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THEORY OF THE CAPITAL DECISION IN

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PREFACE

Several years ago, the writer, then employed as a planning engineer by a large natural-gas transmission and distribution company, became aware of the possibility that the regulated public utility might rationally make investments that do not minimize costs. This possibility has its origin in the fact, peculiar to regulated firms, that cost minimization need not maximize the wealth of the owners of the firm. Accordingly, the regulated firm--given the manner of regulation which has become institutionalized by American public utility regulatory bodies--may logically choose some mix of operating and capital costs that does not result in minimal total costs to produce required output. The writer has observed utility operating personnel who demonstrated various degrees of awareness of this possibility by the observation, "So the investment doesn't save money--it still goes into rate base." The writer is aware of no overt nor general disregard for capital expenditures that results in the firm being calculatedly more capital intensive than is "socially desirable." Yet some discerning people, who were within the observation of the writer, knew that even for minimum-cost conditions, a dollar expended for plant and equipment was more valuable to the owners than a dollar expended for operations and maintenance. This knowledge, or "intuition," had a way of obliquely influencing the solutions of capital-budgeting problems without ever being marshalled into some tractable form--even in theory.

This study is addressed to the problem of determining how, in theory, the regulated firm may rationally make its capital-budgeting (or asset-acquisition) decisions. If the regulated firm's return--and profit--is a function of investment in plant and equipment, then it is plausible to argue that the firm might well use more capital under regulation than it would if not subject to regulation, assuming that output in both cases is the same. If this is true, then what are the constraints that hold in check this propensity for greater use of capital? In this study, it is argued that the constraints are: (1) The demand function facing the firm and (2) the production function which describes how the firm transforms its inputs into output.

The possibility that the regulated firm might be more capital intensive than minimum-cost conditions require has been raised by several observers in recent years. Averch and Johnson stated that the regulated firm can be expected to use more capital than the unregulated firm would use for the same output.[1] Moreover, Averch and Johnson argued that

[1]Harvey Averch and Leland M. Johnson, "Behavior of

the regulated firm would treat its cost of capital for investment in plant as though it were cheaper than its exterior price, by the amount of the rate of pure profit of the firm. Wellicz argued that regulation would lead to inefficient allocation of resources in his study of peak-load pricing for natural-gas pipeline companies.[2] Klevorick discussed a measure of efficiency of input allocation.[3] Westfield developed a persuasive, intriguing argument that the public utility (as opposed to its customers) was not injured--indeed, was profitably aided--by the admitted overpricing by General Electric, Westinghouse, and others in the sale of generating equipment several years ago.[4] Shepherd also saw the possibility of overinvestment by the regulated firm, but also suggested that the customer's insistence on continuous quality service might actually require a more capital-intensive allocation of inputs.[5] Shepherd alone brings to bear a constraint by equating average cost of investment to

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the Firm Under Regulatory Constraint," American Economic Review, Vol. LII, No. 6, pp. 1052-1069.

[2]S.H. Wellicz, "Regulation of Natural-Gas Pipeline Companies: An Economic Analysis," Journal of Political Economy, Vol. LXIX, No. 2, pp. 30-43.

[3]Alvin K. Klevorick, "The Graduated Fair Return: A Regulatory Proposal," American Economic Review, Vol. LVI, No. 2, p. 478.

[4]Fred M. Westfield, "Regulation and Conspiracy," American Economic Review, Vol. LV, No. 2, pp. 424-433.

[5]William G. Shepherd, "Regulatory Constraints and Public Utility Investment," Land Economics, Vol. XLIII, No. 3, pp. 348-355.

average revenue of investment; he assumed that the cost of capital would increase as more of it is used.

In this study, the cost of capital is held constant. Admittedly, Shepherd's point is well taken and a function that describes the increasing cost of capital certainly deserves treatment as a parameter in a theoretical inquiry into capital decisions of the regulated firm. In Chapter III, the cost of capital is treated, even if indirectly, as a factor in capital decisions of the regulated firm. This treatment is incidental to development of the "Lambda" function, which primarily concerns the marginal rate of technical substitutions of inputs. In the writer's view, however, the demand and production functions are more decisive as constraints than an increasing-cost-of-capital function. This study, then, basically concerns the formulation of a model for the regulated firm to describe an optimal allocation of inputs in the face of these two constraints.

In Chapter I, the neoclassical minimum-cost approach to capital decisions is summarized. Current practice of the regulated firm as it adapts real-world cost data to the neoclassical prescription for allocation of inputs is also reviewed. The suggestion is pointedly made that the neoclassical solution (or direct adaptations of it) is not optimal for the regulated firm.

In Chapter II, a model is developed to describe optimal behavior of the regulated firm. Two homogeneous inputs, capital and labor, and a homogeneous output are assumed. A rate of return greater than the cost of capital is assumed. At first, capital is treated as indestructible, but this assumption is in turn relaxed. Chapter II details how the principal constraints, the demand and production functions, are incorporated into the model. The optimal capital decision reflects equilibrium--an equilibrium interplay of inputs, output, and price of output. The model is then examined to show how the equilibrium solutions for capital, labor, output, and price respond to changes in variables within the model. Through Chapter III, and even to a lesser extent in Chapter IV, a study of the changes in the variables is methodologically an exercise in comparative statics.

In the latter part of this investigation, the model is adapted to dynamic conditions. Chapter IV is an inquiry into replacement decisions of the regulated firm in a setting of increasing costs, as the plant ages and technology improves.

Later in Chapter V, the model is adapted to growth, capital deterioration, a range of returns to scale, and changes in price-quantity elasticity. Finally, the model is adapted further to encompass output fluctuations in a sinusoidal pattern, thus accommodating the load-factor problem.

The writer has made but few comments as to the efficiency of the regulated firm and has, for the most part, deliberately refrained from stating welfare implications of the theory of capital decisions under regulation. Much could be said, and deserves to be said, regarding welfare aspects of capital decisions of the regulated firm. But if sound theory really is developed, then and only then, can the welfare problem be defined sharply to permit the subject to receive quantitative rather than qualitative treatment. The writer does not pass judgment on the decisions that lead to greater capital intensity under regulations. This behavior is viewed as "rational," nothing more.

As to the reality of the model, the writer is aware of the possible criticism of homogeneous inputs and homogeneous output. Furthermore, the problem of defining a demand function for public-utility output has been, and still is, a formidable task. Greenhut, without actually developing a model, has discussed the problem of "simple unrealistic theory" versus "complex realistic theory" in relation to the problem of the behavior of the firm under regulation.[6]

Despite the occasionally involved algebra and differentiation, the theory that attends development of the model in this study is simple and straight-forward. The

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[6] M.L. Greenhut, "On the Question of Realism in Economic Theory and the Regulation of Public Utilities," Land Economics, Vol. XLIII, No. 3, pp. 260-267.

writer takes no position as to which of the two camps cited by Greenhut is correct: "the more and more realism school" or the opposition ". . . [which is] in favor of the unrealistic approach in the apparent belief that the simple unrealistic theory opens up the greater number of important vistas and new horizons for economists in general." Greenhut's bias in favor of realism, even complex realism, is clear; but simplicity is not necessarily to be equated to unrealism. The writer is aware of additions and embellishments that would make the model developed in this study more realistically complex, but surely Greenhut would agree that complex realism is an evolutionary process; and, hopefully, the model of this study is an adequate, worthwhile beginning.

TABLE OF CONTENTS

	Page
PREFACE	iv
LIST OF TABLES	xii
LIST OF ILLUSTRATIONS	xiii
Chapter	
I. PROJECT ECONOMY FOR THE REGULATED FIRM	1
II. A STATIC MODEL FOR THE REGULATED FIRM	36
III. ANALYSIS OF THE STATIC MODEL FOR THE REGULATED FIRM	77
IV. REPLACEMENT THEORY AND THE REGULATED FIRM	134
V. CAPITAL REQUIREMENTS OF A DYNAMIC MODEL OF THE REGULATED FIRM	175
VI. POTENTIALITIES FOR FUTURE ELABORATION OF THE MODEL	227
BIBLIOGRAPHY	235
APPENDIX	243

LIST OF TABLES

Table		Page
1.1.	Illustration of Capital Recovery Costs in the Case of a Constantly Diminishing Base for a \$5000 Depreciable Asset.	22
1.2.	Illustration of Capital Recovery Costs in the Case of Fixed Investment Amount.	23
1.3.	Present Worths of Capital Recovery Charges of Tables 1.2 and 1.1	24
1.4.	Analysis of Present Worth of Capital Recovery Costs in the Case of a Diminishing Base (Recast of Table 1.1).	25
1.5.	Depreciation Annuity in Per Cent of Capital Investment for a Range of Probable Lives and Types of Dispersion for a Return of Six Per Cent	30-31
4.1.	Determination of the Economic Life of Equipment that Costs \$1000, that has an Operating Cost Gradient of \$100 a year, and is Burdened with an Interest Rate of 10 Per Cent.	140
4.2.	Numerical Example of Equated Rates of Return Assuming Allowable Return of 6.5 Per Cent and Linear Depreciation for a Range of Asset Lives and Salvage Values	163
5.1(a)	Optimal Size of Plant for Three Planning Periods for Various Returns to Scale (Growth, J=O; Price-Elasticity, σ=.5).	193
5.1(b)	Optimal Size of Plant for Various Returns to Scale (Growth Rate=.07; Price-Elasticity=.5)	194
5.2.	Optimal Size of Plant for Average of Planning Periods, Growth Rates, and Price-Quantity Elasticities.	198

LIST OF ILLUSTRATIONS

Figure		Page
1.1.	Total Costs of a Given Output Produced with Various Combinations of Labor and Capital.	9
1.2.	Graphic Illustration of Gross Revenue Required from a Project to Earn Return, s.	15
1.3.	Revenue Requirements for a Minimum Acceptable Rate of Return, r, and a Fair or Allowable Rate of Return, s.	18
1.4.	Iowa-Type Retirement Dispersion Curves . .	29
1.5.	The Sum of All Fixed Costs, Expressed as a Function of the Present Value of Investment	34
2.1.	Zero excess Profit Function for the Regulated and the Unregulated Firm . . .	44
2.2.	The Regulated and Unregulated Profit Functions in Relation to the Production Function of the Firm	46
2.3.	Profile of the Total-Net-Revenue-Product Surface along Path of Optimum Mix of Labor and Capital as Capital Increases .	48
2.4.	Diagram of Arc of Effective Public-Utility Regulation	48
2.5.	The Path of Short-Run Allocation of Inputs for the Regulated Firm Facing a Log-Linear Demand Function	50
2.6.	The Complete Static Model for the Regulated Firm Facing a Log-Linear Demand Function	62

Figure		Page
2.7.	The Complete Static Model for the Regulated Firm Facing a Linear Demand Function.	71
2.8.	Comparison of Paths of Short-Run Allocation of Inputs for the Linear and Log-Linear Demand Functions	75
3.1.	The Establishment of New Equilibrium Capital Requirements as a Result of a Positive Change in the Demand Function. .	100
3.2.	The Establishment of New Equilibrium Capital Requirements as a Result of an Increase in the "Cost of Service" to Cover an Increase in Return, s.	100
3.3.	Expansion Paths of the Regulated and Unregulated Firms	103
3.4.	Illustration of Excess Profit Resulting from Substitution of Variable Inputs for Fixed Inputs.	110
3.5.	The Interrelation of the λ function and Output, Return, $TNRP_K$ and Profit and Regulation.	125
4.1.	Masse's "Retirement in the Pure Form" . . .	151
4.2.	Optimal Depreciation Schedules for the Regulated Firm under Assumptions of Replacement and Non-Replacement of Plant	156
4.3.	The Effect of the Time-Salvage Relation upon the Uniform Equivalent Total Revenue Required to Earn Return, s. . . .	160
4.4.	Equated Rates of Return for a Range of Asset Lives, Salvage Values and with a Time-Salvage Function xx Imposed.	162
4.5.	Determination of Optimal Service Live of Plant for the Regulated and Unregulated Firm.	169

Figure		Page

4.6. Illustration of the Chain of Replacement Resulting from Increasing Costs, Increasing Technology, and Optimal Depreciation Schedules of the Regulated Firm. 173

5.1. Illustration of Optimal Planning Period Resulting in Optimal Size of Plant. . . . 195

5.2. Illustration of Optimal Planning Period Resulting in Optimal Size of Plant, Given Arbitrary Assumptions as Price-Quantity Elasticity, Returns to Scale . . 200

5.3. The Effect of Price-Quantity Elasticity on the Optimal Size of Plant, for Two Different Returns to Scale. 203

5.4. Effect of the Rate of Growth on the Optimal Size of Plant for Three Planning Periods and Two Price-Quantity Elasticities. 204

5.5. Comparisons of Sizes of Plant for the Regulated and the Unregulated Firm for Various Conditions of Growth, Elasticity, and Returns to Scale. 210

5.6. The Decrease in Inputs for the Regulated Firm as a Result of Sinusoidal Output . . 223

THEORY OF THE CAPITAL DECISION

IN THE REGULATED FIRM

CHAPTER I

PROJECT ECONOMY FOR THE REGULATED FIRM

In recent years, project economy for the regulated firm has become an exercise in measuring and comparing the least revenues required to assure a given rate of return on investment in proposed projects.[1] Conventionally, the relative attractiveness of alternative projects of unregulated firms is assessed by comparisons of present worths, of internal rates of return, or simply by comparisons of minimum costs required for a given output. In the regulated firm, return on investment is fixed; by contrast, in the unregulated firm, it results from a complex of management decisions given the environment of the firm. In fact, the fixed rate of return is the most distinguishing characteristic of the regulated firm.[2] Accordingly, the regulated firm seeks to

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[1] Project economy, a common term in the literature of "engineering economy," is defined as the evaluating procedure for selecting the most attractive of alternative projects competing for adoption in the capital budget of the firm.

[2] Other characteristics associated with regulation and

minimize revenue required--in order to cover total costs--subject to the constraint that a rate of return, s, be earned. This is not the simple minimization problem that it at first appears to be--if inputs are to be allocated optimally.

Conventional planning practice for the regulated firm has been to choose the project for which the present value[3] of revenue collected from its customers is minimized, although permitting the firm to earn a "fair" or allowable return on investment in plant and equipment. The similarities of, and differences between, the minimum-cost approach of the unregulated firm and the minimum-revenue-requirements approach of the regulated firm to project selection is at once apparent in the two functions:

$$C = wL + rWK \qquad (1.1)$$

$$R = wL + sWK \qquad (1.2)$$

where C is the total cost of the project in the unregulated firm,

R is the total revenue required by the project in the regulated firm,

w is the wage rate of labor or, more generally, the price of variable inputs.

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the regulated firm, e.g., price determination, are introduced in later chapters.

[3]Alternately, the uniform equivalent may be minimized. The uniform equivalent is an even flow of funds, which when discounted over a planning period, yields the present worth referred to here.

W is the price of a unit of capital

r is the "rental cost" of capital, expressed in per cent

s is the required, or "fair", rate of return, as allowed by the regulatory authorities

L is the quantity of labor, or more generally, of variable inputs

K is the quantity of capital, or more generally, of fixed inputs.

In either case the firm seeks to minimize the function appropriate to it. If the regulated firm measures its revenue requirements such that only the minimum acceptable rate of return is earned, then functions (1.1) and (1.2) are identical. Implicit here is the identical allocation of inputs for the two functions. Regulation, however, affects optimal allocation of inputs. The differences and similarities of the cost function (1.1) and the revenue function (1.2) are developed more fully later to show that for optimal conditions, the two functions can lead to different, perhaps importantly different, capital decisions.

The Neoclassical Minimum-Cost Case

Neoclassical analysis of project economy generally involves homogeneous inputs, capital and labor. The production function for the project is written in the general form

$$Q = f\,(L,K) \tag{1.3}$$

where Q is the quantity of output

L is the quantity of labor

K is units of capital.

The total cost function is equation (1.1)

$$C = wL + rWK.$$

In this simple cost function, capital is assumed to be indestructible, i.e., has infinite life, and hence no provision for depreciation is made. The term r measures the rental rate on investment WK.

Smith explains r more fully:

> If we imagine the firm selling perpetuities in the bond market at a price 1/r to finance the investment WK then rWK is the annual payment to bondholders of record. In a very real sense we can think of rWK as the cost of "maintaining" the 'presence' of a unit of capital in production.[4]

The problem of minimizing costs has been discussed by a number of writers. Smith,[5] Baumol,[6] and Ferguson,[7] have similar treatments of the problem of minimizing the cost function

$$C = wL + rWK$$

subject to the constraint

$$Q = f\ (L, K).$$

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[4]Vernon L. Smith, Investment and Production (Cambridge, Mass.: Harvard University Press, 1961), 69.

[5]Ibid.

[6]William J. Baumol, Economic Theory and Operations Analysis (Englewood Cliff, N. J.: Prentice-Hall, 1961), 65.

[7]C. E. Ferguson, Microeconomic Theory (Homewood, Ill.: Irwin, 1966), 156.

A classic treatment of this problem makes use of the rule that, in equilibrium, the marginal rate of substitution of labor for capital is equal to the ratio of the wage rate to the rental rate of one unit of capital. This rule results from minimizing the function:

$$\emptyset = wL + rWK - \lambda[f(L,K) - Q] \tag{1.4}$$

where $\emptyset$ is the objective function to be minimized and λ is the Lagrange multiplier.

Smith minimizes $\emptyset$ to obtain the marginal productivity conditions

$$\frac{\partial \emptyset}{\partial L} = w - \lambda f_L = 0 \tag{1.5}$$

$$\frac{\partial \emptyset}{\partial K} = rW - \lambda f_K = 0 \tag{1.6}$$

$$\frac{\partial \emptyset}{\partial \lambda} = Q - f(L,K) = 0. \tag{1.7}$$

where $f_L = \frac{\partial f}{\partial L}$, the marginal product of labor

$f_K = \frac{\partial f}{\partial K}$, the marginal product of capital

The neoclassical choice of project is, then, a simple constrained minimization problem, wherein the cost function to be minimized is subject to the maintenance of some output.

To probe further the necessary conditions for selecting the minimum-cost project, differential variations in the cost and production functions are examined. Total differentiation of (1.3) yields:

$$dQ = \frac{\partial f_L}{\partial L} dL + \frac{\partial f_K}{\partial K} dK = 0 \tag{1.8}$$

$$= f_L dL + f_K dK = 0 \tag{1.9}$$

for constant output.

The necessary condition for minimum cost, i.e., the equality of the ratios of the price of each factor to its respective marginal product, is stated:

$$\lambda = \frac{W}{f_L} = \frac{rW}{f_K} \tag{1.10}$$

where again f_L and f_K are $\frac{\partial f}{\partial L}$ and $\frac{\partial f}{\partial K}$, respectively.

The necessary condition for minimum cost describes the point of "economic balance" of engineering economy.[8] The term economic balance means here that for most projects an allocation of inputs is possible such that an increase in either L or K increases total cost. Engineering or project economy is concerned with identifying and measuring variable operating costs that decrease as fixed investment costs increase so that an optimal mix of variable and fixed costs may be established. This problem is illustrated graphically in Fig. 1.1.

In Fig. 1.1 a given output Q is assumed required.

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[8]Herbert E. Schweyer, Process Engineering Economics (New York: McGraw-Hill, 1955), 208. W.D. Harbert, "Economic Process Operations," Industrial and Engineering Chemistry, Vol. XXXIX, No. 7, 940-949.

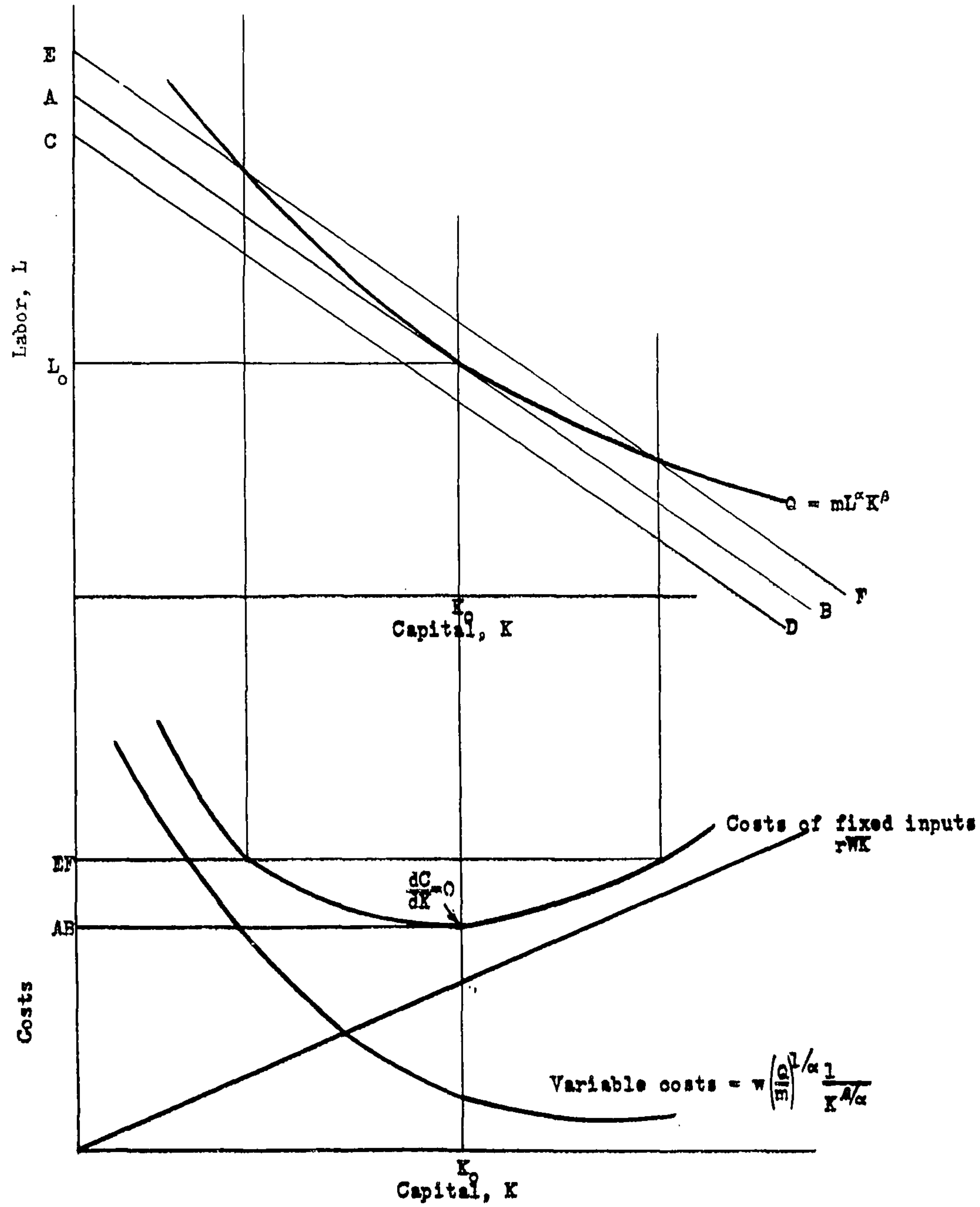


Fig. 1.1--Total costs of a given output produced with various combinations of labor and capital

This output may be described by any number of production functions, but for simplicity is assumed here to be the familiar Cobb-Douglas function:

$$Q = mL^{\alpha}K^{\beta} \tag{1.11}$$

where L is the quantity of labor

K is the units of capital

m is a coefficient, usually associated with a state of technology

α is elasticity of output with respect to the quantity of labor, L

β is the elasticity of output with respect to units of capital, K.

Fig. 1.1 shows that an infinite number of combinations of L and K will produce required output Q. If the price of L and K are defined, then a unique combination of L and K exists such that conditions of minimum cost are achieved to produce output Q.

The isocost lines AB, CD, EF, etc. describe the cost of output for combinations of L and K that lie on those lines. Many combinations of L and K described by isocost line EF can produce output Q, but cost is not minimal for any of them. No combination of L and K described by line CD can produce required output Q. Line AB, then, is unique in that it represents one combination of L and K that is capable of producing output Q and yet line AB represents lower costs than any other isocost line for combinations of L and K capable of producing output Q.

The slope of the isocost line is obtained by total differentiation of Equation (1.1)

$$dC = wdL + rWdK = 0.$$

$$\frac{dK}{dL} = \frac{w}{rW} = \frac{f_L}{f_K}$$

when $\frac{w}{rW}$ exceeds $\frac{f_L}{f_K}$, the mix of inputs lies to right of the point of "economic balance" of Fig. 1.1. Conversely, when $\frac{w}{rW}$ is less than $\frac{f_L}{f_K}$, the mix of inputs lies to left of this optimal point. The terms $\frac{dK}{dL}$, $\frac{w}{rW}$, $\frac{f_L}{f_K}$ are all expressions for the marginal rate of technical substitution, $MRTS_{L \text{ for } K}$, of labor for capital.

Labor, L, or variable inputs, is by its nature a function of output, or,

$$L = g\ (Q,K). \qquad (1.12)$$

The cost function may now be written

$$C = w\ g(Q,K) + rWK. \qquad (1.13)$$

Minimum cost obtains when

$$\frac{dC}{dK} = 0. \qquad (1.14)$$

Equation (1.14) is another way of describing the conditions required for the optimal mix of labor, L^o, and capital, K^o, or the point of "economic balance."

Equation (1.1) may be rewritten, substituting for L from the Cobb-Douglas production function

$$L = \left(\frac{Q}{m}\right)^{1/\alpha} \frac{1}{K^{\beta/\alpha}} \tag{1.15}$$

$$C = w\left(\frac{Q}{m}\right)^{1/\alpha} \frac{1}{K^{\beta/\alpha}} + WK \tag{1.16}$$

$$\frac{dC}{dK} = -\frac{\beta}{\alpha}\left(\frac{Q}{m}\right)^{1/\alpha} K^{-\frac{\beta}{\alpha}-1} + rW = 0. \tag{1.17}$$

Solving for optimal capital, K^o

$$K^o = \left[\frac{\beta w}{\alpha r W}\left(\frac{Q}{m}\right)^{1/\alpha}\right]^{\frac{\alpha}{\alpha+\beta}}. \tag{1.18}$$

And substituting for K in (1.15) and solving for optimal labor, L^o

$$L^o = \left[\frac{\beta w}{\alpha r W}\left(\frac{Q}{m}\right)^{-1/\beta}\right]^{-\frac{\beta}{\alpha+\beta}}. \tag{1.19}$$

Equations (1.18) and (1.19) describe the K and L of Fig. 1.1 for the minimum-cost requirement shown by isocost line <u>AB</u> and by the point where

$$\frac{dC}{dK} = 0$$

for output Q.

Costs are minimal when total investment in plant is

$$WK^o = W\left[\frac{\beta w}{\alpha r W}\left(\frac{Q}{m}\right)^{1/\alpha}\right]^{\frac{\alpha}{\alpha+\beta}}. \tag{1.20}$$

Equation (1.20) represents idealized conditions of indestructible capital stock, no income tax, no property tax, and no insurance. The real-world solution to the minimum-cost problem abstracts easily from the idealized case. Cost of capital, r, may be adjusted to cover real-world charges for depreciation, income tax, property tax and insurance. The required adjustment of r to cover these charges for the cost-minimizing firm parallels the adjustment required for return, s, to cover the same charges for the regulated firm.

Project Economy in the Regulated Firm

The revenue function

$$R = wL + sWK$$

for the project in the regulated firm is minimized but not in the same way as is the cost function for the regulated firm. Substituting $g(Q,K)$ for L and then differentiating R with respect to K yields an erroneous solution. In the normal case of $s > r$, the traditional cost-minimization approach results in the use of less capital for the simple reason that this solution reflects the fact that s makes capital more expensive than r. Since return sWK--and pure profit, $(s-r)WK$--are so clearly greater when more capital is used, it then intuitively makes good sense to argue that rational behavior on the part of the regulated firm calls for greater, not less, use of capital in its production process. Such an argument would imply that the regulated firm somehow uses capital as though it were less expensive than for the unregulated firm. Unfortunately, no measure is available as to just how the regulated firm treats its interior, as opposed to its exterior, cost of capital, until a comparison is made of the regulated firm's use of capital with that of the cost-minimizing unregulated firm.

Project economy for the regulated firm has been largely a matter of comparing revenue requirements based on the minimum acceptable rate of return, or alternately, of finding the neoclassical minimum-cost solution and then adding something to the minimum acceptable rate of return to obtain a "fair" or allowable rate of return. In Chapter II, it will be demonstrated that such a solution is not optimal for the regulated firm. Indeed, most of the work done to date on project or engineering economy has been directed toward refinements to reflect real-world facts of depreciation, income tax, insurance, and property tax on revenue requirements. In short, most of the literature and practically all the project manuals[9] for regulated firms are guides

[9]Examples of these manuals are: American Telephone and Telegraph Company, "Engineering Economy," 2nd. ed. 1963; Central Hudson Gas and Electric Corporation, "Economic Principles as Applied to Public Utility Engineering," 1961; Edison Electric Institute, "Economic Comparison of Alternative Plans," 1959; Iowa-Illinois Gas Company, "Engineering Economics"; Long Island Lighting Company, "Engineering Studies of Economy," 1963; Northern Natural Gas Company and Subsidiaries," Investment Analysis Procedure," 1963; Public

as to how r and s of the cost and revenue functions

$$C = wL + rWK$$

$$R = wL + sWK$$

are to be adjusted to cover the workaday realities of depreciation, income taxes, insurance, property tax, and dispersion of asset life. This part of project economy may be used in, or adapted to, a theoretical treatment of the optimal allocation of inputs.

Adjustments of r and s to Cover All Fixed Costs

The solution to the problem of optimal allocation of inputs for the regulated firm is the subject of Chapter II. The necessary adjustments for r and s are clearly necessary to obtain meaningful real-world minimums for the cost and the revenue functions. This adjustment is required regardless of whether inputs are allocated in accord with the cost-minimizing concept or in accord with the static model for the regulated firm to be developed in Chapter II and modified in Chapters IV and V. All variable inputs are covered in the expression wL; but, as indicated, the expression sWK of the revenue function must be modified to cover not only the cost of capital and pure profit but must cover as well the other named components of fixed inputs: depreciation, income taxes, property taxes, and insurance.

Service Electric and Gas Company, "An Abbreviated Course in Engineering Economics," 2 vols., 1960.

Such comprehensive coverage is achieved without altering the basic character of the revenue function. The revenue function is shown diagramatically in Fig. 1.2. The cost of debt is shown as interest on debt; the cost of equity capital is shown as the larger part of net income. The remaining part, (s-r)WK, of net income is pure profit in the classic sense. A rate of return, s, which exactly offsets the cost of capital has been characterized as the minimum acceptable rate of return. Fig. 1.2 shows that income taxes, property taxes, depreciation, and insurance may be expressed as functions of investment, WK. In turn, these costs may be expressed as functions of absolute return sWK, or of pure profit (s-r)WK. A rate of return less than r may be equated to a kind of confiscation, and may be expected to lead to disinvestment if the owners did not view such a return other than temporary. Under such conditions low bids for the firm's common stock may be expected. A higher rate of return than the cost of capital is compelling incentive for new investment and serves to attract capital for projects as required in subsequent periods. The proper role for the regulatory commission is to place a fair ceiling on pure profit by setting a rate of return, s, on "used and useful" plant, so that the firm will attract capital for projects required for growth and for the quality in service desired by the customer. The commission, however, cannot regulate the minimum acceptable rate of return. Only the investor

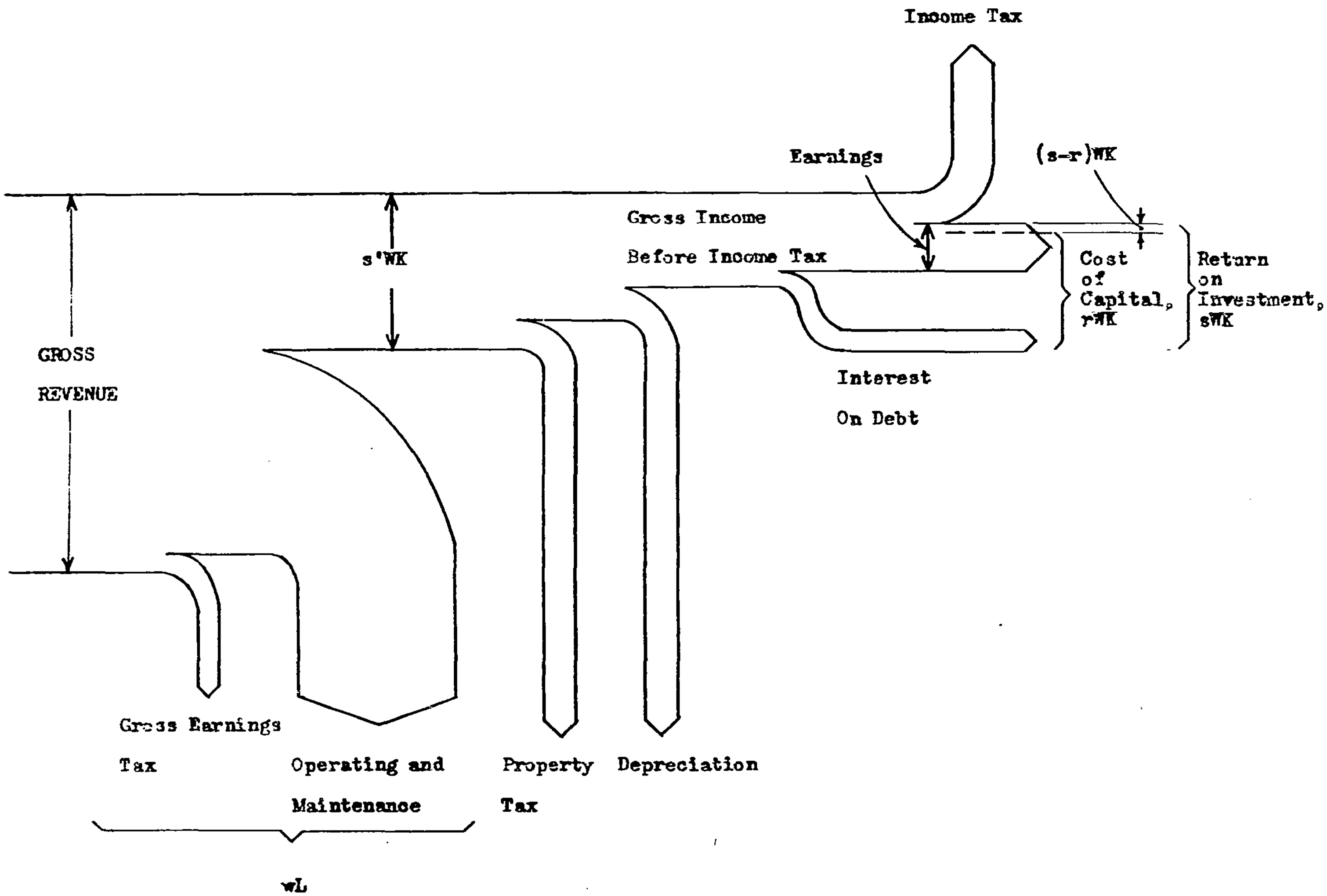


Fig. 1.2.—Graphic illustration of gross revenue required from a project to earn "fair" return, s

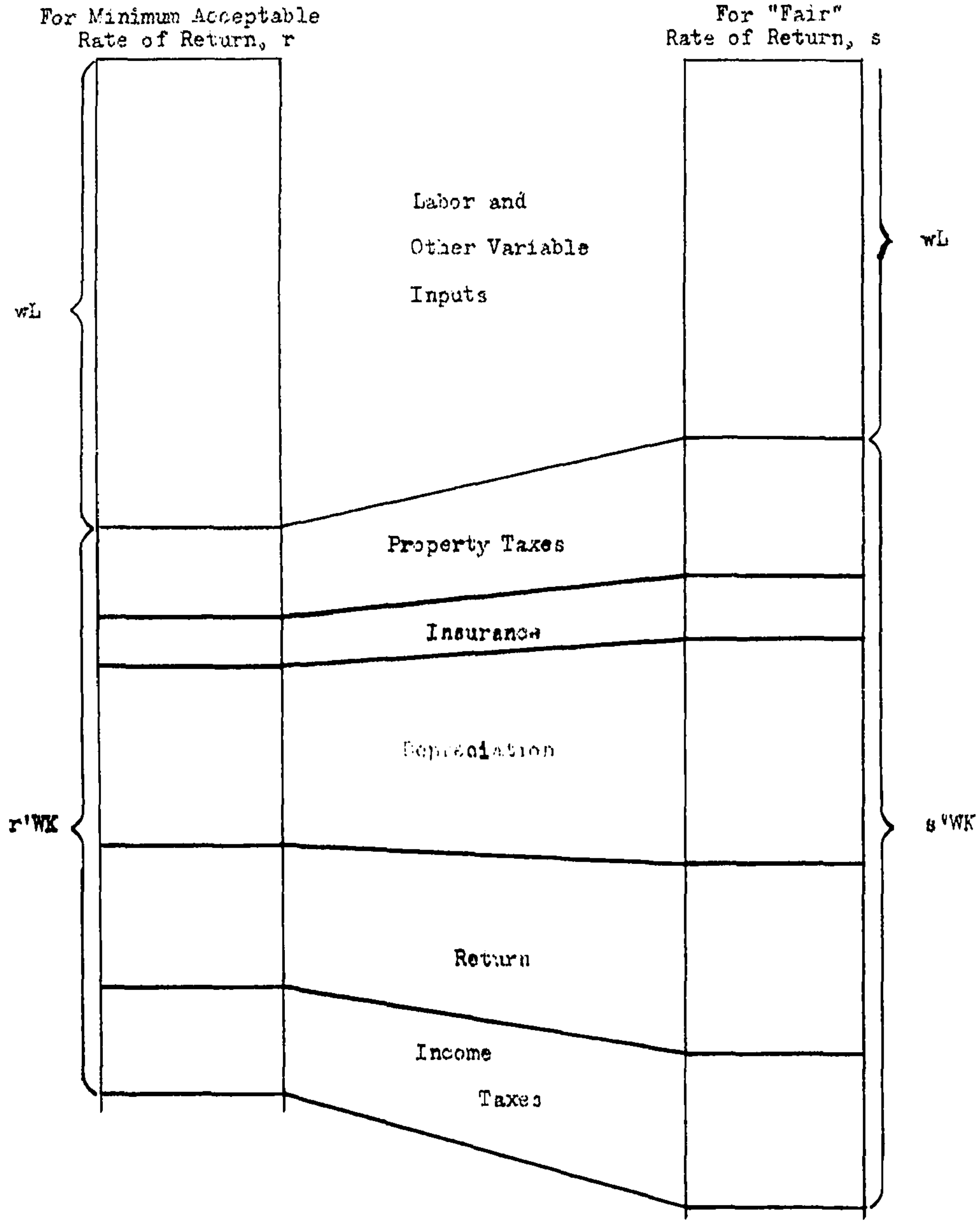
who measures his opportunity costs can set such a rate, but both regulatory commissions and the managements of regulated firms must estimate this rate as well, perhaps, as anticipate changes in it over time.

The minimum acceptable rate of return is expected to be higher than the cost of senior capital issues and less than the actual rate of return. The minimum rate of return may be affected by cyclical attitudes of investors and by the level of interest rates but is apparently remarkably stable over time. The more important consideration, perhaps, is that the minimum acceptable rate decreases as actual earnings increase. Correspondingly, the minimum rate increases as actual earnings decrease.

Comparison of projects in the regulated firm becomes an exercise in reducing to present worth the revenue requirements for a planning period. Alternatively, a comparison may be made of the "levelized" or uniform equivalent of revenue requirements of competing projects. If the discount rate is the minimum acceptable rate of return, the results in project selection are identical to those of minimum-cost case. If the discount rate is a "fair" or allowable rate of return, or if pure profit is simply added to revenue applicable to the minimum-cost case, then significantly different results may obtain.

Fig. 1.3 shows diagrammatically the difference between the revenue requirements for the case of minimum acceptable

REVENUE REQUIREMENTS*



* For the case where

$s > r$

Fig. 1.3.—Revenue requirements for a minimum acceptable rate of return, r, and for a fair or allowable rate of return, s.

return, which is also the minimum-cost case, and for the case where regulated return, s, is earned. Both cost of capital, r, and rate of return, s, have been adjusted to r′ and s′ to encompass other real-world capital charges of income tax, depreciation, property tax, and insurance. Income taxes and depreciation require special treatment to permit their incorporation in the r′WK and s′WK functions.

The two topics will be considered in separate subsections immediately below. Fig. 1.3 does not now show the inputs for the cost function and the revenue function to be the same. This difference in the allocation of inputs is a departure from the assumptions underlying many capital-budgeting manuals for public utilities. These manuals lead the planner to seek the neoclassical minimum-cost solution for the regulated firm and then to add some value equal to pure profit rate (s′-r′) to obtain total revenue requirements.

Income Taxes

The income-tax liability resulting from the year's net taxable income is a function of return, investment, debt-equity ratio, and the tax rate:

$$FIt_x = f(s, WK, L/E, t_x)$$

where FIt_x is the amount of income taxes

L/E is the debt equity ratio, with L and E the amounts of debt and equity, respectively

t_x is the income tax rate.

To develop more clearly that income taxes, FIt_x, is a function of investment, consider the following abbreviated income statement:

Line Symbol		
a,	Operating Expense	$5000.00
b,	Operating Revenue Deductions	3400.00
c,	Net Operating Revenue	1600.00
d,	Other Income	100.00
e,	Gross Income Before Taxes	1700.00
f,	Interest on Debt	200.00
g,	Taxable Income	1500.00
h,	Federal Income Tax (given t_x=.50)	750.00
i,	Net Income After Taxes and Interest	$ 750.00
j,	Return for Payment of Interest and Dividends (line i + line f)	$ 950.00

From the statement above, the following relationships are apparent if the tax rate is t_x:

$$h = gt_x \text{ but } g = e - f$$

$$h = (e - f)\ t_x$$

$$j = (i + f) \text{ but } i = g - h$$

$$j = g - h + f = e - f - h + f = e - h$$

$$e = j + h$$

$$h = jt_x + ht_x - ft_x$$

$$h = ht_x + (j-f)t_x$$

$$h(1 - t_x) = (j - f)t_x$$

$$h = (j - f)\ \frac{t_x}{1 - t_x}.$$

The last of the foregoing equations may be written in more general form:

$$\text{Income Taxes} = \left(\begin{array}{c}\text{Total Dollar}\\ \text{Return on}\\ \text{Investment}\end{array} - \begin{array}{c}\text{Interest}\\ \text{on Debt}\end{array}\right) \times \left(\frac{\text{Income Tax Rate}}{1 - \text{Income Tax Rate}}\right) \qquad (1.21)$$

Utilities are usually characterized by a capitalization with considerably more debt than found in the capital structure of unregulated firms. The debt-capitalization ratio of most utilities ranges from 1/3 to 2/3. Most regulated firms are represented in the upper half of this range. Fixing the debt-equity ratio permits income taxes to be stated as a function of investment. To develop this function, let

$L = \frac{L}{L + E}$, percent debt

$E = \frac{E}{L + E}$, percent equity

s = return on investment

i = interest rate

WK = investment

sWK = return on investment

iLWK = interest payment on debt.

The percent of the return dollar consumed by debt is $\frac{iLWK}{sWK}$ or $\frac{iL}{s}$. This percentage is sometimes referred to as "debt drain." By rearranging and substituting in equation (1.21) the income tax function may be written

$$FIt_x = sWK\left(\frac{100-\frac{iL}{s}}{100}\right)\left(\frac{t_x}{1-t_x}\right). \qquad (1.22)$$

To illustrate, arithmetically, let

Debt, L = 60 percent of total capitalization = .6

Return, s = .065

Interest rate, $i = .05$

Tax rate, $t_x = .52$.

The proportion of return, s, consumed by interest payments is

$$\frac{iL}{s} = \frac{.05 \times .6}{.065} = .415 \text{ or } 41.5 \text{ percent}$$

$$FIt_x = .065WK(1.083)\left(\frac{100 - 41.5}{100}\right) \qquad (1.23)$$

$$= .042\ WK$$

This simple relation expresses the uniform equivalent of income taxes in terms of original investment.

Depreciation and Annuity Depreciation

Provisions for capital recovery, or for replacement of destructible equipment, is achieved by some systematic, periodic charge which presumably is directly related to the rate of deterioration of the capital goods employed. The incorporation of a depreciation component for r' and s' in determining a rental charge for investment WK poses no problem. The depreciation charge (expressed in percent as d) may simply be stated $d \cdot WK$; but cost of capital, r, and rate of return, s, are then applicable to a constantly diminishing base. The quantities rWK and sWK require that some average, or some equivalent, be found to reflect the fact of capital deterioration. A more convenient solution lies in finding an equivalent for r and s rather for WK.

Consider the case of a $5,000 for a planning period or useful life of 5 years, as illustrated in Table 1.1.

Table 1.1.--Illustration of capital recovery costs in the case of a constantly diminishing base for a $5,000 depreciable asset

Year	(1) Net Investment (Initial Cost Less Depreciation Reserve)	(2) Return on Net Investment at 6.5 per cent	(3) Addition to Depreciation Reserve	(4) Total Annual Charge (Col. 2+ Col. 3)
1	$5,000	$325	$1,000	$1,325
2	4,000	260	1,000	1,260
3	3,000	195	1,000	1,195
4	2,000	130	1,000	1,130
5	1,000	65	1,000	1,065

Now, to illustrate the concept of uniform equivalence, suppose that, instead of making straight-line depreciation charges each year and reducing the net investment by the same constant amount, a separate "fund" is established in which equal deposits are made. The fund is assumed to earn interest at 6.5 percent annually. The return on investment must be obtained on the original cost of $5,000 over the entire five-year period. The accumulated sum, five years hence, of the annual "deposits" plus interest compounded annually must equal the original investment; therefore, $5,000 must be multiplied by some present-value factor which will reduce future worth to an annuity equivalent. Thus,

$$\$5000 \times \frac{s}{(1+s)^n-1} = \$5000 \times \frac{.065}{(1.065)^5-1}$$

$$= \$5000 \times .1756 \quad \$878.$$

Table 1.1 may now be recast as Table 1.2.

