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GRADUATE COLLEGE

ON USE OF THE ANELASTIC VERTICAL VORTICITY EQUATION IN
DUAL-DOPPLER ANALYSES OF THE VERTICAL VELOCITY FIELD

A Dissertation

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

Doctor of Philosophy

By

JOHN JEREMY MEWES

Norman, Oklahoma

2001

UMI Number: 3009544



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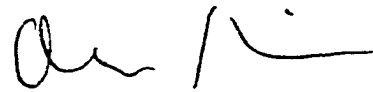
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DUAL-DOPPLER ANALYSES OF THE VERTICAL VELOCITY FIELD

A Dissertation APPROVED FOR THE
SCHOOL OF METEOROLOGY

BY



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ACKNOWLEDGMENTS

My greatest thanks go out to Alan Shapiro for presenting me with such a rewarding topic for my dissertation research, and for giving me the freedom to pursue much of my research while living away from Oklahoma. (Richelle thanks you as well!) Thanks also for keeping me focused on seeing this work through to completion, and for the many insightful discussions along the way. I would also like to extend my thanks to my committee members for their guidance, and to David Dowell for his insight on dual-Doppler analyses and use of the vorticity equation in them.

I would also like to thank my friends at OU for a very fulfilling 7 years of my life (did it really take that long?). Ed, Mark, Curtis, Clinton, J.J. & Karen – I will always consider you among my best friends regardless of where life takes each of us.

Finally, my love and gratitude go out to my family and to my wife, Richelle. Thanks to my family for their support, and for being eternally good role models. Special thanks to Richelle for sticking with me through a long-distance relationship for 3 ½ years, and then for putting up with the late nights since we've been married. You are an incredibly unselfish person, and I love you more each day. Oh yeah, and thanks for making me a daddy-to-be!

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Chapter 1: Introduction

Dual-Doppler analysis is a general term for the process of constructing orthogonal components of the wind field utilizing radial wind observations from two Doppler radars. Dual-Doppler analyses generally seek full reconstruction of the three-dimensional wind field, but must be carried out with only two observations. Without further information the problem is underdetermined, and no solution can be found. Therefore, the mass continuity equation is typically brought into these analyses as a flow constraint in order to bring closure to the system. Since radar scan geometries typically dictate that the vertical velocity is the most poorly observed component of the three-dimensional wind field, it is most frequently reconstructed through a vertical integration of the mass continuity equation.

However, application of the mass continuity equation is not without drawbacks. Analyses based upon this constraint suffer from two common problems. First, application of the mass continuity equation to construct the vertical velocity requires that the vertical velocity be known on at least one boundary of the data domain. For reasons that will be discussed in more detail shortly, this boundary condition is frequently not well known. Second, errors in the interpolated radial wind observations at analysis points (which may stem from the raw observations as well as the interpolation process itself) manifest themselves as errors in the horizontal divergence fields from which the vertical velocity is later constructed. Even seemingly small relative errors in the horizontal wind estimates are magnified in fields based upon derivatives, such as divergence. These divergence errors can accumulate

as the mass continuity equation is integrated vertically, leading to disastrous estimates of the vertical velocity field after integration over a significant depth.

In this work, the anelastic vertical vorticity equation is introduced as a further analysis constraint with the goal of alleviating the aforementioned problems associated with application of the mass continuity equation to construct the vertical velocity field. The anelastic vertical vorticity equation is appealing for use in dual-Doppler analyses since it contains information relating the horizontal and vertical wind components, and because the information it requires is readily available.

Before proceeding with development of techniques utilizing the vorticity equation, a brief history of the evolution of dual-Doppler analyses will be presented in Chapter 2. Then, in Chapter 3, the error sensitivities of the anelastic vertical vorticity equation will be explored by analyzing its exact solution. In Chapter 4, two new techniques for using the anelastic vertical vorticity equation to improve dual-Doppler analyses will be presented. The performance of these new techniques on simulated datasets is then gauged in Chapter 5, after which the results of this work are briefly summarized in Chapter 6.

Chapter 2: A Review and History of Dual-Doppler Analysis Techniques

Section A: Dual-Doppler analysis techniques and the vertical velocity problem

Many variations of dual-Doppler wind analysis techniques have been developed during the past three decades. As mentioned in Chapter 1, most utilize the physical constraint of mass continuity to bring closure to a system that otherwise contains two equations (the geometric relations between the observed radial winds and the three orthogonal wind components) in three unknowns (the wind components - u, v, w in a Cartesian coordinate system). Temporal and horizontal variations of density are typically neglected in the forms of the mass continuity equation used in dual-Doppler analyses. The three-dimensional divergence of the velocity field is therefore often assumed to be either (i) zero due to an assumption that density is constant in the atmosphere (the Boussinesq form – eq. 2.1a) or (ii) a small quantity owing its existence to the vertical advection of a vertically varying mean density field (the anelastic form – eq. 2.1b):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

(2.1a,b)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{w}{\bar{\rho}} \frac{d\bar{\rho}}{dz} = kw$$

In (2.1) $\bar{\rho} = \bar{\rho}(z)$ is the vertically varying density profile and u , v , and w are the Cartesian components of the wind field at point (x,y,z) .

Armijo (1969; henceforth referred to as AR69) was the first to derive the unique solution for a velocity field satisfying the observed radial winds and the anelastic mass continuity equation (eq. 2.1b) exactly. The AR69 technique proceeds as follows. Consider a Cartesian coordinate system (x,y,z) with the origin coinciding with one of the two Doppler radars and oriented such that the x axis passes through the second Doppler radar (the coordinates of which are then $(x_2, 0, 0)$). For this derivation a flat earth will be assumed, although in general this assumption is not necessary. Now, consider an air parcel moving with a velocity (u,v,w) and a hydrometeor falling under the force of gravity (in the $-\hat{k}$ direction) with velocity V_t relative to the air parcel (such that V_t is negative). The velocity of the hydrometeor relative to the fixed coordinate system is then given by:

$$(2.2) \quad \vec{V} = u\hat{i} + v\hat{j} + (w + V_t)\hat{k}.$$

Although we cannot observe w independently of V_t using Doppler radars, it can be estimated using one of several somewhat crude terminal velocity / radar reflectivity parameterizations. Next, assume the first Doppler radar (stationed at the origin of the coordinate system) is sampling a group of hydrometeors at a position (x,y,z) at time τ . The distance (R_1) of this sample volume from the first radar satisfies:

$$(2.3) \quad R_1^2 = x^2 + y^2 + z^2.$$

Differentiation of (2.3) results in:

$$(2.4) \quad R_1 \frac{D(R_1)}{Dt} = x \frac{Dx}{Dt} + y \frac{Dy}{Dt} + z \frac{Dz}{Dt}.$$

The derivatives of R_1 , x , y , and z in (2.4) respectively represent the radial velocity (V_{r1}) of the hydrometeors as observed by the first Doppler radar and the u , v , and $(w+V_t)$ components of the hydrometeor velocity in the Cartesian coordinate system. Thus (2.4) can be rearranged to show:

$$(2.5) \quad V_{r1} = \frac{1}{R_1} [xu + yv + z(w + V_t)].$$

Performing a similar analysis for the second Doppler radar (whose coordinates in the Cartesian grid are $(x_2, 0, 0)$) we find the similar relation:

$$(2.6) \quad V_{r2} = \frac{1}{R_2} [(x - x_2)u + yv + z(w + V_t)].$$

The expressions for the radial velocities as observed by each Doppler radar (eqs. (2.5) and (2.6)) comprise a system of two equations in three unknowns (assuming V_t is known through parameterizations). Solving for u and v in terms of the radial velocities and w we find:

$$(2.7) \quad \begin{aligned} u &= \frac{1}{x_2} (R_1 V_{r1} - R_2 V_{r2}) = \Gamma(x, y, z) \\ v &= \frac{1}{yx_2} [(x_2 - x)R_1 V_{r1} + xR_2 V_{r2}] - \frac{z}{y} (w + V_t) = \Psi(x, y, z) - \frac{z}{y} w \end{aligned}$$

where the expressions Γ and Ψ have been introduced to simplify further manipulations of these equations.

Examining (2.7) it is immediately evident that the system is underdetermined, and we cannot solve for the three-dimensional wind field without more information. In order to close the system AR69 chose to incorporate the anelastic mass continuity equation (eq. (2.1b)) as a constraint on the wind field. Substituting the expressions for u and v from (2.7) into (2.1b) we find:

$$(2.8) \quad \frac{\partial \Gamma}{\partial x} + \frac{\partial \Psi}{\partial y} - z \frac{\partial}{\partial y} \left(\frac{w}{y} \right) + \frac{\partial w}{\partial z} = kw,$$

or, simplifying:

$$(2.9) \quad -z \frac{\partial w}{\partial y} + y \frac{\partial w}{\partial z} + \left(\frac{z}{y} - ky \right) w = -y \left(\frac{\partial \Gamma}{\partial x} + \frac{\partial \Psi}{\partial y} \right) = y\beta .$$

This equation is an inhomogeneous 1st order linear partial differential equation (PDE) for w . One common method for solving hyperbolic PDEs is the *method of characteristics*. The method of characteristics allows us to employ a change of variables that transforms the problem of solving a PDE to one of solving ordinary differential equations (ODEs) along special curves called *characteristic curves*.

In order for the PDE to transform into an ODE the partial derivative terms on the left hand side of the equation must combine into a total derivative with respect to some as yet unknown variable (call it Θ). For this to happen we need (using the chain rule):

$$(2.10) \quad \frac{dw}{d\Theta} = -z \frac{\partial w}{\partial y} + y \frac{\partial w}{\partial z} = \frac{\partial w}{\partial y} \frac{\partial y}{\partial \Theta} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial \Theta} .$$

From (2.10) we can easily see that:

$$(2.11) \quad \frac{\partial y}{\partial \Theta} = -z \text{ and } \frac{\partial z}{\partial \Theta} = y ,$$

which can be written as:

$$(2.12) \quad \frac{dy}{-z} = \frac{dz}{y} = \frac{d\Theta}{1} .$$

The first two expressions in this equality can be rearranged and integrated to yield:

$$(2.13) \quad (y^2 + z^2) = \text{constant} = r^2,$$

where r is the radial distance from the baseline. This equation describes the *characteristic curves*, along which the original PDE reduces to an ODE without approximation. Using the two other possible combinations in eq. (2.12) expressions for y and z in terms of Θ can also be derived. The final expressions are the characteristic (cylindrical) coordinates of the problem and are given as:

$$(2.14) \quad \begin{aligned} x &= s \\ y &= r \cos \Theta, \\ z &= r \sin \Theta \end{aligned}$$

where s is the distance along the baseline from the radar positioned at the origin of the Cartesian coordinate system to the point on the baseline nearest (at a distance r) the observational volume, r is the radial (perpendicular) distance of the observational volume from the baseline, and Θ is the azimuthal angle of the observational volume measured from the plane formed by the x and positive y axes. This special cylindrical coordinate system is often called a *coplane* coordinate system.

Substituting the expressions (2.14) into (2.9) we obtain the ODE:

$$(2.15) \quad \frac{dw}{d\Theta} = (kr \cos \Theta - \tan \Theta) w + \beta r \cos \Theta .$$

Multiplying (2.15) by an integrating factor μ , where:

$$(2.16) \quad \mu = \frac{\exp(-k r \sin \Theta)}{\cos \Theta} ,$$

and integrating, we find the solution for w is given by:

$$(2.17) \quad w = \frac{1}{\mu} \int_{\Theta_{bottom}}^{\Theta} \beta r \exp[-k r \sin \phi] d\phi + \frac{const}{\mu} .$$

Equation (2.17) requires integration upward from the lowest Θ for which data are available, with w specified at this Θ . If it is assumed that data extend to ground level (where $\Theta=0$), the impermeability condition of $w = 0$ can be safely applied and the constant of integration in (2.17) is zero. In general, however, radar data will not extend down to ground level due to ground clutter, beam blockage, and/or the effects of Earth's curvature. Alternatively, if data are available to storm top, $w = 0$ can often also be assumed there and (2.17) can be integrated downward instead of upward. However, data frequently fail to span to this level either, prompting the question of what values of w to impose as a boundary condition. This is as yet an unsolved problem in radar meteorology, and is one of the primary motivations for this work.

In practice the gridded representation of the radial winds observed by each Doppler radar will contain errors (e.g., sampling errors, instrument errors, errors of representativeness, interpolation errors). For example, a distance-weighted interpolation scheme may change both the amplitude and phase of the harmonics in the true signal. Reflectivity gradients may cause one portion of an observational volume to be preferentially sampled by the radar, so that the radial wind observed may not be representative of the true radial wind velocity at the center of the observation volume. In addition, many small scale features may be lost within the observational volume, or may be aliased onto larger scales through the use of ‘picket-fence’ scanning strategies (Carbone et al., 1985). All of these errors invariably lead to errors in the synthesis of the three-dimensional wind field. Although the errors in the interpolated radial wind observation are often small relative to the magnitude of the radial winds, they may not be small relative to the true variations of the radial winds over the scales at which spatial derivatives are calculated. Since the vertical velocity in (2.17) is dependent upon spatial derivatives of the radial wind observations, it is apparent that these errors may severely degrade the accuracy of the synthesized vertical velocity field. It is convenient to show how these errors may affect vertical velocity analyses by introducing the following, more simplified dual-Doppler analysis.

If it is assumed that the $z(w+V_r)$ term in eqs. (2.5) and (2.6) is small compared to the u and v terms (usually a good assumption at lower levels, less so aloft) then a simpler system of two equations in two unknowns results:

$$\begin{aligned}
(2.18) \quad u &= \frac{1}{x_2} (R_1 V_{r1} - R_2 V_{r2}) = \Gamma(x, y, z) \\
v &= \frac{1}{yx_2} [(x_2 - x) R_1 V_{r1} + x R_2 V_{r2}] = \Psi(x, y, z)
\end{aligned}$$

In this case the horizontal wind field is fully determined with radial wind observations from two Doppler radars (at least within ‘lobes’ where the angle of intersection of the two radar beams is within specified bounds). The anelastic mass continuity equation can once again be introduced into the analysis in order to determine w . Integration with respect to z of the anelastic mass continuity equation (eq. (2.1b)) yields the expression:

$$(2.19) \quad \rho w = \rho_0 w_0 - \int_{z_0}^z \left(\frac{\partial}{\partial x} \rho u + \frac{\partial}{\partial y} \rho v \right) dz'.$$

The vertical velocity is therefore dependent upon spatial derivatives of the horizontal wind components and the mean vertical density profile and is uniquely determined by specifying the vertical velocity at the lower boundary of the data volume. From (2.19) we can see that the negative effects of observational errors on the estimates of the horizontal wind components get *amplified* in the derivatives and then *summed* through the vertical integration when determining w . If the observational errors are entirely random the effects of one error are often negated by an error that has an opposing effect elsewhere, and the severity of the error accumulation during integration of the continuity equation is lessened. However, in the presence of non-random (bias or trend) errors, the estimates of w can become severely inaccurate, especially away from the data surface

where the boundary condition for w has been specified. Non-random errors can be introduced through (amongst other sources) errors in boundary condition specification, incorrect terminal velocity parameterizations, and radar reflectivity gradients. In addition, if the true AR69 technique is used instead of this simplified analysis, errors in w feed back into the estimates of u and v (and vice-versa) so that the likelihood of having vertically correlated (bias) errors in the wind fields is dramatically increased.

Some of the first attempts to counter error accumulation during integration of the mass continuity equation were introduced by Lateef (1967) and O'Brien (1970), although neither of these techniques were developed specifically for application in dual-Doppler analyses. Both techniques control the aforementioned errors in w by allowing w to be specified at both the top and bottom of each data column. The Lateef technique imposes a 2nd order ODE (the anelastic mass continuity equation (2.1b) differentiated with respect to z) in order to obtain w . This is equivalent to imposing mass continuity as a weak constraint. Solution of the 2nd order equation introduces a linearly varying 'correction term' to w that is dependent upon the vertical distance from the lower data boundary, and acts to cause w at the upper boundary to agree with the value specified. (In effect, this is equivalent to applying a uniform adjustment to each horizontal divergence estimate within a column in order to cause the calculated vertical velocity at the top the column to match the boundary value specified). In the absence of an extra adjustment step to the horizontal winds (after complete retrieval of w) the synthesized wind field from the Lateef technique will generally not satisfy (2.1b) exactly.

Alternatively, the O'Brien technique implicitly applies mass continuity as a strong constraint and attempts to optimally correct the horizontal divergence estimates at each level based upon the expected error variance of the divergence estimates at that level. The desired effect is once again to acquire a calculated value of w at the top of the data column that agrees with the boundary value of w specified there. Although mass continuity is satisfied exactly by the adjusted divergence and vertical velocity fields, O'Brien gives no indication as to how the corrections to the divergence field should be re-applied to the underlying u and v fields in dual-Doppler analyses. (Recall that neither the O'Brien or Lateef techniques were developed with dual-Doppler analyses in mind.)

The two approaches are similar in effect. In fact, they are identical if the errors in the horizontal divergence estimates are assumed to be independent of height. Due to their similarities, only the Lateef technique will be discussed here (since it is conceptually similar). When introduced as a weak constraint the anelastic mass continuity equation takes the form:

$$(2.20) \quad \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = \varepsilon \neq 0$$

where ε is a small quantity referred to as the *equation error*. In order to quantify the total mean-squared equation error in a vertical column we introduce a *cost functional* (or alternatively an *error functional*) J , defined as:

$$(2.21) \quad J \equiv \int_{z_0}^{z_T} \left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right)^2 dz' = \int_{z_0}^{z_T} \varepsilon^2 dz',$$

where z_0 and z_T are the heights of the lower and upper boundaries of the data column, respectively. In the Lateef technique we seek to minimize J while specifying w at both the lower and upper boundaries of the column. If we chose to minimize J while specifying only one boundary condition we effectively revert back to (2.19), which would yield $J=0$ but give unrealistic vertical velocities within and at the top of the data column. To find the w profile that minimizes (2.21) while meeting the specified boundary conditions we can use the calculus of variations. To find a minimum of the cost function we take the first variation of J with respect to w and set it equal to zero. The first variation (δ) of J with respect to w is given by:

$$(2.22) \quad \delta J = 2 \int_{z_0}^{z_T} \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right) \frac{\partial}{\partial z} (\rho \delta w) dz' = 0.$$

Integration by parts of (2.22) yields:

$$(2.23) \quad \left[\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right) \rho \delta w \right]_{z_0}^{z_T} - \int_{z_0}^{z_T} \left[\frac{\partial}{\partial z} \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right) \right] \rho \delta w dz' = 0.$$

The variation of w (δw) in (2.23) is arbitrary (so that it will not necessarily be zero) indicating that the following conditions must be true for w to minimize J :

$$(2.24a) \quad \frac{\partial}{\partial z} \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right) = \frac{\partial}{\partial z} (\nabla \cdot (\rho \vec{V})) = 0,$$

$$(2.24b) \quad \delta w(z_0) = 0 \quad \text{or:} \quad \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right)_{z=z_0} = \nabla \cdot (\rho \vec{V})_{z=z_0} = 0,$$

$$(2.24c) \quad \delta w(z_T) = 0 \quad \text{or:} \quad \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right)_{z=z_T} = \nabla \cdot (\rho \vec{V})_{z=z_T} = 0.$$

The first equation (2.24a) is the *Euler-Lagrange* equation for the problem, while the latter two sets of equations comprise the necessary boundary conditions. The second set of boundary condition equations in (2.24b) and (2.24c) (i.e., the equations following the “or:”) are often called the *natural boundary conditions*, and are Neumann boundary conditions in this case. Notice that the Euler-Lagrange equation (2.24a) can be integrated to $\nabla \cdot (\rho \vec{V}) = \text{constant}$. Therefore, if the natural boundary conditions in (2.24b) and (2.24c) are applied at any boundary then we find that $\nabla \cdot (\rho \vec{V}) = 0$ throughout the column and we will once again be applying mass continuity as a strong constraint. So, excluding the natural boundary conditions, (2.24) states that with w specified at the column boundaries the divergence of mass flux must be constant with height within the data column in order to minimize the squared error in the continuity equation (i.e., the equation error ε in (2.20) must be constant within each data column).

As it turns out, this minimization process has revealed a useful (and more general) fact that minimization of any squared error measure is accomplished by spreading the magnitude of the errors out as evenly as possible.

Integrating the first equation of (2.24) twice with respect to z yields an equation similar to (2.19):

$$(2.25) \quad \rho w = \rho_0 w_0 + \varepsilon(z - z_0) - \int_{z_0}^z \left(\frac{\partial}{\partial x} \rho u + \frac{\partial}{\partial y} \rho v \right) dz'$$

with the only difference being the addition of an $\varepsilon(z - z_0)$ “correction term”. This correction term is zero at the lower boundary of the data column and increases linearly with height. It acts to correct for errors in the column-integrated divergence so that the vertical velocity boundary condition at the top of the data column is satisfied. If generic values of $w(z_0) = w_0$ and $w(z_T) = w_T$ are inserted into (2.25) we find:

$$(2.26) \quad \varepsilon = \frac{\rho_T w_T - \rho_0 w_0}{z_T - z_0} + \frac{1}{z_T - z_0} \int_{z_0}^{z_T} \left(\frac{\partial}{\partial x} \rho u + \frac{\partial}{\partial y} \rho v \right) dz'.$$

Use of (2.25) with ε determined from (2.26) is essentially what was proposed by Lateef, although Lateef’s formulation proceeds in pressure coordinates. Its success is critically dependent upon selecting proper boundary conditions for w . However, as mentioned

earlier in discussion of the AR69 technique, determination of an appropriate procedure for specifying these boundary conditions is still an active area of research.

Ray et al. (1975) employed the AR69 technique in an observational analysis of an April 20, 1974 tornadic thunderstorm over central Oklahoma. Ray et al. also introduced the idea of applying a two-dimensional Shuman filter (Shuman, 1957) to the horizontal wind fields prior to obtaining the vertical velocity field. Random observational errors are primarily manifested in the smallest scales representable on the grid, so filtering the smallest scales out of the horizontal wind fields prior to integration of the mass continuity equation can dramatically reduce the error variance in the synthesized vertical velocity field. This is a technique that has since been commonly applied.

Brandes (1977; henceforth referred to as BR77) introduced a modification to the AR69 technique in a study of a June 6, 1974 severe thunderstorm over central Oklahoma. This study utilized the same constraints as the AR69 technique but solved for the wind components on a Cartesian grid instead of the coplane grid proposed by AR69. This technique involves iterative solution of the system of equations (2.1b) and (2.7). In other words, a first guess w field was inserted into (2.7) yielding solutions for u and v . These horizontal velocity fields were then used in (2.1b) to deduce a new estimate of the w field. This new estimate was then used to deduce new estimates of the horizontal wind components and subsequently a new estimate of the vertical velocity field. The iterative process continues until convergence to a final three-dimensional wind field is attained.

BR77 was also one of the first authors to present a technique for estimating the vertical velocity boundary condition at the lowest data level, which often does not coincide with the ground. Using surface winds obtained from a local mesonetwork, BR77 estimated the mean divergence field below the lowest radar data level (located at about 0.3 km AGL) to be about 90% of that at the lowest data level. This assumption and application of the impermeability condition at ground level were used to construct an estimate of the vertical velocity at the lowest radar data level.

Ray et al. (1980; henceforth referred to as RY80) outlined and compared a number of possible methods for deducing the three-dimensional wind field utilizing a variety of constraints and methodologies and the radial wind observations from two to four Doppler radars. The methods in which only two Doppler radars are used are outlined in the following paragraphs.

The first two dual-Doppler techniques tested in RY80 were AR69-type techniques using the iterative solution approach of BR77. The differences in these two techniques were in boundary condition specification. In one technique, a value of w is specified on the lower boundary and (2.1b) is integrated upward for each iteration. In the other technique, a value of w on the upper boundary is specified and (2.1b) is integrated downward during each iteration. RY80 found the latter approach to yield better results. Downward integration of the mass continuity equation causes noise from accumulating divergence errors to be placed primarily at low levels, where the higher density causes vertical mass flux errors to be realized as smaller w errors.

Further, RY80 proposed an adjustment to the velocity fields produced from the iterative analyses discussed in the previous paragraphs. This adjustment, similar in effect to the Lateef and O'Brien adjustments discussed earlier, employs the vertically integrated Boussinesq form of the continuity equation as a strong integral constraint:

$$(2.27) \quad C = \int_{z_0}^{\bar{z}} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz = - \int_{z_0}^{\bar{z}} \left(\frac{\partial w}{\partial z} \right) dz ,$$

requiring that the vertically integrated horizontal divergence be equal to a constant C . C is chosen to cause the vertical velocity to meet some specific criteria at boundaries of the data column. The Cartesian horizontal wind components from the aforementioned iterative analysis are applied as weak constraints. Although RY80 note that the anelastic continuity equation is more appropriate when applied to deep volumes of the atmosphere, they justify use of the Boussinesq approximation by noting that the resulting w profiles are similar regardless of which form of the continuity equation is used since both upper and lower boundary conditions are being imposed. The cost functional resulting from these constraints is given by:

$$(2.28) \quad J = \int_V \left[\alpha^2 (u - u^0)^2 + \beta^2 (v - v^0)^2 + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] dV ,$$

where α and β are predetermined weights, λ is the Lagrange multiplier, and u^0 and v^0 are the horizontal Cartesian wind velocities determined a priori by the iterative AR69-type technique. The Euler-Lagrange equations resulting from this cost functional are:

$$\begin{aligned}
 (2.29) \quad & 2\alpha^2(u - u^0) - \frac{\partial \lambda}{\partial x} = 0 \Rightarrow u = u^0 + \frac{1}{2\alpha^2} \frac{\partial \lambda}{\partial x} \\
 & 2\beta^2(v - v^0) - \frac{\partial \lambda}{\partial y} = 0 \Rightarrow v = v^0 + \frac{1}{2\beta^2} \frac{\partial \lambda}{\partial y} \\
 & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
 \end{aligned}$$

Combining these three equations and integrating vertically from $z=z_0$ to $z=z_T$ we find (utilizing (2.27)):

$$(2.30) \quad \int_{z_0}^{z_T} \left[\frac{\partial}{\partial x} \left(\frac{1}{2\alpha^2} \frac{\partial \lambda}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{2\beta^2} \frac{\partial \lambda}{\partial y} \right) \right] dz = - \left[\int_{z_0}^{z_T} \left(\frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y} \right) dz - C \right].$$

Equation (2.30) is an elliptic partial differential equation for $\lambda(x,y)$. It can be solved by a successive overrelaxation (SOR) technique. Once λ has been determined the horizontal Cartesian wind components are corrected following (2.29) and the vertical velocity w is diagnosed by vertically integrating the continuity equation. The vertical velocities at both $z=z_0$ and $z=z_T$ should be match the specified boundary values after the correction procedure has been applied to u and v . When compared to a ‘best’ analysis produced using three radars and the anelastic continuity equation as a constraint, RY80 found that

the variationally constrained techniques yielded more accurate vertical velocity fields than the unconstrained techniques. However, as will be pointed out shortly, the degree to which this is true is dependent upon the amount of systematic error present.

The variational adjustment procedures of Lateef (1967), O'Brien (1970), and RY80 are all members of one class of approaches taken in order to improve the derived vertical velocity field through limitation of divergence related noise. Spatial filtering of the horizontal (or quasi-horizontal) winds prior to integration of the mass continuity equation is another. However, improvements to the synthesized vertical velocity fields can also be made through more accurate specification of the boundary conditions for w . Recapping the dilemma, due primarily to ground clutter, beam blockage, and the curvature of the earth's surface, radar data often fail to extend downward to ground level where the impermeability condition of $w=0$ can safely be applied (Ray, 1976). Applying the $w=0$ boundary condition at the lowest data level (which can be hundreds if not thousands of feet above ground level) can be very dangerous due to the fact that strong convergence or divergence often occurs near the ground, causing the vertical velocity at this lowest data level to be far from zero (Ray, 1976). Similarly, radar data frequently fail to extend up to storm top where the $w=0$ condition might also be safely applied. Even when data are available to storm top, the vertical velocity there may be far from zero in some cases (especially during storm development). Consequently, ad-hoc approaches to specifying boundary conditions for w have been taken within the multitude of dual-Doppler based observational studies.

The most common ad-hoc approach (by far) is the BR77 technique that has already been discussed, where the horizontal convergence below the lowest radar data level was assumed to equal a fraction of the horizontal convergence at the lowest data level. The vertical velocity at the lowest data level was then derived using mass continuity and the impermeability condition. However, this technique is also adversely affected by the fact that convergence or divergence may be strongest in the lowest few hundred meters of the atmosphere, which is below the lowest data level in many cases. Unfortunately, few other practical alternatives to the BR77 approach have surfaced in the literature. Most alternatives require that radial wind observations from at least three radars be available.

Chong and Testud (1983; henceforth referred to as CT83) presented a technique for boundary condition specification that actually seeks to address both the problem of accumulating divergence errors noted with upward integration techniques and the lack of reliable boundary condition data near the ground. CT83 used variational techniques to adjust the boundary condition field at ground level so as to maximize the “mathematical regularity” of the vertical velocity field. With this technique a subsidiary condition ensures that the vertical velocity preserves a statistical mean of zero at ground level while being allowed a prespecified variance. The technique was found to exhibit statistically similar errors to downward integration of the anelastic mass continuity equation from storm top, with the advantage that it preserves the physically-based mean of zero vertical velocity at the ground (provided data are available down to ground level).

The CT83 technique actually takes advantage of the same properties of density stratification that appear to make downward integration of the mass continuity equation preferable to upward integration. By maximizing the regularity of the w field, noise associated with the accumulation of divergence errors is preferentially distributed to lower levels where the vertical mass flux (ρw) errors are realized as smaller w errors. Similar results could be obtained by an upward integration technique which identifies upper-level noise, then makes high wavenumber adjustments in the original lower boundary condition to negate the noise in the upper levels.

Building upon previous discussion in this chapter, downward integration usually appears preferable to upward integration for two reasons. First, density stratification dictates that w errors at a level will be amplified (diminished) as one integrates upward (downward) from that level. This is true only for w errors, however, as the vertical mass flux (ρw) errors do not systematically change with height. Secondly, the geometrical feedback of w into the horizontal divergence estimates is typically greatest at upper levels with ground-based radars, so it is desirable that divergence-related noise be distributed toward lower levels to diminish the potential for undesirable feedback. However, in both cases it is important to note that it is not the direction of integration, but rather where the noise associated with this integration is distributed, that ultimately determines the accuracy of the final vertical velocity field.

If this noise can be distributed to lower levels regardless of the direction of integration (as with the CT83 technique), the ‘best’ direction for integration is

determined only by the location at which a ρw boundary condition (not including noise adjustments) can be specified most accurately. For example, an auxiliary condition of ρw maintaining a statistical mean of zero at storm top (rather than the ground) in the CT83 technique may yield superior results if that boundary condition were more accurate or applicable in a given case. This subtle distinction about the preference toward downward integration is important for this work since the techniques to be presented here will be shown to be much less sensitive to the direction of integration than traditional methods. Since traditional approaches to boundary condition specification have not attempted to counter divergence-related noise, it is not surprising that upward integration has long appeared vastly inferior to downward integration techniques.

A major drawback of the CT83 technique is that it still requires that w be prescribed (at least within a specified error variance) over a significant portion of the horizontal footprint of the data domain. If data are not available to very near the ground or to storm top the technique cannot yield a unique vertical velocity field.

Gal-Chen and Kropfli (1984) utilized the RY80 constrained technique to construct wind fields from which buoyancy and pressure perturbations within the planetary boundary layer (PBL) were derived. They applied the technique on a special, high-resolution dual-Doppler dataset collected during project PHOENIX (Kropfli and Hildebrand, 1980). The small spatial scale ($\sim 25 \text{ km}^2$) of project PHOENIX and the associated radar datasets caused curvature effects to be minimal over Gal-Chen and Kropfli's dual-Doppler analysis region. However, although data were available to very

near the earth's surface (~100 m) Gal-Chen and Kropfli nonetheless noted that sampling errors in radial wind observations were most severe within the lowest portions of the PBL due to the effects of ground clutter and the predominance of small scale turbulent eddies unresolvable by the radars at low-levels (due primarily to frictional effects). They showed that these sub-grid contributions are generally of secondary importance above 300 meters elevation and chose to modify the vertical velocity field obtained from straightforward integration of the Boussinesq continuity equation in order to take advantage of this fact. The modification required that the following conditions be met:

$$(2.31a) \quad w(0) = w(z_T) = 0,$$

$$(2.31b) \quad \frac{\partial w}{\partial z} = \frac{\partial w_0}{\partial z}; \quad z_M \leq z \leq z_T,$$

$$(2.31c) \quad \int_0^{z_M} \left(\frac{\partial w}{\partial z} - \frac{\partial w_0}{\partial z} \right)^2 dz = \text{minimum},$$

where z_M is the height below which the horizontal divergence is to be adjusted (300 meters in this case) and w_0 is the unadjusted vertical velocity profile obtained by an upward integration of the mass continuity equation using the unadjusted horizontal wind field. Thus, Gal-Chen and Kropfli adjusted the horizontal wind field below 300 meters in such a way that the vertical velocity at the top of the domain (the PBL in this case) was zero and the vertical gradients of vertical velocity above 300 meters were unchanged after the adjustment process. The solution of the minimization problem (2.31) is:

$$(2.32) \quad w(z) = \begin{cases} w_0(z) - w_0(z_T) \left[1 - \left(\frac{z}{z_M} - 1 \right)^2 \right] & 0 \leq z \leq z_M \\ w_0(z) - w_0(z_T) & z_M \leq z \leq z_T \end{cases}.$$

Scialom and Lemaitre (1990) presented another new approach to dual-Doppler analysis of the three-dimensional wind field. This approach, termed MANDOP (for Multiple ANalytical DOPpler), consists of expanding the three wind components in terms of products of orthonormal functions on each axis, allowing the analyzed wind field to be expressed analytically. Physical constraints (such as specification of the boundary condition at ground level and application of the continuity equation) are included in the analysis as variational constraints in order to improve the retrieval of the vertical velocity. The analysis results in a system of algebraic equations for the expansion coefficients. The analytical nature of the derived wind field also allows for analytic expression of the derived fields such as vorticity and divergence.

Sun and Crook (1996) retrieved the thermodynamic and velocity fields from an observed gust front and a simulated collapsing cold pool by fitting radial wind observations from two Doppler radars to a dry, Boussinesq model using an adjoint technique. The vertical velocity field was thus constrained to satisfy the continuity equation and the equations of motion. The adjoint technique was found to yield lower RMS errors in the vertical velocity field when compared to a traditional upward integration of the continuity equation, largely because the technique does not suffer from divergence error accumulation. However, the adjoint technique is much more

complex to implement than traditional dual-Doppler analysis techniques, and introduces pressure and temperature as extra control variables. If the adjoint technique was applied in a moist-convective case, moisture would also need to be added as a control variable. The increased number of control variables generally increases the possibility for non-uniqueness in solutions. Extra constraints such as smoothness or background constraints can diminish the uniqueness concerns, but may sacrifice resolution in the solution, or may introduce errors into the solution when the background field contains significant errors.

The most recently proposed analysis technique for reducing vertical velocity errors comes from Protat and Zawadzki (1999), which presents an alternative to the constraining techniques of Lateef (1967), O'Brien (1970), and RY80 for combating accumulating divergence errors. In this technique the vertical velocity field at a given level is a linear combination of interim vertical velocity fields obtained from an upward and a downward integration of the continuity equation, with greater weight being placed on the integration that is expected to have accumulated fewer errors at that height. However, this technique also requires that vertical velocity boundary conditions be specified at both the top and bottom of the analysis domain, similar to more conventional techniques for constraining the vertical velocity.

Matejka and Bartels (1998; henceforth referred to as MB98) have recently conducted a thorough examination of the errors in the vertical velocity fields from various multiple-Doppler analysis techniques. Using a Monte Carlo method they examined the responses of eight different analysis techniques to different magnitudes

and behaviors of errors in the input data, including random, bias, and trend errors. For dual-Doppler analyses, MB98 found that the downward integration implementation of the AR69 or BR77 techniques were best when errors were most random. However, when systematic errors were present, a method similar to that of Lateef (1967), O'Brien (1970), or RY80 was preferable. Errors in vertical velocity boundary condition specification were identified as a major remaining obstacle for both of these techniques. This problem is the exact problem that will be addressed within this work.

Section B: Applications

Observational studies employing dual-Doppler wind analysis techniques began to appear regularly after 1974, the year in which the National Severe Storms Laboratory (NSSL) began operating two Doppler radars geometrically separated so as to be conducive to dual-Doppler analyses. Although dual-Doppler analysis techniques other than those presented in Section 2A have been presented in the literature, the techniques discussed thus far have been the most widely applied to date. The range of problems addressed with these techniques over the past three decades is immense.

Kropfli and Miller (1976) published a study on the kinematic structure of a decaying thunderstorm in a weakly sheared environment as determined from the AR69 analysis technique. Reinking et al. (1981) used an airborne gust probe and dual-Doppler analyses in a study of clear-air roll vortices and turbulent motions within the boundary layer. Carbone (1982, 1983) utilized the iterative technique of BR77 with the upper

boundary condition specified in order to study the hydrodynamic structure of and the circulation associated with the parent vortex of a damaging tornado within a severe frontal rainband. Due to poor data at low levels, Knupp and Cotton (1982) also employed the iterative technique with upper boundary condition of $w=0$ applied at echo top while studying an intense, quasi-steady thunderstorm that occurred over South Park, Colorado on July 19, 1977. Rabin et al. (1982) used a low-level dual-Doppler analysis technique in order to derive an average hodograph for use in studying momentum fluxes in a cloudless convective boundary layer. Ziegler et al. (1983) utilized the RY80 constrained technique in combination with a numerical hail growth and decay model to study hail growth in an Oklahoma multicell thunderstorm.

Brandes et al. (1988) utilized the RY80 constrained dual-Doppler technique in a study of streamwise vorticity effects on supercell morphology and persistence. Eymard and Weill (1988) employed the statistical-variational techniques of Chong and Testud (1983) in an investigation of the tropical convective boundary layer. Parsons and Kropfli (1990) utilized the low-level dual-Doppler analysis technique of (2.18) to derive horizontal wind fields, followed by application of a Schuman filter and upward integration of the anelastic mass continuity equation in order to study the dynamics and structure of a Colorado microburst.

Lin et al. (1993) used the RY80 constrained technique in combination with rainwater estimates derived from the observed radar reflectivity fields and thermodynamic variables obtained via a thermodynamic retrieval technique to initialize a numerical model simulation of a convective storm. Zhang and Gal-Chen (1996) utilized

a low-level dual-Doppler analysis technique (similar to (2.18)) as a means of verifying a new single-Doppler velocity retrieval scheme. Dowell and Bluestein (1997) utilized the RY80 downward integration procedure with a linear O'Brien (Lateef) adjustment in an analysis of an Oklahoma supercell during tornadogenesis.

Most recently, Protat and Zawadzki (2000) introduced the vertical vorticity equation as a dynamic flow constraint to improve retrievals of the perturbation pressure and temperature fields in multi-Doppler analyses. Their basic analysis framework imposed the horizontal momentum equations as weak constraints, while introduction of the vertical vorticity equation ensures equality of the horizontal cross-derivatives of pressure (a requirement in order for an exact solution of the momentum equations to exist). Although introduction of the vorticity equation as a constraint left the 3-D wind solution largely unchanged, the first- and second-order derivatives of the three wind components were changed slightly in a way that allowed for substantial improvement in the retrieval of the perturbation pressure and temperature fields.

This concludes the literature review for dual-Doppler analysis techniques. Although this review has not been exhaustive, the intent here has been to provide an overview of the evolution of dual-Doppler analysis techniques and their applications, and to identify some of the lingering problems with the techniques. It is apparent that accurate synthesis of the vertical velocity field continues to be one of the most elusive problems of dual-Doppler analyses. The goal of this work is to make considerable progress in addressing this long-standing problem by incorporating new information – namely the anelastic vertical vorticity equation – into the analysis procedure.

Chapter 3: The Anelastic Form of the Vertical Vorticity Equation: Exact Solutions and Error Analysis

Section A: Motivation

In Chapter 2, accurate synthesis of the vertical velocity field was identified as the most significant problem for traditional analysis techniques. Reliable knowledge of the vertical velocity field is desirable in many of the situations for which dual-Doppler analyses are commonly applied, such as in studies of the microphysical (Ziegler et al., 1983), kinematic (Brandes, 1977), dynamic (Parsons and Kropfli, 1990), and thermodynamic (Gal-Chen and Kropfli, 1984) structures of thunderstorms, in studies of tornadogenesis (Dowell and Bluestein, 1997), and in initialization for numerical weather prediction models (Lin et al., 1993).

Two primary sources for errors in synthesis of the vertical velocity field in dual-Doppler analyses were identified in Chapter 2: (i) accumulation of errors in the horizontal divergence estimates during integration of the mass continuity equation, and (ii) misspecification of the required vertical velocity boundary condition(s) for the mass continuity equation. In order to improve synthesis of the vertical velocity in dual-Doppler analyses it is useful to seek an additional equation or equations with which the vertical velocity can be further constrained. The desired constraint equation(s) should be conducive to use with the types of data typically available with dual-Doppler datasets, namely quasi-horizontal wind fields and limited atmospheric density information. The most promising constraint equations would appear to be

dynamic constraints arising from Newton's second law. In particular, dynamic constraints arising from the principle of conservation of momentum may contain information on w as well as some or all 1st derivatives of w . The three equations arising from the principle of conservation of momentum are called the *equations of motion* (or, under the inviscid assumption, the *Euler equations*).

Direct application of the equations of motion in dual-Doppler analyses is complicated by the need for detailed information about the atmospheric pressure and density fields. Fortunately, application of the anelastic approximation followed by few simple manipulations of the horizontal equations of motion yields a single equation that serves our needs well. This equation – the anelastic vertical vorticity equation – is well suited for use in dual-Doppler analyses. It bypasses the need for any pressure and density information, yet still contains useful information relating the vertical velocity field to the horizontal wind fields. The vertical vorticity equation assumes the same form under either the Boussinesq or anelastic approximations to mass conservation, but is herein referred to as the anelastic vertical vorticity equation to emphasize its accuracy. Under the anelastic approximation the u and v Euler equations are:

$$(3.1) \quad \begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - fv + \frac{1}{\rho} \frac{\partial p}{\partial x} &= 0 \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + fu + \frac{1}{\rho} \frac{\partial p}{\partial y} &= 0 \end{aligned}$$

where f is the Coriolis parameter and p is pressure. Taking the partial derivative with respect to y of the 1st equation of (3.1) and subtracting it from the partial derivative with respect to x of the 2nd equation of (3.1) we arrive at the anelastic form of the vertical vorticity equation:

$$(3.2) \quad \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} + (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) = 0 ,$$

where ζ is the vertical component of the relative vorticity and the remaining variables are as defined earlier. The fact that the baroclinic terms involving the unknown pressure and density fields do not appear in (3.2) (they act only in the horizontal direction) is a primary motivation for its use.

The anelastic approximation is frequently invoked when studying deep circulations in the atmosphere, and applies equally well to shallower circulations. Although often used in numerical modeling to eliminate sound waves as possible solutions, in this case the primary benefit of its application is that it causes the baroclinic term present in the more general form of the vertical vorticity equation to disappear. Scale analysis reveals that its application will generally not be problematic for application in dual-Doppler analyses of commonly observed phenomena. Assuming the horizontal and vertical scales ($L \sim H \sim 10$ km) and velocities ($U \sim W \sim 10$ m/s) of observed feature are similar, scale analysis shows each of the terms in (3.2) to be representable by the ratio $U^2/L^2 \approx 10^{-6} s^{-2}$. The baroclinic term in the vertical vorticity equation (which disappears under the anelastic approximation) is given by

$\hat{k} \cdot (\bar{\nabla} \rho \times \bar{\nabla} p) / \rho^2$. Under the assumptions $\delta \rho \sim 0.01 \text{ kg/m}^3$, $\rho \sim 1.0 \text{ kg/m}^3$, and $\delta p \sim 10 \text{ mb}$ ($=1000 \text{ Pa}$), this term is representable by the ratio $\delta \rho \delta p / \rho^2 L^2 \approx 10^{-7} \text{ s}^{-2}$. Thus the baroclinic generation of vertical vorticity should generally be an order of magnitude smaller than the terms of (3.2) and the anelastic form of the vertical vorticity equation should suffice for use in most dual-Doppler analyses. Note that this assertion does not imply that baroclinic processes are unimportant in the generation of vorticity, only that they are of secondary importance in the generation of *vertical* vorticity. Tilting of baroclinically generated horizontal vorticity is widely considered to be an important process, especially in thunderstorms, and is still represented in the anelastic system.

The vertical velocity and its horizontal derivatives appear explicitly in (3.2). An analysis using (3.2) as a constraint will therefore require specification of w (or enforcement of the natural boundary conditions) on the horizontal boundaries of the data volume as opposed to the vertical boundaries posed by the mass continuity constraint. A possible benefit of using (3.2) is that the natural boundary conditions will be ‘mixed’ or Robin boundary conditions as opposed to Neumann boundary conditions. This fact has implications for the uniqueness of solutions obtained with the natural boundary conditions, an issue that will be discussed in more detail in Chapter 4.

Section B: Exact solutions of the anelastic vertical vorticity equation

Before applying the anelastic vertical vorticity equation in dual-Doppler analyses, an investigation into the sensitivities of (3.2) is in order. To this end, it is useful to first derive an analytical solution to the vorticity equation and then examine the situations under which the solution may be most prone to errors.

With u and v regarded as known, (3.2) is a first-order linear PDE for w . A common method for solving such equations is the *method of characteristics* (recall discussion of this method in the Armijo technique derivation of Chapter 2, Section A). In principle, the method of characteristics says that this PDE can be reduced to a series of ODEs to be solved along special curves, called *characteristic curves*. To foster visualization of this idea, (3.2) can be rearranged as:

$$(3.3) \quad \frac{\partial v}{\partial z} \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \frac{\partial w}{\partial y} + w \frac{\partial \zeta}{\partial z} + \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0.$$

The leading terms of (3.3) can then be rewritten as:

$$(3.4) \quad \frac{\partial v}{\partial z} \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \frac{\partial w}{\partial y} = \bar{\epsilon} \cdot \nabla w, \quad \bar{\epsilon} = \left(\frac{\partial v}{\partial z}, -\frac{\partial u}{\partial z}, 0 \right).$$

If the vector $\bar{\epsilon}$ is normalized by its magnitude (which in this case turns out to be the magnitude of the vertical wind shear, $|\bar{S}|$), we effectively have an expression for a directional derivative:

$$(3.5) \quad \frac{\partial v}{\partial z} \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \frac{\partial w}{\partial y} = |\bar{S}| (\hat{\epsilon} \cdot \nabla w) = |\bar{S}| \frac{dw}{ds}, \quad \text{where: } \hat{\epsilon} = \frac{\bar{\epsilon}}{|\bar{\epsilon}|} = \frac{\bar{\epsilon}}{|\bar{S}|}.$$

In (3.5), $|\bar{S}|$, the magnitude of the vertical wind shear, is not to be confused with the distance parameter, s , which denotes distance along the characteristic curve and uses the same metric as the original Cartesian coordinate system. The direction $\hat{\epsilon}$, called the characteristic direction, defines the direction of the characteristic curve at a particular point in the Cartesian grid. The equations defining the characteristic curve are:

$$(3.6) \quad \begin{aligned} \frac{dx}{ds} &= \frac{\partial v / \partial z}{|\bar{S}|} \\ \frac{dy}{ds} &= \frac{-\partial u / \partial z}{|\bar{S}|} \\ \frac{dz}{ds} &= 0 \end{aligned}$$

Along curves defined by (3.6) (with appropriate boundary data), solution of the anelastic vertical vorticity equation is reduced to solution of a linear first-order ODE:

$$(3.7) \quad \frac{dw}{ds} + w \left(\frac{\partial \zeta / \partial z}{|\bar{S}|} \right) + \left(\frac{\partial \zeta / \partial t + \bar{V}_h \cdot \nabla \zeta + \delta_h (\zeta + f)}{|\bar{S}|} \right) = 0 ,$$

where $\delta_h = \partial u / \partial x + \partial v / \partial y$. It is interesting to note that the characteristic direction at any point is perpendicular and to the right of the shear vector (or, more accurately, the vertical shear vector of the horizontal winds). It points in the direction of a pseudo horizontal vorticity vector consisting of that portion of the horizontal vorticity vector arising solely from vertical shear of the horizontal winds (i.e., neglecting the contribution of horizontal variations in w to the horizontal vorticity vector). What is gained by putting the anelastic vertical vorticity equation into this canonical form is the ability to keep track of the horizontal vorticity available for tilting into the vertical with a single parameter, $|\bar{S}|$. The vertical vorticity equation then allows us to simply vary the vertical velocity along the characteristic curves, tilting this pseudo horizontal vorticity into and out of the vertical direction in order to satisfy the constraints of the anelastic vertical vorticity equation. Alternatively, the method of characteristics could be thought of as transferring the vertical vorticity equation in the Cartesian coordinate system into a new curvilinear coordinate system where the horizontal vorticity available for tilting is always confined to one coordinate direction. This causes one of the terms multiplying the horizontal derivatives of w to be zero in the vorticity equation, reducing the PDE to an ODE.

It is immediately apparent that (3.7) is singular when the vertical wind shear $|\vec{S}|$ is zero, and from (3.6) we can see that a characteristic curve entering such an area will end abruptly at that location. It is then possible that ‘orphaned’ points could appear within the data domain that are effectively unreachable by any characteristics extending from the data boundaries. Since the characteristics cannot reach these points, no solution can be obtained in these areas. In this case, assuming that there is still a vertical variation in ζ , w could be determined by the algebraic formula that remains when (3.7) is multiplied by $|\vec{S}|$.

To proceed with the error analysis it is useful to derive the exact solution of (3.7). Conveniently, (3.7) is integrable with the aid of an integrating factor. In order to simplify further analysis the following substitutions will be made:

$$(3.8) \quad \begin{aligned} a(s) &= \frac{\partial \zeta / \partial z}{|\vec{S}|} = \frac{A(s)}{|\vec{S}|} \\ b(s) &= \frac{\partial \zeta / \partial t + \bar{v}_h \cdot \nabla \zeta + \delta_h(\zeta + f)}{|\vec{S}|} = \frac{B(s)}{|\vec{S}|} \end{aligned}$$

If we introduce an integrating factor μ , where μ is given by:

$$(3.9) \quad \mu = \exp\left(\int a(s) ds\right),$$

then we can rewrite (3.7) as:

$$(3.10) \quad \frac{d}{ds} \left(w \exp \left(\int a(s) ds \right) \right) = -b(s) \exp \left(\int a(s) ds \right) .$$

Integrating with respect to s from $s=0$ to s , simplifying, and rearranging we find:

$$(3.11) \quad w(s) = - \int_0^s b(s') \exp \left(- \int_{s'}^s a(s'') ds'' \right) ds' + w_0 \exp \left(- \int_0^s a(s'') ds'' \right)$$

where $w_0 = w(s=0)$ is the boundary condition for w .

Section C: Error analysis

The presence of errors in real data makes it unlikely that a realistic solution for $w(s)$ could be obtained using (3.6) and (3.11). However, assessing the effects errors will have on the behavior of this analytic solution may provide insight into the sensitivities of dual-Doppler analysis methods using the vorticity equation as a constraint.

In order to evaluate how (3.11) may respond to errors in the forcing terms ($a(s)$ and $b(s)$) and w_0 , each of these terms can be split up into their truth and error components. We can then find expressions for the true vertical velocity $w_T(s)$ and the error vertical velocity $w_E(s)$. To this end write:

$$\begin{aligned}
w_E(s) &= w(s) - w_T(s) \\
w_{0_E} &= w_0 - w_{0_T} \\
(3.12) \quad a_E(s) &= a(s) - a_T(s) = \frac{A_T(s) + A_E(s)}{|\bar{S}_T + \bar{S}_E|} - \frac{A_T(s)}{|\bar{S}_T|}, \\
b_E(s) &= b(s) - b_T(s) = \frac{B_T(s) + B_E(s)}{|\bar{S}_T + \bar{S}_E|} - \frac{B_T(s)}{|\bar{S}_T|}
\end{aligned}$$

where the subscripts T and E indicate ‘truth’ and ‘error’ terms, respectively. Inserting these expressions into (3.11) we find that the true vertical velocity is given by:

$$(3.13) \quad w_T(s) = -\int_0^s b_T(s') \exp(-\int_{s'}^s a_T(s'') ds'') ds' + w_{0_T} \exp(-\int_0^s a_T(s'') ds'').$$

Interactions between the error and truth terms cause the expression for the error vertical velocity $w_E(s)$ to be considerably more complex. The total error vertical velocity can be expressed as the sum of four individual error terms (ET):

$$(3.14) \quad w_E(s) = w(s) - w_T(s) = \text{ET}_1 + \text{ET}_2 + \text{ET}_3 + \text{ET}_4,$$

where:

$$\begin{aligned}
ET_1 &= -\int_0^s \left(b_T(s') \exp(-\int_{s'}^s a_T(s'') ds'') \right) \left(\exp(-\int_{s'}^s a_E(s'') ds'') - 1 \right) ds' \\
ET_2 &= -\int_0^s b_E(s') \exp(-\int_{s'}^s a_T(s'') + a_E(s'') ds'') ds' \\
ET_3 &= w_{0_T} \exp(-\int_0^s a_T(s'') ds'') \left(\exp(-\int_0^s a_E(s'') ds'') - 1 \right) \\
ET_4 &= w_{0_E} \exp(-\int_0^s a_T(s'') + a_E(s'') ds'')
\end{aligned}
\tag{3.15}$$

Equations (3.14) and (3.15) regulate the growth and decay of errors in the solution for $w(s)$ obtained along the characteristics. However, it should be noted that errors in the data will also affect the calculated positions of the characteristic curves. The locations of the characteristic curves in the Cartesian grid must be obtained by integrating (3.6) with respect to s . Errors in the calculated shear values will cause errors in the estimation of the local characteristic direction and henceforth in the calculated locations of the characteristic curves. Upon integration over a significant distance in s it is possible for a calculated characteristic curve location to differ significantly from the position of the true characteristic curve originating at the same boundary point.

