_1(x) & m_1(-x) \end{bmatrix} \begin{bmatrix} \bar{m}_0(x) & \bar{m}_1(x) \\ \bar{m}_0(-x) & \bar{m}_1(-x) \end{bmatrix} = \begin{bmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{bmatrix}$$

$$\begin{bmatrix} \bar{m}_0(x) & \bar{m}_1(x) \\ \bar{m}_0(-x) & \bar{m}_1(-x) \end{bmatrix} \begin{bmatrix} m_0(x) & m_0(-x) \\ m_1(x) & m_1(-x) \end{bmatrix} = \begin{bmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{bmatrix}$$

To check (CK1) for Laurent polynomials:

$$\begin{aligned} f(z)e_i^*e_j &= (\lambda^{-1}m_i(z)f(z^2))e_j \\ &= \lambda^{-1}(m_i(z^{1/2})f(z)m_j(z^{-1/2}) + m_i(-z^{1/2})f(z)m_j(-z^{-1/2})) \\ &= \lambda^{-1}\frac{1}{2}(m_i(z^{1/2})\bar{m}_j(z^{1/2}) + m_i(-z^{1/2})\bar{m}_j(z^{1/2}))f(z) \\ &= \lambda^{-1}\frac{1}{2}2\lambda\delta_{ij}f(z) \\ &= \delta_{ij}f(z) \end{aligned}$$

To check (CK2):

$$\begin{aligned}
f(z)(e_0 e_0^* + e_1 e_1^*) &= \frac{1}{2} (m_0(z^{-1/2})f(z^{1/2}) + m_0(-z^{-1/2})f(-z^{1/2})) e_0^* \\
&\quad + \frac{1}{2} (m_1(z^{-1/2})f(z^{1/2}) + m_1(-z^{-1/2})f(-z^{1/2})) e_1^* \\
&= \frac{1}{2} \lambda^{-1} (m_0(z^{-1})f(z)m_0(z) + m_0(-z^{-1})f(-z)m_0(z)) \\
&\quad + \frac{1}{2} \lambda^{-1} (m_1(z^{-1})f(z)m_1(z) + m_1(-z^{-1})f(-z)m_1(z)) \\
&= \frac{1}{2} \lambda^{-1} f(z) (\bar{m}_0(z)m_0(z) + \bar{m}_1(z)m_1(z)) \\
&\quad + \frac{1}{2} \lambda^{-1} f(-z) (\bar{m}_0(-z)m_0(z) + \bar{m}_1(-z)m_1(z)) \\
&= \frac{1}{2} \lambda^{-1} f(z)(2\lambda) + \frac{1}{2} \lambda^{-1} f(-z)(0) = f(z)
\end{aligned}$$

The action on $\mathbb{Z}$ -indexed sequences is analogous.

We now have a way of generating representations of $L(1, 2)$. Let's look at the representation that is defined by $m_0 = 1 + x$ (and thus $m_1 = 1 - x$). Regardless of whether you look at this in the space of Laurent Polynomials or bi-infinite 2^k periodic sequences, if you want to produce simple representations, you find that you need to restrict to one of three spaces: infinite sequences (indexed by natural numbers), $\mathbb{F}[x]$, or $x^{-1}\mathbb{F}[x^{-1}]$. For the action on infinite sequences, it is helpful to look at the basis of rows of the infinite Hadamard matrix constructed recursively:

$$H_0 = \begin{bmatrix} 1 \end{bmatrix} \text{ and } H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$$