Table 1.2.--Illustration of capital recovery costs in the case of fixed initial investment amount

Year	Initial Investment	Return at 6.5 per cent	Equivalent Addition to Depreciation Reserve	Total Annual Charges
1	$5,000	$ 325	$ 878	$1,203
2	5,000	325	878	1,203
3	5,000	325	878	1,203
4	5,000	325	878	1,203
5	5,000	325	878	1,203
		$1,625	$4,390	$6,015

The example cited in Table 1.1 is correct but is difficult and cumbersome to use in project evaluation studies. In the example of Table 1.2, depreciation charges are determined by "setting aside" equal annual payments, which when compounded at 6.5 per cent per year would accumulate to $5000 in five years.

The following computation is presented to demonstrate that the present worths of the total annual charges in the two preceding illustrations are equal, even though the sums of the charges over five years differ slightly.

Table 1.3.--Present worths of capital recovery charges of Tables 1.2 and 1.1

Year	Total Annual Charges		Future Worth to Present Worth Factor	Present Worth	
	Table 1.2	Table 1.1		Table 1.2	Table 1.1
1	$1,203	$1,325	.9390	$1,129.62	$1,244.18
2	1,203	1,260	.8817	1,060.69	1,110.94
3	1,203	1,195	.8278	995.84	989.22
4	1,203	1,130	.7773	935.09	878.34
5	1,203	1,065	.7299	878.07	777.34
	$6,015	$5,975		$4,999.31	$5,000.02

As previously suggested, determination of the average return over the life of the investment, or the planning period, becomes tedious if the method of Table 1 is used. This approach can thus be replaced by a more manageable method involving an "equated rate of return" which, when multiplied by the initial investment, will yield the average return over the life of the investment. Such a procedure is significantly time-saving in calculations when proposed projects possess different useful lives and different salvage values.

This treatment becomes clearer when Table 1.1 is re-examined and recast in terms of present worth as in

Table 1.4.

Table 1.4.--Analysis of present worth of capital recovery costs in the case of a diminishing base (recast of Table 1.1)

Year	Annual Charges: Return on Investment	Annual Charges: Straight Line Depreciation	Future Worth to Present Worth Factor	Present Worth: Return on Investment	Present Worth: Straight Line Depreciation
1	$325.00	$1,000.00	.9390	$305.18	$ 939.00
2	260.00	1,000.00	.8817	229.14	881.70
3	193.00	1,000.00	.8278	161.42	827.80
4	130.00	1,000.00	.7773	101.05	777.30
5	65.00	1,000.00	.7299	41.44	729.90
Total	$975.00	$5,000.00		$844.33	$4,155.70

The present worth of the annual charges may be equated to an annuity covering the five-year period for both return and depreciation. By reference to a table for present worth to annuity, the factor

$$\frac{s}{1-(1+s)^{-n}}$$

for five years and 6.5 per cent is found to be .2406. Thus:

Uniform equivalent of present worth of return on investment	= .2406 x	\$844.33 =	\$203.15 =	4.06% of \$5,000
Uniform equivalent of present worth of straight line depreciation	= .2406 x	4,155.70 =	999.86 =	20.00% of \$5,000
Total		\$5,000.03	\$1,203.01 =	24.06% of \$5,000

Clearly, the annual charge for depreciation is 20 per cent, but the charge for return on investment is only \$203.15, which is equal to an average annual charge of 4.06 per cent on the initial investment of \$5000. This average charge is often called the <u>equated rate of return on investment</u>. This concept can be clarified by returning again to the example of Table 1.2.

Return on Investment =	\$5,000 x .065 =	\$ 325.00
Annuity Depreciation =	5,000 x .1756 =	878.00
Total Annual Charge	\$5,000 x .2406 =	\$1,203.00

The following identity may now be stated:

$$\begin{bmatrix}\text{Equated Return on Investment}\\ \text{Plus}\\ \text{Straight Line Depreciation}\end{bmatrix} \equiv \begin{bmatrix}\text{Return on Investment}\\ \text{Plus}\\ \text{Annuity Depreciation}\end{bmatrix} \quad (1.25)$$

or, alternatively,

$$\text{Equated Return on Investment} \equiv \left[\begin{array}{l}\text{Return on Investment plus Annuity} \\ \text{Depreciation minus Straight Line} \\ \text{Depreciation}\end{array}\right] \qquad (1\ 26)$$

By illustration,

$$.0406 = .065 + .1756 - .20.$$

The depreciation annuity is a method for reducing total cost of plant to be recovered to a level per cent of initial plant investment. Either side of the above identity is a means of providing for return of, and return on, invested capital. This sum is commonly referred to as the "capital recovery factor"[10] and may appear either as

$$\frac{s\ (1 + s)^t}{(1 + s)^t - 1} \quad \text{or} \quad \frac{se^{st}}{e^{st} - 1}$$

to permit return and depreciation to be stated as a combined constant function of investment WK.[11]

Retirement Dispersion

An important consideration relative to annuity depreciation is the implicit certainty attributed to the life of the project. Plant retirement in the real world, however, is characterized by uncertainty. Public-utility plant is unique in that so much is known of the probability of length of its

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[10] Paul H. Jeynes, "The Depreciation Annuity," Transactions of American Institute of Electrical Engineers, LXXV, part III, 1956, pp. 1398-1415.

[11] Paul H. Jeynes and L. Van Nimwegen, "The Criterion of Economic Choice," Transactions of American Institute of Electrical Engineers, LXXVII, part III, 1958, pp. 606-632.

useful life. The dispersion of useful life estimates of utility plant about some mean and hence the dispersion of depreciation costs can be estimated in much the same manner as insurance premiums are calculated for human life.

The Engineering Experiment Station of Iowa State University has developed a "codified" system that describes patterns of retirement dispersion for various types of public-utility plant. Figure 1.4 shows several types of dispersion patterns described by letters, S, L, and R and by subscripts. The letter S designates symmetrical dispersion; letters L and R refer to asymmetry where the maximum rate of retirement occurs before and after, respectively, the average life of plant. If retirements occur at nearly the same rates at all ages, a subscript near zero is applied. A higher subscript is applied if nearly all retirements occur in a brief period near average life.[12,13,14,15]

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[12] R. Winfrey, "Statistical Analyses of Industrial Property Retirements," Bulletin 125, Iowa Engineering Experiment Station, Iowa State University, Ames, Iowa, 1939.

[13] R. Winfrey, "Depreciation of Group Properties," Bulletin 155, Iowa Engineering Experiment Station, Iowa State University, Ames, Iowa, 1942.

[14] R. Winfrey, "Condition-Percent Tables and Depreciation of Unit and Group Properties," Bulletin 156, Iowa Engineering Experiment Station, Iowa State University, Ames, Iowa, 1943.

[15] J. Jeming, "Estimates of Average Service Life," Econometrica, 1943. Vol. XI, No. 2, pp. 141-150.

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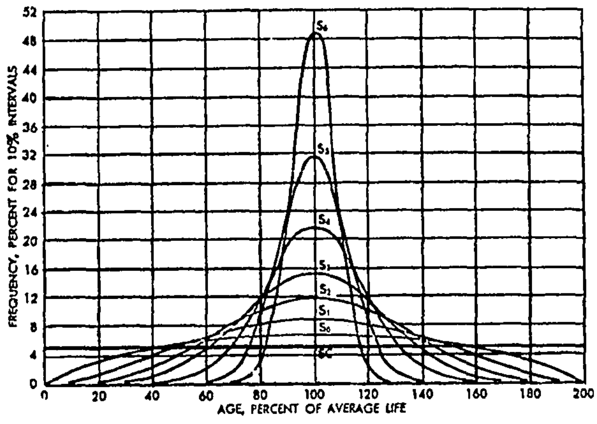

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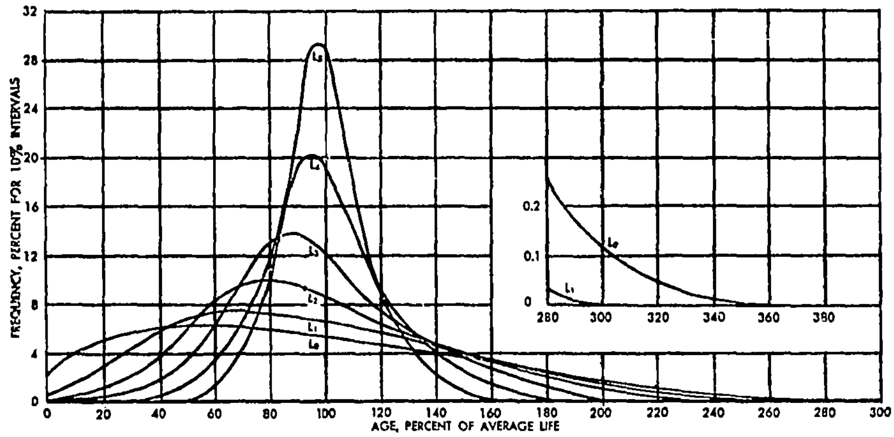

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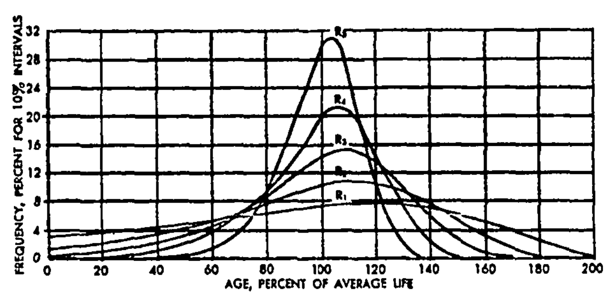

Fig. 1.4 - Iowa-Type Retirement Dispersion Curves

Source: Reproduced from Iowa State Experiment Station Bulletin No. 155. R. Winfrey, "Depreciation of Group Properties."

Table 1.5.--Depreciation annuity in per cent of capital investment for a range of probable lives and types of dispersion for a return of six per cent[a]

Probable Average Life, Years	I_O	L_1	L_3	L_5	SC	S_O
5	.1897	.1857	.1808	.1781	.1880	.1843
6	.1555	.1517	.1467	1442	1540	1503
7	.1313	.1274	.1224	.1200	.1298	.1261
8	.1131	.1092	.1043	.1019	.1117	.1081
9	.0991	.0952	.0903	.0878	.0977	.0939
10	.0878	.0840	.0791	.0767	.0866	.0828
15	.0547	.0509	.0461	.0437	.0538	.0499
20	.0386	.0349	.0301	.0279	.0379	.0340
25	.0292	.0256	.0210	.0189	.0288	.0248
30	.0232	.0197	.0152	.0133	.0230	.0190
40	.0159	.0127	.0087	.0070	.0162	.0122
60	.0093	.0065	.0033	.0022	.0100	.0064

[a]See P.H. Jeynes, American Institute of Electrical Engineers Transactions, Vol. LXXV, Part 3, pp. 1398-1415.

Table 1.5--Continued

S_1	S_3	S_5	R_1	R_3	R_5	SQ
.1821	.1793	.1779	.1843	.1799	.1781	.1774
.1480	.1452	.1438	.1503	.1458	.1439	.1434
.1238	.1210	.1196	.1260	.1215	.1197	.1191
.1056	.1029	.1015	.1079	.1034	.1016	.1010
.0916	.0888	.0875	.0940	.0894	.0876	.0870
.0805	.0777	.0163	.0828	.0783	.0764	.0759
.0475	.0447	.0434	.0499	.0453	.0435	.0430
.0316	.0288	.0276	.0341	.0295	.0277	.0272
.0225	.0198	.0186	.0250	.0205	.0187	.0182
.0167	.0160	.0130	.0193	.0148	.0131	.0126
.0101	.0077	.0067	.0127	.0084	.0068	.0065
.0045	.0027	.0020	.0070	.0033	.0021	.0019

The annuity depreciation can be adjusted to reflect the effect of retirement dispersion as estimated in the Iowa State curves. Jeynes and others have developed the following relation to achieve the required adjustment:[16,17]

$$ {}_{r}^{\beta}d = \frac{\sum v^x \text{ Retirements}_x}{\sum v^x \ \bar{y}x} = \frac{1}{\sum v^x \ \bar{y}x} - r \qquad (1.27)$$

where ${}_{r}^{\beta}d$ is the present-worth group-basis depreciation annuity

β indicates that the annuity is the level percentage applicable to the group of units whose retirements are dispersed about average life

v^x is present worth factor for age x

x is age in years

r is the minimum acceptable rate of return

$\bar{y}_x$ is the mean annual survivors in year x

Retirements_x is the retirements in year x.

For "fair" or allowable rate of return s

$$ {}_{s}^{\beta}d = \frac{1}{\sum v^x \ \bar{y}x} - s . \qquad (1.28)$$

Table 1.5 shows a range of depreciation annuities for various types of dispersions. Different dispersions could as much as double the revenue required for depreciation in

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[16] P.H. Jeynes and C.J. Baldwin, "Financial Mathematics for Economic Studies," The Westinghouse Engineer, XI, No. 3, 1964, pp. 41-47.

[17] Iowa-Illinois Gas and Electric Company, "Engineering Economics," 1962.

certain cases. Jeynes called attention the extreme example of the case of SQ (no dispersion) and R_1 (right-skewed dispersion) for a plant having an average life of 40 years. The resulting depreciation annuity rates are .065 and 1.27 per cent, respectively.

Summary Statement: Adjustments r and s to Cover All Real-World Fixed Charges

Once the retirement dispersion is estimated, it is possible then to calculate an annuity depreciation as a component of r' and s' in both the cost and revenue functions

$$C = wL + r'WK \tag{1.29}$$

$$R = wL + s'WK. \tag{1.30}$$

By definition, $r' = r + t_{rx} + d_r + F$

where r is the cost of capital

t_{rx} is the tax factor expressed as a function of WK

d_r is the annuity depreciation adjusted for dispersion

F is a factor for property taxes and insurance.

Also, by definition,

$$s' = s + t_{sx} + d_s + F.$$

All fixed charges may now be expressed as a function of plant investment and of plant life. Fig. 1.5 shows one such function for specified conditions of return, taxes, salvage, and debt-drain (which is the result of a specified debt-equity ratio).

$$C = wL + r'WK$$

$$R = wL + s'WK$$

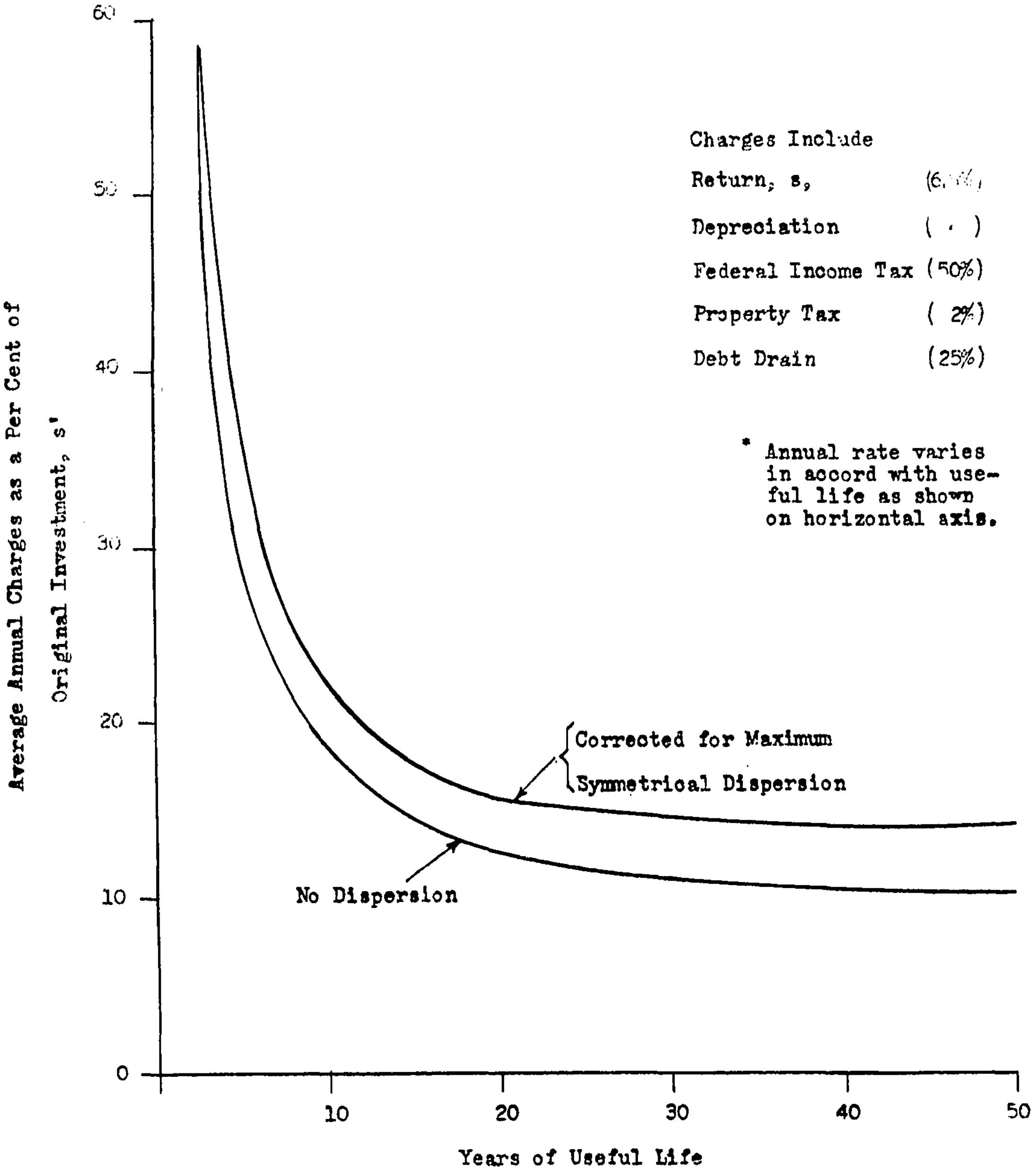

Fig. 1.5.—The sum of the cost of all fixed inputs, expressed as a function of the present value of investment.

Minimizing the cost function classically results in optimal allocation of inputs for the unregulated firm. The mechanics of allocating inputs for the regulated firm, however, are yet to be developed. Conventional minimization of the revenue-requirements function,however, does not result in optimal allocation of inputs for the regulated firm. Accordingly, some other means must be found to allocate inputs for the regulated firm. A model for optimal allocation of inputs, *inter alia*, for the regulated firm is the substance of Chapter II.

CHAPTER II

A STATIC MODEL FOR THE REGULATED FIRM

In Chapter I, discussion of the minimum-cost and revenue-requirements approach to project selection was clear with respect as to what constituted optimal allocation of inputs, labor and capital, in the minimum-cost case; but was much less so in the revenue-requirements approach that is applicable to the regulated firm. In Chapter I, it was suggested that the regulated firm might rationally allocate its inputs, variable and fixed, differently from the unregulated firm. This view of the firm's behavior is a result of conventional regulatory practice that makes return--and pure profit--a function of the firm's investment in plant and equipment. This regulatory convention led to the observation that the regulated firm might logically be expected to make its production process as capital intensive as is rationally possible. It becomes at once apparent that the regulated firm may well select a mix of inputs that results in greater pure profit, (s-r)WK, even though this mix also results in greater costs. The question now arises: How far can the regulated firm push its propensity for substituting capital for labor in its production process? Principally, the constraint is the demand function faced by the firm; but another constraint is also posed by the production function itself. The revenue required for the optimizing regulated firm, then, is not determined simply by adding pure profit (s-r)WK to the costs that would apply under neoclassical minimum-cost conditions.

If the regulated firm substitutes more capital for labor than indicated by the minimum-cost solution to the problem of allocating inputs, then costs are greater and the output is sold at a higher price if return, s, is to be earned and pure profit, (s-r), is to accrue to the owners of the firm. If the output is sold at a higher price, however, less output is sure to be demanded. In this chapter, inputs for the production process are analyzed for the regulated firm and a model developed to determine the equilibrium interplay of inputs, output, and price of output.

A static model for the regulated firm evolves logically from the one-project situation of Chapter I. In Chapter I, perfect knowledge of prices of inputs was assumed. In this chapter and in subsequent chapters, perfect knowledge continues to be assumed not only of prices of inputs but of the firm's production and demand functions as well. The static model for the regulated firm is, then, developed for conditions of certainty. Such a model is the simplest if the assumption is made that one project generates the entire output of the firm. As before, two homogeneous inputs, labor and capital, are assumed. The model is a bit more realistic if the inputs are treated more encompassingly as variable and fixed inputs. Initially, capital is treated as indestructible. Output is treated as smooth and continuous, i.e., all cyclical patterns of output are assumed away.[1] While output is one of the variables determined by the equilibrium model developed here, the demand function upon which output, in part, depends is exogenously determined. In accord with established public-utility practice, all customers within service range who seek to purchase output at regulated price, P, must be supplied.

In the static model, the cost of capital, r, is assumed to be fixed. This assumption requires that the debt-equity ratio also be fixed. Under dynamic conditions this requirement may be relaxed. Similarly, the rate of return, s, is fixed at some level greater than the cost of capital.

Since the model to be developed concerns the firm rather than the project, a profit function may be defined. Two profit functions for the regulated firm are relevant. These two profit functions may be called simply the "regulated" and the "unregulated." The regulated profit function, by convention in the United States, means that profit is a direct function of invested capital. Unregulated profit, however, is determined by classical considerations of supply and demand. The unregulated profit function serves to

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[1]In utility operating parlance, smooth continuous output is characterized by the phrase "100 per cent load factor."

constrain profit resulting from the regulated profit function. In effect, then, regulated profit is the ceiling profit for the regulated firm; and since regulated profit is a function of investment, rational behavior of the firm calls for as much capital in the production process as is feasible. How much capital is feasible? The answer lies in the dimensions of the production and demand functions. The following static model is developed as a general solution to the problem of measuring the degree of capital intensity that can be achieved rationally in the regulated firm.

The Production Function

As in Chapter I, a production function of the Cobb-Douglas type is utilized:

$$Q = mI^{\alpha}K^{\beta}, \tag{2.1}$$

The Revenue Function

The revenue function is expressed simply as

$$R = PQ \tag{2.2}$$

where R is revenue

P is price

Q is output.

The Profit Function

The unregulated profit function is stated

$$\pi = PQ - wL - rWK \tag{2.3}$$

where π is profit

PQ is revenue (price times output)

w is the wage rate, or price of variable inputs

L is the quantity of labor, or variable inputs

r is the cost of capital, or fixed inputs

W is the price of a unit of capital

K is the units of capital.

Unregulated profit typically increases to some peak value as more capital is used. The level of capital where

$$\frac{d\pi}{dK} = 0$$

defines the optimal level of capital use by the unregulated firm. When more than optimal amounts of capital are used, unregulated profit declines curvilinearly until at some level of capital use all profit disappears.

The regulated profit function may now be stated as

$$\pi = (s - r)WK \qquad (2.4)$$

where s is the "fair" or allowable rate of return

r is the cost of capital.

Obviously, equation (2.4) is a linear relation, i.e. profit, π increases linearly as more and more capital is used.

Profit, π, is equation (2.3) and (2.4) is pure profit. Since pure profit reflects opportunity costs, π is a measure of the profit attainable from the regulated firm's investment in public-utility plant and not attainable elsewhere.

At some level of capital use, curvilinearly decreasing unregulated profit is equal to linearly increasing profit. It is this level of capital use that the unregulated profit function constrains the regulated firm's propensity for greater use of capital. This point may be determined by equating the regulated and unregulated profit function:

$$(s - r)WK = PQ - wL - rWK$$

to form the telescoped zero "excess" profit function

$$0 = PQ - wL - sWK. \qquad (2.5)$$

Equation (2.5) means simply that zero "excess" profit results when a return exactly equal to s, is earned. Rate of return, s, may now be specified as

$$s = \frac{PQ - wL}{WK}. \qquad (2.6)$$

The net represented by (PQ - wL) is the familiar $TNRP_K$, or total net revenue product of capital. Equation (2.6) may be restated as

$$sWK = PQ - wL$$

which means simply that return in dollars is equal to total revenue minus variable costs.

The Demand Function

The demand function may be represented by several functions, two of which are the familiar linear relation

$$P = a - bQ$$

where a and b are constants determining the intercept and slope of the demand curve, and by the log-linear function,

$$P = HQ^{-\sigma}$$

where H and σ are constants,

σ being a measure of price-quantity elasticity,[2]

---

[2]Note that $-\sigma$ is not the familiar elasticity of

P is price

Q is output.

---

demand, but rather is its reciprocal. This expression

$$P = HQ^{-\sigma} \quad (1)$$

means that $(-\sigma)$ is a constant signifying elasticity of price with respect to quantity. The more conventional expression of elasticity of demand means the elasticity of quantity with respect to price is $(-1/\sigma)$.
Elasticity of quantity with respect to price is expressed as

$$\frac{dQ}{dP} \cdot \frac{P}{Q} = \varepsilon. \quad (2)$$

Thus the term $-\sigma$ in (1) above is really $\frac{1}{\varepsilon}$ as in

$$\frac{dP}{dQ} \cdot \frac{Q}{P} = \frac{1}{\varepsilon}. \quad (3)$$

To show this, equation (1) is differentiated with respect to quantity Q:

$$\frac{dP}{dQ} = -\sigma HQ^{-\sigma-1} = -\sigma HQ^{-(\sigma+1)}. \quad (4)$$

Both sides of (1) are divided into Q

$$\frac{Q}{P} = \frac{Q}{HQ^{-\sigma}}$$

$$\frac{Q}{P} = \frac{Q^{\sigma+1}}{H}. \quad (5)$$

Now each side of (4) is multiplied by the respective sides of (5)

$$\frac{dP}{dQ} \cdot \frac{Q}{P} = -\sigma HQ^{-(\sigma+1)} \cdot \frac{Q^{\sigma+1}}{H}$$

$$\frac{dP}{dQ} \cdot \frac{Q}{P} = -\sigma$$

and

$$\frac{dQ}{dP} \cdot \frac{P}{Q} = -\frac{1}{\sigma}.$$

See George J. Stigler, The Theory of Price (New York: Macmillan Company, Third Edition), Appendix B.

In this and subsequent chapters, the log-linear demand function is used for development of a model for the regulated firm. The log-linear demand function is considerably more tractable mathematically than the linear function. The linear case is discussed in detail in the Appendix of this chapter. Conclusions for conditions of equilibrium are the same for both cases. The problems of managing the linear case are reported in the Appendix.

The "Zero" Profit Functions for the Unregulated and the Regulated Firm

The special case where

$$0 = PQ - wL - rWK \tag{2.8}$$

and

$$0 = PQ - wL - sWK \tag{2.9}$$

may now be examined. These two situations are illustrated graphically and compared in Fig. 2.1.[3]

---

[3]Note that if Q of the production function is substituted in 0 = PQ-wL-rWK the result is $PmL^{\alpha}K^{\beta} - wL - rWK$. Now if price P is set, say at 10, m at 1, price, w, of labor at 4, price, W, of a unit of capital at 100, α and β at 1/2, then,

$$0 = 10L^{1/2}K^{1/2} - 4L - 4K$$

$$10\ L^{1/2}K^{1/2} = 4L + 4K$$

Arbitrarily set K at 9, then,

$$30\ L^{1/2} = 4L + 36$$

$$0 = L^2 - \frac{612L}{16} + 81$$

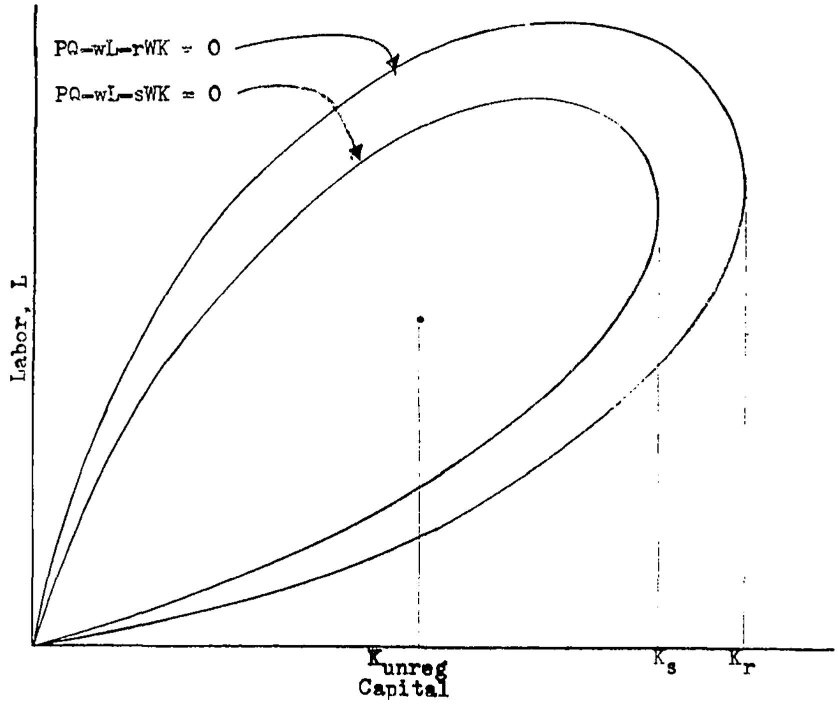

Fig. 2.1.--The zero excess profit functions for the regulated and the unregulated firm.

The locus of points described by

$$\pi = 0 = PQ - wL - rWK$$

defines the limits of the input mix where all profits disappear. Similarly, the locus of points described by

$$\pi = 0 = PQ - wL - sWK$$

defines the limits of the input mix where all profits in excess of those allowed by regulation disappear.

$$L = \frac{38.25 \pm \sqrt{38.25^2 - 4(81)}}{2} = 2.17,\ 36.03$$

Clearly, K can be chosen so large, or to the right of K_r so that the quantity under the radical has a minus sign, i.e., the solution for L is then indeterminate.

Since pure profit $(s-r)wK$ is maximum at the point on the locus of

$$0 = PQ - wL - sWK$$

where K is largest, the regulated firm may be expected to seek, and operate at, this point namely, where $K = K_s$. Fig. 2.1 shows the limits of the mix of input of capital and labor for the zero and zero excess profit functions.

Clearly, regulated profit $(s-r)WK$ is greatest at K_s of Fig. 2.1. All profit vanishes at K_r. The unregulated optimum is at $K_{unreg.}$, which is the minimum-cost solution (pages 115 and 125). In geometric illustrative terms, the locus of points described by the function $PQ - wL - sWK = 0$ is formed by the intersection of the plane $(s-r)WK$ intersecting the unregulated profit "hill," $\pi = PQ - wL - rWK$. This interpretation is shown in Fig. 2.2. Unregulated profit is shown rising above the $(s-r)WK$ plane. The peak of this hill describes the coordinates of L and K for the minimum-cost case. Fig. 2.2a shows the production surface. Output Q_o, resulting from inputs K_s and L_s, is the optimal output for the regulated firm. Output $Q_{unreg.}$ is the optimal output for the unregulated firm. At output Q_r, all profit vanishes. Fig. 2.2 illustrates graphically the compelling incentive for the regulated firm to be as capital intensive as is prudently possible.[4]

[4]One advantage of a theory study is that such real-world situations as relationships with public utility commissioners may be ignored.

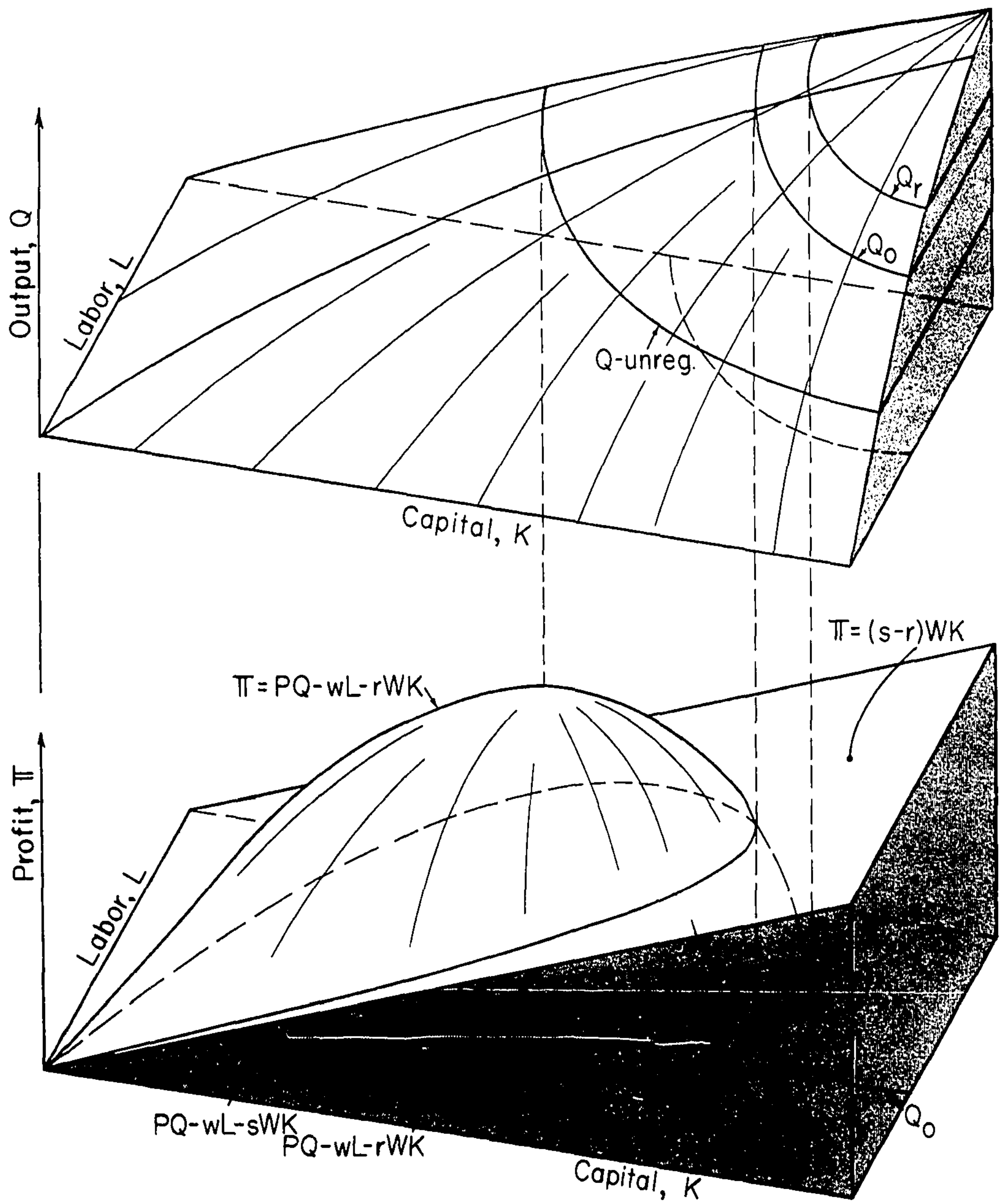

Fig. 2.2 – The regulated and unregulated profit functions in relation to the production function of the regulated firm.

The profit function may be shown graphically in another way. Figs. 2.3 and 2.4, allow two planes sWK and rWK, both hinged on the L axis to slice, this time, through the total-net-revenue-product, $TNRP_K$, surface. The profiles shown in Figs. 2.3 and 2.4 apply to the path of optimal mix of labor and capital as capital increases. This path will be described more precisely later.

The arc through which the regulatory authority can swing the ray sWK is shown as delimited by $K_{unreg.}$ on the left and by K_r on the right, in Fig. 2.4. If the ray sWK intersects the $TNRP_K$ arc to the left of $K_{unreg.}$, the "fair" or allowable price would be set above the unregulated price.[5] If the ray sWK intersects $TNRP_K$ to the right of K_r, the rate of return is less than the cost of capital and the firm would disinvest. This point is the "threshold of confiscation" referred to by some students of public-utility rate making. If sWK intersects $TNRP_K$ to the left of K_r, but quite close to it, new capital for growth may be difficult to attract at a "satisfactory" price.

The Short-run Optimal Allocation of Inputs

The solution to the problem of optimal allocation of inputs, labor and capital, in the regulated firm makes use of the rule that labor (or variable inputs) is used up to

[5]Fred M. Westfield, "Regulation and Conspiracy," American Economic Review. Vol. LV, No. 2, pp. 414-423.

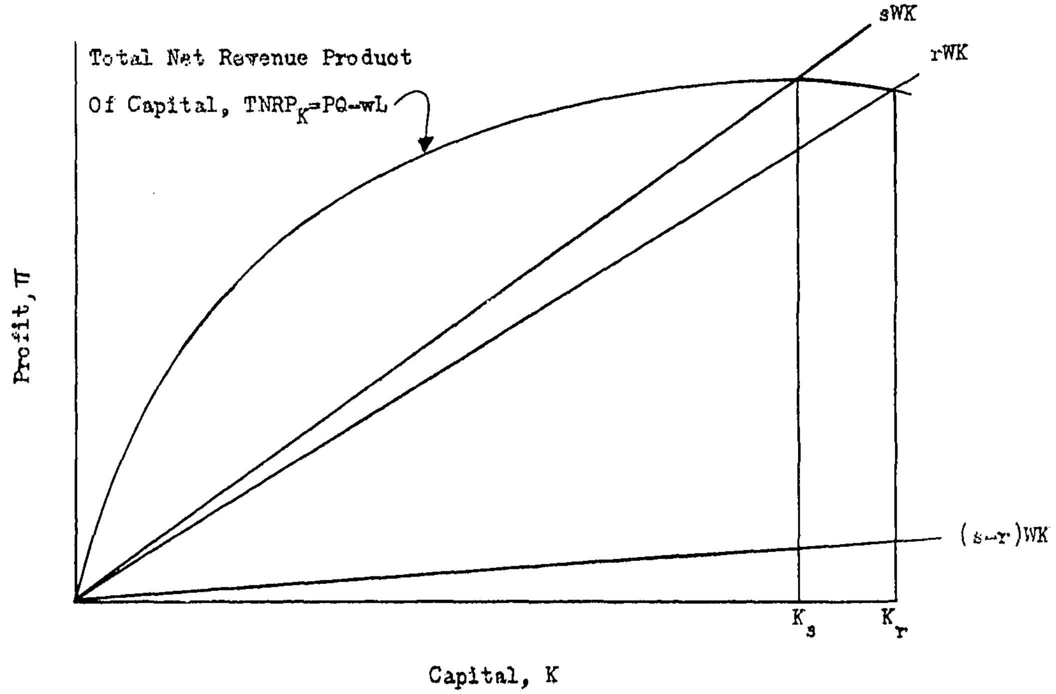

Fig. 2.3. —Profile of total-net-revenue-product surface along path of optimum mix of labor and capital as capital increases.

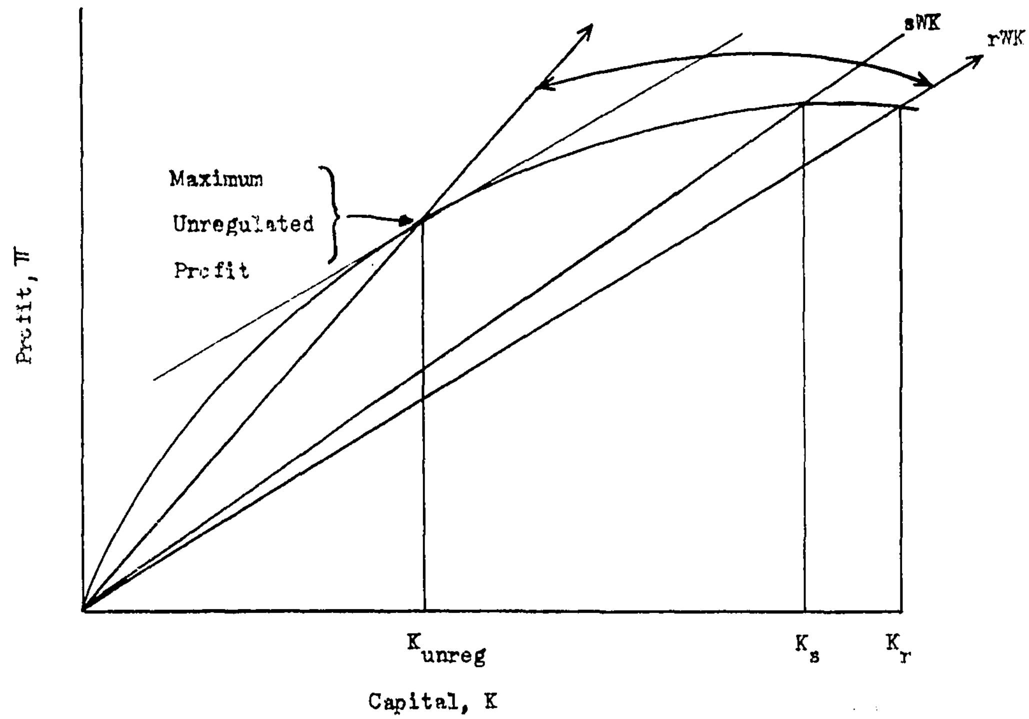

Fig. 2.4 .—Diagram of "arc of effective public-utility regulation"

the point where the addition to total revenue is just equal to the wage rate. Capital is used until the resulting addition to total revenue is just equal to the "fair" or allowable return on the price of one unit of capital. Maximum profits are realized when

$$MR_Q \; f_L = w \tag{2.10}$$

and

$$MR_Q \; f_K = sW \tag{2.11}$$

where MR_Q is marginal revenue

f_L, the marginal product of labor, is

$$\frac{\partial Q}{\partial L} = \alpha m L^{\alpha - 1} K^{\beta} \tag{2.12}$$

f_K, the marginal product of capital, is

$$\frac{\partial Q}{\partial K} = \beta m L^{\alpha} K^{\beta - 1} \tag{2.13}$$

and hence, the left side of (2.10) and (2.11) are, respectively the marginal revenue products of labor and capital.

The function, $MR_Q \; f_L = w$, defines the short-run path of the optimal use of labor and capital. This path is shown graphically in Fig. 2.5. The log-linear demand function may be brought to bear to define further the optimal short-run path by converting the demand function

$$P = HQ^{-\sigma}$$

to a revenue function by multiplying both sides by Q:

$$PQ = HQ^{1-\sigma}. \tag{2.14}$$

Marginal revenue is obtained by differentiating the revenue function (2.14) with respect to Q to obtain

$$MR_Q = (1-\sigma)HQ^{-\sigma}.$$

Substituting from the production function for Q

$$MR_Q = (1-\sigma)H(mL^{\alpha}K^{\beta})^{-\sigma}$$

$$MR_Q = (1-\sigma)Hm^{-\sigma}L^{-\alpha} K^{-\alpha\sigma}. \quad (2.15)$$

Substituting for MR_Q and f_L in equation (2.10)

$$MR_Q f_L = \alpha (1-\sigma)Hm^{1-\sigma}L^{\alpha(1-\sigma)}K^{\beta(1-\sigma)} = w. \quad (2.16)$$

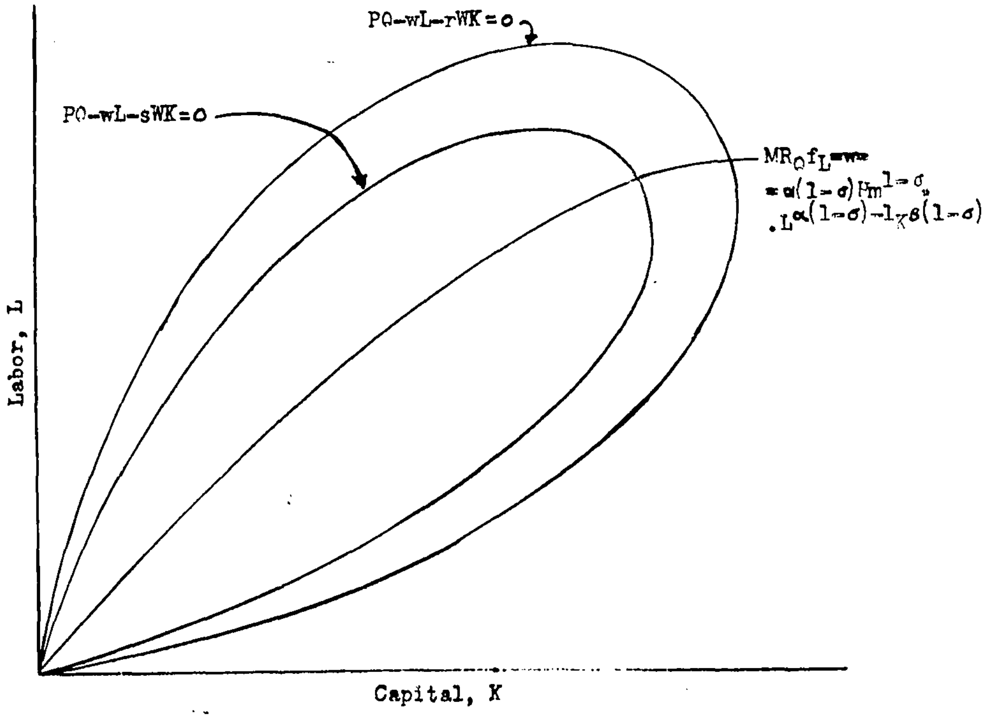

Fig. 2.5.--The path of short-run allocation of inputs for the regulated firm facing a log-linear demand function.

Equation (2.16) defines the path of optimal short-run allocation of inputs, labor and capital as more and more capital is used. The equilibrium point for the regulated firm is yet to be defined. In the unregulated case, the firm merely finds the point on the path

$$(1-\sigma)m^{1-\sigma}L^{\alpha(1-\sigma)-1}K^{\beta(1-\sigma)} = w$$

that produces the output of greatest profit. The regulated firm is not free to restrict its output to a level of greatest profit. All customers who are within service range and who are prepared to pay regulated price, P, for output must be supplied. Significantly, both the regulated and unregulated firms find their optimum allocation of inputs somewhere along the path defined by equation (2.16).