Unfortunately, a direct interpretation of the physical meaning of the error terms in (3.14) and (3.15) is not readily apparent. However, note that all four of the terms involve exponentials of integrals of $a_E(s)$ and/or $a_T(s)$. In particular, ET_1 and ET_3 will virtually disappear if the integrals of $a_E(s)$ contained in these terms remain near zero (so that the exponentials of these integrals remain near 1.0). These integrals can be approximated as summations over discrete elemental segments of the integration path (the limits of integration are being left off here for generality):

$$\begin{aligned}
(3.16) \quad \int a(s) ds &\approx \sum_i a_i(s) \Delta s \\
e^{\int a(s) ds} &\approx e^{\sum_i a_i(s) \Delta s} = \prod_i e^{a_i(s) \Delta s} .
\end{aligned}$$

Assuming the errors $a_E(s)$ are unbiased and normally distributed, the expected value of any integral of $a_E(s)$ will also be expected to be zero (although the variance of the value of the integrals will grow larger along the characteristic as distance away from the boundary where w_0 was specified increases). From (3.16) it is immediately apparent that the probability of an integral of $a_E(s)$ retaining a value near zero will be dramatically reduced if an extreme value of $a_E(s)$ has been encountered somewhere along the integration path. An obvious question is then under what conditions can $a_E(s)$ become anomalously large? Recall that $a_E(s)$ is given by:

$$(3.17) \quad a_E(s) = a(s) - a_T(s) = \frac{A_T(s)}{|\vec{S}_T + \vec{S}_E|} + \frac{A_E(s)}{|\vec{S}_T + \vec{S}_E|} - \frac{A_T(s)}{|\vec{S}_T|} ,$$

where $A_T(s)$ and $A_E(s)$ are as defined in (3.8) and (3.12). The most plausible way for $a_E(s)$ to become anomalously large is for $|\vec{S}_T + \vec{S}_E|$ to approach zero. In practice we will not know what the vectors $\vec{S}_T$ and $\vec{S}_E$ are individually, but we will know what the vector $(\vec{S}_T + \vec{S}_E)$ is. Although we cannot isolate the two vectors in practice, if the threshold magnitude for $|\vec{S}_T + \vec{S}_E|$ is larger than a couple of standard deviations for the

expected error in the shear magnitude ($\sigma_{|s|}$ -- see Appendix I), then we will largely be assured that $a_E(s)$ (and integrals of $a_E(s)$) will not become anomalously large because of the denominator alone. Based upon this error analysis, limiting use of the vorticity equation to areas where the shear exceeds a threshold magnitude would be expected to yield more favorable results in dual-Doppler analyses. However, as will be shown in the next chapter, the dangers of using the vorticity equation in areas of low shear can be circumvented by applying analysis approaches which tie the vertical velocity field in areas of low shear to the vertical velocity field in other areas where the shear is more favorable (e.g., by using mass continuity to tie regions of higher shear together).

ET_2 and ET_4 will also benefit from the integrals of $a_E(s)$ remaining near zero since this term will not contribute to magnification of errors in $b(s)$ and w_0 . Note that if the integrals of $a_E(s)$ become large *and* positive, the error terms ET_2 and ET_4 will actually be reduced. However, this situation would lead to amplification of error terms ET_1 and ET_3 . Further, there is no way to control the sign of integrals of $a_E(s)$, only its magnitude, so it would be impossible to assure that it remains positive. Limitation of error terms ET_2 and ET_4 is therefore best accomplished by limiting the errors $b(s)$ and w_0 .

Other than assuring that w_{0_E} is specified as accurately as possible, little can be done to minimize error term ET_4 . The final error source, $b_E(s)$, appears only in ET_2 of (3.14) and (3.15). In order to minimize the errors arising from this term we will want to minimize $b_E(s)$. Expanding $b_E(s)$ we find:

$$(3.18) \quad b_E(s) = b(s) - b_T(s) = \frac{B_T(s)}{|\vec{S}_T + \vec{S}_E|} + \frac{B_E(s)}{|\vec{S}_T + \vec{S}_E|} - \frac{B_T(s)}{|\vec{S}_T|},$$

similar to (3.17). Aside from the shear magnitude considerations discussed earlier, errors in the forcing for the vorticity equation ($B_E(s)$) will be responsible for regulating the growth of error term ET_2 . When expanded into its contributing terms $B_E(s)$ can be written as:

$$(3.19) \quad B_E(s) = \left(\frac{\partial \zeta}{\partial t} \right)_E + \bar{v}_{h_T} \cdot \nabla \zeta_E + \bar{v}_{h_E} \cdot \nabla \zeta_E + \bar{v}_{h_E} \cdot \nabla \zeta_T + \delta_{h_E} (\zeta_T + f) + \delta_{h_T} \zeta_E + \delta_{h_E} \zeta_E.$$

In (3.19), the first term on the right hand side is the error in the calculated temporal derivative of ζ . The second and third terms on the right hand side of (3.19) represent advection of errors in the calculated vertical vorticity by both the ‘true’ horizontal wind and the ‘error’ horizontal winds. The fourth term represents the error due to advection of the ‘true’ vertical vorticity by the ‘error’ horizontal wind. The fifth term represents the stretching of the ‘true’ vertical vorticity by the ‘error’ divergence. Finally, the sixth and seventh terms represent the stretching of the ‘error’ vertical vorticity by both the ‘true’ and ‘error’ divergence. Obviously we cannot hope to eliminate all of these errors in an analysis. However, steps can be taken to minimize them.

Before discussing how these errors may be minimized, the issue of what exactly is meant by an ‘error’ must be clarified. The vertical vorticity equation is a linear PDE for w , but the forcing for the equation contains nonlinear interactions in u and v . In order to prevent unresolvable features (those having scales less than twice the Cartesian grid spacing) from appearing in the solution for w , features of scales smaller than $4\Delta x$ (or $4\Delta y$) in u and v must be filtered out prior to obtaining a solution for w . An ‘error’ at a particular location is then the deviation of the calculated value of that entity from the ‘true’ value of the entity in harmonics of scales of $4\Delta x$ or larger. Deviation from ‘true’ features of scales less than $4\Delta x$ is then not considered ‘error’. Filtering is necessary only prior to application of the vorticity equation, so the finer scales in u and v could be allowed back into a final analysis once a solution has been obtained from the vorticity equation.

Although a necessity, filtering of scales less than $4\Delta x$ in the horizontal wind fields turns out to be very useful in controlling errors in the forcing term $b_E(s)$. Assuming random observational errors and an objective analysis procedure that does significantly smooth observational errors beforehand, errors in the horizontal wind fields should primarily lie in the $2\Delta x$ scales. Therefore, filtering of scales less than $4\Delta x$ should dramatically reduce the magnitude of the error terms in (3.19). For instance, the error vorticity and divergence terms can be represented in scale analysis by the magnitude of the velocity errors multiplied by the wavenumber of the scale of the errors. Before filtering, the errors should lie primarily in the $2\Delta x$ scales and the error vorticity and divergence can be represented by $|\bar{v}_{h_E}| \cdot (2\pi/2\Delta x)$. However, after

filtering the magnitude of $\bar{v}_{h_E}$ will be reduced and additionally the errors will lie primarily in the $4\Delta x$ scales. The error vorticity and divergence after filtering could then be represented by $|\bar{v}_{h_E}| \cdot (2\pi/4\Delta x)$ so that the contribution of the individual terms to $b_E(s)$ should be reduced by *at least* a factor of two.

The most prominent source of errors in (3.19) is apt to be the term representing errors in the temporal derivative of ζ . This includes errors due to misspecification of ζ at either of the times used in the centered difference approximation as well as errors introduced by discretization of the temporal derivative with a finite difference over a considerable span of time (which could approach 10-12 minutes using centered differences with the operational NEXRAD network; Klazura and Imy, 1993). The first of these error sources can be diminished by the filtering process. However, the second source of errors in this term can be considerably more difficult to overcome. If the local evolution between the two volume scans used to calculate this derivative occurs in a purely linear fashion (i.e., varies as a linear function of time) there should be little or no error introduced in this manner. Unfortunately, this will rarely be the case. However, these errors can be minimized by performing analyses in a reference frame in which the linearity of the local evolution is maximized. Since one of the primary causes of nonlinear local evolution is advection of small scale features, it is probably most useful to redefine the observations in a reference frame within which the features are most stationary. This ‘stationary’ reference frame will move with the primary feature present in the data rather than being fixed with respect to the ground so that advection within the analysis domain as a whole is minimized. One method for

determining an appropriate reference frame using radial wind data from two Doppler radars was presented by Gal-Chen (1982). More discussion on the Gal-Chen technique can be found in Appendix II.

The severity of errors in the temporal derivative resulting from the nonlinear effects of advection is a function of the advection speed of a particular feature as well as the feature's wavelength and the time span between observations. If a feature is entirely stationary in the analysis reference frame the contribution of the nonlinear effects of advection should be nil. Figures (3.1) and (3.2) depict the relative errors expected in the discretized temporal derivative as a result of the nonlinear effects of advection. Figure (3.1) illustrates the expected errors when new observations are available every minute (so that the centered difference occurs over a two-minute time span) while Fig. (3.2) illustrates the situation when new observations become available every five minutes. These figures were constructed using the wave equation:

$$(3.20) \quad \frac{\partial \phi}{\partial t} = -c \frac{\partial \phi}{\partial x} ,$$

with a particular solution of $\phi = \Phi \sin[k(x-ct)]$. Errors in the discretized temporal derivative were determined by comparing the analytic value of the derivative to the following discretized approximation to the derivative:

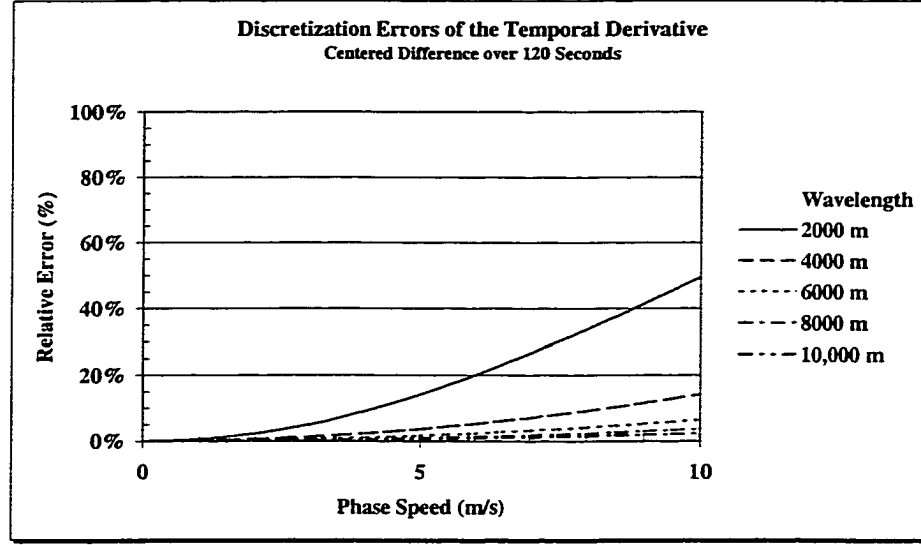


Figure 3.1: Expected relative errors in the discretized temporal derivative as a function of advection speed and feature wavelength. A 2nd order centered difference function is used to approximate the derivative. The time span between observational updates is 60 seconds.

$$(3.21) \quad \frac{\partial \phi}{\partial t} = \frac{\phi[k(x - c(t + \Delta t))] - \phi[k(x - c(t - \Delta t))]}{2\Delta t} + Error$$

$$Error = -\frac{\partial^3 \phi}{\partial t^3} \frac{\Delta t^2}{6} - O(\Delta t^4) = -\frac{\Phi k^3 c^3 \Delta t^2 \cos[k(x - ct)]}{6} - O(\Delta t^4)$$

Notice from the figures that the expected errors in the temporal derivative worsen as the advection speed of a feature increases and as the wavelength of a feature decreases. Note also the considerable improvement in the expected accuracy of the temporal derivative when the observational frequency is increased.

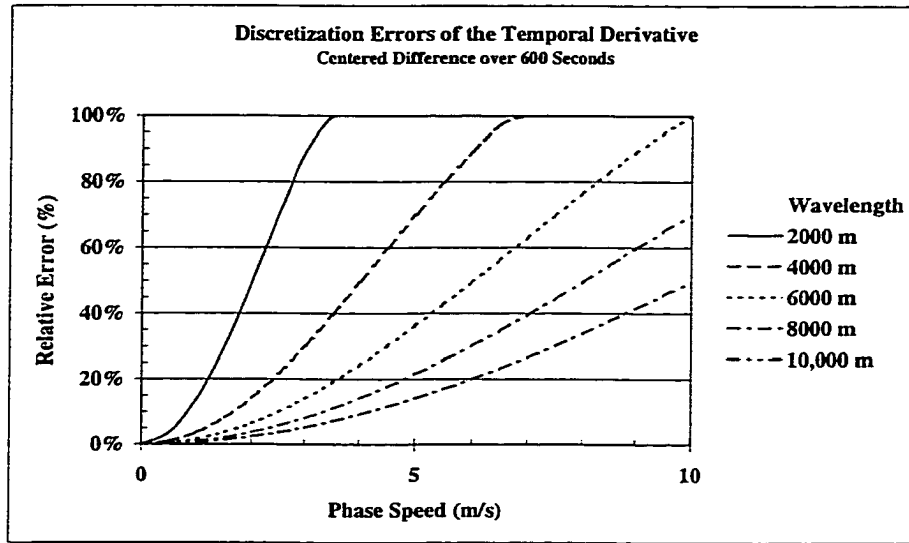


Figure 3.2: Expected relative errors in the discretized temporal derivative as a function of advection speed and feature wavelength. A 2nd order centered difference function is used to approximate the derivative. The time span between observational updates is 300 seconds.

It is well known that geometry plays a key role in determining the spatial accuracy of the wind field in dual-Doppler analyses (Doviak et al., 1976). In general, the synthesis of the wind field will be most accurate in regions where the beam intersection angles between the two radars is near 90°. Accuracy is reduced significantly at beam intersection angles less than 30° or more than 150°. Also, synthesized horizontal wind fields typically decrease in accuracy at higher elevation angles due to feedback from the uncertainties involved in synthesizing w (as discussed in the last chapter). Unfortunately, explicit expressions for the spatial variations of error variance in u , v , and w from dual-Doppler analyses in a Cartesian coordinate system are very difficult to obtain. However, Doviak et al. have derived expressions for the error variances of the wind components in the coplane (recall the Armijo technique) coordinate system:

$$\begin{aligned}
\sigma_s^2 &= \frac{R_1^2 \sigma_1^2 + R_2^2 \sigma_2^2}{s_2^2} \\
(3.22) \quad \sigma_r^2 &= \frac{R_1^2 \left(s + \frac{1}{2}s_2\right)^2 \sigma_1^2 + R_2^2 \left(s - \frac{1}{2}s_2\right)^2 \sigma_2^2}{s_2^2 r^2} + \sigma_t^2 \sin^2(\Theta) \quad , \\
\sigma_\Theta^2 &= \frac{\Delta \Theta r^2}{2(\Delta r)^2 \rho^2} \left[\frac{R_1^2 R_2^2}{s_2^2 r^2} \int_0^\Theta \rho^2 (\sigma_1^2 + \sigma_2^2) d\Theta + \int_0^\Theta \rho^2 \sigma_t^2 \sin^2 \Theta d\Theta \right]
\end{aligned}$$

where σ_s^2 , σ_r^2 , σ_Θ^2 are the error variances of the coplane wind components, s , r , Θ , R_1 , R_2 , and ρ are as defined in (2.1) – (2.14), Δ represents the grid spacing of the quantity it precedes, σ_1 and σ_2 are the standard deviations of errors in the observed radial winds, σ_t is the standard deviation of the error in the terminal velocity estimate, and s_2 is the location of the 2nd Doppler radar in the coplane coordinate system (recall $s_1=0$). Upon determination of the coplane wind component error variances the variances of the Cartesian wind components (σ_u^2 , σ_v^2 , σ_w^2) can be obtained through a coordinate transformation:

$$\begin{aligned}
(3.23) \quad \sigma_u^2 &= \sigma_s^2 \\
\sigma_v^2 &= \sigma_r^2 \cos^2 \Theta + \sigma_\Theta^2 \sin^2 \Theta \quad , \\
\sigma_w^2 &= \sigma_r^2 \sin^2 \Theta + \sigma_\Theta^2 \cos^2 \Theta
\end{aligned}$$

where it has once again been assumed that the coplane s coordinate corresponds to the Cartesian x coordinate.

Errors due to geometrical considerations tend to vary over much larger spatial scales than the errors in Doppler radial wind measurements. Therefore, these errors are expected to be of secondary importance in creating small-scale, local errors in solutions to the vorticity equation obtained from eq. (3.11). However, the manner in which the vorticity equation will be integrated into dual-Doppler analyses in the next chapter makes errors due to geometry a primary concern. In these new techniques the vorticity equation will be utilized to retrieve appropriate vertical velocity boundary conditions for traditional analyses that apply mass continuity as a constraint. When utilized in this manner, information from the vorticity equation at all levels is combined to estimate the vertical velocity boundary condition on the bottom and/or top boundary of the data region. Therefore, given a mechanism for applying spatially varying weights on information from the vorticity equation, (3.22) and (3.23) can be applied to preferentially weight regions expected to have smaller errors.

Section D: Weak constraint formulations

The exact solution of the vorticity equation (3.11) requires only one horizontal boundary condition in order to obtain a solution. However, due to errors in the data, it is almost certain that the vertical velocity solution along the characteristic will become unrealistic well before the characteristic returns to a data boundary (i.e., before the end of the characteristic curve is reached). Application of the vertical vorticity equation as a *weak constraint* to retrieve w can be useful in this regards as it allows specification of w boundary conditions at both ends of a characteristic curve. This is similar to the

situation that arises when mass continuity is applied as a weak constraint, where w boundary conditions can be specified at both ends of a quasi-vertical column. When applied as a weak constraint the Euler-Lagrange equation for w can be solved to yield the vertical velocity field that best satisfies the vertical vorticity equation while also still satisfying both boundary conditions.

Complete derivations of several new dual-Doppler analysis formulations applying the vorticity equation as a weak constraint will be presented in the Chapter 4. In these formulations the vorticity equation will be combined with the mass continuity equation to retrieve an optimal w field. It is interesting to note that the new formulations actually give rise to elliptic equations, which conveniently circumvent the need to utilize characteristic curves. However, analysis of the expected error sensitivities of the weak constraint formulations is simplified if the anelastic vertical vorticity equation is applied alone as a weak constraint to retrieve w . In this case we seek to minimize the cost functional:

(3.24)

$$J = \int_{x,y,z} \left(\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} + (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right)^2 dx' dy' dz'.$$

Regarding the u and v fields as fixed, we then take the 1st variation of (3.24) with respect to w . Setting the result equal to zero yields (noting that the variational operator δ commutes with both the derivative and integral), we get:

$$(3.25) \quad \delta_w J = \int_{x,y,z} VE \left(\delta w \frac{\partial \zeta}{\partial z} + \frac{\partial(\delta w)}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial(\delta w)}{\partial y} \frac{\partial u}{\partial z} \right) dx' dy' dz' = 0,$$

where:

$$(3.26) \quad \begin{aligned} VE &= w \frac{\partial \zeta}{\partial z} + \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} + VE_k \\ VE_k &= \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \end{aligned}$$

Integration by parts of the δw tilting terms in (3.25) yields:

$$(3.27) \quad \begin{aligned} & \int_{x,y,z} \left(VE \frac{\partial \zeta}{\partial z} - \frac{\partial}{\partial x} \left[VE \frac{\partial v}{\partial z} \right] + \frac{\partial}{\partial y} \left[VE \frac{\partial u}{\partial z} \right] \right) \delta w dx' dy' dz' \\ & + \int_{y,z} VE \frac{\partial v}{\partial z} \delta w dy' dz' - \int_{x,z} VE \frac{\partial u}{\partial z} \delta w dx' dz' = 0 \end{aligned}$$

For the equality to hold for any arbitrary δw , each of the integrals in (3.27) must be equal to zero. Neglecting the trivial solution $\delta w=0$ in the first integral leaves the Euler-Lagrange equation:

$$(3.28) \quad a \frac{\partial^2 w}{\partial x^2} + 2b \frac{\partial^2 w}{\partial x \partial y} + c \frac{\partial^2 w}{\partial y^2} + d \frac{\partial w}{\partial x} + e \frac{\partial w}{\partial y} + fw + g = 0,$$

where the coefficients are given by:

$$\begin{aligned}
a &= -\left(\frac{\partial v}{\partial z}\right)^2, \quad b = \frac{\partial u}{\partial z} \frac{\partial v}{\partial z}, \quad c = -\left(\frac{\partial u}{\partial z}\right)^2, \\
d &= -\frac{\partial v}{\partial z} \frac{\partial \zeta}{\partial z} - \frac{\partial v}{\partial z} \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial u}{\partial z} \frac{\partial^2 v}{\partial y \partial z}, \quad e = \frac{\partial u}{\partial z} \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial^2 u}{\partial x \partial z} - \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial y \partial z}, \\
(3.29) \quad f &= \frac{\partial u}{\partial z} \frac{\partial^2 \zeta}{\partial y \partial z} - \frac{\partial v}{\partial z} \frac{\partial^2 \zeta}{\partial x \partial z}, \quad g = -\frac{\partial v}{\partial z} \frac{\partial (VE_k)}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial (VE_k)}{\partial y}.
\end{aligned}$$

Eq. (3.28) is parabolic since $(b^2 - ac) = 0$ (Reddy and Rasmussen, 1990), and hence does have characteristic curves. For minimization of the functional (3.24) to be ensured, the boundary integrals must also be equal to zero. This can be accomplished in one of two ways. The first option is to specify w on the boundaries of the data volume (in which case $\delta w = 0$ on the boundaries). The second option is to apply the *natural boundary conditions* of the problem. In this formulation the natural boundary conditions are:

$$(3.30) \quad \left(\frac{\partial v}{\partial z} \left(\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} \right) + (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) = 0,$$

on the boundaries in the east and west directions, and:

$$(3.31) \quad \left(\frac{\partial u}{\partial z} \left(\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} \right) + (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) = 0,$$

on the boundaries in the north and south directions. Equations (3.30) and (3.31) are linear first order PDEs for w (Robin boundary conditions). The form of these natural boundary conditions indicates that satisfying the vorticity equation exactly on the horizontal boundaries of the domain is an alternative to specifying w on those boundaries. The presence of the shear terms which multiply the vorticity equation in (3.30) and (3.31) can be interpreted in terms of the characteristic curves of the Euler-Lagrange equation (3.28) (which happen to be the same as the characteristic curves of the exact solution, and are therefore defined by (3.6)). If either of these shear terms is equal to zero no characteristic curves reach the corresponding boundary in that region and the interior solution is insensitive to w along that boundary.

For the sake of simplifying the error analysis, it is useful to restrict ourselves to the special case where $u=u(x,y,t)$ and $v=v(x,y,z,t)$. In this special case, all of the vertical shear lies in one coordinate direction so that the characteristics of the vorticity equation are parallel to the x -axis. It can be shown that, under these assumptions, (3.28) reduces to:

$$(3.32) \quad -\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial z} \frac{\partial w}{\partial x} + \frac{\partial^2 v}{\partial x \partial z} w + \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + \delta_h(\zeta + f) \right) = 0.$$

Equation (3.32) can be integrated immediately, resulting in:

$$(3.33) \quad \frac{dw}{dx} + w \left(\frac{\partial \zeta / \partial z}{\partial v / \partial z} \right) + \left(\frac{\partial \zeta / \partial t + \bar{v}_h \cdot \nabla \zeta + \delta_h(\zeta + f)}{\partial v / \partial z} \right) = \frac{F(y, z, t)}{\partial v / \partial z},$$

where F in (3.33) is a function of integration. The solution (3.33) is remarkably similar to (3.7) with the exception of the added forcing term (appearing on the right hand side of (3.33)). Similar to (3.7), equation (3.33) is also integrable. However, the solution of (3.33) contains two constants (functions) of integration (as opposed to one in the solution of (3.7)), allowing specification of the vertical velocity at both ends of a given characteristic.

Since (3.24) is parabolic in nature it can be reduced to a canonical form involving a 2nd partial derivative with respect to only one independent variable (similar to reduction of (3.2) to (3.7)). Noting that the characteristics of the weak constraint formulation are the same as the characteristics of the anelastic vertical vorticity equation itself, it is possible to derive the canonical form by deriving the Euler-Lagrange equation in the characteristic coordinates rather than in the Cartesian coordinate system. In this characteristic coordinate system, the Euler-Lagrange equation for the weak constraint formulation will reduce to the linear 2nd order ODE:

$$(3.34) \quad \frac{d^2 w}{ds^2} + c_1 \frac{dw}{ds} + c_2 w + c_3 = 0,$$

where:

$$(3.35) \quad c_1 = \frac{2}{|\bar{S}|} \frac{\partial(|\bar{S}|)}{\partial s}, \quad c_2 = \frac{1}{|\bar{S}|^2} \frac{\partial(a|\bar{S}|^2)}{\partial s} - a^2, \quad c_3 = -\frac{1}{|\bar{S}|^2} \frac{\partial(b|\bar{S}|^2)}{\partial s} - ab,$$

and $a=a(s)$ and $b=b(s)$ are as defined earlier in (3.8). It can be shown that under the special conditions $u=u(x,y,t)$ and $v=v(x,y,z,t)$ (3.34) is immediately integrable to the same solution as that of (3.33), likely indicating that (3.34) is indeed the canonical form of (3.28). However, if vertical shear is allowed in both Cartesian wind components (i.e., $u=u(x,y,z,t)$ and $v=v(x,y,z,t)$) (3.28) will have mixed partial derivatives of both 1st and 2nd order, while (3.34) maintains its form of a linear 2nd order ODE for w (although it is no longer immediately integrable to a linear 1st order ODE). The simplicity of (3.34) as compared to (3.28) is the obvious benefit of converting the equation to its canonical form. Further, the similarities of the equations for the exact and weak constraint applications of the anelastic vertical vorticity equation would appear to indicate that the error analysis of the exact solution can be generalized to the weak constraint formulations.

Section E: Error magnitudes

The conditions under which dual-Doppler analyses utilizing the anelastic vertical vorticity equation may exhibit the most severe errors have now been established. Unfortunately, the error analysis has failed to provide guidelines on the expected severity (or magnitude) of the errors in the retrieved vertical velocity given errors in the forcing for the vorticity equation. In reality, the magnitude of solution errors may be boundless given integration over a sufficiently long characteristic curve in the presence of errors in the forcing for the equation. In spite of this fact, it is useful

to obtain some estimate of the magnitude of these errors by making the simplifying assumption that $u=0$ (we could just as easily assume $v=0$). Under this assumption, both the anelastic vertical vorticity equation and the weak constraint formulation given by (3.28) integrate to:

$$(3.36) \quad w = -\frac{1}{\partial v / \partial z} \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial y} \right),$$

where the constants of integration have been neglected. Equation (3.36) is reflective of the y-component equation of motion under the assumption $u=0$ (and with no forcing). Examining (3.36) it is apparent that the amplitude of the errors in w relative to the errors in v are related to the ratio of the horizontal wind shear to the vertical shear. If the vertical wind shear is of the same order of magnitude as the horizontal shear, no significant amplification of the errors in the estimated horizontal wind fields would be expected to appear in the retrieved vertical velocity field. As has been done previously in this chapter, the terms in (3.36) can be parsed into truth and error terms in order to assess the scale of errors that may occur in the retrieved vertical velocities. After subtracting out the ‘truth’ from the expression for the total vertical velocity we find that the error in the retrieved vertical velocity in this special case is given by:

$$(3.37) \quad w_E = -\frac{\frac{\partial v_E}{\partial t} + v_T \frac{\partial v_E}{\partial y} + v_E \frac{\partial v_T}{\partial y} + v_E \frac{\partial v_E}{\partial y} + w_T \frac{\partial v_E}{\partial z}}{\frac{\partial v_T}{\partial z} + \frac{\partial v_E}{\partial z}},$$

where the subscripts ‘ T ’ and ‘ E ’ indicate truth and error terms, respectively. Errors in the representation of continuous derivatives by finite difference approximations will be neglected (although, as discussed earlier, this may be a poor assumption for the temporal derivative). The denominator of (3.37) is a 1-dimensional quantity analogous to the 2-dimensional shear magnitude discussed earlier. The error analysis of Section C has shown that this is a quantity to which we may wish to assign a lower limit, thus avoiding major amplification of w_E in (3.37) by a small denominator.

Using (3.37) as a starting point an expression for the expected error variance of w (σ_w^2) can be derived. In order to derive an expression in which σ_w^2 is dependent upon only the true wind field and σ_v^2 it is useful to assume that v_E , $\partial v_E / \partial t$, $\partial v_E / \partial y$, and $\partial v_E / \partial z$ are independent of one another. Since each of these derivatives are calculated using centered finite-differences none of them use the value of v_E at the particular grid point, and each of the derivatives derives its value from the values of v_E at a different set of surrounding grid points. If the analysis is set up so as to keep the correlation of the observational errors over $2\Delta y$ (or $2\Delta z$ or $2\Delta t$) scales to a minimum, the variance of the numerator of (3.37) can be approximated by the sum of the variances of the individual terms in the numerator. Upon making this assumption the expression for the error variance of w as retrieved from (3.37) can be shown to be:

$$(3.38) \quad \sigma_w^2 = \frac{\sigma_{v_t}^2 + v_T^2 \sigma_{v_y}^2 + \sigma_v^2 \left(\frac{\partial v_T}{\partial y} \right)^2 + \sigma_v^2 \sigma_{v_y}^2 + w_T^2 \sigma_{v_z}^2}{\left(\frac{\partial v_T}{\partial z} \right)^2 + (1 + r^2) \sigma_{v_z}^2},$$

where r is the correlation coefficient between w_E and $\partial v_E / \partial z$. In (3.38), subscripts in lower-case letters indicate the relevant dependent variable and subscripts of subscripts indicate the 1st derivative of the first subscripted dependent variable (v or w in this case) with respect to the second subscripted independent variable (y , z , or t). For example, $\sigma_{v_r}^2$ represents the error variance in the temporal derivative of v . The variances of this and other derivatives can be parameterized in terms of the variance of v (see Appendix D). The value of the correlation coefficient r in (3.38) will vary within the analysis domain. However, the error variance in (3.38) is bounded (maximized) by the case $r=0$. Following this simplification and parameterization of derivatives, (3.38) reduces to:

$$(3.39) \quad \sigma_w^2 = \sigma_v^2 \left[\frac{\frac{1}{2(\Delta t)^2} + \frac{v_r^2}{2(\Delta y)^2} + \left(\frac{\partial v_r}{\partial y} \right)^2 + \frac{\sigma_v^2}{2(\Delta y)^2} + \frac{w_r^2}{2(\Delta z)^2}}{\left(\frac{\partial v_r}{\partial z} \right)^2 + \frac{\sigma_v^2}{2(\Delta z)^2}} \right],$$

where Δt is the time span between volume scans, and Δy and Δz are the Cartesian grid spacings. The term in square brackets on the right hand side of (3.39) can be thought of as a ‘variance amplifier’, meaning that it controls the error variance of the retrieved vertical velocity relative to the error variance of the input horizontal velocity. Keep in mind that errors due to approximation of continuous derivatives by finite differences have been neglected in the derivation of (3.39). As mentioned previously, this type of error may be the dominant error for the temporal derivative.

Equation (3.39) can be used to construct contour plots of the expected error variance in w as a function of the free parameters v_T , $\partial v_T / \partial y$, $\partial v_T / \partial z$, σ_v , Δt , Δy , and Δz . Note that w_T is not classified as a free parameter since it is fully determined by the true values of v and its derivatives via (3.36). For a given analysis σ_v , Δt , Δy , and Δz will be fixed quantities, leaving only v_T , $\partial v_T / \partial y$, and $\partial v_T / \partial z$ as truly free parameters within that analysis. Figures (3.3) - (3.5) explore the relationship between these three free parameters and the error standard deviation of w . The most notable feature in these figures is the rapid increase in the error standard deviation of w as the true vertical shear becomes small. Also, as suggested earlier, the errors in w are smallest when the magnitude of the vertical shear exceeds that of the horizontal shear.

This concludes the analysis of the expected sensitivities of solutions of the vorticity equation to errors in its forcing. Unfortunately, the restricted nature of the analysis prohibits the development of firm guidelines for safe application of the vorticity equation in all situations. However, the general conditions under which we can expect the most useful information from the vorticity equation have been established. Building upon this information, it is possible to develop methods for incorporating the vorticity equation into dual-Doppler analyses that will take advantage of the equation's inherent strengths, while remaining robust against its weaknesses.

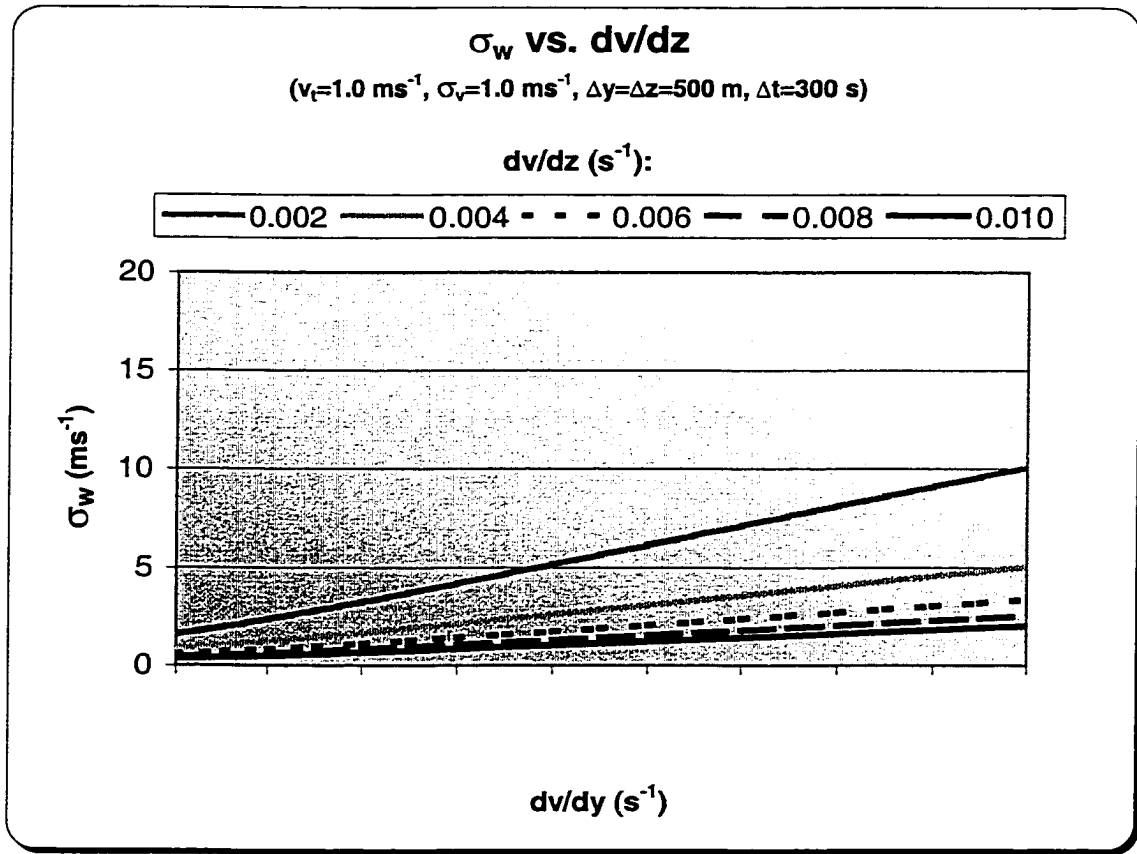


Figure 3.3: Standard deviation (m/s) for the retrieved vertical velocity as determined by eq. (3.39) with $v_T=1 \text{ m/s}$, $\Delta y=\Delta z=500 \text{ m}$, $\Delta t=300 \text{ s}$, and $\sigma_v=1.0 \text{ m/s}$ as a function of the vertical shear magnitude.

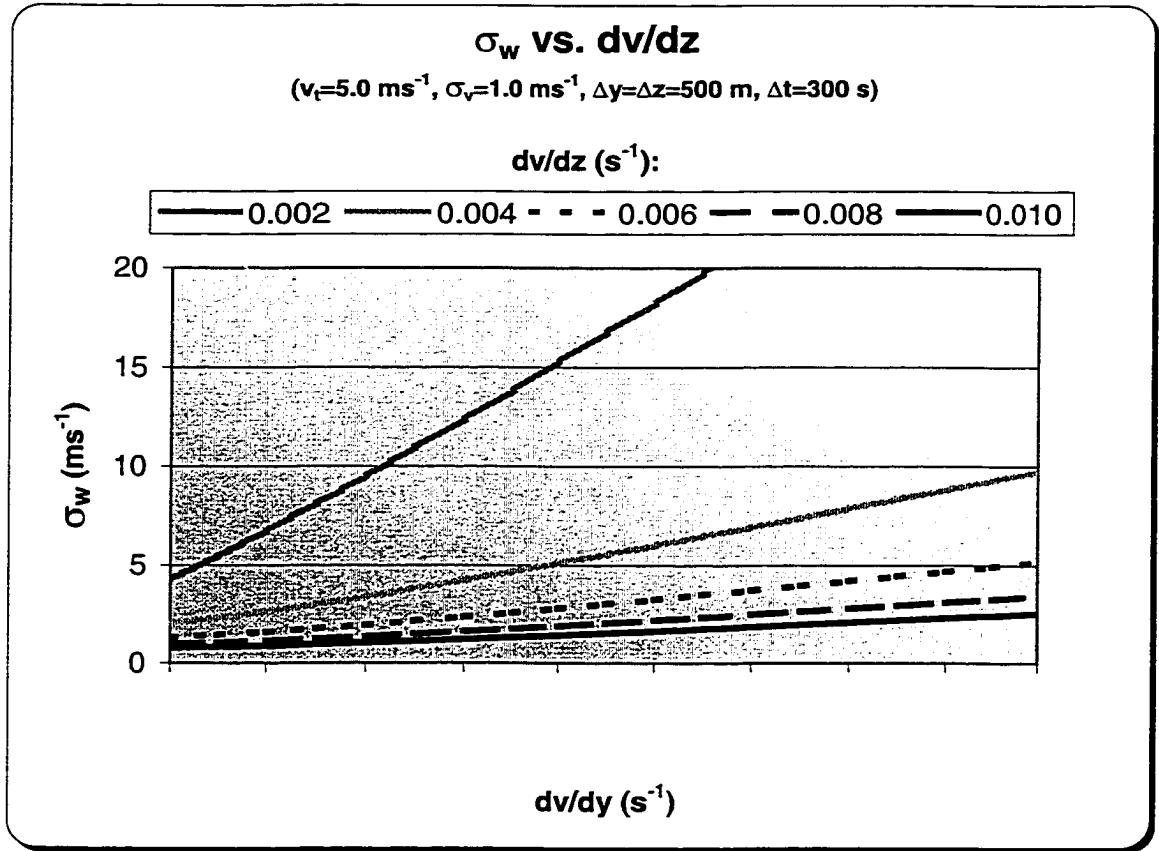


Figure 3.4: Standard deviation (m/s) for the retrieved vertical velocity as determined by eq. (3.39) with $v_T=5 \text{ m/s}$, $\Delta y=\Delta z=500 \text{ m}$, $\Delta t=300 \text{ s}$, and $\sigma_v=1.0 \text{ m/s}$ as a function of the vertical shear magnitude.

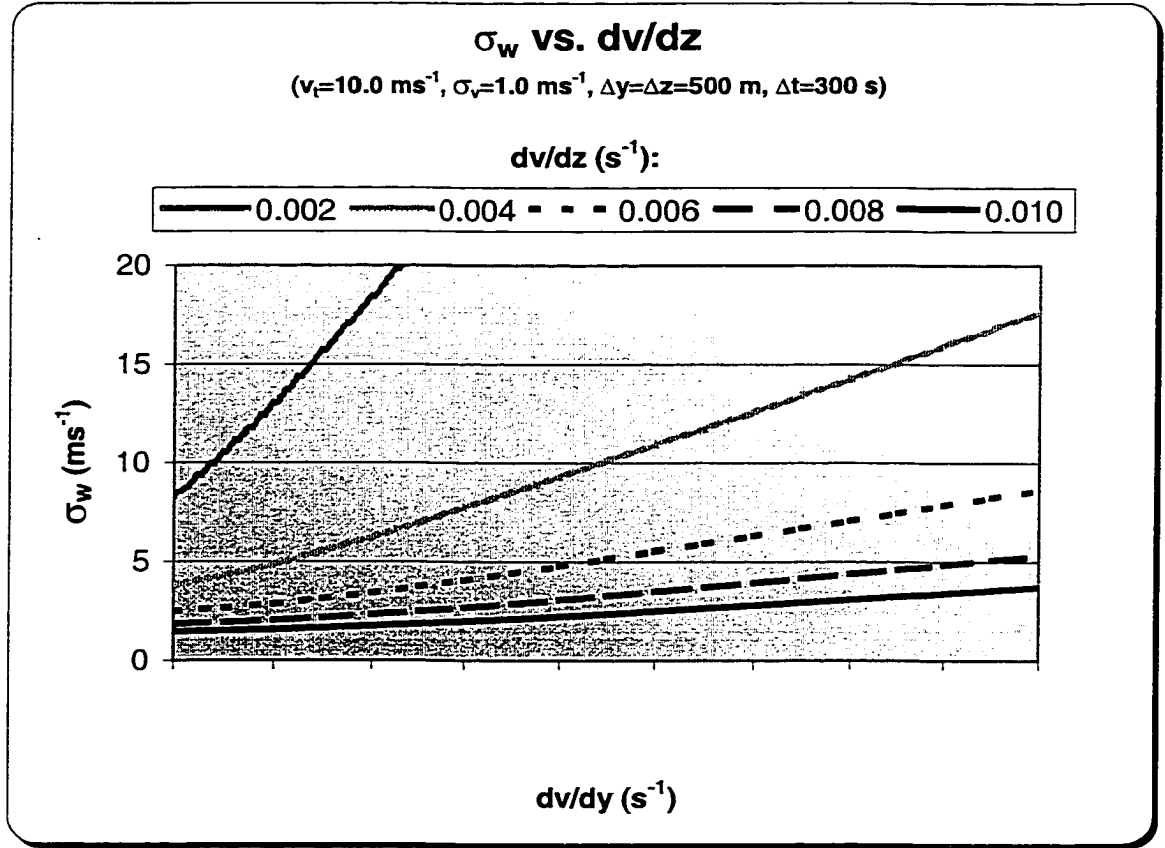


Figure 3.5: Standard deviation (m/s) for the retrieved vertical velocity as determined by eq. (3.39) with $v_T=10 \text{ m/s}$, $\Delta y=\Delta z=500 \text{ m}$, $\Delta t=300 \text{ s}$, and $\sigma_v=1.0 \text{ m/s}$ as a function of the vertical shear magnitude.

Chapter 4: New Formulations of Dual-Doppler Analysis using the Vertical Vorticity Equation

Section A: General theory

There are a number of different ways in which the vertical vorticity equation could be incorporated as a flow constraint in dual-Doppler analyses. The general goal of any dual-Doppler analysis is to find a three-dimensional wind field that is in the best possible agreement with the radial wind observations from both Doppler radars as well as any desired flow constraints. However, when applying these constraints using the calculus of variations (Courant and Hilbert, 1953), the tractability of the Euler-Lagrange equations for minimizing a cost functional is dependent upon both the number of components of the wind allowed to vary and the number of constraints invoked within the cost functional. If we seek to minimize a functional F (where F is a function of u , v , and w , as well as the 1st and 2nd derivatives of u , v , and w with respect to the independent variables x , y , z , and t) while allowing each of u , v , and w to vary during the minimization process, the resulting cost functional J is of the form:

$$(4.1) \quad J = \int_{x,y,z,t} F(x, y, z, t, \lambda_1, \lambda_2, \dots, \lambda_n, u, u', u'', v, v', v'', w, w', w'') dx' dy' dz' dt',$$

where the ' and '' symbols are used in a most general sense and collectively represent the 1st and 2nd partial derivative(s) of the given dependent variable with respect to any or all

of the independent variables x, y, z , and t , and λ_i are Lagrange multipliers associated with each of the n strong constraints contained in F . All of the cost functionals developed in this work will take on a form similar to (4.1). Taking the first variation of (4.1) with respect to u, v, w , and each of the Lagrange multipliers λ_i yields $(3 + n)$ equations of the form (the variation of (4.1) with respect to w is shown here):

$$\begin{aligned}
 \delta_w J &= \int_{x, y, z, t} (\delta_w F) dx' dy' dz' dt' \\
 (4.2) \quad &= \int_{x, y, z, t} \left(\frac{\partial F}{\partial w} \delta w + \frac{\partial F}{\partial w_x} \delta w_x + \frac{\partial F}{\partial w_y} \delta w_y + \frac{\partial F}{\partial w_z} \delta w_z + \frac{\partial F}{\partial w_t} \delta w_t \right. \\
 &\quad + \frac{\partial F}{\partial w_{xx}} \delta w_{xx} + \frac{\partial F}{\partial w_{yy}} \delta w_{yy} + \frac{\partial F}{\partial w_{zz}} \delta w_{zz} + \frac{\partial F}{\partial w_{tt}} \delta w_{tt} + \frac{\partial F}{\partial w_{xy}} \delta w_{xy} \\
 &\quad \left. + \frac{\partial F}{\partial w_{xz}} \delta w_{xz} + \frac{\partial F}{\partial w_{xt}} \delta w_{xt} + \frac{\partial F}{\partial w_{yz}} \delta w_{yz} + \frac{\partial F}{\partial w_{yt}} \delta w_{yt} + \frac{\partial F}{\partial w_{zx}} \delta w_{zx} \right) dx' dy' dz' dt'
 \end{aligned}$$

Equation (4.2) must be integrated by parts to convert the variations of the derivatives of w (e.g. $\delta w_x, \delta w_{xy}$, etc.) into variations of w (δw), and the result must be set equal to zero in order to locate the minimum of the cost functional (similar to the manner in which setting a derivative of a function equal to zero locates an extremum of that function). This process leads to general Euler-Lagrange equations of this generic functional (once again, only the Euler-Lagrange equation resulting from taking the 1st variation of J with respect to w is shown here):

$$\begin{aligned}
(4.3) \quad \delta_w J = 0 = & \int_{x,y,z,t} \left[\frac{\partial F}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial w_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial w_y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial F}{\partial w_z} \right) - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial w_t} \right) \right. \\
& + \frac{\partial^2}{\partial x^2} \left(\frac{\partial F}{\partial w_{xx}} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial F}{\partial w_{yy}} \right) + \frac{\partial^2}{\partial z^2} \left(\frac{\partial F}{\partial w_{zz}} \right) + \frac{\partial^2}{\partial t^2} \left(\frac{\partial F}{\partial w_{tt}} \right) \\
& + \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial F}{\partial w_{xy}} \right) + \frac{\partial^2}{\partial x \partial z} \left(\frac{\partial F}{\partial w_{xz}} \right) + \frac{\partial^2}{\partial x \partial t} \left(\frac{\partial F}{\partial w_{xt}} \right) \\
& \left. + \frac{\partial^2}{\partial y \partial z} \left(\frac{\partial F}{\partial w_{yz}} \right) + \frac{\partial^2}{\partial y \partial t} \left(\frac{\partial F}{\partial w_{yt}} \right) + \frac{\partial^2}{\partial z \partial t} \left(\frac{\partial F}{\partial w_{zt}} \right) \right] \delta w \, dx' dy' dz' dt' \\
& + \text{Boundary Integrals}
\end{aligned}$$

where the 'Boundary Integrals' term is a result of the integration by parts and represents surface integrals which must be satisfied on the boundaries of the data volume. The boundary integrals are given by:

(4.4)

Boundary Integrals =

$$\begin{aligned}
& \int_{y,z,t} \left\{ \left[\frac{\partial F}{\partial w_x} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial w_{xx}} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial w_{xy}} \right) - \frac{\partial}{\partial z} \left(\frac{\partial F}{\partial w_{xz}} \right) - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial w_{xt}} \right) \right] \delta w + \frac{\partial F}{\partial w_{xx}} \delta w_x \right\} dy' dz' dt' + \\
& \int_{x,z,t} \left\{ \left[\frac{\partial F}{\partial w_y} - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial w_{yy}} \right) - \frac{\partial}{\partial z} \left(\frac{\partial F}{\partial w_{yz}} \right) - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial w_{yt}} \right) \right] \delta w + \frac{\partial F}{\partial w_{xy}} \delta w_x + \frac{\partial F}{\partial w_{yy}} \delta w_y \right\} dx' dz' dt' + \\
& \int_{x,y,t} \left\{ \left[\frac{\partial F}{\partial w_z} - \frac{\partial}{\partial z} \left(\frac{\partial F}{\partial w_{zz}} \right) - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial w_{zt}} \right) \right] \delta w + \frac{\partial F}{\partial w_{xz}} \delta w_x + \frac{\partial F}{\partial w_{yz}} \delta w_y + \frac{\partial F}{\partial w_{zz}} \delta w_z \right\} dx' dy' dt' + \\
& \int_{x,y,z} \left\{ \left[\frac{\partial F}{\partial w_t} - \frac{\partial}{\partial t} \left(\frac{\partial F}{\partial w_{tt}} \right) \right] \delta w + \frac{\partial F}{\partial w_{xt}} \delta w_x + \frac{\partial F}{\partial w_{yt}} \delta w_y + \frac{\partial F}{\partial w_{zt}} \delta w_z + \frac{\partial F}{\partial w_{tt}} \delta w_t \right\} dx' dy' dz'
\end{aligned}$$

In a similar manner, taking variations of J with respect to u , v , and λ_1 through λ_n results in a closed system of $(3 + n)$ equations in $(3 + n)$ unknowns.

Fortunately, the horizontal wind fields from dual-Doppler analyses can be obtained with reasonable accuracy through application of simple geometric relationships, especially at low elevation angles. The primary goal of this work is to improve retrievals of the vertical velocity in dual-Doppler analyses. Since the value of the anelastic vertical vorticity equation in dual-Doppler analyses has not yet been established, and the primary concern here is improvement of the vertical velocity retrieval, u and v will be regarded as known parameters and only w will be allowed to vary within the cost functionals presented within this work. This simplification could be lifted in future work provided the vertical vorticity equation can be shown to be a valuable flow constraint.

Section B: Using the vorticity equation as a weak constraint with mass continuity as a strong constraint

Recall that specification of the vertical velocity boundary condition(s) is a prominent source of vertical velocity errors when mass continuity is used in dual-Doppler analyses. Conveniently, decomposing w into a form consistent with the anelastic mass continuity equation allows the vorticity equation to be used in specifying an appropriate boundary condition field. This boundary condition field will be optimal in the sense that it will minimize the squared error in the anelastic vertical vorticity equation over the entire analysis domain (when considered in combination

with a first-guess horizontal wind field). To this end, note that if anelastic mass continuity is imposed as a strong constraint, the vertical velocity can be parsed as:

$$\rho w(x, y, z) = \rho_0 w_0(x, y) - \int_{z_0}^z \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz' \quad (4.5)$$

or :

$$w^*(x, y, z) = w_0^*(x, y) + w_k^*(x, y, z)$$

where $\rho = \rho(z)$ is the mean density profile, $z_0 = z_0(x, y)$ is the level where the boundary condition is specified, and $w_0 = w(z_0)$ is the unknown vertical velocity boundary condition field. The presentation of this technique is simplified by working with the vertical mass flux w^* instead of the vertical velocity w ($w^* = \rho w$). The quantity w_k^* represents that portion of the total vertical mass flux field known from horizontal divergence. Equation (4.5) is valid regardless of whether z_0 is taken to be the bottom, top, or somewhere in the middle of the analysis domain.

In this approach minimization of the cost functional:

$$J = \int_V \alpha^2 \left[\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + \frac{w^*}{\rho} \frac{\partial \zeta}{\partial z} + (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{1}{\rho} \left(\frac{\partial w^*}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w^*}{\partial y} \frac{\partial u}{\partial z} \right) \right]^2 dV' \quad (4.6)$$

is sought over the analysis domain V . Provision for the weighting function α^2 in the cost functional could lead to improved results through preferential weighting of areas expected to have smaller retrieval errors (or, as an example at the other extreme, giving little or no weight to regions where hole-filling techniques have been applied).

Taking the first variation of this cost functional with respect to w_0^* , and setting the result equal to zero yields:

$$(4.7) \quad \int_V 2\alpha^2 VE \left[\delta w_0^* \left(\frac{1}{\rho} \frac{\partial \zeta}{\partial z} \right) + \frac{\partial(\delta w_0^*)}{\partial x} \left(\frac{1}{\rho} \frac{\partial v}{\partial z} \right) - \frac{\partial(\delta w_0^*)}{\partial y} \left(\frac{1}{\rho} \frac{\partial u}{\partial z} \right) \right] dV' = 0,$$

where:

$$(4.8) \quad \begin{aligned} VE &= VE_k + w_0^* \left(\frac{1}{\rho} \frac{\partial \zeta}{\partial z} \right) + \frac{\partial w_0^*}{\partial x} \left(\frac{1}{\rho} \frac{\partial v}{\partial z} \right) - \frac{\partial w_0^*}{\partial y} \left(\frac{1}{\rho} \frac{\partial u}{\partial z} \right) \\ VE_k &= \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w_k^* \left(\frac{1}{\rho} \frac{\partial \zeta}{\partial z} \right) + (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{1}{\rho} \left(\frac{\partial w_k^*}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w_k^*}{\partial y} \frac{\partial u}{\partial z} \right) \end{aligned}$$

.