Using the action on $\underline{f} = (\dots, f_0, f_1, f_2, \dots, f_{2^k-1} \dots)$, which simplifies as:

$$\underline{f} e_0^* = \frac{1}{2} (\dots, f_0, f_0, f_1, f_1, f_2, f_2, \dots, f_{2^k-1}, f_{2^k-1}, \dots)$$

$$\underline{f}e_1^* = \frac{1}{2}(\dots, f_0, -f_0, f_1, -f_1, f_2, -f_2, \dots, f_{2^k-1}, -f_{2^k-1}, \dots)$$

$$\underline{f}e_0 = (\dots, f_0 + f_1, f_2 + f_3, f_4 + f_5 \dots, f_{2^k-2} + f_{2^k-1}, f_0 + f_1, \dots)$$

$$\underline{f}e_1 = (\dots, f_0 - f_1, f_2 - f_3, f_4 - f_5 \dots, f_{2^k-2} - f_{2^k-1}, f_0 - f_1, \dots)$$

We get that that rows go to scalar multiples of rows under the action of e_0^* and e_1^* . This allows us to see that we can obtain all 2^k periodic sequences (as rows of the Hadamard transform form a basis of 2^k periodic sequences). As all sequences can be reduced to the constant sequence (e_0, e_1 divide the period of any sequence by 2, and they cannot both annihilate a given element), and the constant sequence survives (up to constant multiple) only along the infinite path $e_0 e_0 e_0 \dots$, this is a Chen Module (one can see from a future lemma that the feature of an element only surviving along one path uniquely defines a Chen Module).

On polynomials, the action is markedly different. This module is simple because all nontrivial polynomials can be reduced by repeated action of e_0, e_1 to a nontrivial constant polynomial. By repeated action of $e_0^* + e_1^*, e_0^* - e_1^*$, we are able to take x^n and obtain x^{2n} or x^{2n+1} . We are able to obtain all monomials of power greater than or equal to 0 this way. This module is the same as Papillon - 1 corresponds to $(1, 1, 1, 1, \dots)$ and monomials correspond to the rows of the infinite Hadamard transform.

Lastly, the action of $L(1, 2)$ on $x^{-1}\mathbb{F}[x^{-1}]$ is a nontrivial twist of the Papillon. We will show that all twists by the gauge action are nontrivial.

Lemma 19. *For all $(a, b) \in \mathbb{F}^\times \times \mathbb{F}^\times$, Papillon twisted by the gauge action of (a, b) is a distinct simple module up to isomorphism.*

Proof. We will check that the only thing that stabilizes Papillon is $(1, 1)$. We will consider the twisted action of $L(1, 2)$ on Papillon, where $\tilde{e}_0 = ae_0$ and $\tilde{e}_1 = be_1$, for $(a, b) \in \mathbb{F}^\times \times \mathbb{F}^\times$.

Consider that $(1, 1, 1, 1, \dots)$ is fixed under the action of e_0 and e_1 . Scalar multiples of this sequence are the only ones fixed under the action of e_0 and e_1 . However, there is no nontrivial constant sequence fixed by $\tilde{e}_0$ and $\tilde{e}_1$, with the exception of when $(a, b) = (1, 1)$. Thus, the action is faithful. $\square$

The action of $L(1, 2)$ on $x^{-1}\mathbb{F}[x^{-1}]$ is a nontrivial twist of the Papillon, by the scalars $(1, -1)$.

Other modules can be constructed by considering the action defined $m_0 = 1 + x^{2n+1}$, $m_1 = 1 - x^{2n+1}$ on Laurent polynomials ($n \in \mathbb{N}$). On monomials, the action simplifies as follows:

$$x^k e_0 = \begin{cases} x^{k/2} & k \text{ is even} \\ x^{(k-2n-1)/2} & k \text{ is odd} \end{cases}$$

$$x^k e_1 = \begin{cases} x^{k/2} & k \text{ is even} \\ -x^{(k-2n-1)/2} & k \text{ is odd} \end{cases}$$

$$x^k e_0^* = x^{2k} + x^{2k+2n+1}$$

$$x^k e_1^* = x^{2k} - x^{2k+2n+1}$$

Given this action we have this decomposition of $\mathbb{F}[x, x^{-1}]$ into these simple modules:

$$\mathbb{F}[x, x^{-1}] = \mathbb{F}[x^{2n+1}] \oplus x^{-2n-1} \mathbb{F}[x^{-2n-1}] \bigoplus_O \left(\bigoplus_{k \in \tilde{O}} \mathbb{F} x^k \right)$$