Total differentiation of the zero excess profit function

$$O = PQ - wL - sWK$$

yields the slope of this function at the point of maximum capital employed by the regulated firm:

$$[MR_Q f_L - w]\,dL + [MR_Q f_L - sW]\,dK = O = -\frac{dL}{dK} - \frac{MR_Q f_K - sW}{MR_Q f_L - w} \,.$$

Since the firm allocates its inputs such that

$$MR_Q f_L = w$$

then,

$$\lim_{(MR_Q f_L - w)\to 0} -\frac{dL}{dK} = \infty \,.$$

Solving for optimal allocations of labor and capital under regulation is now possible. The solution lies on the path described by the function $MR_Q f_L = w$. But the optimum for the unregulated firm is also found on this path. The optimal solution for the unregulated firm has already been described. The solution for the regulated case requires finding the limit, established by the demand and production functions, on the firm's desire to make the production process as capital intensive as is rationally possible.

Solution of the Model: A Specific Case

A solution is developed first for the simplest, but specific, case of constant returns to scale (that is, where both exponents of the production function, α and β, equal 1/2) and for $\sigma = 1/2$ in the demand function.[6] Later, a general solution is developed. Consider the specific case where the production function is

$$Q = 100L^{1/2}K^{1/2}, \tag{2.17}$$

the demand function is

$$P = 10Q^{-1/2}, \tag{2.18}$$

and the profit function is

$$O = PQ-4L-(.05)(100)K. \tag{2.19}$$

In (2.19), 4 is the wage rate; .05 is the rate of return, and 100 is the price of a unit of capital.

---

[6] Solution for equilibrium quantities for K, L, Q and P for this specific case is simple and straight-forward. Solution of the simple, specific case facilitates presentation later of the general, and more complex, case.

$$PQ = 10Q^{1/2}$$

$$PQ = 10(100L^{1/2}K^{1/2})^{1/2} \qquad (2.20)$$

$$PQ = 100L^{1/4}K^{1/4}.$$

Substituting in the zero excess profit function (2.19),

$$0 = 100L^{1/4}K^{1/4}-4L-5K. \qquad (2.21)$$

Since regulated profit is a linear function of K, profit is maximum when

$$\frac{dK}{dL} = 0.$$

Differentiating (2.21) with respect to L,

$$0 = \frac{100}{4}L^{-3/4}K^{1/4}-4$$

$$L = \frac{100^{4/3}}{16}K^{1/3}. \qquad (2.22)$$

Substituting for L in equation

$$0 = 100 \frac{100}{16}^{1/3}K^{1/12}-4 \frac{100}{16}^{4/3}K^{1/3}-5K,$$

and solving for K,

$$K = \frac{138.4}{5}^{3/2} = 144. \qquad (2.23)$$

Substituting in equation (2.22) permits solving for L,

$$L = \frac{100}{16}^{4/3}(144)^{1/3} = 60.7.$$

Now that K and L are known, output, Q, may be solved for:

$$Q = mL^{1/2}K^{1/2} = 100x(144)^{1/2}(60.7)^{1/2} = 9360. \qquad (2.24)$$

In turn, substituting in the demand function permits solving for price, P

$$P = HQ^{-\sigma}$$

$$P = 10(9360)^{-1/2} = .1032. \qquad (2.25)$$

General solutions for K, L, Q and P represent equilibrium interplay of inputs, output, and price for the regulated firm having the specific production function and facing the specific demand function described.

Solution of the Model: The General Case

The general solution is achieved in the same procedure as the specific case starting with

$$Q = mL^{\alpha}K^{\beta} \qquad (2.26)$$

$$P = HQ^{-\sigma} \qquad (2.27)$$

$$0 = PQ-wL-sWK\,. \qquad (2.28)$$

Solution for Capital, K

As in the specific case, equilibrium solutions for K, L, Q, and P start with a solution for K. An intermediate solution for L in terms of K, however, is required for substitution in the zero excess profit function (equation 2.28).

Substituting in the revenue function,

$$PQ = HQ^{1-\sigma}$$

from the production function obtains

$$PQ = Hm^{1-\sigma}L^{(1-\sigma)}K^{(1-\sigma)}. \qquad (2.29)$$

Differentiating the revenue function with respect to Q,

$$MR_Q = (1-\sigma)HQ^{-\sigma} \tag{2.30}$$

and substituting again from the production function obtains

$$= (1-\sigma)Hm^{-\sigma}L^{-\alpha\sigma}K^{-\beta\sigma}.$$

Since f_L, the marginal productivity of labor, is

$$f_L = \alpha m L^{\alpha-1}K^{\beta} \tag{2.31}$$

substituting for MR_Q and f_L from (2.30) and (2.31) into (2.28),

$$MR_Q f_L = \alpha(1-\sigma)m^{1-\sigma}HL^{\alpha(1-\sigma)-1}K^{\beta(1-\sigma)} = w. \tag{2.32}$$

Solving for, labor, L,

$$L^{[1-\alpha(1-\sigma)]} = \frac{\alpha(1-\sigma)m^{1-\sigma}HK^{\sigma(1-\sigma)}}{w}.$$

$$L = \left[\frac{\alpha(1-\sigma)m^{1-\sigma}HK^{\beta(1-\sigma)}}{w}\right]^{\frac{1}{1-\alpha(1-\sigma)}}. \tag{2.33}$$

Inasmuch as equation (2.33) contains the term K, it represents an intermediate, not a final, solution for labor, L.

Substituting for PQ from equation (2.29) into the zero excess profit function,

$$PQ - wL - sWK = 0$$

eliminates Q, so that

$$Hm^{(1-\sigma)}\ L^{\alpha(1-\sigma)}\ K^{\beta(1-\sigma)} - wL - sWK = 0. \tag{2.34}$$

Substituting for L from (2.33) into (2.34) eliminates L and thus permits solving the zero excess profit function on capital, K:

$$Hm^{1-\sigma}\ K^{\beta(1-\sigma)}\left[\frac{\alpha(1-\sigma)m^{1-\sigma}HK^{\sigma(1-\sigma)}}{w}\right]^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}$$

$$-w\left[\frac{\alpha(1-\sigma)m^{1-\sigma}HK^{\beta(1-\sigma)}}{w}\right]^{\frac{1}{1-\alpha(1-\sigma)}} - sWK = 0. \qquad (2.35)$$

Solution of equation (2.35) for K permits K to be expressed in terms of internal variables of the model. The final, general solution for capital, K, is given in equation (2.36). Intermediate steps in the solution are shown in (2.35a, b, and c).

$$sWK = \frac{H^{\frac{1}{1-\alpha(1-\sigma)}}\ m^{\frac{(1-\sigma)}{1-\alpha(1-\sigma)}}\ K^{\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}}}{w^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}}\left[\frac{1-\alpha(1-\sigma)}{[\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}}\right] \qquad (2.35a)$$

$$K^{1-\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}} = \frac{H^{\frac{1}{1-\alpha(1-\sigma)}}\ m^{\frac{(1-\sigma)}{1-\alpha(1-\sigma)}}}{w^{\frac{(1-\sigma)}{1-\alpha(1-\sigma)}}\ sW}\left[\frac{1-\alpha(1-\sigma)}{[\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}}\right] \qquad (2.35b)$$

$$K = \frac{H^{\frac{1}{1-\alpha(1-\sigma)}} \; m^{\frac{1-\sigma}{1-\alpha(1-\sigma)}}}{w^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}} \; sW} \left[\frac{1-\alpha(1-\sigma)}{[\alpha(1-\sigma)]^{-\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}}\right]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \tag{2.35c}$$

$$K = \frac{[1-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \; [\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}} \; m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \tag{2.36}$$

Equation (2.36) is the final general solution for K in terms of the internal variables of the model. The solution for K may now be used to generate general solutions for labor, L, output, Q, and price, P.

Solution for Labor, L

The general solution for L requires substitution for K from equation (2.36) into the intermediate solution for L of equation (2.33). The intermediate L function may now be solved in terms of the internal variables of the model:

$$L = \left[\frac{(1-\sigma) \quad m^{1-\sigma} \quad HK^{\beta(1-\sigma)}}{w}\right]^{\frac{1}{1-\alpha(1-\sigma)}} \tag{2.33}$$

or, restating equation (2.33)

$$L = \left[\frac{[\alpha(1-\sigma)]^{\frac{1}{1-\alpha(1-\sigma)}} \; m^{\frac{1-\sigma}{1-\alpha(1-\sigma)}} \; H^{\frac{1}{1-\alpha(1-\sigma)}}}{w^{\frac{1}{1-\alpha(1-\sigma)}}} \right] K^{\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}} .$$

(2.37)

Substituting for K from equation (2.36) into the intermediate solution for L (equation 2.33)

$$L = \frac{[\alpha(1-\sigma)]^{\frac{1}{1-\alpha(1-\sigma)}} \; m^{\frac{1-\sigma}{1-\alpha(1-\sigma)}} \; H^{\frac{1}{1-\alpha(1-\sigma)}}}{w^{\frac{1}{1-\alpha(1-\sigma)}}} \, .$$

$$\cdot \left\{ \frac{[1-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \; [\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{1} \right. \, .$$

$$\left. \cdot \; \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}} \; m^{\frac{(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \; w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \right\}^{\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}} .$$

(2.38)

Equation 2.38 is a general solution for L, but the solution for L may be expressed in briefer, cleaner form in equation 2.39. Equation (2.38a) is an intermediate, clarifying step.

$$L = \frac{[1-\alpha(1-\sigma)]^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\,[\alpha(1-\sigma)]^{\frac{1-(\alpha+\beta)(1-\sigma)+\beta(1-\sigma)^2}{[1-\alpha(1-\sigma)][1-(\alpha+\beta)(1-\sigma)]}}}{(sW)^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}\,\circ$$

$$\circ\,\frac{H^{\frac{1-(\alpha+\beta)(1-\sigma)+\beta(1-\sigma)}{[1-\alpha(1-\sigma)][1-(\alpha+\beta)(1-\sigma)]}}\,m^{\frac{1-(\alpha+\beta)(1-\sigma)+\beta(1-\sigma)}{[1-\alpha(1-\alpha)][1-(\alpha+\beta)(1-\sigma)]}}}{w^{\frac{[1-(\alpha+\beta)(1-\sigma)]-\alpha\beta(1-\sigma)^2}{[1-\alpha(1-\sigma)][1-(\alpha+\beta)(1-\sigma)]}}} \tag{2.38a}$$

$$L = \frac{[1-\alpha(1-\sigma)]^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\,[\alpha(1-\sigma)]^{\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{1}\,\circ$$

$$\circ\,\frac{m^{\frac{[1-\sigma]}{1-(\alpha+\beta)(1-\sigma)}}\,H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\,w^{\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}\,. \tag{2.39}$$

Equation (2.39) is the final general solution for Labor, L. It may be used with the general solution for capital, K, to generate general solutions for output, Q, and price, P.

Solution for Output, Q

Output, Q, may now be solved for by substituting for K and L, equations (2.36) and (2.39) respectively, in the production function

$$Q = mL^{\alpha}K^{\beta} \tag{2.40}$$

$$Q = \frac{m\,[1-\alpha(1-\sigma)]^{\frac{\alpha\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\,[\alpha(1-\sigma)]^{\frac{\alpha[1-\beta(1-\sigma)]}{1-(\alpha+\beta)(1-\sigma)}}}{I}\,\circ$$

$$\circ\,\frac{m^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\,H^{\frac{\alpha}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{\alpha\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\,w^{\frac{\alpha[1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}\,\circ$$

$$\circ\,\frac{[1-\alpha(1-\sigma)]^{\frac{\beta[1-\alpha(1-\sigma)]}{1-(\alpha+\beta)(1-\sigma)}}\,[\alpha(1-\sigma)]^{\frac{\alpha\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{I}\,\circ$$

$$\circ\,\frac{m^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\,H^{\frac{\beta}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{\beta[1-(1-\sigma)]}{1-(\alpha+\beta)(1-\sigma)}}\,w^{\frac{\alpha\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \tag{2.41}$$

$$Q = \frac{[1-\alpha(1-\sigma)]^{\frac{\beta}{1-(\alpha+\beta)(1-\sigma)}}\,[\alpha(1-\sigma)]^{\frac{\alpha}{1-(\alpha+\beta)(1-\sigma)}}}{I}\,\circ$$

$$\frac{H^{\frac{\alpha+\beta}{1-(\alpha+\beta)(1-\sigma)}}\,m^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{\beta}{1-(\alpha+\beta)(1-\sigma)}}\,w^{\frac{\alpha}{1-(\alpha+\beta)(1-\sigma)}}}\,\circ \tag{2.42}$$

Solution for Price, P

The solution for equilibrium Price, P, may be obtained by substituting for K, L, and Q in the function

$$0 = PQ - wL - sWK$$

or more conveniently, by merely substituting for Q in the demand function

$$P = HQ^{-\sigma}.$$

$$P = \frac{H[1-\alpha(1-\sigma)]^{\frac{-\beta\sigma}{1-(\alpha+\beta)(1-\sigma)}}\,[\alpha(1-\sigma)]^{\frac{-\alpha\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{1} \circ \frac{H^{\frac{1-(\alpha+\beta)}{1-(\alpha+\beta)(1-\sigma)}}\,m^{\frac{\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{-\beta\sigma}{1-(\alpha+\beta)(1-\sigma)}}\,w^{\frac{-\alpha\sigma}{1-(\alpha+\beta)(1-\sigma)}}} \tag{2.43}$$

$$P = \frac{[1-\alpha(1-\sigma)]^{\frac{-\beta\sigma}{1-(\alpha+\beta)(1-\sigma)}}\,[\alpha(1-\sigma)]^{\frac{-\alpha\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{1} \circ \frac{H^{\frac{1-(\alpha+\beta)}{1-(\alpha+\beta)(1-\sigma)}}\,m^{\frac{-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{-\beta\sigma}{1-(\alpha+\beta)(1-\sigma)}}\,w^{\frac{-\alpha\sigma}{1-(\alpha+\beta)(1-\sigma)}}}. \tag{2.44}$$

The Graphic Solution of the Model

Equilibrium inputs, output and price are shown graphically in Fig. 2.6. Fig. 2.6 is a graphical solution of the static model for the regulated firm.

The difference between the rational solution for inputs for the regulated firm on the one hand and the solution for inputs for the unregulated firm on the other becomes graphically clear in Fig. 2.6. The optimal solution for

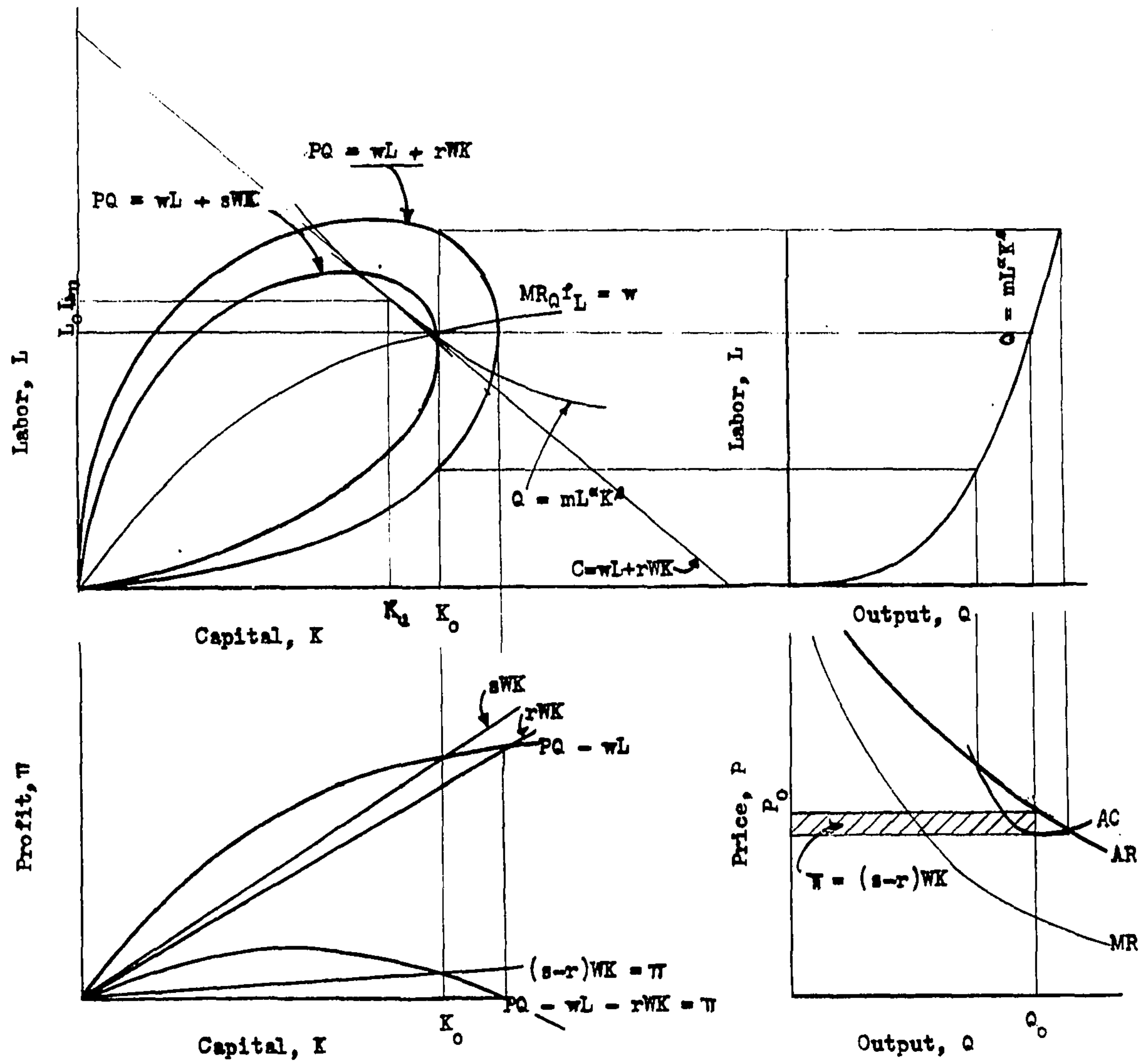

Fig. 2.6.—The complete static model for the regulated firm facing a log-linear demand function

inputs is shown as K_o and L_o. Equilibrium output is shown as Q_o, to be sold at price, P_o. The demand curve is shown as applicable to only one capital requirement, equilibrium capital, K_o. The unregulated firm would allocate its inputs as K_u and L_u--if it were required to produce Q_o. The unregulated firm does not produce Q_o, however, but reduces output to that level where

$$\frac{d\pi}{dK} = 0,$$

or where the profit curve

$$\pi = PQ - wL - rWK$$

is at its highest level.

A Comparison with the Minimum-Cost Case: An Example

For cost of capital, r, and using the same numerical values cited in the example of allocating inputs for the regulated firm (page 53), the corresponding solution for the cost-minimizing unregulated firm develops as follows:

$$C = wL + rWK \tag{2.45}$$

$$= w\left(\frac{Q}{m}\right)^{1/2} \frac{1}{K^{\beta/\alpha}} + rWK. \tag{2.46}$$

If α and $\beta = 1/2$, then

$$C = 4\left(\frac{9360}{100}\right)^2 \frac{1}{K} + (.04)(100)K \tag{2.47}$$

$$\frac{dC}{dK} = -\frac{(93.6)^2}{K^2} + 1 = 0 \tag{2.48}$$

$$K = 87.6\ . \tag{2.49}$$

Output remains the same as in the regulated case, or $Q = 9{,}360$, so,

$$9{,}360 = 100L^{1/2}(87.6)^{1/2} \tag{2.50}$$

$$L = 87.6 \tag{2.51}$$

Price, P, for the minimum-cost case where return, s, is earned, is not determined by the demand function, but by the profit function

$$O = PQ - wL - sWK$$

$$O = P(9{,}360) - 4(87.6) - 5(87.6) \tag{2.52}$$

$$P = .0842 \tag{2.53}$$

The corresponding rational solutions for the regulated case are K = 144, L = 60.7, P = .1032--that is, the regulated firm uses more capital and less labor to generate the same output than the cost-minimizing unregulated firm. Moreover, to earn the same return on investment in plant and equipment, the regulated firm charges a higher price than the unregulated firm.

Summary Statement: The Complete Static Model

The general static model for the regulated firm is now complete. When the production function,

$$Q = mL^{\alpha}K^{\beta},$$

the demand function,

$$P = HQ^{-\sigma},$$

the unregulated profit function,

$$\pi = PQ - wL - rWK,$$

and the regulated profit function,

$$\pi = (s-r)WK,$$

are specified, solution of the static model developed in this chapter describes the optimal allocation of inputs for the regulated firm, the output of the firm, and the price at which that output is sold.

Appendix to Chapter II

The Static Model and the Linear Demand Function

If the demand function is expressed in the more familiar linear form instead of the log-linear form, then,

$$P = a - bQ \tag{1}$$

The demand function is converted to a revenue function by multiplying both sides of (1) by Q

$$PQ = aQ - bQ^2 \tag{2}$$

Marginal revenue, MR_Q, is obtained by differentiating (2) with respect to Q :

$$MR_Q = a - 2bQ \tag{3}$$

$$= a - 2bmL^{\alpha}K^{\beta} \tag{4}$$

$$f_L = \alpha mL^{\alpha-1}K^{\beta} \tag{5}$$

$$MR_Q f_L = (a - 2bmL^{\alpha}K^{\beta})(\alpha mL^{\alpha-1}K^{\beta}) = w \tag{6}$$

$$= a\alpha mL^{\alpha-1}K^{\beta} - 2\alpha bm^2L^{2\alpha-1}K^{2\beta} = w. \tag{7}$$

For the linear demand function, marginal revenue becomes zero when $Q = a/2b$.

Calculations for allocation of inputs in case of a linear demand function parallel those for the log-linear demand function.

The fundamental relations for the static model are the same as before except for the demand function

$$P = a - bQ \tag{8}$$

$$Q = mL^{\alpha}K^{\beta} \tag{9}$$

$$\pi = PQ - wL - rWK \tag{10}$$

$$\pi = (s-r)WK \tag{11}$$

$$PQ = aQ - bQ^2 \tag{12}$$

$$= amL^{\alpha}K^{\beta} - bm^2L^{2\alpha}K^{2\beta}$$

$$amL^{\alpha}K^{\beta} - bm^2L^{2\alpha}K^{2\beta} - wL - sWK = 0. \tag{13}$$

Differentiating with respect to L,

$$a\alpha mL^{\alpha-1}K^{\beta} - 2\alpha bm^2L^{2\alpha-1}K^{2\beta} - w = 0 \tag{14}$$

or

$$K^{2\beta} - \frac{aK}{2bmL^{\alpha}} + \frac{w}{2\alpha bmL^{2\alpha-1}} = 0.$$

Solving the quadratic,

$$K^{\beta} = \frac{\frac{a}{2bmL^{\alpha}} \pm \sqrt{\frac{a^2}{4b^2m^2L^{2\alpha-4}} - \frac{w}{2\alpha bmL^{2\alpha-1}}}}{2}$$

$$K = \left\{\frac{\frac{1}{mL^{\alpha}}\left[\frac{a}{2b} \pm \sqrt{\frac{a^2}{4b^2} - \frac{2wL}{\alpha b}}\right]}{2}\right\}^{\frac{1}{\beta}}. \tag{15}$$

Equation (15) poses no special problem when numerical values are substituted for the internal variables of the model; but in general form, this expression for K is most unwieldly when used to attempt, say, a general solution for labor, L, by substituting in equation (13). To solve for K, the required value for substitution in equation (13) must be in terms of L, not K. Equation (15) may be recast as

$$2K^{\beta}L^{\alpha} = \frac{1}{m}\left[\frac{a}{2b} \pm \sqrt{\frac{a^2}{4b^2} - \frac{2wL}{\alpha b}}\right]. \tag{16a}$$

$$mK^{\beta}L^{\alpha} - \frac{a}{2b} = \pm\sqrt{\frac{a^2}{4b^2} - \frac{2wL}{\alpha b}}\ . \tag{16b}$$

Squaring both sides of (16a),

$$m^2K^{2\beta}L^{2\alpha} - \frac{2maK^{\beta}L^{\alpha}}{2b} + \frac{a^2}{4b^2} = \frac{a^2}{4b^2} - \frac{2wL}{\alpha b} \tag{16c}$$

$$m^2K^{2\beta}L^{2\alpha} - \frac{a}{b}mK^{\beta}L^{\alpha} + \frac{2wL}{\alpha b} = 0\ . \tag{17}$$

Solution of equation (17) in terms of L requires working with a polynomial. The solution may be approximated by use of Hoerner's or Newton's method.[1] The difficulties of managing either equation (15) or (17) to obtain a general solution in terms of L are obvious.

Conceptually, equations (15) and (17) are the same as their counterparts in the case of the log-linear demand function. The difficulties of managing these equations are surmountable--but obviously, the log-linear demand function more easily facilitates development of a general model for the regulated firm.

Management of equations (15) and (17) pose no such problems in the specific case where constant returns to scale apply. Restating equation (13)

$$amL^{1/2}K^{1/2} - bm^2LK - wL - sWK = 0. \tag{18}$$

Differentiating with respect to L

---

[1]John Brixey and Richard Andree, "College Mathematics" Newton: Holt, 1954), pp. 243-247.

$$1/2amL^{-1/2}K^{1/2} - bm^2K - w = 0 \tag{19}$$

$$L = \left[\frac{amK^{1/2}}{2(bm^2K + w)}\right]^2 . \tag{20}$$

Substituting in equation (9)

$$\frac{a^2m^2K}{2(bm^2K + w)} - \frac{a^2m^2K(bm^2K + w)}{4(bm^2K + w)^2} - sWK = 0 \tag{21}$$

$$K = \frac{a^2m^2 - 4sWw}{4bm^2 sW} . \tag{22}$$

Substituting for K in equation 11

$$L = \frac{sW(a^2m^2 - 4sWw)}{a^2bm^4} . \tag{23}$$

Output, Q, may now be solved for by substituting for K and L in the production function,

$$Q = mL^{\alpha}K^{\beta}$$

$$Q = m\left[\frac{sW(a^2m^2 - 4sWw)}{a^2bm^4}\right]^{1/2}\left[\frac{a^2m^2 - 4sWw}{4bm^2sW}\right]^{1/2}$$

$$Q = \frac{(a^2m^2 - 4sWw)}{2m^2ab} . \tag{24}$$

Price, P, may be solved for by substituting in the demand function

$$P = a - bQ$$

$$P = a - b\left[\frac{a^2m^2 - 4sWw}{2m^2ab}\right]$$

$$P = \frac{2m^2a^2b - m^2a^2b - 4sWw}{2m^2ab}$$

$$= \frac{m^2a^2b - 4sWw}{2m^2ab}. \qquad (25)$$

The graphical solution to the problem of determining equilibrium inputs, output and price of output for the linear demand function is shown in Fig. 2.7. Fig. 2.7 is a graphic representation of the static model for the regulated firm facing a linear demand function and is the linear counterpart for the log-linear version of Fig. 2.6. The short-run path of optimum allocation of inputs

$$MR_Q f_L = w$$

for the linear demand function differs substantially from the optimal path of inputs for the log-linear demand function. When[2]

$$P = a - bQ$$

then

$$(a - 2bmL^{\alpha}K^{\beta})\ (\alpha mL^{\alpha-1}K^{\beta}) = w$$

$$a\alpha mL^{\alpha-1}K^{\beta} - 2\alpha bm^2L^{2\alpha-1}K^{2\beta} - w = 0. \qquad (26)$$

---

[2]From equation (3), Q is determinable when $MR_Q = 0$

$$MR_Q = a - 2bQ$$
$$0 = a - 2bQ$$
$$Q = \frac{a}{2b}$$

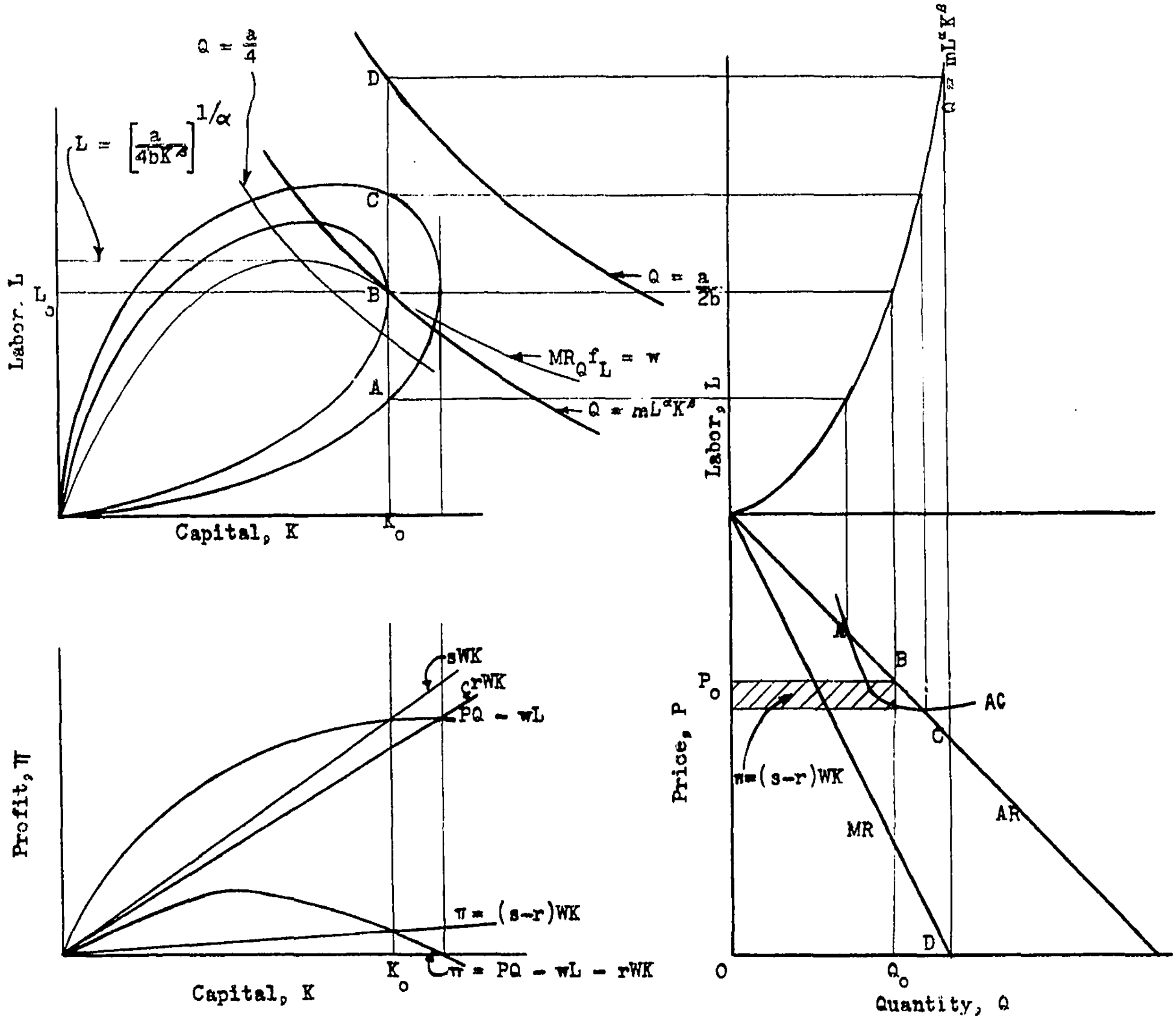

Fig. 2.7.--The complete static model for the regulated firm facing a linear demand function

Total differentiation yields

$$-\frac{dL}{dK} = \frac{\beta a\alpha mL^{\alpha-1}K^{\beta-1} - 4\alpha\beta m^2L^{2\alpha-1}K^{2\beta-1}}{(\alpha-1)a\alpha mL^{\alpha-2}K^{\beta} - (2\alpha-1)2\alpha bm^2L^{2\alpha-2}K^{2\beta}}. \quad (27)$$

Marginal revenue is zero when

$$Q = a/2b.$$

Therefore, the firm may be expected to operate when

$$Q < a/2b.$$

An inspection of equation (27) shows that for small inputs the slope of the path of $MR_Qf_L = w$ will be positive. The slope of that path becomes zero when the numerator of the right side of equation (27) becomes zero. If the numerator is set equal to zero, a solution for L at its maximum is obtained. This is equivalent to setting

$$-\frac{dL}{dK} = 0$$

or,

$$a\alpha mL^{\alpha-1}K^{\beta-1} - 4\alpha m^2L^{2\alpha-1}K^{2\beta-1} = 0. \quad (28)$$

Solving for L,

$$L = \left(\frac{a}{4mK^{\beta}}\right)^{1/\alpha}. \quad (29)$$

From the production function,

$$L = \left(\frac{Q}{mK^{\beta}}\right)^{1/\alpha}. \quad (30)$$

Output Q at the point of maximum L is obtained by substituting from (30) into (29), or

$$Q = \frac{a}{4}. \quad (31)$$

The slope of the path of optimum allocation of inputs for output greater than a/4 is negative. The question now arises as to whether the slope of this path is more or less than the slope of the isoquant itself. The slope of an isoquant is $\frac{\beta L}{\alpha K}$.

The slope of the path of $MR_Q f_L = w$ for output greater than a/4 may be shown to be less than the slope of the isoquant by first rearranging equation (27).

$$-\frac{dL}{dK} = \frac{\beta L}{\alpha K} \frac{[aL^{\alpha-1}K^{\beta-1} - 4mL^{2\alpha-1}K^{2\beta-1}]}{[aL^{\alpha-1}K^{\beta-1} - 4mL^{2\alpha-1}K^{2\beta-1} + \frac{2bm}{\alpha} L^{2\alpha-1}K^{2\beta-1} - \frac{aL}{\alpha}^{\alpha-1}K^{\beta-1}]} . \tag{32}$$

The slopes of the isoquant and the path of $mR_Q f_L = w$ are equal when

$$\frac{2bm}{\alpha} L^{2\alpha-1}K^{2\beta-1} - \frac{aL}{\alpha}^{\alpha-1}K^{\beta-1} = 0 \tag{33}$$

or

$$2bmL^{\alpha}K^{\beta} = a$$

or

$$2bQ = a$$

which is

$$Q = \frac{a}{2b}$$

the output where marginal revenue is zero. Accordingly, the firm will operate at a lesser output, or when

$$Q < \frac{a}{2b}$$

which implies

$$2bQ < a$$

and

$$2bmL^{\alpha}K^{\beta} < a$$

and

$$2bmL^{2\alpha-1}L^{2\beta-1} < aL^{\alpha-1}K^{\beta-1}. \qquad (34)$$

So, the sign of the algebraic sum of

$$\frac{2bm}{\alpha}L^{2\alpha-1}K^{2\beta-1} - \frac{aL}{\alpha}^{\alpha-1}K^{\beta-1} \qquad (35)$$

must be negative.

And since it is known that the sign of the algebraic sum of

$$a\alpha mL^{\alpha-1}K^{\beta-1} - 4\alpha\beta mL^{2\alpha-1}K^{2\beta-1} \qquad (36)$$

is negative when output is greater than a/4, the signs of both numerator and denominator of equations (32) and (27) are established as negative. And since the denominator of equation (32) is larger than the numerator when

$$\frac{a}{4} < Q < \frac{a}{2b},$$

the slope

$$\frac{\beta}{\alpha}\frac{L}{K} > \frac{\beta}{\alpha}\frac{L}{K}\frac{[aL^{\alpha-1}K^{\beta-1}-4mL^{2\alpha-1}K^{2\beta-1}]}{[aL^{\alpha-1}K^{\beta-1}-4mL^{2\alpha-1}K^{2\beta-1}+\frac{2bm}{\alpha}L^{2\alpha-1}K^{2\beta-1}-\frac{a}{\alpha}L^{\alpha-1}K^{\beta-1}]}. \qquad (37)$$

This interpretation is shown graphically in Fig. 2.8.

The equilibrium output for the regulated firm is to be found at some level less than a/2b and probably greater than a/4. The precise level of equilibrium output is determined by the values of H and σ of the demand function, by m, α, and β of the production function, and by w, W, and s of the profit function.

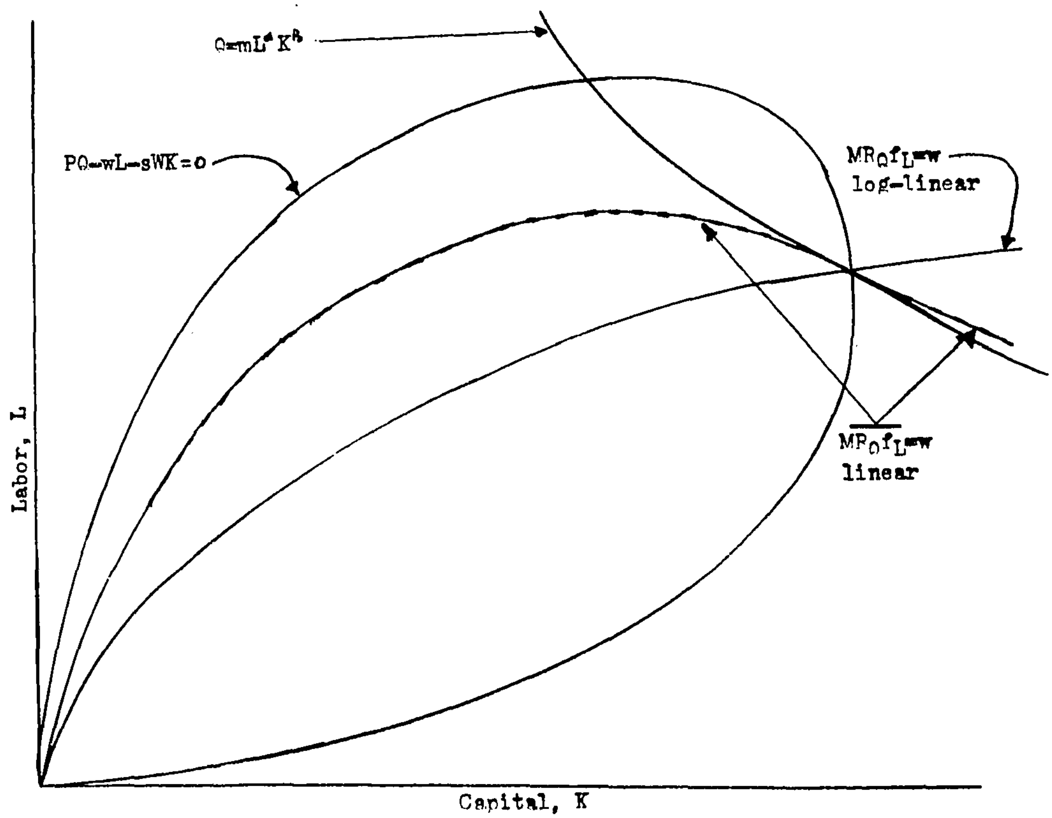

Fig. 2.8.--Comparisons of short-run paths of allocation of inputs for the regulated firm facing linear and log-linear demand functions.

In the short run, the production process of the regulated firm becomes more capital intensive as output increases regardless of the specification of the demand function. The path of optimum allocation of inputs for the log-linear demand function is continually rising, i.e., both L and K increase as output increases. Significantly, for the linear demand function, input L actually decreases as short-run output is expanded beyond $Q = a/4$. This difference in allocation of inputs imposed by the two demand functions may not be as important as it would first appear. The paths may have no divergence at the equilibrium output for the firm. The output where the two paths intersect may be obtained by equating the marginal revenues for the two demand functions and solving the resulting polynomial for output Q, which is the equilibrium output:

$$(1 - \sigma)\ HQ^{-\sigma} = a - 2bQ$$

$$(1 - \sigma)\ H = aQ^{\sigma} - 2bQ^{1+\sigma}\ .$$

CHAPTER III

ANALYSIS OF THE STATIC MODEL FOR THE REGULATED FIRM

In this chapter, the static model for the regulated firm developed in Chapter II is analyzed and interpreted. The model retains its static character even though some aspects of the model under conditions of change resemble dynamic behavior. No inter-period analysis is made here. The capital stock is again assumed to be indestructible but the effects of changes of variables within the model are analyzed, and their results interpreted.

In Chapter II, equilibrium solutions for capital, labor, output, and price for the regulated firm were developed. These solutions were:

Capital,

$$K = [1-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} [\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}} m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \qquad (3.1)$$

Labor,

$$L = [1-\alpha(1-\sigma)]^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} [\alpha(1-\sigma)]^{\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)1-\sigma)}} \cdot$$

$$\cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}} m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} w^{\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \cdot \tag{3.2}$$

Output,

$$Q = [1-\alpha(1-\sigma)]^{\frac{\beta}{1-(\alpha+\beta)(1-\sigma)}} [\alpha(1-\sigma)]^{\frac{\alpha}{1-(\alpha+\beta)(1-\sigma)}} \cdot$$

$$\cdot \frac{H^{\frac{\alpha+\beta}{1-(\alpha+\beta)(1-\sigma)}} m^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{\beta}{1-(\alpha+\beta)(1-\sigma)}} w^{\frac{\alpha}{1-(\alpha+\beta)(1-\sigma)}}} \cdot \tag{3.3}$$

Price,

$$P = [1-\alpha(1-\sigma)]^{\frac{-\beta\alpha}{1-(\alpha+\beta)(1-\sigma)}} [\alpha(1-\sigma)]^{\frac{-\alpha\sigma}{1-(\alpha+\beta)(1-\sigma}},$$

$$\cdot \frac{H^{\frac{1-(\alpha+\beta)}{1-(\alpha+\beta)(1-\sigma)}} m^{\frac{-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{-\beta\sigma}{1-(\alpha+\beta)(1-\sigma)}} w^{\frac{-\alpha\sigma}{1-(\alpha+\beta)(1-\sigma)}}} \cdot \tag{3.4}$$

If exponents α, β, and σ of the production and demand functions are held constant, i.e.,

$$\frac{d\alpha}{\alpha} = \frac{d\beta}{\beta} = \frac{d\sigma}{\sigma} = 0 \tag{3.5}$$

then,

$$\frac{dK}{K} = \left[\frac{1}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{dH}{H} + \left[\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{dm}{m} -$$

$$- \left[\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{d(sW)}{(sW)} - \left[\frac{1\cdot\sigma}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{dw}{w} \qquad (3.6)$$

$$\frac{dL}{L} = \left[\frac{1}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{dH}{H} + \left[\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{dm}{m} -$$

$$- \left[\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{d(sW)}{(sW)} - \left[\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{dw}{w} \qquad (3.7)$$

$$\frac{dQ}{Q} = \left[\frac{\alpha+\beta}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{dH}{H} + \left[\frac{1}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{dm}{m} -$$

$$- \left[\frac{\beta}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{d(sW)}{(sW)} - \left[\frac{\alpha}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{dw}{w} \qquad (3.8)$$

$$\frac{dP}{P} = \left[\frac{1-(\alpha+\beta)}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{dH}{H} - \left[\frac{\sigma}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{dm}{m} +$$

$$+ \left[\frac{\beta\sigma}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{d(sW)}{(sW)} + \left[\frac{\alpha\sigma}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{dw}{w}\ . \qquad (3.9)$$

Effect of Changes in Demand

The effect of a change in demand, i.e., a change in variable H, the coefficient of the log-linear demand function, on the four equilibrium products of the static model: capital, labor, output, and price may be expressed

$$\frac{\partial K}{K} = \left[\frac{1}{1-(\alpha+\beta)(1-\sigma)}\right]\frac{\partial H}{H} \qquad (3.10)$$

$$\frac{\partial L}{L} = \left[\frac{1}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial H}{H} \tag{3.11}$$

$$\frac{\partial Q}{Q} = \left[\frac{\alpha\cdot\beta}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial H}{H} \tag{3.12}$$

$$\frac{\partial P}{P} = \left[\frac{1-(\alpha+\beta)}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial H}{H}\,. \tag{3.13}$$

A change in variable H, among all the variables of the model, is unique in that a change in H represents a shift of the demand function rather than movement along it as induced by changes in m, s, W, and w. A change in H of the log-linear demand function corresponds to a change in the Y-axis intercept of the linear demand function. Exponents α, β, and σ are still assumed to be constant. Because a change in H represents a shift of the demand function, it can be expected to have a greater impact on the equilibrium products of the model; capital, labor, output, and price; than the other variables of the model. Equations (3.10) and (3.11) show that changes in capital and labor requirements as a result of changes in variable H are identical. Equations (3.6), (3.7), (3.8), and (3.9) show that the log-linear factor [the elements within brackets in equations (3.10) through (3.13)] for all variables have in common the term $1-(\sigma+\beta)(1-\sigma)$ in the denominator of the factor. This term approaches zero as $(\alpha+\beta)(1-\sigma)$ approaches one and accordingly, the log-linear factor becomes indeterminate.

For all determinate cases,

$$\frac{1}{1-(\alpha+\beta)(1-\sigma)} > 1.$$

Therefore, a one per cent change in H may be expected to result in more than a one per cent change in capital and labor.[1] The profound effect of a change in price-quantity elasticity, σ, on capital and labor requirements becomes clear as the effect of even a small change on the denominator, $1-(\alpha+\beta)(1-\sigma)$, is observed.

The effect of a change in variable H on output is conditioned, in part, by returns to scale. The expression

$$\frac{\partial Q}{Q} = \left[\frac{\alpha+\beta}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial H}{H}$$

shows this to be true. Only for constant returns to scale would a change in H have the same effect on output as on capital and labor requirements. Obviously, a decrease in σ (or an increase in the elasticity of demand) has the same explosive effect on output as it has on capital and labor requirements. In the case of increasing returns to scale, a one per cent increase in demand causes a greater than one per cent increase in output. A one per cent increase in demand increases output more than proportionately even for decreasing returns to scale if

$$\alpha+\beta > 1-(\alpha+\beta)(1-\sigma).$$

Equation (3.13) relating changes in price to changes

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[1]In Chapter II, p. 42, it was shown that price-quantity elasticity, σ, is the reciprocal of the more familiar concept, elasticity of demand.

in demand may be restated

$$\frac{\partial P}{\partial H} = \left[\frac{1-(\alpha+\beta)}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{P}{H} . \qquad (3.14)$$

Equation (3.14) conforms to widely held notions as to the shape of the long-run cost curve for increasing and decreasing returns to scale. The numerator tells whether the long-run cost falls or rises; and the equation (3.14) also precisely defines the sign and slope of the cost curve and, hence, whether the latter is rising or falling.