Integrating (4.7) by parts yields:

$$\begin{aligned}
(4.9) \quad & \int_V \left[\alpha^2 VE \frac{1}{\rho} \frac{\partial \zeta}{\partial z} - \frac{\partial}{\partial x} \left(\alpha^2 VE \frac{1}{\rho} \frac{\partial v}{\partial z} \right) + \frac{\partial}{\partial y} \left(\alpha^2 VE \frac{1}{\rho} \frac{\partial u}{\partial z} \right) \right] \delta w_0^* dV' \\
& + \int_{y,z} \alpha^2 VE \frac{1}{\rho} \frac{\partial v}{\partial z} \delta w_0^* dy' dz' - \int_{x,z} \alpha^2 VE \frac{1}{\rho} \frac{\partial u}{\partial z} \delta w_0^* dx' dz' = 0
\end{aligned}$$

Therefore, since δw_0^* is arbitrary, minimization of (4.6) requires that each of the three integrals of (4.9) be independently equal to zero. Concentrating first on the volume integral of (4.9), if we neglect the trivial solution of $\delta w_0^* = 0$ and note that $w_0 = w_0(x, y)$ is not a function of z , we can deduce that:

$$(4.10) \quad \int_z \left[\alpha^2 VE \frac{1}{\rho} \frac{\partial \zeta}{\partial z} - \frac{\partial}{\partial x} \left(\alpha^2 VE \frac{1}{\rho} \frac{\partial v}{\partial z} \right) + \frac{\partial}{\partial y} \left(\alpha^2 VE \frac{1}{\rho} \frac{\partial u}{\partial z} \right) \right] dz' = 0$$

must be satisfied in order for the volume integral to be equal zero over an arbitrary domain. Manipulation of (4.10) yields the Euler-Lagrange equation for the vertical mass flux boundary field:

$$(4.11) \quad a \frac{\partial^2 w_0^*}{\partial x^2} + 2b \frac{\partial^2 w_0^*}{\partial x \partial y} + c \frac{\partial^2 w_0^*}{\partial y^2} + d \frac{\partial w_0^*}{\partial x} + e \frac{\partial w_0^*}{\partial y} + f w_0^* + g = 0,$$

where:

$$a = - \int_z \frac{\alpha^2}{\rho^2} \left(\frac{\partial v}{\partial z} \right)^2 dz'$$

$$\begin{aligned}
b &= \int_z \frac{\alpha^2}{\rho^2} \left(\frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right) dz' \\
c &= - \int_z \frac{\alpha^2}{\rho^2} \left(\frac{\partial u}{\partial z} \right)^2 dz' \\
d &= \int_z \frac{\alpha^2}{\rho^2} \left(- \frac{\partial v}{\partial z} \frac{\partial \zeta}{\partial z} - \frac{\partial v}{\partial z} \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial u}{\partial z} \frac{\partial^2 v}{\partial y \partial z} + \sigma \frac{\partial v}{\partial z} \right) dz' \\
e &= \int_z \frac{\alpha^2}{\rho^2} \left(\frac{\partial u}{\partial z} \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial^2 u}{\partial x \partial z} - \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial y \partial z} - \sigma \frac{\partial u}{\partial z} \right) dz' \\
f &= \int_z \frac{\alpha^2}{\rho^2} \left(- \frac{\partial v}{\partial z} \frac{\partial^2 \zeta}{\partial x \partial z} + \frac{\partial u}{\partial z} \frac{\partial^2 \zeta}{\partial y \partial z} + \sigma \frac{\partial \zeta}{\partial z} \right) dz' \\
g &= \int_z \frac{\alpha^2}{\rho} \left(- \frac{\partial v}{\partial z} \frac{\partial}{\partial x} (VE_k) + \frac{\partial u}{\partial z} \frac{\partial}{\partial y} (VE_k) + \sigma (VE_k) \right) dz' \\
(4.12) \quad \sigma &= \frac{1}{\alpha^2} \left(\frac{\partial u}{\partial z} \frac{\partial \alpha^2}{\partial y} - \frac{\partial v}{\partial z} \frac{\partial \alpha^2}{\partial x} \right).
\end{aligned}$$

Equation (4.11) is a linear second order PDE for w_0^* . It can be shown to be elliptic through application of Schwarz's inequality (as in Shapiro and Mewes, 1999). The discriminant of (4.11) is given by:

$$(4.13) \quad b^2 - ac = \left(\int_z \frac{\alpha^2}{\rho^2} \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} dz' \right)^2 - \int_z \frac{\alpha^2}{\rho^2} \left(\frac{\partial v}{\partial z} \right)^2 dz' \int_z \frac{\alpha^2}{\rho^2} \left(\frac{\partial u}{\partial z} \right)^2 dz'.$$

Application of Schwarz's inequality gives:

$$(4.14) \quad \left(\int_z \frac{\alpha^2}{\rho^2} \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} dz' \right)^2 \leq \int_z \frac{\alpha^2}{\rho^2} \left(\frac{\partial u}{\partial z} \right)^2 dz' \int_z \frac{\alpha^2}{\rho^2} \left(\frac{\partial v}{\partial z} \right)^2 dz',$$

so that:

$$(4.15) \quad b^2 - ac \leq 0$$

and the equation is elliptic with the exception of vertical columns where the equality in (4.15) holds true (in which case the problem is parabolic). This occurs if $\left| \frac{\partial u}{\partial z} \right| = \left| \frac{\partial v}{\partial z} \right|$ at each level within the vertical integration depth or if $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$ are constant throughout the depth of integration. The probability of this occurring in any given column is minimal, especially if a sufficient number of levels of data are included in the integration.

Minimization of (4.6) also requires that the two surface (boundary) integrals of (4.9) be set equal to zero as well. This yields the natural boundary conditions:

$$(4.16) \quad -a \frac{\partial w_0^*}{\partial x} - b \frac{\partial w_0^*}{\partial y} + w_0^* \int_z \frac{\alpha^2}{\rho^2} \frac{\partial v}{\partial z} \frac{\partial \zeta}{\partial z} dz' + \int_z \frac{\alpha^2}{\rho} \frac{\partial v}{\partial z} (v E_k) dz' = 0$$

on the east and west boundaries of the data domain, and:

$$(4.17) \quad b \frac{\partial w_0^*}{\partial x} + c \frac{\partial w_0^*}{\partial y} + w_0^* \int_z \frac{\alpha^2}{\rho^2} \frac{\partial u}{\partial z} \frac{\partial \zeta}{\partial z} dz' + \int_z \frac{\alpha^2}{\rho} \frac{\partial u}{\partial z} (VE_k) dz' = 0$$

on the north and south boundaries. Equations (4.16) and (4.17) are Robin boundary conditions. Alternatively, if w_0^* is known on the lateral boundaries it could be specified as a boundary condition, in which case $\delta w_0^* = 0$ on the boundaries and the surface integrals of (4.9) are automatically equal to zero. The vertical limits of integration in (4.12), (4.16), and (4.17) can vary horizontally, which makes the method well-suited for the irregular upper and lower data boundaries encountered with real data.

This method can be applied to either an upward or downward integration technique applying mass continuity as a strong constraint (such as the method of BR77, or methods U or D of MB98). As indicated earlier, the equation set consisting of (4.11)-(4.12) and (4.16)-(4.17) is valid regardless of whether the vertical velocity boundary condition field is chosen to be valid at the bottom, top, or at some arbitrary height in middle of the data volume.

There are several advantages to applying the vorticity equation in this manner. Using the vorticity equation to obtain a vertical velocity boundary condition allows us to take advantage of the error filtering effects of the vertically integrated coefficients in (4.12), (4.16), and (4.17). The problems associated with misplaced characteristic curves and ‘orphaned regions’ mentioned in Chapter 3 are avoided altogether ((4.11) in an elliptic equation, which has no characteristic curves). Further, minimization of the cost functional is naturally dominated by regions which exhibit large vertical shear

and/or large magnitudes of $\partial\zeta/\partial z$, since the cost functional is punished most by an inappropriate w field in these regions. Since these regions naturally play the most significant role in determining the appropriate boundary condition field, it is not necessary to have significant shear and/or $\partial\zeta/\partial z$ present throughout the entire depth of a vertical column.

It should be noted that, in general, the variable sign of the coefficient f in (4.12) prevents solution uniqueness from being guaranteed except for “sufficiently small areas” that depend upon the form of that coefficient, that is, are situation dependent (Petrovsky, 1991). This situation applies to all vorticity based methods presented within this work. Fortunately, in practice, the solutions appear to be unique in all experiments (see Chapter 5). The solution fields presented in Chapter 5 have been tested for uniqueness by supplying a randomly changing initial field to the elliptic solver, then comparing the solutions obtained from each initial field.

This technique can easily be adapted into the BR77 or other iterative techniques applying mass continuity as a strong constraint. To proceed, a first-guess estimate of the horizontal wind field must first be constructed. The ‘known’ portion of the vertical mass flux field, $w_k^*(x,y,z)$, can then be deduced from this first-guess field using (4.5). This can be done, for instance, by taking a provisional estimate $w_0^*(x,y)=0$ and applying the BR77 technique (solving for w^* instead of w). Equation (4.11) can then be used to solve for the optimal $w_0^*(x,y)$, after which the total three-dimensional wind field can be constructed by reapplying the BR77 technique with the new boundary condition field and dividing through by ρ . In principle, the technique is

insensitive to the level at which the boundary condition is applied, since noise from accumulating divergence errors will be distributed to areas where disagreement with the vorticity equation constraint is minimized. However, imperfect filters applied to u and v dampen the response of the technique to $2\Delta x$ scales, limiting the technique's ability to distribute noise. Therefore, in the absence of a perfect filtering scheme, downward integration is often still preferable (although to a lesser degree than with traditional methods of boundary condition specification).

Section C: Using both the vorticity and mass continuity equations as weak constraints

Recall from Chapter 2 that the vertical velocity fields in dual-Doppler analyses often suffer from two distinct problems. The first problem, specification of a vertical velocity boundary condition field, was at least partially addressed in the previous section. However, the technique of the Section B does little to counter the problem of accumulating divergence errors.

As indicated in Chapter 2, density stratification dictates that the problem of accumulating divergence errors is lessened if a downward integration technique is applied (at least when viewed strictly in terms of w and not vertical mass flux). Since the technique outlined in Section B can provide an upper boundary condition just as easily as a lower boundary condition, the technique can take advantage of this property. In fact, given a perfect filtering scheme (i.e., one that leaves scales of $4\Delta x$ and larger in the horizontal wind fields unscathed) the technique of Section B should (in principle) distribute noise in the vertical mass flux field in such a way that it

minimizes disagreement with the vorticity equation constraint. Since the anelastic mass continuity equation relates u and v to w (not ρw), it is likely that this would usually result in a distribution of noise to low levels *regardless* of which direction the mass continuity equation is being integrated. Therefore, the technique of Section B would appear to have a similar desirable quality to that of CT83, i.e. the divergence related noise can be distributed to low levels (where it does the least measurable damage to w) even if the mass continuity equation is integrated upward. Unfortunately, the imperfections of the filtering schemes applied to the horizontal wind field cause features of scales $4\Delta x$ and larger to be damped somewhat from their true magnitudes. Since the $2\Delta x$ and $3\Delta x$ scales in the vertical velocity boundary condition (which are capable of redistributing the bulk of the divergence related noise) are created entirely through wave interactions on these larger, damped scales, the technique of Section B is unable to make sufficient small scale adjustments to the boundary field to allow the technique to be truly direction independent. Therefore, in the absence of a perfect filtering scheme, downward integration is still likely to be preferable to upward integration with the technique of Section B for controlling divergence related noise in the vertical velocity field.

Although downward integration is preferable to upward integration for control of accumulating divergence related noise, it does not alleviate the noise problem entirely. Further, MB98 have pointed out that the downward integration technique has little tolerance for systematic (bias or trend) errors in the divergence fields. Therefore, another common approach has been to constrain the vertical velocity field to meet prescribed boundary conditions on both upper and lower data boundaries. MB98 have

shown that this technique is resistant to both random errors and bias errors in the divergence field. However, to utilize this technique we must trade the problem of accumulating divergence errors for an equally difficult problem of determining two (as opposed to one) vertical velocity boundary conditions. Since typical radar scan geometries do not provide data from ground up through storm top, this has often meant that at least one of the boundary conditions must be estimated using some ad-hoc technique. Fortunately, the vorticity equation appears to be very well suited to address this problem.

One technique for constraining the vertical velocity field using two boundary conditions is to apply mass continuity as a weak constraint for the first step in a two step process. This can be followed by a second correction step to the horizontal wind fields which ensures that mass continuity is met exactly in the final analysis (the correction step generally adjusts the horizontal wind fields to offset the accumulation of divergence errors). If the anelastic mass continuity equation is applied as a weak constraint in a cost functional, it is easy to show that the resulting Euler-Lagrange equation is the anelastic mass continuity equation differentiated with respect to z (or p , if working in pressure coordinates). This equation can be integrated twice, yielding two constants of integration (per column) which can be determined through the application of boundary conditions. If general expressions are substituted for these boundary conditions we find that the vertical velocity field can be parsed into a “weak-constraint form” in terms of upper and lower boundary fields (w_T and w_B):

$$\begin{aligned}
(4.18) \quad w(x, y, z) &= \frac{\rho_B}{\rho} \left(\frac{z_T - z}{z_T - z_B} \right) w_B(x, y) + \frac{\rho_T}{\rho} \left(\frac{z - z_B}{z_T - z_B} \right) w_T(x, y) + \\
&\quad \frac{1}{\rho} \left(\frac{z - z_B}{z_T - z_B} \right) \int_{z_B}^{z_T} \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz' - \frac{1}{\rho} \int_{z_B}^z \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz' , \\
&= F_B(x, y, z) w_B(x, y) + F_T(x, y, z) w_T(x, y) + w_k(x, y, z)
\end{aligned}$$

where $\rho = \rho(z)$ is the mean density profile, w_B and ρ_B (w_T and ρ_T) are the vertical velocity and mean density at a lower (upper) boundary located at $z_B(x, y)$ ($z_T(x, y)$), and w_k represents that portion of the vertical velocity known from horizontal divergence.

Substituting (4.18) into the cost-functional:

$$\begin{aligned}
(4.19) \quad J &= \int_V \left\{ \alpha^2 \left[\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} + (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right) \right]^2 \right. \\
&\quad \left. + \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right)^2 \right\} dV' ,
\end{aligned}$$

and setting the first variations of J with respect to w_B and w_T equal to zero yields a set of two equations similar to (4.7) (the subscripts B and T here correspond to separate but similar equations):

$$\begin{aligned}
(4.20) \quad \int_V \left\{ 2\alpha^2 VE \left[\delta w_{B,T} \left(F_{B,T} \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial F_{B,T}}{\partial x} - \frac{\partial u}{\partial z} \frac{\partial F_{B,T}}{\partial y} \right) + \frac{\partial(\delta w_{B,T})}{\partial x} \left(F_{B,T} \frac{\partial v}{\partial z} \right) \right. \right. \\
\left. \left. - \frac{\partial(\delta w_{B,T})}{\partial y} \left(F_{B,T} \frac{\partial u}{\partial z} \right) \right] + 2MC \left[\delta w_{B,T} \frac{\partial(\rho F_{B,T})}{\partial z} \right] \right\} dV' = 0
\end{aligned}$$

where:

$$(4.21) \quad VE = \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + (w_k + F_B w_B + F_T w_T) \frac{\partial \zeta}{\partial z} + (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ + \frac{\partial (w_k + F_B w_B + F_T w_T)}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial (w_k + F_B w_B + F_T w_T)}{\partial y} \frac{\partial u}{\partial z}$$

Equations (4.20) can be integrated by parts (in a similar fashion to (4.7)), yielding:

(4.22)

$$\iint_{x,y} \left\{ \int_z \left[\alpha^2 VE \left(F_{B,T} \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial F_{B,T}}{\partial x} - \frac{\partial u}{\partial z} \frac{\partial F_{B,T}}{\partial y} \right) \right. \right. \\ \left. \left. - \frac{\partial}{\partial x} \left(\alpha^2 (VE) F_{B,T} \frac{\partial v}{\partial z} \right) + \frac{\partial}{\partial y} \left(\alpha^2 (VE) F_{B,T} \frac{\partial u}{\partial z} \right) + MC \frac{\partial (\rho F_{B,T})}{\partial z} \right] \delta w_{B,T} dz' \right\} dy' dx' \\ + \int_{y,z} \left(\alpha^2 (VE) F_{B,T} \frac{\partial v}{\partial z} \right) \delta w_{B,T} dy' dz' - \int_{x,z} \left(\alpha^2 (VE) F_{B,T} \frac{\partial u}{\partial z} \right) \delta w_{B,T} dx' dz' = 0$$

Since $\delta w_{B,T}$ is arbitrary, minimization of the equations represented in (4.22) requires that each of the three integrals of each equation represented in (4.22) be independently equal to zero. Keeping in mind that w_B and w_T are not functions of z , (4.22) yields the system of Euler-Lagrange equations:

$$\begin{aligned}
& a_B \frac{\partial^2 w_B}{\partial x^2} + 2b_B \frac{\partial^2 w_B}{\partial x \partial y} + c_B \frac{\partial^2 w_B}{\partial y^2} + d_B \frac{\partial w_B}{\partial x} + e_B \frac{\partial w_B}{\partial y} + f_B w_B + g_B = 0, \\
(4.23) \quad & a_T \frac{\partial^2 w_T}{\partial x^2} + 2b_T \frac{\partial^2 w_T}{\partial x \partial y} + c_T \frac{\partial^2 w_T}{\partial y^2} + d_T \frac{\partial w_T}{\partial x} + e_T \frac{\partial w_T}{\partial y} + f_T w_T + g_T = 0,
\end{aligned}$$

where:

$$a_{B,T} = - \int_z \alpha^2 F_{B,T}^2 \left(\frac{\partial v}{\partial z} \right)^2 dz'$$

$$b_{B,T} = \int_z \alpha^2 F_{B,T}^2 \left(\frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right) dz'$$

$$c_{B,T} = - \int_z \alpha^2 F_{B,T}^2 \left(\frac{\partial u}{\partial z} \right)^2 dz'$$

$$\begin{aligned}
d_{B,T} = \int_z \alpha^2 F_{B,T} \left[F_{B,T} \left(- \frac{\partial v}{\partial z} \frac{\partial \zeta}{\partial z} - \frac{\partial v}{\partial z} \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial u}{\partial z} \frac{\partial^2 v}{\partial y \partial z} + \sigma \frac{\partial v}{\partial z} \right) + \right. \\
\left. 2 \left(\frac{\partial F_{B,T}}{\partial y} \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} - \frac{\partial F_{B,T}}{\partial x} \frac{\partial v}{\partial z} \frac{\partial v}{\partial z} \right) \right] dz'
\end{aligned}$$

$$\begin{aligned}
e_{B,T} = \int_z \alpha^2 F_{B,T} \left[F_{B,T} \left(\frac{\partial u}{\partial z} \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial^2 u}{\partial x \partial z} - \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial y \partial z} - \sigma \frac{\partial u}{\partial z} \right) - \right. \\
\left. 2 \left(\frac{\partial F_{B,T}}{\partial y} \frac{\partial u}{\partial z} \frac{\partial u}{\partial z} - \frac{\partial F_{B,T}}{\partial x} \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right) \right] dz'
\end{aligned}$$

$$\begin{aligned}
f_{B,T} &= \int_{\mathbb{Z}} \alpha^2 F_{B,T} \left[F_{B,T} \left(\frac{\partial u}{\partial z} \frac{\partial^2 \zeta}{\partial y \partial z} - \frac{\partial v}{\partial z} \frac{\partial^2 \zeta}{\partial x \partial z} + \sigma \frac{\partial \zeta}{\partial z} \right) + \left(\frac{\partial(\rho F_{B,T})}{\partial z} \right)^2 \right. \\
&\quad \left. - \frac{\partial^2 F_{B,T}}{\partial x^2} \left(\frac{\partial v}{\partial z} \right)^2 + 2 \frac{\partial^2 F_{B,T}}{\partial x \partial y} \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} - \frac{\partial^2 F_{B,T}}{\partial y^2} \left(\frac{\partial u}{\partial z} \right)^2 \right. \\
&\quad \left. + \frac{\partial F_{B,T}}{\partial x} \left(-\frac{\partial v}{\partial z} \frac{\partial \zeta}{\partial z} - \frac{\partial v}{\partial z} \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial u}{\partial z} \frac{\partial^2 v}{\partial y \partial z} + \sigma \frac{\partial v}{\partial z} \right) \right. \\
&\quad \left. + \frac{\partial F_{B,T}}{\partial y} \left(\frac{\partial u}{\partial z} \frac{\partial \zeta}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial^2 u}{\partial x \partial z} - \frac{\partial u}{\partial z} \frac{\partial^2 u}{\partial y \partial z} - \sigma \frac{\partial u}{\partial z} \right) \right] dz' \\
g_{B,T} &= \int_{\mathbb{Z}} \left[\alpha^2 F_{B,T} \left(\frac{\partial u}{\partial z} \frac{\partial(VE_{B,T})}{\partial y} - \frac{\partial v}{\partial z} \frac{\partial(VE_{B,T})}{\partial x} + \sigma(VE_{B,T}) \right) + \right. \\
&\quad \left. \frac{\partial(\rho F_{B,T})}{\partial z} \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial(\rho(w_k + F_{T,B} w_{T,B}))}{\partial z} \right) \right] dz' \\
\sigma &= \frac{1}{\alpha^2} \left(\frac{\partial u}{\partial z} \frac{\partial \alpha^2}{\partial y} - \frac{\partial v}{\partial z} \frac{\partial \alpha^2}{\partial x} \right) \\
VE_{B,T} &= \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + (w_k + F_{T,B} w_{T,B}) \frac{\partial \zeta}{\partial z} + (\zeta + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\
&\quad + \frac{\partial(w_k + F_{T,B} w_{T,B})}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial(w_k + F_{T,B} w_{T,B})}{\partial y} \frac{\partial u}{\partial z}.
\end{aligned}
\tag{4.24}$$

The equations (4.23) comprise a system of coupled linear elliptic second order PDEs for the vertical velocity boundary fields, w_B and w_T . The coupling appears in the inhomogeneous terms g_B and g_T . The natural boundary conditions of (4.23) are the Robin conditions:

$$(4.25) \quad -a_{B,T} \frac{\partial w_{B,T}}{\partial x} - b_{B,T} \frac{\partial w_{B,T}}{\partial y} + w_{B,T} \int_{\mathbb{Z}} \alpha^2 F_{B,T}^2 \frac{\partial v}{\partial z} \frac{\partial \zeta}{\partial z} dz' + \int_{\mathbb{Z}} \alpha^2 F_{B,T} \frac{\partial v}{\partial z} (VE_{B,T}) dz' = 0$$

on the east and west boundaries, and:

$$(4.26) \quad b_{B,T} \frac{\partial w_{B,T}}{\partial x} + c_{B,T} \frac{\partial w_{B,T}}{\partial y} + w_{B,T} \int_z \alpha^2 F_{B,T}^2 \frac{\partial u}{\partial z} \frac{\partial \zeta}{\partial z} dz' + \int_z \alpha^2 F_{B,T} \frac{\partial u}{\partial z} (v E_{B,T}) dz' = 0$$

on the north and south boundaries. Once again, if w is available on the lateral boundaries it can be specified in lieu of the natural boundary conditions. In this case δw_B and δw_T equal zero and the latter two surface integrals of (4.22) are also equal to zero.

Note that anelastic mass continuity has been added as a separate weak constraint in the cost-functional (4.19). This optional constraint limits the degree to which mass continuity is violated during the process of finding the upper and lower boundary fields. In regions where the vorticity equation contains significant errors, absence of this extra weak constraint could yield vertical velocity fields that severely violate mass continuity (when viewed in concert with the first-guess horizontal wind fields). This extra constraint was unnecessary in the previous method since mass continuity was implicitly used as a strong constraint through the form of w assumed.

This technique is also straightforward to implement. A simple dual-Doppler analysis technique can first be used to estimate u and v (and w_k). Using a provisional value of $w_T = 0$, (4.23) can then be solved for w_B . Using this w_B field, we can solve (4.23) for w_T . The system as a whole can be solved by continuing to iterate back and forth between interim solutions of w_B and w_T in (4.23) until no further significant

adjustments are made to either solution. Using these w_B and w_T fields, the total vertical velocity field can be constructed from (4.18). A particularly useful modification of this technique arises when one of the boundary fields can be estimated reliably using another technique (e.g., applying the impermeability condition at ground level). In this case, one of equations (4.23) can still be used to retrieve the opposing boundary condition. For example, if the data region spans downward to the ground but ends significantly below the storm top, the latter equation of (4.23) could be used to retrieve an optimal vertical velocity boundary condition for the upper boundary of the data region while the impermeability condition is applied at the ground. In this situation we are no longer solving a coupled system of equations so there is no need to iterate back and forth between interim solutions for w_B and w_T . In any case, the three-dimensional wind field at this point in the multi-step process will generally not satisfy the radial wind observations or the anelastic mass continuity equation without an extra correction step to u and v , since mass continuity has been applied as a weak constraint. A correction step is therefore desirable upon retrieval of the vertical velocity field.

One means of correcting u and v (for fixed w) is to apply the radial wind observations as strong constraints, in which case anelastic mass continuity will not be satisfied exactly in the final analysis fields. Alternatively, it is possible to correct u and v by applying the radial wind observations as weak constraints while imposing mass continuity as a strong constraint with w fixed from the previous step (Shapiro and Mewes, 1999). The final analysis field would then have a reasonably accurate vertical velocity field, would satisfy the anelastic mass continuity equation exactly,

and would exhibit minimal disagreement with the radial wind observations given the vertical velocity field and the mass continuity constraint.

Toward this end, the geometric relations between the radial wind observations from each Doppler radar and the Cartesian wind components at a point (x,y,z) are given by:

$$(4.27) \quad \begin{aligned} V_{r1} &= \frac{1}{R_1} [(x-x_1)u + (y-y_1)v + (z-z_1)(w+V_t)] = [X_1u + Y_1v + Z_1W] \\ V_{r2} &= \frac{1}{R_2} [(x-x_2)u + (y-y_2)v + (z-z_2)(w+V_t)] = [X_2u + Y_2v + Z_2W] \end{aligned}$$

where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of the two Doppler radars and R_1 and R_2 are the radial distances from the respective Doppler radars to the analysis point (x,y,z) . The latter equalities in (4.27) are simplifying substitutions that have been made for notational convenience in the remaining derivations. Note that the quantities $X_{1,2}$, $Y_{1,2}$, and $Z_{1,2}$ in these latter equalities represent the direction cosines of the individual radar beams. Note also that the quantity $W=w+V_t$ is the vertical velocity of the observed hydrometeors. The cost functional we wish to minimize during this correction step is:

$$(4.28) \quad \begin{aligned} J = \int_{x,y,z} & \left[\alpha^2 (V_{r1} - X_1u - Y_1v - Z_1W)^2 + \beta^2 (V_{r2} - X_2u - Y_2v - Z_2W)^2 \right. \\ & \left. + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{w}{\rho} \frac{d\rho}{dz} \right) \right] dx' dy' dz' \end{aligned}$$

where α and β are spatially varying weight functions (not related to those presented in earlier derivations) and λ is a Lagrange multiplier. In order to find the u and v wind fields which minimize this cost functional we must solve the system of equations that results from taking the 1st variations of (4.28) with respect to u , v , and λ and setting the results individually equal to zero. The resulting equations are:

$$\begin{aligned}
 0 &= 2(\alpha^2 X_1^2 + \beta^2 X_2^2)u + 2(\alpha^2 X_1 Y_1 + \beta^2 X_2 Y_2)v \\
 &\quad + 2\alpha^2 X_1 (Z_1 W - V_{r1}) + 2\beta^2 X_2 (Z_2 W - V_{r2}) - \frac{\partial \lambda}{\partial x}, \\
 (4.29) \quad 0 &= 2(\alpha^2 X_1 Y_1 + \beta^2 X_2 Y_2)u + 2(\alpha^2 Y_1^2 + \beta^2 Y_2^2)v \\
 &\quad + 2\alpha^2 Y_1 (Z_1 W - V_{r1}) + 2\beta^2 Y_2 (Z_2 W - V_{r2}) - \frac{\partial \lambda}{\partial y}, \\
 0 &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{w}{\rho} \frac{d\rho}{dz},
 \end{aligned}$$

which can be put into the simpler form:

$$\begin{aligned}
 0 &= A_1 u + B_1 v + C_1 - \frac{\partial \lambda}{\partial x}, \\
 (4.30) \quad 0 &= A_2 u + B_2 v + C_2 - \frac{\partial \lambda}{\partial y}, \\
 0 &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{w}{\rho} \frac{d\rho}{dz},
 \end{aligned}$$

where the coefficients in (4.30) are related in an obvious manner to the coefficients in (4.29). The first two expressions in (4.30) can be solved to yield explicit expressions for u and v in terms of the radial wind observations:

$$(4.31) \quad \begin{aligned} u &= \frac{1}{\Theta} \left(\Phi + B_2 \frac{\partial \lambda}{\partial x} - B_1 \frac{\partial \lambda}{\partial y} \right), \\ v &= \frac{1}{\Theta} \left(\Omega - A_2 \frac{\partial \lambda}{\partial x} + A_1 \frac{\partial \lambda}{\partial y} \right), \end{aligned}$$

where:

$$(4.32) \quad \begin{aligned} \Theta &= A_1 B_2 - A_2 B_1 = \alpha^2 \beta^2 (X_1 Y_2 - X_2 Y_1)^2, \\ \Omega &= -A_1 C_2 + A_2 C_1, \\ \Phi &= -B_2 C_1 + B_1 C_2. \end{aligned}$$

Substitution of these expressions for u and v into the last equation of (4.30), followed by algebraic manipulation, yields one equation in one unknown (λ):

$$(4.33) \quad \begin{aligned} & B_2 \frac{\partial^2 \lambda}{\partial x^2} - (A_2 + B_1) \frac{\partial^2 \lambda}{\partial x \partial y} + A_1 \frac{\partial^2 \lambda}{\partial y^2} + \left(\frac{\partial B_2}{\partial x} - \frac{\partial A_2}{\partial y} - \frac{B_2}{\Theta} \frac{\partial \Theta}{\partial x} + \frac{A_2}{\Theta} \frac{\partial \Theta}{\partial y} \right) \frac{\partial \lambda}{\partial x} \\ & + \left(\frac{\partial A_1}{\partial y} - \frac{\partial B_1}{\partial x} + \frac{B_1}{\Theta} \frac{\partial \Theta}{\partial x} - \frac{A_1}{\Theta} \frac{\partial \Theta}{\partial y} \right) \frac{\partial \lambda}{\partial y} + \frac{\Theta}{\rho} \frac{\partial(\rho w)}{\partial z} + \frac{\partial \Phi}{\partial x} + \frac{\partial \Omega}{\partial y} - \frac{\Phi}{\Theta} \frac{\partial \Theta}{\partial x} - \frac{\Omega}{\Theta} \frac{\partial \Theta}{\partial y} = 0 \end{aligned}$$

This equation can be shown to be elliptic as long as (i) α and β are universally non-zero, (ii) the two Doppler radars are not co-located, and (iii) observational points directly overlying the baseline are not included in the analysis domain. The natural boundary conditions of the problem conveniently turn out to be $\lambda=0$. Upon solution of (4.33) on a particular analysis level, the ‘optimal’ u and v fields can be constructed using (4.31). Although not apparent upon inspection, if λ is set to zero across the domain, (4.31) yields the u and v fields that would result from using the radial wind observations as strong geometric constraints in the absence of any mass continuity constraint. Thus, the derivatives of λ in (4.31) can be thought of as ‘correction terms’ which adjust the u and v fields to satisfy anelastic mass continuity while maintaining minimal disagreement of the analysis wind field with the radial wind observations.

To implement this correction step, the system (4.31) and (4.33) must be solved separately on each vertical level of the analysis grid. It is similar in spirit to the u and v correction technique of Ray et al. (1980), with the exceptions that the adjustments here vary with height, and the minimization is performed against the actual radial wind observations (as opposed to first guesses of the u and v fields). Also, note that the anelastic mass continuity equation provides only nz - I relationships between the wind fields on the nz grid levels. Therefore the horizontal wind field on one vertical grid level must be left unadjusted or must be corrected using alternative assumptions (e.g., accepting the radial wind observations as strong constraints).

Section D: Using the anelastic vertical vorticity equation as a strong constraint

New dual-Doppler analysis techniques using the anelastic vorticity equation as a weak constraint have now been introduced. Just as we can use the continuity equation as either a strong or weak constraint in dual-Doppler analyses, so can we use the anelastic vertical vorticity equation as a strong or weak constraint. An obvious question to ask is what type of a problem results from using the anelastic vertical vorticity equation alone as a strong constraint? The answer is simply that we obtain the anelastic vertical vorticity equation as the Euler-Lagrange equation, and it can be solved by the method of characteristics as described in Chapter 3. However, as discussed in Chapter 3, it is unlikely that a realistic vertical velocity field could be obtained in this manner and therefore no such analysis method will be tested.

Chapter 5: Tests on Numerically Simulated Data

Section A: Procedure

The new techniques presented in Chapter 4 were tested on simulated data sampled from the Advanced Regional Prediction System (ARPS; Xue et al., 1995). The first set of experiments involve data sampled from a simulation of the 20 May 1977 Del City, Oklahoma supercell thunderstorm (herein referred to as the *Del City storm*). Data for the second set of experiments were sampled from an ARPS simulation of a thunderstorm in a typical dry microburst environment (as in Brown et al, 1982; herein referred to as the *microburst storm*). The soundings supplied for each simulation are shown in Figs. 5.1 and 5.2. The average storm motion has been subtracted from the sounding wind fields from the outset to keep the primary storms of each simulation as stationary as possible.

The ARPS simulations were performed on a 101 x 101 x 35 point vertically stretched Arakawa C grid with uniform 500 m horizontal grid spacing. Vertical grid spacing ranged from 25 m near the ground up to 1000 m near the top of the domain. Prior to performing the dual-Doppler analyses, the simulated wind fields were interpolated from the staggered and stretched ARPS grid to a non-staggered, non-stretched analysis grid with 500 m grid spacing in all directions. The vertical structures of the simulation and analysis grids are therefore very different.

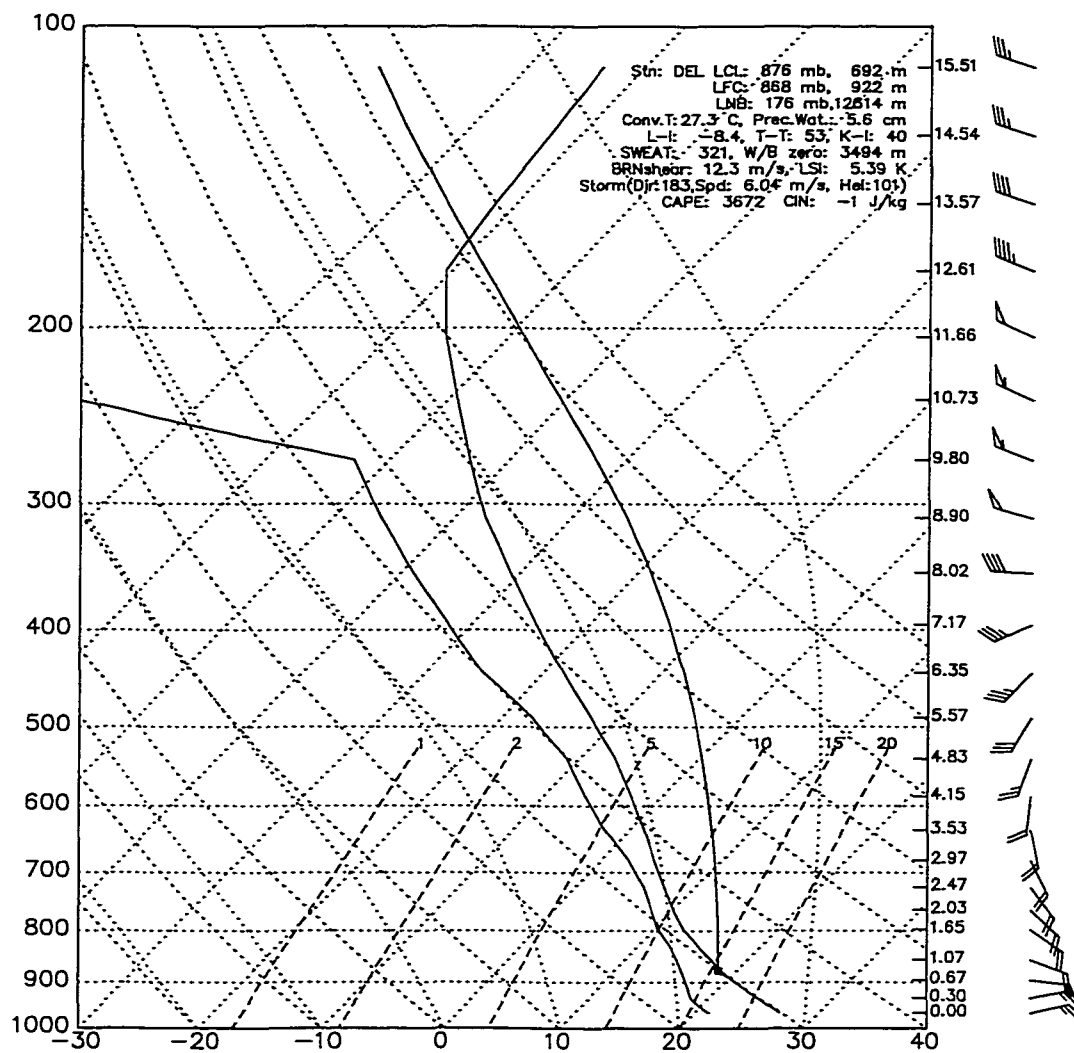


Figure 5.1: Sounding supplied to initialize ARPS simulation of the Del City supercell thunderstorm. The simulated storm motion has been subtracted from the sounding.

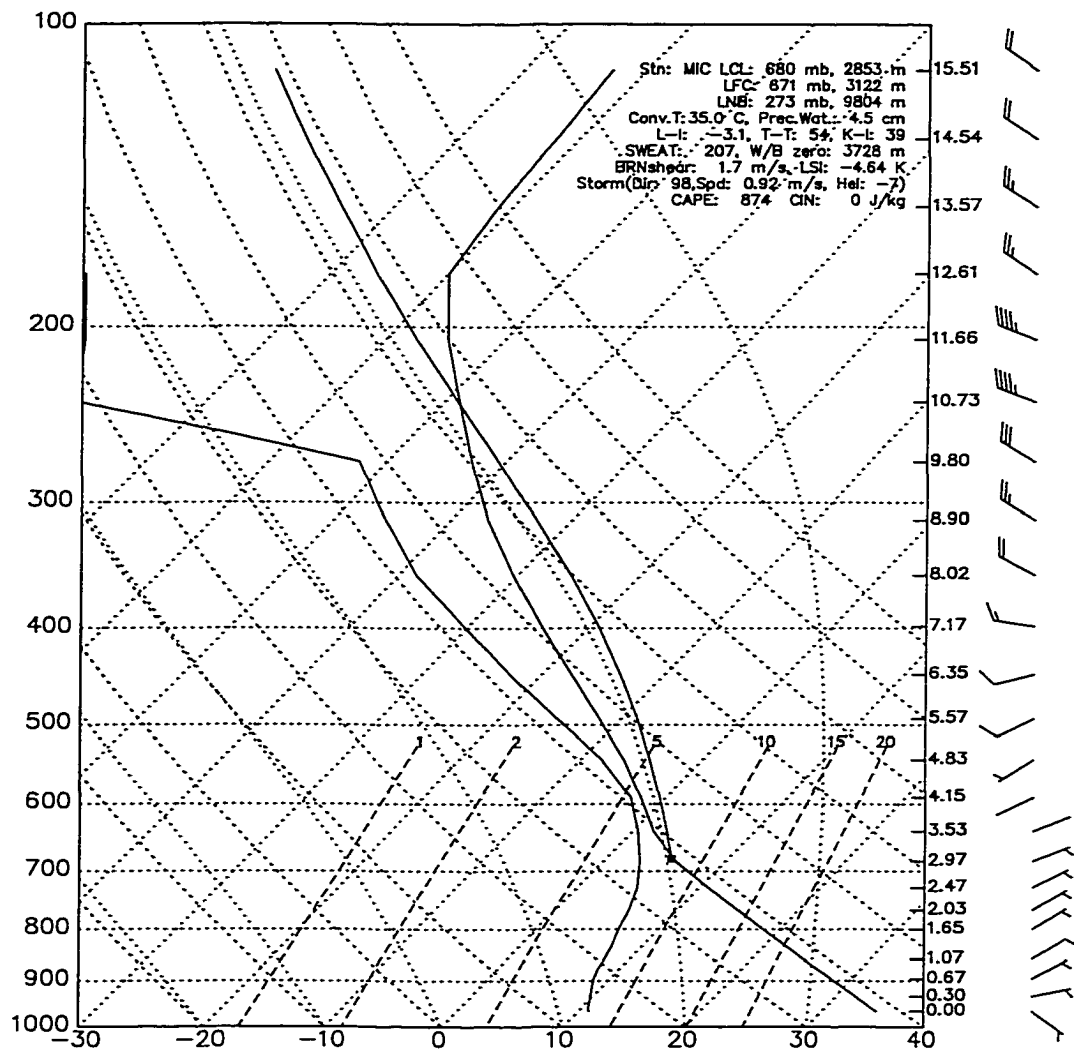


Figure 5.2: Sounding supplied to initialize ARPS simulation of the microburst thunderstorm. The simulated storm motion has been subtracted from the sounding.

For all experiments Doppler radars were assumed to lie in the lower left (SW) and upper left (NW) corners of the analysis grid. Radial wind observations were simulated by projecting the interpolated Cartesian wind fields at each grid point into the directions of the radars. Since the wind field was sampled directly, errors due to inaccurate radar reflectivity / terminal velocity parameterizations are absent from these experiments. However, random errors with standard deviation of $1.0 \text{ m}\cdot\text{s}^{-1}$ and magnitudes ranging up to $3.0 \text{ m}\cdot\text{s}^{-1}$ were explicitly introduced into the simulated observations at each analysis point. No bias or trend errors were explicitly introduced.

The practical data availability problems resulting from the earth's curvature, ground clutter contamination, and scan geometry limitations were addressed in these experiments by limiting the spatial availability of the data. In addition, observations were made available only at discrete intervals of 90 and 300 s, intended to be representative of the volume scan periods of the Dopplers on Wheels (DOWs) research radars (Wurman, 1997) and the WSR-88D network (Klazura and Imy, 1993). No attempts were made to mimic the non-simultaneous nature of radar observations in real data.

The labeling system used in the experiments is as follows: U, D, and C represent upward, downward, and constrained integration techniques (in U & D mass continuity is applied as a strong constraint; in C it is applied as a weak constraint while retrieving w – although it can later be applied as a strong constraint when correcting u and v); I, E, and V represent application of impermeability, extrapolated divergence (followed by impermeability), or the vorticity equation, respectively, to obtain the

required vertical velocity boundary condition(s) at the top (t) and / or bottom (b) of the data domain; and 90 and 300 represent the volume scan periods (in seconds) used with the vorticity equation. For example, a technique labeled C-Ib-Vt-90 would utilize mass continuity as a weak constraint in the step where w is obtained, with impermeability applied as the lower boundary condition and the vorticity equation (using 90 s volume scan periods) used to obtain the upper boundary condition. The natural boundary conditions are imposed for the vorticity-based techniques in the following results. It will be useful to note that techniques U and D in this work are identical to the U and D techniques of MB98, and technique C is similar to MB98's technique B.

For all of the experiments the weight function α^2 in cost-functionals (4.6) and (4.19) have been treated as constants (1.0 in U & D, and 250.0 in C). The actual value of the constant is irrelevant in techniques U and D, since any constant value yields the same solution. However, in method C, this constant determines the relative weights put on the mass continuity and vorticity equations, and does affect the results. A series of experiments (not shown) suggested that the method C retrievals are only weakly dependent on the value of this weight for a large range of weights. The value of 250.0 was chosen arbitrarily, but generally caused the value of the vorticity portion of the cost functional to dominate the mass continuity portion by a 5:1 ratio. This means the vorticity equation has about 5 times as much influence as mass continuity in determining opposing vertical velocity boundary conditions. It is noteworthy that if α^2 is set too low mass continuity is effectively forced to be met exactly (when taken in

combination with the filtered u and v fields) in a technique where mass continuity is intended to be held as a weak constraint during construction of the w field.

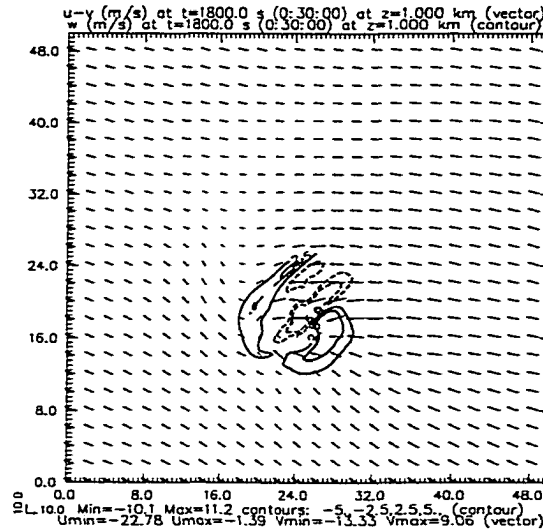
For comparison, control analyses were constructed using common traditional dual-Doppler analysis techniques that would be appealing in each situation. For example, impermeability is applied as a vertical velocity boundary condition at the ground and/or at storm top whenever data extend to those levels. When data fail to extend downward to the ground, extrapolated divergence followed by application of the impermeability condition was used to determine a lower boundary condition.

In all the following experiments, a 1-2-1 Shuman (Shuman, 1957; Ray et al., 1975) filter was used to remove $2\Delta x$ scales from the horizontal wind fields prior to integration of the mass continuity equation, thus aiding in control of cumulative integration errors. Although the vorticity equation requires further filtering of the $3\Delta x$ scales prior to retrieval of the boundary condition fields, the retrieved boundary fields themselves do contain information in the $2\Delta x$ scale (although in a damped fashion, as pointed out in Chapter 4). After retrieving these boundary fields the three-dimensional wind fields have been reconstructed using only a 1-2-1 filter combined with the radial wind observations, the newly obtained boundary condition fields, and the noted analysis framework. Applying the same filtering technique in all analyses makes comparisons of the various techniques legitimate.

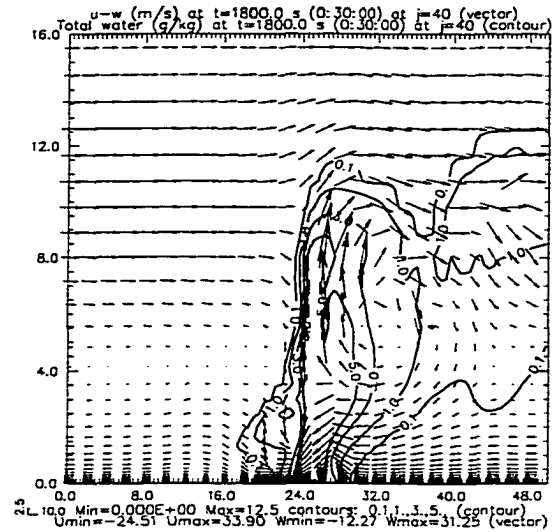
Section B: Analyses of the Del City storm

The primary storm of this simulation was initiated by releasing a thermal bubble over the southeastern corner of the domain. As the induced updraft evolves, it is also advected northwestward, becoming established and nearly stationary just south of the center of the domain by $t = 1800$ s (Fig. 5.3). The precipitation core and associated downdraft become organized around this time and the resulting cold pool begins to spread northwestward from the primary updraft. A series of weaker and less steady storms form along the advancing boundaries of this cold pool. Between $t = 1800$ and $t = 4200$ s the primary storm remains nearly stationary while it evolves a quasi-steady rotating updraft and rear flank downdraft typical of a supercell thunderstorm. Finally, between $t = 4200$ and $t = 7200$ s the storm begins to propagate southeastward into the low-level inflow, nearly reaching the southeastern boundary of the domain by $t = 7200$ s. Therefore, although the average storm motion through the 7200 s simulation was removed from the sounding, the storm actually traverses from the southeast corner of the domain to the center of the domain, then back again, over the course of the 2-hour simulation.

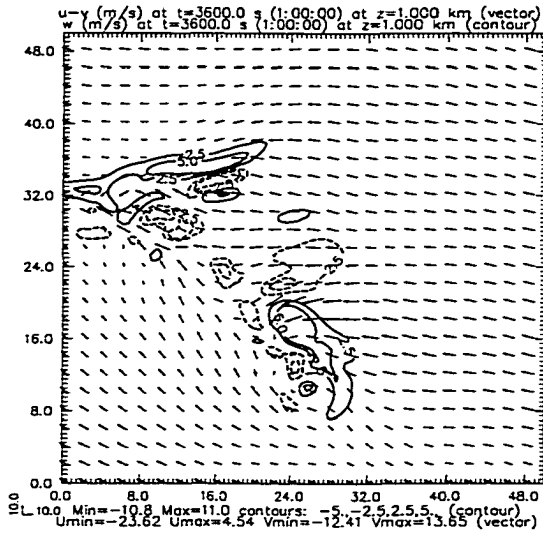
Recall from Chapter 3 that the accuracy of the discretized estimates of $\frac{\partial \zeta}{\partial t}$ can be increased by performing the analyses in a reference frame in which the effects of advection are minimized. Since the average primary storm motion in these experiments has been subtracted from the velocity field at the outset, the Gal-Chen (1982) or related



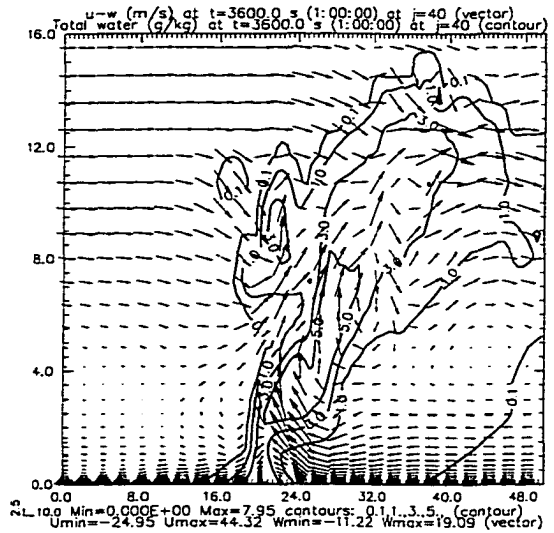
(a)



(b)

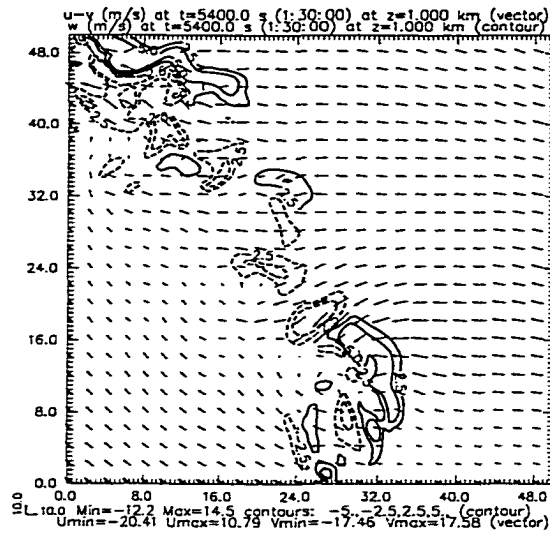


(c)

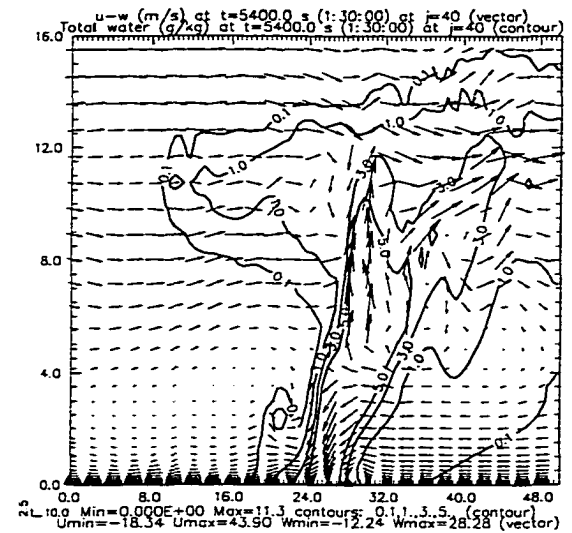


(d)

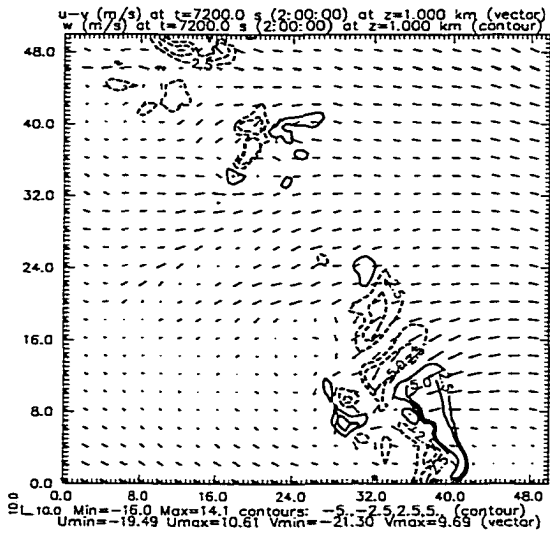
Figure 5.3: Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell thunderstorm at (a) 1800 s and (c) 3600 s. Also shown are x - z vertical cross sections near the center of the storm at (b) 1800 s and (d) 3600 s. Contours of 0.1, 1.0, 3.0, and 5.0 $\text{g}\cdot\text{kg}^{-1}$ of total (non-vapor) water mixing ratio are shown.