Where O is a nontrivial orbit of $\langle 2 \rangle$ (the multiplicative group generated by 2 in $\mathbb{Z}/(2n+1)\mathbb{Z}$), and $\tilde{O}$ is the preimage of O under the map from $\mathbb{Z}$ to $\mathbb{Z}/(2n+1)\mathbb{Z}$. This is because the action of e_0, e_0^*, e_1, e_1^* on a x^k yields terms whose degree is either $2k, 2k + 2n + 1, k/2$, or

$(k - 2n - 1)/2$. All of the algebraic operations involved (multiplying or dividing by 2, adding or subtracting $2n + 1$) leave invariant the orbits of $\langle 2 \rangle$ in $\mathbb{Z}/(2n + 1)\mathbb{Z}$.

As far as why each piece is simple - notice that any Laurent polynomial under the repeated action of e_0 and e_1 becomes a polynomial of reduced degree and increased order until the order is greater than or equal to $-2n - 1$ and the degree is less than or equal to 0. It just so happens that 1 and x^{-2n-1} are fixed under the action of e_0, e_1 . Repeated application of e_0, e_1 (such that the action does not yield the 0 polynomial) to a polynomial of order strictly greater than $-2n - 1$ and degree less than 0 keeps multiplying the powers of each term by $1/2$, and adjusting to where each term is strictly between $-2n - 1$ and 0 - thus giving a basis of polynomials of terms with powers in the same orbit of $\langle 2 \rangle$. As 2 and 2^{-1} are units in $\mathbb{Z}/(2n + 1)\mathbb{Z}$, all elements of the orbit are achieved through repeated application of e_0, e_1 .

While $\mathbb{F}[x^{2n+1}]$ and $x^{-2n-1}\mathbb{F}[x^{-2n-1}]$ are recognizable as Papillon and the Papillon twisted by $(1, -1)$ (this via the vector isomorphism $x^k \mapsto x^{k(2n+1)}$), the modules indexed by the orbits of $\langle 2 \rangle$ in $\mathbb{Z}/(2n + 1)\mathbb{Z}$ are different. It is clear that these modules are not Chen, as monomials are all nonzero under the action of e_0 and e_1 forever. At the same time, these modules are not Papillon- this is clear when you consider the O always have even and odd elements. Look at the action of e_0, e_1 on odd powered monomials. If there was a polynomial such that it was fixed (up to scalar multiple) under that action of e_0, e_1 , it would need to have all powers between $-2n - 1$ and 0. The polynomial must have all even or odd degree (as e_0 and e_1 differ only by a scalar on even or odd power terms - but not on polynomials with mixed even and odd power terms). This cannot happen, as all orbits of $\langle 2 \rangle$ have even and odd numbers:

Lemma 20. *All orbits of $\langle 2 \rangle$ in $\mathbb{Z}/(2n + 1)\mathbb{Z}$ have even and odd numbers*

Proof. Given any even element of an orbit, if you multiply by 2 enough, you get something larger than $2n + 1$. When you mod by $2n + 1$, you will get something odd. If you start with something odd instead, multiplication by 2 will either yield an even number (if your starting number was less than or equal to n) or an odd number less than n (if your starting number was greater than or equal to $n + 1$). When in the latter case, multiply by 2 again, giving an

even number. □

There is no element fixed up to scalar by e_0, e_1 . Thus, these modules are not the same as Papillon (where the constant sequence is fixed). They cannot be any module generated by an indicator function on P_∞ , as we have just shown that this module is not the one generated by $\mathbb{1}_{P_\infty}$.

Thus, we have constructed a module which is neither a Chen module, nor a module generated by an indicator function on P_∞ .

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