Equation (3.14) shows that for constant returns to scale a change in demand has no effect on price. This situation must be considered a special case. Clearly, for all other returns to scale, increasing and decreasing, a change in demand does affect price.

<u>Effect of Change in Technology</u>

The coefficient m of the production function

$$Q = mL^{\alpha}K^{\beta}$$

is usually considered a measure of the state of technology of the production process. The effect of a change in technology on capital, labor, output, and price may be expressed

$$\frac{\partial K}{K} = \left[\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial m}{m} \qquad (3.15)$$

$$\frac{\partial L}{L} = \left[\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial m}{m} \qquad (3.16)$$

$$\frac{\partial Q}{Q} = \left[\frac{1}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial m}{m} \qquad (3.17)$$

$$\frac{\partial P}{P} = - \left[\frac{\sigma}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial m}{m} . \qquad (3.18)$$

A change in technology causes identical changes in capital and labor. The term $(1-\sigma)$ first appeared in the revenue function

$$PQ = HQ^{1-\sigma} = Hm^{1-\sigma}L^{\alpha(1-\sigma)}K^{\beta(1-\sigma)}.$$

The term $(1-\sigma)$ of equations (3.15) and (3.16) indicates that one per cent change in technology causes a less than proportionate change in capital and labor. If price-quantity elasticity is one, the numerator of (3.15) and (3.16) becomes zero; hence a change in technology has no effect on capital and labor requirements. A positive change in technology has a positive effect on capital and labor (3.15 and 3.16) for the simple reason that improving technology decreases the cost of output. Decreased cost of output serves to increase--by way of the demand function--requirements of inputs capital and labor.

A change in technology causes the same change in output (3.17) that a change in demand causes in changes of capital and labor; i.e., the elasticity of output to technology is determined by the dimensions of the denominator, $1-(\alpha+\beta)(1-\sigma)$, as in equation (3.17). The elasticity of price to technology (3.18) has σ alone in the numerator; but σ, as for the equations relating to changes in variables H, above, w (3.19) through (3.22), and (sW) (3.23) through (3.26), also appears in the denominator. Whether or not a change of one per cent increase in technology causes more than a one per cent decrease in price depends in large measure on returns to scale, $(\alpha+\beta)$.

Effect of Change in the Wage Rate

The effect of a change in the wage rate on capital, labor, output, and price may be expressed.

$$\frac{\partial K}{K} = - \left[\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial w}{w} \tag{3.19}$$

$$\frac{\partial L}{L} = - \left[\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial w}{w} \tag{3.20}$$

$$\frac{\partial Q}{Q} = - \left[\frac{\alpha}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial w}{w} \tag{3.21}$$

$$\frac{\partial P}{P} = \left[\frac{\alpha\sigma}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial w}{w} \tag{3.22}$$

An increase in the wage rate, w, does not result in an increase in capital requirements. An increase in the wage rate makes output more expensive. The demand function, then, acts to decrease output, which, in turn, causes a decrease in capital. An increase in the wage rate causes changes in the same negative direction for labor and output, but the magnitude of the change is clearly different. For most cases, it seems likely that

$$1-\beta(1-\sigma) > \alpha(1-\sigma)$$

which would mean that a decrease in the wage rate causes a greater positive change in requirements of labor than of capital. As for output, it is not at all surprising to find that the elasticity of output to labor is represented in the numerator of the log-linear factor describing the change in output in response to a change in the price of labor. This characteristic emerges when K and L are substituted in

$$Q = mL^{\alpha}K^{\beta},$$

$$Q = \left[\frac{C'}{w^{\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}\right]^{\alpha} \cdot \left[\frac{C''}{w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}\right]^{\beta}$$

$$Q = \frac{C'''}{w^{\frac{\alpha[1-\beta(1-\sigma)]}{1-(\alpha+\beta)(1-\sigma)}} \cdot w^{\frac{\beta\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} = \frac{C'''}{w^{\frac{\alpha}{1-(\alpha+\beta)(1-\sigma)}}}$$

where C', C'', C''' are inclusive constants substituted for all internal variables and constants except w.

Similarly, the product $\alpha\sigma$ appears in the numerator of the factor describing the elasticity of price of output to the wage rate.

Capital, labor, and output decrease as the wage rate increases. Price increases as the wage rate increases. The relations developed here measure the magnitude of the changes in capital, labor, output and price in response to a change in the wage rate.

Changes in Return and the Price of a Unit of Capital

Return on, and the price of, a unit of capital are treated together in the term, (sW). The effect of changes in (sW) on capital, labor, output, and price may be expressed

$$\frac{\partial K}{K} = - \left[\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial(sW)}{(sW)} \qquad (3.23)$$

$$\frac{\partial L}{L} = - \left[\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial(sW)}{(sW)} \qquad (3.24)$$

$$\frac{\partial Q}{Q} = - \left[\frac{\beta}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial(sW)}{(sW)} \qquad (3.25)$$

$$\frac{\partial P}{P} = \left[\frac{\beta\sigma}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\partial(sW)}{(sW)}. \qquad (3.26)$$

An increase in return, s, does not result in the use of more capital. An increase in return, like an increase in the price of a unit of capital or an increase in the wage rate, makes the product more expensive. This change causes a decrease in output, which, in turn, causes a decrease in requirements of capital and labor. Obviously, an increase in (sW) results in an increase in price. The terms, $1-\alpha(1-\sigma)$ and $\beta(1-\sigma)$, are less than one; β and $\beta\sigma$ are also likely to be less than one. Accordingly, a one per cent change in s, W, or the product sW is expected to cause a less than proportionate change in K, L, Q, and P.

Some Comparisons of, and Combinations of, Changes in Internal Variables

A change in (sW) causes changes in K, L, Q and P that are in the same direction as those caused by changes in the wage rate. The question arises as to whether a one per cent increase in the wage affects capital requirements more than a one per cent increase, in the price, W, of unit of capital or in the rate of return, s. The answers cannot be definitely stated until α, β, and σ are known; but later, in the discussion of the expansion path of the firm, it is established that $\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}$ may be expected to be less than one. If so, then clearly

$$\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)} > \frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}.$$

Accordingly, an increase of one per cent in the price of a unit of capital (or in the rate of return) decreases capital requirements more than a one per cent increase in the wage rate. As for the relative effect of changes in (sW) and w on a change in labor, no general statement may be made as to whether $1-\beta(1-\sigma)$ is greater than $\beta(1-\sigma)$.

A one per cent increase in demand affects changes in capital by a factor $\frac{1}{1-\sigma}$ greater than changes in technology. Again the larger part of this greater response of capital to a change in H than in m lies in the fact that the former represents a shift of the demand function while the latter represents movement along it.

The effect of a one per cent change in technology on capital requirements may be offset by a $\frac{1}{\alpha}$ per cent change in the wage rate or by a $\frac{(1-\sigma)}{1-\alpha(1-\sigma)}$ per cent change in (sW).

Combinational changes may be studied, e.g., the effect of a one per cent increase in the price of both capital and labor on the use of capital (or use of labor, or in output, or price of output) may be investigated. A change in capital, labor, output, or price may be explained not only by the values of the variables within the model (H, m, w, W, and s) but by changes in any one, or all, of the remaining three of the solutions for K, L, Q, and P of the model. For example, output may be explained

$$\frac{dQ}{Q} = \frac{dm}{m} + \frac{\alpha dL}{L} + \frac{\beta dK}{K} . \qquad (3.27)$$

This expression may be modified to explain changes in capital another way,

$$\frac{dK}{K} = \frac{dm}{\beta m} + \frac{\alpha}{\beta}\frac{dL}{L} - \frac{1}{\beta}\frac{dQ}{Q} . \qquad (3.27a)$$

Price changes may be explained by

$$\frac{dP}{P} = \frac{dH}{H} - \sigma \frac{dQ}{Q} \qquad (3.28)$$

or by,

$$\frac{dP}{P} = \frac{dH}{H} - \sigma\frac{dm}{m} - \frac{\alpha\sigma dL}{L} - \frac{\beta\sigma dK}{K} . \qquad (3.28a)$$

Intermediate relations may also explain changes, in solutions for K, L, Q, and P. For example, since

$$MR_Q f_L = w$$

then

$$\frac{d(MR_Q)}{MR_Q} + \frac{df_L}{f_L} = \frac{dw}{w} .$$

Similarly,

$$\frac{d(MR_Q)}{MR_Q} + \frac{df_K}{f_K} = \frac{d(sW)}{sW} .$$

The intermediate solution for labor is another example. Here changes in labor may be explained by combinational changes of variables within the model plus a change in capital, K.

Since,

$$L = \left[\frac{Hm^{1-\sigma}\alpha(1-\sigma)K^{\beta(1-\sigma)}}{w}\right]^{\frac{1}{1-\alpha(1-\sigma)}}$$

or

$$L = \frac{H^{\frac{1}{1-\alpha(1-\sigma)}} m^{\frac{1-\sigma}{1-\alpha(1-\sigma)}} [\alpha(1-\sigma)]^{\frac{1}{1-\alpha(1-\sigma)}} K^{\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}}}{w^{\frac{1}{1-\alpha(1-\sigma)}}}$$

then, assuming that α, β, and σ remain constant,

$$\frac{dL}{L} = \left[\frac{1}{1-\alpha(1-\sigma)}\right]\frac{dH}{H} + \left[\frac{1-\sigma}{1-\alpha(1-\sigma)}\right]\frac{dm}{m} +$$

$$+ \left[\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}\right]\frac{dK}{K} - \left[\frac{1}{1-\alpha(1-\sigma)}\right]\frac{dw}{w} .$$

The examples above (assuming α, β, and σ are constant) are only a few of the many possible combinations of changes in the internal variables and changes in the solutions for K, L, Q, and P that may explain changes in a selected single solution of the model.

Changes in Capital in Response to Changes in σ, α, and β

Attention is now directed toward explanations of the changes in capital where all internal variables H, m, w, and sW are held constant, but where σ of the demand function and α and β of the production function are allowed to change. Differentiation of K with respect to α, β, and σ is a considerably more involved exercise than for the variables previously investigated. Under this assumption all changes in capital may be expressed

$$dK = \frac{\partial K}{\partial \sigma} d\sigma + \frac{\partial K}{\partial \alpha} d\alpha + \frac{\partial K}{\partial \beta} d\beta. \qquad (3.29)$$

Restating the capital function,

$$K = [1-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} [\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-(\sigma+\beta)(1-\sigma)}} \cdot$$

$$\cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}} \, m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \, w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \cdot$$

Differentiation with respect to σ, α, and β requires that the capital function be expressed in logarithmic form:

$$\log K = \left[\frac{1}{1-(\alpha+\beta)(1-\sigma)}\right] \log H + \left[\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}\right] \log m$$

$$+ \left[\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \log\,[1-\alpha(1-\sigma)] + \left[\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \log\,[\alpha(1-\sigma)]$$

$$- \left[\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \log(sW) - \left[\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \log w \tag{3.30}$$

The Change in Capital in Response to a Change in the Price-Quantity Elasticity, σ

Changes in capital, K, in response to a change in price-quantity elasticity, σ, is expressed[2]

$$\frac{\partial K}{\partial \sigma} = \frac{[1-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\ [\alpha(1-\sigma)]^{\frac{(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{[1-(\alpha+\beta)(1-\sigma)]^2} \cdot$$

$$\cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}}\ m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\ w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \cdot$$

$$\cdot \left| -(\alpha+\beta)\log H - \log m - \beta\log[1-\alpha(1-\sigma)] - \alpha\log[\alpha(1-\sigma)] + \alpha\log w - \beta\log(sw) \right| \tag{3.31}$$

All terms in last bracketed collection of terms in (3.31) are negative in sign except the wage rate. An increase in σ, then, has a drastic negative effect on capital.

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[2]Details of the logarithmic differentiation are supplied in the appendix to this chapter.

The corollary observation is that an increase in the elasticity of demand has an equally drastic positive effect on capital requirements.

The Change in Capital in Response to a Change in the Elasticity of Output to Labor

The change in capital, K, in response to a change in α, the elasticity of output to labor, may be expressed

$$\frac{\partial K}{\partial \alpha} = \frac{[1-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\,[\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{[1-(\alpha+\beta)(1-\sigma)]^2} \cdot$$

$$\cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}}\; m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\; w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \cdot$$

$$\cdot \Big[(1-\sigma)\log H + (1-\sigma)^2\log m + \beta(1-\sigma)^2\log[1-\alpha(1-\sigma)] + [(1-\sigma)-\beta(1-\sigma)^2]\log\alpha(1-\sigma) - [(1-\sigma)-\beta(1-\sigma)^2]\log w - \beta(1-\sigma)^2\log(sW)\Big] \;. \tag{3.32}$$

The signs of the first four terms of the latter bracketed collection of terms in (3.32) are all positive. The fifth term has a negative sign but because of the subtractions within brackets is very small. The sign of logw could even be positive. The last term is unmistakably negative but the coefficient of log(sW) is clearly very small. Accordingly, $\frac{\partial K}{\partial \alpha}$ may be expected to be positive; that is, an increase in elasticity of output with respect to labor increases capital requirements.

The Change in Capital in Response to a Change in the Elasticity of Output to Capital

The change in capital, K, in response to a change in β, the elasticity of output with respect to capital, may be expressed

$$\frac{\partial K}{\partial \beta} = \frac{[-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}[\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{[1-(\alpha+\beta)(1-\sigma)]^2} \cdot$$

$$\cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}}\, m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\, w^{\frac{(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \cdot$$

$$\cdot \Big[(1-\sigma)\log H + (1-\sigma)^2\log m + [(1-\sigma)-\alpha(1-\sigma)^2]\log(1-\alpha(1-\sigma)] -$$

$$- [(1-\sigma)-\alpha(1-\sigma)^2](sW)+\alpha(1-\sigma)^2\log[\alpha(1-\sigma)]-\alpha(1-\sigma)^2\log w\Big] \quad . \tag{3.33}$$

The sign of $\frac{\partial K}{\partial \beta}$ is almost sure to be positive. The term $\alpha(1-\sigma)^2$ is very small, so the effect of the wage rate is minimal. The price of a unit of capital would have to be extremely high to make the sign of $\frac{\partial K}{\partial \beta}$ negative. In Chapter V, changes of capital are investigated in relation to the sum of α and β, or returns to scale, in a dynamic setting. Positive changes in α and β, then, may be expected to have a positive effect on changes in capital. It is difficult to imagine a case where an increase in α or β would have a negative effect on the change in capital. The expressions $\frac{\partial K}{\partial \alpha}$ and $\frac{\partial K}{\partial \beta}$ have common terms for logH and logm so the relative effect of $\partial\alpha$ versus $\partial\beta$ rests in a comparison of

$$\beta(1-\sigma)^2\log[1-\alpha(1-\sigma)] + [(1-\sigma)-\beta(1-\sigma)^2]\log\alpha(1-\sigma) -$$

$$- [(1-\sigma)-\beta(1-\sigma)^2]\log w-\beta(1-\sigma)^2\log(sW)$$

with

$$[1-\sigma)-\alpha(1-\sigma)^2]\log[1-\alpha(1-\sigma)] + \alpha(1-\sigma)^2\log\alpha(1-\sigma) -$$

$$- [\alpha(1-\sigma)^2]\log w-[(1-\sigma)-\alpha(1-\sigma)^2]\log(sW).$$

No general statement may be made here as to the relative effect of changes in α and β on K; but in Chapter V, where some specific numerical values for α, β, and σ are supplied, changes in capital in response to changes in $(\alpha+\beta)$ are shown to be greatest when the composition of this sum is most heavily weighted by β rather than α.

The Process of Attaining Equilibrium

Instead of considering wL+sWK as an expression for required revenue of the regulated firm, this sum may be considered a "quasi-cost" function. This sum is a quasi-cost in that it represents a collection of charges without being exclusively costs in the strictest sense. Indeed, this expression simulates the sum of "costs" to be covered by revenues and is referred to as "cost of service" in public-utility rate making. The symbol C' represents cost of service in the quasi-cost function

$$C' = wL + sWK. \tag{3.34}$$

This function may be used to provide for conditions of disequilibrium. Only at equilibrium is the cost of service function equal to the revenue function

$$R = PQ = HQ^{1-\sigma}. \tag{3.35}$$

A quasi-revenue function may also be devised to represent conditions of disequilibrium, i.e., in cases where revenue is not equal to revenue requirements, the symbol R' represents such conditions, or,

$$R' = HQ^{1-\sigma}.$$

Disequilibrium conditions may be investigated to gain insight into, for example, changes in capital with respect to changes in other variables of the model. Parallel investigations may also be made of changes in labor, output, and price with respect to changes in the same variables of the model.

To pursue this line of investigation, the quasi-cost and quasi-revenue functions must be expressed in terms of capital K. From the static model of Chapter II, it is known that

$$L = f\ (K).$$

To obtain this function more precisely, use may be made of the dimensions, defined in Chapter II, of L and K.

$$L = [1-\alpha(1-\sigma)]^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} [\alpha(1-\sigma)]^{\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \cdot$$

$$\cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}} m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} w^{\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}$$

and

$$K = [1-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} [\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \cdot$$

$$\cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}} m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \cdot$$

The labor-capital ratio, $\frac{L}{K}$, then, is

$$\frac{L}{K} = \frac{[\ \alpha(1-\sigma)\ ]}{[1-\alpha(1-\sigma)]} \frac{sW}{w}. \qquad (3.36)$$

Substitution from equation (3.36) may be made into equation (3.34)

$$C' = w \frac{[\ \alpha(1-\sigma)\ ]}{[1-\sigma(1-\sigma)]} \frac{sWK}{w} + sWK. \qquad (3.37)$$

Rearranging,

$$C' = \frac{sWK}{1-\alpha(1-\sigma)}. \qquad (3.38)$$

Equation (3.35) must also be expressed in terms of K and L

$$R' = HQ^{1-\sigma}. \qquad (3.39)$$

Substituting for Q from the Cobb-Douglas production,

$Q = m\ L^{\alpha}K^{\beta}$,

$$R' = H\ m^{1-\sigma}\ L^{\alpha(1-\sigma)}\ K^{\beta(1-\sigma)} \qquad (3.40)$$

$$R' = H\ m^{1-\sigma}\ K^{\beta(1-\sigma)} \left[\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}\ \frac{sWK}{w}\right]^{\alpha(1-\sigma)}$$

$$= \frac{H\ m^{1-\sigma}(sW)^{\alpha(1-\sigma)}K^{(\alpha+\beta)(1-\sigma)}}{w^{\alpha(1-\sigma)}} \left[\frac{(1-\sigma)}{1-\alpha(1-\sigma)}\right]^{\alpha(1-\sigma)}. \qquad (3.41)$$

Equating the quasi-cost and quasi-revenue functions results in the familiar equilibrium expression for capital, K, developed in Chapter II.

To equate the changes in the quasi-cost function to changes in the quasi-revenue function, it is necessary to restate the functions in the differential logrithmic form:

$$\mathrm{dlog}\ C' = -\ \mathrm{dlog}\ [1-\alpha(1-\sigma)] + \mathrm{dlog}\ (sW) + \mathrm{dlog}\ K \qquad (3.42)$$

$$\mathrm{dlog}\ R' = \mathrm{dlog}\ H + (1-\sigma)\mathrm{dlog}\ m + \alpha(1-\sigma)\mathrm{dlog}\ (sW) -$$

$$-\ \alpha(1-\sigma)\mathrm{dlog}\ w + (\alpha+\beta)(1-\sigma)\mathrm{dlog}\ K +$$

$$[\alpha(1-\sigma)]\mathrm{dlog}\ [\alpha(1-\sigma)] - [\alpha(1-\sigma)]\mathrm{dlog}[1-\alpha(1-\sigma)]. \qquad (3.43)$$

Under equilibrium conditions, the changes in the cost function may be equated to changes in the revenue function.

$$\begin{aligned}\mathrm{dlog}\ R' - \mathrm{dlog}\ C' = {} & \mathrm{dlog}\ H + (1-\sigma)\ \mathrm{dlog}\ m + \alpha(1-\sigma)\ \mathrm{dlog}(sW) \\ & - \alpha(1-\sigma)\ \mathrm{dlog}\ w + (\alpha+\beta)(1-\sigma)\ \mathrm{dlog}\ K \\ & + [\alpha(1-\sigma)]\ \mathrm{dlog}\ [\alpha(1-\sigma)] - [\alpha(1-\sigma)]\ \mathrm{dlog}\ [1-\alpha(1-\sigma)] \\ & + \mathrm{dlog}\ [1-\alpha(1-\sigma)] - \mathrm{dlog}\ (sW) - \mathrm{dlog}\ K = 0.\end{aligned} \tag{3.44}$$

Since at equilibrium,

$$R' - C' = 0$$

equation (3.44) may be rearranged:

$$\begin{aligned}[1-(\alpha+\beta)(1-\sigma)]\mathrm{dlog}\ K = {} & \mathrm{dlog}\ H+(1-\sigma)\log m+[\alpha(1-\sigma)-1]\log\ (sW) \\ & [1-\alpha(1-\sigma)]\ \mathrm{dlog}\ [1-\alpha(1-\sigma)]+[\alpha(1-\sigma)]\ \mathrm{dlog}\ [\alpha(1-\sigma)] \\ & - [\alpha(1-\sigma)]\ \mathrm{dlog}\ w.\end{aligned} \tag{3.45}$$

Dividing (3.45) by $1-(\alpha+\beta)(1-\sigma)$ and taking the antilogarithm of both sides of the equation results in the familiar expression for capital, K, developed in Chapter II.

A change in any of the variables destroys the equality of R′ and C′, but a kind of dynamic process is set in motion to restore equilibrium. The upsetting, and restoration, of equilibrium may be illustrated by examining, for example, the change in capital induced by a change in the demand function. Clearly, equilibrium is upset by this change; but immediately the adjustment process begins. First, the quasi-revenue, R′, is shifted by an amount equal to dR as shown in Fig. 3.1. Gradually the dynamic process, illustrated by the sawtooth steps, works itself out in the change of capital requirements that is shown in Fig. 3.1. The factor $\frac{1}{1-(\alpha+\beta)(1-\sigma)}$ of the log-linear function acts in a manner that parallels the action of the familiar investment multiplier $\frac{1}{1-\Delta C/\Delta Y}$ of macroeconomic theory.

Similarly an increase in quasi-cost, C', because of an increase in either return, s, or in the price, W, of a unit of capital, causes, by the same dynamic process, a decrease in capital requirements. This process is shown graphically in Fig. 3.2.

The log-linear factor $\frac{1}{1-(\alpha+\beta)(1-\sigma)}$ may be examined a bit further. Since σ of the demand function, $P = HQ^{-\sigma}$, is equal to $\frac{1}{e_d}$ where e_d is the elasticity of demand, marginal revenue MR_Q may be defined as

$$MR_Q = P\ (1 - \frac{1}{e_d}).$$

Thus, $(1-\sigma)$ becomes equivalent to marginal revenue divided by price,

$$(1-\sigma) = \frac{MR_Q}{P}.$$

For constant returns to scale and when $\sigma=1$ (and $\frac{1}{e_d}=1$), then

$$\frac{1}{1-(\alpha+\beta)(1-\sigma)} = 1.$$

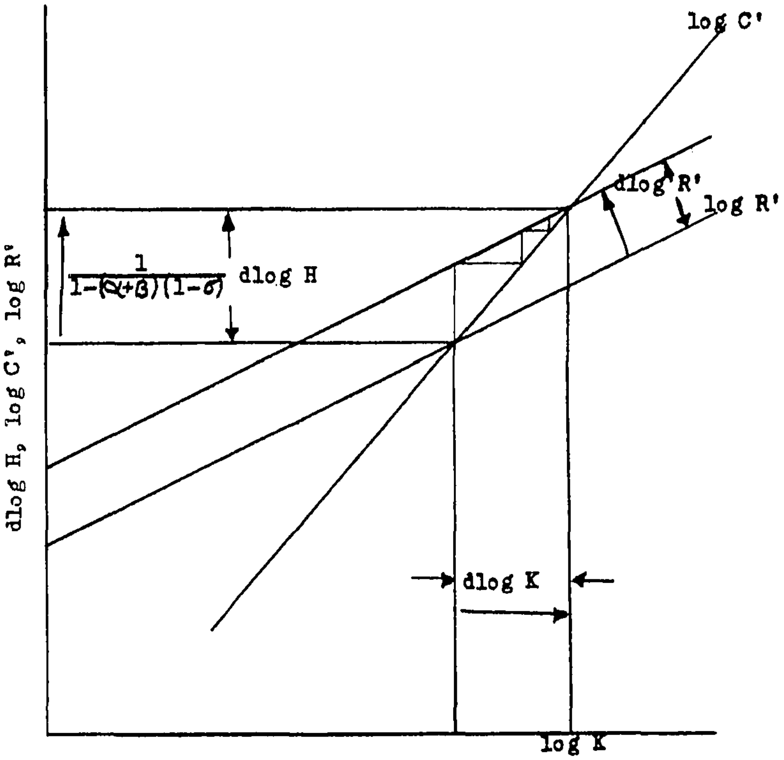


Fig. 3.1--The establishment of new equilibrium capital requirements as a result of a positive change in the demand function

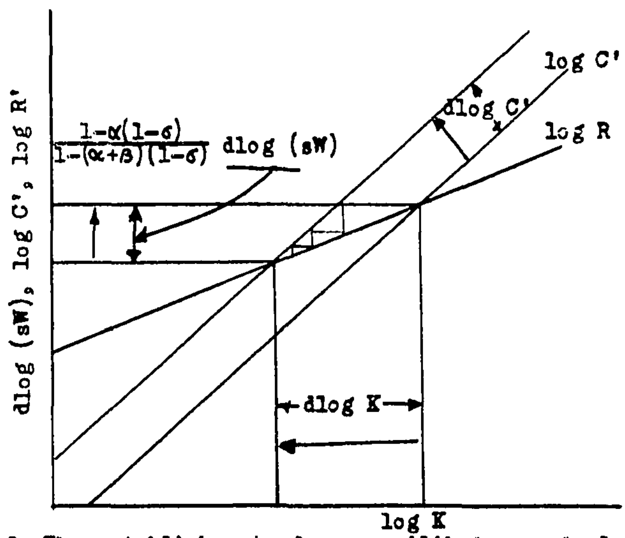


Fig. 3.2--The establishment of new equilibrium capital requirements as a result of an increase in the "cost-of-service" to cover an increase in the rate of return, s

Obviously, for increasing returns to scale, σ need not equal one for $1-(\alpha+\beta)(1-\sigma)$ to equal one. For example when $\alpha+\beta=2$, $1-(\alpha+\beta)(1-\sigma)$ is equal to one when $\sigma=1/2$. The explosive nature of this multiplier may be grasped by considering the case of increasing returns to scale and a small price-quantity elasticity, σ, (large elasticity of demand). Capital requirements approach infinity as the denominator approaches zero. No meaning is attached to $1-(\alpha+\beta)(1-\sigma) \leq 0$; that is, when $(\alpha+\beta)(1-\sigma) \geq 0$.

<u>Expansion Path of the Firm</u>

The function

$$\frac{L}{K} = \left[\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}\right] \frac{sW}{w}$$

is of greater importance than a mere device to permit substitution for Labor, L, in the cost-of-service function. The $\frac{L}{K}$ function defines the long-run expansion path of the regulated firm. The path so defined is an equilibrium path.

Since the regulated firm is rationally more capital intensive in its production process than is the unregulated firm, and since $s>r$, then the ratio $\frac{(1-\sigma)}{1-\alpha(1-\sigma)}$ may be expected to be less than one.

The ratio of the expansion paths of the regulated and unregulated firms is a constant. Therefore, the expansion path of the unregulated firm may be obtained without solving for cost-minimizing inputs. The marginal rate of technical substitution of capital for labor in the cost-minimizing firm is defined

$$MRTS_{K*} \text{ for } L* = \frac{rW}{w}$$

or

$$\frac{f_K}{f_L} = \frac{rW}{w} \text{ .}$$

But

$$f_K = \frac{BQ}{K*}$$

and

$$f_K = \frac{\alpha Q}{L*} \text{ .}$$

Thus

$$\frac{\frac{BQ}{K*}}{\frac{\alpha Q}{L*}} = \frac{rW}{w}$$

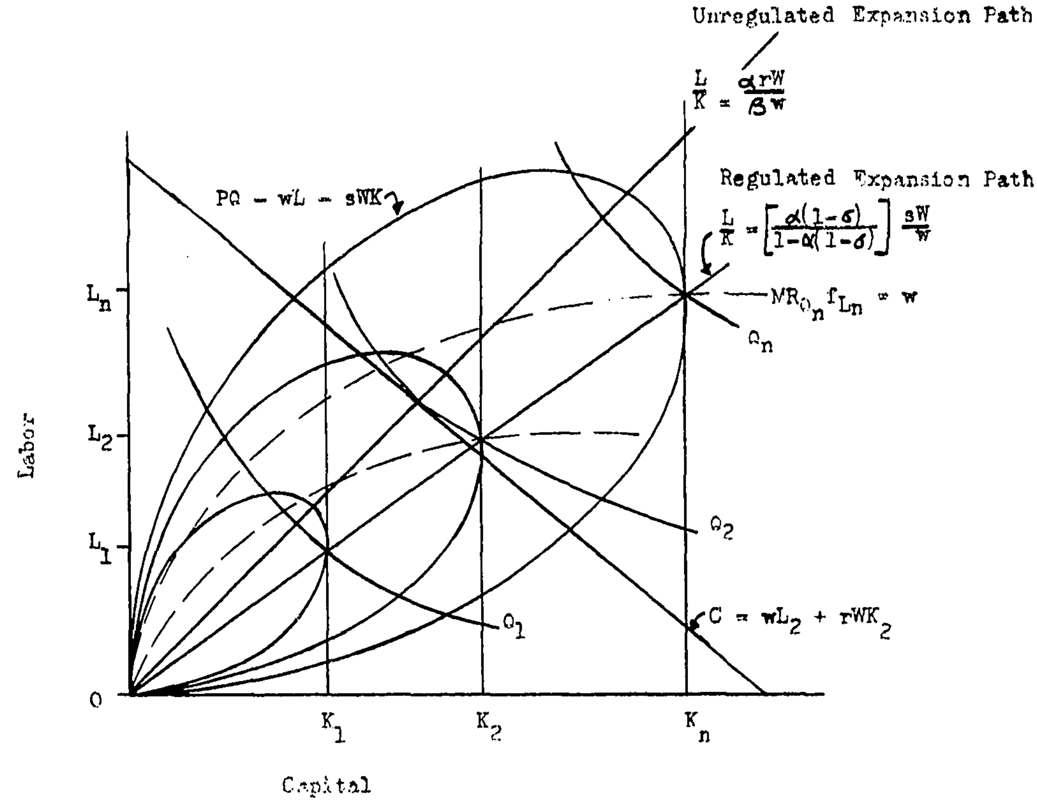
$$\frac{L*}{K*} = \frac{\alpha \; r \; W}{\beta \; w} \qquad (3.46)$$

where L* and K* are optimal inputs for the unregulated firm.

Equation (3.46) defines the expansion path of the unregulated firm. The ratio of the unregulated and regulated expansion paths then is

$$\frac{\frac{L*}{K*}}{\frac{L}{K}} = \frac{\frac{\alpha \; r \; W}{\beta \; w}}{\frac{[\;\alpha(1-\sigma)\;]}{[1-\alpha(1-\sigma)]} \frac{sW}{w}}$$

$$= \frac{r}{s} \frac{[1-\alpha(1-\sigma)]}{[\;\beta(1-\sigma)\;]} \text{ .} \qquad (3.47)$$



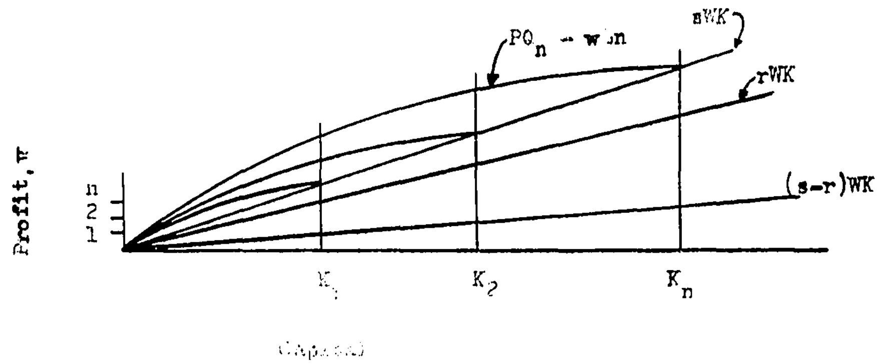


Fig. 3.3.—Expansion paths of the regulated and the unregulated firms

The expansion path defined by $\frac{\alpha}{\beta}\frac{r W}{w}$ has been described as "efficient."[3] Expansion paths for the regulated and unregulated firm are shown in Fig. 3.3. The divergence between these two paths is a central theme in proposals to improve the "quality of regulation." To the extent that divergence from cost minimization is a measure of efficiency, the expression

$$E = \frac{s}{r}\,\frac{[\ \beta(1-\sigma)]}{[1-\alpha(1-\sigma)]} \qquad (3.48)$$

determines, in per cent, the efficiency of the regulated firm. Equation (3.48) has significant welfare implications to those who believe any departure from cost minimization is socially undesirable. Equation (3.48) precisely defines the extent to which the regulated firm digresses from cost minimization.

Capital Requirements in Response to Changes in Output and to the Rate of Change of Output

For the static model developed in the last chapter, changes in capital requirements may be expressed in several ways. The expression, $\frac{dK}{dQ}$ may take different forms

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[3]See Alvin K. Klevorick, "The Graduated Fair Return: A Regulatory Proposal," American Economic Review, Vol. LVI, No. 2, p. 478. Klevorick defines efficiency as

$$E = \frac{L/\ K}{\frac{rW}{w}}$$

which is a ratio of $MRTS_{K \text{ for } L}$ for conditions that do not minimize costs, with rW/w.

because changes in output may be explained by changes in one or more variables of the model itself.

If changes in output are explained by changes in the demand function, then from equations (3.6) and (3.8)

$$\frac{dK}{K} = \left[\frac{1}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{dH}{H} \tag{3.49}$$

and

$$\frac{dQ}{Q} = \left[\frac{\alpha+\beta}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{dH}{H}\,. \tag{3.50}$$

Eliminating $\frac{dH}{H}$ from equations (3.49) and (3.50),

$$dK = \left[\frac{1}{\alpha+\beta}\right] \frac{K}{Q}\, dQ\,. \tag{3.51}$$

The capital-output ratio is obtained from equations (3.1) and (3.3):

$$\frac{K}{Q} = \frac{[1-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)-\beta}{1-(\alpha+\beta)(1-\sigma)}}}{[\alpha(1-\sigma)]^{\frac{\alpha\sigma}{1-(\alpha+\beta)(1-\sigma)}}}\,.$$

$$\cdot\ \frac{H^{\frac{1-(\alpha+\beta)}{1-(\alpha+\beta)(1-\sigma)}}\ m^{\frac{-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)-\beta}{1-(\alpha+\beta)(1-\sigma)}}\ w^{\frac{\alpha\sigma}{1-(\alpha+\beta)(1-\sigma)}}} \tag{3.52}$$

and may be substituted in equation (3.51) to obtain the change in capital in response to a change in output.

If an increase in output is explained by an increase in technology m, then, from equations (3.26) and (3.34)

$$dK = (1-\sigma)\ \frac{K}{Q}\ dQ. \tag{3.53}$$

Similarly if an increase in output is explained by a decrease in the wage rate, w, then again,

$$dK = (1-\sigma)\ \frac{K}{Q}\ dQ \tag{3.54}$$

and if an increase in output is explained by a decrease in (sW), then,

$$dK = \frac{1-\alpha(1-\sigma)}{\beta}\ \frac{K}{Q}\ dQ. \tag{3.55}$$

Equations (3.53), (3.54), and (3.55) are not satisfactory, because they all contain H of the demand function in the capital-output ratio. If H is eliminated, then, the absolute level of capital, K, as well as the change in K may be expressed in terms of technology, m, wage rate, w, return on the price of a unit of capital, (sW), α and β of the production function, and σ of the demand function. Changes in all these variables represent movement along the demand curve rather than a shift of the demand function.

Starting with the production function,

$$Q = m\ L^{\alpha}\ K^{\beta}$$

or,

$$Q = m\left(\frac{L}{K}\right)^{\alpha} K^{\alpha+\beta}. \tag{3.56}$$

From equation (3.36):

$$\frac{L}{K} = \frac{[\ \alpha(1-\sigma)\ ]}{[1-\alpha(1-\sigma)]}\ \frac{sW}{w}\ .$$

Substituting for $\frac{L}{K}$ in (3.56),

$$Q = m \left\{ \frac{[\ \alpha(1-\sigma)\ ]}{[1-\alpha(1-\sigma)]}\ \frac{sW}{w} \right\}^{\alpha} K^{\alpha+\beta} . \qquad (3.57)$$

Solving for capital, K,

$$K = \left\{ \frac{[1-\alpha(1-\sigma)]}{\alpha(1-\sigma)}\ \frac{w}{sW} \right\}^{\frac{\alpha}{\alpha+\beta}} \left(\frac{Q}{m}\right)^{\frac{1}{\alpha+\beta}} . \qquad (3.58)$$

Equation (3.58) makes K a function of Q. If output, Q, is a function solely of H of the demand function, then both H and Q should not appear in the same equation. Equation (3.58) permits treating output as a constant to permit, in turn, investigation of the effect of a change in m, w, sW, α, β and σ on capital requirements.

The rate of change of output may be investigated for its effect on investment, I:

$$\frac{dK}{dt} = \frac{1}{\alpha+\beta} \left\{ \frac{[1-\alpha(1-\sigma)]}{[\ \alpha(1-\sigma)\ ]}\ \frac{w}{sW} \right\}^{\frac{\alpha}{\alpha+\beta}} m^{-\frac{1}{\alpha+\beta}}\ Q^{\frac{1}{\alpha+\beta}-1}\ \frac{dQ}{dt} \qquad (3.59)$$

or, by multiplying both sides by W

$$\frac{WdK}{dt} = \frac{W}{\alpha+\beta} \left\{ \frac{[1-\alpha(1-\sigma)]}{[\ \alpha(1-\sigma)\ ]}\ \frac{w}{sW} \right\}^{\frac{\alpha}{\alpha+\beta}} m^{-\frac{1}{\alpha+\beta}}\ Q^{\frac{1-(\alpha+\beta)}{\alpha+\beta}}\ \frac{dQ}{dt} . \qquad (3.60)$$

But investment, I, is $W\frac{dK}{dt}$; therefore equation (3.60) becomes an expression for investment in terms of the rate of change of output, and from equations (3.58) and (3.59),

$$\frac{dK}{dt}\frac{1}{K} = \left[\frac{1}{\alpha+\beta}\right]\frac{dQ}{dt}\frac{1}{Q}. \tag{3.61}$$

Like equation (3.51), equation (3.61) shows the change in capital to be an increasing function of the change in output for decreasing returns to scale. But for the more interesting case, the change in capital is a decreasing function of output for increasing returns to scale. This finding for the regulated firm is in accord with Smith's conclusion for the unregulated cost-minimizing firm.[4] Like Smith's finding for the cost-minimizing firm, the investment implications of equations (3.60) and (3.61) are not generally believed.

The relations investigated in this section are those to which, in other works, the terms accelerator, accelerator coefficient, and the acceleration principle are frequently applied, both in macro- and microeconomic studies. Because these terms are not uniform in specific relations, the writer has avoided their use in this section.

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[4]Vernon L. Smith, Investment and Production (Cambridge, Mass.: Harvard University Press, 1960), pp. 313-318 and Vernon L. Smith, "Theory of Investment and Production," Quarterly Journal of Economics, Vol. LXXIII, No. 1, pp. 83-86.

The Case Where Capital Employed Falls Short of Its Regulated Optimum When the Regulated Firm Maintains a Specified Output

The static model developed in Chapter II may be used to explain what happens to profits, regulated and unregulated, when the firm uses capital in an amount less than the optimal level determined by the equilibrium solution of the model. The situation where the amount of capital falls short of the optimal amount permits profit, π, to rise even though a constant output is maintained. The increased profit, however, results from the unregulated profit function. Regulated profit, on the other hand, declines linearly as the amount of capital employed decreases. Unregulated profit rises as the amount of capital employed is reduced from the regulated firm's optimal amount to that level which would be used by the cost-minimizing firm. Classically the cost-minimizing firm increases profits as it reduces output to some optimal level. The case examined here assumes constant output. In this case, unregulated profit also rises--until a maximum profit is attained. Thereafter, as the amount of capital employed is further decreased, profit declines and ultimately disappears altogether. See Fig. 3.4.

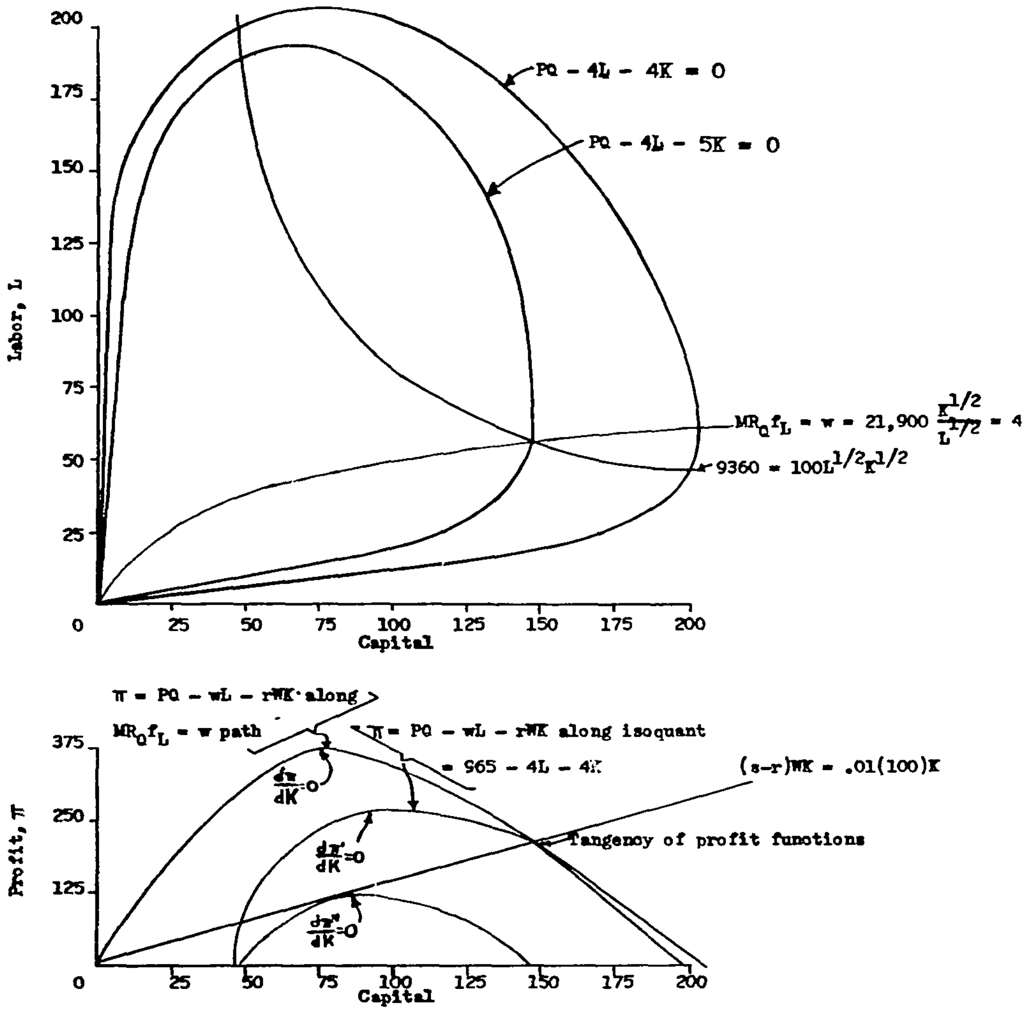
The unregulated profit function, restated, is

$$\pi = PQ-wL-rWK$$

where π is profit

r is the cost of capital

Fig. 3.4.—Illustration of excess profit resulting from substitution of variable inputs for fixed inputs



Changes in profit, π, may then be described:

$$d\pi = MR_Q dQ - wdL - rWdK. \tag{3.62}$$

Since for the case under scrutiny output does not change,

$$MR_Q dQ = 0.$$

Then,

$$d\pi = wdL - rWdk$$

$$\frac{d\pi}{dK} = w\frac{dL}{dK} - rW.$$

Along the isoquant,

$$\frac{dL}{dK} = -\frac{f_K}{f_L}$$

$$\frac{d\pi}{dK} = -w\frac{dL}{dK} - rW$$

$$\frac{d\pi}{dK} = w\frac{f_K}{f_L} - rW$$

or,

$$\frac{d\pi}{dK} = f_K\left[\frac{w}{f_L} - \frac{rW}{f_K}\right]. \tag{3.63}$$

The expression

$$\frac{w}{f_L} - \frac{rW}{f_K} = 0$$

represents conditions of maximum profit for the unregulated firm. Substitution for f_L and f_K yields the minimum-cost solution of Chapter I:

$$\frac{w}{\alpha m L^{\alpha-1} K^{\beta}} = \frac{rW}{\beta m L^{\alpha} K^{\beta-1}}. \tag{3.64}$$

Substituting for L from the production function and solving for K,

$$K = \left(\frac{w\beta}{rW\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{Q}{m}\right)^{\frac{1}{\alpha+\beta}} . \tag{3.65}$$

If price is set for a certain constant output, Q, (to be produced with the optimal allocation of inputs under regulation), the amount of profit that results when the amount of capital used decreases is designated as profit, π'. A portion, π'', of profit, π', is "excess" in that fair or allowable return is exceeded. This "excess" is subject to refund unless the regulatory authority elects to attribute the excess to chance. If the excess is attributed to chance, it presumably would be offset by a corresponding deficiency in profit during some other period. The amount of the excess profit, π'', is

$$\pi'' = PQ-wL-sWk. \tag{3.66}$$

If the regulated firm allocates its inputs optimally and if rate of return, s, is perfectly set, then excess profit, π'', is zero.