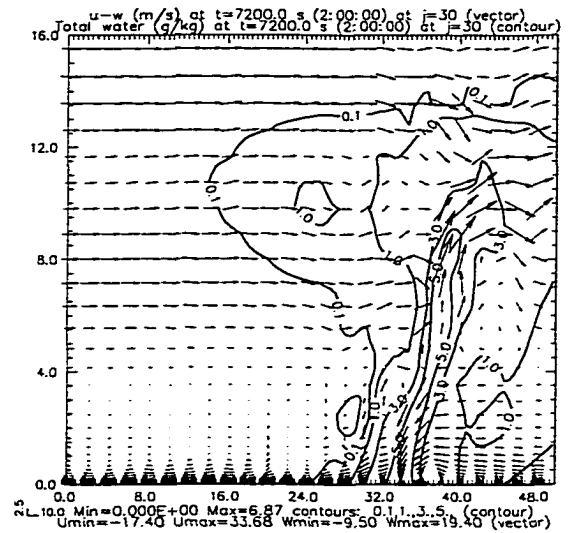
(e)



(f)



(g)



(h)

Figure 5.3 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell thunderstorm at (a) 5400 s and (c) 7200 s. Also shown are x - z vertical cross sections near the center of the storm at (b) 5400 s and (d) 7200 s. Contours of 0.1, 1.0, 3.0, and 5.0 $\text{g}\cdot\text{kg}^{-1}$ of total (non-vapor) water mixing ratio are shown.

techniques have not been utilized in the following results. Unfortunately, the secondary storms and other features present in the simulation domain propagate at significantly different speeds and in different directions than the primary updraft. For example, the elongated updraft over the northwestern portions of the domain in Figs. 5.3c and 5.3e shows northwestward propagation / advection speeds of up to $15 \text{ m}\cdot\text{s}^{-1}$. Considering the width of this updraft in the direction of propagation ranges between only 2 and 4 km, the updraft will entirely traverse across a given point within a period of slightly more than 3 minutes (on average). Given this fact, the finite-difference estimates of the temporal derivative of vorticity in this storm are likely to be completely unreliable with 300 s volume scans, and of only limited accuracy even with 90 s volume scan periods.

This behavior is illustrated in Fig. 5.4. Fig. 5.4a, the true field, is simply close-up view of the northwestern quadrant of the ARPS grid at $t = 3600 \text{ s}$ and $z = 1 \text{ km}$ (similar to Fig. 5.3c). The locations of two prominent features in Fig. 5.4a are marked with X's to facilitate intercomparison with Figs. 5.4b and 5.4c. Figures 5.4b and 5.4c show the results of dual-Doppler analyses using the vorticity equation to estimate the lower boundary condition. In Fig. 5.4c, technique U-Vb-300, a 300 s volume scan period has been used. Notice that the core of the strongest updraft in Fig. 5.4a is collocated with a region of vertical velocities estimated to be near zero in Fig. 5.4c. Further, the downdraft field from U-Vb-300 is grossly overestimated and bears little spatial resemblance to that of the true field. Although the downdraft field is still too large with technique U-Vb-90 (Fig. 5.4b), the errors are much smaller than with 300 s volume scan periods. Furthermore, the updraft is much better resolved with the 90 s volume scan periods. The w RMS errors at this level from the U-Vb techniques are $1.67 \text{ m}\cdot\text{s}^{-1}$ with 90 s volume

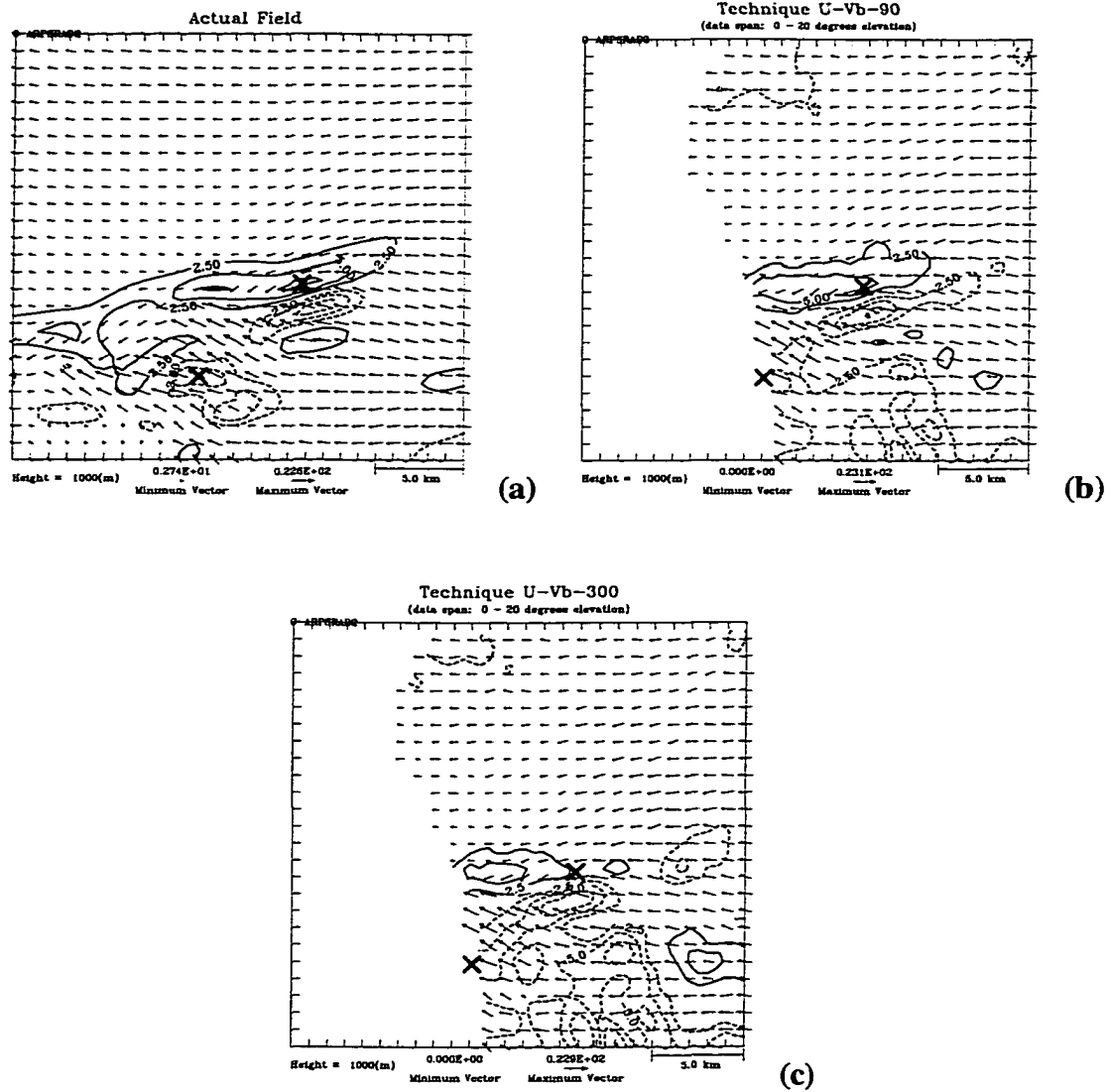


Figure 5.4: Horizontal wind vectors and contours of w at $t = 3600$ s and $z = 1.0$ km for the Del City storm. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Depicted are the (a) true field as well as those from (b) technique U-Vb-90 and (c) technique U-Vb-300. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).

scans and $2.57 \text{ m}\cdot\text{s}^{-1}$ with 300 s volume scan periods, indicating that better resolution of the temporal derivative has lead to a 35% improvement in the analysis (differences in the temporal derivative estimates account for all differences between the two retrievals).

Given this rapid advection speed, further improvement would be likely with even shorter volume scan. Similar improvements could most likely be made through a simple redefinition of the observations in a reference frame moving with this transient storm. However, given the widely varying advection speeds and directions of the secondary storms of this simulation it is not possible to define a single reference frame in which all features are reasonably stationary. Instead, it is useful to focus only upon a 50×50 grid point subdomain surrounding the quasi-stationary primary storm of the simulation that excludes most of the highly transient features. All remaining discussion of the Del City storm pertains only to data within this subdomain.

The first three experiments using the Del City storm assume that the volume coverage pattern (VCP) of the scanning radars provide data up to elevation angles of 20° . This is considered to be a very typical situation. For example, in severe-weather VCP mode, a WSR-88D radar completes sweeps of 14 elevation angles in 5 minutes, with the top of the beam just reaching 20° in the final sweep. For research radars such as the DOWs, complete coverage of twenty 90° azimuthal sector scans with a 1° beamwidth would require a 20° s^{-1} azimuthal scan rate to completely encompass the volume below 20° elevation in 90 seconds. This is, in fact, comparable to the current scan rate capabilities of the DOWs (Wurman, 1999). In the first of these three experiments, Case I, all data between 0° and 20° elevation for both radars were included within the analysis. In the second and third experiments (Cases II and III), data below 1° ($\sim 500 \text{ m}$) and 2°

(~1500 m), respectively, were rejected. These values were intended to represent typical and severe cases of beam blockage and/or data contamination at low levels by terrain and other ground clutter. The 20° maximum elevation angle falls far below (3 – 5 km) the actual storm top for all but the first 20 minutes of the simulation, leaving the upper vertical velocity boundary condition unknown. Historically, this lack of boundary condition information would likely prevent application of the downward (D) or constrained (C) integration techniques. Therefore, the researcher encountering this situation would probably resort to upward integration of the mass continuity equation (technique U) from the lower boundary where w appears to be known with greater precision. As mentioned previously, this technique can suffer severely from accumulating divergence errors in some circumstances.

When data are available down to the surface (Case I) the lower boundary condition is known exactly ($w = 0$, or as determined from orographic effects with real data). In this case the primary concern is finding a way to negate the effects of accumulating divergence errors. A vorticity equation constraint can be very useful in this regard. This can be seen in Fig. 5.5a by comparing curve U-Ib (control) with curves C-Ib-Vt-90 and C-Ib-Vt-300. Reiterating the naming convention, U-Ib applies a simple upward integration of the continuity equation (as in Brandes, 1977) with impermeability applied as a lower boundary condition. Techniques C-Ib-Vt-90 and C-Ib-Vt-300 constrain the vertical velocity field to meet the impermeability boundary condition at the ground as well as a vorticity based boundary condition at the top of the domain, with 90 and 300 s volume scan periods, respectively, used in evaluating the vorticity equation. Techniques C-Ib-Vt can therefore negate the accumulation of divergence errors.

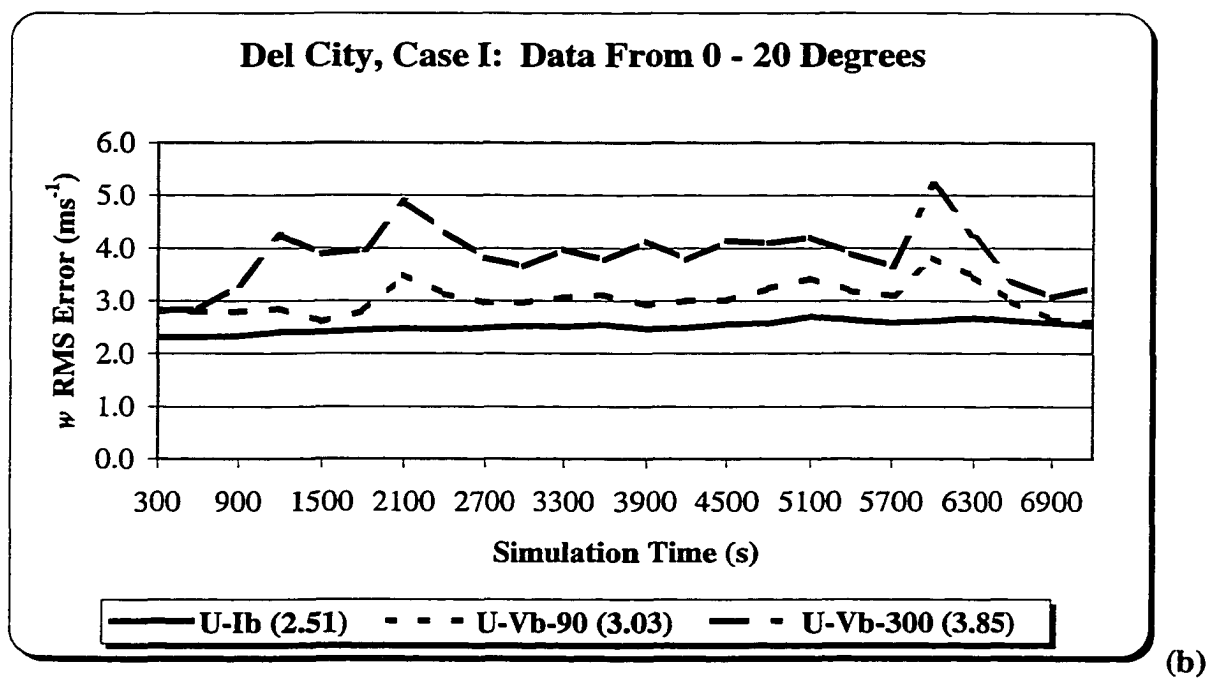
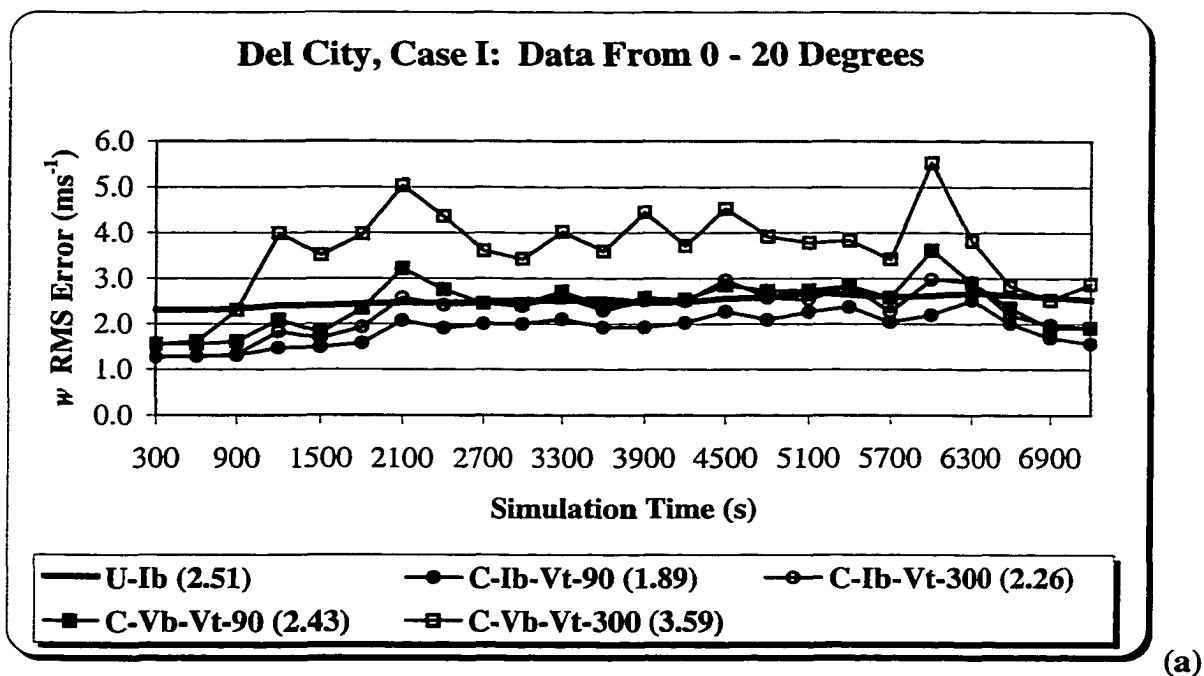


Figure 5.5: Root mean square vertical velocity retrieval errors (a,b) from experiments on the Del City supercell simulation using volume scan geometries providing data between 0° and 20° (Case I).

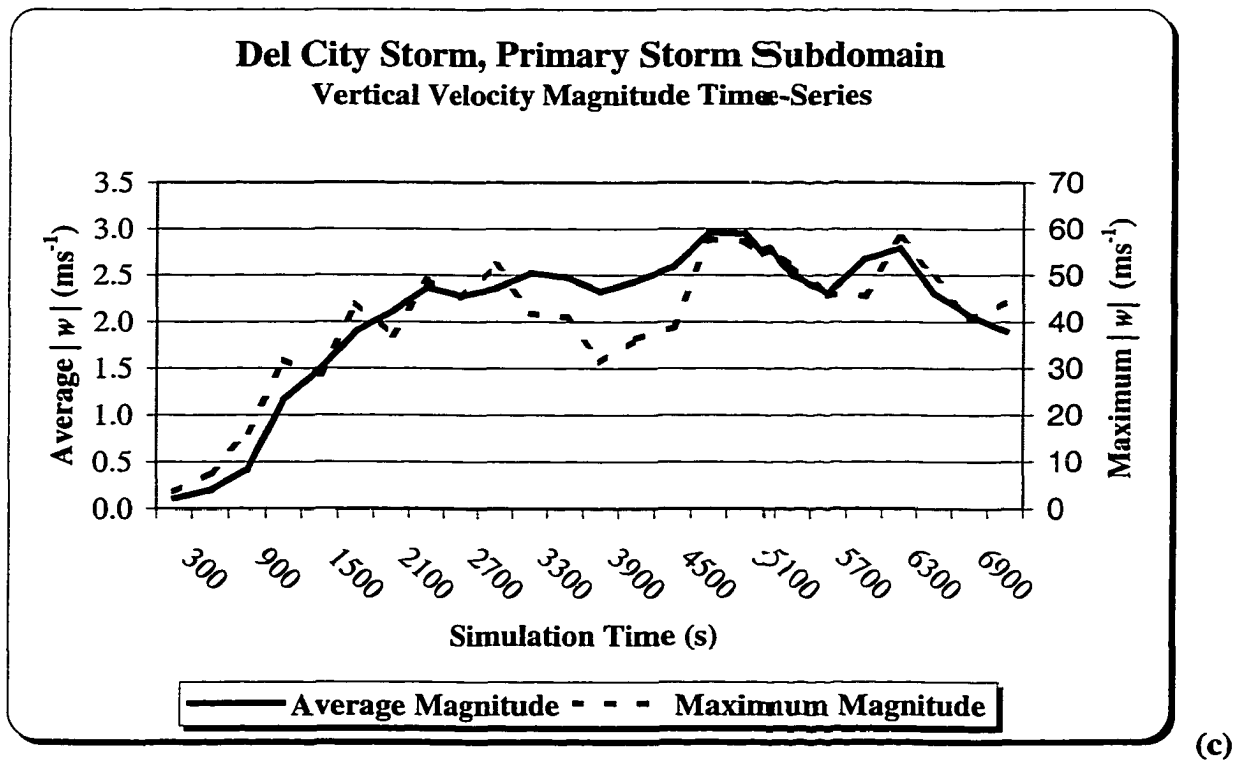


Figure 5.5 (continued): Average and maximum vertical velocity magnitudes (c) over the primary storm subdomain throughout the course of 7200 s simulation. Panel (c) is useful for interpreting the error magnitudes in terms of the true vertical velocities, and can be applied in interpretation of Figs. 5.9, 5.13, and 5.17 as well. Note that the presence of large areas of near-zero vertical velocity skew the average updraft magnitudes downward.

Averaged across all 24 retrieval times, the RMS vertical velocity error of control technique U-Ib is $2.51 \text{ m}\cdot\text{s}^{-1}$. This is a relatively low RMS error for technique U given the considerable integration depth and specified error magnitudes, and is likely attributable to the fact that the Shuman filter is particularly efficient at removing purely random errors in the horizontal divergence field. The control technique would likely have performed much worse were non-random errors applied (as noted in Matejka and Bartels, 1998; henceforth referred to as MB98). When constraining the vertical velocity field using an upper boundary condition retrieved from the vorticity equation, the RMS errors are reduced to $2.26 \text{ m}\cdot\text{s}^{-1}$ and $1.89 \text{ m}\cdot\text{s}^{-1}$ with techniques C-Ib-Vt-90 and C-Ib-Vt-300, respectively (up to a 25% reduction). Since technique C-Ib-Vt is similar to MB98's technique B, which was shown to be robust in the presence of bias errors, the error reductions noted here may be understated in cases where these errors are present. However, this has yet to be verified experimentally with a vorticity based boundary condition(s). Another interesting feature of Fig. 5.5a is that technique C-Vb-Vt-90, which makes no use of the known $w = 0$ lower boundary condition, is actually able to outperform the control technique. In this case, a set of two reasonable (but not perfect) boundary conditions proves superior to application of one perfect boundary condition by controlling accumulating divergence errors.

Figure 5.5b explores the results from vorticity techniques that share the same analysis framework as the control technique (U-Ib). Differences between the analyses shown here arise solely from differences in the boundary condition. Thus, Fig. 5.5b is useful for isolating the accuracy of the different methods of boundary condition specification (i.e., in the absence of the divergence error control that technique C

provides). Curves U-Vb-90 and U-Vb-300 make no use of impermeability and instead retrieve an appropriate boundary condition using the vorticity equation (with the volume scan period indicated). For Case I it is apparent that application of the impermeability condition (which is met exactly in this idealized case) is preferable to a vorticity-based boundary condition, an unsurprising result. However, the vorticity equation will be shown to be more useful in other cases, thus Fig. 5.5b has been included for completeness.

A cross-section through the upper portion of the subdomain ($z = 10$ km) from each of these techniques at the midpoint of the simulation ($t = 3600$ s) reveals where the benefits of two boundary conditions are realized (Fig. 5.6). Comparison of the true field, Fig. 5.6a, with the field from the control technique U-Ib shown in Fig. 5.6b reveals that the control field contains a significant number of spurious small scale features that in many cases mask the true vertical velocity field. Physical interpretation of portions of the vertical velocity field in Fig. 5.6b would be very difficult. On the other hand, the vertical velocity fields from each of the aforementioned vorticity techniques are much better representations of the true vertical velocity field. Nearly all of the features present in the true vertical velocity field are easily identifiable in Figs. 5.6c-5.6e. Note that the missing data in the lower left and extreme upper left corners of Figs. 5.6b-5.6e demarcates the 20° elevation angle used as an upper data cutoff level for Cases I-III.

The wind fields at $t = 1800$ s, $t = 3600$ s, $t = 5400$ s, and $t = 7200$ s from the control technique (U-Ib) and the ‘best’ (as determined from Fig. 5.5) vorticity based technique (C-Ib-Vt-90) are compared against the true wind field in Figs. 5.7 and 5.8. Figure 5.7 depicts the wind fields at $z = 1.0$ km, and is intended to show how each

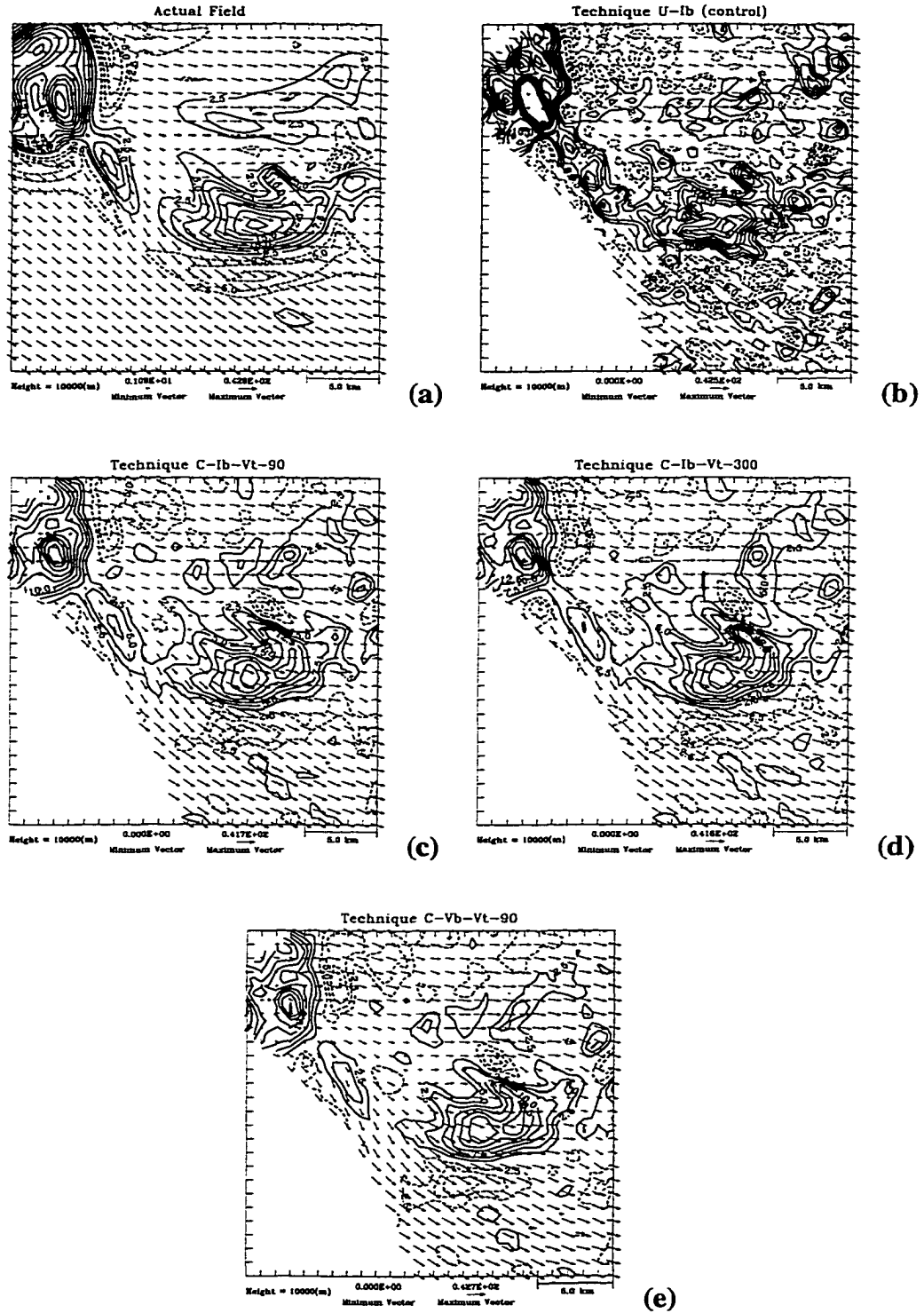


Figure 5.6: Horizontal wind vectors and contours of w at $z = 10.0$ km in the Del City supercell storm, taken from (a) the true field, (b) control technique U-Ib, (c) technique C-Ib-Vt-90, (d) technique C-Ib-Vt-300, and (e) technique C-Vb-Vt-90. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).

technique performs in retrieving the appropriate lower boundary condition. In this case, since both techniques apply impermeability as the lower boundary condition, the fields are largely similar. However, this will situation will change in other scan geometry scenarios. Thus, Figure 5.7 is still shown for completeness. Figure 5.8 is valid at $z = 10.0$ km, and highlights the advantages of applying a second boundary condition determined from the vorticity equation to control accumulating divergence errors.

If data below 1° are rejected due to ground clutter or other data contamination (Case II), we are faced with the usual situation of not having reliable knowledge of either an upper or lower boundary condition. In this case it is common to apply technique U using the extrapolated divergence technique to estimate the lower boundary condition. However, the vorticity equation can offer information regarding one or both of the required boundary conditions for common techniques in this situation. For example, extrapolated divergence can be used to estimate the lower boundary condition while using the vorticity equation to retrieve an upper boundary condition in technique C, controlling accumulating divergence errors. The vorticity equation could also be used alone to retrieve both the upper and lower boundary conditions in technique C. The benefits of the vorticity equation in this second experiment are explored in Fig. 5.9a. Method U-Eb (control) yields an average RMS error of $3.34 \text{ m}\cdot\text{s}^{-1}$ under these circumstances. Methods C-Eb-Vt-90 and C-Eb-Vt-300 show up to a 23% improvement over the control method, with 2.57 and $2.84 \text{ m}\cdot\text{s}^{-1}$ RMS errors, respectively. Method C-Vb-Vt-90 also shows a 21% improvement over the traditional technique with an RMS error of $2.64 \text{ m}\cdot\text{s}^{-1}$. Again, these error reductions are noted using random errors. As mentioned previously, based on

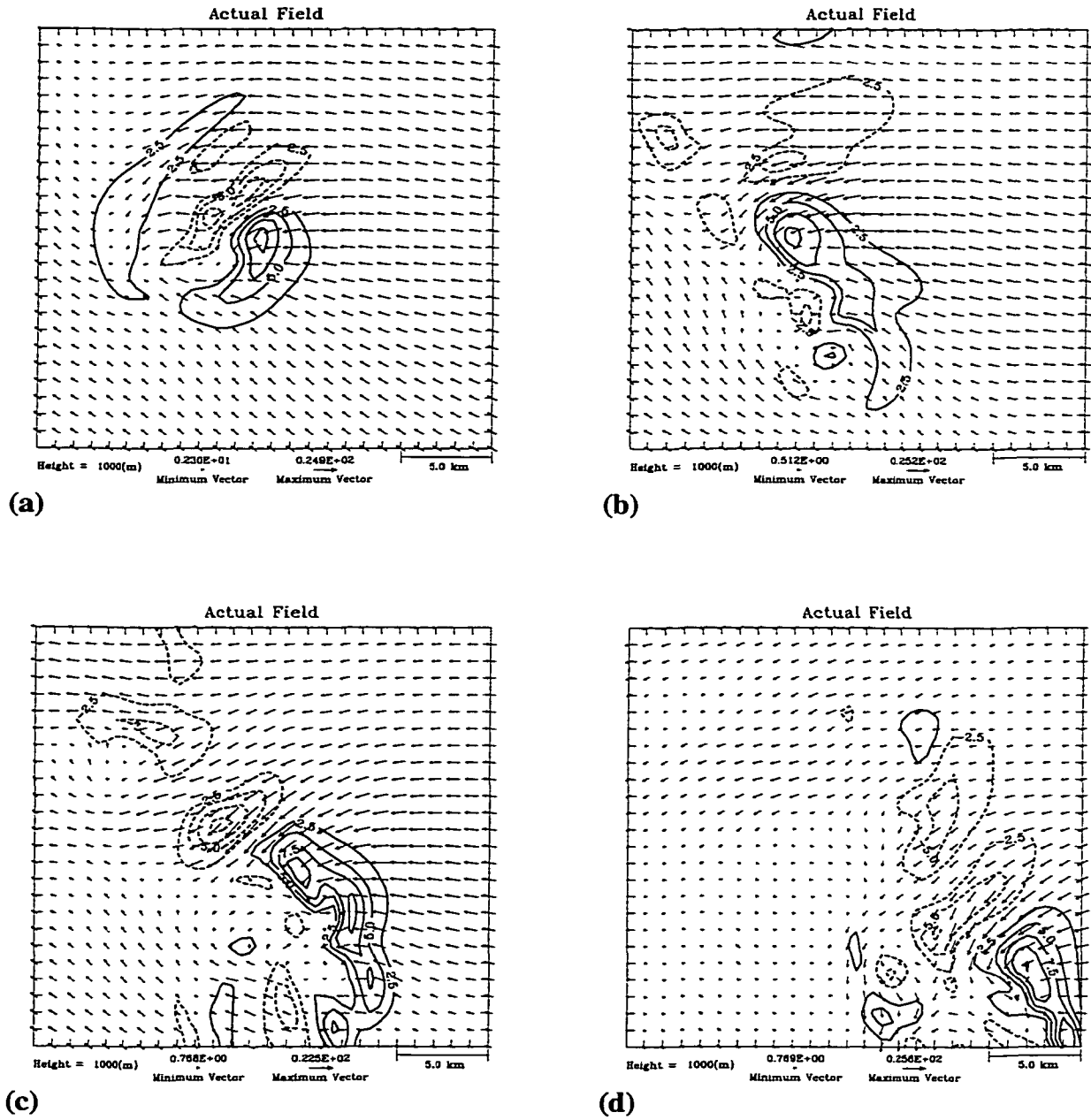


Figure 5.7: Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell storm, taken from the true field at (a) $t = 1800$ s, (b) $t = 3600$ s, (c) $t = 5400$ s, and (d) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).

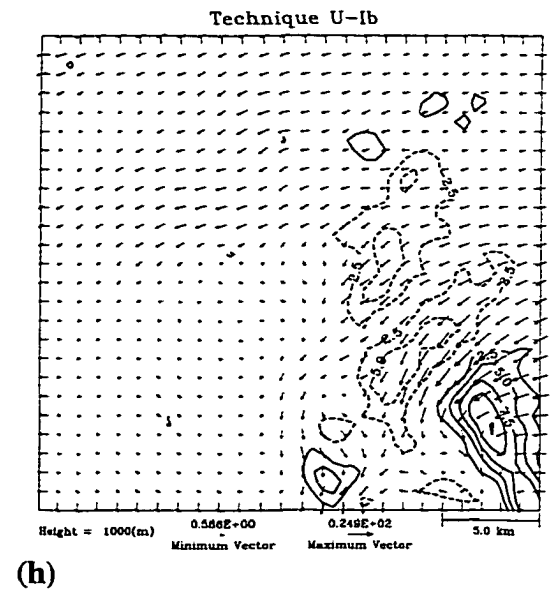
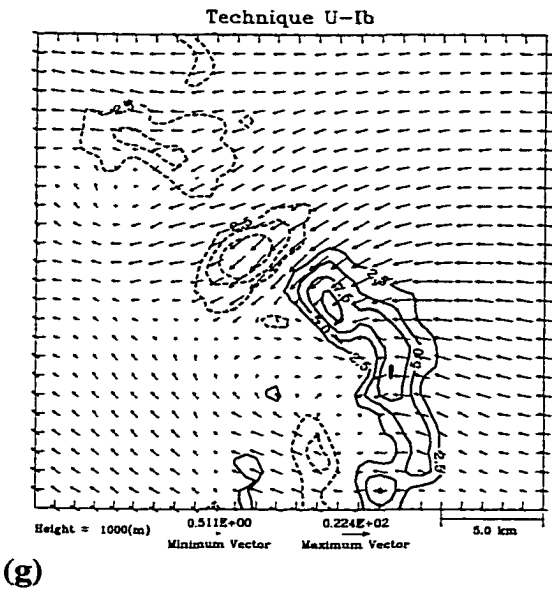
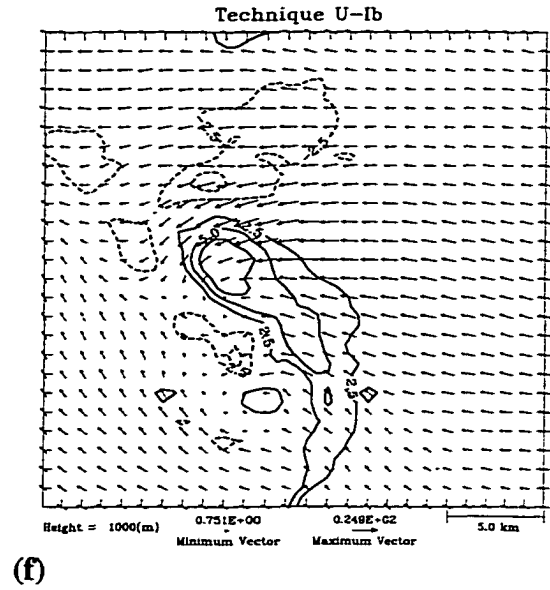
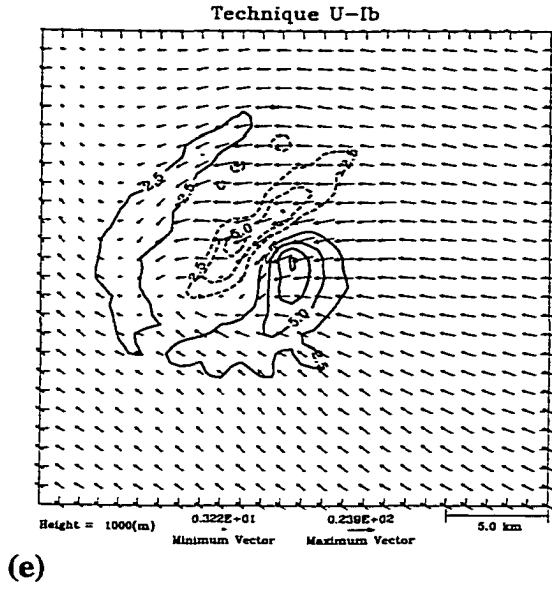


Figure 5.7 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell storm, taken from control technique U-Ib at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).

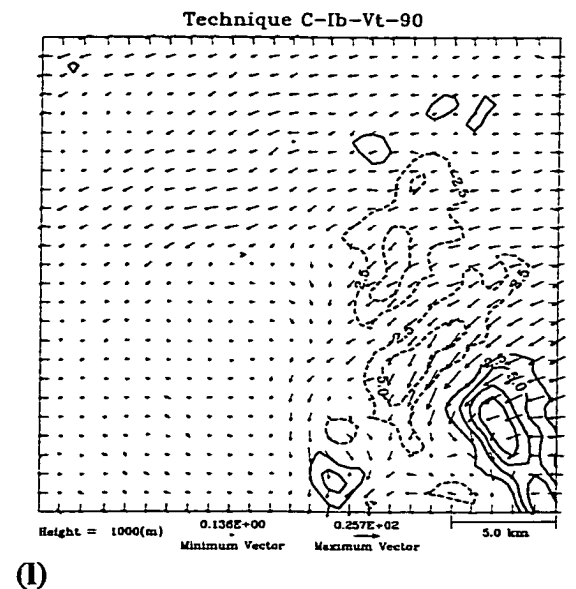
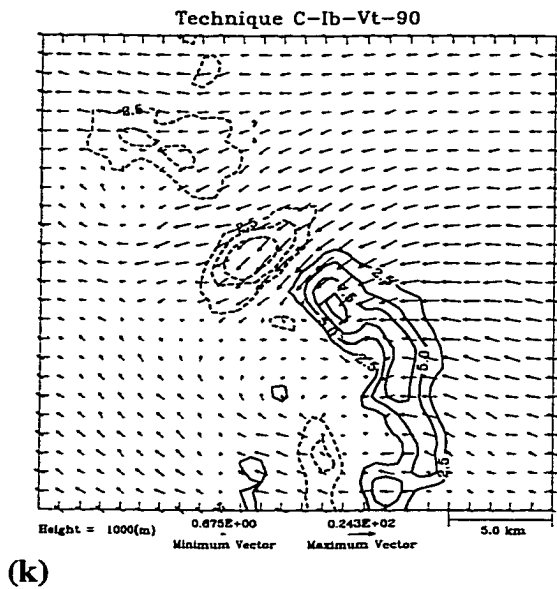
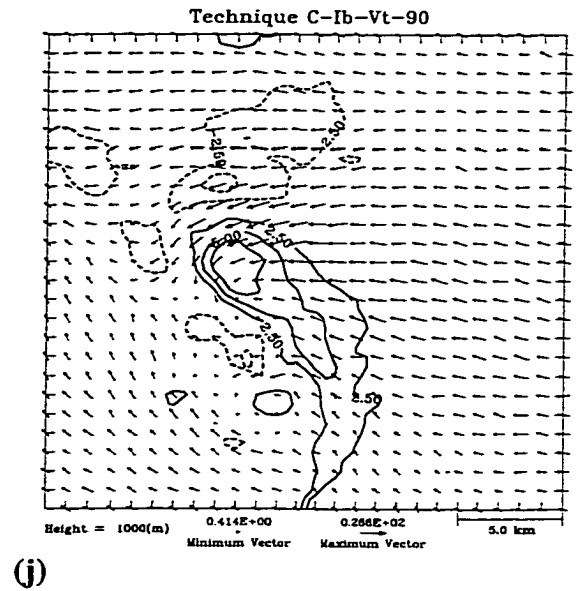
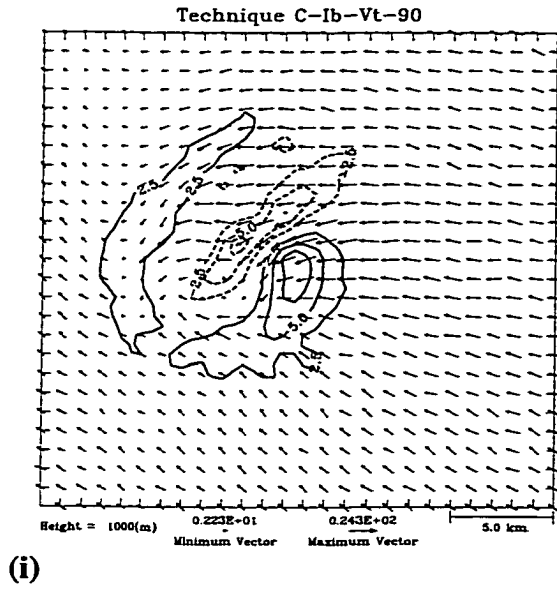


Figure 5.7 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell storm, taken from the ‘best’ vorticity technique C-Ib-Vt-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).

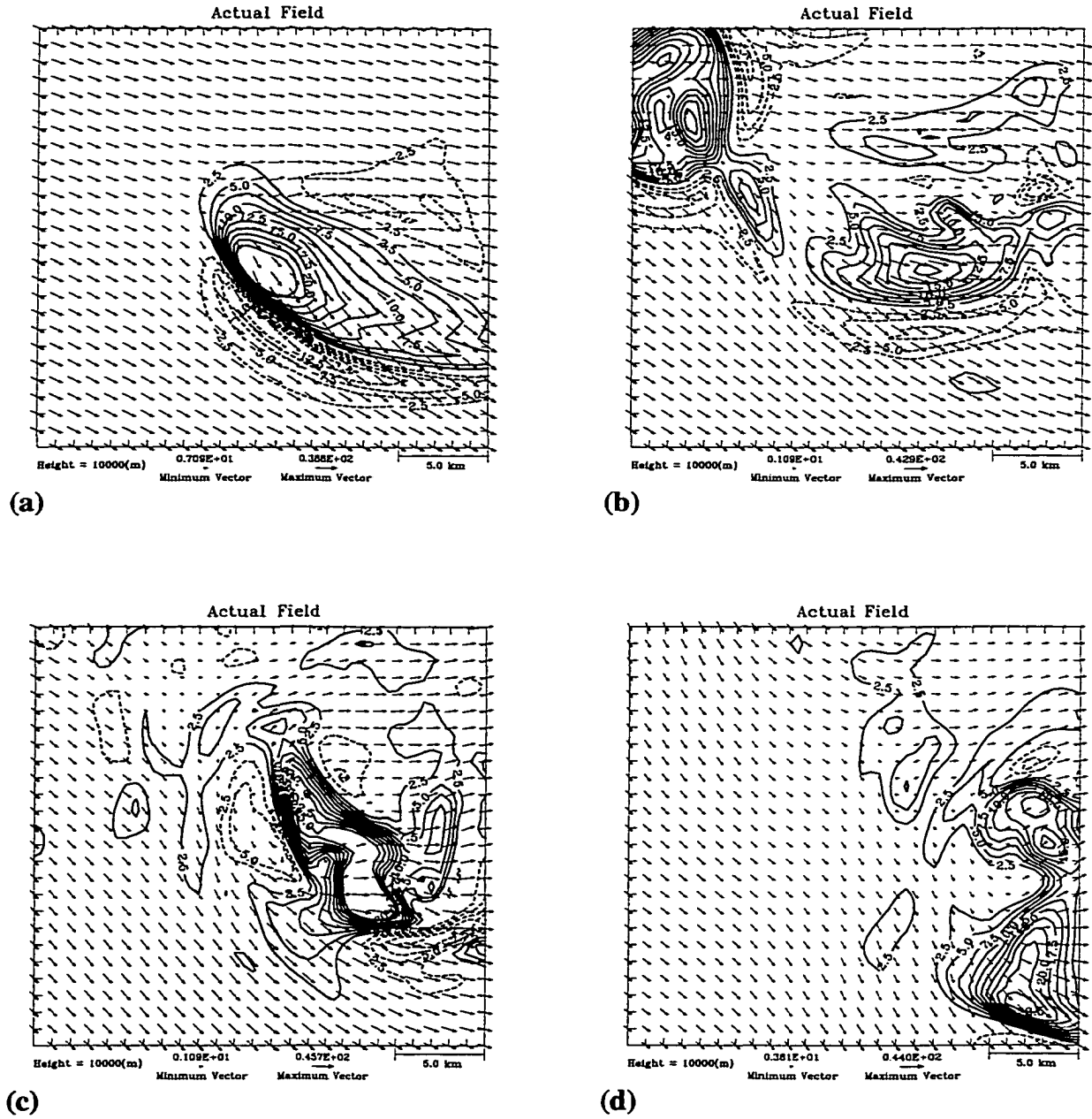


Figure 5.8: Horizontal wind vectors and contours of w at $z = 10.0 \text{ km}$ in the Del City supercell storm, taken from the true field at (a) $t = 1800 \text{ s}$, (b) $t = 3600 \text{ s}$, (c) $t = 5400 \text{ s}$, and (d) $t = 7200 \text{ s}$. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).

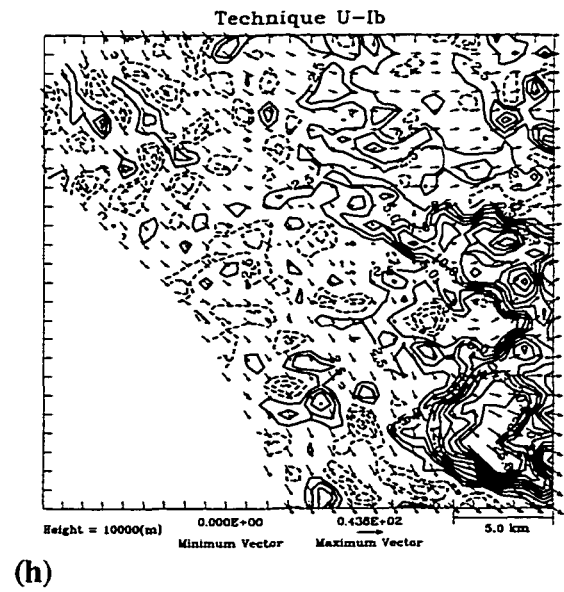
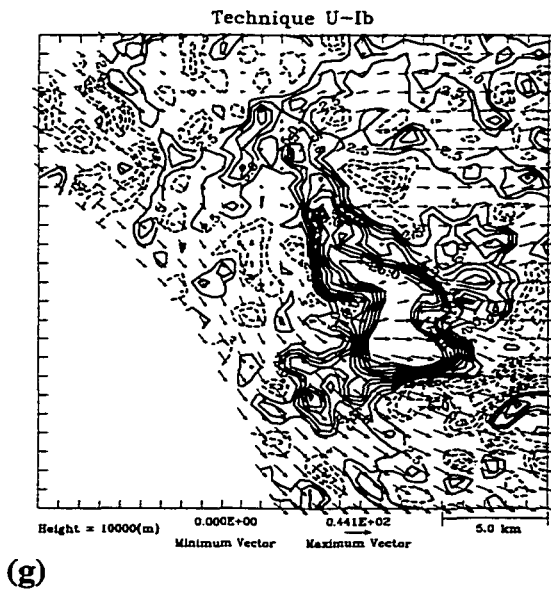
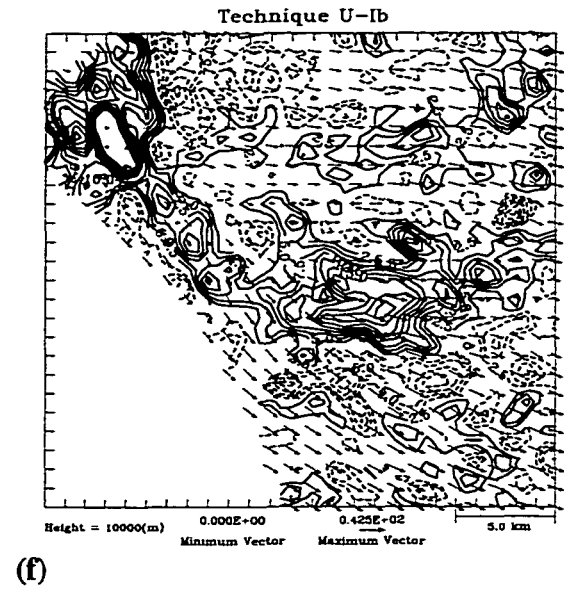
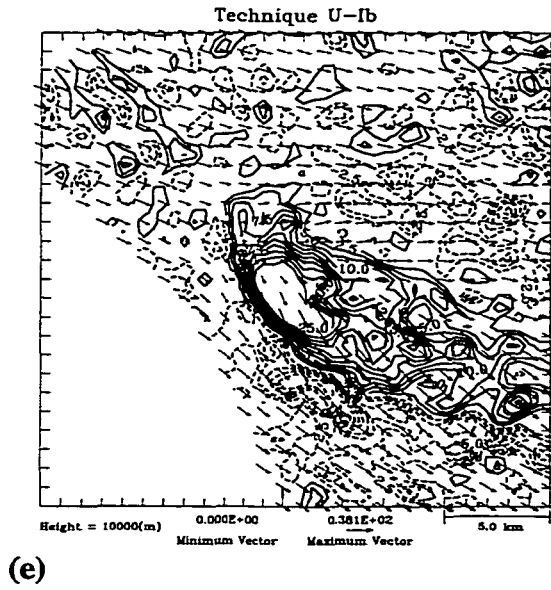
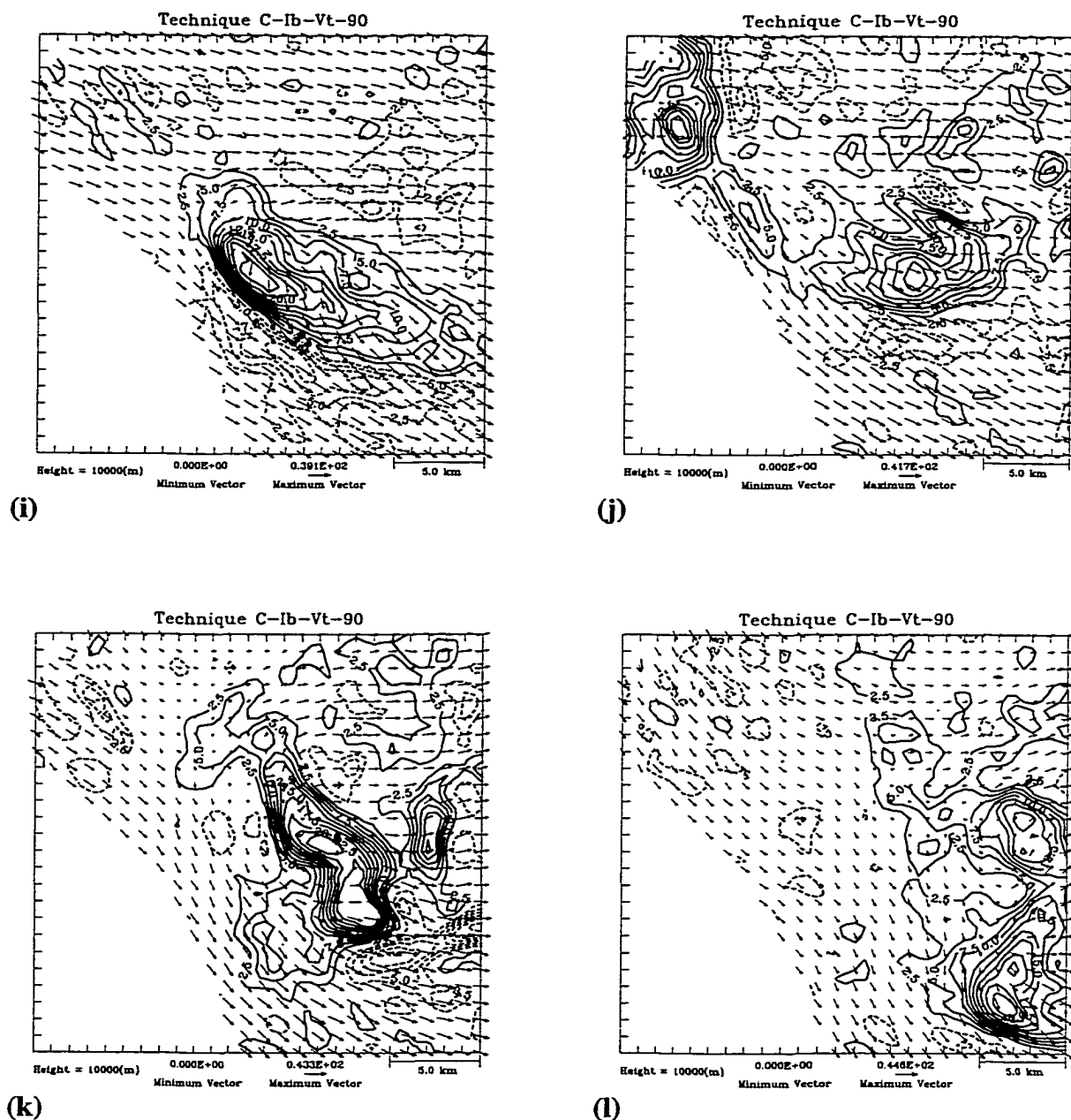


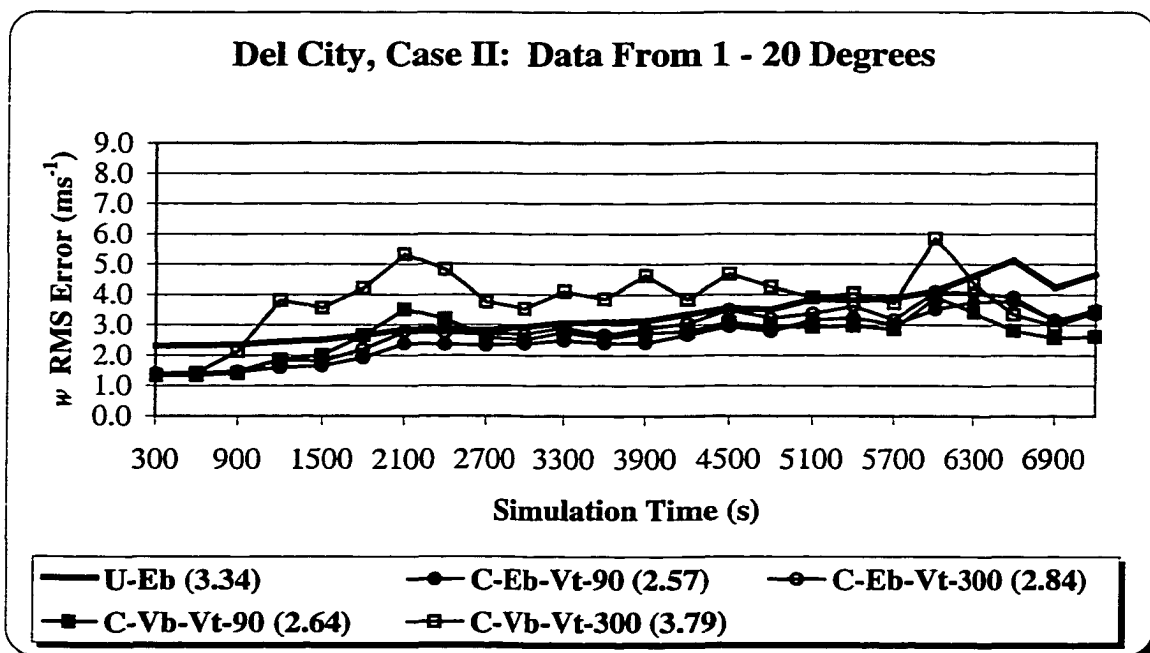
Figure 5.8 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the Del City supercell storm, taken from control technique U-Ib at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).



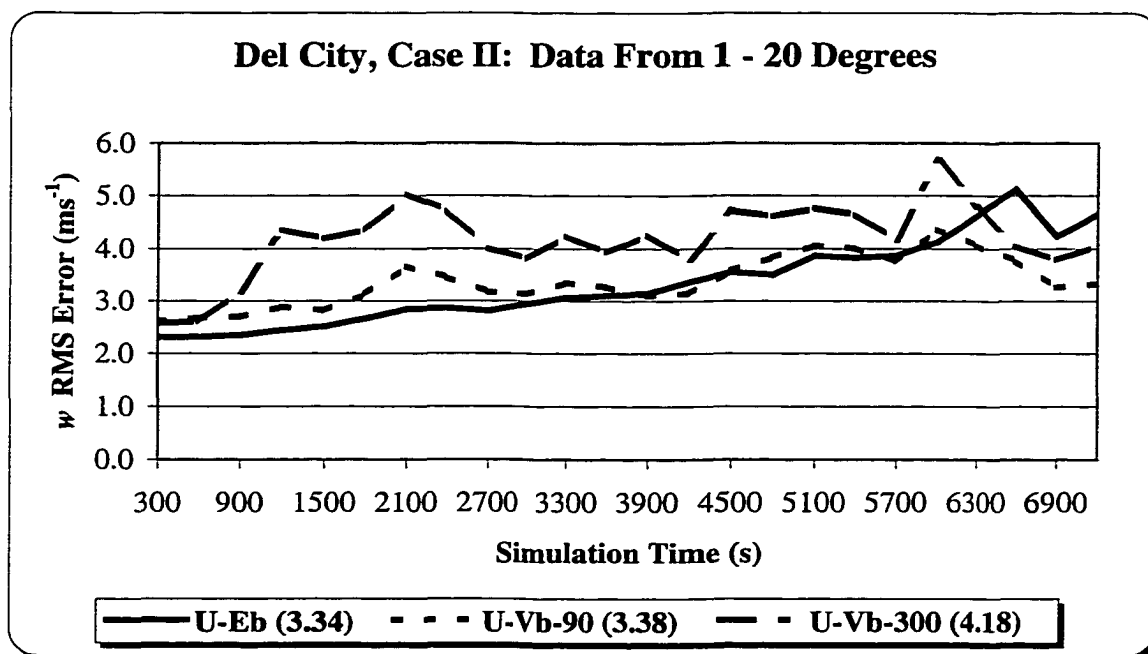
the results of MB98 the control technique U would be expected to suffer severely were bias and/or trend errors introduced.

Figure 5.9b shows that, given the ~500 m data void below the lowest radar data level in Case II, extrapolating divergence and application of the vorticity equation (with 90 s volume scan periods) yield boundary condition fields that provide statistically similar accuracy for technique U. Averaged across all 24 retrieval times, the RMS vertical velocity errors from techniques U-Eb and U-Vb-90 are 3.34 and 3.38 $\text{m}\cdot\text{s}^{-1}$, respectively. However, when the volume scan periods are increased to 300 s, extrapolating divergence appears to be a better means of estimating the lower boundary condition, at least for this Del City storm simulation.

Figure 5.10 explores the spatial differences between the true vertical velocity field near the lower boundary ($z = 1 \text{ km}$) and the analysis fields using extrapolated divergence (technique C-Eb-Vt-90) and the vorticity equation (C-Vb-Vt-90) to estimate the lower boundary condition. (Since the height of the lower data boundary varies from 500 m to 1000 m across the subdomain of Fig. 5.10 it is useful to view the vertical velocity at 1000 m so that data are present everywhere.) The fields are again presented at $t = 3600 \text{ s}$ to facilitate comparison with figures from the other cases discussed here. Comparison of Figs. 5.10a and 5.10b shows that extrapolating divergence yields an updraft that is misshaped, with the southern half of the updraft missing completely. The downdraft is retrieved much better, with the primary problem being a slight underestimation of the downdraft magnitudes. Ray et al. (1976) have noted that divergence in downdrafts may reach a maximum in the lowest few hundred meters, which is often not sampled by radars. In this case, failure to sample data from the lowest few hundred meters would



(a)



(b)

Figure 5.9: Root mean square vertical velocity retrieval errors from experiments on the Del City supercell simulation using volume scan geometries providing data between 1° and 20° (Case II).

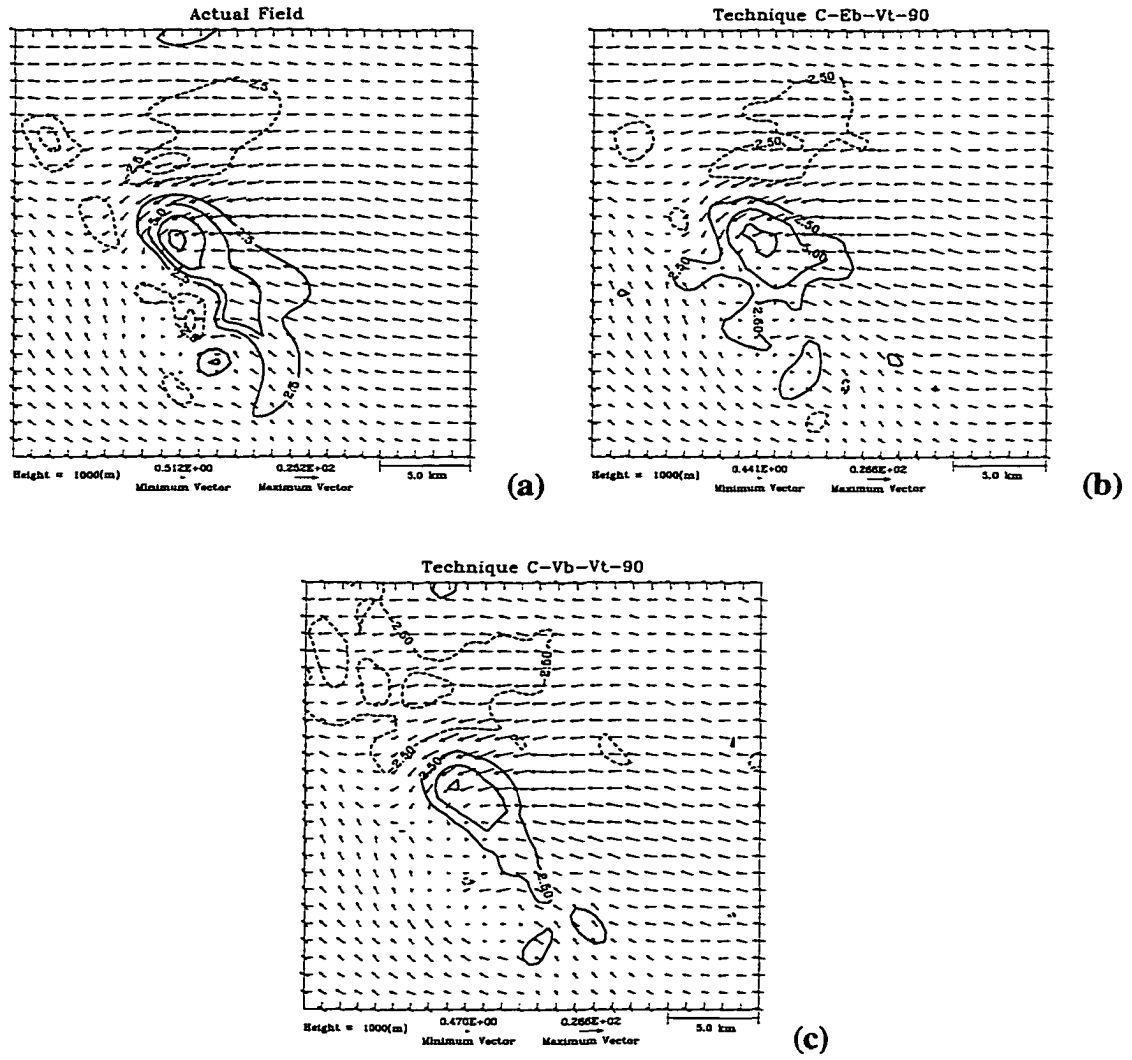


Figure 5.10: Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell storm, taken from (a) the true field, (b) technique C-Eb-Vt-90, and (c) technique C-Vb-Vt-90. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 20° elevation for each radar (Case II).

lead to an underestimation of the low-level divergence and a subsequent underestimation of the true downdraft velocity at the lowest radar data level when using extrapolated divergence to estimate the boundary condition. Thus, the field in Fig. 5.10b is not unexpected. Figure 5.10c shows that the vorticity equation does a (qualitatively) better job of delineating the true updraft region. However, the vertical velocities are slightly underestimated, and the spatial extent of the downdraft region is exaggerated. All features considered, a comparison of the quality of the fields from each technique would likely be considered a 'draw'. This is verified by the similarity of the RMS vertical velocity errors of the two fields.

Slight underestimation of magnitudes of vertical velocity maxima is a persistent problem with the vorticity techniques in this work. However, the problem seems to be attributable to imperfect filters rather than a property of the techniques. Although the filters applied to the horizontal wind field prior to application of the vorticity equation are intended to remove only $2\Delta x$ and $3\Delta x$ scales, the filtering schemes also cause significant damping in scales of $4\Delta x$ and larger as well. Since wave interactions on the smallest remaining scales are responsible for creating the small scale features in the boundary condition fields, these features are inevitably damped in magnitude. This problem is closely related to the discussion at the end of Section A of Chapter 4.

The wind fields at $t = 1800$ s, $t = 3600$ s, $t = 5400$ s, and $t = 7200$ s from the control technique (U-Eb) and what could be called the 'nearly best' (as determined from Fig. 5.9) vorticity based technique (C-Vb-Vt-90) are compared against the true wind field in Figs. 5.11 and 5.12. In this case it seems more logical to show the 'nearly best' technique instead of the best technique (which is actually C-Eb-Vt-90) since C-Eb-Vt-90

would share the same lower boundary condition with the control technique U-Eb, yielding figure panels which look identical. Further, C-Eb-Vt-90 is only marginally preferable to C-Vb-Vt-90 for Case II with this storm, and will actually be shown to be inferior to C-Vb-Vt-90 in the upcoming microburst storm. Figure 5.11 depicts the wind fields at $z = 1.0$ km, and shows how extrapolating divergence and application of the vorticity equation perform in retrieving the appropriate lower boundary condition over a range of times. As indicated in Fig. 5.9, the overall accuracy of the two techniques is similar in this case. Figure 5.12 is valid at $z = 10.0$ km, and continues to reveal the advantages of applying a second boundary condition determined from the vorticity equation to control accumulating divergence errors.

In extreme cases where data below 2° (or even higher) are unavailable, such as in Case III, there has traditionally been little hope of retrieving accurate vertical velocity fields since reliable estimation of the boundary condition was exceedingly difficult. This is reflected in Fig. 5.13a by curve U-Eb (control), which now exhibits an average RMS vertical velocity error of $5.01 \text{ m}\cdot\text{s}^{-1}$ across all 24 retrieval times. Although applying extrapolated divergence in method C with the vorticity equation providing the upper boundary condition does show some improvement over the traditional technique (techniques C-Eb-Vt), the most significant improvement is evident in curves C-Vb-Vt-90 and C-Vb-Vt-300, which show up to a 44% improvement over the traditional technique with $2.79 \text{ m}\cdot\text{s}^{-1}$ and $3.91 \text{ m}\cdot\text{s}^{-1}$ RMS errors, respectively. It is interesting to note that the RMS error from the C-Vb-Vt technique grew only slightly between Cases I and III, from 2.43 to $2.79 \text{ m}\cdot\text{s}^{-1}$ for 90 s volume scans and from 3.59 to $3.91 \text{ m}\cdot\text{s}^{-1}$ for 300 s volume scans, in spite of the fact that the lower boundary was moved approximately 1500 m away

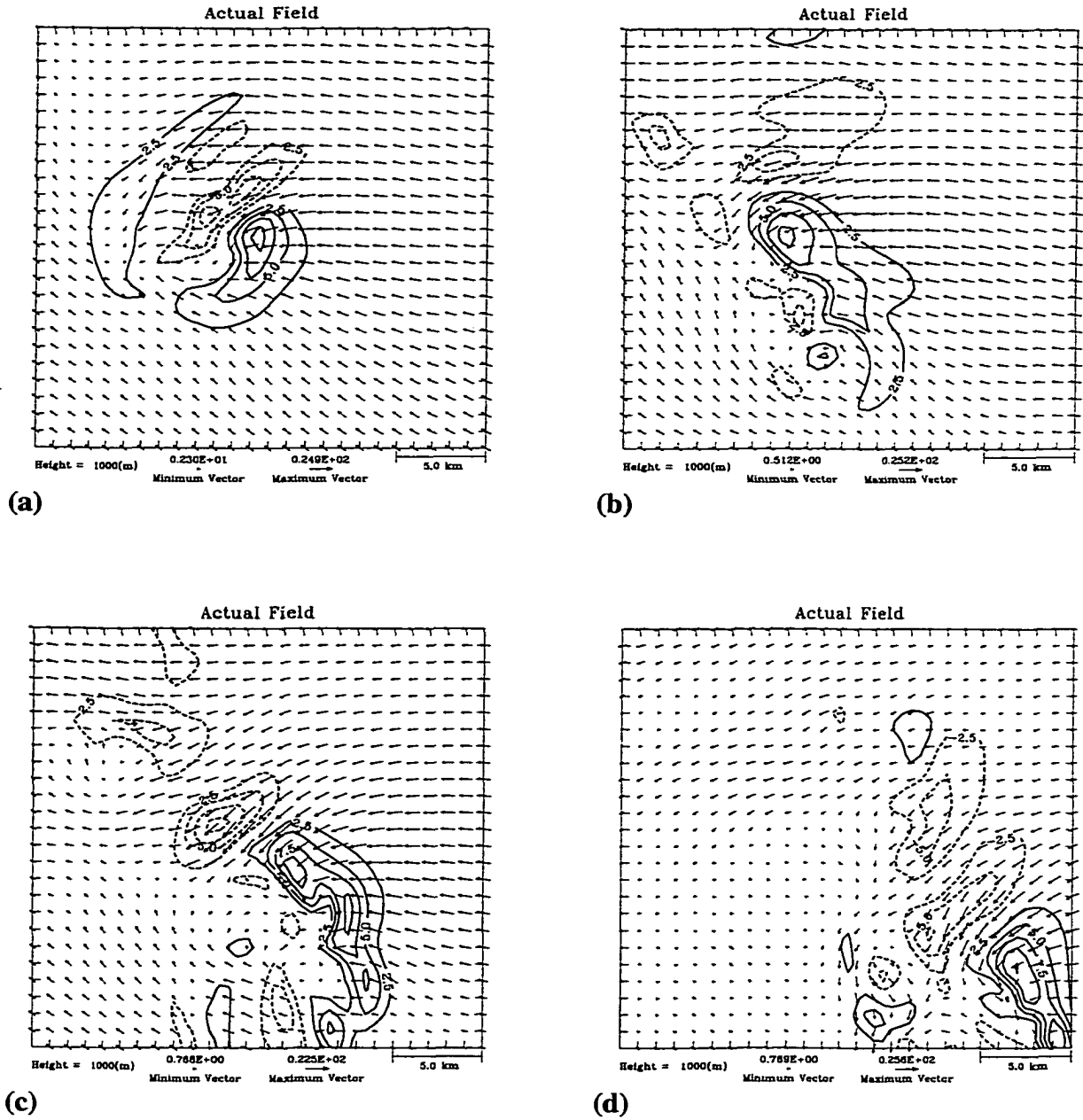


Figure 5.11: Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell storm, taken from the true field at (a) $t = 1800$ s, (b) $t = 3600$ s, (c) $t = 5400$ s, and (d) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 20° elevation for each radar (Case II).

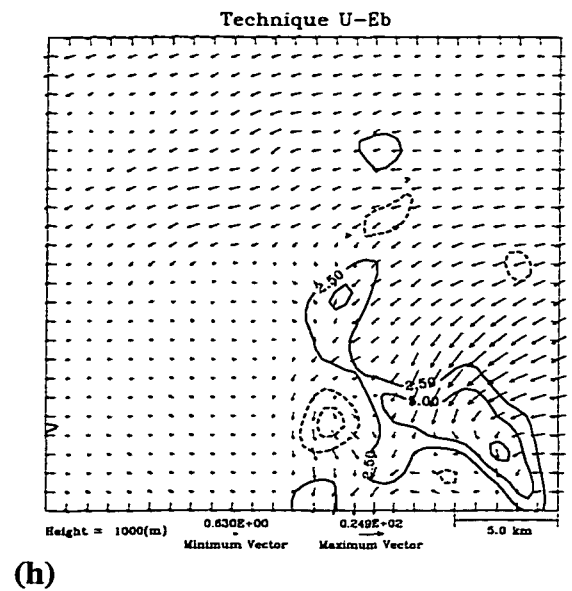
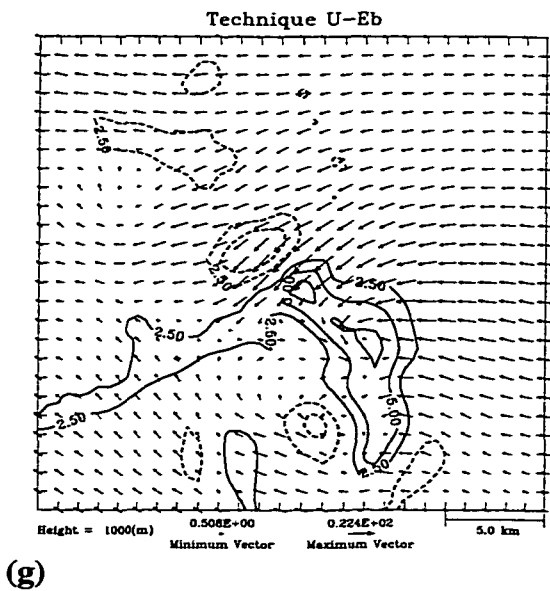
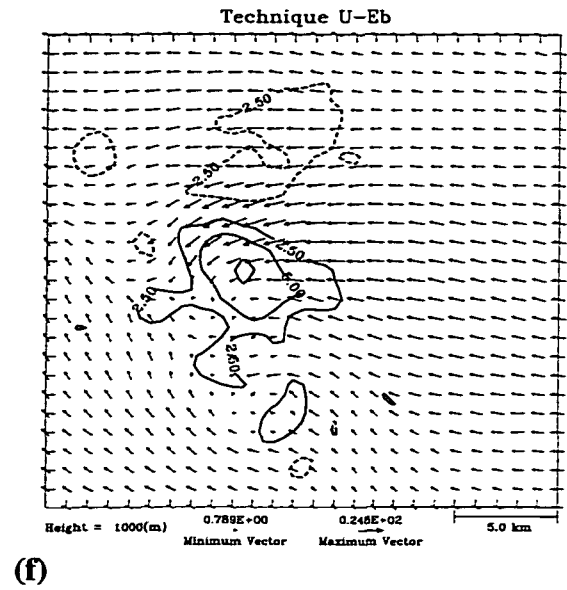
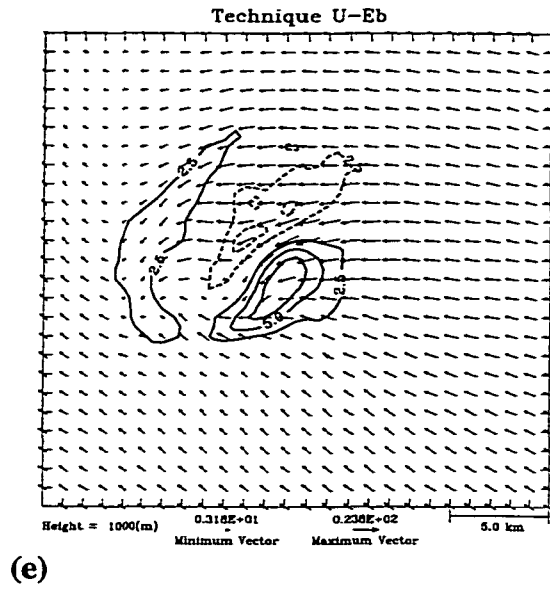


Figure 5.11 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell storm, taken from control technique U-Eb at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 20° elevation for each radar (Case II).

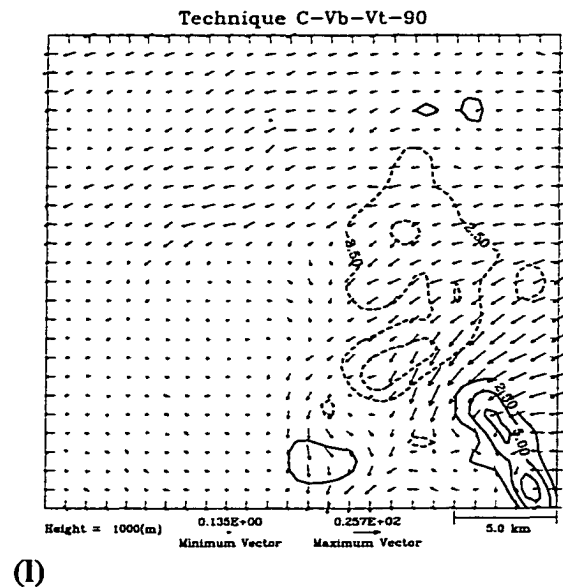
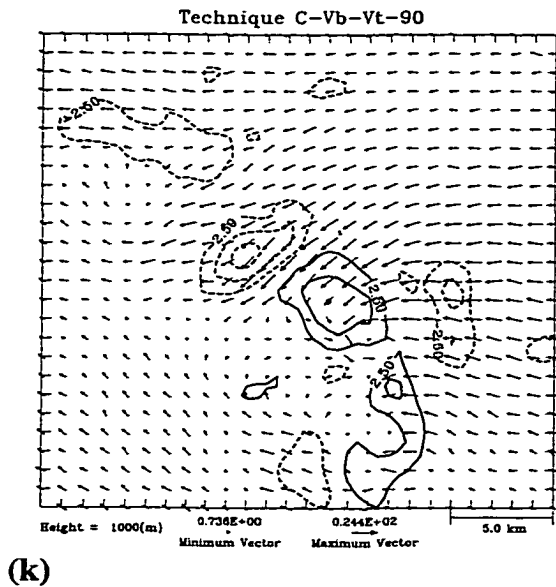
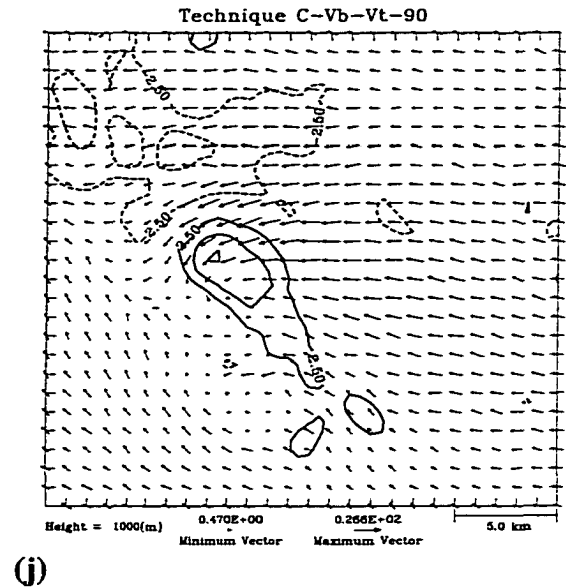
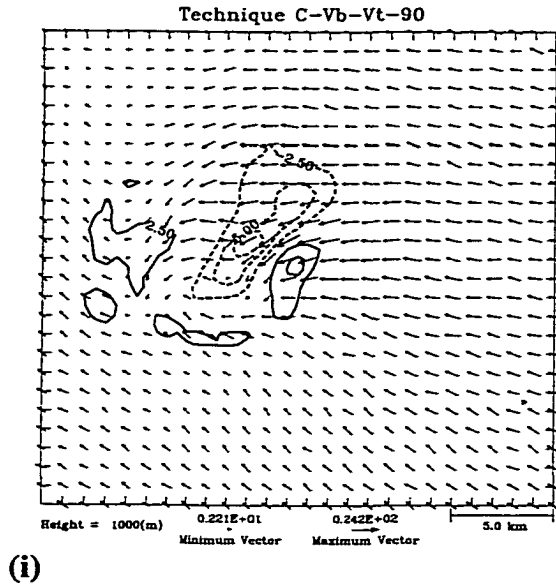


Figure 5.11 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell storm, taken from the ‘nearly best’ vorticity technique C-Vb-Vt-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 20° elevation for each radar (Case II).

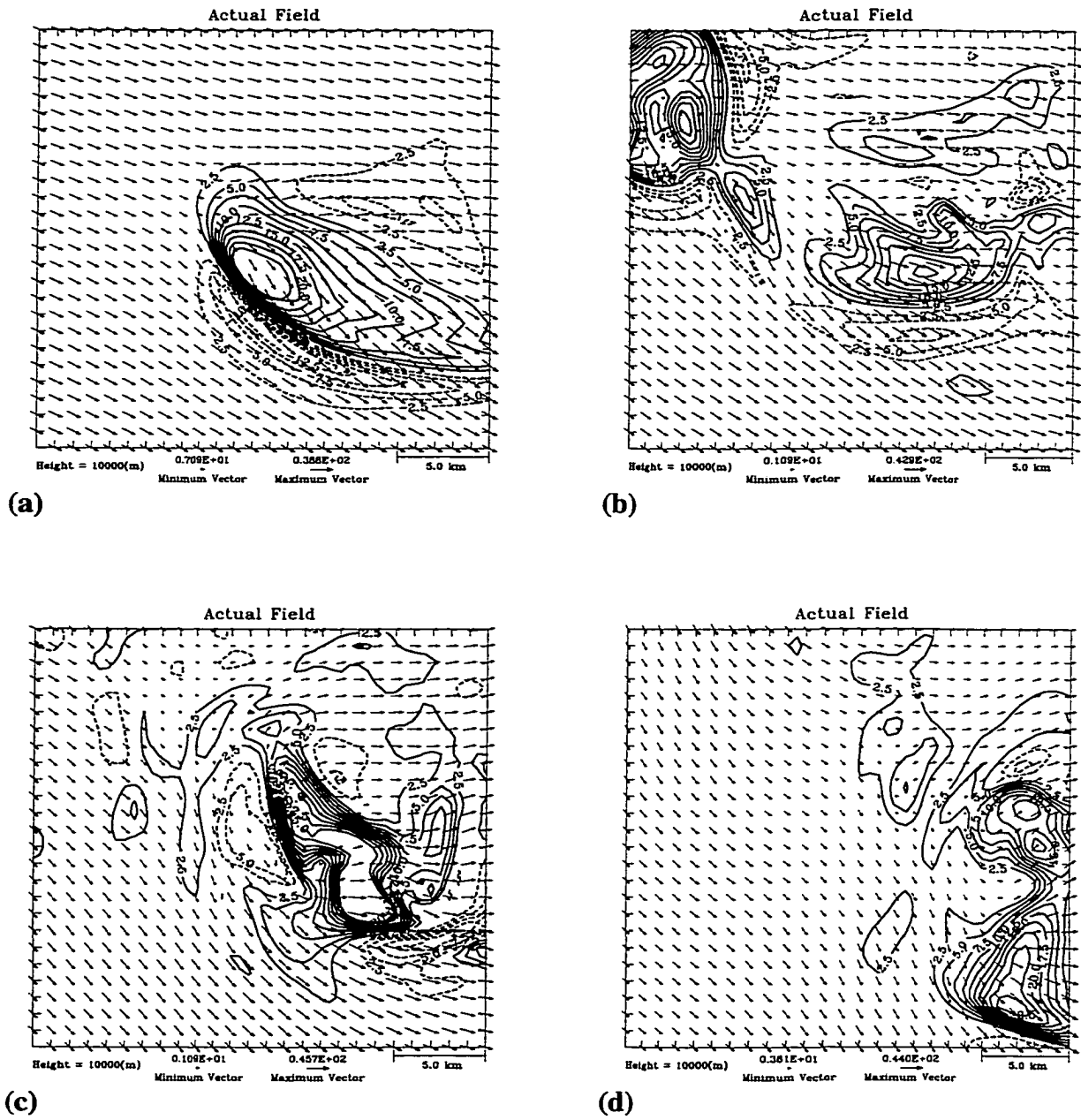


Figure 5.12: Horizontal wind vectors and contours of w at $z = 10.0$ km in the Del City supercell storm, taken from the true field at (a) $t = 1800$ s, (b) $t = 3600$ s, (c) $t = 5400$ s, and (d) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 20° elevation for each radar (Case II).

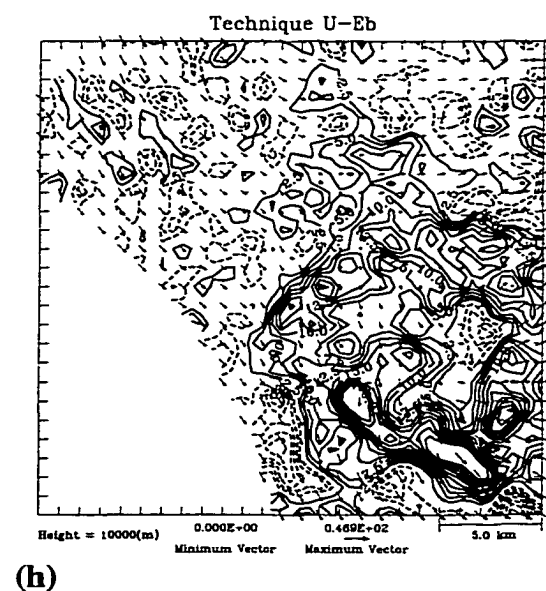
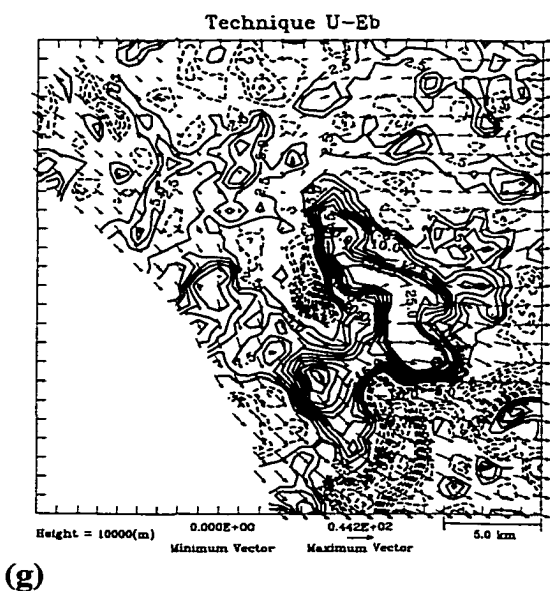
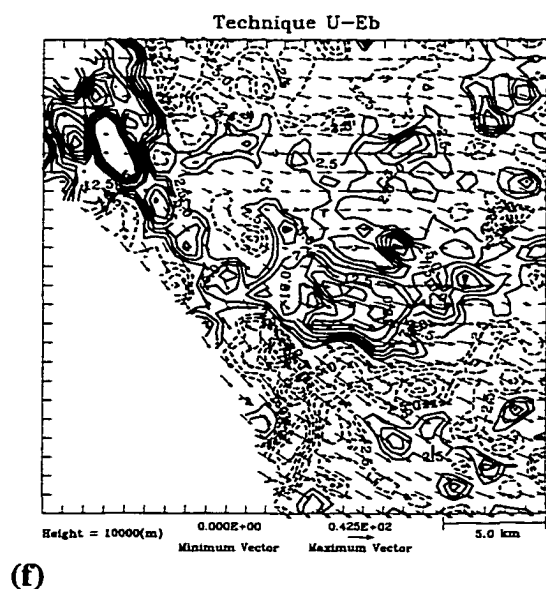
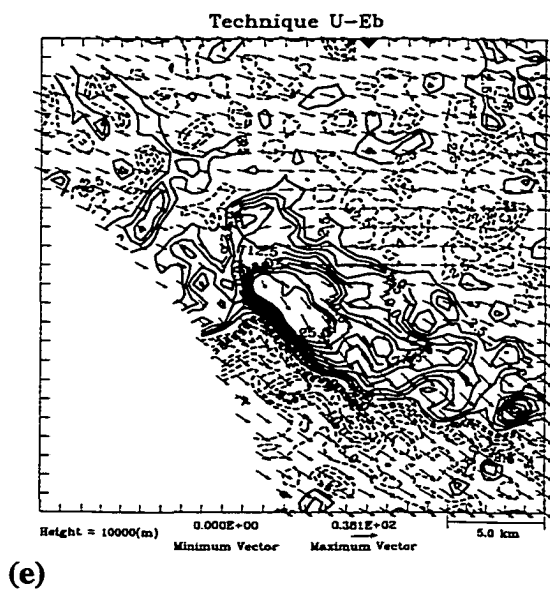


Figure 5.12 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the Del City supercell storm, taken from control technique U-Eb at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 20° elevation for each radar (Case II).

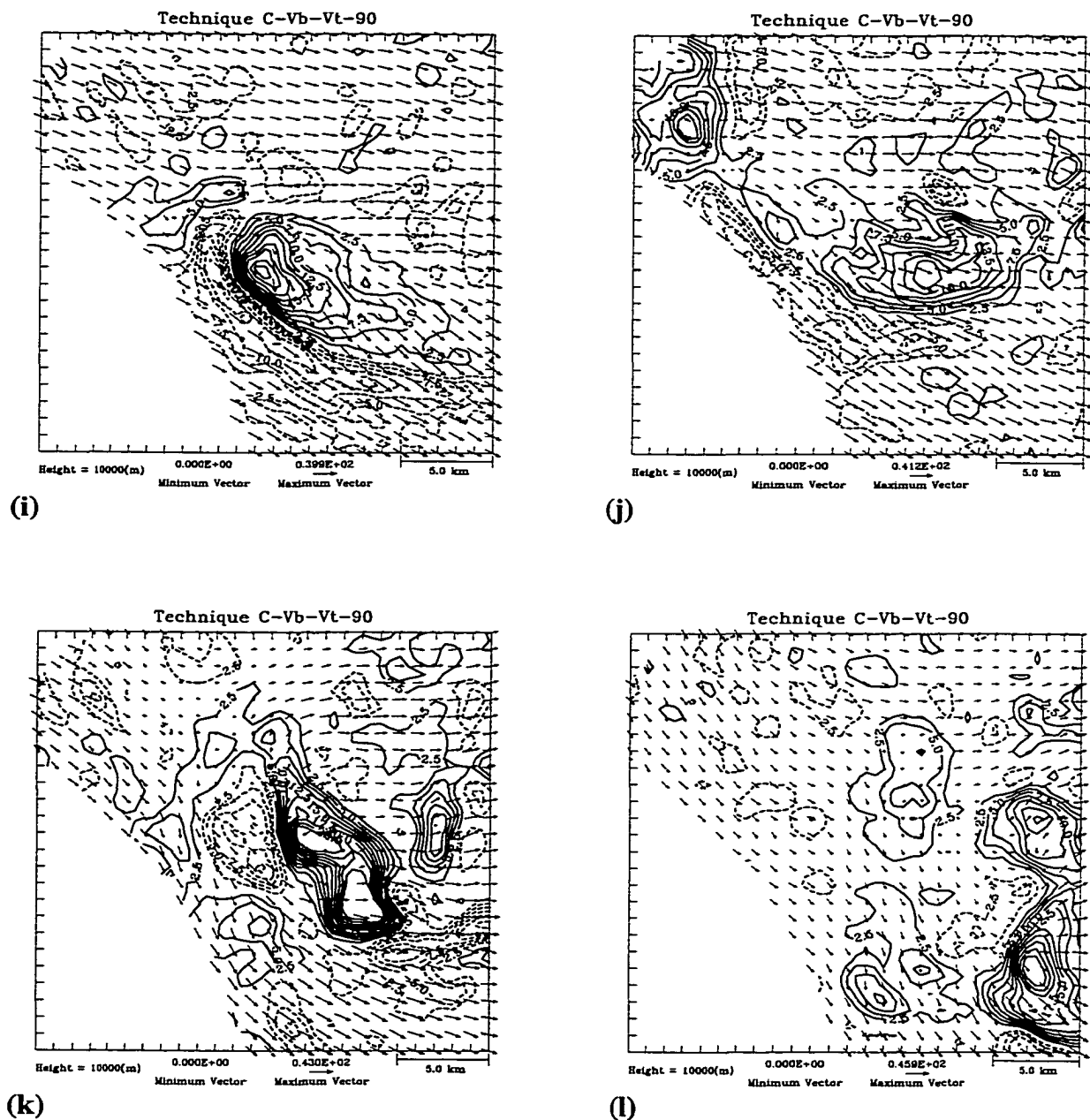


Figure 5.12 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the Del City supercell storm, taken from the 'nearly best' vorticity technique C-Vb-Vt-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 20° elevation for each radar (Case II).

from the surface. This fact bodes well for use of method C-Vb-Vt in cases of severe scan geometry limitations due to blockage, contamination, or other factors.

In Case III the vorticity equation appears to be preferable to extrapolating divergence as a means of lower boundary condition specification, even with volume scan periods of 300 s (see Fig. 5.13b, and recall that differences between the three techniques shown are due entirely to boundary condition differences). Techniques U-Vb-90 and U-Vb-300 now show average RMS vertical velocity errors of 3.64 and 4.42 $\text{m}\cdot\text{s}^{-1}$, compared to 5.01 $\text{m}\cdot\text{s}^{-1}$ for control technique U-Eb. Interestingly, the RMS error of technique U-Eb increased 50%, from 3.34 to 5.01 $\text{m}\cdot\text{s}^{-1}$, as the lowest data level was increased from 1° to 2° . However, the vorticity techniques showed minimal error increases of 6-7%. Once again, this fact bodes well for application of the vorticity equation to retrieve boundary condition fields under conditions of severe data blockage.

Figure 5.14 shows one spatial example of how the vorticity equation can be preferable to extrapolating divergence in a case of severely lacking low-level data. Figure 5.14a shows the true vertical velocity field at $z = 1.5 \text{ km}$ and $t = 3600 \text{ s}$. For reference, four significant features of the true vertical velocity field are marked with X's. Technique C-Eb-Vt-90 (Fig. 5.14b) shows a significant degradation in the vertical velocity field obtained through extrapolation of the divergence at the lowest radar data level when compared to Case II (Fig. 5.10b). The updraft region using extrapolated divergence to estimate the lower boundary condition is now displaced to the west of its true position. This has very negative effects on the vertical velocity errors. Although the downdraft placement is still reasonable in Fig. 5.14b, spurious updrafts show up to the north and west of the downdraft. Examination of the retrieval from technique C-Vb-Vt-

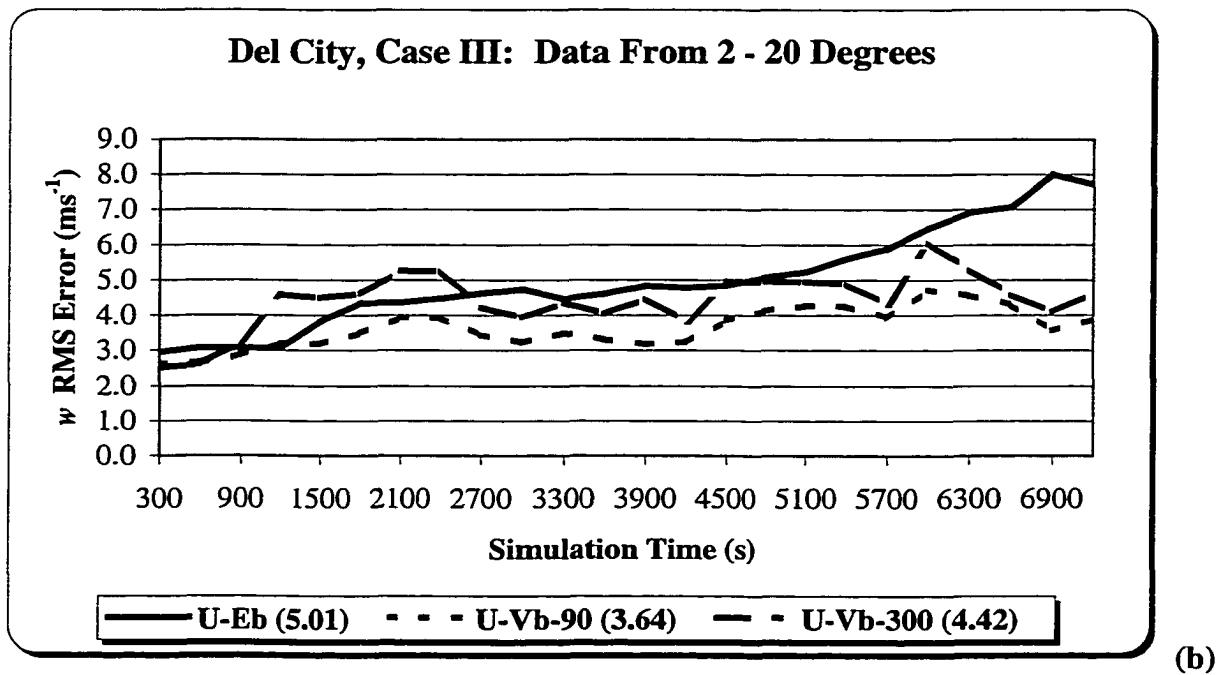
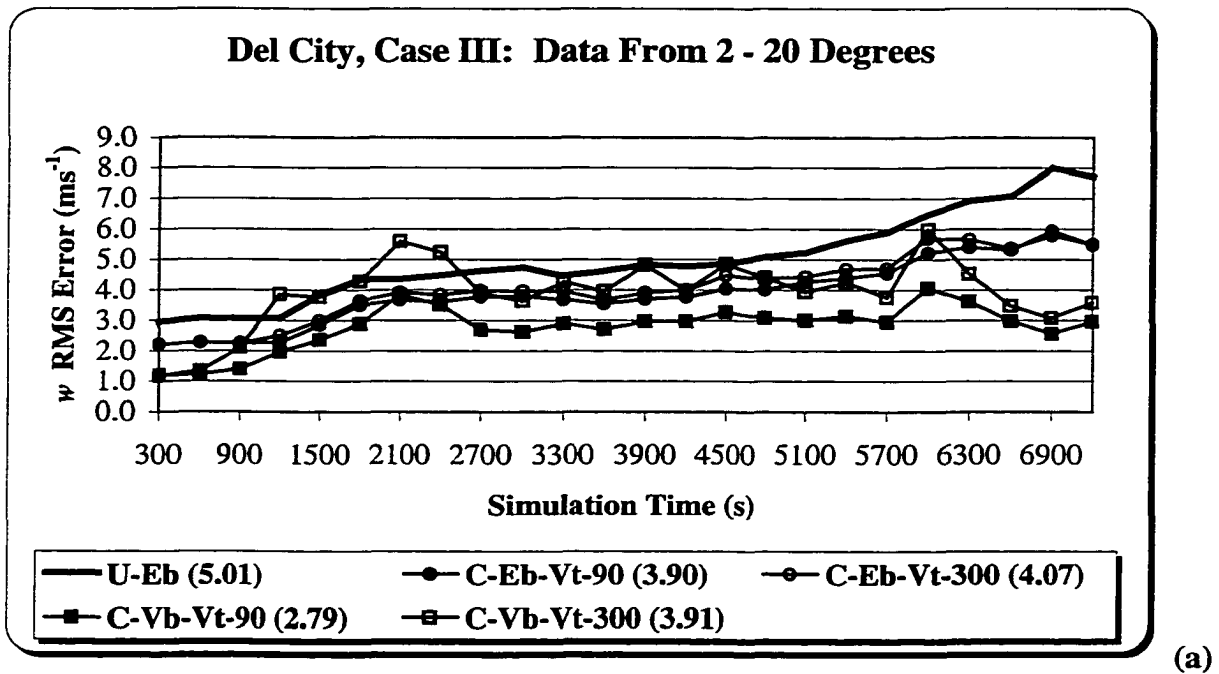


Figure 5.13: Root mean square vertical velocity retrieval errors from experiments on the Del City supercell simulation using volume scan geometries providing data between 2° and 20° (Case III).

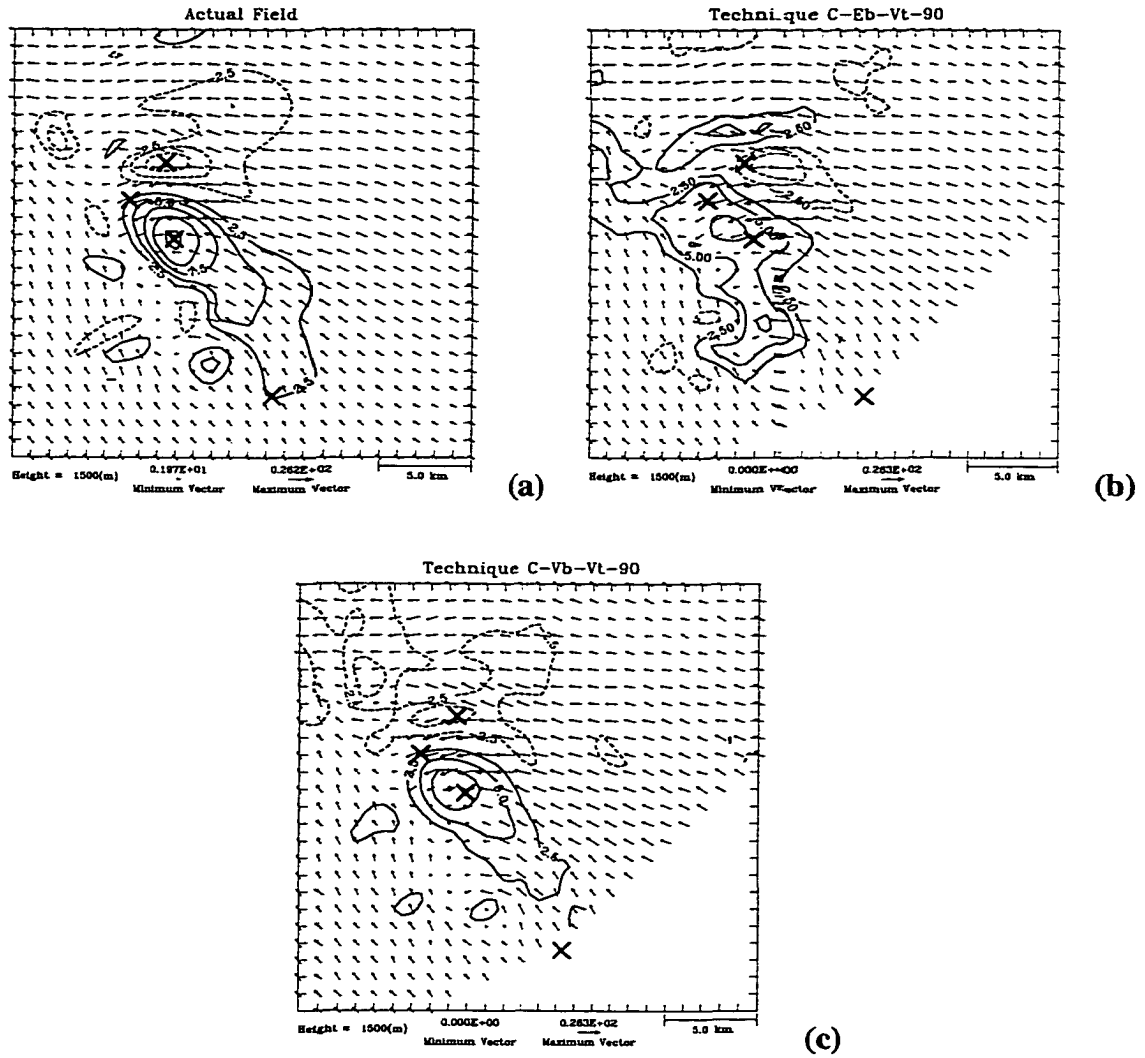


Figure 5.14: Horizontal wind vectors and contours of w at $z = 1.5$ km in the Del City supercell storm, taken from (a) the true field, (b) technique C-Eb-Vt-90, and (c) technique C-Vb-Vt-90. Contour interval is 2.5 m/s . Scan geometry is limited to regions between 2° and 20° elevation for each radar (Case III).

90, shown in Fig. 5.14c, clearly shows a significant improvement in the spatial representation of the true vertical velocity field. Reexamining Fig. 5.13 it is apparent that this trend is even more pronounced later in the simulation, as extrapolated divergence appears to become an increasingly poor estimator of the lower boundary condition. On the other hand, the errors from the techniques using the vorticity equation to estimate the lower boundary condition are much more stable over time.

Figures 5.15 and 5.16 again explore the wind fields at $t = 1800$ s, $t = 3600$ s, $t = 5400$ s, and $t = 7200$ s from the control technique (U-Eb) and the ‘best’ (as determined from Fig. 5.13) vorticity based technique (C-Vb-Vt-90). With the lower data boundary now residing 1000 to 2000 m above the ground over the subdomain shown, application of the vorticity equation to retrieve the lower boundary condition has clearly become preferable to extrapolating divergence (notice that many of the features in Fig. 5.15e-5.15h are misplaced and/or misshaped). Figure 5.16 is valid at $z = 10.0$ km, and continues to reveal the advantages of applying a second boundary condition determined from the vorticity equation to control accumulating divergence errors. However, with the problematic lower boundary condition from the control technique, the vertical velocity at 10.0 km is now much worse than that of technique C-Vb-Vt-90 (compare Figs. 5.16e-5.16h with Figs. 5.16i-5.16l).