Excess profit disappears when capital, K, is decreased to a certain level, e.g. at approximately 46 of Fig. 3.4(a). Capital, K, at the point where profit disappears, may be determined by simultaneous solution of the equations

$$0 = PQ-wL-sWk$$

and

$$Q = mL^{\alpha}K^{\beta}.$$

A simultaneous solution is possible here because both output, Q, and its price, P, are fixed.

For the example in Chapter II (page 53) where regulated optimal inputs, L and K, were 144 and 607 units, respectively (and α, β, and σ each were 0.5), the output is

$$Q = 100\ L^{1/2}\ K^{1/2} = 9{,}360 \text{ units}$$

and the price is

$$P = 10\ Q^{-5} = .103.$$

Where wage rate, w, = 4

Price of a unit of capital = 100

Rate of return, s, = .05

substituting in the zero excess profit function,

$$0 = PQ\text{-}wL\text{-}sWk$$

$$0 = 963\text{-}4L\text{-}5K. \qquad (3.67)$$

From the production function,

$$L = \frac{(93.6)^2}{K}. \qquad (3.68)$$

Substituting in equation (3.67) and solving the resulting quadratic[5]

$$K = \frac{193 \pm \sqrt{193 - 4(.8)(93.6)^2}}{2} \qquad (3.69)$$

$$K = 144, 46$$

---

[5]Solutions to the equations for conditions other than constant returns to scale require working with a polynomial. Approximation by Hoerner's or Newton's Method are required. See J.C. Brixey and R.C. Andree, Fundamentals of College Mathematics (New York: Holt, 1954), pp. 243-245.

The first of these two roots is the optimal solution for capital, K, already solved for; the second is the reduced level of capital where all excess profit disappears.

Similarly, substitution in the function

$$0 = PQ-wL-rWK$$

yields the levels of capital where all "ordinary" profit disappears. For cost of capital, r, = .04; wage rate, w, = 4; and price of a unit of capital, W, = 100, the simultaneous solution of the profit function and the production function yields the two roots to the resulting quadratic, K = 197, 44. Clearly, these two levels of capital represent the outer limits of the profit "hill" for an output of 9,360 units sold at a price, P, of .103. All these solutions for capital, K, are shown graphically in Fig. 3.4.

Capital used by the unregulated firm while maintaining output of 9,360 units when

$$\frac{d\pi'}{dK} = 0$$

is obtained by differentiating the unregulated profit function

$$\pi' = PQ-wL-rWK$$

$$= 963-4\left(\frac{9360}{100}\right)^2 \frac{1}{K} = 4K$$

$$\frac{d\pi'}{dK} = \frac{93.60^{\ 2}}{K^2} -1 = 0$$

$$K = 93.6$$

This solution means that profit, π', increases as capital is decreased from the regulated firm's optimal level of 144 to 93.6 units. Thereafter profit, π', decreases until it disappears altogether at a capital use of 44 units.

A part, π'', of profit, π', is "excess" profit. Excess profit π'' is maximum when

$$\frac{d\pi''}{dK} = 0$$

This solution is obtained by differentiating the excess profit function with respect to K:

$$\pi'' = PQ-wL-sWK$$

$$\frac{d\pi''}{dK} = 963-4\left(\frac{9,360}{100}\right)^2 \frac{1}{K} - 5 = 0$$

$$K = 84.1.$$

The two profit functions, π = PQ-wL-rWK, one for optimal output along the path $MR_Q f_L = w$ and the other for constant output, are shown as tangent at the point of optimum employment of capital for the regulated firm. The point determined by

$$\frac{d\pi}{dK} = 0$$

on the path $MR_Q f_L = w$ indicates the cost-minimizing amount of capital to be used. This amount of capital is used to generate the output that equates marginal revenue and marginal cost.

But setting

$$\frac{d\pi'}{dK} = 0$$

for the π' profit function merely establishes the cost-minimizing amount of capital to be used when a particular output must be maintained. The amount of capital used when excess profit π'' is greatest, i.e., when

$$\frac{d\pi''}{dK} = 0$$

involves the same output. All three solutions are shown in Fig. 3.4.

The "Interior" Cost of Capital of the Regulated Firm

A principal theme of this study is that rational behavior of the regulated firm calls for the use of more capital in producing output than is used by the unregulated firm; furthermore, given demand and production functions, it is possible to measure how much more capital intensive the production process is in the regulated firm than in the unregulated firm. If this margin of relative capital intensity is indeed measurable, then it is possible to solve for the interior cost of capital of the regulated firm.[6]

If the optimum solution for the regulated firm calls for capital K_s, an artificial "interior" cost function may be described as

$$C_s = wL_s + \rho W K_s \tag{3.70}$$

where L_s and K_s represent optimal solution for inputs in the regulated firm

C_s is total "interior" cost

ρ is the "interior" cost of capital .

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[6]Here the term <u>internal cost of capital</u> might have been used. The term <u>internal cost of capital</u> implies some assessment of future <u>earnings to provide the</u> equity component of cost of capital. The word <u>internal</u> also is applied to funds originating from <u>retained earnings</u> and depreciation. Such funds may be regarded as having a "price" different from equity funds generated outside the firm. The word <u>interior</u> is used here to call attention to the <u>regulated firm's use</u> of capital, the "rate base," as though in effect, the rental rate of capital were lower than its market or "exterior rate."

Substituting from the production function for L_s,

$$C_s = w\left(\frac{Q}{m}\right)^{1/\alpha} \frac{1}{K_s{}^{\beta/\alpha}} + \rho W K_s . \qquad (3.71)$$

Differentiating the artificial cost function with respect to K_s,

$$\frac{dC_s}{dK_s} = - \frac{\beta w}{\alpha}\left(\frac{Q}{m}\right)^{\frac{1}{\alpha}} K^{\cdot \frac{\beta}{\alpha} -1} + \rho W = 0. \qquad (3.72)$$

The interior cost of capital ρ then, is

$$\rho = \frac{\beta w}{\alpha W}\left(\frac{Q}{m}\right)^{\frac{1}{\alpha}} K^{-\frac{\beta+\alpha}{\alpha}} . \qquad (3.73)$$

Again, in the numerical example of Chapter II, solutions for K and L were found to be 144 and 60.7 units, respectively, for the situation where the production was defined as

$$Q = 100\ L^{1/2}K^{1/2} = 9360 \text{ units}$$

the demand function was defined as

$$P = 10\ Q^{-1/2}$$

and where the rate of return, s, was .05.

The corresponding solutions for K and L for the unregulated cost-minimizing firm were both found to be 93.6 units where the cost of capital was .04. The wage rate, w, and the cost, W, of a unit of capital, were assumed at \$4 and \$100, respectively, for both the regulated and the unregulated case.

Substituting in equation (3.73)

$$\rho = \frac{(1/2)(4)(9360)^2}{(1/2)(100)(100)} 146^{-2} = .0169 \qquad (3.74)$$

In the example chosen, the margin between the "interior" cost of capital, ρ, and the exterior cost of capital (in this case, .04) appears large; but the magnitude of the margin depends upon the values attributed to the variables of the model and to the spread between the cost of exterior cost of capital, r, and rate of return, s. The findings here confirm the view that regulation serves, in effect, to cheapen the cost of capital for the firm. The findings here, however, do not confirm the findings of Averch and Johnson who held that the interior cost of capital to the regulated firm is cheapened by the amount of the difference between the rate of return, s, and the cost of capital, r.[7]

If the regulated firm knew its interior cost of capital, the solution for optimal allocation of inputs would proceed exactly as for the cost-minimizing firm. Unfortunately, the interior cost of capital cannot be known until the optimal allocation of inputs under regulation are known. In other words, the interior cost of capital to the

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[7]This pricing of the interior cost of capital is equivalent to [r-(s-r)] or [2r-s]. Only for a special contrived case would ρ be equal to [2r-s]. See Harvey Averch and Leland M. Johnson, "Behavior of the Firm Under Regulatory Constraint," American Economic Review, Vol. LII, No. 4, pp. 1052-69.

regulated firm can only be known "ex post."

The Marginal Rate of Technical Substitution, the "Lambda" Function, and their Significance to the Regulated Firm

The marginal rate of technical substitution of capital for labor for the cost-minimizing firm is

$$MRTS_{K \text{ for } L} = \frac{f_K}{f_L} = \frac{rW}{w} . \qquad (3.75)$$

Equation (3.75), however, does not describe the marginal rate of technical substitution for the regulated firm. Averch and Johnson, however, have developed a MRTS function for the regulated firm.[8] The static model developed in Chapter II may be used to more sharply define the MRTS function developed by Averch and Johnson.

Averch and Johnson evolve the MRTS function as follows [Equations (3.76) through (3.82)]:

$$Z = PQ-wL-rWK-\lambda[PQ-wL-sWK] \qquad (3.76)$$

where Z is the objective function to be minimized--and, because of λ and the expression in brackets, is an "artificial function"; λ is a coefficient, or technically the LaGrange multiplier required by the expression of Z as a constrained minimum.

Now, set

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[8]Harvey Averch and Leland L. Johnson, "Behavior of the Firm under Regulatory Restraint," American Economic Review, Vol. LII, No. 4, pp. 1052-1069.

$$\frac{dZ}{dL} = MR_Q f_L - w - \lambda MR_Q f_L + \lambda w = 0. \tag{3.77}$$

Minimizing Z with respect to L results in

$$f_L = \frac{w}{MR_Q}. \tag{3.78}$$

For this case λ is one; but for the more interesting case--under regulation--where Z is minimized with respect to K,

$$\frac{dZ}{dK} = MR_Q f_K - rW - \lambda MR_Q f_K + \lambda sW = 0. \tag{3.79}$$

$$f_K = \frac{W(r-\lambda s)}{(1-\lambda)MR_Q}. \tag{3.80}$$

Solving for λ,

$$\lambda = \frac{MR_Q f_K - rW}{MR_Q f_K - sW}. \tag{3.81}$$

Obviously, λ is not one when Z is minimized with respect to K--except for the case where regulation is absolute, i.e., all pure profit is eliminated.

Since

$$MRTS_K \text{ for } L = -\frac{dL}{dK} = \frac{f_K}{f_L},$$

$$MRTS_K \text{ for } L = \frac{\frac{W(r-\lambda s)}{(1-\lambda)MR_Q}}{\frac{w}{MR_Q}}$$

$$= \frac{W(r-\lambda)s}{(1-\lambda)w} \tag{3.82}$$

Equation (3.82) is conceptually adequate as a device to describe $MRTS_{K \text{ for } L}$, but the static model of Chapter II may be used to lend more meaning, and precision, to the term λ than is afforded by equation (3.81). Accordingly, attention is now directed to use of the static model to define λ. The procedure adopted here to define λ, is to use the model to obtain an expression for $MRTS_{K \text{ for } L}$ and then equate it to right-hand side of equation (3.82) to permit solving the resulting equation for λ.

The development of an expression for $MRTS_{K \text{ for } L}$ starts with the optimal labor-capital ratio (equation 3.36), which represents the ratio of equilibrium solutions of the model for labor, L, and capital, K:

$$\frac{L}{K} = \left[\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}\right] \frac{sW}{w}. \tag{3.83}$$

Since

$$f_L = \frac{\alpha Q}{L} \tag{3.84}$$

and

$$f_K = \frac{\beta Q}{K}, \tag{3.85}$$

substitution for f_L and f_K from equations (3.84) and (3.85) into

$$MRTS_{K \text{ for } L} = - \frac{dI}{dK} = \frac{f_K}{f_L}$$

yields

$$\text{MRTS}_{K \text{ for } L} = \frac{\frac{\beta Q}{K}}{\frac{\alpha Q}{L}} = \frac{\beta L}{\alpha K} . \tag{3.86}$$

Substituting for $\frac{L}{K}$ in (3.86) from the expression for the optimal labor-capital ratio [equation (3.83)], in turn, yields

$$\text{MRTS}_{K \text{ for } L} = \frac{sW}{w} \frac{[\ \beta(1-\sigma)\ ]}{[1-\alpha(1-\sigma)]} . \tag{3.87}$$

The $\text{MRTS}_{K \text{ for } L}$ function above resulting from the basic static model of this study may now be equated to the Averch and Johnson solution for $\text{MRTS}_{K \text{ for } L}$

$$\frac{sW}{w} \frac{[\ \beta(1-\sigma)\ ]}{[1-\alpha(1-\sigma)]} = \frac{W(r-\lambda s)}{(1-\lambda)w} . \tag{3.88}$$

Equation (3.88) permits solving for λ in terms of the variables of the static model:

$$\lambda = \frac{\frac{r}{s}[1-\alpha(1-\sigma)] - \beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)} . \tag{3.89}$$

If the cost of capital, r, is assumed fixed, then changes in λ induced by changes in the rate of return, s, may be expressed,

$$\frac{\partial \lambda}{\partial s} = - \frac{r}{s^2} \frac{[1-\alpha(1-\sigma)] - \beta(1-\sigma)}{[1-(\alpha+\beta)(1-\sigma)]} . \tag{3.90}$$

From equation (2.36), rate of return, s, may be expressed

$$s = \frac{H^{\frac{1}{1-\alpha(1-\sigma)}} \, m^{\frac{1-\sigma}{1-\alpha(1-\sigma)}} \, [1-\alpha(1-\sigma)] \, [\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}}{w^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}} \, W \, K^{\frac{1-(\alpha+\beta)(1-\sigma)}{1-\alpha(1-\sigma)}}} \, . \tag{3.91}$$

Substituting for s in equation (3.89)

$$\lambda = \frac{r[1-\alpha(1-\sigma)]-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)} \, .$$

$$\cdot \frac{H^{\frac{1}{1-\alpha(1-\sigma)}} \, m^{\frac{1-\sigma}{1-\alpha(1-\sigma)}} \, [1-\alpha(1-\sigma)][\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}}{w^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}} \, W \, K^{\frac{1-(\alpha+\beta)(1-\sigma)}{1-\alpha(1-\sigma)}}} \, . \tag{3.92}$$

Equation (3.92) expresses λ in terms of capital, K, so that changes in λ in response to changes in K may be expressed:

$$\frac{\partial\lambda}{\partial K} = - \frac{[1-(\alpha+\beta)(1-\sigma)]}{[1-\alpha(1-\sigma)]} \cdot \frac{r[1-\alpha(1-\sigma)] - \beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)} \, .$$

$$\cdot \; [1-\alpha(1-\sigma)][\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}} \, .$$

$$\cdot \; \frac{H^{\frac{1}{1-\alpha(1-\sigma)}} \, m^{\frac{1-\sigma}{1-\alpha(1-\sigma)}} \, K^{\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}}}{w^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}} \, K} \tag{3.93}$$

$$\frac{\partial\lambda}{\partial K} = - \frac{\{r[1-\alpha(1-\sigma)]-\beta(1-\alpha)\}\ [\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}\ H^{\frac{1}{1-\alpha(1-\sigma)}}}{W^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}} \cdot$$

$$\cdot \frac{M^{\frac{1-\sigma}{1-\alpha(1-\sigma)}}\ K^{\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}}}{W} . \tag{3.94}$$

Since $\beta(1-\sigma) > r[1-\alpha(1-\sigma)]$, the sign of $d\lambda/dK$ is unambiguously positive and is so shown graphically in Fig. 3.5(b). Requirements of capital K, however, are decreased by increases in the rate of return, s. It is then not surprising to find the sign of $d\lambda/ds$ to be negative.

Figs. 3.5(a) and 3.5(b) show the relation of allocation of inputs to λ. Fig. 3.5(b) is the graphic solution of equation (3.92). Changes in λ are caused by changes in rate of return, s. Equation (3.89) states λ in terms of rate of returns, s. Equation (3.90) shows the changes in λ in response to changes in rate of return, s. Fig. 3.5(d) shows how the rate of return, s, determines λ as the ray sWK swings through the regulatory "arc" of Fig. 3.5(d). Output is also related to λ. When λ is zero, output is restricted to that of the cost-minimizing firm. When λ=1, output is expanded to the level where all profit disappears. Finally, regulated profit, (s-r)WK, of Fig. 3.5(e) may be graphically related to the λ function of Fig. 3.5(b) and the return function of Fig. 3.5(c).

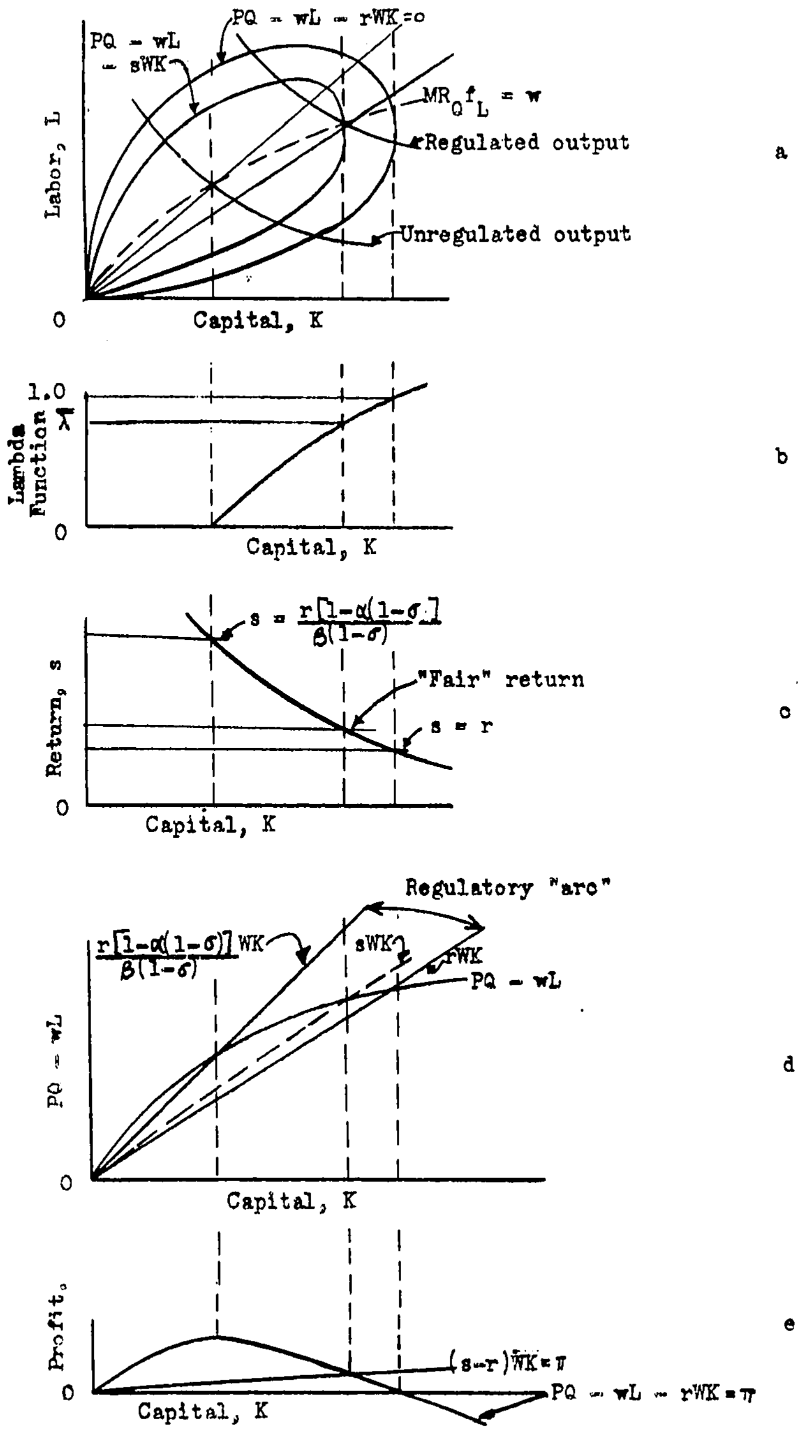

Fig. 3.5 —The interrelation of the λ function and output, return, $TNRP_K$, profit and regulation

For $\lambda=0$, or when the marginal productivity of capital is equal to its cost, the rate of return may be determined from equation (3.89). If $\lambda=0$, then

$$\frac{r}{s}\,[1-\alpha(1-\sigma)]-\beta(1-\sigma) = 0 \qquad (3.95)$$

$$s = \frac{r[1-\alpha(1-\sigma)]}{\beta(1-\sigma)}\,. \qquad (3.96)$$

Equation (3.81) shows

$$\lim_{MR_Q f_K \to rW.} \lambda=o \qquad (3.97)$$

But, from equations (3.89) and (3.95),

$$\lim_{s \to \frac{r[1-\alpha(1-\sigma)]}{\beta(1-\sigma)}} \lambda=o \qquad (3.98)$$

Again, from equations (3.81) and (3.89)

$$\lim_{s \to r} \lambda=1 \qquad (3.99)$$

If λ were known, return could be specified in terms of r, α, and β. But λ is probably more properly considered as the result of setting return, s, rather than the determinant of s. Figs. 3.5(c) and 3.5(d) are more likely to represent the deliberation of the regulatory authority than Fig. 3.3(b).

Whether the λ function is a cause of, or the effect of, the capital decision of the regulated firm may be a moot issue. The λ function is so generic in character that it can encompass the whole range of behavior of the firm from no regulation to absolute regulation. The λ function is useful, and perhaps is really required in determining the marginal rate of technical substitution, MRTS, of inputs for the regulated firm, i.e., for the case where MRTS of inputs is not described by the inverse ratio of the costs of inputs. Moreover, the λ function as used here serves another purpose: It permits the cost of capital, r, to re-enter the static model of the regulated firm. It will be remembered that cost of capital, r, was eliminated as a variable early in the development of the static model.

The subject of changes in capital in response to change in the cost of capital, r, is not investigated here for the reason that r is arbitrarily held constant; but the λ function developed here provides an opportunity to investigate the impact of changes in the cost of capital, r, and of changes in pure profit rate (s-r) on capital use, on output and price of output. The all-encompassing character of the λ function makes it a most useful tool in probing the subject of the pure theory of regulation especially in those cases where cost of capital, r, and return, s, are correlated.

This subsection concludes the analysis and interpretation of the static model. The following two chapters are devoted to adapting the static model to dynamic conditions. Chapter IV treats on the problem of replacement. Chapter V treats on the subject of optimal size of plant for a range of conditions of growth, returns to scale, elasticity, capital durability, and patterns of output.

Appendix to Chapter III

Logarithmic Differentiation of the Capital Function with Respect to σ, α, and β

Differentiation of the capital function

$$K = [1-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} [\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \cdot$$

$$\cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}} m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \tag{1}$$

with respect to σ, α and β requires that the capital function be expressed in logarithmic form and that partial differentiation of log K with respect to σ, α, and β be completed. The capital function in logarithmic form is

$$\log K = \left[\frac{1}{1-(\alpha+\beta)(1-\sigma)}\right] \log H + \left[\frac{(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \log m +$$

$$+ \left[\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \log [1-\alpha(1-\sigma)] + \left[\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \log \alpha(1-\sigma) -$$

$$- \left[\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \log w + \left[\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \log (sW) \tag{2}$$

$$\frac{\partial \log K}{\partial \sigma} = \left[\frac{-(\alpha+\beta)}{[1-(\alpha+\beta)(1-\sigma)]^2}\right] \log H +$$

$$+ \frac{-[(1-(\alpha+\beta)(1-\alpha)]-(1-\sigma)(\alpha+\beta)}{[1-(\alpha+\beta)(1-\sigma)]^2} \log m +$$

$$+ \frac{[1-(\alpha+\beta)(1-\sigma)]\alpha+\alpha(1-\sigma)][\alpha+\beta]}{[1-(\alpha+\beta)(1-\sigma)]^2} \log[1-\alpha(1-\sigma)] +$$

$$+ \left[\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right] \frac{\alpha}{1-\alpha(1-\sigma)} +$$

$$+ \frac{[1-(\alpha+\beta)(1-\sigma)][-\alpha]-\alpha(1-\sigma)(\alpha+\beta)}{[1-(\alpha+\beta)(1-\sigma)]^2} \log[\alpha(1-\sigma)] +$$

$$+ \left[\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right]\left[\frac{-\alpha}{\alpha(1-\sigma)}\right] +$$

$$+ \frac{[1-(\alpha+\beta)(1-\sigma)]\alpha+\alpha(1-\sigma)][\alpha+\beta)]}{[1-(\alpha+\beta)(1-\sigma)]^2} \log w +$$

$$+ \frac{[1-(\alpha+\beta)(1-\sigma)]\alpha+[-1+\alpha(1-\sigma)][\alpha+\beta]}{[1-(\alpha+\beta)(1-\sigma)]^2} \log (sW) \tag{3}$$

$$\frac{[1-(\alpha+\beta)(1-\sigma)]^2 \partial \log K}{\partial\sigma} = -(\alpha+\beta)\log H - \log m - \beta\log[1-\alpha(1-\sigma)]$$
$$-\alpha\log[\alpha(1-\sigma)]+\alpha\log w-\beta\log(sW) \tag{4}$$

$$\frac{\partial K}{\partial\sigma} = \frac{K}{[1-(\alpha+\beta)(1-\sigma)]^2}[-(\alpha+\beta)\log H-\beta\log m-\beta\log[1-\alpha(1-\sigma)]$$
$$-\alpha\log[\alpha(1-\sigma)]+\alpha\log w-\beta\log(sW), \tag{5}$$

Substituting for K,

$$\frac{\partial K}{\partial \sigma} = \frac{[1-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}[\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{[1-(\alpha+\beta)(1-\sigma)]^2} \cdot$$

$$\cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}} m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \cdot$$

$$\cdot \Big[-(\alpha+\beta)\log H-\log m-\beta\log[1-\alpha(1-\sigma)]-\alpha\log[\alpha(1-\sigma)] +$$

$$+ \alpha\log w-\beta\log (sW)\Big]. \tag{6}$$

Similarly, changes in K resulting from a change in α may be investigated.

$$\frac{\partial \log K}{\partial \alpha} = \frac{1-\sigma}{[1-(\alpha+\beta)(1-\sigma)]^2} \log H + \frac{(1-\sigma)^2}{[1-(\alpha+\beta)(1-\sigma)]^2} \log m$$

$$+ \frac{[1-(\alpha+\beta)(1-\sigma)][-(1-\sigma)]+[1-\alpha(1-\sigma)](1-\sigma)]}{[1-(\alpha+\beta)(1-\sigma)]^2} \log[1-\alpha(1-\sigma)]$$

$$+ \frac{(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)} + \frac{[1-(\alpha+\beta)(1-\sigma)][1-\sigma]+\alpha(1-\sigma)^2}{[1-(\alpha+\beta)(1-\sigma)]^2} \log[\alpha(1-\sigma)]$$

$$+ \frac{(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)} - \frac{[1-(\sigma+\beta)(1-\sigma)][1-\sigma]+\alpha(1-\sigma)^2}{[1-(\alpha+\beta)(1-\sigma)]^2} \log w$$

$$- \frac{[1-(\alpha+\beta)(1-\sigma)][-(1-\sigma)]+[1-\alpha(1-\sigma)][1-\sigma]}{[1-(\alpha+\beta)(1-\sigma)]^2} \log(sW)$$

$$\tag{7}$$

$$[1-(\alpha+\beta)(1-\sigma)]\ \frac{\partial K}{\partial \alpha} = (1-\sigma)\log H+(1-\sigma)^2\log m+\beta(1-\sigma)^2$$

$$\log[1-\alpha(1-\sigma)]+[(1-\sigma)-\beta(1-\sigma)^2]\log\alpha(1-\sigma)$$

$$-\ [(1-\sigma)-\beta(1-\sigma)^2]\log w-[\beta(1-\sigma)^2]\log(sW) \tag{8}$$

$$\frac{\partial K}{\partial \alpha} = \frac{[1-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}[\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{[1-(\alpha+\beta)(1-\sigma)]^2}$$

$$\frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}}\ m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\ w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}$$

$$\cdot\ \Big[(1-\sigma)\log H+(1-\sigma)^2\log m+\beta(1-\sigma)^2\log[1-\alpha(1-\sigma)]$$

$$+\ [(1-\sigma)-\beta(1-\sigma)^2]\log\alpha(1-\sigma)$$

$$-\ [(1-\sigma)-\beta(1-\sigma)^2]\log w-\beta[1-\sigma)^2]\log(sW)\Big]. \tag{9}$$

Finally changes in Capital K resulting from a change in β are investigated.

$$\frac{\partial \log K}{\partial \beta} = \frac{1-\sigma}{[1-(\alpha+\beta)(1-\sigma)]^2}\log H + \frac{(1-\sigma)^2}{[1-(\alpha+\beta)(1-\sigma)]^2}\ \log m$$

$$\frac{[1-\alpha(1-\sigma)][1-\sigma]\log[1-\alpha(1-\sigma)]}{[1-(\alpha+\beta)(1-\sigma)]^2} - \frac{(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}$$

$$+ \frac{\alpha(1-\sigma)^2}{[1-(\alpha+\beta)(1-\sigma)]^2} \log[\alpha(1-\sigma)] + \frac{(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}$$

$$- \frac{\alpha(1-\sigma)^2}{[1-(\alpha+\beta)(1-\sigma)]^2} \log w + \frac{[1-\alpha(1-\sigma)][1-\sigma]}{1-(\alpha+\beta)(1-\sigma)} \log(sW) \tag{10}$$

$$[1-(\alpha+\beta)(1-\sigma)]^2 \frac{\partial \log K}{\partial \beta} = (1-\sigma)\log H + (1-\sigma)^2 \log m +$$

$$[1-\alpha(1-\sigma)][1-\sigma]\log[1-\alpha(1-\sigma)] + [\alpha(1-\sigma)^2]\log[\alpha(1-\sigma)]$$

$$- [\alpha(1-\sigma)]^2 \log w - [1-\alpha(1-\sigma)][1-\sigma]\log(sW) \,. \tag{11}$$

$$\frac{\partial K}{\partial \beta} = \frac{[1-\alpha(1-\sigma)]^{\frac{1-\sigma(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} [\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{[1-(\alpha+\beta)(1-\sigma)]^2} \circ$$

$$\cdot \frac{H^{\frac{1}{1+(\alpha+\beta)(1-\sigma)}} \; m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \; (sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \circ$$

$$\circ \Big[(1-\sigma)\log H + (1-\sigma)^2 \log m + [1-\alpha(1-\sigma)][1-\sigma]\log[1-\alpha(1-\sigma)] +$$

$$+ \alpha(1-\sigma)^2 \log \alpha(1-\sigma) - [\alpha(1-\sigma)^2]\log w -$$

$$- [1-\alpha(1-\sigma)][1-\sigma]\log(sW)\Big] \,. \tag{12}$$

CHAPTER IV

REPLACEMENT THEORY AND THE REGULATED FIRM

In previous chapters a model was developed to show that rational behavior of the regulated firm calls for a more capital-intensive production process than for the unregulated firm. The model developed for the regulated firm is best presented, and most easily interpreted, in a setting of indestructible capital goods. But capital goods are destructible and must be replaced. In this chapter, the behavior of the regulated firm is investigated in relation to plant retirement and replacement. Behavior of the regulated firm is investigated first for constant and then for improving technology.

Modern replacement theory is principally concerned with the optimal timing of replacement of plant and equipment. To date, such theory has been directed almost exclusively toward timing replacement to minimize costs in the neoclassical tradition. At once, the question arises as to whether or not the regulated firm might rationally adopt replacement timing different from that of the unregulated cost-minimizing firm. The problem of replacement timing is a problem in dynamics, however; the model as developed thus far in this study has been a static model. Even the problem of determining the expansion path of the regulated firm was essentially an exercise in comparative statics, not dynamics. But relaxation of the assumption of the indestructibility of capital goods--and the introduction in turn of the problem of timing of replacement of capital goods in the regulated firm--requires that some consideration of dynamics be incorporated into the model. Accordingly then, time as a variable must be related systematically to other variables of the model.

Procedurally, some of the most useful approaches to replacement theory are reviewed below; the model of this study will then be adapted to incorporate the replacement problem--in a dynamic setting.

Review of Current Replacement Theory

In probing the problem of replacement timing for the regulated firm some of the minimum-cost treatments that are most relevant are first reviewed. Probably the most important name in replacement theory is George Terborgh.[1]

[1]George Terborgh, Business Investment Policy (Washington, D.C.: Machinery and Allied Products Institute, 1958). George Terborgh, Dynamic Replacement Policy (New York: McGraw-Hill, 1959). George Terborgh, Realistic Replacement Policy (Washington, D.C.: Machinery and Allied Products Institute, 1954). George Terborgh, MAPI Replacement Manual (Washington, D.C.: Machinery and Allied Products Institute, 1950).

Other significant contributors to replacement theory were made by Vernon Smith,[2] Pierre Massé,[3] and Frederich and Vera Lutz.[4] Significantly, Smith and Massé draw upon the work of Terborgh in their development of replacement theory.

The Work of George Terborgh

Terborgh used two words, defender and challenger, to illustrate, and to dramatize, his approach to the problem of determining optimum replacement policy. Terborgh envisaged a contest between the asset in use, the "defender," and the prospective replacement, the "challenger." Terborgh's analysis required finding an "adverse minimum" posed by a continually improving production process. The adverse minimum, or the discounted annual cost of replacement equipment which is assumed to be installed optimally, is compared with the discounted annual cost of continuing to operate existing equipment. Terborgh's special and enduring contribution was the extension of this

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[2]Vernon L. Smith, Investment and Production (Cambridge, Mass.: Harvard University Press, 1961), Chapters V and VI. Vernon L. Smith, "Problems in Production-Investment Planning Over Time," International Economic Review, Vol. I, No. 3, pp. 198-216. Vernon L. Smith, "Economic Equipment Policies," Management Science, Vol. IV, No. 3, pp. 20-37.

[3]Pierre Massé, Optimal Investment Decisions, Trans., Scripta Technica, Inc. (Englewood Cliffs, N.J.: Prentice-Hall, 1962).

[4]Frederich and Vera Lutz, The Theory of Investment of the Firm (Princeton, N.J.: Princeton University Press, 1951).

marginal replacement requirement to the case of improving technology. Terborgh made his concept easy to use (even to the point of incorporating it in shop manuals) by deriving simple formulas for computing the life of replacement equipment, based upon simple time averages of equipment cost.

Terborgh projected past trends indefinitely into a future of constantly recurring replacement chains. Terborgh's adverse minimum provided a measure of the "operating inferiority" of existing machines. Terborgh added the operating inferiority at time, T, to all other costs including capital recovery costs from time T_0 to T_1[5] to establish the real cost of keeping old equipment in service.

Terborgh took some license with truth in assuming linearity to obtain his measure of operating inferiority of existing equipment. But this error is of minor significance. Indeed, Terborgh recognized the error and used Cotter's arithmetic average, which increases linearly from zero to z(T-1) to obtain an average $z\left\{\frac{T-1}{2}\right\}$ to which he adds an average of the capital charges, interest and depreciation. The term z is the amount by which operating cost is assumed to increase per year to time T. Terborgh's sum

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[5]In this section the symbol, T, refers to specific time period which would serve to establish a specific date. The symbol, t, is used in a more generic sense, e.g., a period for which a maximum or a minimum is to be determined.

to be minimized is the approximate annual cost inflow to be minimized. Terborgh allowed his capital costs to decline linearly as the service life increases:

$$\text{Capital costs} = \frac{WK}{T} + \frac{rWK}{2}$$

where WK is investment

T is the service life

r is the rate of interest.

The sum to be minimized is

$$m' = z\left(\frac{T-1}{2}\right) + \frac{WK}{T} + \frac{rWK}{2} .$$

when T is optimal, m' becomes Terborgh's "adverse minimum," or when

$$\frac{dm'}{dt} = \frac{z}{2} - \frac{WK}{T^2} = 0.$$

Solving for T,

$$T = \sqrt{\frac{2WK}{z}} .$$

where T is the optimal service life

z is the annual increase in operating costs .

This is the simple expression Terborgh used for determining the optimal length of service in an endless chain of replacements.

To illustrate Terborgh's thesis, assume that (1) the operating inferiority of existing equipment increases at the rate of $100 a year, (2) the prospective replacing investment is $1,000, and (3) the interest rate is 10 per cent. Service life, T, then, is

$$T = \sqrt{\frac{2WK}{z}} = \sqrt{\frac{2000}{100}} = 4.47 \text{ years.}$$

Table 4.1 shows the solution in tabular form. The column entitled "total" is the tabular solution for Terborgh's "adverse minimum." The service life and minimum cost is shown to be between four and five years. Grant and Ireson have developed a refinement for the column entitled "excess of annual disbursements." Their refinement is a tabulated equivalent to the expression[6]

$$R = \frac{z}{r} - \frac{nz}{r}\left[\frac{r}{(1+r)^n-1}\right]$$

which yields a uniform equivalent to a linear gradient where R is the uniform equivalent,

z is the annual increase in operating costs,

n is the service life, and

r is the interest rate.

Derivations of this function are given below under the discussion of Smith's work. Further refinement may be obtained by continuous, rather than discrete, discounting.

Terborgh's adverse minimum, m′, in the case of the previous arithmetic example is

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[6] E. L. Grant and W. G. Ireson, Principles of Engineering Economy (New York: The Ronald Press, 1960), 50-52 and Appendix E, Table E-23.

Table 4.1.--Determination of the economic life of equipment that costs $1,000, that has an operating cost gradient of $100 a year, and is burdened with an interest rate of 10 per cent

Year T	Excess Disbursement for Year Indicated over First Year's Disbursement for New Equipment	Equivalent Annual Cost		
		Capital Recovery Cost for T Years[a]	Excess of Annual Disburse-ments[b]	Total
1	$ 0	$1050	$ 0	$1050
2	100	576	48	624
3	200	402	94	496
4	300	316	138	454
5	400	265	181	446
6	500	230	222	452
7	600	205	262	467
8	700	187	300	487

[a] $1000\left(\frac{re^{rT}}{e^{rT}-1}\right)$

[b] 100 x Factor to convert from linear gradient to a uniform equivalent. See Eugene L. Grant and W. Grant Ireson, Principles of Engineering Economy (New York: The Ronald Press, 1960), Table E-23, p. 560.

$$m' = z\left(\frac{T-1}{2}\right) + \frac{WK}{T} + \frac{rWK}{2}$$

$$= 100\frac{(4.47-1)}{2} + \frac{1000}{4.47} + \frac{(.10)(1000)}{2}$$

$$= \$442 .$$

This solution approximates the tabular solution of Table 4.1.

Here, term z includes opportunity costs that are caused by improving technology. Terborgh's simple adverse-minimum relation represents an important contribution to replacement in that it was the first to reckon with improving technology.

The Approach of Vernon L. Smith[7]

Smith's contribution to replacement theory is largely an embellishment of the simpler treatment of Terborgh. Smith does not use a linear obsolescence rate as does Terborgh, but he does defend Terborgh's linear approximations as being satisfactory in most cases.

Smith makes his operating cost a function of both utilization, u, and equipment age, t. After a period $\bar{L}_o$, operating costs, then, will have increased to $\phi(u_o \bar{L}_o)$. The optimal time for replacing existing equipment is the time that minimizes annual unit equipment cost. The existing equipment, however, cannot be studied apart from the

[7]Vernon L. Smith, "Economic Equipment Policies: An Evaluation," Management Science, Vol. IV, No. 3, 20-29.

equipment that will prospectively replace it.

Smith makes bolder use of Terborgh's point that delaying the purchase of new equipment will allow, ultimately, the purchase of more and more advanced models. He describes the operating expense of the first replacement as $\emptyset(u_1, \bar{L}_o, t)$. Smith interpreted this expression to mean that operating expense is an increasing function of output, a decreasing function of the length of time the installation of more advanced equipment is delayed, and an increasing function of the age of the incumbent equipment. Similarly, Smith describes the operating cost of the second chain of replacements as $\emptyset(u_2, \bar{L}_o + \bar{L}_1, t)$, the third as $\emptyset(u_3, \bar{L}_o + \bar{L}_1 + \bar{L}_2, t)$ and so on.

The function to be minimized is the uniform equivalent of all costs, fixed and variable. To set up the function to be minimized, Smith lets ψ_o be the rate at which operating costs increase with age; E_o, the present or initial operating expense; E_1, the first-year operating expense of replacement equipment; and ζ, the rate of technological change, or the rate at which the first-year operating cost of new equipment decreases each year. The utilization rate, u, is held constant for reasons of simplicity.

The function to be minimized is then

$$h = r\left\{\int_o^{\bar{L}_o} \emptyset_o(t)e^{-rt}dt - \overline{SV}_o(\bar{L}_o)e^{-rL_o}\right\} + h'e^{-r\bar{L}_o}$$

where ϕ_o relates operating expenses to $\bar{L}_o$ and t h is the uniform annual equivalent of all operating and capital costs,

h′ is the uniform equivalent of all future equipment expenses starting with the first replacement, and

$\overline{SV}_o$ is salvage at the end of period $\overline{L}_o$.

Substituting from the linear approximations

$$\phi_o(t) = E_o + \psi_o t$$

$$\phi(\overline{L}_o + k\overline{L}_1 t) = E - \zeta(L_o + k\overline{L}) + \psi t$$

where E is operating expense

ψ is the rate at which operating cost increase with age

ζ is the rate of technological change

Smith obtains

$$h = r\left\{\int_o^{L_o} (E + \psi_o t)e^{-rt}dt - \overline{SV}_o e^{-r\overline{L}_o}\right\} + h' e^{-r\overline{L}_o}$$

with k replacements,

$$h' = r\sum_{k=0}^{\infty} e^{-rk\overline{L}} \left\{\int_o^{\overline{L}} E - \zeta(\overline{L}_o + k\overline{L}) + \psi t\right\} e^{-rt}dt + WK - \overline{SV}_o{}^{-r\overline{L}_o}$$

$$= E - \zeta L_o + \frac{\psi}{r} + \frac{r(WK - \overline{SV}_o e^{-r\overline{L}}) - (\zeta + \psi)\overline{L}e^{-rt}}{1 - e^{-r\overline{L}}} .$$

Smith minimizes the h function by setting $\frac{\partial h}{\partial \overline{L}} = 0$ and $\frac{\partial h}{\partial \overline{L}} = e^{-rL} \frac{\partial h'}{\partial \overline{L}}$ to obtain

$$E_o + \psi_o\overline{L}_o + r\overline{SV}_o - \frac{\zeta}{r} = h'$$

$$(\zeta + \psi)(1 - r\overline{L}_o - e^{-rL_o}) + r^2(WK - \overline{SV}) = 0.$$

Smith's sum

$$(E_o + \psi_o\overline{L}_o + r\overline{SV}_o - \frac{\zeta}{r}) = h'$$

represents the direct and the opportunity cost of continuing existing equipment in operation another year. Now L_o may be solved for

$$\overline{L}_o{}^o = \frac{h' - E_o - r\overline{SV}_o + \frac{\zeta}{r}}{\psi}$$

which is the optimal service life of future replacement equipment, but the solution contains the term h'.

Solutions of the equation

$$(\zeta + \psi)(1 - r\overline{L}^o - e^{-r\overline{L}^o}) + r^2(WK - \overline{SV}) = 0$$

require an approximation of the Taylor expansion

$$e^{-r\overline{L}} = 1 - r\overline{L}^o + \frac{(r\overline{L}_o{}^o)^2}{2!} - \frac{(r\overline{L}_o{}^o)^3}{3!} + \frac{(r\overline{L}_o{}^o)^4}{4!} - \ldots$$

which Smith achieves by discarding all terms after the third. Now solving for $\overline{L}_o{}^o$

$$\overline{L}_o{}^o = \left(\frac{2(WK - \overline{SV})}{\zeta + \psi}\right)^{1/2}.$$

The term h' may now be solved for by substituting for $\overline{L}_o{}^o$ in

$$\bar{L}_o^{\,o} = h' - E_o - r\overline{SV}_o + \frac{\zeta}{r} .$$

Solutions for $L_o^{\,o}$ above are the same as that obtained by Terborgh except that Smith makes provision for salvage, $\overline{SV}$.

The Contribution of Pierre Massé[8]

In recent years a group of French research workers engaged in theoretical analysis have made significant contributions to the theory of investment decisions. Among these contributions to investment theory, the work of Pierre Massé is probably the most noteworthy. Massé gives credit to Terborgh for the first and, to date, the most significant work in replacement theory.

Massé, like Terborgh and Smith, views the solution of the service-life problem as one that minimizes costs. Massé first discusses "Retirement in the Pure Form." Here, he develops a simple function to determine the optimum service life when replacement is not contemplated.

Massé optimizes the relation

$$G(T) = \int_0^T (PQ - wL)(t)e^{-rt} + (\overline{SV})(T)e^{-rt} - WK$$

where G(T) is discounted profit

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[8]Pierre Massé, <u>Optimal Investment Decisions Rules for Action and Criteria for Choice</u>. Trans. Scripta Technica, Inc. (Englewood Cliffs, New Jersey: Prentice-Hall, 1962).

(PQ-wL)t is the gross income earned during the period between t and t+dt,

WK is the initial investment,

r is the interest rate,

$\overline{SV}$ is salvage, and

T is the retirement date.

The optimum may be obtained by differentiating and setting the result equal to zero.

$$0 = \frac{dG}{dt} = \left[(PQ-wL)(T) - r(T)\overline{SV}T + \frac{d\overline{SV}}{dt}\right]e^{-rt}.$$

This relation simply means the equipment should be retired when the gross profit on continuing the service life is equal to the interest on the retirement value plus the loss in retirement value.

Massé considers the problem of partial replacement. Here he proposes something short of total replacement to extend the service life of the initial equipment. For the extended service life, T′, the revised discounted profit

$$G' = \int_{0}^{T'} (PQ-wL)(t)e^{-rt}dt - ce^{-rt} + \int_{\bar{\theta}}^{T'} (Pq-wL)(\bar{\theta},t)e^{-rt}dt$$

where G′ is again total discounted profit,

(PQ-wL)tdt is the profit yield by the initial equipment,

c is the interim investment made at time t

q is the output of the interim investment

$\bar{\theta}$ is the period of partial replacement.