In Case IV data are assumed to be available between 1° and 30° for each radar. This scan geometry provides data to above storm top, allowing for the application of a downward integration technique (D). Recall that with technique D the noise resulting from accumulation of mass divergence errors is realized at lower levels, where density stratification causes the resulting w errors to be significantly smaller than those observed

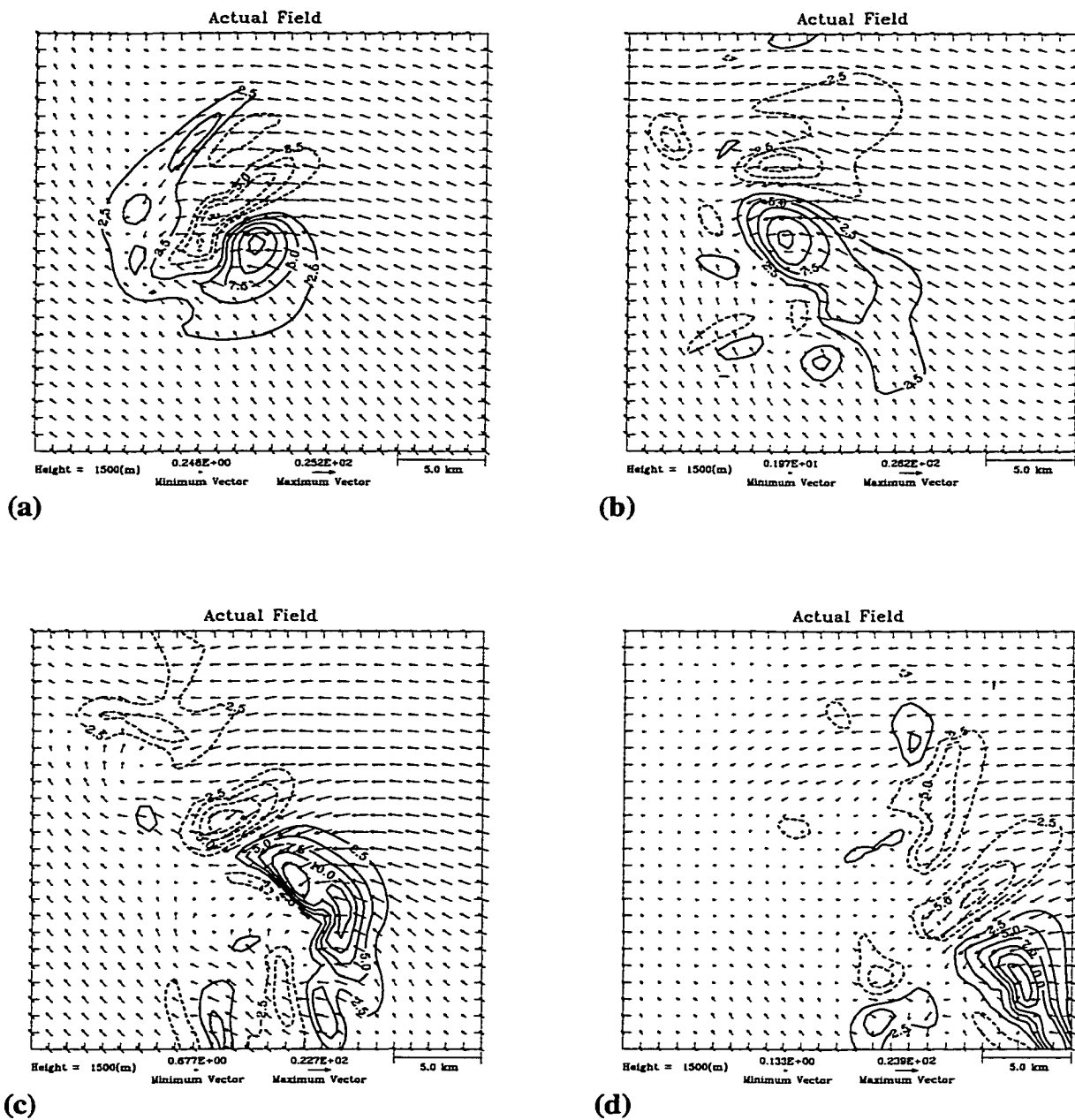


Figure 5.15: Horizontal wind vectors and contours of w at $z = 1.5$ km in the Del City supercell storm, taken from the true field at (a) $t = 1800$ s, (b) $t = 3600$ s, (c) $t = 5400$ s, and (d) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 2° and 20° elevation for each radar (Case III).

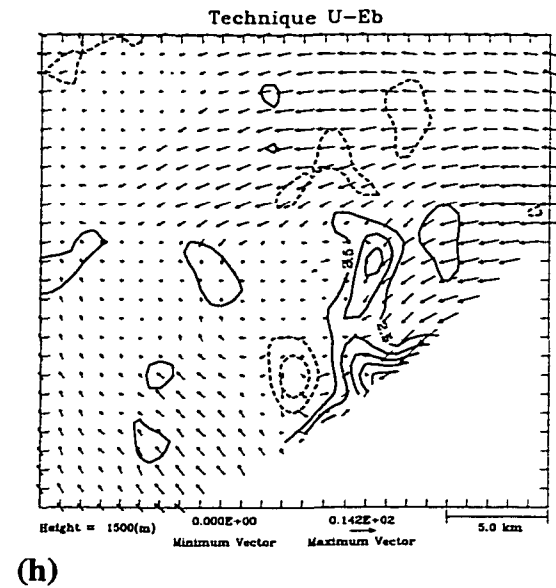
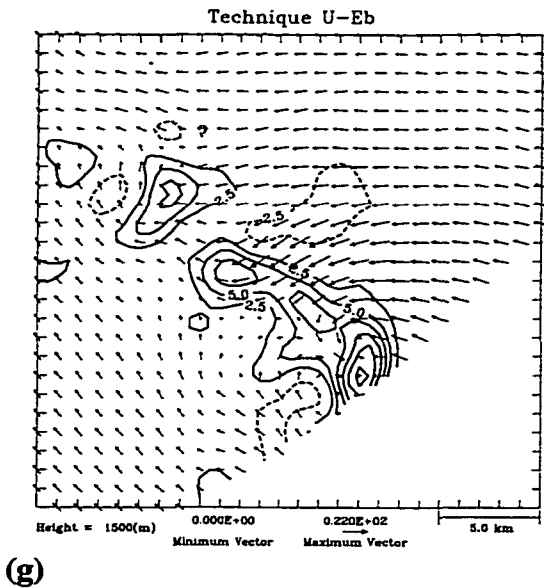
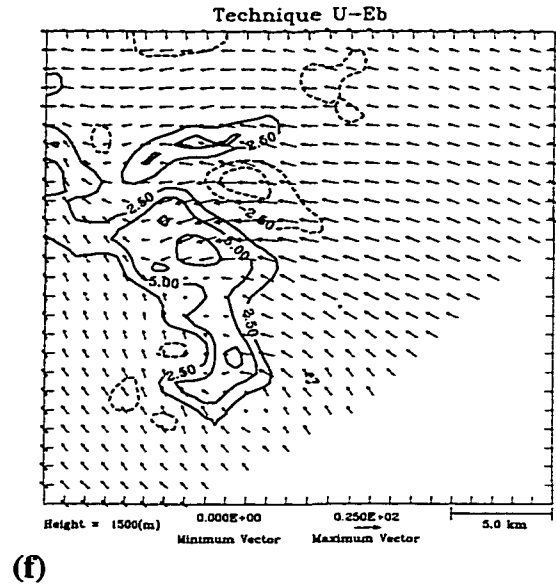
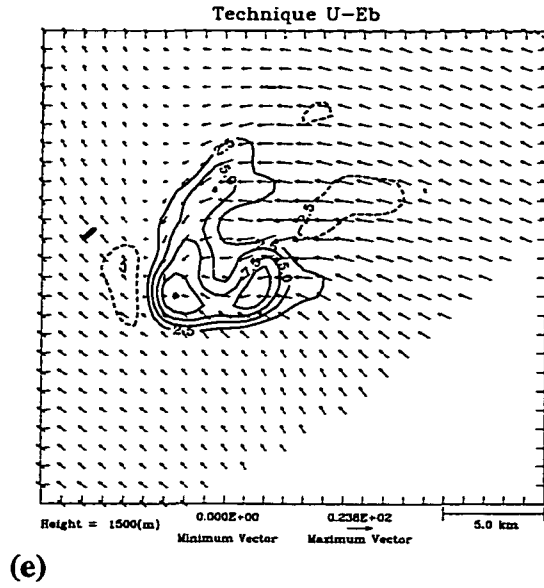
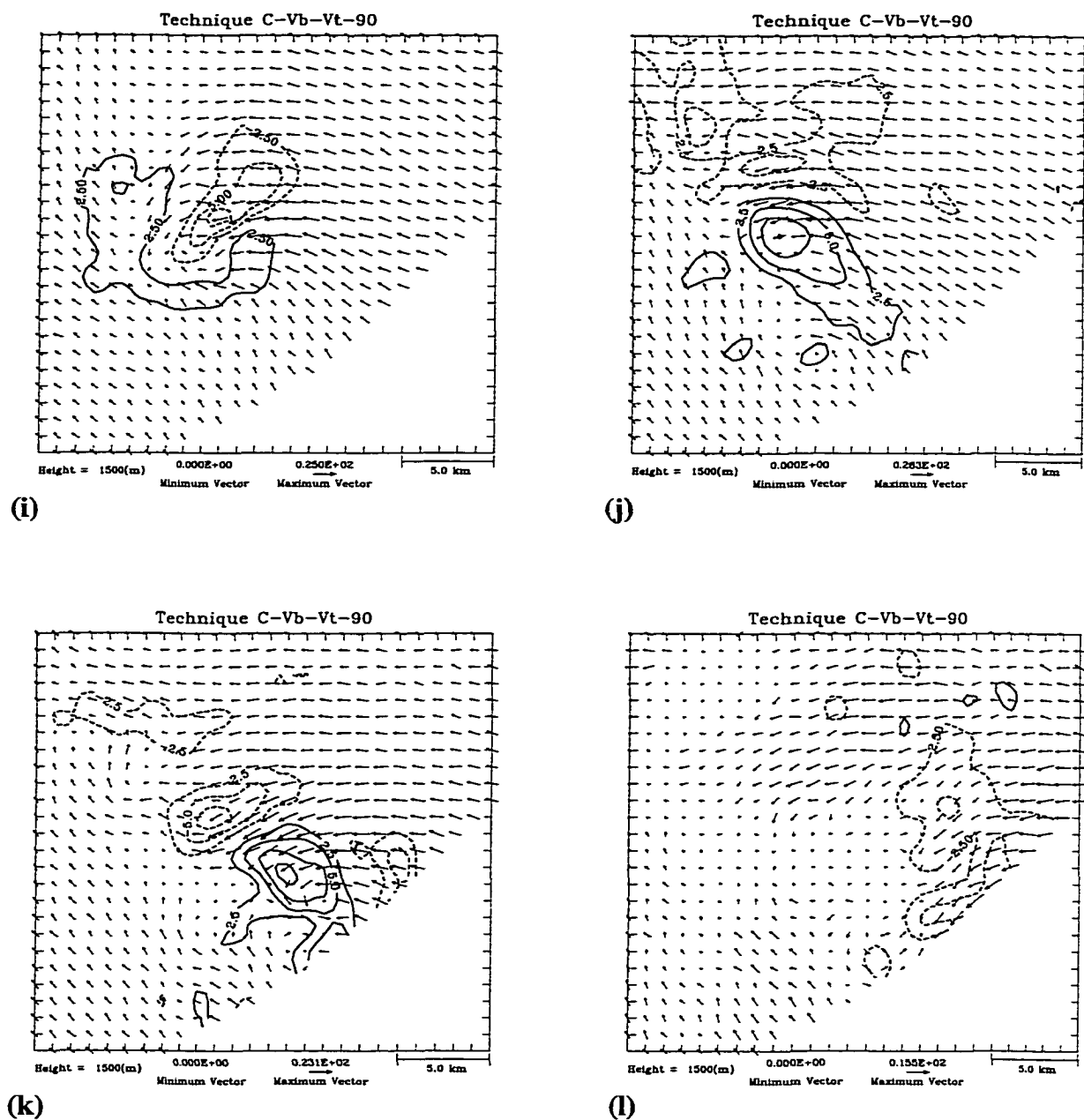


Figure 5.15 (continued): Horizontal wind vectors and contours of w at $z = 1.5$ km in the Del City supercell storm, taken from control technique U-Eb at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 2° and 20° elevation for each radar (Case III).



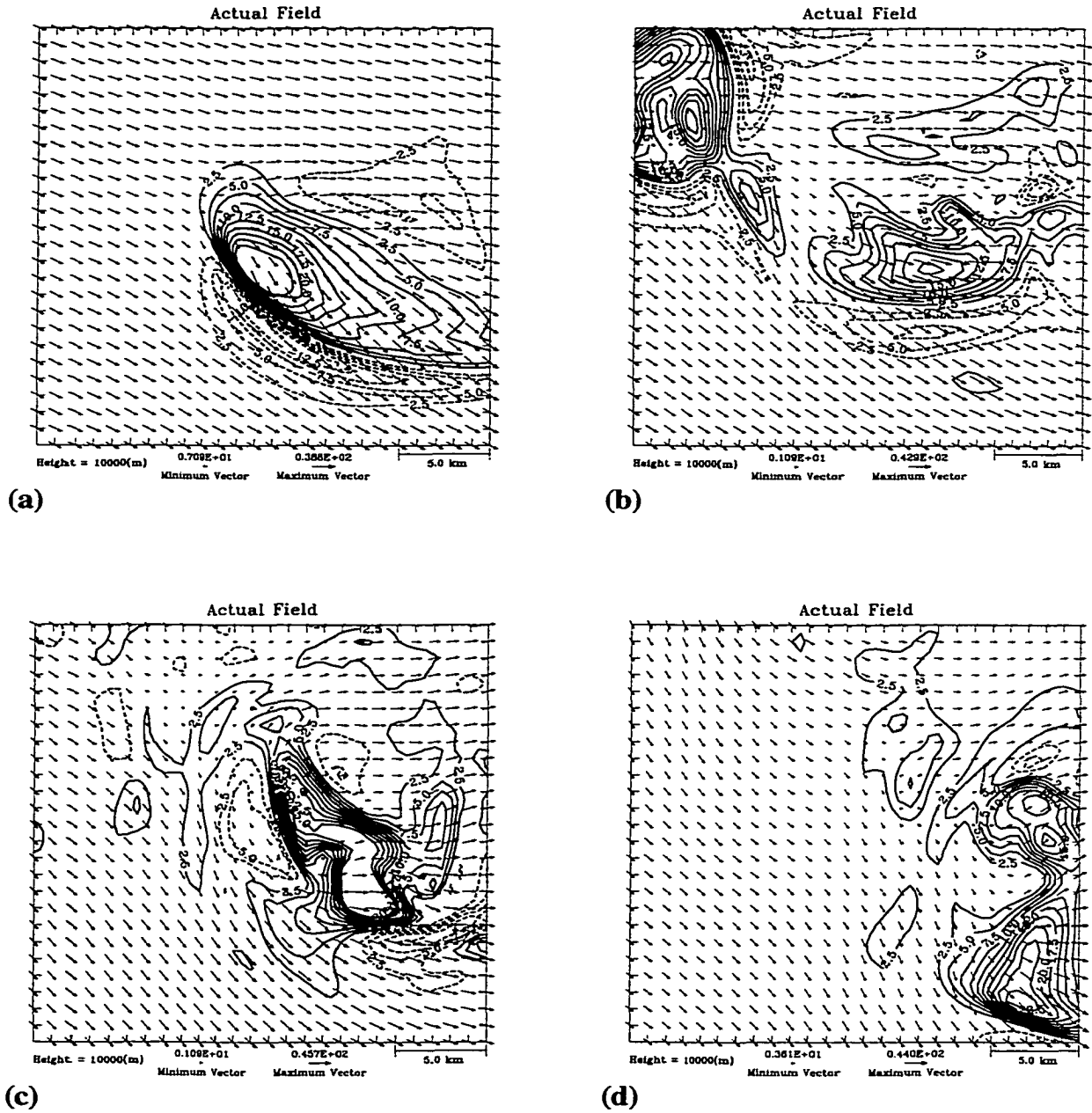
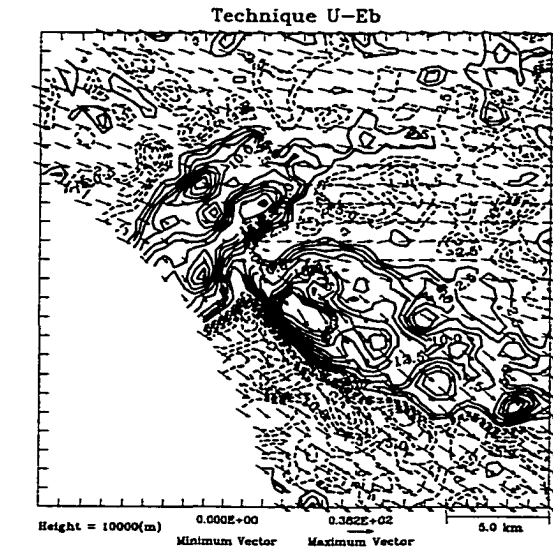
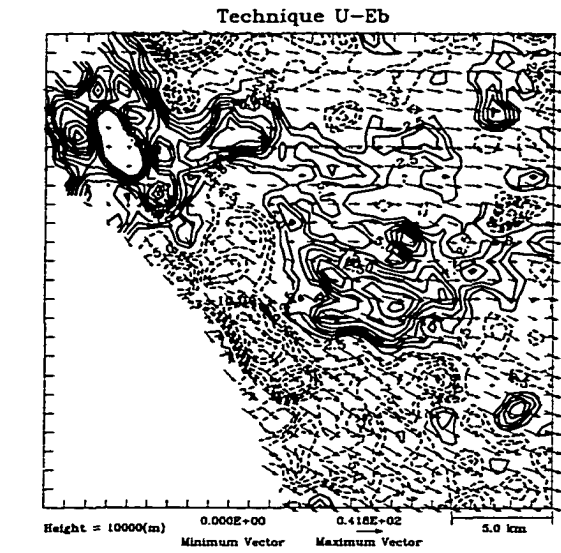


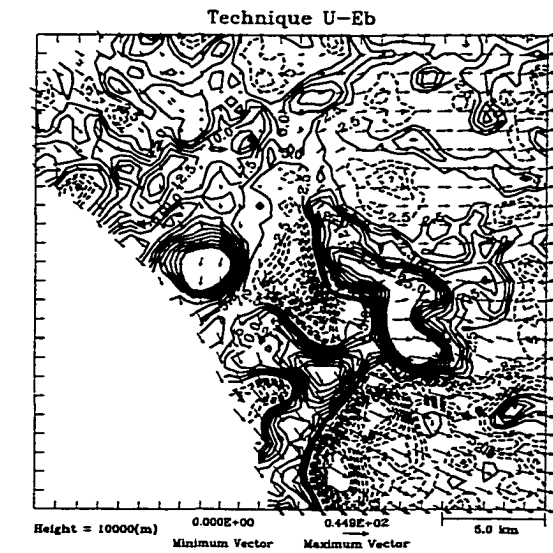
Figure 5.16: Horizontal wind vectors and contours of w at $z = 10.0$ km in the Del City supercell storm, taken from the true field at (a) $t = 1800$ s, (b) $t = 3600$ s, (c) $t = 5400$ s, and (d) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 2° and 20° elevation for each radar (Case III).



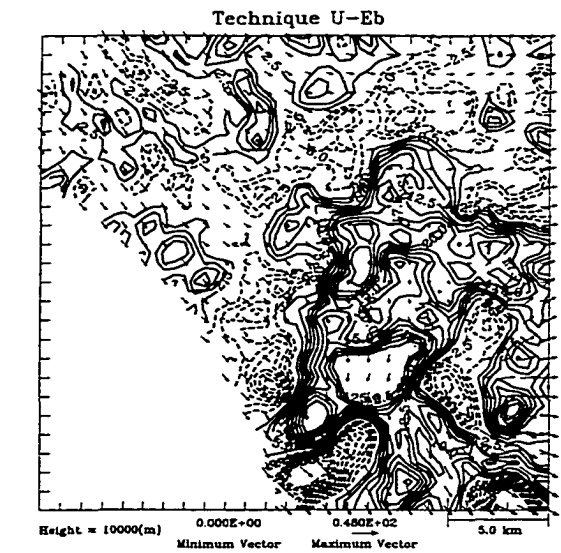
(e)



(f)



(g)



(h)

Figure 5.16 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the Del City supercell storm, taken from control technique U-Eb at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 2° and 20° elevation for each radar (Case III).

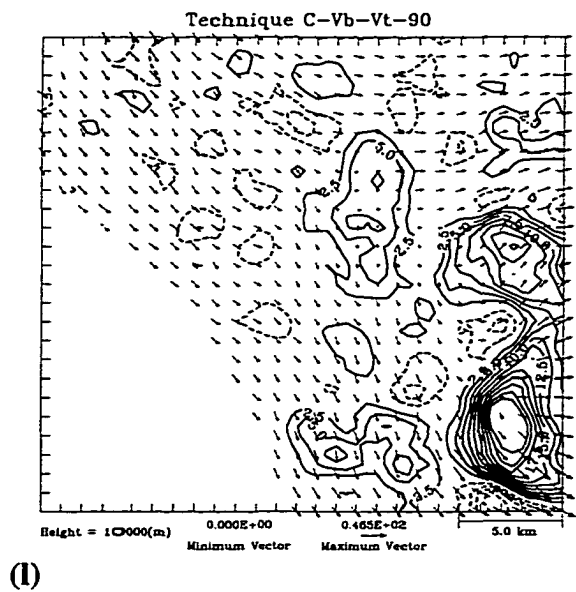
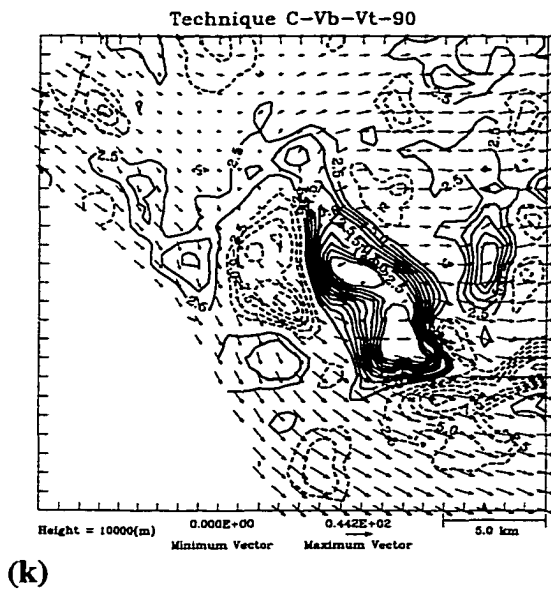
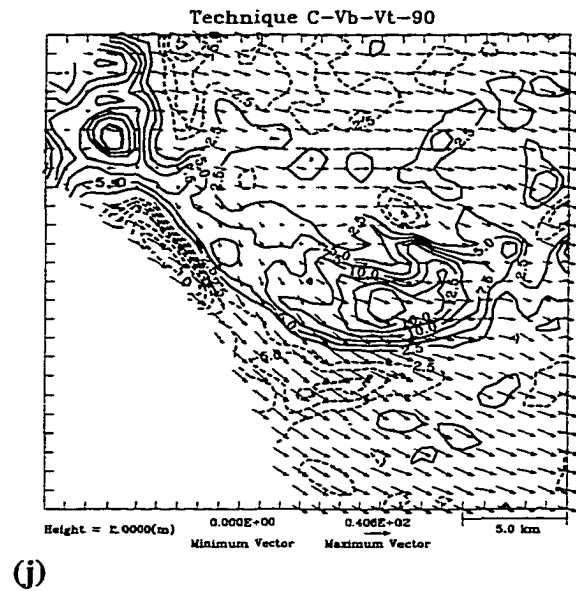
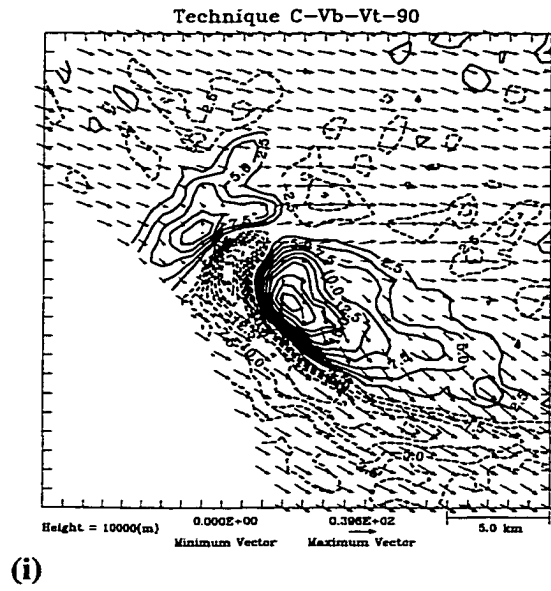


Figure 5.16 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the Del City supercell storm, taken from the ‘best’ vorticity technique C-Vb-Vt-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 2° and 20° elevation for each radar (Case III).

with the upward integration technique. For this case a lower elevation limit of 1° was chosen (instead of 0°) so that the impermeability condition cannot be applied at both the upper and lower boundaries. The vorticity equation would not be necessary were data available to 0° , since it is not likely to provide better information than the impermeability condition on either of these boundaries (this is verified for the lower boundary in Fig. 5.5b).

Since the accumulation of divergence errors should be significantly reduced, and since $w = 0$ is likely a safe assumption on the upper data boundary here, the vertical velocity retrievals from the control technique should be much improved in Case IV. Figure 5.17a shows that this is indeed the case, as the average RMS error from the control technique (D-It) has dropped to $1.60 \text{ m}\cdot\text{s}^{-1}$, 36% better than the control technique in any of the upward integration control cases. Reduction of w errors through use of the vorticity equation proves to be very difficult in this case. Only technique C-Vb-It-90 is able to show any improvement over the control technique, with a slight 2% error reduction to $1.57 \text{ m}\cdot\text{s}^{-1}$. The true vertical velocity field at $z = 1 \text{ km}$ and $t = 3600 \text{ s}$ and those retrieved from techniques D-It, C-Vb-It-90, C-Vb-It-300, and C-Eb-It are shown in Fig. 5.18. Clearly the accumulation of divergence errors has a much smaller effect with downward integration as compared to upward integration (compare, for example, Fig. 5.18b with Fig. 5.6b). Also, notice that using extrapolated divergence from 1° ($\sim 500 \text{ m}$) in combination with a $w = 0$ storm top boundary condition in technique C (curve C-Eb-It) actually produces worse results than control technique D-It (see Figs. 5.17a and 5.18). In other words, specifying a second, somewhat reasonable boundary condition actually deteriorates results. Based upon MB98 this finding is expected only when the degree of

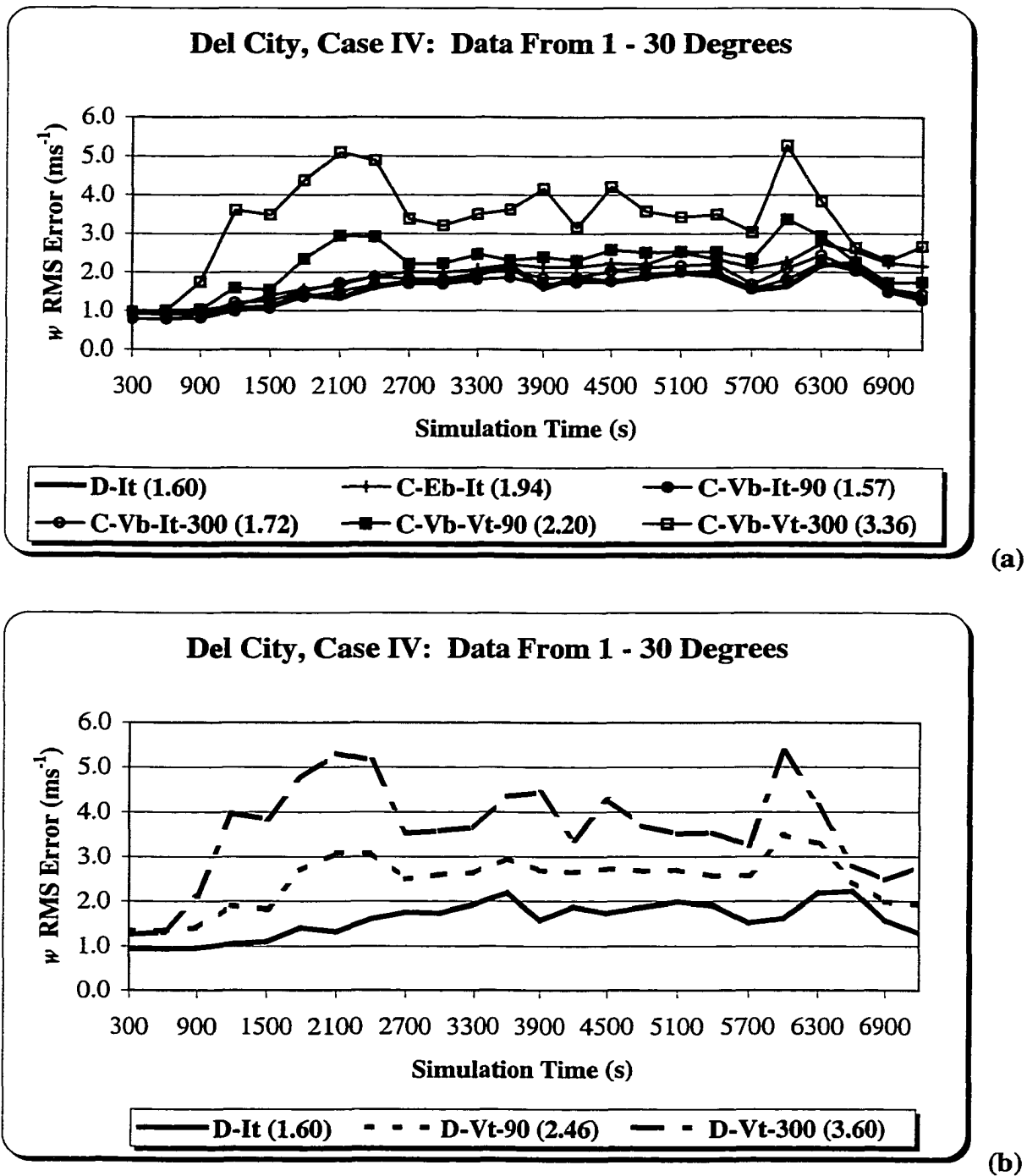


Figure 5.17: Root mean square vertical velocity retrieval errors from experiments on the Del City supercell simulation using volume scan geometries providing data between 1° and 30° (Case IV).

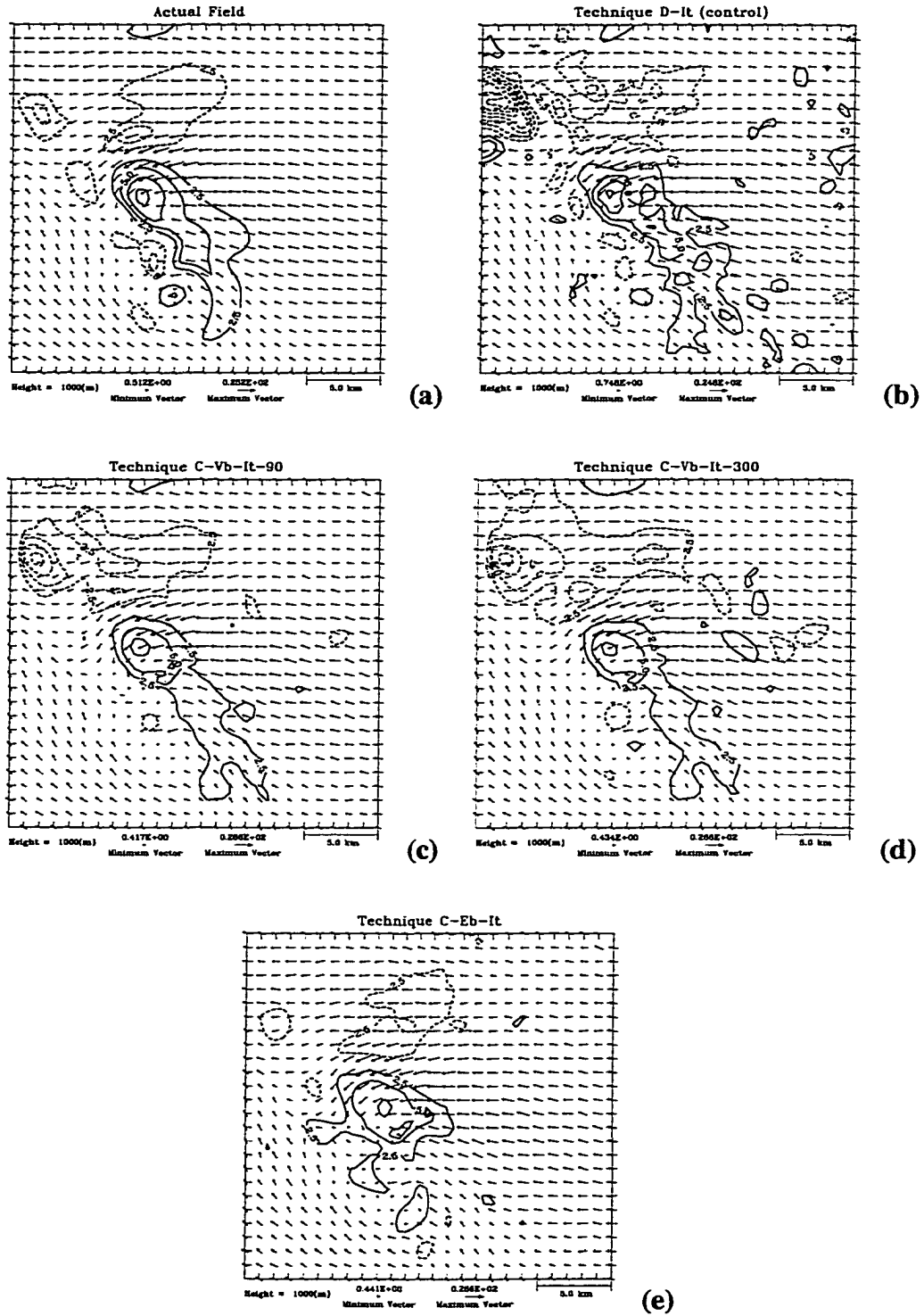


Figure 5.18: Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell storm, taken from (a) the true field, (b) control technique D-It, (c) technique C-Vb-It-90, (d) technique C-Vb-It-300, and (e) technique C-Vb-Vt-90. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).

error randomness is high, as is the case here. In the presence of systematic errors MB98 showed that specification of the second boundary condition can be very beneficial, even over the control downward integration technique used here. Again, this finding suggests that a second boundary condition derived from the vorticity equation may offer hope of further improvement over the control technique when non-random errors are present. However, as stated earlier, this has yet to be verified experimentally.

If viewed in terms of vertical mass flux (ρw) instead of vertical velocity, the error reduction with technique C-Vb-It-90 is a slightly more significant 9% (from 0.93 to 0.85 $\text{kg}\cdot\text{m}^{-2}\cdot\text{s}^{-1}$). Recall from the discussion in Chapter 2 that, unlike vertical velocity errors, vertical mass flux errors should be relatively insensitive to the direction of integration. Unfortunately, none of the cases presented here span the entire depth from 0° to 30° to permit fair comparison of techniques U and D with known $w = 0$ boundary conditions. However, average vertical mass flux errors of $1.13 \text{ kg}\cdot\text{m}^{-2}\cdot\text{s}^{-1}$ and $0.93 \text{ kg}\cdot\text{m}^{-2}\cdot\text{s}^{-1}$ from the control techniques in Cases I (U-Ib) and IV (D-It), respectively, are statistically similar results considering the average density within the data region of Case I is higher (so that ρw errors should be larger as well).

Figures 5.19 and 5.20 explore the wind fields at $t = 1800 \text{ s}$, $t = 3600 \text{ s}$, $t = 5400 \text{ s}$, and $t = 7200 \text{ s}$ from the control technique (which is now downward technique D-It) and the ‘best’ (as determined from Fig. 5.17) vorticity based technique (C-Vb-It-90). Clearly the fields are much improved for the control technique here as compared to the earlier cases, when upward integration was applied as the control. Few significant spatial differences can be found between the fields from techniques D-It and C-Vb-It-90. This is not surprising in light of the very similar average errors for the two techniques noted in

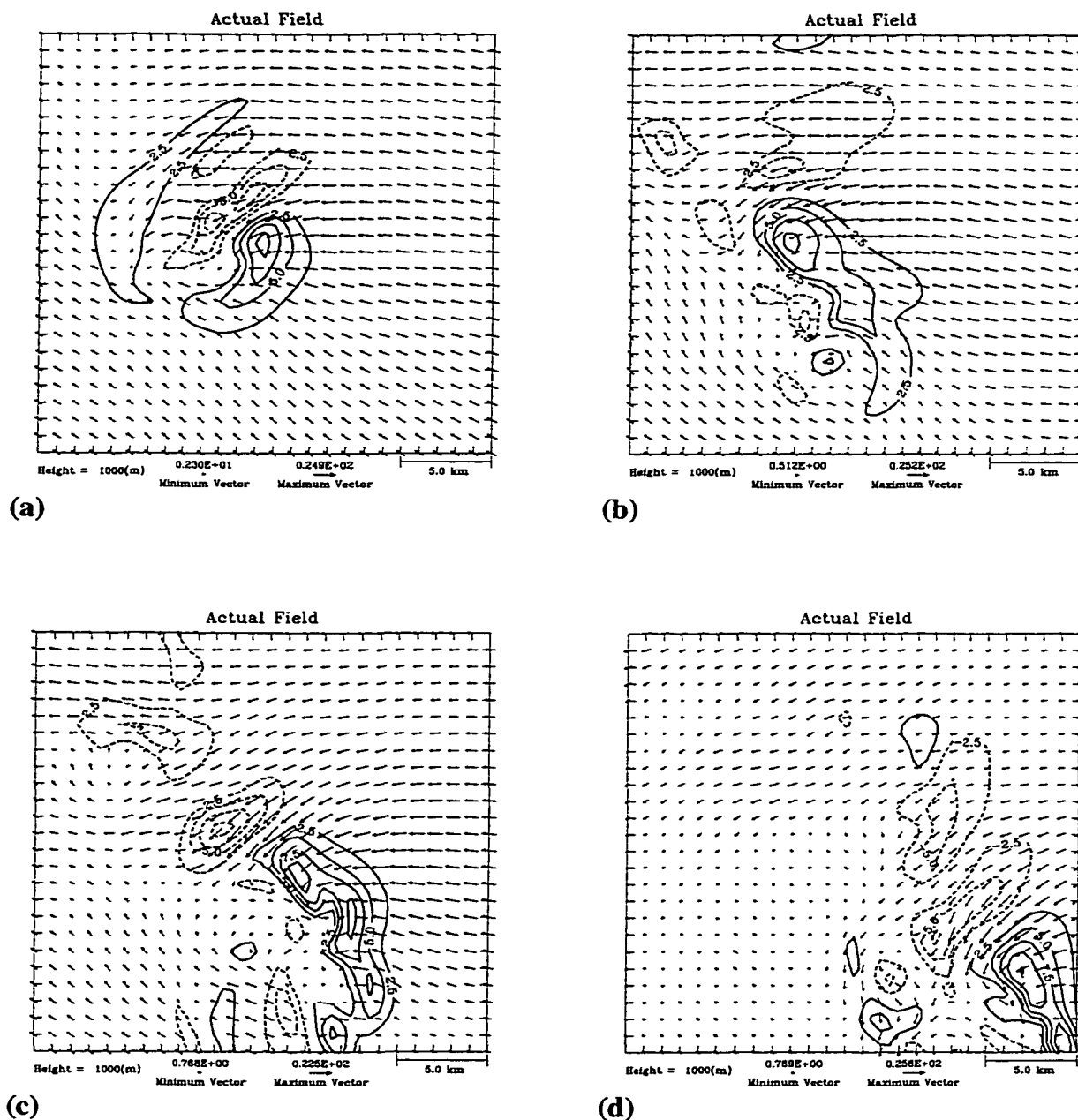


Figure 5.19: Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell storm, taken from the true field at (a) $t = 1800 \text{ s}$, (b) $t = 3600 \text{ s}$, (c) $t = 5400 \text{ s}$, and (d) $t = 7200 \text{ s}$. Contour interval is $2.5 \text{ m} \cdot \text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).

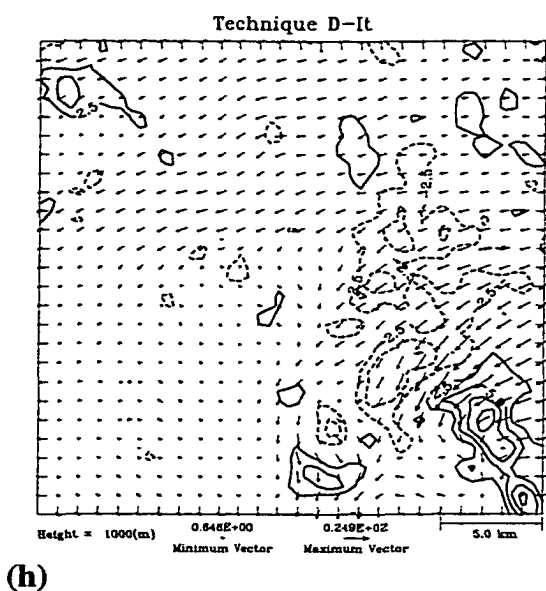
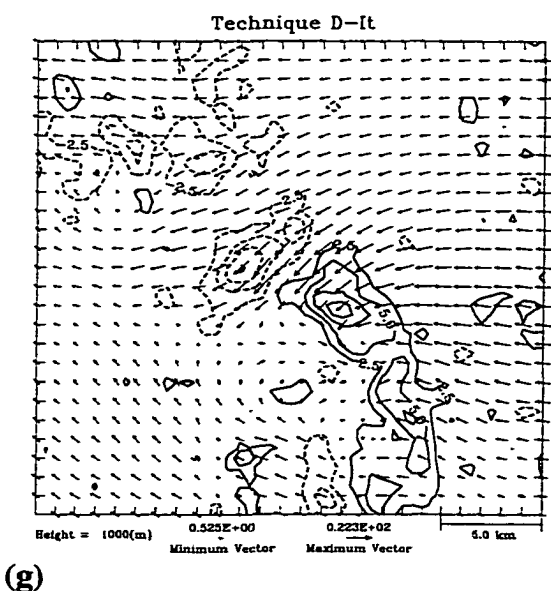
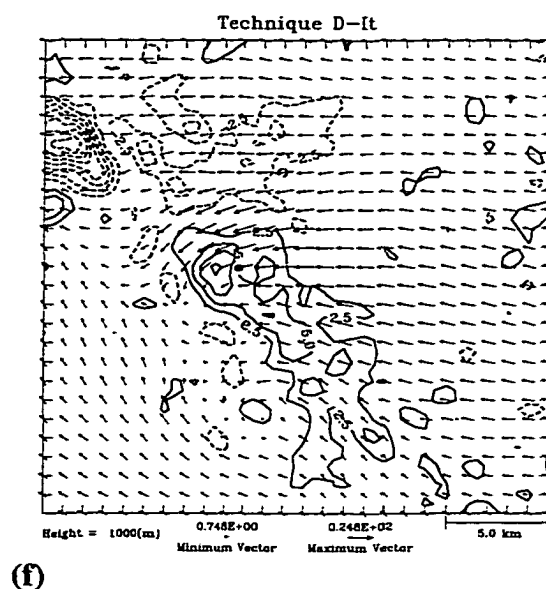
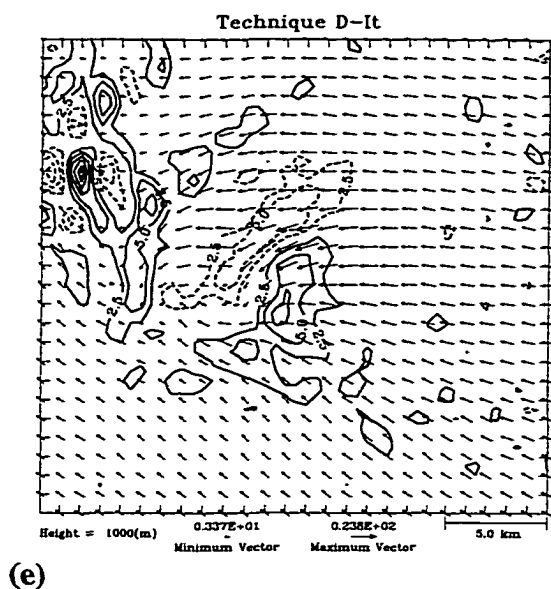


Figure 5.19 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell storm, taken from control technique D-It at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).

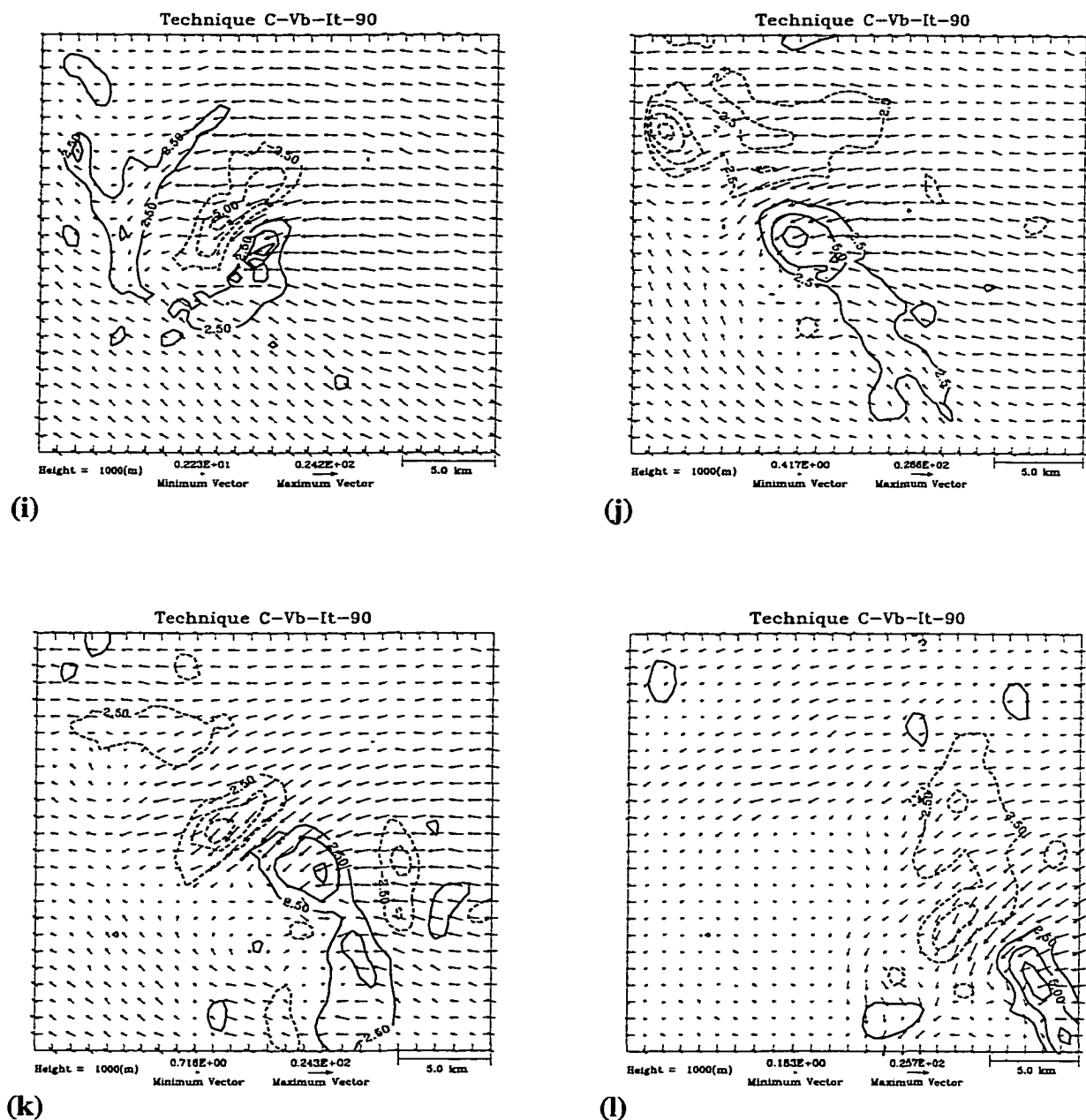


Figure 5.19 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the Del City supercell storm, taken from the 'best' vorticity technique C-Vb-It-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).

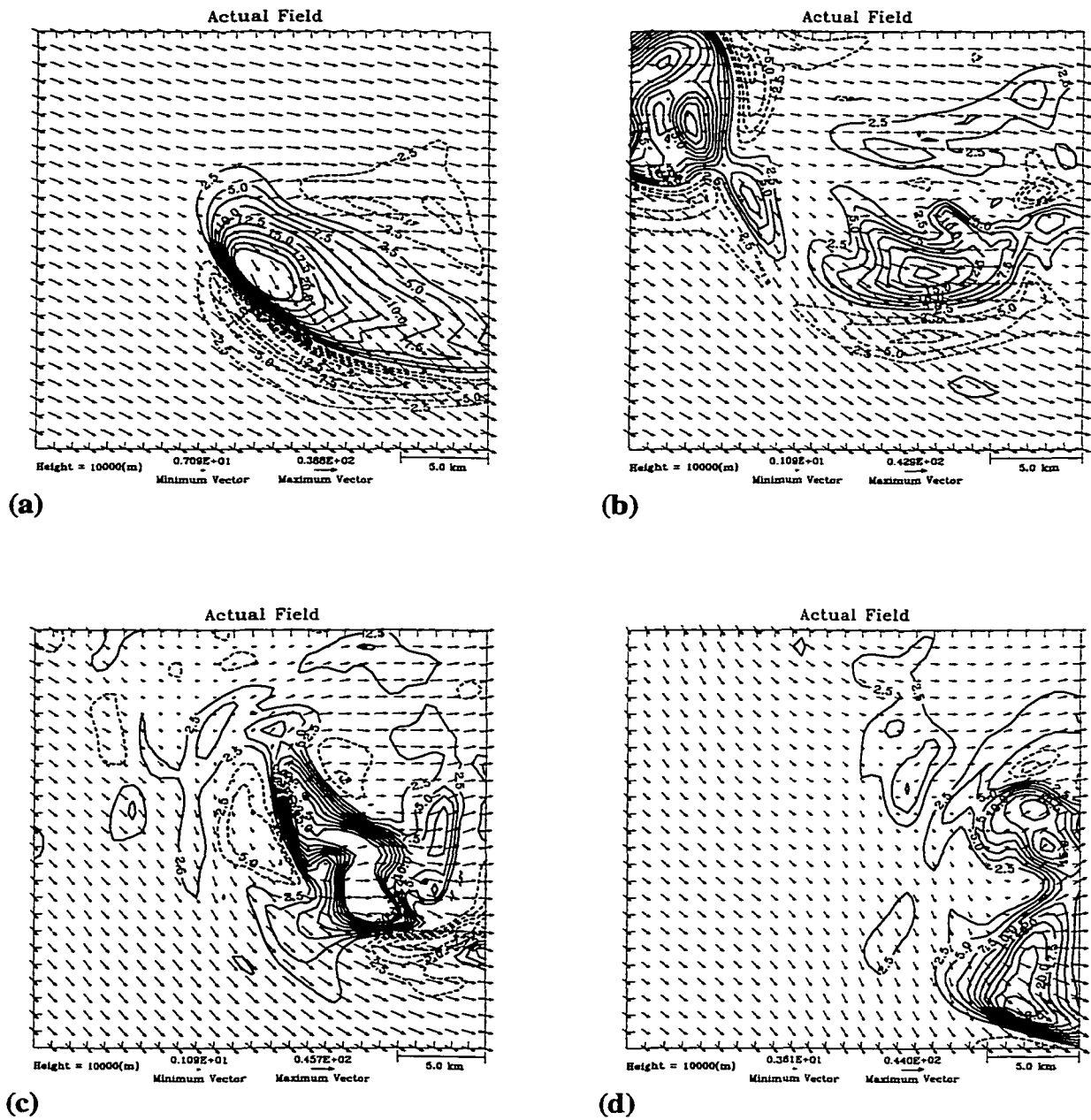


Figure 5.20: Horizontal wind vectors and contours of w at $z = 10.0$ km in the Del City supercell storm, taken from the true field at (a) $t = 1800$ s, (b) $t = 3600$ s, (c) $t = 5400$ s, and (d) $t = 7200$ s. Contour interval is $2.5 \text{ m} \cdot \text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).

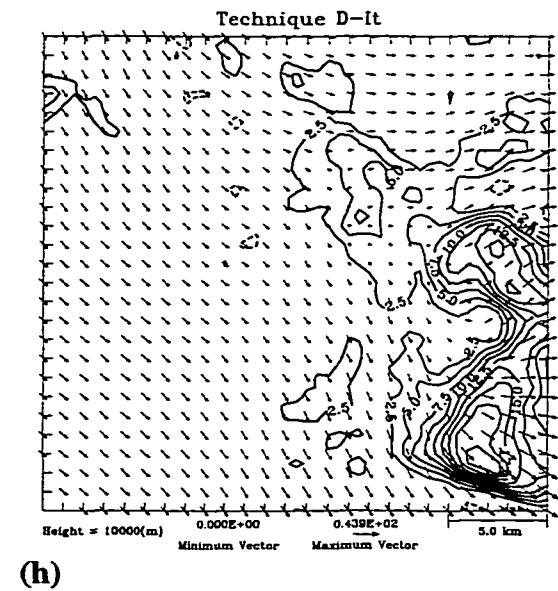
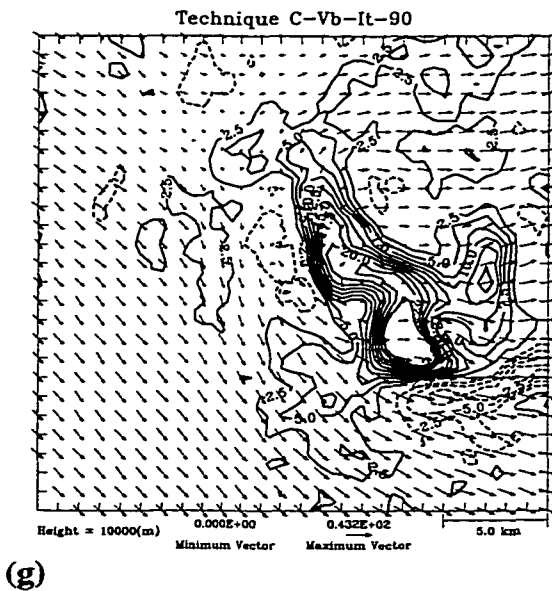
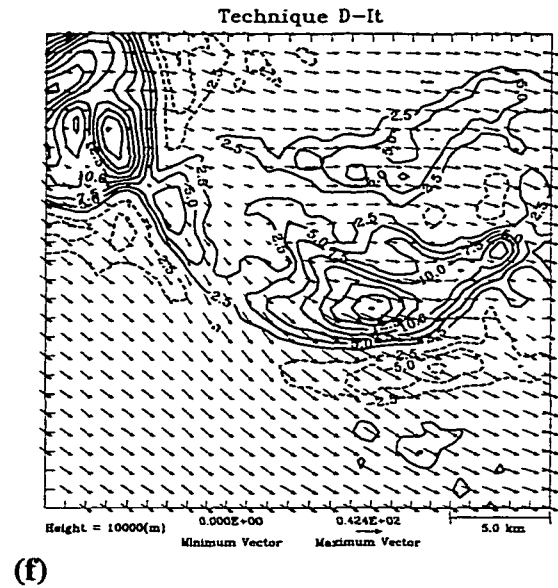
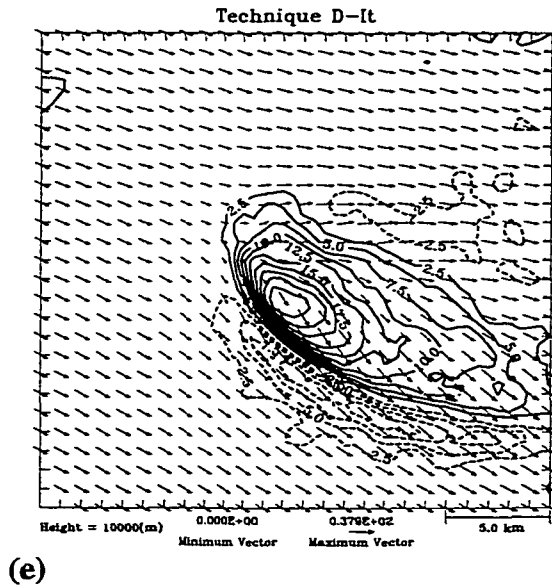


Figure 5.20 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the Del City supercell storm, taken from control technique D-It at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).

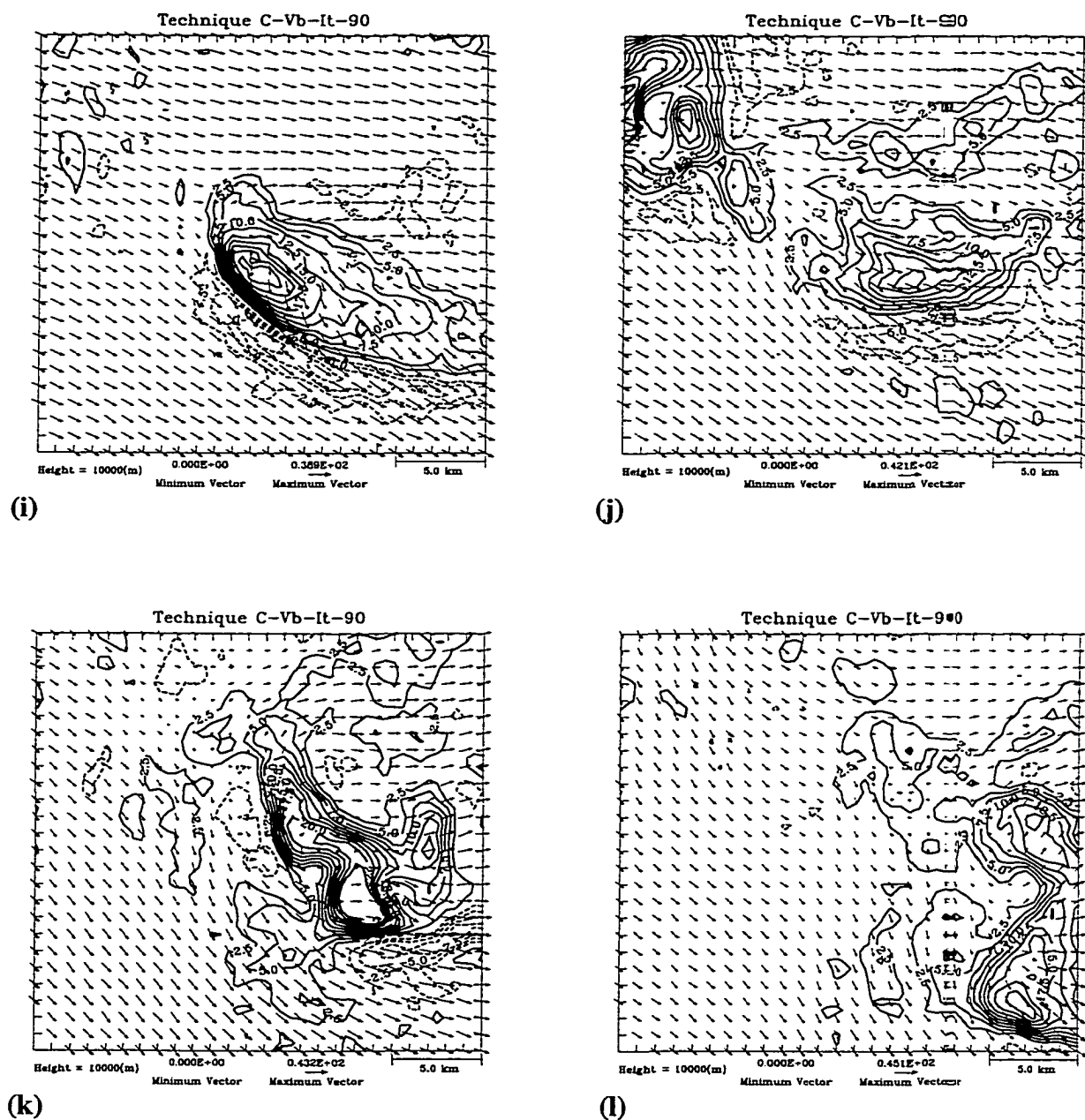


Fig. 5.17. However, as indicated several times previously, the results of MB98 suggest that the control technique D-It would be expected to suffer more severely than C-Vb-It-90 were non-random errors introduced.

A summary of the errors from the various techniques applied to the Del City storm is shown in Table 5.1. This table shows the RMS vertical velocity errors for each of the techniques discussed in each of the four data coverage scenarios presented, averaged across all 24 retrieval times. Hyphens are shown in place of RMS errors for techniques that were not applied in each case. The control technique for each case is labeled accordingly. The ‘best’ technique for each case is shown in a bold, underlined font. This table clearly shows that, with 90 s volume scan periods, the vorticity equation offers the potential for improvement upon traditional analysis techniques in all of the cases studied. In Case III the improvement is nearly 50%. When the volume scan periods are lengthened to 300 s, the vorticity equation still appears to offer significant value in dual-Doppler analyses. Only when data are available to storm top (Case IV) and the volume scan period length is taken to be 300 s is the vorticity equation unable to add value to the analyses. However, as previously mentioned, in this final scenario the vorticity equation may still provide analysis improvements if non-random errors are present in the data.

Section C: Analyses of the microburst storm

The apparent sensitivity of the vorticity equation to the magnitude of the vertical wind shear was established in Chapter 3. Given this sensitivity, it is not surprising that

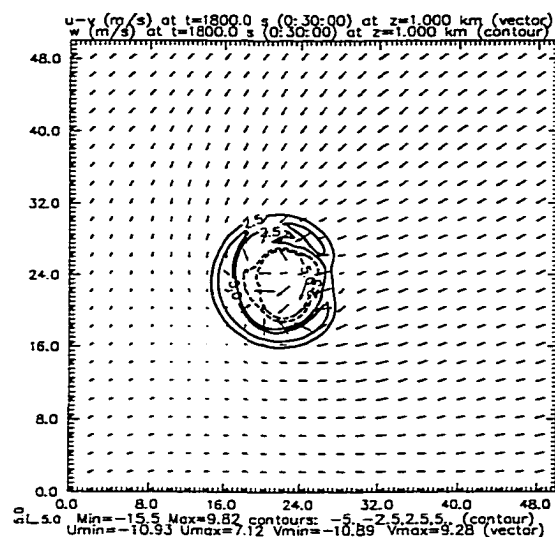
	Case I (0° to 20°)	Case II (1° to 20°)	Case III (2° to 20°)	Case IV (1° to 30°)
U-Ib	2.51 (Control)	-	-	-
U-Eb	-	3.34 (Control)	5.01 (Control)	-
U-Vb-90	3.03	3.38	3.64	-
U-Vb-300	3.85	4.18	4.42	-
D-It	-	-	-	1.60 (Control)
D-Vt-90	-	-	-	2.46
D-Vt-300	-	-	-	3.60
C-Eb-It	-	-	-	1.94
C-Ib-Vt-90	<u>1.89</u>	-	-	-
C-Ib-Vt-300	2.26	-	-	-
C-Eb-Vt-90	-	<u>2.57</u>	3.90	-
C-Eb-Vt-300	-	2.84	4.07	-
C-Vb-Vt-90	2.43	2.64	<u>2.79</u>	2.20
C-Vb-Vt-300	3.59	3.79	3.91	2.36
C-Vb-It-90	-	-	-	<u>1.57</u>
C-Vb-It-300	-	-	-	1.72

Table 5.1: RMS vertical velocity errors for each of the techniques and data coverages discussed in Section B (the Del City supercell storm), averaged across all 24 retrieval times. Hyphens are shown in place of RMS errors for techniques that were not applied in each case. The control technique for each case is labeled as such, and the ‘best’ technique for each case is underlined.

the vorticity equation appears to be a useful constraint in a high-shear case such as the Del City storm. However, to be widely applicable the vorticity equation must hold promise of improving vertical velocity retrievals under much more common environmental shear profiles.

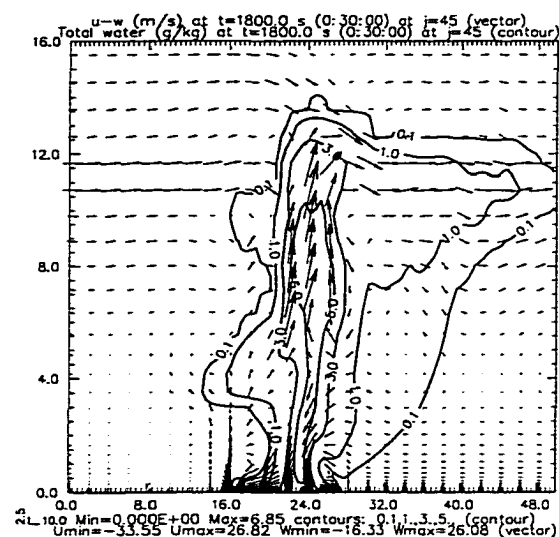
To examine this potential it is useful to test the new techniques using a simulation of a typical dry microburst producing thunderstorm. This storm was initiated by releasing a thermal bubble into an environment similar to the composite dry microburst environment of Brown et al. (1982), depicted in Fig. 5.2. The environmental shear profile for this case is much weaker than that of the Del City storm. This type of storm is also attractive because it tends to occur frequently in the western United States, where beam blockage in complex terrain and the low-precipitation nature of the storms often lead to sparse radar data at low elevation angles. As discussed earlier, since the divergence associated with microbursts may be strongest in the lowest few hundred meters of the atmosphere, sparse data at low levels may be especially problematic for extrapolated divergence techniques (Ray, 1976).

To initiate the storm, a thermal bubble was released over the center of the simulation domain. A thunderstorm develops rapidly upon release of this thermal bubble, with the heaviest precipitation and first significant microburst occurring around $t = 1800$ s (see Fig. 5.21). The storm continues to evolve and produce weaker microbursts between $t = 1800$ and $t = 7200$ s while exhibiting a gradual decrease in intensity. Although much of the average storm motion was subtracted from the sounding at the outset, the storm still drifts slowly southward at a rate of approximately $5 \text{ km}\cdot\text{hr}^{-1}$ throughout the course of the simulation.

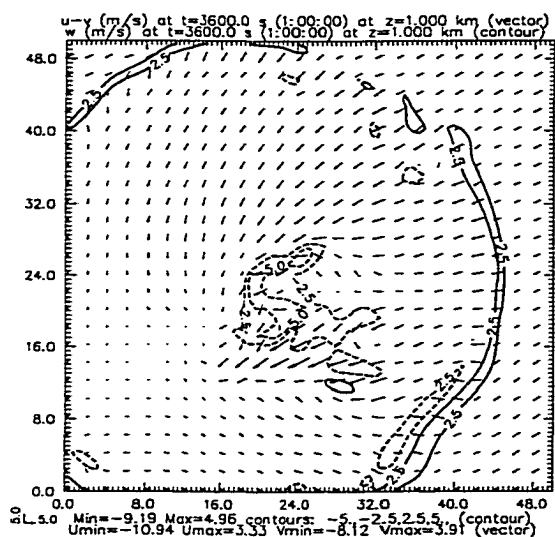


(a)

micro, Version 4.3.0, Dry Microburst Sounding
ARPSPLT/ZXPLOT

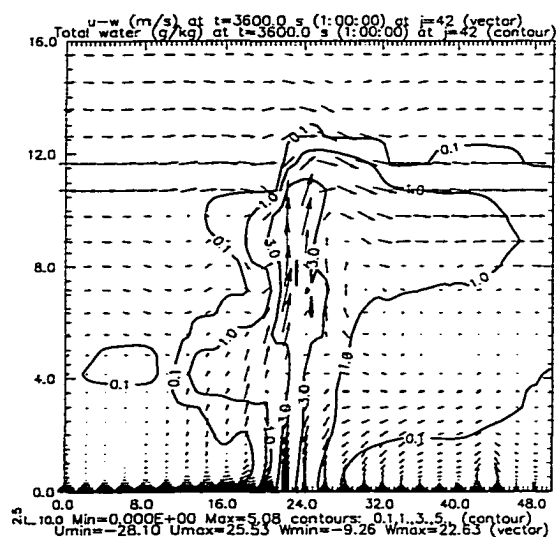


(b)



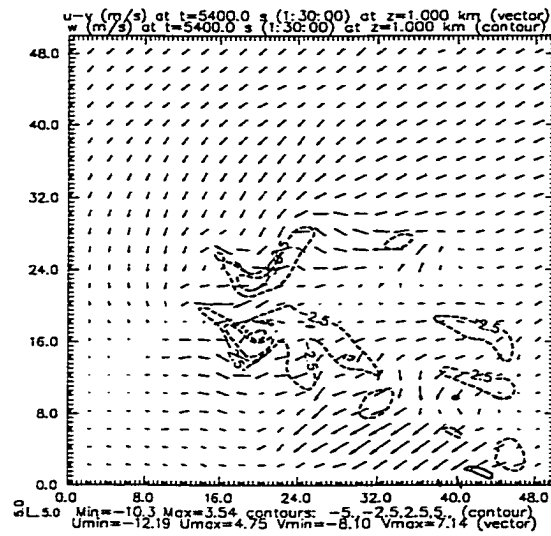
(c)

micro, Version 4.3.0, Dry Microburst Sounding
ARPSPLT/ZXPLOT



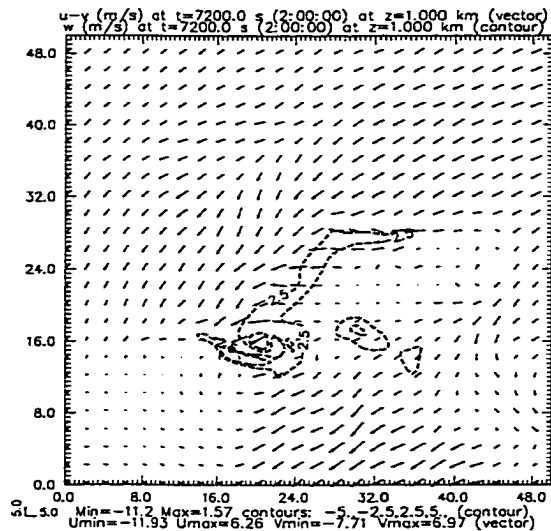
(d)

Figure 5.21: Horizontal wind vectors and contours of w at $z = 1.0$ km in the microburst thunderstorm at (a) 1800 s and (c) 3600 s. Also shown are x-z vertical cross sections near the center of the storm at (b) 1800 s and (d) 3600 s. Contours of 0.1, 1.0, 3.0, and 5.0 $\text{g}\cdot\text{kg}^{-1}$ of total (non-vapor) water mixing ratio are shown.



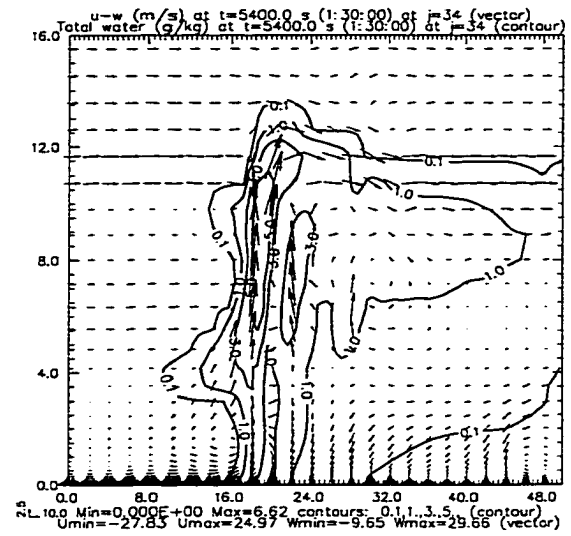
(e)

micro, Version 4.3.0, Dry Microburst Sounding
ARPSPLT/ZXPLOT

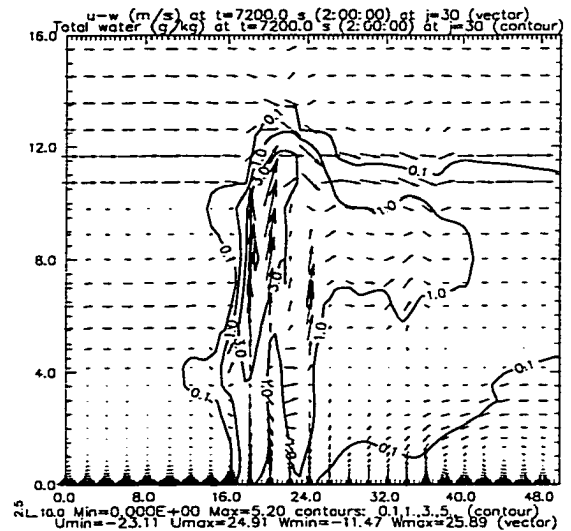


(g)

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ARPSPLT/ZXPLOT



(f)

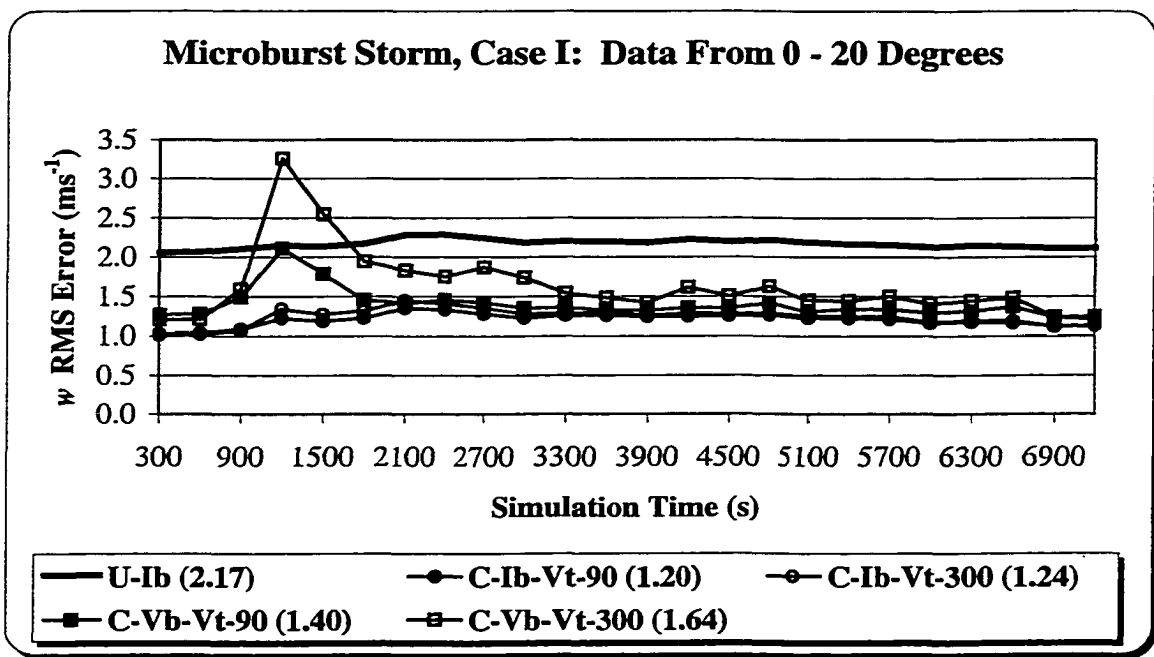


(h)

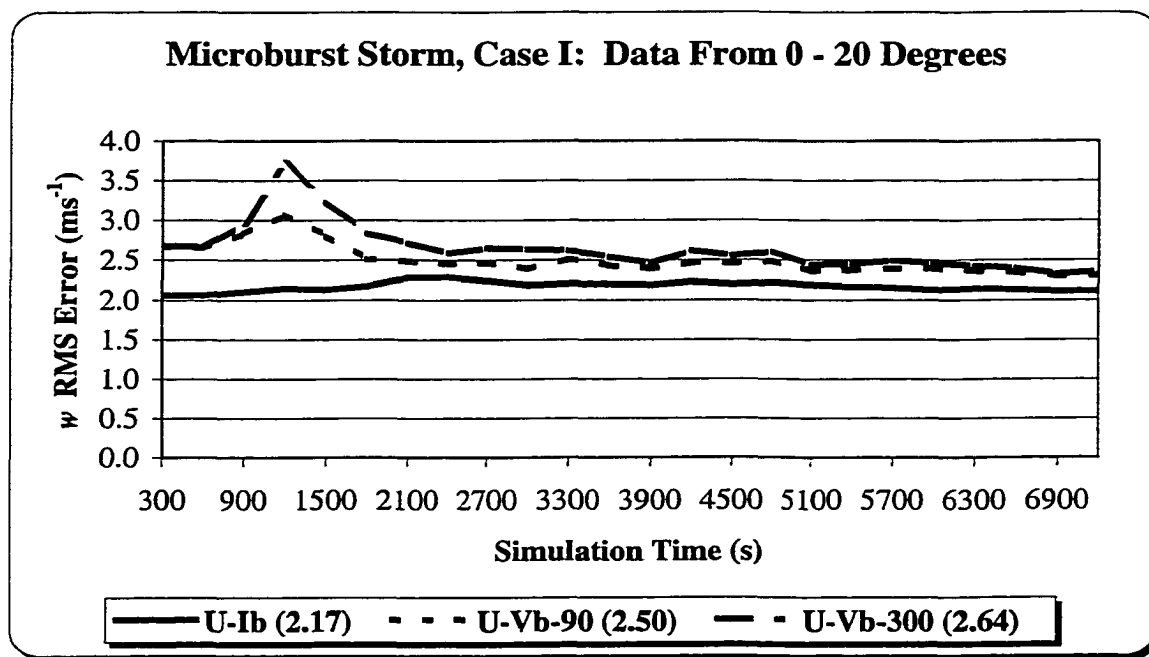
Figure 5.21 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the microburst thunderstorm at (a) 5400 s and (c) 7200 s. Also shown are x - z vertical cross sections near the center of the storm at (b) 5400 s and (d) 7200 s. Contours of 0.1, 1.0, 3.0, and 5.0 $\text{g}\cdot\text{kg}^{-1}$ of total (non-vapor) water mixing ratio are shown.

The retrieval experiments for this case used the same four VCPs used in the Del City storm retrievals. The 20° maximum elevation angle for Cases I, II, and III again falls well below the actual storm top for all but the first 20 minutes of the simulation, and therefore necessitates an upward integration technique. In Case IV, data were assumed to span from 1° to 30° (above storm top) so that a downward integration technique using a zero vertical velocity at storm top could be safely applied.

Recall that in Case I the impermeability condition can be safely applied at the ground, so our primary concern is limiting the detrimental effects of accumulating divergence errors. In spite of the weaker shear profile of this storm, the vorticity equation appears to be even more valuable to the analyses than was the case with the Del City storm. Technique C-Ib-Vt again appears to be the preferred analysis method for Case I (see Fig. 5.22a). This technique yields RMS vertical velocity errors of 1.20 to 1.24 m·s⁻¹ for 90 and 300 s volume scan periods, respectively, when averaged across all 24 retrievals from the microburst storm. This amounts to approximately 45% error reduction relative to control technique U-Ib for Case I, which exhibits an average RMS error of 2.17 m·s⁻¹. Comparison of Figs. 5.5 and 5.18 reveals that the RMS vertical velocity errors are lower for all techniques in this storm as compared to the Del City storm. However, it can be noted that the vorticity equation techniques all show substantial improvement over the Del City storm retrievals in spite of the weaker shear profile, whereas the conventional technique shows much more limited improvement. Also interesting is the increased similarity between the retrieval quality with 90 and 300 s volume scans in this case as compared to the Del City storm, likely indicating more linear local evolution in ζ with this storm.

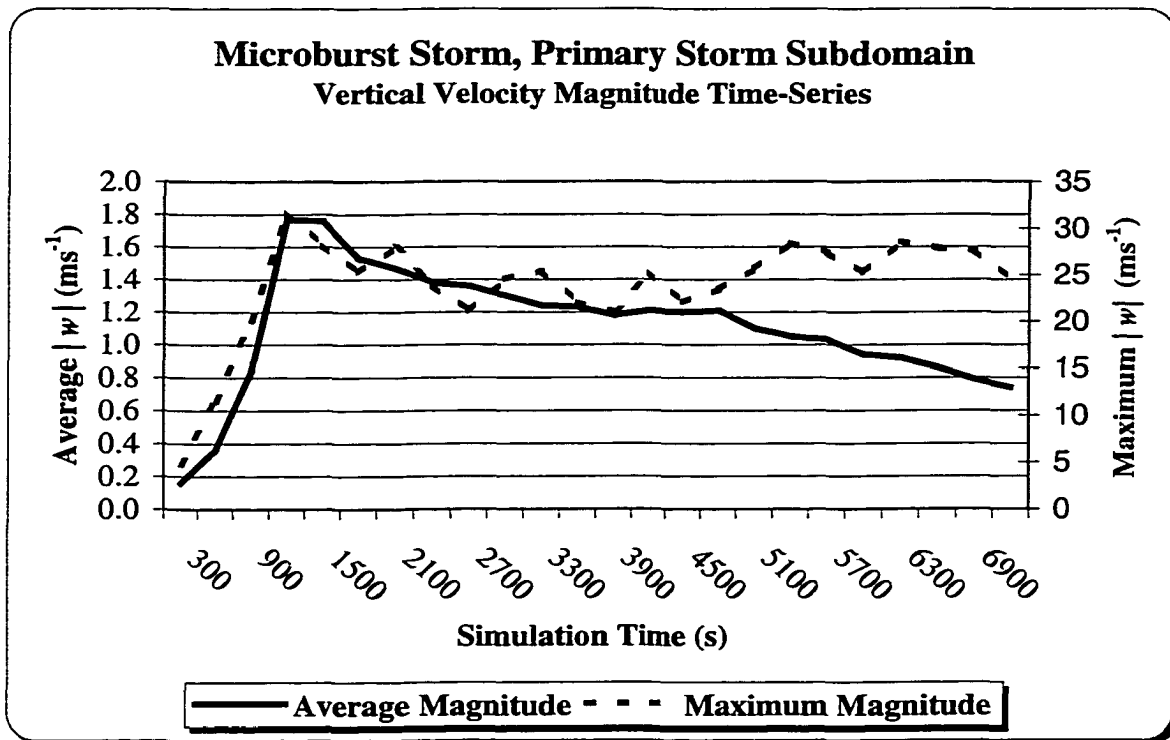


(a)



(b)

Figure 5.22: Root mean square vertical velocity retrieval errors (a,b) from experiments on the microburst storm simulation using volume scan geometries providing data between 0° and 20° (Case I).



(c)

Figure 5.22 (continued): Average and maximum vertical velocity magnitudes (c) over the primary storm subdomain throughout the course of 7200 s simulation. Panel (c) is useful for interpreting the error magnitudes in terms of the true vertical velocities, and can be applied in interpretation of Figs. 5.26, 5.27, and 5.33 as well. Note that the presence of large areas of near-zero vertical velocity skew the average updraft magnitudes downward.

As noted earlier, the value of the vorticity equation in boundary condition estimation can be isolated from its ability to control divergence errors (through application of a second boundary condition) by applying the vorticity equation in the same analysis framework as the control technique (U-Ib for Case I). Analysis differences between the U-Vb techniques and the U-Ib technique are due entirely to boundary condition differences. With Case I for this microburst storm we find very similar results to those noted for Case I of the Del City storm (Fig. 5.22b). Once again, impermeability (assuming it can be applied safely) appears to be uniformly preferable to application of the vorticity equation to estimate a lower boundary condition.

Examination of a slice through the upper portion of the subdomain ($z = 10$ km) from each of these techniques at the midpoint of this simulation ($t = 3600$ s) again reveals where the benefits of two boundary conditions are realized (Fig. 5.23). Comparison of the true field, Fig. 5.23a, with the field from the control technique U-Ib shown in Fig. 5.23b reveals that the control field suffers significantly from divergence error accumulation. Interpretation of the vertical velocity field in Fig. 5.23b could be difficult, depending upon the spatial scale of the feature of interest. The vorticity techniques offer much better representations of the true vertical velocity field. All features of the true vertical velocity field are easily identifiable in Figs. 5.23c-5.23e, with relatively few spurious features present. Note that the missing data in the lower left and upper left corners of Figs. 5.23b-5.23e demarcate the 20° elevation angle used as an upper data cutoff level for Cases I-III.

The wind fields at $t = 1800$ s, $t = 3600$ s, $t = 5400$ s, and $t = 7200$ s from the control technique (U-Ib) and the 'best' (as determined from Fig. 5.22) vorticity based

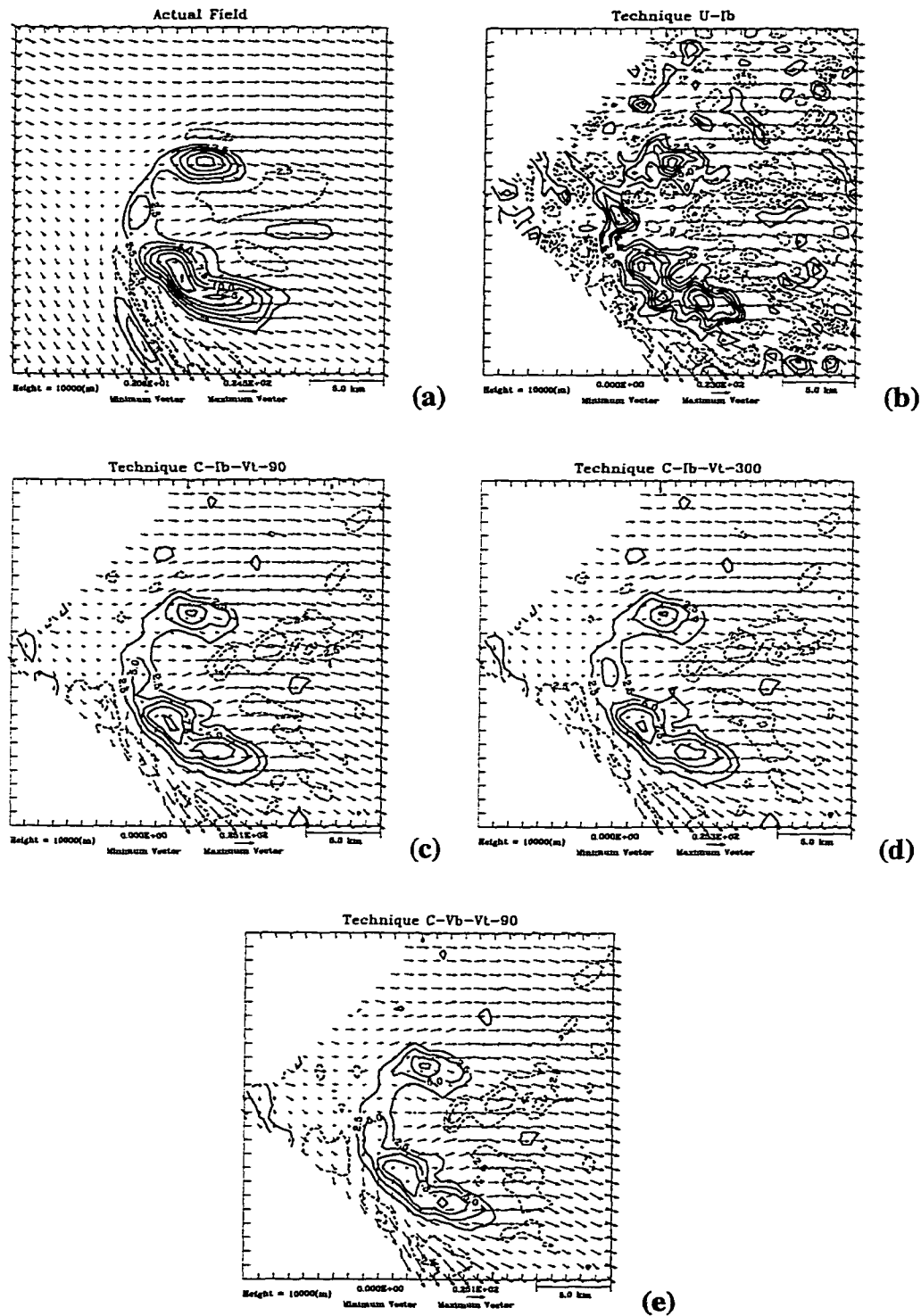


Figure 5.23: Horizontal wind vectors and contours of w at $z = 10.0$ km in the microburst storm, taken from (a) the true field, (b) control technique U-Ib, (c) technique C-Ib-Vt-90, (d) technique C-Ib-Vt-300, and (e) technique C-Vb-Vt-90. Contour interval is 2.5 m s^{-1} . Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).

technique (C-Ib-Vt-90) are compared against the true wind field in Figs. 5.24 and 5.25. Since the same lower boundary condition was applied in both techniques, few differences can be found at $z = 1.0$ km (Fig. 5.24). However, Fig. 5.25, valid at $z = 10.0$ km, continues to highlight the distinct advantages of applying a second boundary condition determined from the vorticity equation to control accumulating divergence errors.

The vorticity equation continues to show promise for improving vertical velocity retrievals in Cases II and III. Figures 5.26 and 5.27 illustrate the dangers of extrapolating divergence in microburst situations, as the RMS errors for any techniques applying extrapolated divergence (denoted “-Eb”) increase dramatically after the storm first begins producing microbursts shortly after $t = 1500$ s. In Cases II and III technique C-Vb-Vt yields the best results, with RMS errors ranging between 1.67 to 1.73 $\text{m}\cdot\text{s}^{-1}$ with 90 s volume scan periods and between 1.92 and 1.98 $\text{m}\cdot\text{s}^{-1}$ with 300 s volume scan periods. This amounts to as much as a 44% reduction in RMS errors relative to control technique U-Eb for Case II, and up to a 61% reduction in RMS errors for Case III. The techniques utilizing only the vorticity equation to obtain the boundary condition(s) are again relatively insensitive to movement of the lower data boundary away from the earth’s surface. This behavior is similar to that noted with the Del City storm, and suggests that these techniques may provide reliable vertical velocity retrievals even in the most severe cases of beam blockage.

The benefits of utilizing the vorticity equation to estimate the lower boundary condition in this situation are explored spatially in Fig. 5.28. For consistency, the fields shown are again valid at $t = 3600$ s. Figures 5.28a and 5.28b depict the true vertical velocity fields at $z = 1.0$ km and $z = 1.5$ km. These levels are the lowest data levels for

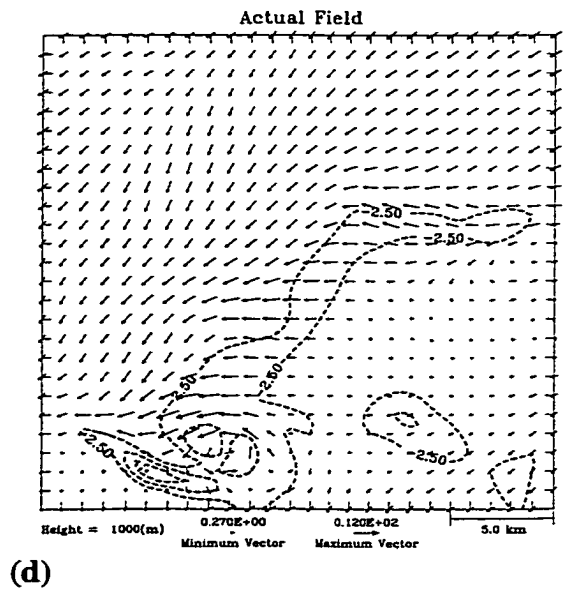
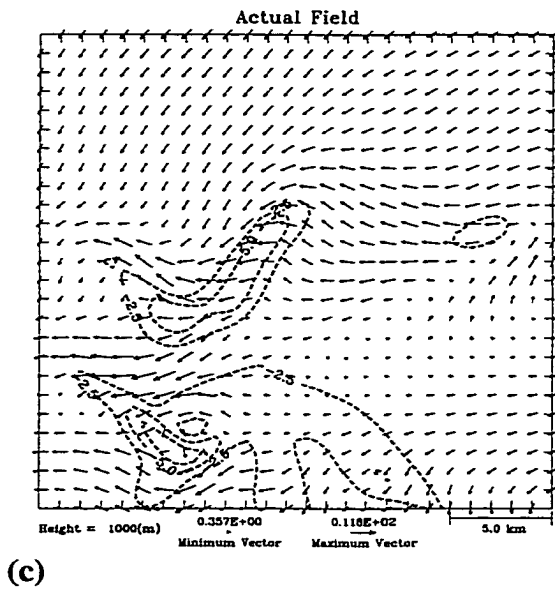
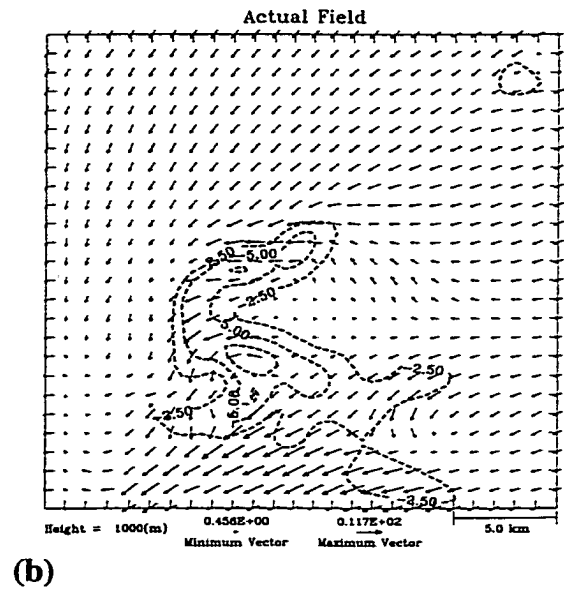
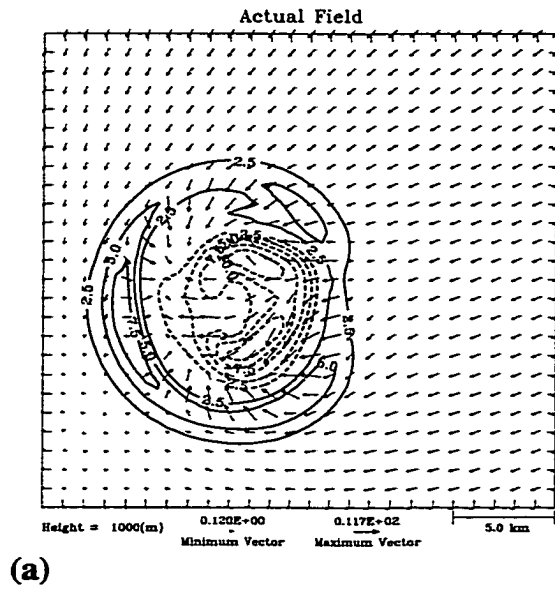


Figure 5.24: Horizontal wind vectors and contours of w at $z = 1.0$ km in the microburst storm, taken from the true field at (a) $t = 1800$ s, (b) $t = 3600$ s, (c) $t = 5400$ s, and (d) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).

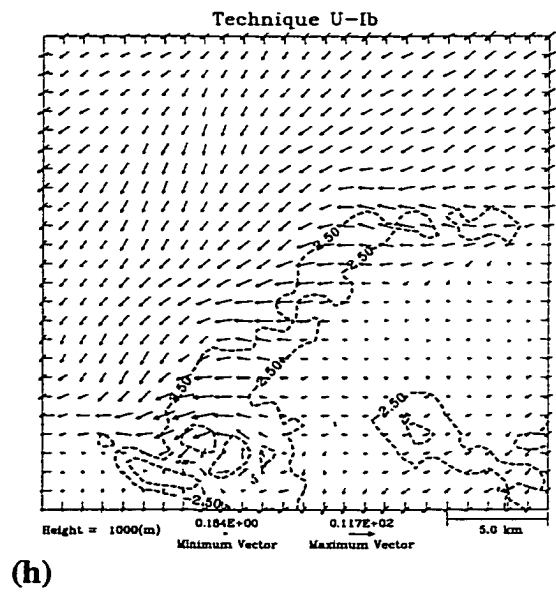
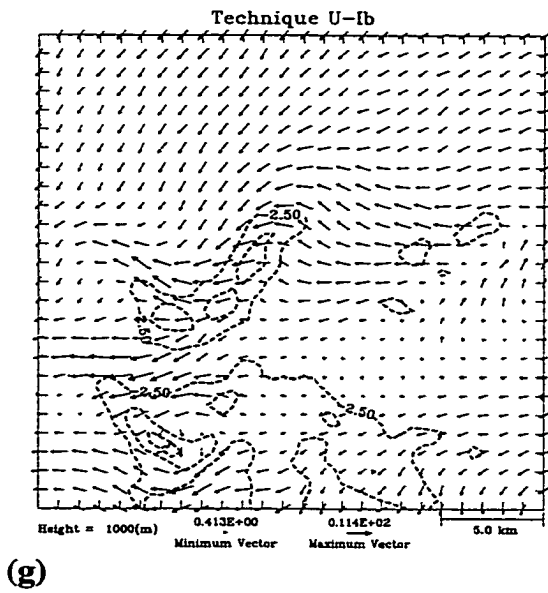
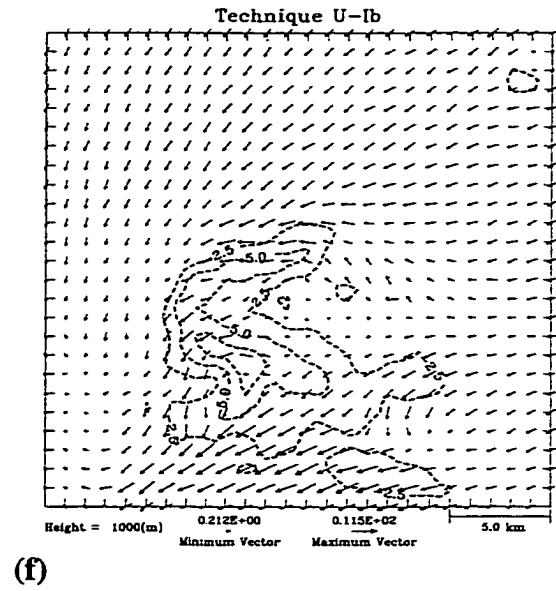
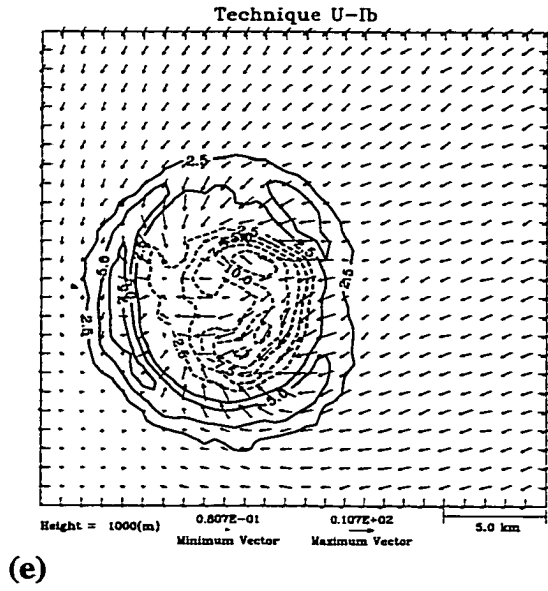


Figure 5.24 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the microburst storm, taken from control technique U-Ib at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).

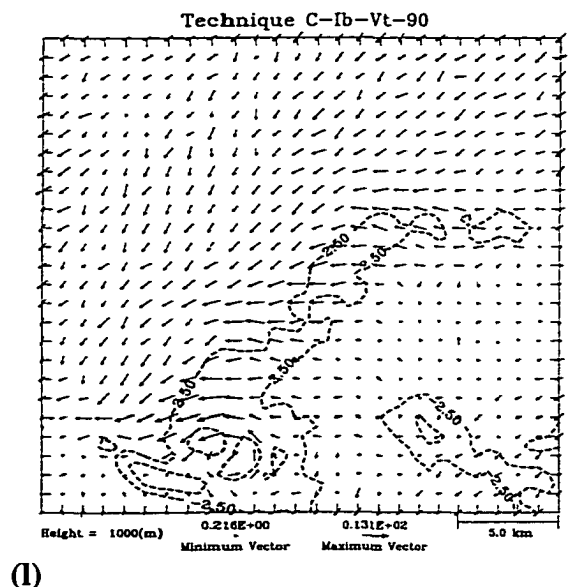
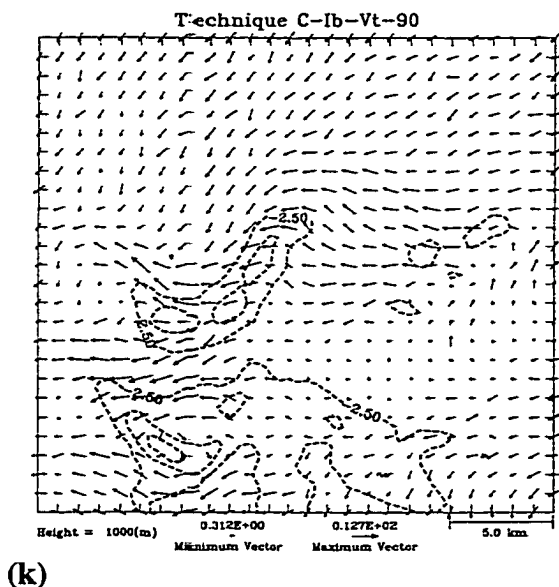
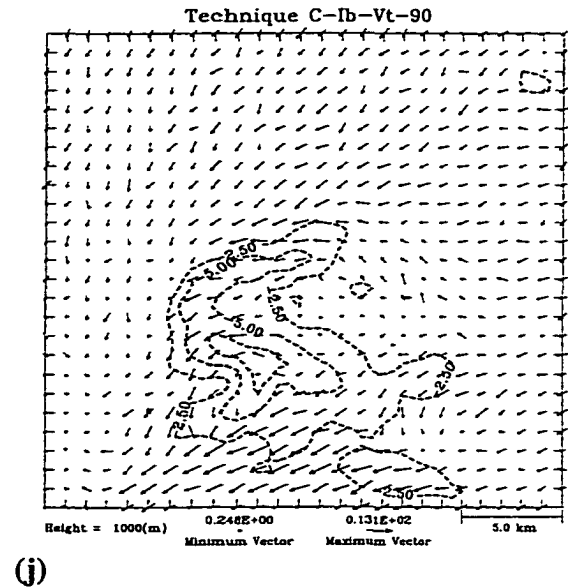
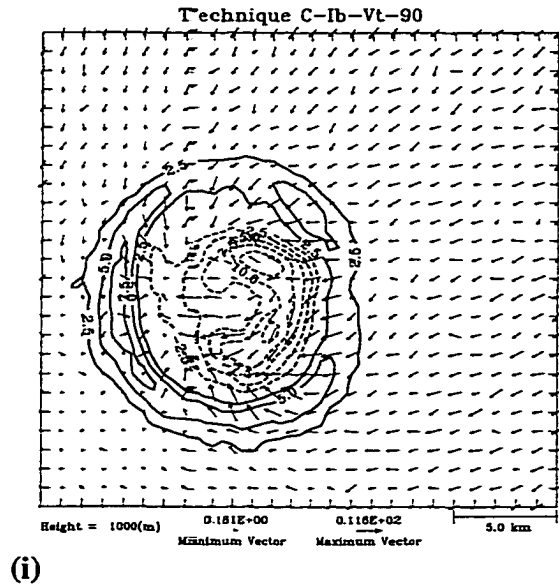


Figure 5.24 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the microburst storm, taken from the ‘best’ vorticity technique C-Ib-Vt-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).

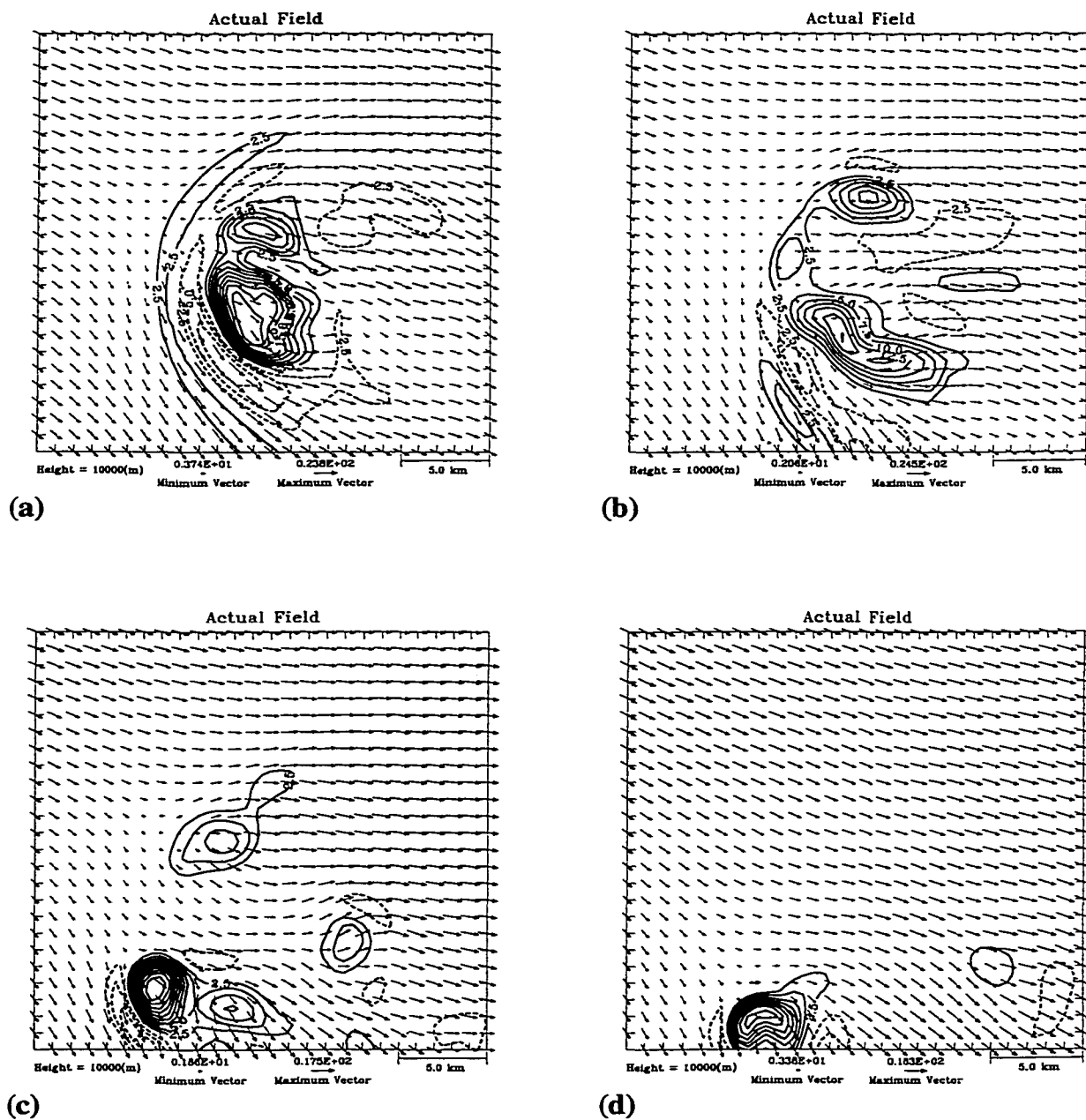


Figure 5.25: Horizontal wind vectors and contours of w at $z = 10.0$ km in the microburst storm, taken from the true field at (a) $t = 1800$ s, (b) $t = 3600$ s, (c) $t = 5400$ s, and (d) $t = 7200$ s. Contour interval is 2.5 m s^{-1} . Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).