Masse´ obtains partial derivations to find the optimum time $\bar{\theta}$ for the interim investment and optimum time T´ for the extended service life of the original equipment:

$$0 = \frac{\partial G'}{\partial \bar{\theta}} = cr(\bar{\theta})e^{-r\bar{\theta}} - (Pq-wL)(\bar{\theta},t)e^{-r\bar{\theta}} + \int_{\bar{\theta}}^{T'} \frac{\partial(Pq-wL)}{\partial \bar{\theta}} \cdot$$

$$\cdot\ (\bar{\theta},t)e^{-rt}dt$$

$$0 = \frac{\partial G'}{\partial T'} = (PQ-wL)T'e^{-rT'} + (Pq-wL)(\bar{\theta},T')e^{-rT'}.$$

After solving for T´ and $\bar{\theta}$, the discounted profit G´ can be determined and compared with discounted profit for which no interim investment is made.

Masse´'s work in chain replacement retraces the work of Terborgh; like Smith, he arrives at the same conclusions but by a more elegant process.

His expression for the optimal chain replacement period, τ, is

$$\tau = \sqrt{\frac{2\,(WK - \overline{SV})}{z}}$$

where τ is the service life of the chain replacement

WK is initial investment,

$\overline{SV}$ is salvage, and

z is a measure of the adverse position per year (of variable costs of the incumbent equipment compared to total cost of the prospective replacement).

This expression is identical in form to that developed by Terborgh and Smith.

Masse´ also abstracts from his function defining optimum retirement in the pure form to define optimum replacement in a constant chain. To do this he restates his basic discounted function to form

$$G(T) = \int_{o}^{T} (PQ-wL)(T)e^{-rT}dt + [\overline{SV}(T)+G]e^{-rT} - WK.$$

From this expression he defines his optimum point for retirement-replacement by the expression

$$(PQ-wL)T = r(T)[\overline{SV}(T)+G] - \frac{d\overline{SV}}{dT}.$$

In the above, Masse´ delineates the conditions that apply for retirement of incumbent equipment when a constantly recurring replacement chain is involved. Since

$$r(T)[\overline{SV}(T)+G] > r(T)\overline{SV}(T)$$

the service life is shortened. A parallel for this condition exists for the regulated firm as is developed in the next section below.

Masse´ demonstrates that, at the optimum replacement time τ , the variable costs of the old equipment is equal to total costs, variable and fixed, of the new. The assumption is made that discounted profit of the recurring replacement chain is $G(\tau)$. Then total net revenue product is

$$(PQ-wL)T = rG(\tau)$$

and

$$D_o(T) = rM(\tau)$$

where $D_o(T)$ is the variable cost of the equipment in use and $rM(\tau)$ is the total discounted cost of the replacement chain

$$rM(\tau) = \frac{r}{1-e^{-r\tau}}\left[\int_o^\tau D(t)dt + WK\right]$$

where WK is investment.

This expression may be restated

$$D_o(T) = \frac{\int_o^\tau D(t)e^{-rt}dt}{\int_o^\tau e^{-rt}dt}.$$

Massé uses this relation to show that the optimum service life, τ, defines the time when variable costs of incumbent equipment is equal to the uniform equivalent of the total cost of a new machine. This relation, expressed in less general terms, is almost an axiom in engineering economy. This concept is not quite as simple as it would at first appear, because the capital costs of the replacement equipment cannot be known until its service life is known.

Replacement of Equipment in the Regulated Firm

In Chapters II and III, a model was developed to show that rational behavior on the part of the regulated firm requires a more capital-intensive production process than for the unregulated firm. In this section, an investigation is made as to whether or not this same propensity for greater use of capital affects the solution to the problem of determining the optimum service life and replacement policy of the regulated firm.

Retirement in the Pure Form

Conventional behavior requires that equipment be retired from service when the return on the retirement value equals the interest on the retirement value plus the loss in salvage value per unit of time. Obviously, for this equality to have meaning the loss in salvage value must be expressed in the same time units as return and interest. If, in the case of the regulated firm, return is held to a "fair" or allowable level, then,

$$s\overline{SV} = r\overline{SV} + \frac{\Phi WK}{n'} \qquad (4.1)$$

where s is the allowable rate of return

r is the interest rate, or cost of capital

$\frac{\Phi WK}{n'}$ is the loss of retirement, or salvage, value per unit of time

n' is the period required for salvage value to equal zero.

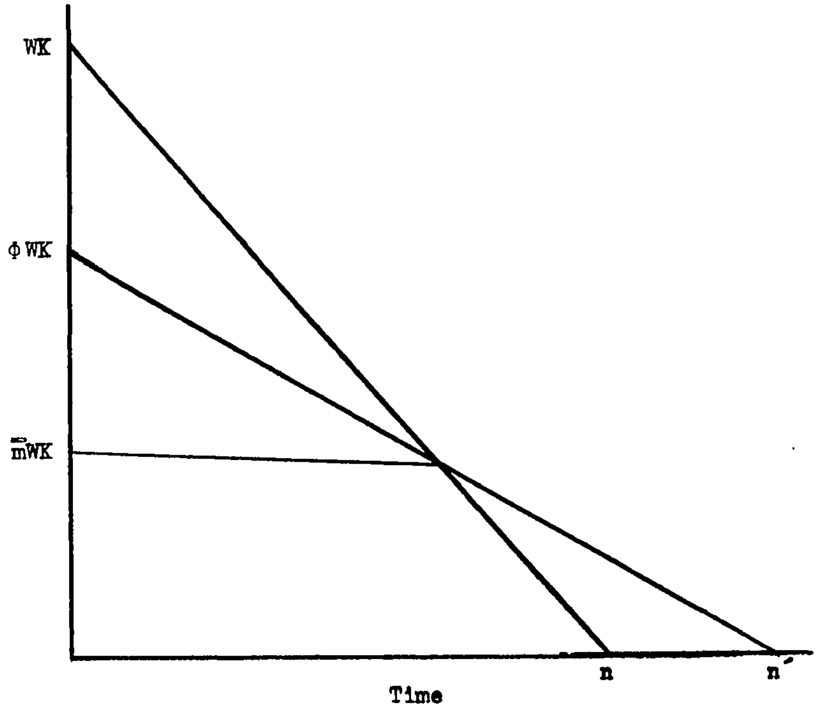

Fig. 4.1.--Masse´'s "retirement in the pure form"

Here, following the argument of Masse´,[9] the decline in earnings is a linear function, as is the decline in salvage value:

$$\frac{\phi WK}{n'} = \frac{\phi WK - \overline{SV}}{n} = \frac{WK(\phi - \bar{m})}{n} \qquad (4.2)$$

where ϕ is the ratio of salvage $\overline{SV}$ to initial cost, WK at time zero.

[9] Pierre Masse´, Optimal Investment Decisions (Englewood Cliffs, N.J.: Prentice-Hall, 1962), 54-56.

Now,

$$\overline{SV} = \overline{m}WK$$

$$\overline{SV} = \Phi WK - \delta t$$

$$= \frac{WK(\Phi - \overline{m})}{n}$$

where t is year when salvage and book values are equal

n is the economic service life

$\overline{m}$ is the per cent of initial investment represented by salvage.

From equations (4.1) and (4.2)

$$(s-r)\overline{SV} = \frac{WK(\Phi-\overline{m})}{n}$$

$$\overline{SV} = \frac{WK(\Phi-\overline{m})}{(s-r)n} = \frac{WK}{n'(s-r)}. \qquad (4.3)$$

Then,

$$\frac{\Phi}{n'} = \frac{\Phi-\overline{m}}{n}. \qquad (4.4)$$

For a specific example, let

$$WK = 750$$

$$\Phi WK = 375$$

$$s = .065$$

$$r = .05$$

$$n' = 25 \text{ years}$$

$$n = \text{service life}$$

$$\overline{SV} = \frac{\Phi WK}{n'(s-r)} = \frac{375}{25(.065-.05)} = 100.$$

Then,

$$\bar{m} = \frac{100}{750} = .133$$

$$\frac{\Phi}{n'} = \frac{\Phi-\bar{m}}{n}$$

$$\frac{30}{25} = \frac{.50-.133}{n}$$

Service life, n = 18.33 years.

This solution is equivalent to Masse',s "retirement in the pure form" and is not the optimum solution to the problem of recurring replacement.

Determination of Service Life When Replacement is Contemplated

If replacement is contemplated, or required, rational behavior on the part of the regulated firm requires that pure profit foregone because of delayed replacement be measured against the loss caused by retiring the equipment before the end of period n. Here, n is still the optimal service life, as in the preceding discussion of "retirement in the pure form."

For constant technology and where the rate of depreciation, $\bar{\gamma}$, on book value,[10] or "rate base," is expressed

---

[10]The term book value conforms to conventional accounting usage, i.e., the undepreciated value, or value as per accounting records, is determined in accord with some prearranged schedule to depreciate, or "write off," the asset.

$$\bar{\gamma} = \frac{WK-\overline{SV}}{n} = \frac{WK-\bar{m}WK}{n} = \frac{WK(1-\bar{m})}{n}. \quad (4.5)$$

The rate of depreciation, δ, of salvage value is expressed

$$\delta = \frac{\Phi WK-\overline{SV}}{n} = \frac{\Phi WK-\bar{m}WK}{n} = \frac{WK(\phi-\bar{m})}{n}. \quad (4.6)$$

If t is the service life of plant and equipment which permits book value and salvage realization value to be equal,

$$(s-r)[WK-(WK-\bar{\gamma}t)] = [WK-\bar{\gamma}t] - [WK-\bar{\delta}t] \quad (4.7)$$

$$(s-r)\bar{\gamma}t = WK(1-\phi) + t(\delta-\gamma)$$

$$t = \left[\frac{(s-r)\bar{\gamma}+\bar{\gamma}-\delta}{n}\right] = WK(1-\phi). \quad (4.8)$$

Substituting for $\bar{\gamma}$ and δ

$$t \ (s-r)\frac{WK(1-\bar{m})}{n} + \frac{WK(1-\bar{m})}{n} - \frac{WK(\phi-\bar{m})}{n} = WK(1-\phi)$$

$$t = \frac{(1-\phi)n}{(s-r)(1-\bar{m}) + 1-\phi}. \quad (4.9)$$

To correct for income-tax reduction because of the capital loss on retirement of the asset when the income tax rate is t_x, the indicated loss is decreased, or

$$(s-r)\bar{\gamma}t = \frac{WK(1-\phi) + t(\delta-\bar{\gamma})}{(1-t_x)}.$$

Then t, of equation (4.9) corrected for the income-tax effect, is

$$t = \frac{n(1-\phi)}{2(s-r)(1-\overline{m}) + 1-\phi} . \qquad (4.10)$$

For the conditions used in the example where replacement was not contemplated, (page 152), the optimal service life for an asset which depreciates at a rate, $\overline{\gamma}$, is shortened to 17.9 years for the case of no income tax and to 17.6 years for the case where the income tax rate, t_x, is .50--when replacement is planned.

Neither of the two preceding solutions for service life, t, is truly optimal, because the firm could have shortened its depreciation schedule so that no capital loss would have occurred at the end of the service life. This approach is shown graphically in Fig. 4.2. The solution for service life, n, for the no-replacement case is shown in relation to the preceding solution for service life, t, for the replacement case. If the firm were to change its depreciation schedule (represented by the line AB to the line AC), the consequences would be: (1) no capital loss occurs on retirement of the equipment at the end of period, t; (2) the regulated firm earns on a larger rate base, $WK-\overline{\gamma}t$, during the service life of the equipment; and (3) the equipment is retired at a higher salvage value.

The shorter the service life of the productive equipment of the regulated firm, the greater is its average

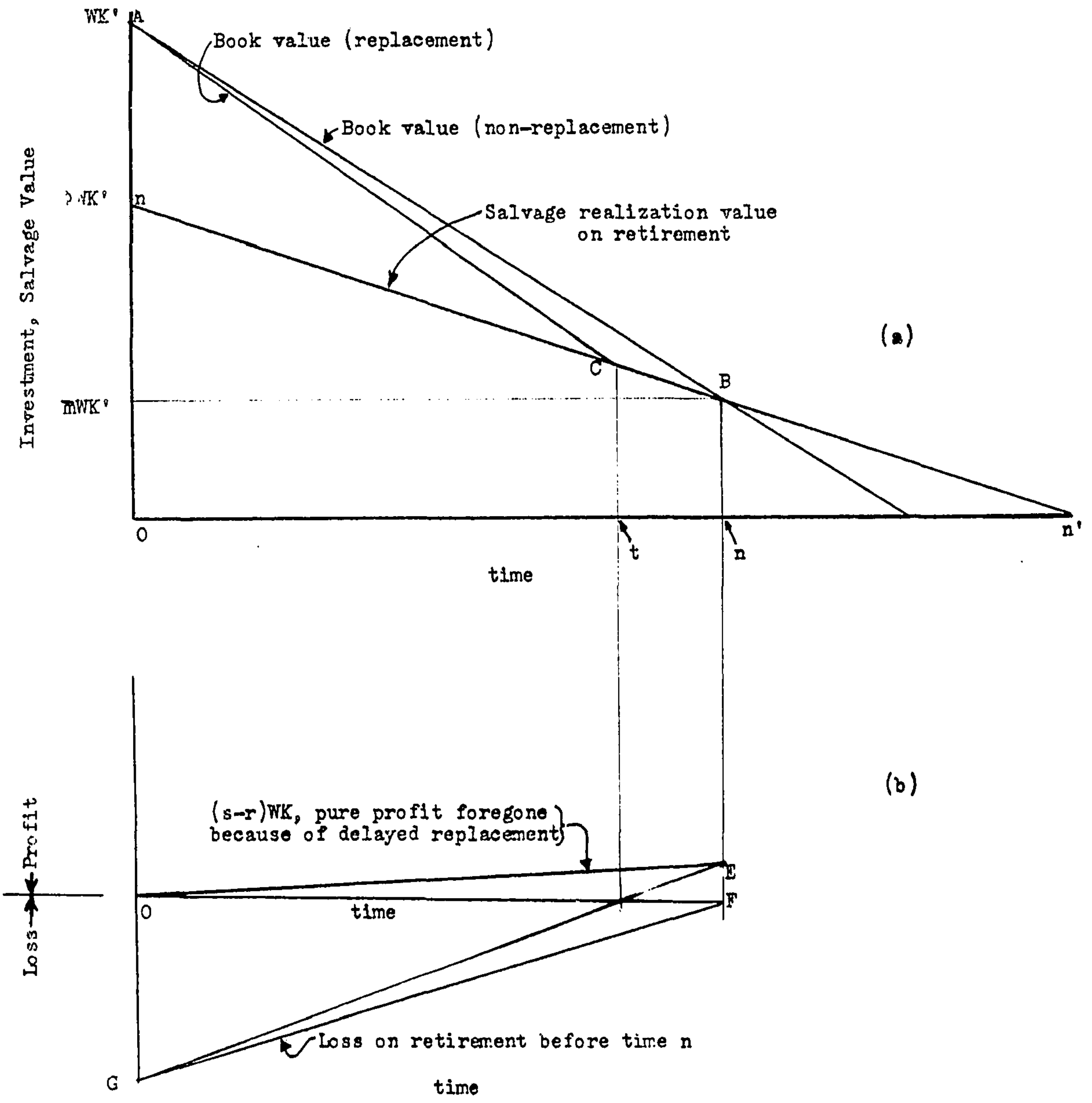

Fig. 4.2—Optimal depreciation schedules for the regulated firm under assumptions of replacement and non-replacement of plant

rate base.

$$\text{Average Rate Base} = \frac{WK-\overline{SV}}{2} = \frac{WK(1-\overline{m})}{2}, \qquad (4.11)$$

Unless some constraint is brought to bear, the regulated firm would logically retire its plant and equipment in the year of its installation if the depreciation schedule permitted the firm to recover full cost minus salvage: $WK-\overline{SV}$, or $WK(1-\overline{m})$. Apart from the consideration that plant retirement is subject to regulatory approval, the demand function faced by the regulated firm serves to define the limit to which the firm may shorten the service life of its plant and equipment. Clearly, unless the salvage value is compensatingly high, a shortened service life increases the cost of output because of higher depreciation charges. Most realistic time-salvage-realization functions do not permit the higher salvage values attending shortened plant life to be great enough to obviate higher depreciation charges.

The solution to the problem of determining a truly optimal service life of plant and equipment for the regulated firm requires that a time-salvage function be described, that a function for the increase in cost of variable inputs as the plant ages be defined, and that the demand function faced by the firm be known.

The salvage function may be restated:

$$\text{Salvage, } \overline{SV} = \frac{\phi WK}{n} = \frac{\phi WK-\overline{m}WK}{n} = \frac{WK(\phi-\overline{m})}{n}$$

where $\overline{SV}$ is salvage

n' is the period required for salvage to acquire zero value

ϕWK is the salvage value immediately after construction of the plant

$\bar{m}$ is the ratio of salvage value to initial cost

n is the period, the end of which, book value becomes zero

t is the period, the end of which, salvage values and book value are equal.

The function for the increase in cost of variable inputs is

$$\text{Variable Inputs,} \quad wL_t = wLo+zt. \qquad (4.12$$

The increase in cost of variable inputs as the plant ages could have been expressed exponentially rather than linearly, but a linear function was chosen here, partly because the depreciation function is linear.

The demand function is the same as used in Chapters II and III

$$P = HQ^{-\sigma}.$$

Unlike the cost and revenue functions of Chapters II and III, the cost and revenue function for deteriorating capital stock must provide for the return of, as well as the return on, capital. Accordingly, these functions must now provide for the annuity of depreciation in order to recover capital. The expression (for zero salvage) to achieve this end was developed in Chapter I to be

$$\frac{se^{st}}{e^{st}-1}.$$

The expression that provides for an increase in the cost of variable inputs as the plant ages, plus the capital recovery factor for an asset with realizable salvage value, is:

$$C = [wL_o+zt] + WK\left[r + \frac{r(1-\mu)}{e^{rt}-1}\right], \qquad (4.13)$$

Similarly, the corresponding expression for revenue requirements is

$$R = [wL_o+zt] + WK\left[s + \frac{s(1-\mu)}{e^{st}-1}\right]. \qquad (4.14)$$

The term, μ, in equations (4.13) and (4.14) is the ratio at time, t, of salvage value, $\overline{SV}$, to initial investment, WK; and thus, μ represents a special case of $\overline{m}$.

The total revenue requirements function is shown diagrammatically in Fig. 4.3. The height of line AB is a measure of the return on salvage at the time of retirement. Obviously, the shorter the service life, the greater is AB. While the rate of pure profit (s-r)WK remains unchanged, the absolute value represented by (s-r)WK becomes exponentially greater as the service life is shortened.

In Chapter I, an expression for an equated average rate of return on capital stock that is expiring linearly

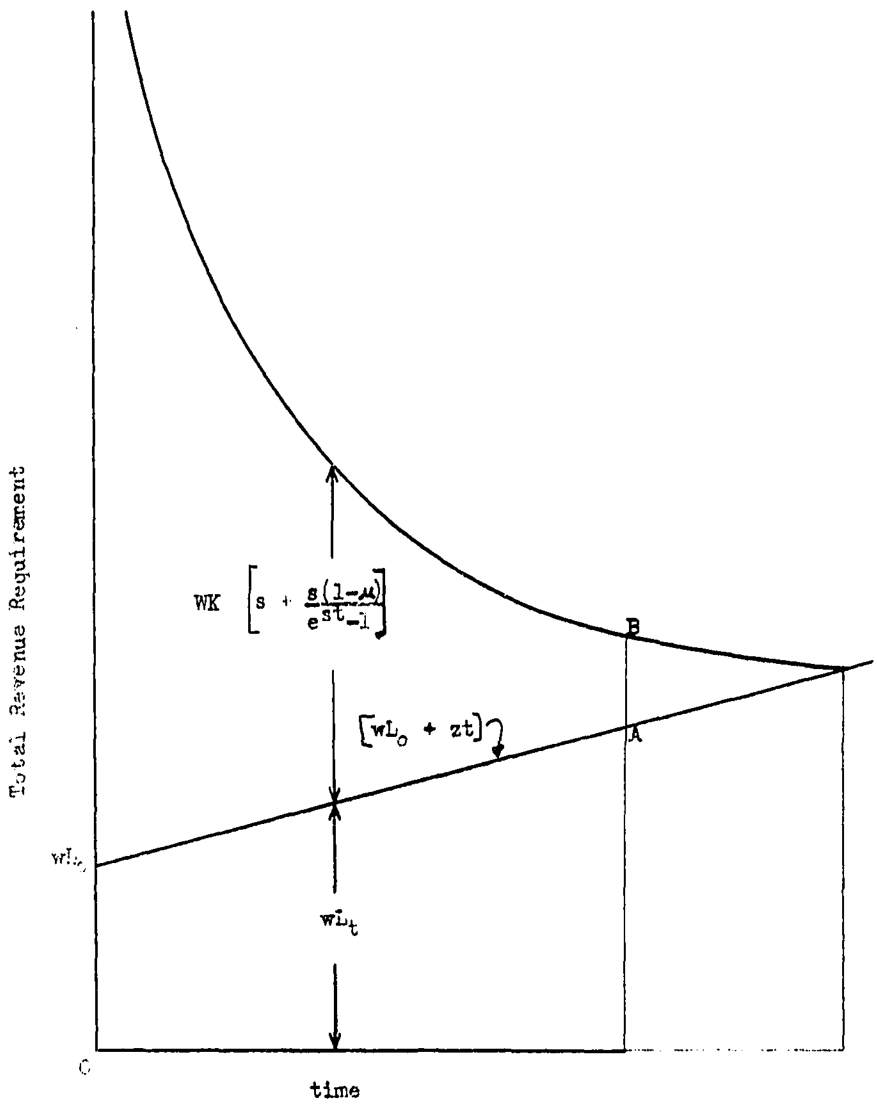

Fig. 4.3--The effect of the time-salvage relation upon the uniform equivalent revenue required to earn return, s

was developed to be

$$\begin{array}{c}\text{Equated Average}\\ \text{Rate of Return}\end{array} = s + \frac{s(1-\mu)}{e^{st}-1} - \frac{1}{t} \quad (4.15)$$ [11]

where s is the rate of return

μ is the ratio of salvage value to initial investment at retirement date.

Hypothetical data for a range of t in the service life, or planning period, and salvage values are shown in Table 4.1.

Fig. 4.4 shows the increase in the equated average rate of return as the service life is shortened and as the salvage value is increased. The example illustrated in Fig. 4.4 and Table 4.2 reflects a rate of return, s, of .065 and a specific salvage function

$$\overline{SV} = \frac{WK}{n'} = \frac{.6WK}{25} = \frac{WK(\phi-\bar{m})}{n} = \frac{WK(.6-\bar{m})}{n}.$$

The path x-x describes how the equated rate of return increases as the service life is shortened. A commensurate increase in the equated average pure profit rate (s-r)WK occurs as the service life is shortened.

The cost and revenue functions must be expressed in terms of their present value or in terms of their uniform equivalent. The latter form is more convenient,

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[11]From the identity developed in Chapter I,

$$\left[\begin{array}{c}\text{Equated Rate of Return}\\ +\\ \text{Straight Line Depreciation}\end{array}\right] = \left[\begin{array}{c}\text{Rate of Return}\\ +\\ \text{Annuity Depreciation}\end{array}\right].$$

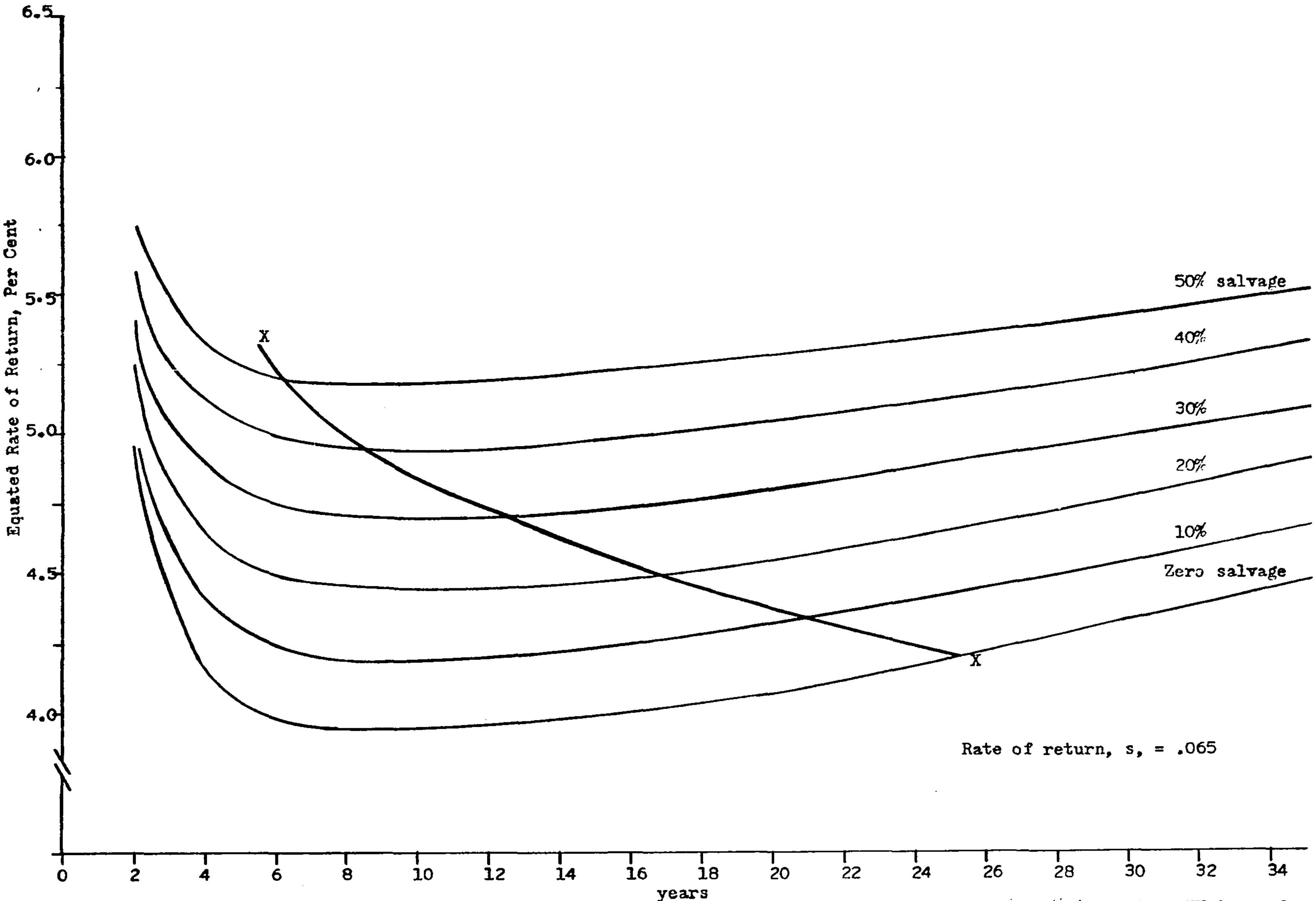

Fig. 4.4—Equated rates of return for a range of asset lives, salvage values, and with a time-salvage function XX imposed

Table 4.2.--Numerical example of equated rates of return, assuming an allowable return of 6.5 per cent and linear depreciation for a range of asset lives and salvage values

Year	Salvage, Per cent of Investment					
	0	10	20	30	40	50
1	6.50	6.50	6.50	6.50	6.50	6.50
2	4.93	5.09	5.24	5.40	5.56	5.72
3	4.43	4.63	4.85	5.05	5.26	5.46
4	4.19	4.42	4.65	4.88	5.11	5.35
5	4.06	4.30	4.55	4.79	5.04	5.28
10	3.91	4.17	4.43	4.69	4.95	5.21
15	3.97	4.23	4.47	4.73	4.98	5.23
20	4.08	4.32	4.56	4.81	5.05	5.29
25	4.20	4.43	4.66	4.89	5.12	5.35
30	4.33	4.54	4.77	4.98	5.30	5.41
40	4.51	4.76	4.96	5.15	5.34	5.54
50	4.79	4.96	5.13	5.30	5.47	5.65

partly because so many of the tables and much of the literature of engineering economy is expressed in terms of the uniform equivalent.

The cost function

$$C = wL_t + rWK$$

and revenue function

$$R = wL_t + sWK$$

expressed in terms of the uniform equivalent are[12]

$$C = wL_o + \frac{zt}{r} - \frac{tz}{r}\left(\frac{r}{e^{rt}-1}\right) + WK\left[r + \frac{r(1-\mu)}{e^{rt}-1}\right] \tag{4.16}$$

and

$$R = wL_o + \frac{zt}{s} - \frac{tz}{s}\left(\frac{s}{e^{st}-1}\right) + WK\left[s + \frac{s(1-\mu)}{e^{st}-1}\right] \tag{4.17}$$

---

[12]The expression $\frac{z}{r} - \frac{tz}{r}\left(\frac{r}{e^{rt}-1}\right)$ for the uniform annual equivalent to a gradient was derived by Grant and Ireson. See E.L. Grant and W.G. Ireson, "Principles of Engineering Economy" (New York: The Ronald Press, 1960), Chapters IV and Appendix D.

Essentials of the derivation are:

$$\sum_0^t z = z\left[\frac{e^{r(t-1)}-1}{r} + \frac{e^{r(t-2)}-1}{r} + \ldots \frac{e^{r\cdot 2}-1}{r} + \frac{e^{r}-1}{r}\right]$$

$$= \frac{z}{r}\left[e^{r(t-1)} + e^{r(t-2)} + \ldots e^{r.2} + e^{r} + 1 - (t-1)\right]$$

$$= \frac{z}{r}\left[e^{r(t-1)} + e^{r(t-2)} + \ldots e^{r.2} + e^{r} + 1\right] - \frac{tz}{r}.$$

The expression in brackets is the compound amount of a sinking fund of one per year for t years. Hence, where $$\mu = \frac{\overline{SV}}{WK} = \frac{WKt}{n'WK} = \frac{WK(\phi-\overline{m})t}{WKn} \quad (4.18)$$

ϕ is the per cent salvage upon completion of construction of plant

$\overline{m}$ is the ratio of salvage $\overline{SV}$ to initial investment, WK

μ is the ratio of salvage $\overline{SV}$ to initial investment WK when salvage value equals book value

t is the year when salvage value equals book value

n' is the period for which salvage value became zero

n is the period for which salvage value became $\overline{m}$WK.

r is the cost of capital

s is the rate of return

z is the annual increase in the cost of variable inputs

WK is the initial investment

L_o is variable inputs

w is the wage rate

$$\text{the sum of compound amounts} = \frac{z}{r}\ \frac{e^{rt}-1}{r} - \frac{tz}{r}\,.$$

The uniform equivalent for t years is determined by multiplying by the sinking-fund factor for t years, $\frac{r}{e^{rt}-1}$ to obtain

$$\text{the uniform equivalent} = \frac{z}{r}\left(\frac{e^{rt}-1}{r}\right)\left(\frac{r}{e^{rt}-1}\right) - \frac{tz}{r}\left(\frac{r}{e^{rt}-1}\right)$$

$$= \frac{z}{r} - \frac{tz}{r}\left(\frac{r}{e^{rt}-1}\right).$$

The optimum service life, t, for the unregulated firm occurs when[13]

$$\frac{dC}{dt} = 0 \quad \text{and} \quad \frac{d^2C}{dt^2} > 0.$$

Differentiation of the revenue function, like differentiation of the cost function, yields a unique solution for t, or when

$$\frac{dR}{dt} = 0 \quad \text{and} \quad \frac{d^2R}{dt^2} > 0.$$

The solution for optimum service life, t, for the regulated and the unregulated firm is shown in Figs. 4.5(a) and 4.5(b).

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[13]Differentiating the cost function yields a polynomial and accordingly, requires approximating techniques to achieve a solution, e.g., Newton's method. See John Brixey and Richard Andree, Fundamentals of College Mathematics (New York: Holt, 1954), pp. 243-246.

$$C = wL_o + \frac{zt}{r} - \frac{tz}{r}\left(\frac{r}{e^{rt}-1}\right) + WK\left[r + \frac{r(1-\mu)}{e^{rt}-1}\right]$$

$$\frac{dC}{dt} = \frac{z}{r} - z\,\frac{(e^{rt}-1)-tre^{rt}}{(e^{rt}-1)^2} - \frac{WKr^2(1-\mu)e^{rt}}{(e^{rt}-1)^2} = 0$$

$$\frac{\frac{z}{r}(e^{rt}-1)^2}{(e^{rt}-1)^2} - \frac{z(e^{rt}-1)+tzre^{rt}}{(e^{rt}-1)^2} - \frac{WKr^2(1-\mu)e^{rt}}{(e^{rt}-1)^2} = 0$$

$$\frac{z}{r}(e^{rt}-1)^2 - z(e^{rt}-1) + ztre^{rt} - WKr^2(1-\mu)e^{rt} = 0$$

The profit function for the unregulated firm is stated in terms of the uniform equivalent

$$\pi = HQ^{1-\sigma} - \left\{WL + \frac{z}{r} - \frac{tz}{r}\left(\frac{r}{e^{rt}-1}\right) + WK\left[r + \frac{r(1-\mu)}{e^{rt}-1}\right]\right\}. \tag{4.19}$$

The profit function under regulation is also expressed in terms of the uniform equivalent

$$\pi = \left[(s-r) + \frac{(s-r)(1-\mu)}{e^{(s-r)t}-1}\right]WK. \tag{4.20}$$

Equation 4.20 states pure profit in terms of the uniform equivalent to the pure profit rate (s-r) applied to capital stock that is depreciating at the rate $\frac{1}{t}$.

The two profit functions may be equated, as was done for the static model, to yield the zero excess profit function

$$0 = HQ^{1-\sigma} - \left\{wL + \frac{z}{r} - \frac{tz}{r}\left(\frac{r}{e^{rt}-1}\right) + WK\left[s + \frac{s(1-\mu)}{e^{st}-1}\right]\right\}$$

or

$$R = wL + \frac{z}{r} - \frac{tz}{r}\left(\frac{r}{e^{rt}-1}\right) + WK\left[s + \frac{s(1-\mu)}{e^{st}-1}\right].$$

So equation (4.21) becomes the revenue function already described in equation (4.17). Thus, the profit function in a dynamic setting parallels the behavior of the static case of Chapter II.

Graphic Solution to the Problem of Determining Optimal Service Life of Plant

The problem of determining optimal service life in the regulated firm is illustrated graphically in Fig. 4.5--the solution obtained by the simultaneous solution of the two profit functions [equations (4.19) and (4.20)]. The interesting solution is at point A of Fig. 4.5(a). Another solution exists at point B, but because it does not maximize profit--or anything else--is of no more interest than the second and minimal solution for capital K obtained by simultaneous solutions of the production and zero excess profit functions of Chapter III, page 114.

Either a decrease in demand, which means a decrease in unregulated profit, or an increase in the pure profit rate, s-r, moves the solutions for optimal service life of plant and equipment of the regulated and of the unregulated firms closer together. If the regulated profit function lies entirely above that of the regulated, then the unregulated profit function solely determines optimal service life.

The depreciation schedules for the three solutions are shown in Fig. 4.5(c). Depreciation schedules C and D are optimal, when replacement is planned for the regulated and unregulated firms, respectively. Depreciation schedule E is optimal when no replacement is to occur. Constant technology is assumed for all cases. All three

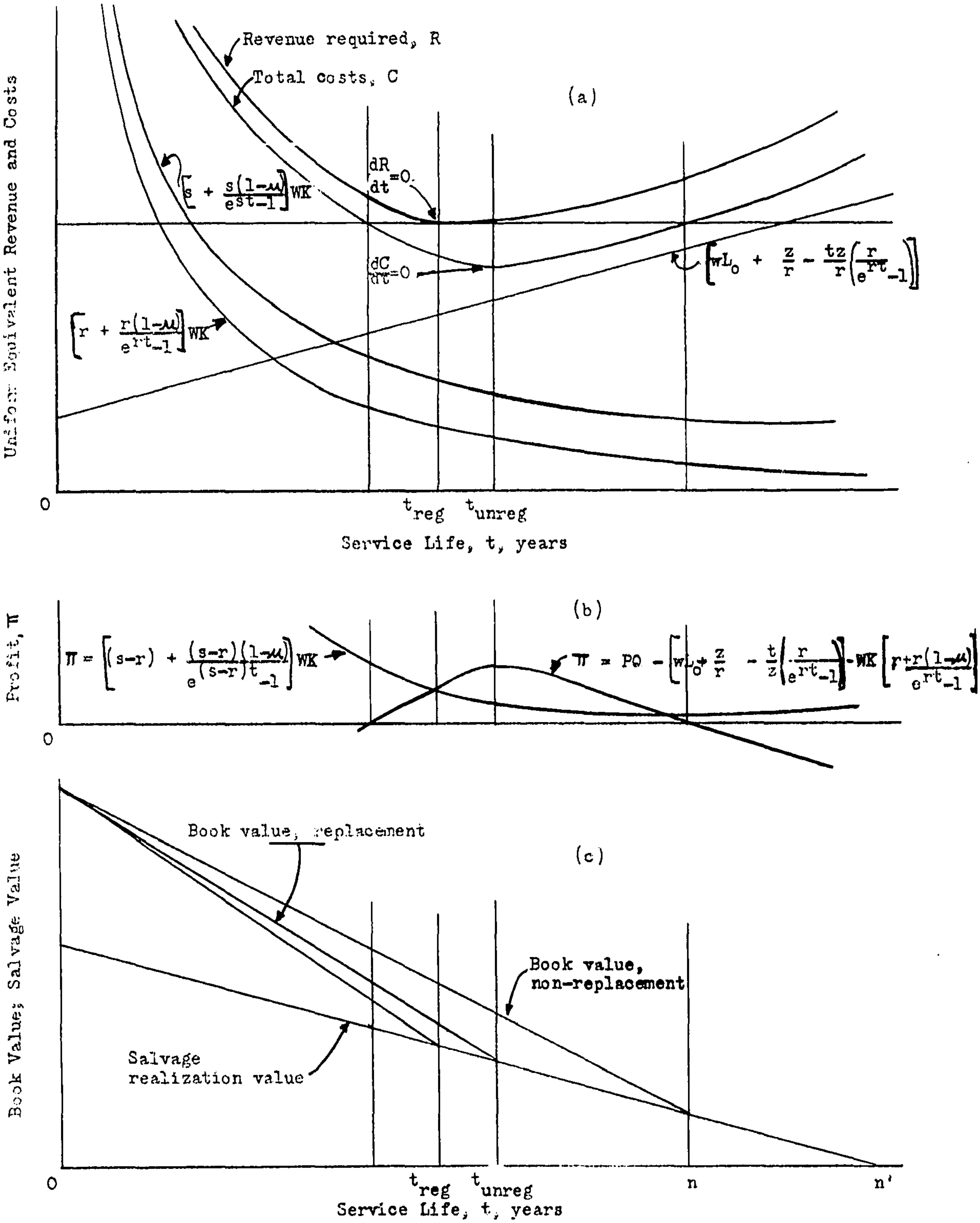

Fig. 4.5--Determination of optimal service life of plant for the regulated and unregulated firm

depreciation schedules require a time-salvage function for optimizing service life of the firm's plant and equipment.

The Replacement Chain

In the case of technological improvement, Smith[14] adds a measure of the reduction in costs that would be realizable if new equipment were purchased to the cost of operating incumbent equipment. Smith, then, adds opportunity cost to current cost to obtain the total cost of continuing to operate existing equipment. Addition of opportunity costs to increasing variable costs moves the points of the minimum uniform equivalents of cost and revenue to the left for both the unregulated and the regulated firms. An increase in costs would ordinarily move the solutions for optimum service life of the regulated and of the unregulated firm closer together; but an increase in technology decreases the cost of output, which, in turn, causes output to be increased. In Chapter III, the changes in price, P, output, Q, and capital, K, that resulted from changes in technology, m, were reported to be

$$\frac{\partial P}{\partial m} = - \frac{\sigma}{1 - (\alpha+\beta)(1-\sigma)} \frac{P}{m}$$

$$\frac{\partial Q}{\partial m} = \frac{1}{1 - (\alpha+\beta)(1-\sigma)} \frac{Q}{m}$$

---

[14] Vernon L. Smith, <u>Investment and Production</u> (Cambridge, Mass.: Harvard University Press, 1961), 146-152.

$$\frac{\partial K}{\partial m} = \frac{1 - \sigma}{1 - (\alpha+\beta)(1-\sigma)} \frac{K}{m} .$$

One of the axioms of engineering economy is that an asset is replaced when the variable costs of inputs increase to a level equal to the total cost of improved equipment. This axiom is sometimes equated to the dictum, "Sunk costs don't count."[15]

The rationally managed regulated firm adapts its depreciation scheduled in accord with its optimal service life and its time-salvage function. For constant technology, the firm measures its return foregone because of delayed replacement by triangle FBA of Fig. 4.6. But for improving technology, the measure of return foregone is now triangle FDA. Line FA is a measure of the level of decreased total costs prospectively realizeable through constantly improving technology. Line FA permits the total cost of new equipment to be equated to the variable cost of the old, shown by line EA. Line EB is the opportunity cost of the old equipment as Smith saw the problem. Line FC is the service life of the equipment if permitted to wear out completely. Line FG represents economic life if no

[15] In the regulated public utility, sunk costs do count on some occasions. If technology improvement is great enough, the regulatory authority, in a number of cases, has permitted the firm to write off the old asset--and then to amortize and earn on the unamortized after - tax loss, according to some artificially contrived schedule which is usually shorter than would have been applicable had the old asset remained in service.

replacement is contemplated, i.e., "retirement in the pure form."

Fig. 4.6 shows technology improving linearly (line FA), but the illustration could have shown technology improving exponentially. The replacement chain is now established and may be expected to maintain the illustrated behavior until the variables of the model assume new dimensions. Obviously, the investment that is made at the beginning of each chain is greater and the optimum service life shorter than for constant technology. Despite the shorter service life and attendant increase in depreciation charges, the cost of output is lower--not merely because of improvement in technology but also because of concommitant increasing returns to scale. Moreover, as a result of growth, cost of capital r is likely to decrease.

The rationally managed firm, then, may be expected to keep its plants and equipment in service a shorter period, *ceteris puribus*, than the unregulated firm. The traditionally long-lived character of public utility properties does not upset this proposition. Explicit in the replacement model of this chapter is a time-salvage function and a demand function. Perhaps the time-salvage function--apart from the inherent durability and more slowly improving technology--of public-utility properties is the compelling reason for their relatively longer life. But the incentive to shorten service life--subject

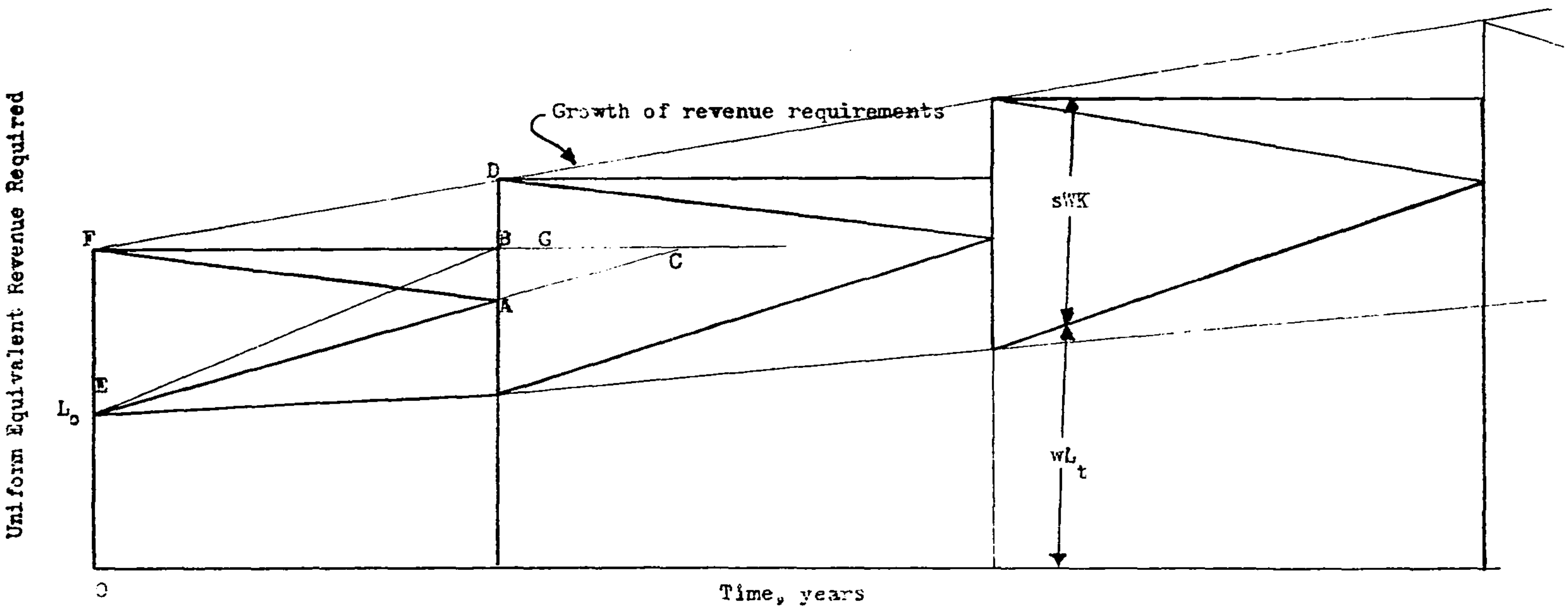

Fig. 4.6 --Illustration of the chain of replacement resulting from increasing costs, increasing technology, and optimal depreciation schedules for the regulated firm

to the constraint of the time-salvage function and the demand function--is demonstrably present and, in theory, is measurable.

CHAPTER V

CAPITAL REQUIREMENTS OF A DYNAMIC MODEL OF THE REGULATED FIRM

In this chapter, investigation of the use of capital by the regulated firm in a dynamic setting is continued. The rate of growth of the firm, rate of capital deterioration, returns to scale, and price-quantity elasticity are the parameters investigated. In Chapter IV, a model for capital replacement in the regulated firm was developed. Chapter IV was concerned with (1), increasing cost of variable inputs as the plant ages, and (2), improving technology. The object of Chapter IV, like most replacement theory, was the determination of optimal service life of plant and equipment. In this chapter, as previously stated, concern is for the amount of capital used as determined by growth, the planning period, returns to scale and price-quantity elasticity. Adaptations developed here of the static model to determine capital use in a dynamic setting, however, may be meshed with replacement theory for the regulated firm.

Technology and the cost of variable inputs are held constant throughout this chapter. This treatment of technology and variable costs is deliberate; such treatment facilitates the investigation of the parameters that are of primary concern. Accordingly, the planning period is held to be determined solely by the rate of destructibility of capital stock. The planning period, however, may be, and should be, coincident with optimal service life in a setting of improving technology and increasing variable costs.

Growth and returns to scale may cause the firm to invest rationally in "overcapacity." Such investment is independent of the regulated firm's propensity toward capital intensity as developed in the static model of Chapter II. Finally, assuming equal output over a planning period, further investment in capacity is prompted by a fluctuating pattern of output, as opposed to the smooth, continuous output heretofore assumed. The interplay of growth, capital deterioration, returns to scale, and price-quantity elasticity is essentially dynamic in character, but the investigation of these parameters abstracts from the static model of Chapter II.

The planning period as implied above is determined by the destructibility--or its obverse, the durability--of capital stock. The present value of the gross revenue for the planning period discounted at the rate of return, minus the present value of variable costs for the planning period (also discounted at the rate of return), must equal the present value of the investment minus the present value of salvage. In other words, the total net revenue product of capital, $TNRP_K$, discounted at the rate of return must equal the present value of investment minus the present value of salvage. In the simplest case, the present value of investment is the initial investment; and no salvage value is realized at the end of the planning period. Alternately, all revenues and costs may be stated in terms of the uniform equivalent rather than in terms of present value.

Indestructible capital was assumed for the static model. This requirement was relaxed in the investigation of optimal service life of Chapter IV. Equating the present value of $TNRP_K$ to the present value of the investment suggests integration of replacement theory with theory that relates to the amount of capital to be used. Thus, the optimal service life merely becomes the t in the capital recovery factor of Chapter IV, $(s + \frac{s}{e^{st}-1})$, or as more frequently expressed, $\frac{se^{st}}{e^{st}-1}$.

Chenery adopted the planning period as a parameter in the problem of determining the optimal scale of plant in the unregulated firm.[1] Chenery also considered the scale coefficient ($\alpha+\beta$, in the Cobb-Douglas production function) and the discount rate as parameters. Chenery

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[1]Hollis B. Chenery, "Overcapacity and the Acceleration Principle," Econometrica, Vol. XX, No. 1, 1-24.

did not treat the rate of growth as a parameter; and, in a minimum-cost setting, he did not reckon with the elasticity of demand. Indeed, price was held constant in Chenery's study as was production technology. Chenery developed the thesis that the unregulated firm would rationally construct excess capacity in the case of increasing returns to scale even when no growth was anticipated.[2] Smith developed a case for overcapacity in the presence of sinusoidal output from a production process of constant returns to scale, assuming no secular growth to be present.[3]

Excess capacity is clearly logical for constant returns to scale when certain growth is present, as is established below. The extent of overcapacity depends, among other considerations, upon the rate of growth, the planning period, and opportunities for what Chenery calls "flexible capacity."[4] An example of the latter term is the addition of generating capacity after a high-voltage transmission line has been completed and put into use.

In the regulated firm, rational behavior in a dynamic setting continues to require that the firm be as capital intensive as permitted by its production function and demand function. Increasing returns to scale reduces

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[2] Ibid., p. 17.

[3] Vernon L. Smith, Investment and Production (Cambridge, Mass.: Harvard University Press, 1961), 286-292.

[4] Chenery, p. 19.

the cost of the product to permit greater output which, in turn, requires more capital. Obviously, this is also true for the unregulated firm but, as in the static case, the regulated and unregulated firms respond differently to such a situation. Analysis of overcapacity in the regulated firm reveals two components: that attributable to growth and that attributable to increasing returns to scale. Each of these, in turn, may be investigated for the overcapacity inherently attributable to regulation by comparing the capital requirements with those of the unregulated firm (which presumably behaves in accord with practices that achieve minimum cost).

The Static Model Modified for Growth and Capital Deterioration

To measure the overcapacity that would rationally be constructed by the regulated firm, the static model of Chapter II must be modified. A certain total net revenue product function

$$TNRP_K = PQ - wL \tag{5.1}$$

for the planning period, discounted at the rate of return, must equal the present value of a certain investment, which is the initial investment if no interim capacity and no growth in $TNRP_K$ is anticipated; that is,

$$\int_0^T \frac{PQ - wL}{e^{st}} = WK. \tag{5.2}$$

In the more interesting case where a certain rate of growth, J, occurs,[5]

$$\int_0^T \frac{(PQ - wL)e^{Jt}}{e^{st}} = WK. \qquad (5.3)$$

Integration of (5.3) yields

$$(PQ - wL)\left[\frac{e^{(J-s)t} - 1}{J - s}\right] = WK. \qquad (5.4)$$

In the development of the static model of Chapter II, equation (2.7)

$$H_t m_o^{(1-\sigma)} L_t^{\alpha(1-\sigma)} K_o^{\beta(1-\sigma)} - wL_t - sWK_o = 0 \quad (5.5)$$

was differentiated with respect to the static equivalent of L_t. The resulting form was solved for K_o and this value was substituted back in equation (5.5) to obtain the static equivalent of L_t for conditions of equilibrium. Subscripts are added to equation (2.7) of Chapter II to obtain equation (5.5) above. The subscript, t, for H and L shows that these values may change over time. The subscript, o, shows that K and m remain constant until K is replaced or improved. With respect to the term m_o, it should be noted that improving technology does cause

[5]This function may also be expressed in terms of the uniform annual equivalent.

opportunity costs and has implications relative to replacement decisions as developed in the last chapter.

After substituting for L_t, equation (5.5) may be restated

$$H_t m_o^{(1-\sigma)} K_o^{\beta(1-\sigma)} \left[\frac{m_o^{(1-\sigma)} H_t \alpha (1-\sigma) K_o^{\beta(1-\sigma)}}{w} \right]^{\frac{1}{1-\alpha(1-\sigma)}}$$

$$- w \left[\frac{m_o^{(1-\sigma)} H_t \alpha (1-\sigma) K_o^{\beta(1-\sigma)}}{w} \right]^{\frac{1}{1-\alpha(1-\sigma)}} = sWK. \tag{5.6}$$

If the demand function grows at rate, J, and revenue and variable costs are discounted at rate, s, then the discounted revenue minus discounted variable costs must equal the present value of the investment, which in this case is the initial investment. To achieve this equality, integration of the total net revenue product function from time zero to T is required. Equation (5.6) is now rearranged and simplified:

$$\frac{H_t^{\frac{1}{1-\alpha(1-\sigma)}} m_o^{\frac{(1-\sigma)}{1-\alpha(1-\sigma)}} K_o^{\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}}}{w^{\frac{(1-\sigma)}{1-\alpha(1-\sigma)}}} \left[\frac{1-\alpha(1-\sigma)}{[\alpha(1-\sigma)]^{-\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}} \right] = sWK. \tag{5.7}$$

But

$$H_t = H_o e^{Jt}.$$

permits investigation of capital requirements of the regulated firm in response to changes in the growth rate, the planning period, and the rate of return.

Changes in Capital in Response to Changes in the Planning Period, the Growth Rate, and the Rate of Return

In Chapter III the static model was investigated to show how requirements of capital and labor, resulting output, and price responded to changes in the variables of the model. Now changes in K, L, P, and Q may be defined in response to the new variables, planning period, t, and growth rate, J. Another variable, rate of return, s, is not new but its role is different for depreciating investment WK than for a constant WK. In analyzing and interpreting the static model the terms s and W were treated as a unit (sW). This treatment, so logical for indestructible capital stock must, in a dynamic setting, be abandoned in favor of separate treatments of s and W. The separate treatment is required because while return, s, may remain constant, investment WK is decreasing in accord with some depreciation schedule which is presumably determined by the rate of destructibility of capital stock.

In this section the emphasis is on capital requirements. Accordingly, changes in capital only are investigated in response to changes in J, t, and s.

Capital, K, of equations (5.10) may be

differentiated partially with respect to planning period, t, growth rate, J, and rate of return, s.

Changes in Capital in Response to Changes in the Planning Period

The planning period, t, here is assumed to be determined by the durability of the capital stock of the regulated firm. Changes in capital, K, in response to changes in the planning period, t, may be determined by differentiating equation (5.10) with respect to, t.

The term

$$\frac{H_o^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} m_o^{\frac{(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} W^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \left[\frac{1-\alpha(1-\sigma)}{[\alpha(1-\sigma)]^{-\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}}\right]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}$$

of equation (5.10) may be treated as a constant, C, so that

$$K = C\left[\frac{e^{\frac{J}{1-\alpha(1-\sigma)}-s}\, t_{-1}}{\frac{J}{1-\alpha(1-\sigma)}-s}\right]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \qquad (5.11)$$

Differentiating capital, K, with respect to the planning period, t,

$$\frac{\partial K}{\partial t} = C\left[\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right]\left[\frac{e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}-1}{\frac{J}{1-\alpha(1-\sigma)}-s}\right]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}-1} \cdot$$

$$\cdot \frac{\left[\frac{J}{1-\alpha(1-\sigma)}-s\right]e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}}{\frac{J}{1-\alpha(1-\sigma)}}$$

$$\frac{\partial K}{\partial t}=C\left[\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right]\left[\frac{e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}-1}{\frac{J}{1-\alpha(1-\sigma)}-s}\right]^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t} \cdot \tag{5.12}$$

No maximum exists, i.e.,

$$0 \ngtr \frac{\partial^2 K}{\partial t^2}.$$

Clearly, constant C is not zero and $e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}$ is not zero. The bracketed quantity in (5.12) may be equal to zero; but only if t or $\frac{J}{1-\alpha(1-\sigma)}-s$ is equal to zero. The situation where t=0 is uninteresting. The term $\frac{J}{1-\alpha(1-\sigma)}-s$ may possibly equal zero; but in that case, the solution for $\frac{\partial K}{\partial t}$ is indeterminate.

The term $\frac{\partial K}{\partial t}$ is positive for both regulated and unregulated firms. Cost is decreased for the unregulated firm as capital becomes durable. The same is true for the regulated firm but here the effect of capital of greater durability influences the capital decision by way of the demand function.

Capital is shown graphically as a function of the planning period in Figs. 5.1 and 5.2 for a wide range of conditions. Equation (5.12) and Figs. 5.1 and 5.2 show that as capital proves more durable it is used more. The important point to be made here is that, while the unregulated firm uses greater durability of capital to minimize costs, the regulated firm sees greater durability of capital as a means of making the firm still more capital intensive— by way of its demand function.

Changes in Capital in Response to Changes in the Growth Rate

Equation (5.11) permits the changes in capital requirements to expressed in terms of the growth rate, J.

$$\frac{\partial K}{\partial J} = C\left[\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right]\left[\frac{e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}-1}{\frac{J}{1-\alpha(1-\sigma)}-s}\right]^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \cdot$$

$$\cdot\frac{\left[\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)\left(\frac{t}{1-\alpha(1-\sigma)}\,e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}\right)-\left(e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}-1\right)\left(\frac{1}{1-\alpha(1-\sigma)}\right)\right]}{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)^{2}} .$$

(5.13)

Table 5.2 shows the effect of a change in the growth rate on capital requirements. Figs. 5.2 through 5.5 also show graphically this effect for a wide range of conditions. The sign of $\frac{\partial K}{\partial J}$ is unambiguously positive even when

$$s > \frac{J}{1-\alpha(1-\sigma)} .$$

The effect of a change in the growth rate is obviously affected by price-quantity elasticity, σ, by α, and by t. The most important of these is σ. A very slight negative change in σ that occurs simultaneously with a positive change in the growth rate can cause a very large change in capital, K.

Capital requirements as a function of the growth rate, J, is shown graphically for a range of specific conditions of σ and t, in Fig. 5.4.

Changes in Capital and Changes in the Rate of Return

In the static model, rate of return, s, was always associated, and investigated with, the price of capital, W. For deteriorating capital the rate of return must be treated apart from W for the simple reason that the return stays constant while WK does not.

$$\frac{\partial K}{\partial s} = C\left[\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}\right]\left[\frac{e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}-1}{\frac{J}{1-\alpha(1-\sigma)}-s}\right]^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \cdot$$

$$\cdot\left[\frac{\left(\frac{J}{1-(1-\sigma)}-s\right)\left(-te^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}\right)+\left(e^{\left(\frac{J}{1-(1-\sigma)}-s\right)t}-1\right)}{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)^{2}}\right]. \quad (5.14)$$

The sign of $\frac{\partial K}{\partial s}$ in a dynamic setting is unambiguously negative just as it is for static conditions. The precise impact of a change in the rate of return, s, like a change in growth rate, J, is affected by other variables, the most important of which is again price-quantity elasticity.

Solutions for Labor, L; Output, Q, and Price, P, to Provide for Growth and Capital Deterioration

The equilibrium solution for variable input, L, which provides for growth and capital deterioration is obtained by the same procedure adopted for the static model of Chapter II. The solution for fixed input K in equation (5.9) is again substituted into the intermediate solution for L in

$$L = \left[\frac{\alpha(1-\sigma)Hm^{1-\sigma}K^{\beta(1-\sigma)}}{w}\right]^{\frac{1}{1-\alpha(1-\sigma)}}$$

to obtain a solution for L that provides for growth and capital deterioration:

$$L = \frac{1-\alpha(1-\sigma)^{\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}[\alpha(1-\sigma)]^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{w^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}$$

$$\cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}} m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{W^{\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \cdot \left[\frac{e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}-1}{\frac{J}{1-\alpha(1-\sigma)}-s}\right]^{\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \tag{5.15}$$

Output, Q_t, is again obtained by substituting for L and K in the production function

$$Q_t = mL_t^{\alpha}K_o^{\beta}.$$

Thus,

$$Q_t = \frac{1-\alpha(1-\sigma)^{\frac{\beta}{1-(\alpha+\beta)(1-\sigma)}}\alpha(1-\sigma)^{\frac{\alpha}{1-(\alpha+\beta)(1-\sigma)}}}{w^{\frac{\alpha}{1-(\alpha+\beta)(1-\sigma)}}\ W^{\frac{\beta}{1-(\alpha+\beta)(1-\sigma)}}}$$

$$\cdot \frac{H^{\frac{\alpha+\beta}{1-(\alpha+\beta)(1-\sigma)}} m^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}}}{} \cdot \left[\frac{e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}-1}{\frac{J}{1-\alpha(1-\sigma)}-s}\right]^{\frac{\beta}{1-(\alpha+\beta)(1-\sigma)}} \tag{5.16}$$

Price, P, is also again obtained by substituting in the demand function

$$P_t = HQ_t^{-\sigma}$$

$$P_t = \frac{[1-\alpha(1-\sigma)]^{\frac{-\beta\sigma}{1-(\alpha+\beta)(1-\sigma)}}\,[\alpha(1-\sigma)]^{\frac{-\alpha\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{w^{\frac{-\alpha\sigma}{1-(\alpha+\beta)(1-\sigma)}}\,W^{\frac{-\beta\sigma}{1-(\alpha+\beta)(1-\sigma)}}}\,\cdot$$

$$\cdot\, H^{\frac{1-(\alpha+\beta)}{1-(\alpha+\beta)(1-\sigma)}}\,m^{\frac{-\sigma}{1-(\alpha+\beta)(1-\sigma)}}\left[\frac{e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}-1}{\frac{J}{1-\alpha(1-\sigma)}-s}\right]^{-\frac{\beta\sigma}{1-(\alpha+\beta)(1-\sigma)}}\,. \tag{5.17}$$

<u>Capital Requirements in Relation to the Planning Period, the Growth Rate, Returns to Scale, and Price-Quantity Elasticity: An Example</u>

In a dynamic setting, the planning period becomes a significant parameter in the problem of determining the optimal size of plant. Fig. 5.1 shows that even in the no-growth case, the planning period is a parameter. The planning period remains a parameter regardless of returns to scale. In Chenery's investigation, the planning period was a parameter only in the case of increasing returns to scale.[6] The planning period is a parameter for the regulated firm, regardless of returns to scale, for the simple reason that the more durable is the capital stock, the less is the resultant depreciation charge. Greater durability

[6]Hollis B. Chenery, "Overcapacity and the Acceleration Principle," <u>Econometrica</u>, Vol. XV, No. 1, 1-24.

of plant and equipment decreases the cost of the product. Here again, the demand function indicates more output which, in turn, requires more capital. Fig. 5.1 shows that capital requirements, however, are by no means independent of returns to scale. The growth rate too is shown to be a significant parameter. The interdependence of these and other variables is developed more fully in Figs. 5.2, 5.3, 5.4, and 5.5.

Fig. 5.1 is a graphic solution to the equation

$$K = \left[\frac{e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}-1}{\frac{J}{1-\alpha(1-\sigma)}-s}\right]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}}\, m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{w^{\frac{(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\, W^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \cdot \left[\frac{1-\alpha(1-\sigma)}{[\alpha(1-\sigma)]^{-\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}}\right]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} .$$

Table 5.1(a) is a tabular solution for the no-growth case for planning periods of 10, 20, and 30 years, and for a range of returns to scale. Table 5.1(b) is a tabular solution for the same problem except that here the growth rate is 7 per cent per year. In Fig. 5.1, the optimal size of plant is a function of the planning period. In Fig. 5.1,

the wage rate, w, is	4
the price of a unit of capital, W, is	100
rate of return, s, is	.065

price-quantity elasticity, σ, is .5

initial price-quantity coefficient, H, is 10

production function coefficient, m, is 100

the rate of growth, J, is both .00 and .07

α, β range from 1/4 and 1/4, to 3/4 and 3/4.

Fig. 5.1 shows graphically the effect of various returns to scale[7] on the optimum size of plant for conditions of no growth and 7 per cent continuous growth. Price-quantity elasticity, σ, is held constant at 0.5. In Fig. 5.1, longer planning periods reduce depreciation costs to permit a lower price for output. Each line of Fig. 5.1 measures the constraint imposed by the demand function upon the propensity toward capital intensity in the production process.

For the no growth case and for constant returns to scale, the optimal size plant for a planning period of 30 years is 2.4 times that for 10 years. For decreasing returns to scale, e.g., $(\alpha+\beta)=1/2$, the ratio of optimal scale of plant for 30 years to that for 10 years is 2.1; however, for increasing returns $(\alpha+\beta=1.5)$, the corresponding ratio is 4.3. These calculations have some rough parallel to Chenery's findings that overcapacity will attend increasing

[7]Here, as earlier in this study, the measure of returns to scale is the sum of the elasticity of output to labor and elasticity of output to capital, or α plus β. Constant returns to scale is defined as the conditions of the production process that equates $(\alpha+\beta)$ to one. For decreasing returns to scale $1>(\alpha+\beta)$; for increasing returns to scale, $(\alpha+\beta)>1$.

Table 5.1(a).--Optimal size of plant for three planning periods for various returns to scale (growth, J = 0; price-elasticity, σ = .5)

	Optimal Scale of Plant		
	10 years	20 years	30 years
α = 1/4, β = 1/4	10.4	17.2	22.1
α = 1/2, β = 1/2	33.3	60.4	77.8
α = 1/2, β = 3/4	99.6	231.0	324.0
α = 3/4, β = 1/2	114.0	240.0	303.0
α = 3/4, β = 3/4	1,262.0	3,610.0	5,420.0

Table 5.1(b).--Optimal size of plant for various returns to scale (growth rate = .07; price-elasticity = .5)

$\alpha = 1/4, \quad \beta = 1/4$

$$K = \left[\frac{H^{4/3}m^{2/3}7^{7/6}}{w^{1/6}W^{7/6}8^{4/3}}\right]\left[\frac{e^{\left(\frac{J}{7/8}-s\right)t}-1}{\frac{J}{7/8}-s}\right]^{7/6}$$

J=.07, s=.065, t=10: 16.5
t=20: 39.0
t=30: 56.6

$\alpha = 1/2, \quad \beta = 1/2$

$$K = \left[\frac{H^{2}m \quad 3^{3/2}}{w^{1/2}W^{3/2}4^{2}}\right]\left[\frac{e^{\left(\frac{J}{3/4}-s\right)t}-1}{\frac{J}{3/4}}\right]^{3/2}$$

J=.07, s=.065, t=10: 63.8
t=20: 228
t=30: 526

$\alpha = 1/2, \quad \beta = 3/4$

$$K = \left[\frac{H^{8/3}m^{4/3}3^{2}}{w^{2/3}W^{2}4^{8/3}}\right]\left[\frac{e^{\frac{J}{3/4}-s\ t}-1}{\frac{J}{3/4}-s}\right]^{2}$$

J=.07, s=.065, t=10: 246
t=20:1422
t=30:4035

$\alpha = 3/4, \quad \beta = 1/2$

$$K = \left[\frac{H^{8/3}m^{4/3}(5/3)^{5/3}}{w \quad W^{5/3}(8/3)^{8/3}}\right]\left[\frac{e^{\frac{J}{5/8}-s\ t}-1}{\frac{J}{5/8}-s}\right]^{5/3}$$

J=.07, s=.065, t=10: 244
t=20:1031
t=30:2503

$\alpha = 3/4, \quad \beta = 3/4$

$$K = \left[\frac{H^{4} \quad m^{2} \quad (5/3)^{5/2}}{w^{3/2}W^{5/2}(8/3)^{4}}\right]\left[\frac{e^{\frac{J}{5/8}-s\ t}-1}{\frac{J}{5/8}-s}\right]^{5/2}$$

J=.07, s=.065, t=10: 3,990
t=20: 31,040
t=30:128,000

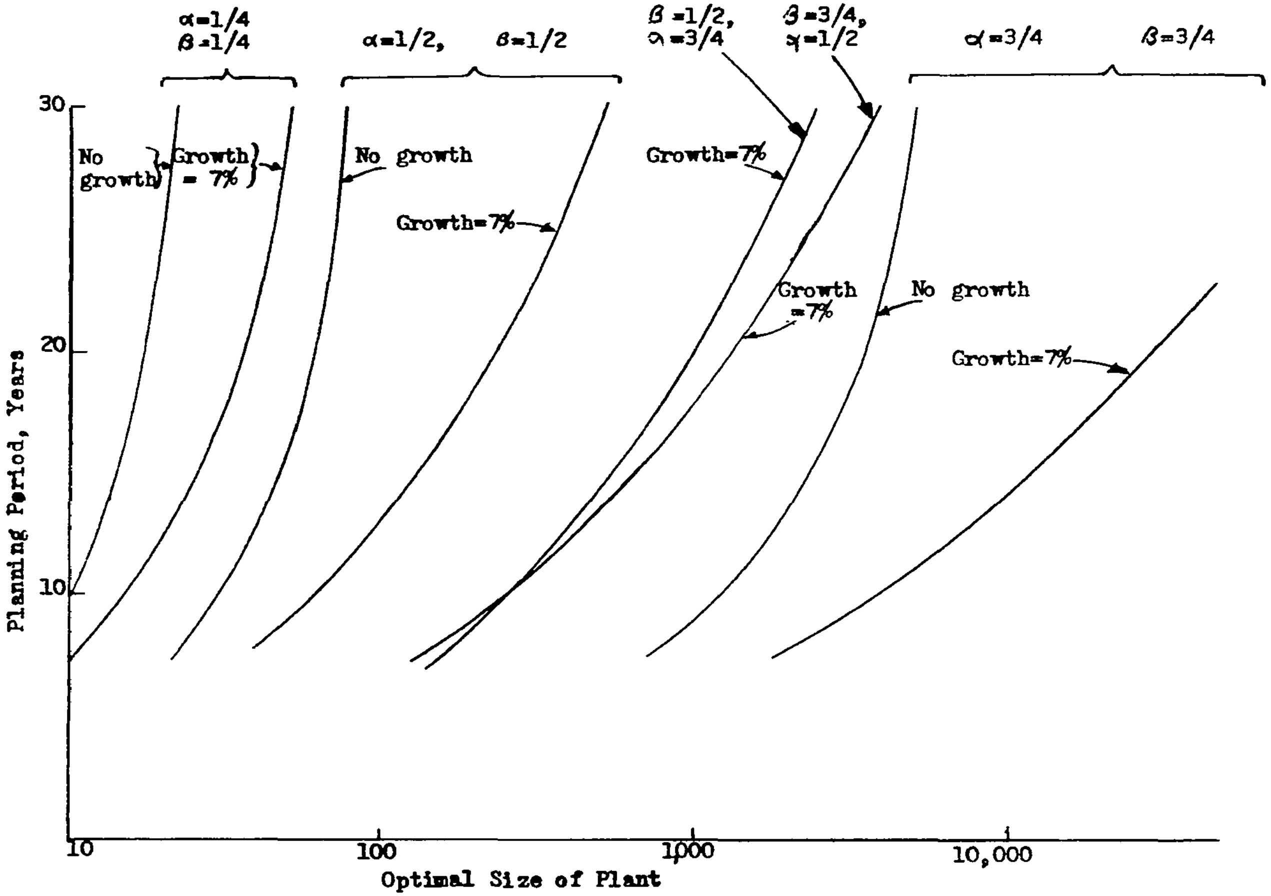

Fig. 5.1.—Illustration of optimal planning period resulting in optimal size of plant.

returns to scale in the unregulated firm. The difference in the two cases is important. In Chenery's case, overcapacity for increasing returns to scale was an instrument for cost reduction. Here, in the regulated case, increasing returns to scale cheapened the cost of output to permit use of more plant upon which a return, s, can be earned. The constraint again is the demand function.

For growth rate $J=.07$, optimal sizes of plant for a range of returns to scale are shown in Fig. 5.1. In each case, plant was presumed constructed for the entire planning period. The margin between the no growth lines and the 7 per cent growth lines for each condition of returns to scale affords certain meaningful comparisons. The margin between optimal size of plant for 7 per cent and no growth for 30 years for increasing returns ($\alpha+\beta=3/4$) is some 250 times that for constant returns.

In Chapter III, the changes in capital in response to changes in α and β were investigated. Equations (3.32) and (3.33) show clearly that $\frac{\partial K}{\partial \alpha}$ is substantially different from $\frac{\partial K}{\partial \beta}$. But Chenery makes capital requirements a function of the "scale" coefficient ($\alpha+\beta$) for his cost-minimizing case without regard to the composition of the sum. Clearly, the regulated firm does not equate K, and K_2

$$K_1 = \phi(\alpha_1, \beta_1)$$

$$\text{where } \alpha_1 = 1/4, \text{ and } \beta_1 = 3/4$$

$$K_2 = \phi(\alpha_2, \beta_2)$$

where $\alpha_2 = 1/2$, and $\beta = 1/2$.

But neither does the unregulated firm equate K_1 and K_2. Chenery merely treated the sum $(\alpha+\beta)$ as though the composition of the sum were fixed. Later, changes in capital in response to changes in the composition of this sum are compared graphically in Fig. 5.5 for the regulated and unregulated firms.

Chenery used the term "flexible capacity" to cover situations where plant is added incrementally as demand for output grows. In his example, a gas pipeline was built with capacity years ahead of requirements, and compressor capacity was installed incrementally as demand for output grew. The electric utility has parallels, for example, in additions to generating capacity once high-voltage main transmission and distribution lines are constructed. Such flexible capacity is generally added in relatively large discrete units. If a minimum size of capital addition can be defined, and if the size of the irreducible starting plant can also be defined, the optimum scale of plant can be recalculated. Flexible capacity would increase the optimum scale of plant for a planning period for the

Table 5.2.--Optimal size of plant for a range of planning periods, growth rates, and price-quantity elasticities

$\alpha = 1/2,\ \beta = 1/2$	$\sigma = 1/4$	$\sigma = 1/2$	$\sigma = 3/4$	$\sigma = 1$
	$\left[\frac{H^4 m^3 (5/3)^{5/2}}{w^{3/2} W^{5/2} (8/3)^4}\right]\left[\frac{e^{\left(\frac{J}{5/8}-s\right)t}-1}{\frac{J}{5/8}-s}\right]^{5/2}$	$\left[\frac{H^2 m\ 3^{3/2}}{w^{1/2} W^{3/2} 4^2}\right]\left[\frac{e^{\left(\frac{J}{3/4}-s\right)t}-1}{\frac{J}{3/4}-s}\right]^{3/2}$	$\left[\frac{H^{4/3} m^{1/3} 7^{7/6}}{w^{1/6} W^{7/6} 8^{4/3}}\right]\left[\frac{e^{\left(\frac{J}{7/8}-s\right)t}-1}{\frac{J}{7/8}-s}\right]^{7/6}$	$\left[\frac{H\ m^0}{w^0 W}\right]\left[\frac{e^{(J-s)t}-1}{J-s}\right]$
	s = .065	s = .065	s = .065	s = .065
J = 0				
t = 10:		33.3	2.3	.75
t = 20:		61.2	3.68	1.12
t = 30:		78.0	4.42	1.32
J = .04				
t = 10:		47.2		
t = 20:		130.0		
t = 30:		206.0		
J = .07				
t = 10:	585,000	63.8	3.55	1.06
t = 20:	7,160,000	241.	8.66	2.16
t = 30:	25,650,000	526.	17.20	2.96
J = .10				
t = 10:		87.		
t = 20:		434.		
t = 30:		1,558.		

$\alpha = 3/4,\ \beta = 3/4$	$\sigma = 1/4$[b]	$\sigma = 1/2$	$\sigma = 3/4$	$\sigma = 1$
		$\left[\frac{H^2 m^{3/2} 3^{3/2}}{w^{1/2} W^{3/2} 4^2}\right]\left[\frac{e^{\left(\frac{J}{5/8}-s\right)t}-1}{\frac{J}{5/8}-s}\right]^{5/2}$	$\left[\frac{H^{3/5} m^{2/5} (13/3)^{13/10}}{w^{3/10} W^{13/10} (16/3)^{8/5}}\right]\left[\frac{e^{\left(\frac{J}{13/10}-s\right)t}-1}{\frac{J}{13/10}-s}\right]^{13/10}$	$\left[\frac{H\ m^0}{w^0 W}\right]\left[\frac{e^{(J-s)t}-1}{J-s}\right]$
	s = .065	s = .065	s = .065	s = .065
J = 0				
t = 10:		1,142		
t = 20:		3,560		
t = 30:		9,920		
J = .04				
t = 10:		2,322		
t = 20:		18,650		
t = 30:		48,300		
J = .07				
t = 10:		3,900	3.75	1.06
t = 20:		31,040	9.60	2.16
t = 30:		128,000	22.27	2.96
J = .10				
t = 10:		6,440		
t = 20:		96,650		
t = 30:		266,000		

[a]The numerical values for parameters J, t, σ, α, and β are as shown in the table; the values for the internal variables are the same as used in previous examples in the text, namely, H=10, m=100, w=4, W=100, s=.065. These are substituted in the equation for the following general solution for K, in order to produce the examples of K values in the table:

$$K = \left[\frac{e^{\left(\frac{J}{1-\alpha(1-\sigma)}-s\right)t}-1}{\frac{J}{1-\alpha(1-\sigma)}-s}\right]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \frac{H^{\frac{1}{1-\alpha(1-\sigma)-\beta(1-\sigma)}}\ m^{\frac{1-\sigma}{1-\alpha(1-\sigma)-\beta(1-\sigma)}}}{w^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)-\beta(1-\sigma)}}\ W^{\frac{1-\alpha(1-\sigma)}{1-\alpha(1-\sigma)-\beta(1-\sigma)}}} \left[\frac{1-\alpha(1-\sigma)}{\alpha(1-\sigma)^{\frac{-\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}}\right]^{\frac{1-\alpha(1-\sigma)}{1-\alpha(1-\sigma)-\beta(1-\sigma)}}$$

[b]For $\sigma = 1/4$, $\alpha = 3/4$, and $\beta = 3/4$ no solution exists.

compelling reason that the cost of excess capacity is reduced.[8,9]

In Fig. 5.1, price-quantity elasticity, σ, is held constant. Fig. 5.2 again shows the optimum scale of plant as a function of the planning period, but this time price-elasticity, σ, is a variable. Here the rate of growth of demand is held constant at $J=.07$.

The optimum scale of plant for price-quantity elasticity, $\sigma=1$, is the same for all combinations of α and β. These identities result when it is assumed that $\sigma=1$ in the capital requirements function,

$$K = \left[\frac{e^{\frac{J}{1-\alpha(1-\sigma)}-s\;t}-1}{\frac{J}{1-(1-\sigma)}-s}\right]^{\frac{1-(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}}\, m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{W^{\frac{(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\, W^{\frac{1-(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}$$

$$\cdot \left[\frac{1-\alpha(1-\sigma)}{[\alpha(1-\sigma)]^{\frac{-\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}}\right]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}$$

or

$$K = \left[\frac{e^{(J-s)t}-1}{J-s}\right]\frac{H}{W} \qquad (5.18)$$

where $\sigma=1$, for all values for α and β.

---

[8]Hollis B. Chenery, "Process and Production Functions from Engineering Data" in Wassily W. Leontief (ed.) <u>Studies in the Structure of the American Economy</u> (Oxford: Oxford University Press, 1953).

[9]Hollis B. Chenery, "Engineering Production Functions," <u>Quarterly Journal of Economics</u>, Vol. LXIII, No. 4, 507-531.

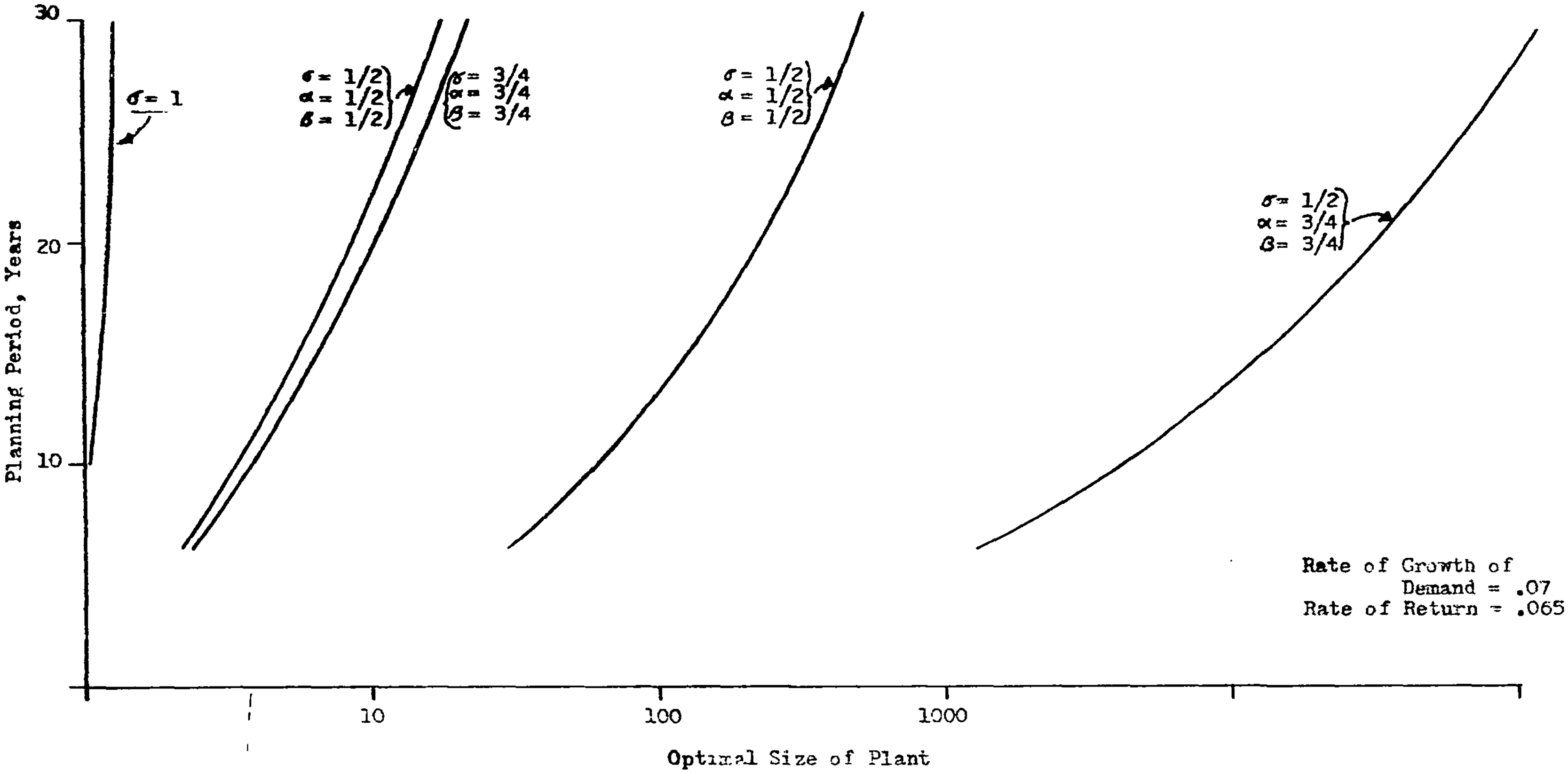

Fig. 5.2--Illustration of optimal planning period resulting in optimal size of plant, given arbitrary assumption of price-quantity elasticity, returns to scale

For the example cited on page 192, in Table 5.1 and illustrated in Fig. 5.1, the optimum scale of plant is 2.96 when the price-quantity elasticity is 1 (and when the planning period is 30 years, for constant returns to scale and for growth rate of .07). The optimal scale of plant increases exponentially as price-quantity elasticity decreases (or as the elasticity of demand increases). For $t=30$, and constant returns to scale (α, $\beta=1/2$), $J=.07$, the optimum scale of plant for $\sigma=3/4$ is 17.2; for $\sigma=1/2$, 52.6; and for $\sigma=1/4$, 25,650,000 (See Table 5.2). Further decreases in the price-quantity elasticity tends to make capital requirements infinitely great, or,

$$\lim_{1-(\alpha+\beta)(1-\sigma)\to 0} K = \infty \quad . \tag{5.19}$$

This characteristic of the model has some resemblance to Gordon's growth stock paradox, which permits the price, P, of a share of stock to approach infinity as the growth rate approaches the discount factor in the valuation model[10]

$$P = \frac{D}{k-g}$$

$$\lim_{g\to k} P = \infty$$

where P is the price of the stock

D is the dividend paid

---

[10]Myron H. Gordon, The Investment, Financing and Valuation of the Corporation (Homewood, Ill.: Irwin, 1962), Chap. 4.

k is the discount rate

g is the growth rate.

As pointed out in Chapter III, this characteristic of the model is present in a static setting; but since, in a dynamic problem, the elasticity term is even more interrelated with the growth rate and the planning period than other variables of the model, changes in capital requirements are made even more responsive to changes in elasticity in the dynamic setting than in the static.

Table 5.2 shows that capital requirements increase much more slowly as the elasticity of demand increases (or as price-quantity elasticity decreases) for the no-growth assumption than for a 7 per cent rate of growth. Fig. 5.3 shows that capital requirements increase much faster for increasing returns as elasticity of demand increases than for constant returns to scale. To a much lesser extent, the planning period also affects the rate of response of capital requirements to changes in elasticity of demand. Fig. 5.4 shows the optimum scale of plant as a function of the rate of growth for constant returns to scale. The planning period and price-quantity elasticity are also parameters in Fig. 5.4. In effect, Fig. 5.4 merely measures how capital requirements respond to a decreasing price of the product because of lesser depreciation charges than attend longer planning periods.

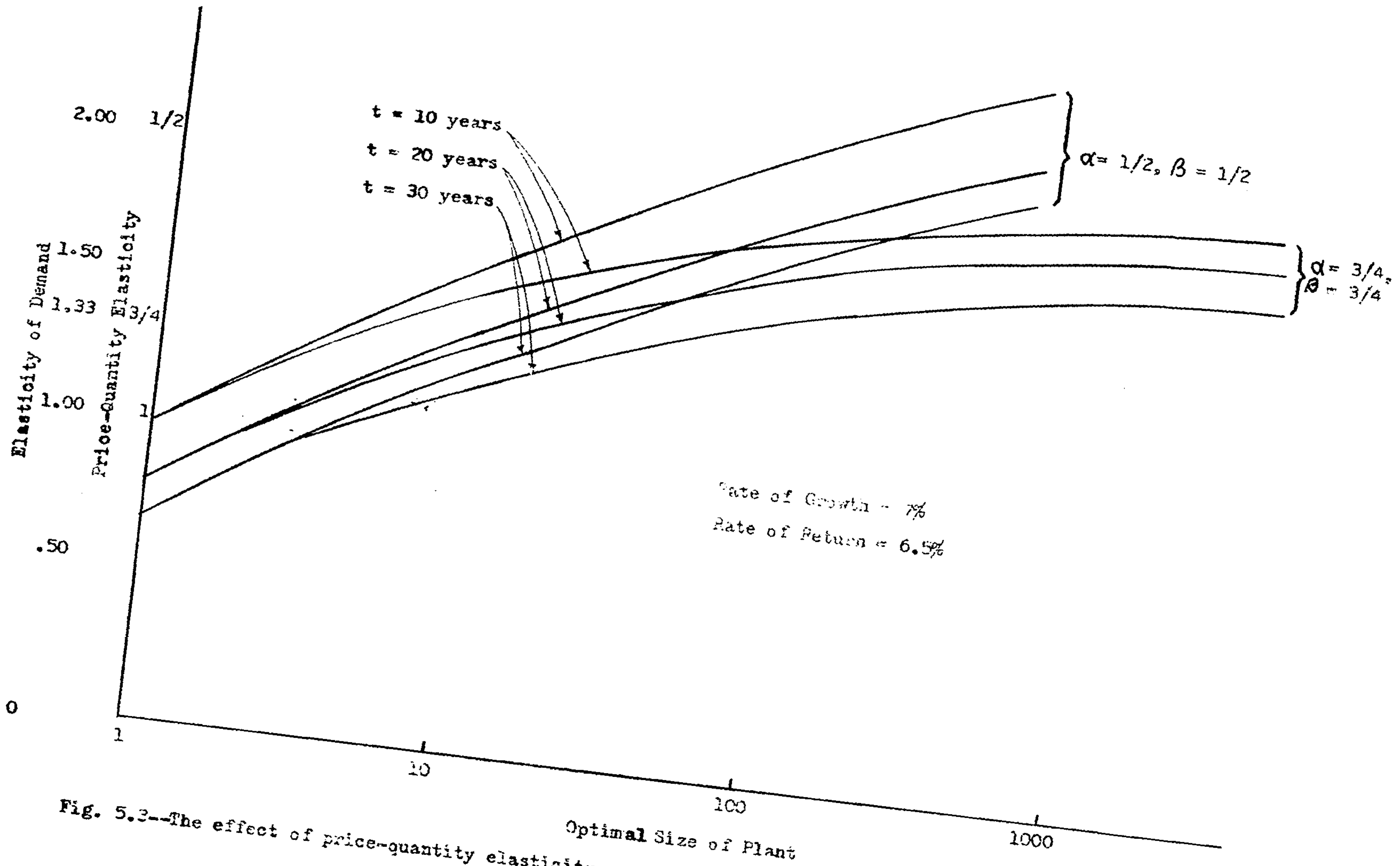

Fig. 5.3--The effect of price-quantity elasticity on the optimal size of plant, for two different returns to scale

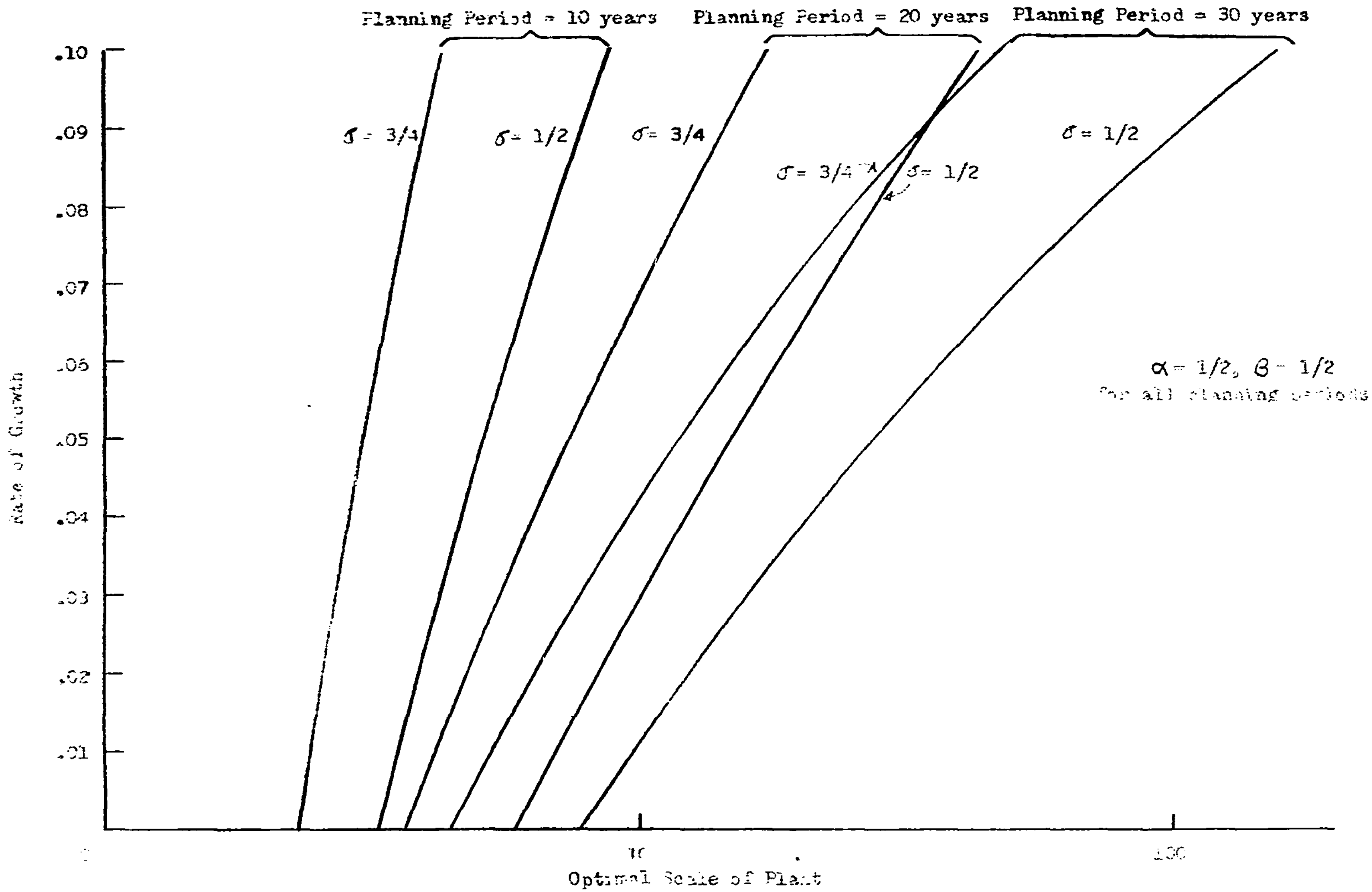

Fig. 5.4.--Effect of the rate of growth on the optimal size of plant for three planning periods and two price-quantity elasticities

Comparisons of Capital Requirements of the Regulated and Unregulated Firm Under Dynamic Conditions

In Chapter II, the difference between capital requirements for the regulated and unregulated firm were developed for the firm in a static setting, i.e., no growth and no capital deterioration. A corresponding difference between capital requirements in the two cases also applies in a dynamic setting. The difference between the two requirements may be demonstrated as in the static case by imposing identical conditions of output, cost of capital, rate of return, wage rate, and price of capital. Additionally, identical conditions of growth in demand and of planning period may be imposed.

Table 5.2 illustrates a solution for capital units, for the regulated firm. Here K is 526 units for a specific numerical example.

Where:	planning period,	$t = 30$ years
	production coefficient,	$m = 100$
	demand coefficient,	$H = 10$
	rate of growth of demand,	$J = .07$
	rate of return,	$s = .065$
	wage rate,	$w = 4$
	price of a unit of capital,	$W = 100$
	production exponents, α and	$\beta = .5$
	price elasticity,	$\sigma = .5$
	cost of capital,	$r = .05$

To obtain output, Q_t, for the above conditions, the required solution for variable input, L_t, was developed initially in Chapter II and adapted to dynamic conditions in Chapter IV. The adapted solution for L_t is

$$L_t = \left[\frac{m^{(1-\sigma)}\ H_t{}^{\alpha(1-\sigma)}\ K_o{}^{\beta(1-\sigma)}}{w}\right]^{\frac{1}{1-\alpha(1-\sigma)}} . \qquad (5.20a)$$

Since

$$H_t = H_o e^{jt}$$

substituting for the internal variables of (5.20a) yields

$$L_t = \left[\frac{100^{1/2} \times \left(10\ e^{2.1}\right)^{1/4} \times 526^{1/4}}{4}\right]^{4/3} = 126$$

$$(5.20b)$$

$$Q_t = m_o L_t^{\alpha} K_o^{\beta} = 100 \times 126^{1/2} \times 526^{1/2} = 28{,}000.$$

Similarly, variable inputs, L_o, and output, Q_o, for time zero is determined to be 62.5 and 19,700 units, respectively. The growth rates for the 30-year period then, for variable inputs and output are .0673 and .008, respectively. If J' is the rate of growth of output

$$Q_t = Q_o e^{J't} \qquad (5.21)$$

To obtain the neoclassical minimum cost capital requirements for output Q_t, L_t is first restated:

$$L_t = \left(\frac{Q_t}{m}\right)^{1/\alpha} \frac{1}{K^{\beta/\alpha}}$$

$$= \left(\frac{Q_o e^{J't}}{m}\right)^{1/\alpha} \frac{1}{K^{\beta/\alpha}}$$

$$L_t = \left(\frac{Q_o}{m}\right)^{1/\alpha} e^{\frac{Jt}{\alpha}} \frac{1}{K^{\beta/\alpha}} . \qquad (5.22)$$

Since $$L_o = \left(\frac{Q_o}{m}\right)^{1/\alpha} \frac{1}{K^{\beta/\alpha}}$$

$$L_t = L_o\, e^{\frac{Jt}{\alpha}} . \qquad (5.23)$$

The classical cost function

$$C = wL + rWK$$

may be restated in terms of present value for the no growth case

$$C = \int_o^T \frac{wL}{e^{rt}} + WK$$

and where growth is present

$$C = \int_o^T \frac{wL_t}{e^{rt}} + WK .$$

Substituting for L_t from (5.23)

$$C = \int_o^T \frac{wL_o e^{\frac{Jt}{\alpha}}}{e^{rt}} + WK . \qquad (5.24)$$

Integrating to obtain the present value of all costs of output (growing at rate J) produced by a plant that deteriorates in t years,

The

$$C = wL_o \left[\frac{e^{\left(\frac{J}{\alpha} - r\right)t} - 1}{\frac{J}{\alpha} - r} \right] + WK. \qquad (5.25)$$

To obtain the neoclassical minimum-cost capital requirements, where growth and capital deterioration are present, the marginal productivity conditions are equated to permit solving for capital, K.

$$\frac{dQ}{dL} = \alpha m L^{\alpha-1} K^{\beta} = w \left[\frac{e^{\left(\frac{J}{\alpha} - r\right)t} - 1}{\frac{J}{\alpha} - r} \right] \qquad (5.26)$$

$$\frac{dQ}{dK} = \beta m L^{\alpha} K^{\beta-1} = W. \qquad (5.27)$$

Equating (5.26) and (5.27) and solving for K,

$$K = \left\{ \frac{\beta w \left[\frac{e^{\left(\frac{J}{\alpha} - r\right)t} - 1}{\frac{J}{\alpha} - r} \right]}{\alpha W} \right\}^{\frac{\alpha}{\alpha+\beta}} \left(\frac{Q_t}{m} \right)^{\frac{1}{\alpha+\beta}} \qquad (5.28)$$

For the case where investment is all made at the start of the planning period, substitution in (5.28) yields the capital requirements for output growing at rate, J.[11]

For the values cited on page 205, and where the growth rate of output is .008, the capital requirements of a cost-minimizing firm may be determined:

$$K = \left[\frac{.5 \times 4\, e^{\left(\frac{.008}{.5} - .05\right)30} - 1}{.5 \times 100\left(\frac{.008}{.5} - .05\right)}\right]^{1/2} \frac{28,000}{100}$$

$$= \left[.04\ (18.82)\right]^{1/2} \times 280$$

$$= 243 \qquad (5.29)$$

The optimal capital requirements for this firm under regulation is 526. See Table 5.2.

As in the static case, no general statement may be made as to how much more capital intensive is the regulated than the unregulated firm until the production, demand, and profit functions are specified.

Fig. 5.5 shows graphically the minimum-cost solutions to the example cited at the beginning of this section. Fig. 5.5 shows the optimal size of plant for the cost-minimizing firm in relation to the optimal size of plant for the regulated firm for a range of t, J, α, β,

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[11]Note that the rate of growth of variable inputs, L, is greater than the growth of output, which means that the K/L and K/Q ratios decline throughout the planning period.

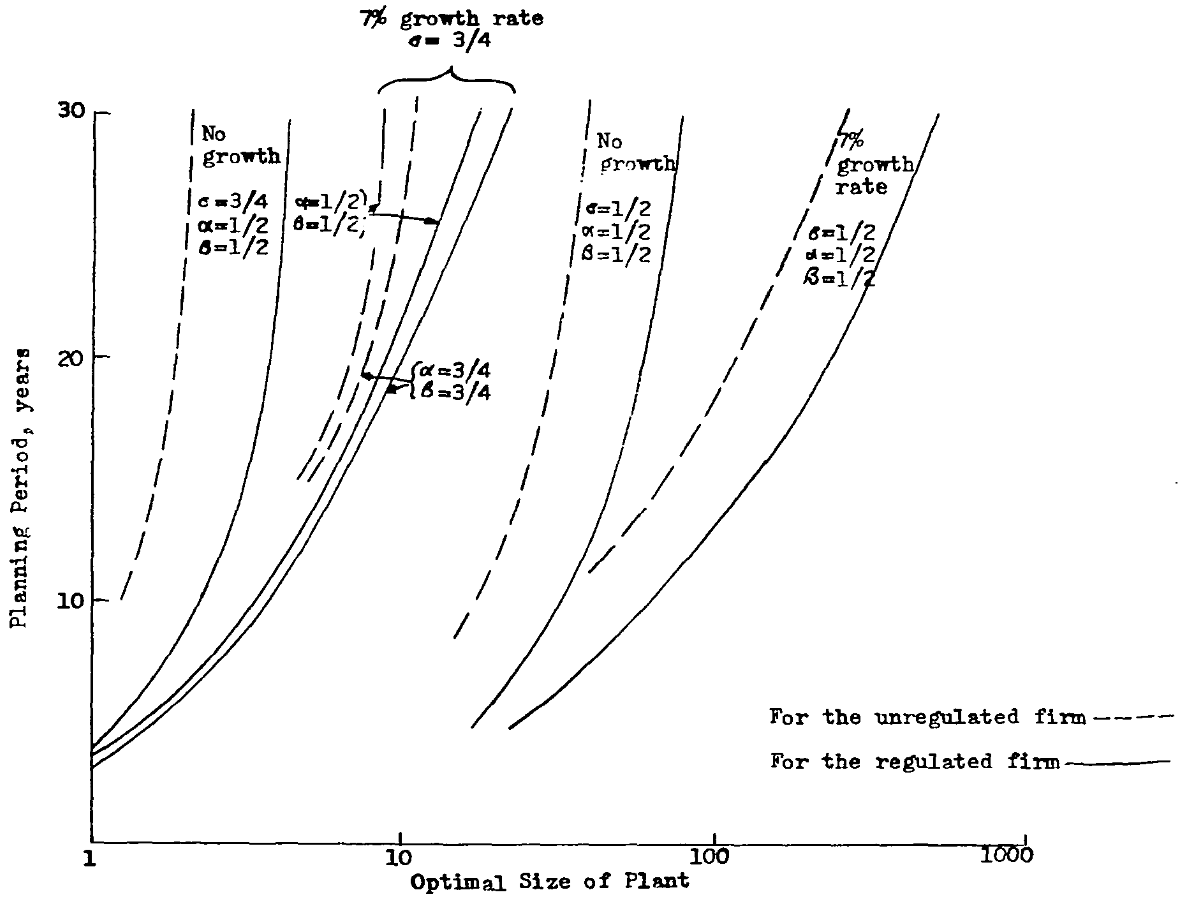

Fig. 5.5.—Comparisons of sizes of plant for the regulated and the unregulated firm for various conditions of growth, elasticity, and returns to scale.

and σ. All minimum-cost solutions shown in Fig. 5.5 lie appreciably to the left of the optimum solutions for the regulated firm.

Capital Requirements for the Regulated Firm that Produces Output Sinusoidally

To this point, all output has been assumed to be smooth and continuous. Patterns of fluctuation, if not pure periodicity, occur in output of most real-world production processes. If output is not storable, time series fluctuations in output are unavoidable. These fluctuations may involve normal seasonality as well as monthly, weekly, daily, and even hourly peaks and lows. Except for gas utilities, output of public utilities is for the most part not storable. Even in the gas-utility industry, the investment required for storage facilities is so enormous that no attempt is made to eliminate seasonality of the pattern of output from the pipeline; rather the firm seeks some optimal amplitude in the fluctuations in output. To a much lesser extent, the electric-utility industry stores output by pumping water to uphill reservoirs during periods of slack demand, for release to produce electricity during peak periods. The familiar "heat pump" is another, but relatively unimportant, example of indirectly stored electric-utility output.

If any contention of relative completeness is to be made for a theoretical model of the regulated firm, the model must in some way encompass fluctuating output. So that these fluctuations in output may be accommodated by the model, reliance will be placed by the present writer on the sinusoidal pattern of fluctuation; that is, the pattern described by a sine wave (i.e., constant frequency with equal, constant positive and negative amplitude) is used to represent unavoidable fluctuations in output characteristic of an operating regulated firm. Although only a simple sinusoidal pattern is employed here, it is possible to simulate complicated recurring patterns by the use of a complex function which, in effect, is achieved by adding sine waves of different frequencies and amplitudes.

Obviously, problems arise out of the necessity that peak capacity be constructed to provide peak output--regardless of the considerable amount of time that peak capacity is not used. A low load factor makes no distinction between, say, 10 per cent use for a day and 10 per cent use for a 3-month season.

It would also be possible to subject the sinusoidal pattern to additional random fluctuations (characterized by known or subjectively established probability distributions). Interesting as this problem is in planning operations, it is a refinement not covered in this study.

Chenery,[12] Goodwin,[13] and Smith[14] have discussed the problem of capital requirements for sinusoidal, rather than smooth continuous output. All three discussed the problem from the neoclassical minimum-cost view. Only Goodwin discussed the problem in a setting of growth. Classically, this problem is and has been integrated into conventional minimum-cost inventory theory in those cases where inputs and outputs are storable. For storable outputs of the regulated firm, the problem becomes a bit muddled by the ability of the utility to earn, not only on the capital required for storage facilities, but also on the value of the stored output itself. Conventional inventory theory is also deficient for the regulated firm in another way. Gas storage projects, and to a much lesser extent projects for the indirect storage of electric-utility output, have production functions of their own and therefore do not readily lend themselves to treatment in accord with the large body of current inventory theory.

Smith covers the case of no growth and constant returns to scale, providing a minimum-cost solution for

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[12]Hollis B. Chenery, "Overcapacity and the Acceleration Principle," Econometrica, Vol. XX, No. 1, 1-28.

[13]Richard M. Goodwin, "Secular and Cyclical Aspects of the Multiplier and Accelerator" in Income, Employment, and Public Policy, Essays in Honor of Alvin M. Hansen (New York: W. W. Norton & Co., 1948), 120.

[14]Vernon L. Smith, Investment and Production (Cambridge, Mass.: Harvard University Press, 1961), 286-290.

fixed and variable inputs for a given output.[15] He concluded that more inputs are required for sinusoidal output than for an equivalent constant output. The amount of output has not been rigidly specified throughout this investigation up to this point; rather, a demand and a production function have been specified and equilibrium inputs, outputs, and price determined by the constraints. The equilibrium output so determined has heretofore been a continuous, non-fluctuating output. To the extent, then, that sinusoidal output is inherently inefficiently produced, it must cost more and accordingly must decrease from levels represented by constant output. Again, the constraint is the demand function.

If the demand function behaves in some sinusoidal pattern and price is held constant over the sine wave, then, perforce, the only degree of freedom is output. So if the amplitude of the demand function is γ and if

$$H_t = H_o(1 + \gamma \sin\theta t) \qquad (5.30)$$

where H_t is the coefficient of demand at time t

H_o is the coefficient of demand at time o

γ is the amplitude of demand

θ is the relative position in the sine wave at time t; $\frac{2\pi}{\theta}$ represents one complete sine wave.

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[15] *Ibid.*

Then

$$Q_t = \left(\frac{H_o}{P}\right)^{1/\sigma} (1+\gamma\sin\theta t)^{1/\sigma}. \tag{5.31}$$

Thus at $\frac{\pi}{2\theta}$ and $\frac{3\pi}{2\theta}$,

$$(1+\gamma') = (1+\gamma)^{1/\sigma} \tag{5.32}$$

where γ' is the amplitude of output.

Since price, P, is constant over the period $\frac{2\pi}{\theta}$, sinusoidal output, Q_t, may now be defined

$$Q_t = Q_o(1+\gamma'\sin\theta t) \tag{5.33}$$

and equated to the production function

$$mL_t^{\alpha}K_o^{\beta} = Q_o(1+\gamma'\sin\theta t).$$

Solving for L_t

$$L_t = \left(\frac{Q_o}{mK_o}\right)^{1/\alpha} (1+\gamma'\sin\theta t)^{1/\alpha}. \tag{5.34}$$

Now let L* be the sum of inputs for a planning period.

$$L^* = \int_0^{\frac{2\pi n}{\theta}} L_t = \left(\frac{Q_o}{mK_o}\right)^{1/\alpha} \int_0^{\frac{2\pi n}{\theta}} (1+\gamma'\sin\theta t)^{1/\alpha}dt \tag{5.35}$$

$$L^* = \left[\frac{mL_oK_o^{\beta}}{mK_o^{\beta}}\right]^{1/\alpha} \int_0^{\frac{2\pi n}{\theta}} (1+\gamma'\sin\theta t)^{1/\alpha}dt. \tag{5.36}$$

Here L^* represents the sum of all variable inputs where $\frac{2\pi n}{\theta}$ is the planning period of n wave lengths.

$$L_o = \frac{L^*}{\int_o^{\frac{2\pi n}{\theta}} (1+\gamma' \sin\theta t)^{1/\alpha} dt} \tag{5.37}$$

where L_o represents the mean of variable inputs for the planning period. The familiar zero excess profit function

$$PQ - wL - sWK = 0$$

may be restated to accommodate the demand function (equation 2.29)

$$Hm^{1-\sigma} L_o^{\alpha(1-\sigma)} K_o^{\beta(1-\sigma)} - wL - sWK = 0 .$$

Substituting for L_o from (5.37),

$$\frac{Hm^{1-\sigma} L^{*\alpha(1-\sigma)} K^{\beta(1-\sigma)}}{\left[\int_o^{\frac{2\pi n}{\theta}} (1+\gamma' \sin\theta t)^{1/\alpha} dt\right]^{\alpha(1-\sigma)}} - wL - sWK = 0. \tag{5.38}$$

Differentiating with respect to L^*,

$$\frac{\alpha(1-\sigma) Hm^{1-\sigma} \; L^{*\alpha(1-\sigma)-1} \; K^{\beta(1-\sigma)}}{\left[\int_o^{\frac{2\pi n}{\theta}} (1+\gamma' \sin\theta t)^{1/\alpha} dt\right]^{\alpha(1-\sigma)}} = w . \tag{5.39}$$

Solving for L*

$$L^* = \left[\frac{\alpha(1-\sigma)Hm^{1-\sigma}}{\left[\int_0^{\frac{2\pi n}{\theta}} (1+\gamma' \sin\theta t)^{1/\alpha} dt\right]^{\alpha(1-\sigma)} w}\right]^{\frac{1}{1-\alpha(1-\sigma)}} \underline{K^{\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}}} \; . \tag{5.40}$$

After substituting for L* from equation (5.37), equation (5.37) may be used to define the short-run path of inputs for the mean output of the regulated firm that produces output sinusoidally.

Substituting for L* from equation (5.40) into equation (5.39) permits solving for capital, K:

$$Hm^{1-\sigma}K^{\beta(1-\sigma)}\left[\frac{\alpha(1-\sigma)Hm^{1-\sigma}K^{\beta(1-\sigma)}}{\left[\int_0^{\frac{2\pi n}{\theta}} (1+\gamma' \sin\theta t)^{1/\alpha} dt\right]^{\alpha(1-\sigma)} w}\right]^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}$$

$$- w\left[\frac{\alpha(1-\sigma)Hm^{1-\sigma}K^{\beta(1-\sigma)}}{\left[\int_0^{\frac{2\pi n}{\theta}} (1+\gamma' \sin\theta t)^{1/\alpha} dt\right]^{\alpha(1-\sigma)}}\right]^{\frac{1}{1-\alpha(1-\sigma)}} - sWK = 0 \tag{5.41}$$

$$K = \frac{[1-\alpha(1-\sigma)]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}[\alpha(1-\sigma)]^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{w^{\frac{\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \cdot$$

$$\cdot \frac{H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}} m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \left\{ \frac{\left[\int_0^{\frac{2\pi n}{\theta}} (1+\gamma' \sin\theta t)^{1/\alpha} dt\right]^{\alpha(1-\sigma)} - 1}{\int_0^{\frac{2\pi n}{\theta}} (1+\gamma' \sin\theta t)^{1/\alpha} dt^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}} \right\}^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}$$

(5.42)

Since $\int_0^{\frac{2\pi n}{\theta}} (1+\sin\theta)^{1/\alpha} dt > 1$

and since $\alpha(1-\sigma)$ and $\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}$ are positive,

then the denominator of the bracketed quantity in equation (5.39) must be less than one. This means that less capital is used for sinusoidal output than for smooth continuous output of the firm with specified demand and production functions.

Similarly in equation (5.41) L* is less for sinusoidal output than for smooth continuous output because again

$$\int_0^{\frac{2\pi n}{\theta}} (1+\gamma' \sin\theta t)^{1/\alpha} dt > 1.$$

Thus, less labor as well as less capital is used for sinusoidal output.

The relative inefficiency of sinusoidal output makes the inputs more expensive in terms of output. Accordingly, less of each is used.[16] Since both L and K are decreased relative to requirements for smooth output, output is less and price is higher. As in the solution for the static model, output is obtained by substitution for L_t and K_o in

$$Q_t = mL_t^{\alpha}K_o^{\beta} .$$

If the mean rate of output is desired, mean input, L_o, must be substituted to obtain Q_o. However, L^* may be substituted to obtain cumulative output, Q^*:

$$L^* = \frac{[\alpha(1-\sigma)Hm^{1-\sigma}]^{\frac{1}{1-\alpha(1-\sigma)}}}{\left[\int_o^{\frac{2\pi n}{\theta}} (1+\gamma'\sin\theta t)^{1/\alpha}dt\right]^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}} \; \frac{K^{\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}}}{W^{\frac{1}{1-\alpha(1-\sigma)}}} \tag{5.43}$$

---

[16] Smith has a solution for inputs for sinusoidal output in the neoclassical minimum-cost tradition. Smith maintained a constant level of output in his problem. Accordingly, his solution required more inputs for a given output. Again, he did not impose a demand function as a constraint. See Vernon L. Smith, Investment and Production (Cambridge, Mass.: Harvard University Press, 1960), 286-290.

$$L^* = \frac{[1-\alpha(1-\sigma)]^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}[\alpha(1-\sigma)]^{\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}}{(sW)^{\frac{\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}} \cdot$$

$$\cdot \frac{m^{\frac{1-\sigma}{1-(\alpha+\beta)(1-\sigma)}} H^{\frac{1}{1-(\alpha+\beta)(1-\sigma)}}}{w^{\frac{1-\beta(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}}\left[\int_0^{\frac{2\pi n}{\theta}}(1+\gamma' \sin\theta t)^{1/\alpha}dt\right]^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}} \cdot$$

$$\cdot \left[\frac{\left[\int_0^{\frac{2\pi n}{\theta}}(1+\gamma' \sin\theta t)^{1/\alpha}dt\right]^{\alpha(1-\sigma)} - 1}{\left[\int_0^{\frac{2\pi n}{\theta}}(1+\theta' \sin\theta t)^{1/\alpha}dt\right]^{\frac{\alpha(1-\sigma)}{1-\alpha(1-\sigma)}}}\right]^{\frac{1-\alpha(1-\sigma)}{1-(\alpha+\beta)(1-\sigma)}} . \tag{5.44}$$

But

$$L^* = L_o \int_0^{\frac{2\pi n}{\theta}}(1+\gamma' \sin\theta t)^{1/\alpha}dt.^{1} \tag{5.45}$$

---

[1]If $\alpha=1/2$, and planning period $T=\frac{2\pi n}{\theta}$ where n represents an integral number of wave lengths, then

$$L^* = L_o \frac{\pi n}{\theta} (2+\gamma'^2)$$

or

$$L_o = \frac{L^*}{\frac{\pi n}{\theta}(2+\theta'^2)} .$$

The expression $\frac{\pi n}{\theta}(2+\gamma'^2)$ evolves from integration of

Now L_o may be substituted in

$$Q_o = mL_o^{\alpha}K^{\beta} \tag{5.46}$$

$$\int_o^{\frac{2\pi n}{\theta}} (1+\gamma'\sin\theta t)^2 dt = \int_o^{\frac{2\pi n}{\theta}} (1+2\gamma'\sin\theta t+\gamma'^2\sin^2\theta t)dt$$

$$=\int_o^{\frac{2\pi n}{\theta}} \left[1+2\gamma'\sin\theta t+\frac{\gamma'^2}{2}(1-\cos 2\theta t)\right] dt$$

$$= \left. t+\frac{2\gamma'}{\theta}(-\cos\theta t)+\frac{\gamma'^2}{2} \cdot t-\frac{\gamma'^2}{4\theta}(\sin 4\theta t)+\frac{2\gamma'}{\theta}\right|_o^{\frac{2\pi n}{\theta}}$$

$$= \frac{2\pi n}{\theta}+\frac{2\gamma'}{\theta}(-1)+\frac{\gamma'^2}{2} \cdot \frac{2\pi n}{\theta}-\frac{\gamma'^2}{4\theta}(0)+\frac{2\gamma'}{\theta}$$

$$= \frac{2\pi n}{\theta}\left[1+\frac{\gamma'^2}{2}\right] = \frac{\pi n}{\theta}\left[2+\gamma'^2\right].$$

When $1/\alpha$ is equal to some value other than 2, say, 3/2, (which means the elasticity of output to labor is 2/3), the problem becomes an exercise in Maclaurin's series where f(t) is stated in terms of a power series in (t):

$$f(t) = (1+\gamma'\sin\theta t)^{3/2}$$

$$f'(t) = 3/2(1+\gamma'\sin\theta t)^{1/2}\gamma'\theta\cos\theta t$$

$$f''(t) = 3/4(1+\gamma'\sin\theta t)^{-1/2}(\gamma'\theta\cos\theta t)^2+3/2(1+\gamma'\sin\theta t)^{1/2}$$
$$(-\gamma'\theta^2\sin\theta t)$$

$$f'''(t) = -3/8(1+\gamma'\sin\theta t)^{-3/2}(\gamma'\theta\cos\theta t)\ \ldots\ .$$

where $f(t) = f(o)+tf'(o)+\frac{t^2f''(o)}{2!}+\frac{t^3f'''(o)}{3!}+\ \ldots\ .\ +\frac{t^nf^n(o)}{n!}$

$f(o) = 1;\ f'(o) = \frac{3\gamma'\theta}{2};\ f''(o) = \frac{3\gamma'^2\theta^2}{4};\ f'''(o) =$

$= -3/8\gamma'^3\theta^3;\ \ldots\ .;\ f^n(o) = -\frac{3\gamma'^n\theta^n}{2^n}.$

to obtain mean equilibrium output, Q_o, to permit substitution in the demand function

$$P = HQ_o^{-\sigma} \qquad (5.47)$$

to obtain equilibrium price, P, for sinusoidal output.

The solution to the problem of sinusoidal output is illustrated in Fig. 5.6. Here mean equilibrium output Q_o is shown as $Q_{A'}$.

Consider the firm that produces smooth continuous output, Q_A. Now assume that the firm must produce sinusoidal output, the mean value of which is still Q_A. The firm now finds that its average level of variable inputs is now L_{ave}, not L_o. This is due to the non-linear transformation of input L into output Q_A (diminishing returns). The firm now finds that required average inputs L_{ave} no longer afford regulated pure profit (s-r)WK. The inefficiency of sinusoidal output causes the product to cost more. If the firm must supply sinusoidal output, it

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$$\int_o^{\frac{2\pi n}{\theta}} \left[1 + \frac{3\gamma'\theta t}{2} + \frac{3\gamma'^2\theta^2 t^2}{4 \cdot 2!} \cdot \cdot \cdot \cdot \cdot \cdot \right] dt$$

$$= t + \frac{3\gamma'\theta t^2}{4} + \frac{\beta}{4}\,\frac{\gamma'^2\theta^2 t^3}{2!} - \frac{3}{8}\,\frac{\gamma'^3\theta^3 t^4}{3!} \cdot \cdot \cdot \cdot$$

$$= \frac{2\pi n}{\theta} + \frac{3}{4}\gamma' \cdot \frac{(2\pi n)^2}{\theta} + \frac{3}{4}\gamma'^2 \frac{(2\pi n)^3}{\theta 2!} - \frac{3}{8}\gamma'^3 \frac{(2\pi n)^4}{\theta 3!} \cdot \cdot \cdot \cdot$$

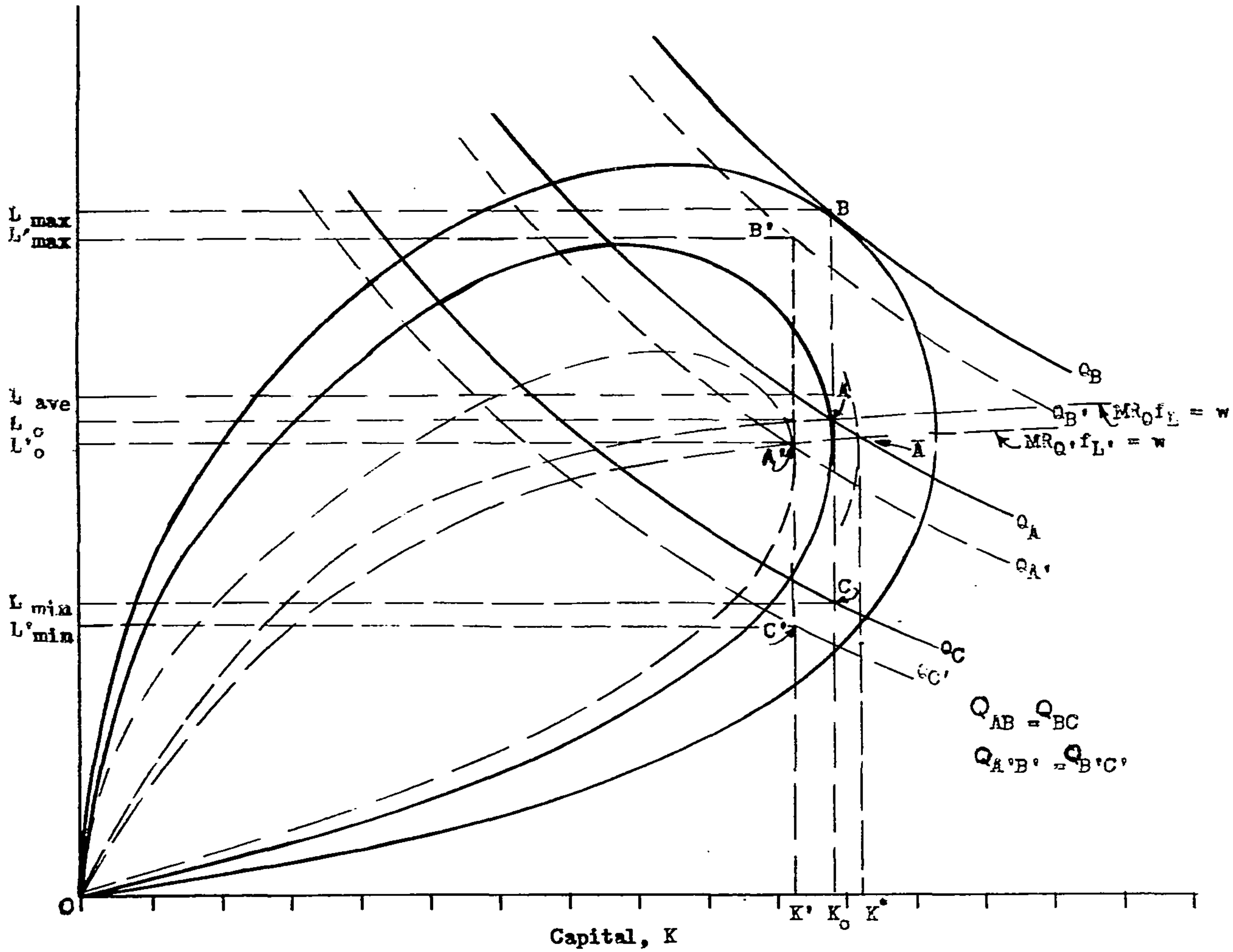

Fig. 5.6—The decrease in inputs for the regulated firm as a result of sinusoidal output

adjusts to this situation by reducing mean output to a new and lower level, $Q_{A'}$, and charging a higher price. Both inputs are decreased from the levels required for smooth, continuous output; but variable inputs are reduced relatively more than fixed inputs. The required cyclic output makes the variable inputs relatively more expensive than for constant output. This may be seen more clearly if it is assumed that the firm must produce a mean sinusoidal output equal to constant output. To earn its pure profit (s-r)WK, the firm would have to substitute fixed inputs for variable until the allocation of inputs conformed to the new input path:

$$MR_{Q_{A'}} f_L = w. \tag{5.48}$$

The simultaneous solution of (5.48) and

$$Q_A = mL^{\alpha}K^{\beta}$$

which represents smooth continuous output, yields the solution for optimal allocation of inputs for the regulated firm that must produce a mean sinusoidal output equal to smooth continuous output Q_A. This solution is shown graphically at $\bar{A}$ in Fig. 5.6.

A general relation for mean variable inputs and capital evolves from equations (5.43) and (5.45),

$$L_o = \frac{\left[\dfrac{\alpha(1-\sigma)Hm^{1-\sigma}}{w\left[\int_o^{\frac{2\pi n}{\theta}}(1+\gamma'\sin\theta t)^{1/\alpha}dt\right]^{\alpha(1-\sigma)}}\right]^{\frac{1}{1-\alpha(1-\sigma)}} K^{\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}}}{\int_o^{\frac{2\pi n}{\theta}}(1+\gamma'\sin\theta t)^{1/\alpha}dt} \tag{5.49}$$

$$L_o = \left[\frac{\alpha(1-\sigma)Hm^{1-\sigma}}{\left[\alpha\int_o^{\frac{2\pi n}{\theta}}(1+\gamma'\sin\theta t)^{1/\alpha}dt\right]}\right]^{\frac{1}{1-\alpha(1-\sigma)}} K^{\frac{\beta(1-\sigma)}{1-\alpha(1-\sigma)}} .$$

Note that as amplitude, γ, approaches zero, inputs L and K approach values that are optimal for constant output as determined in Chapter II.

As for the static model of Chapter II, and for the static model adjusted for dynamic conditions in the first part of this chapter, output, Q_o, may be determined once L_o and K are known by solving the production function. And since

$$Q_t = Q_o\ (1+\gamma'\sin\theta t)$$

and

$$H_t = H_o\ (1+\gamma\ \sin\theta t)$$

where

$$(1+\gamma') = (1-\gamma)^2$$

the time path of output may be determined. A solution may also be obtained for maximum and minimum output levels as well as for maximum and minimum variable inputs. Graphic solutions are shown in Fig. 5.6.

Price, P, may be obtained for either equilibrium mean sinusoidal output or for a mean sinusoidal output equal to equilibrium smooth continuous output by substituting for Q in

$$P = HQ_{A'}^{-\sigma}$$

and

$$P = HQ_A^{-\sigma} ,$$

respectively.

The model for sinusoidal output is thus complete. Using the same variables, except for added variables amplitude, γ, and wave length, $\frac{2\pi}{\theta}$, the sinusoidal model abstracts from the static model of Chapter II.

The sinusoidal adaptation of the static model of the regulated firm shows that sinusoidal output serves to increase the capital-labor and capital-output ratios of the regulated firm. Given values for the internal variables, the model developed here precisely determines these ratios.

CHAPTER VI

POTENTIALITIES FOR FUTURE ELABORATION OF THE MODEL

This theoretical investigation has demonstrated that the regulated firm, as represented by a simple idealized model rationally makes its production process as capital intensive as the firm's constraints (the production function and the demand function) permit. Given specific production and demand functions, equilibrium solutions of the model measure precisely the amounts of capital and labor to be used as inputs by the regulated firm. The model also determines, equally precisely, the firm's output and the price to be charged for it.

The model has been adapted to determine the optimal service life of the regulated firm's plant and equipment. Again, the optimal solution reflects the regulated firm's propensity for use of capital.

The behavior of the regulated firm in relation to its capital decisions has been investigated over a considerable range of conditions including growth, price-quantity elasticity, technology, durability of capital stock, and production characteristics. The capital decisions of the regulated firm have also been investigated in a setting of sinusoidal output. In all these conditions, the regulated firm was found to be more capital intensive in its production process than the unregulated firm.

Despite the wide range of conditions for which rational behavior of the regulated firm has been investigated, the model developed in this study to illustrate and explain this behavior is admittedly a simple one. The question arises, is the model too simple? Whether or not the regulated firm actually acquits itself in accord with the simple model developed in this investigation would appear not to rest upon the simplicity or complexity of the model as developed, but rather upon a number of other factors: difficulties in dealing with uncertainties, upon human behaviorism (which relates, more to the attitudes of the individual in, or near, management) in todays business environment, upon the complexity attending the large number of separate decisions involving heterogeneous capital additions to be made by the firm, and still others. This observation is not made to refute the argument that the model should be made more "realistically complex."[1] If the simple model of this investigation is tenable in its essentials, then indeed the model should be made more and more realistically complex.

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[1]M.L. Greenhut, "On the Question of Realism and the Regulation of Public Utilities," Land Economics, Vol. XLIII, No. 3, pp. 260-267.

What can, and should, be done to make the model of this investigation more realistic? During the course of this investigation, several "areas" that are related to capital decisions of the regulated firm appeared to the writer to merit new or further inquiry. Treatment of these areas would serve to make the model more realistic. Some of these areas suggested below may more properly be called sub-areas in that they may well relate to peripheral treatment of issues that have already been investigated, to some degree, rather than new and separate problems.

Probably the first move to greater realism should be a general treatment, first, of heterogeneous inputs and, then, heterogeneous outputs. Initially these inputs, and outputs, could well apply to only one production function. It would appear that the work of Samuelson[2] and Henderson and Quandt[3] relating to heterogeneous inputs may be adapted for application to the static model for the regulated firm.

Next to be considered is the problem of the multi-production functions, first with homogeneous and then with heterogeneous inputs. The problem of heterogeneous capital inputs for the regulated firm opens up a broad spectrum of problems relating to the selection of particular capital

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[2]Paul A. Samuelson, Foundations of Economic Analysis (Cambridge, Mass.: Harvard University Press, 1947), pp. 57-89.

[3]James M. Henderson and Richard E. Quandt, Microeconomic Theory: A Mathematical Approach (New York: McGraw-Hill Book Co., Inc., 1958), pp. 47-53.

inputs. Selection of particular capital inputs and the timing thereof suggests the kind of capital budgeting problems that concerned Lorie and Savage,[4] Weingartner,[5] et al. Weingartner's solutions to problems relating to ranking of projects, mutual exclusiveness of projects, contingency of projects, interperiod analysis, and parametric programming were linear solutions. The nonlinear character of the production and demand functions that serve as constraints suggests that new techniques or adaptions of existing techniques are needed in order to place less reliance on linear methods. Here project selection should be a fertile area for applications of much of the theory relating to nonlinear programming that has been developed in the past decade.

Probably the next move to greater realism would be to encompass the problem of uncertainty. This entire investigation has been an exercise in certainty. The cost of capital has been fixed in this study, but the suggestion has been made in Chapter III that the cost of capital be made a variable determined within the model itself. Indeed, the "Lambda" function of Chapter III appears ideally suited as a mathematical device to explore the effects of changes

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[4] J.H. Lorie and L.J. Savage, "Three Problems in Rationing Capital," Journal of Business, Vol. XXVIII, No. 4, pp. 229-239.

[5] H. Martin Weingartner, Mathematical Programming and the Analysis of Capital Budgeting Problems (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1963).

in the cost of capital, r, or of changes in pure profit rate (s-r). Changes in both cost of capital and pure profit rate seem likely to be, in part, the products of uncertainty. For example, as the pure profit rate rises in period I, the cost of capital in period II may be expected to decline, and vice versa. The rate of pure profit may be expected to determine the growth rate of the firm's assets. The growth rate of the firm, in turn, may be expected to determine largely the equity component of the cost of capital. This suggests that for a particular pure profit rate, there is an optimal debt-equity ratio. Obviously the pure profit rate is not determined by chance, but the variance in pure profit and particularly the variance in the growth of pure profit has an important, perhaps decisive, effect on the cost of capital to the regulated firm. If, as seems plausible, the growth rate is some function of the pure-profit rate, then prospects for making cost of capital for period II a function of pure profit, debt-equity ratio, and growth rate in period I appear fruitful.

The problem of fluctuations in output deserves much further study. Research may well show that decreasing amplitude (in the sinusoidal pattern) of output of the electric-utility industry has had a large role in decreasing the <u>real</u> price of output during the last two decades. It appears certain to this writer that technology and returns to scale have received credit for cost reduction that really belongs, in part, to decreasing amplitude of the fluctuations in electric-utility output. If empirical studies confirm this hypothesis, then certainly the subject of peak-load pricing for output of the regulated firm deserves further study. The prospects of theoretically linking capital requirements and average price of output to peak-load pricing would appear to be rewarding. The cost of increasing amplitude of output of the natural-gas industry has been offset, in part, by the ability of this industry to store output. This circumstance, largely unique among regulated firms, suggests a model to determine optimal investment in--and output of--storage facilities of the regulated natural-gas firm. Such a model could be integrated with a model for optimal total investment, output, and price for the regulated natural-gas firm.

The problem of replacement deserves to be integrated with the problem of fluctuating output. Aging plant may be relegated for standby or "peak" service rather than actually retired. If so, then there is probably some optimal age at which plant and equipment of the regulated firm is retired from continuous service and relegated to standby service. This suggests that, given a depreciation schedule, a time-salvage realization schedule, an output schedule, and a demand function, there is some optimal average age of plant for the regulated firm.

To the writer, it appears that the subject of theory of regulation should properly start with a model, whether unrealistically simple or realistically complex, of the regulated firm. The writer believes the "Lambda" function of Chapter III may well be a start in at least one treatment of the pure theory of regulation. The Lambda function of Chapter III abstracts from certainty. Probably, this function or some other function that purports to be a central theme in the pure theory of regulation must encompass the greater part of the problems engendered in the real world by uncertainty.

Finally, the area of welfare economics resulting from rational capital decisions under regulation may be investigated. The entire rate-of-return system of compensating the regulated firm would come under scrutiny. Successional treatment of the subject of welfare economics in relation to conventional regulation probably should start from a tenable model for the regulated firm. Indeed, a case--if one exists--against conventional methods of regulation of the public utility might well start with a "complete" model for the regulated firm.

The model of the regulated firm developed in this study--and the various adaptations described herein--is another step in the direction of understanding of both the regulated firm and the theoretical problems in regulation itself. While the study contains several original elements, it is of course firmly established upon the prior works of micro-theorists. It is hopefully suggested that it will provide a foundation for further work in the theory of the regulated firm, at least some of which may be along the lines suggested above.

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APPENDIX

LIST OF SYMBOLS USED

The symbols used in this study are listed alphabetically below. For Greek symbols the order of listing conforms to the English equivalent to the symbol used.

a Y-intercept of linear demand curve

α Elasticity of output with respect to labor

b Slope of linear demand curve

β Elasticity of output with respect to capital

c Interim investment of Masse's investment function, Chapter IV

C Cost, of the cost function

C' Quasi cost of the quasi cost function of Chapter III

${}^{B}_{r}d$ Present worth group-basis annuity depreciation

$D(T)$ Variable cost of Masses chain replacement model, Chapter IV

δ Rate of depreciation of salvage value

E Operating expense, in Smith's replacement model of Chapter IV

E Equity component of capitalization for function relating income taxes to investment, Chapter I

ζ Rate of technological change

f_L Marginal productivity of labor

f_K Marginal productivity of capital

G(T) Discounted profit in Masse's replacement function, Chapter IV

$\bar{\gamma}$ Rate of depreciation of book value

γ Amplitude of sinusoidal output

h Uniform equivalent of all operating and capital costs in Smith's replacement function, Chapter IV

h′ Uniform equivalent of all future expenses starting with the first replacement in Smith's chain replacement model, Chapter IV

H The coefficient of the log-linear demand function

i Rate of interest on debt, Chapter I

I Investment

J Rate of growth

k Number of replacements, in Smith's replacement model, Chapter IV

K Units of capital

L Units of labor

$\bar{L}$ Service life in Smith's replacement function, Chapter IV

λ The LaGrange multiplier--as a multiplier has added meaning in the section on Lambda function of Chapter III

MRTS Marginal rate of technical substitution

M(τ) Discounted cost of Masse's replacement chain Chapter IV

m Coefficient of the Cobb-Douglas production function

m′ Terborgh's adverse minimum, Chapter IV

$\bar{m}$ Ratio of salvage value to initial investment

μ Ratio of salvage value to initial investment when book and salvage values are equal

n Integral number of wave lengths, Chapter V

n Year book value becomes zero, Chapter IV

n' Year salvage value becomes zero, Chapter IV

ψ Rate of increase of operating expense

P Price

π Profit

Q Output

q Interim output in Masse's interim investment function

r Cost of capital, per cent

r' Cost of capital, per cent, to cover real-world charges of income taxes, depreciation, insurance and property tax

R Revenue

R' Quasi-revenue of Chapter III

s Rate of return

s' Rate of return to cover real-world charges of income taxes, depreciation insurance and property taxes

σ Price-quantity elasticity

T Time, generally signifying a particular date

t Time, generally used more for a period

t_x Income tax rate

τ Optimal replacement time, of Masse's replacement chain

$\bar{\theta}$ Time of interim investment in Masse's interim investment relation, Chapter IV

θ Angle subtended by the arc of the circumference equal in length to the radius, Chapter V

u Utilization rate, Smith's replacement model, Chapter IV

v^x Present worth factor for age x for retirement dispersion related to depreciation annuity, Chapter I

ϕ Ratio of salvage value to initial investment at time zero

w Wage rate

W Price of a unit of capital

x Age of plant in years, retirement dispersion, Chapter I

y^x Mean survivors in year x, retirement dispersion, Chapter I

z Late operating costs increase each year, replacement model for the regulated firm, Chapter IV