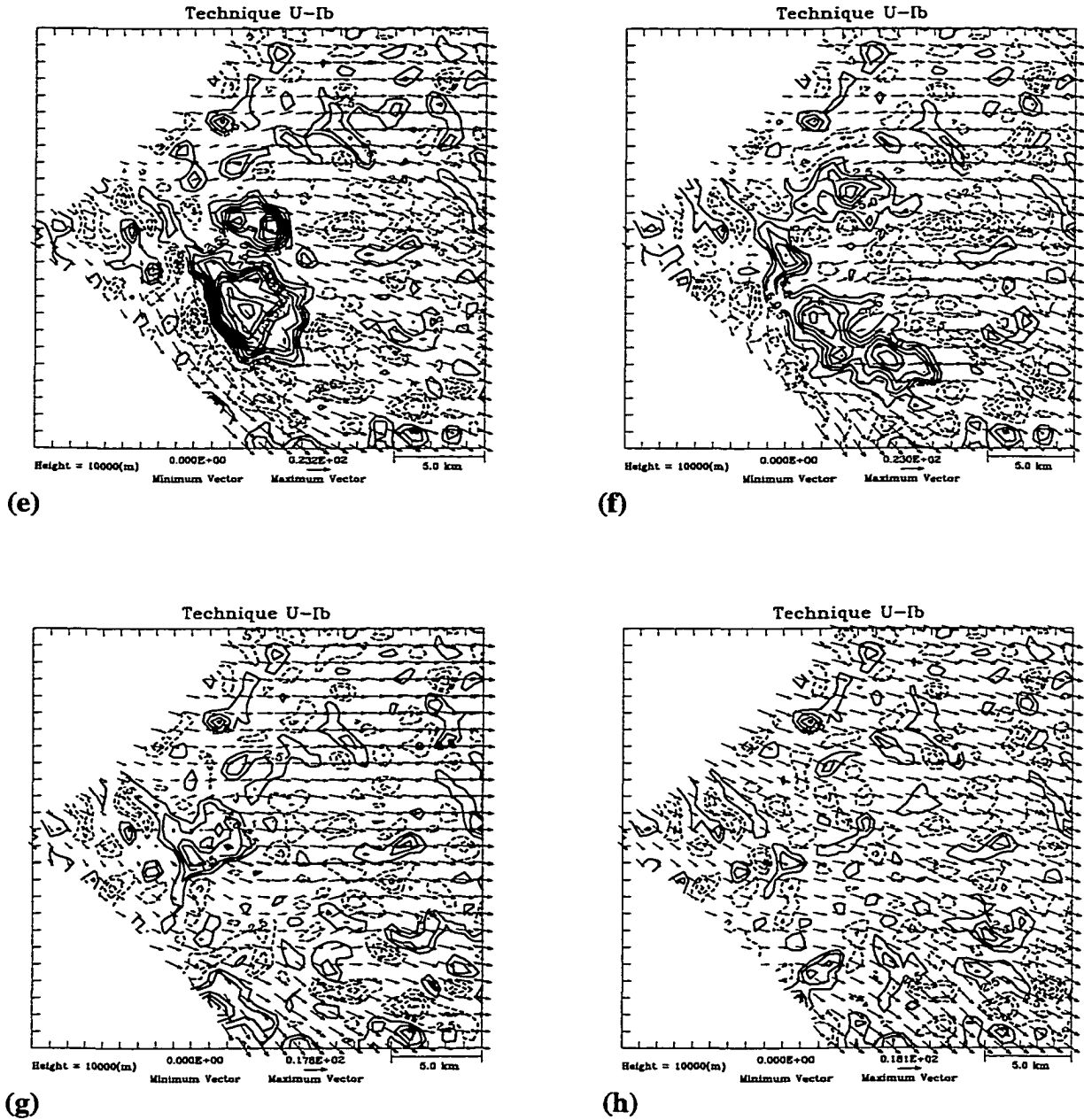
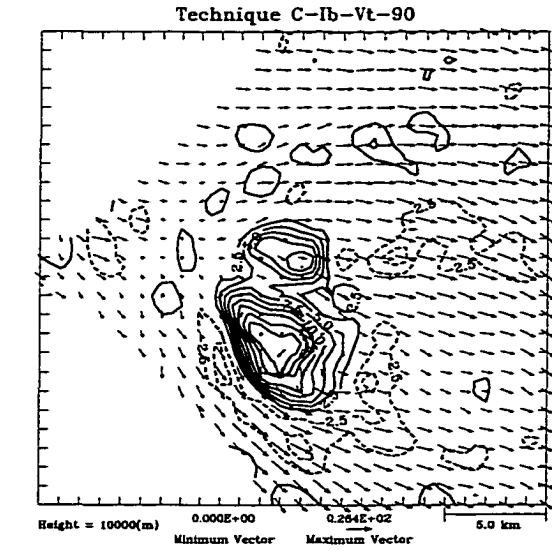
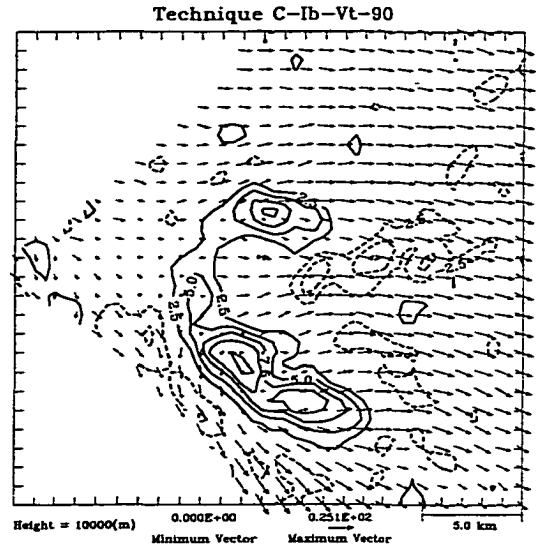


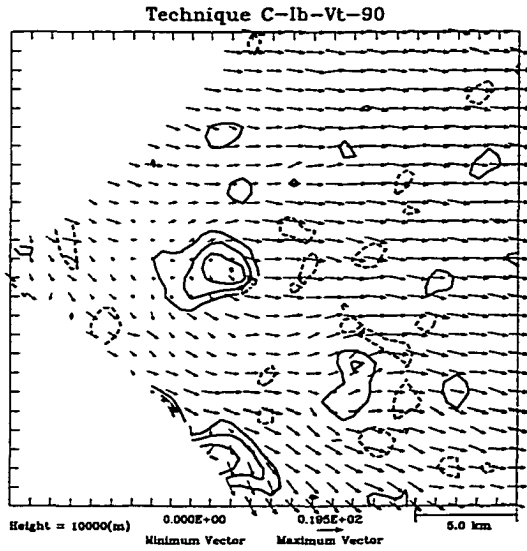
Figure 5.25 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the microburst storm, taken from control technique U-Ib at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).



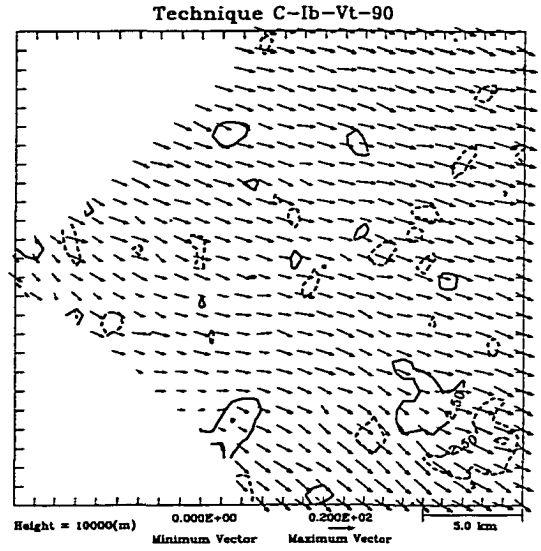
(i)



(j)



(k)



(l)

Figure 5.25 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the microburst storm, taken from the 'best' vorticity technique C-Ib-Vt-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I).

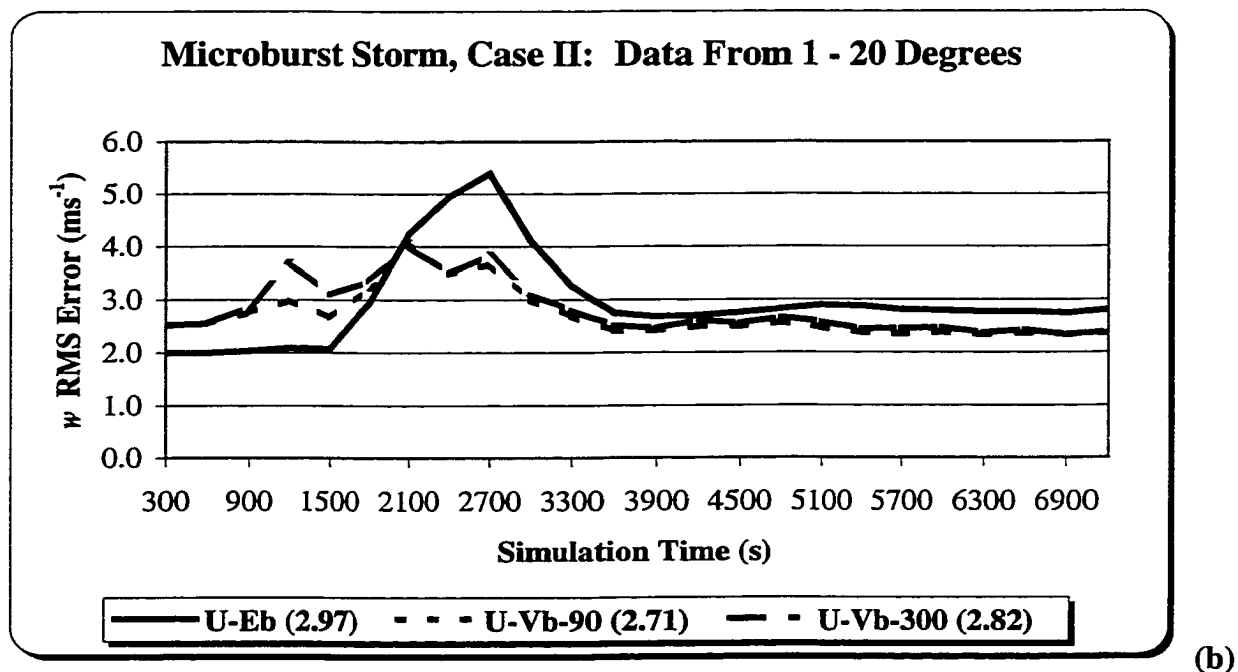
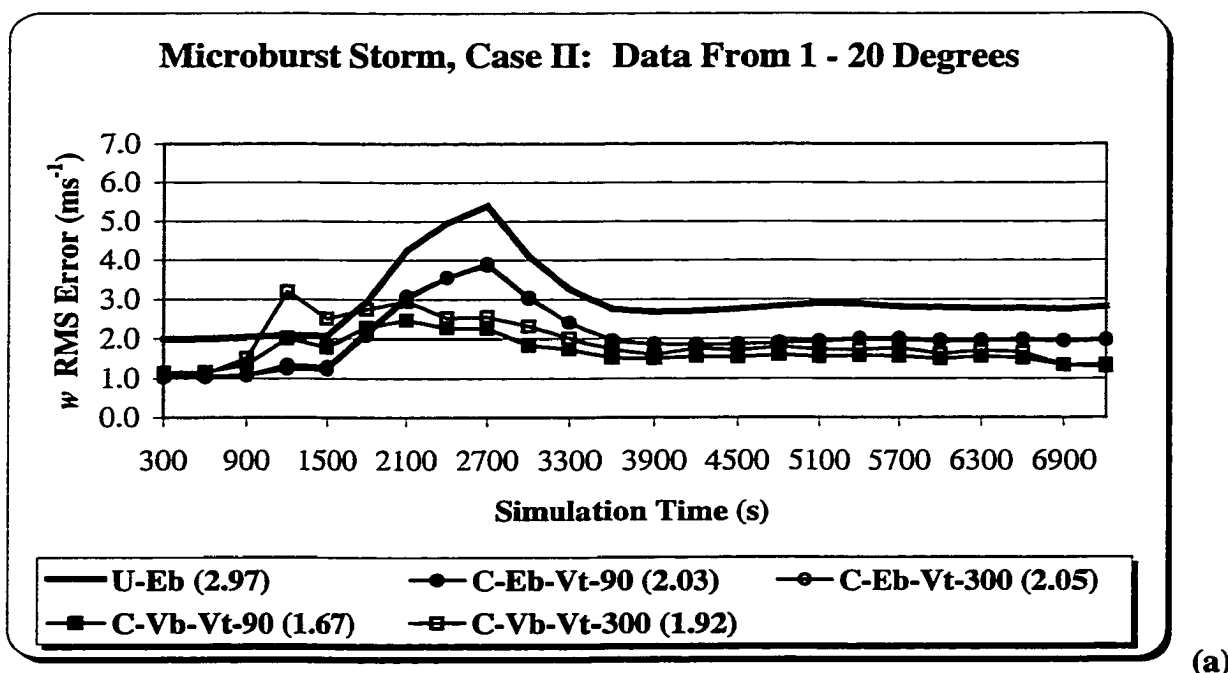


Figure 5.26: Root mean square vertical velocity retrieval errors from experiments on the microburst storm simulation using volume scan geometries providing data between 1° and 20° (Case II).

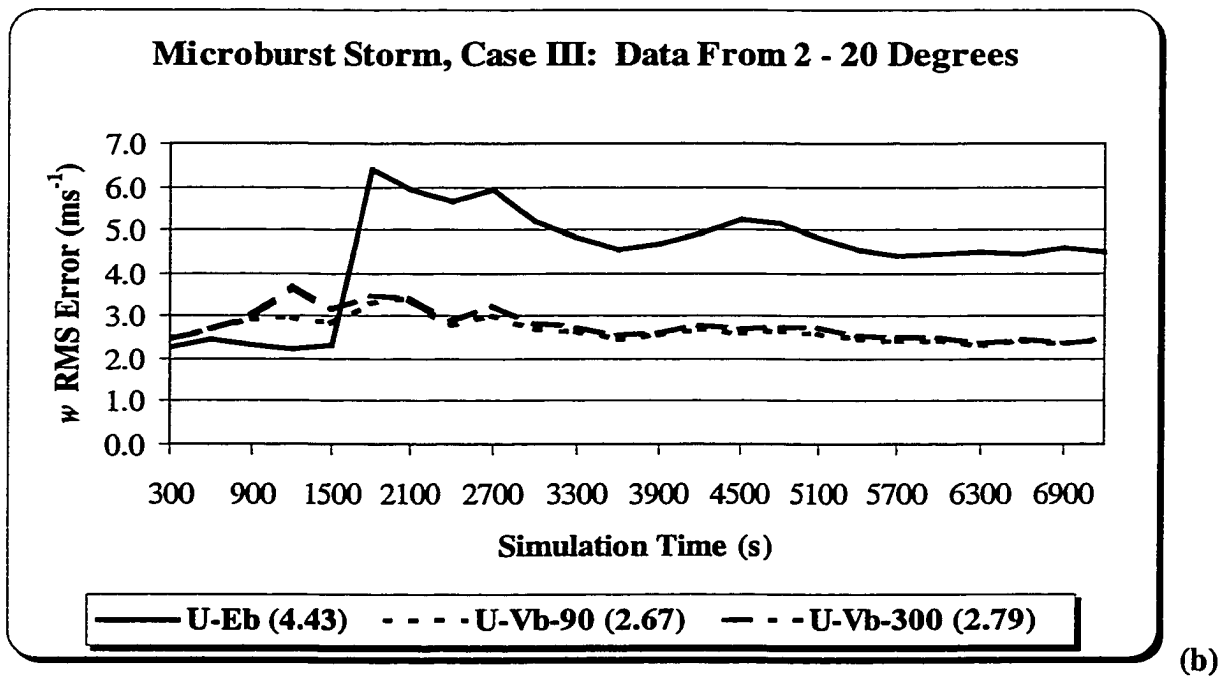
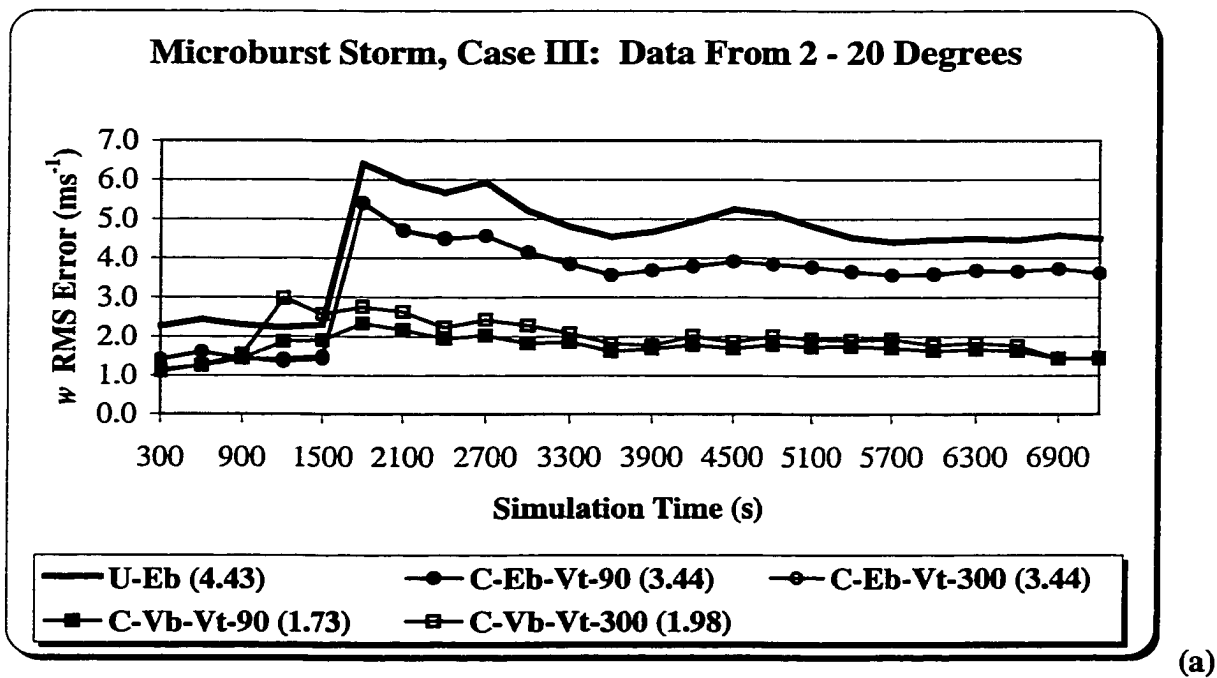


Figure 5.27: Root mean square vertical velocity retrieval errors from experiments on the microburst storm simulation using volume scan geometries providing data between 2° and 20° (Case III).

which the fields shown are largely populated for scan geometry Cases II and III, respectively. Figures 5.28c and 5.28d depict the vertical velocity fields at the same two levels from analysis technique C-Eb-Vt-90, which uses extrapolated divergence to estimate the lower boundary condition. Figures 5.28e and 5.28f depict the corresponding vertical velocity fields from analysis technique C-Vb-Vt-90, which uses the vorticity equation to estimate the lower boundary condition. Note that Figs. 5.28c and 5.28e are constructed from analyses in which the Case II scan geometry limitations were applied, while Case III geometry was applied in the analyses shown in Figs. 5.28d and 5.18f. In Case II (Fig. 5.28c) technique C-Eb-Vt-90 fails to capture the full spatial extent of the downdraft region, and also places spurious updrafts in the region surrounding the downdraft. In Case III the analysis from technique C-Eb-Vt-90 deteriorates drastically, and bears little resemblance to the true field (compare Figs. 5.28b and 5.28d). Extrapolating divergence in this case is clearly dangerous, as most of the true downdraft is instead realized as an updraft in the C-Eb-Vt-90 analysis. Comparison of Figs. 5.28e with 5.28a and 5.28f with 5.28b shows how application of the vorticity equation in the analyses can dramatically improve results. Although minor differences do exist, the spatial structures of the true vertical velocity fields are well preserved for both of Cases II and III (Figs. 5.28e and 5.28f, respectively).

The wind fields at $t = 1800$ s, $t = 3600$ s, $t = 5400$ s, and $t = 7200$ s from the control technique (U-Eb) and the ‘best’ (as determined from Fig. 5.26 and 5.27) vorticity based technique (C-Vb-Vt-90) are compared against the true wind field in Figs. 5.29 and 5.30 for Case II, and in Figs. 5.31 and 5.32 for Case III. Most noteworthy is the dramatic degradation of the lower boundary condition retrievals from control technique U-Eb as

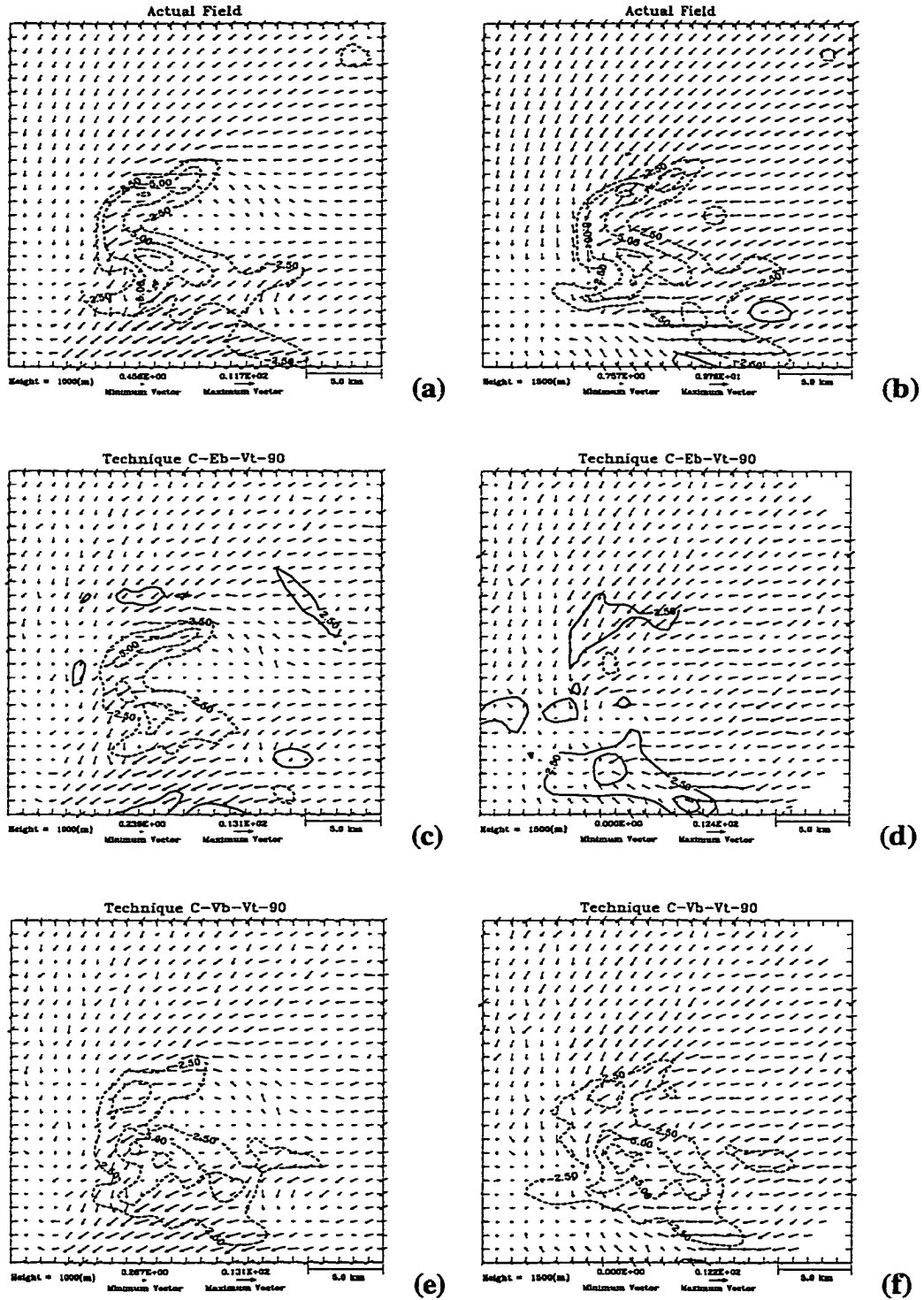


Figure 5.28: Horizontal wind vectors and contours of w at $z = 1.0$ and 1.5 km in the microburst storm, taken from (a,b) the true field, (c,d) technique C-Eb-Vt-300, and (e,f) technique C-Vb-Vt-90. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Case II (1° to 20°) scan geometry was prescribed in panels (a), (c), and (e), while Case III (2° to 20°) scan geometry was prescribed in panels (b), (d), and (f).

the lower data boundary is moved further away from the ground (compare panels e-h of Fig. 5.29 with the same panels of Fig. 5.31). The vorticity equation appears to be widely preferable to extrapolating divergence in each of Cases II and III with this storm. Also notice that applying the vorticity equation to retrieve an upper boundary condition yields excellent control of accumulating divergence errors in this case.

In Case IV we again apply a downward integration technique with $w = 0$ at storm top as the control technique (technique D-It). As in the Del City case, this is a more difficult technique for the vorticity equation to improve upon (see Fig. 5.33). However, with this storm the C-Vb-It technique is able to show improvement over the control technique with either 90 or 300 s volume scans. Technique C-Vb-It also applies $w = 0$ at storm top, but uses a constrained vertical velocity profile with the lower boundary condition determined with the vorticity equation. Error reductions are between 11% and 13%, with $1.02 \text{ m}\cdot\text{s}^{-1}$ and $1.01 \text{ m}\cdot\text{s}^{-1}$ errors for 300 and 90 s volume scan periods, respectively, versus $1.15 \text{ m}\cdot\text{s}^{-1}$ average RMS errors for the control technique. Although this improvement is still only modest, it is more significant than the improvement noted with the Del City storm. Note that curve C-Eb-It is by far the least desirable of the methods explored for Case IV. Clearly, the detrimental effects of errors in lower boundary condition specification often outweigh the effects of accumulating divergence errors in this case. Based on MB98's results, this is to be expected when systematic errors in divergence are not present.

The true vertical velocity field at $z = 1 \text{ km}$ and $t = 3600 \text{ s}$ and those retrieved from techniques D-It, C-Vb-It-90, C-Vb-It-300, and C-Eb-It are shown in Fig. 5.34. Although accumulation of divergence errors is clearly less problematic for the control technique in

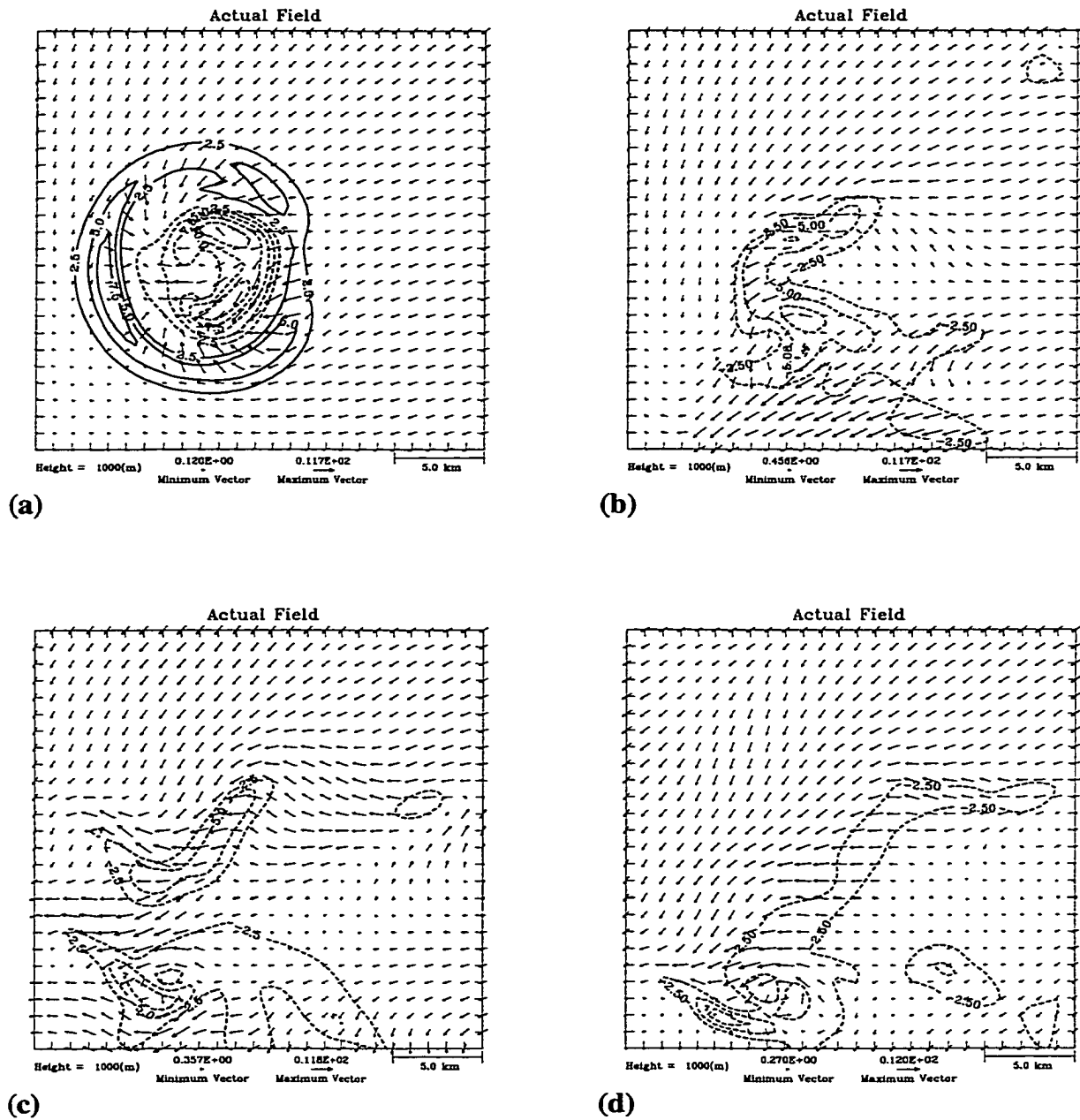


Figure 5.29: Horizontal wind vectors and contours of w at $z = 1.0$ km in the microburst storm, taken from the true field at (a) $t = 1800$ s, (b) $t = 3600$ s, (c) $t = 5400$ s, and (d) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 20° elevation for each radar (Case II).

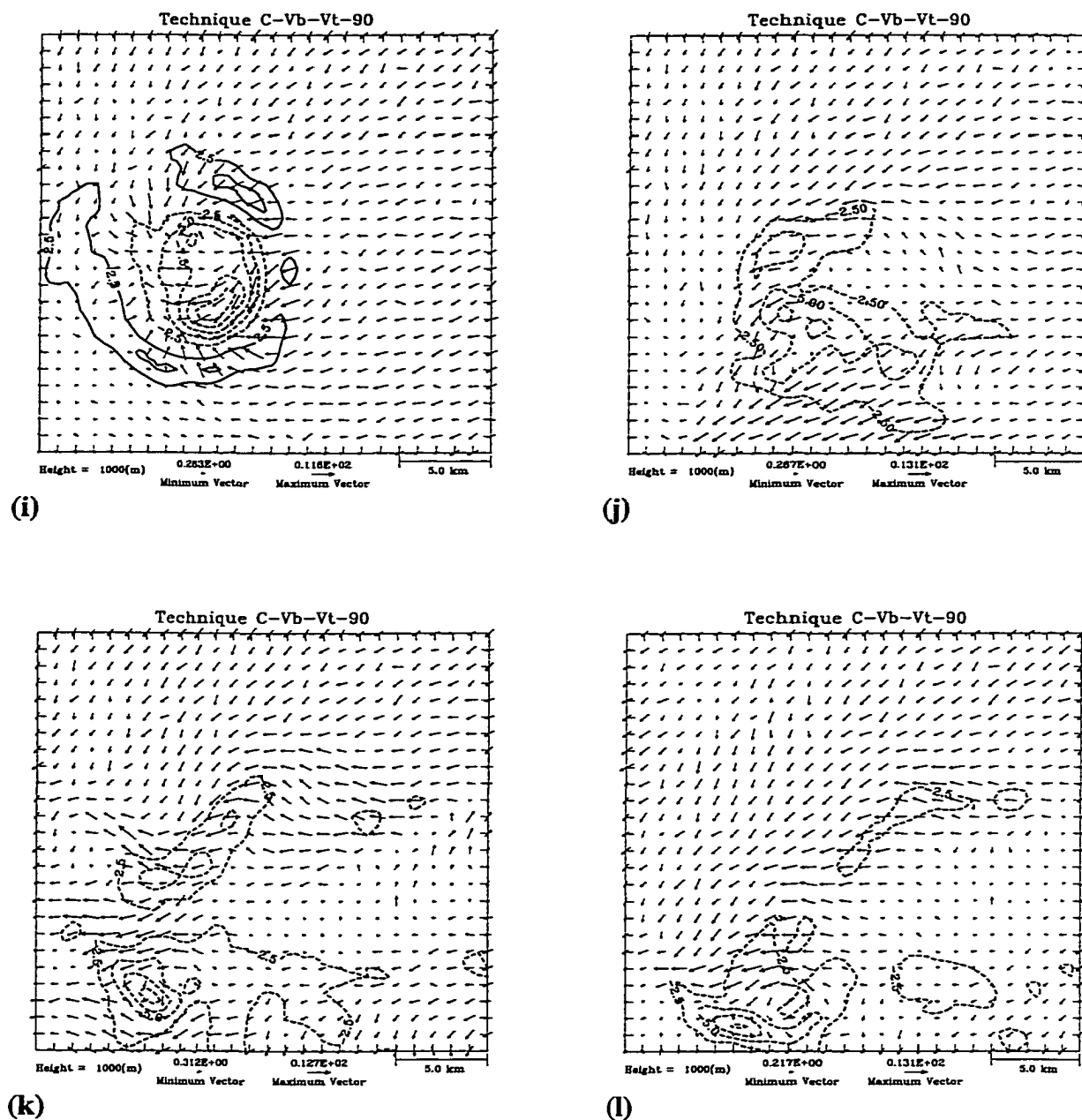


Figure 5.29 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the microburst storm, taken from the 'best' vorticity technique C-Vb-Vt-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 20° elevation for each radar (Case II).

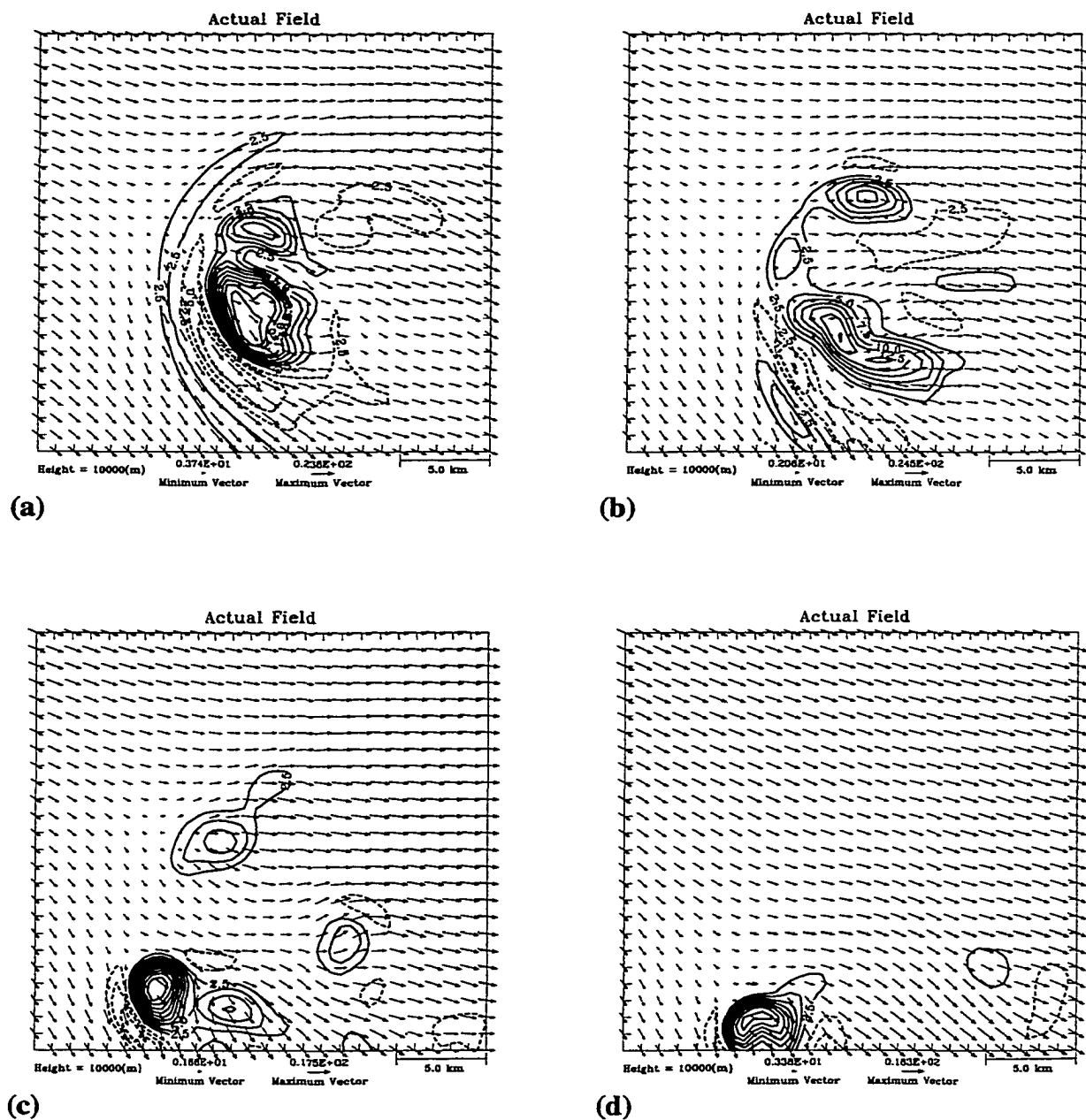


Figure 5.30: Horizontal wind vectors and contours of w at $z = 10.0 \text{ km}$ in the microburst storm, taken from the true field at (a) $t = 1800 \text{ s}$, (b) $t = 3600 \text{ s}$, (c) $t = 5400 \text{ s}$, and (d) $t = 7200 \text{ s}$. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 20° elevation for each radar (Case II).

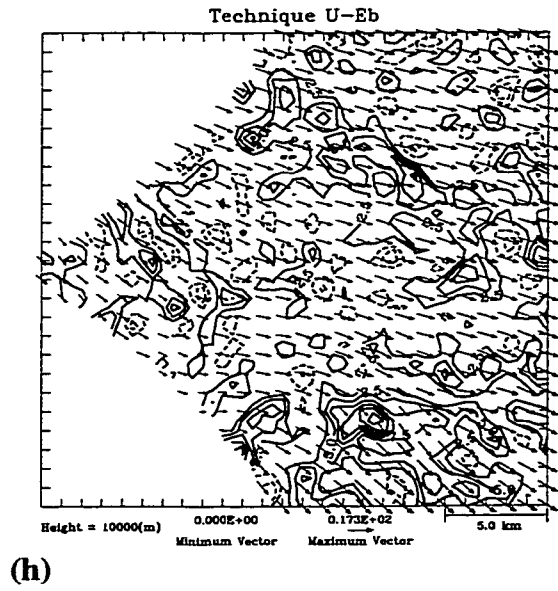
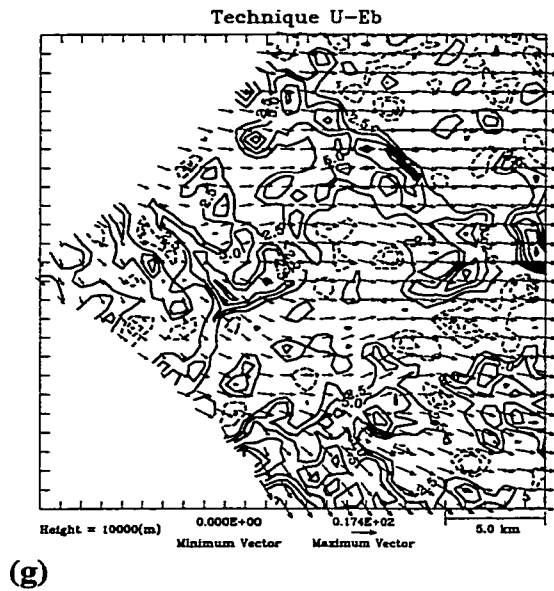
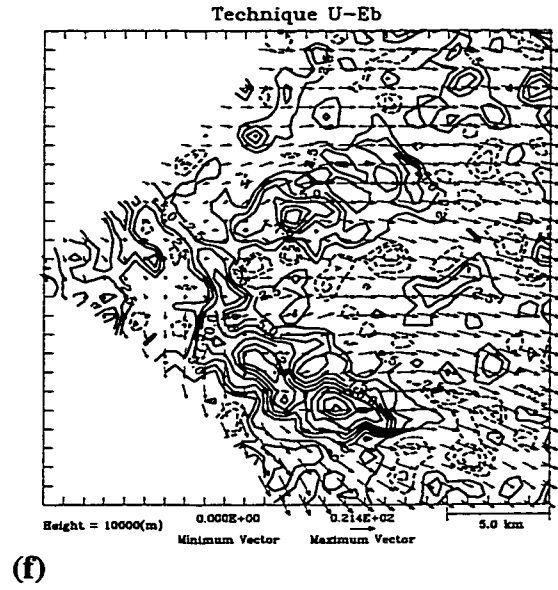
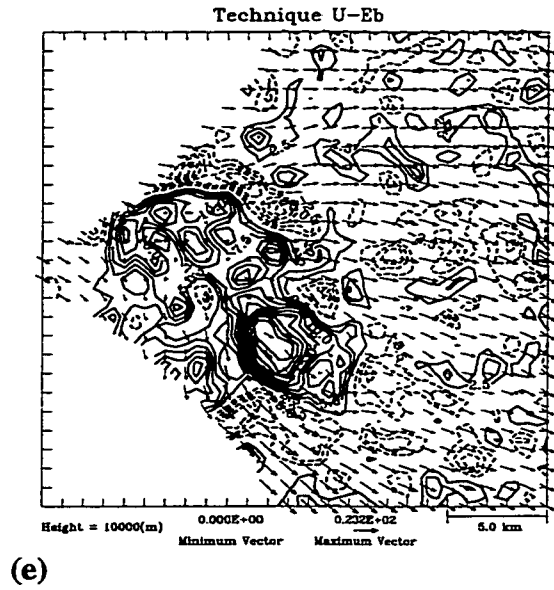


Figure 5.30 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the microburst storm, taken from control technique U-Eb at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 20° elevation for each radar (Case II).

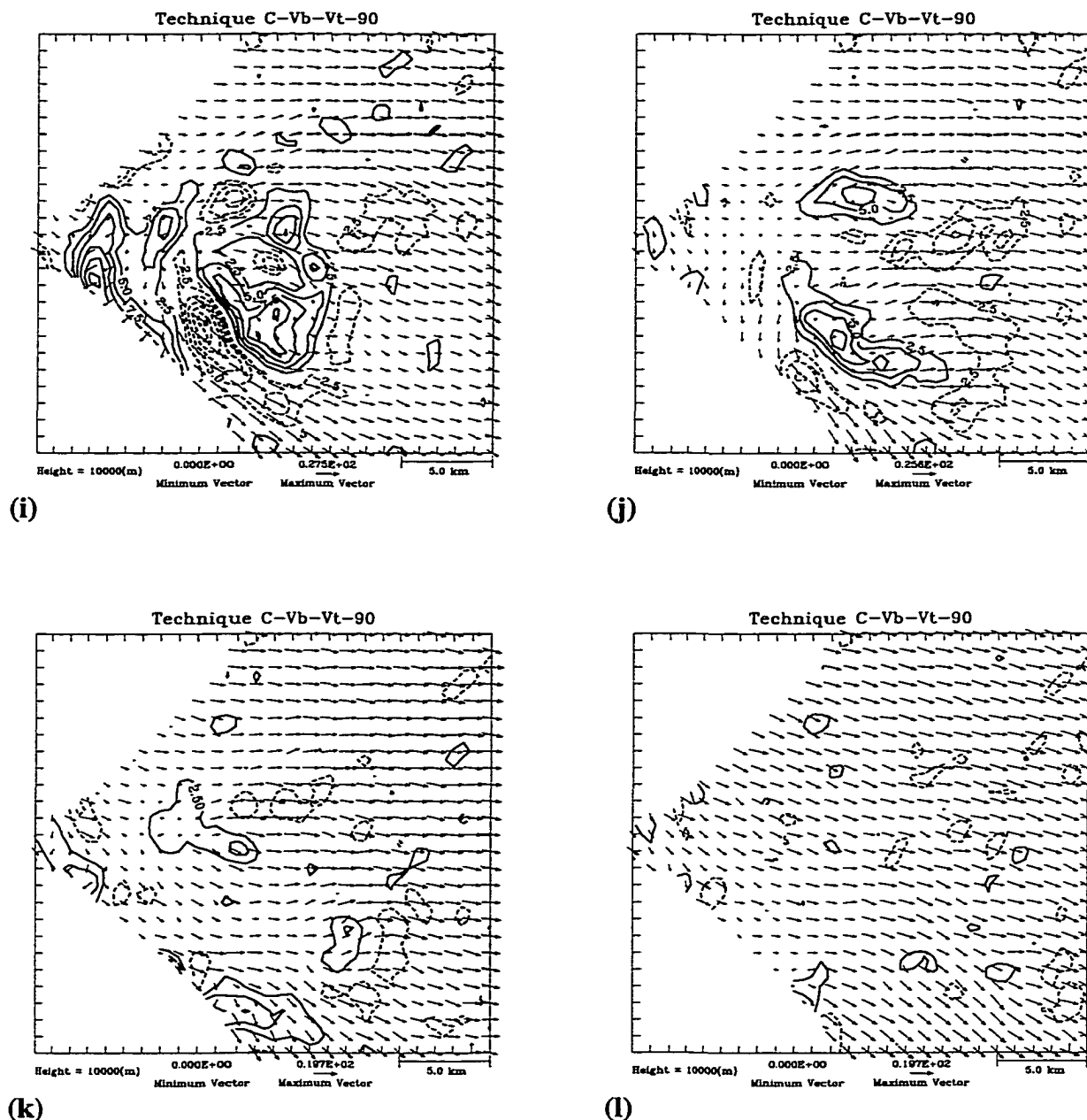
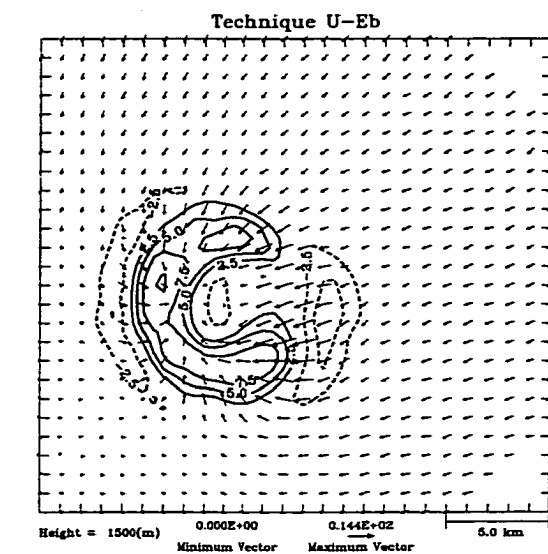
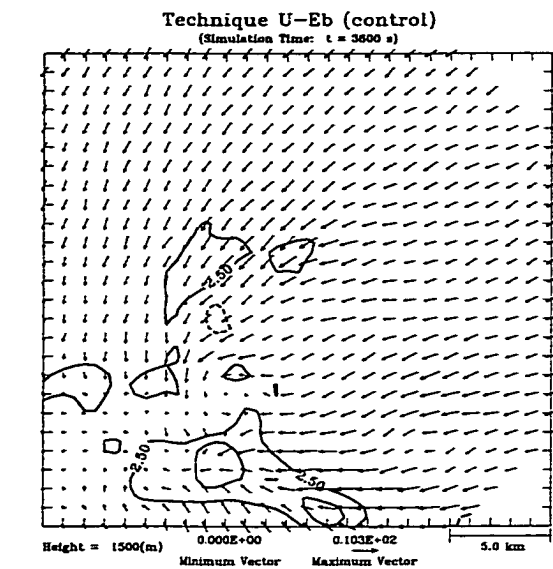


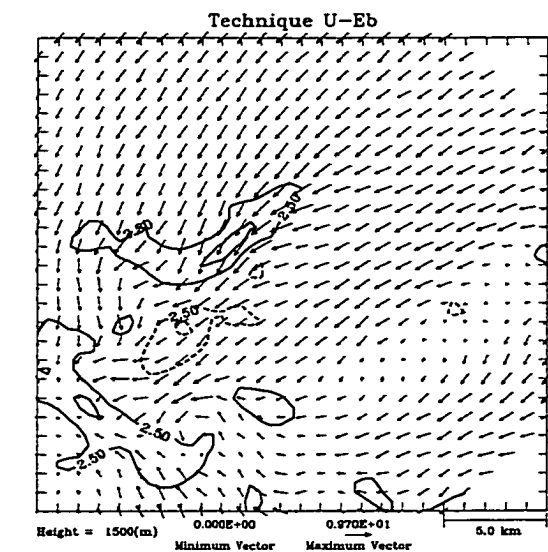
Figure 5.30 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the microburst storm, taken from the 'best' vorticity technique C-Vb-Vt-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 20° elevation for each radar (Case II).



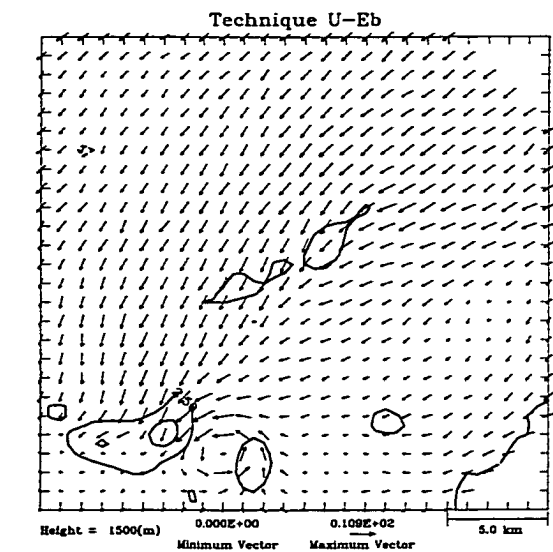
(e)



(f)



(g)



(h)

Figure 5.31 (continued): Horizontal wind vectors and contours of w at $z = 1.5$ km in the microburst storm, taken from control technique U-Eb at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 2° and 20° elevation for each radar (Case III).

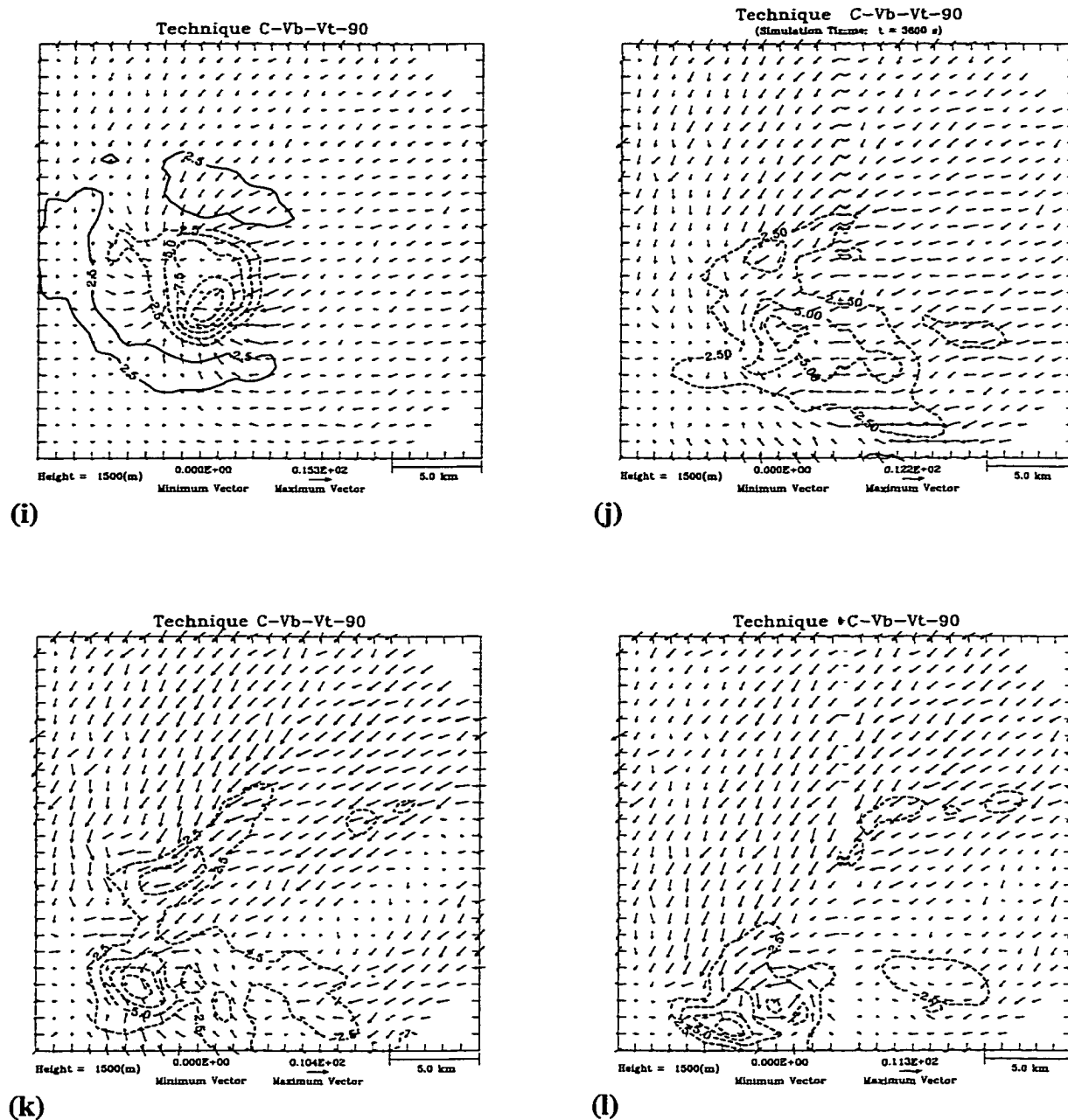


Figure 5.31 (continued): Horizontal wind vectors and contours of w at $z = 1.5$ km in the microburst storm, taken from the 'best' vorticity technique C-Vb-Vt-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is $2.5 \text{ m} \cdot \text{s}^{-1}$. Scan geometry is limited to regions between 2° and 20° elevation for each radar (Case III).

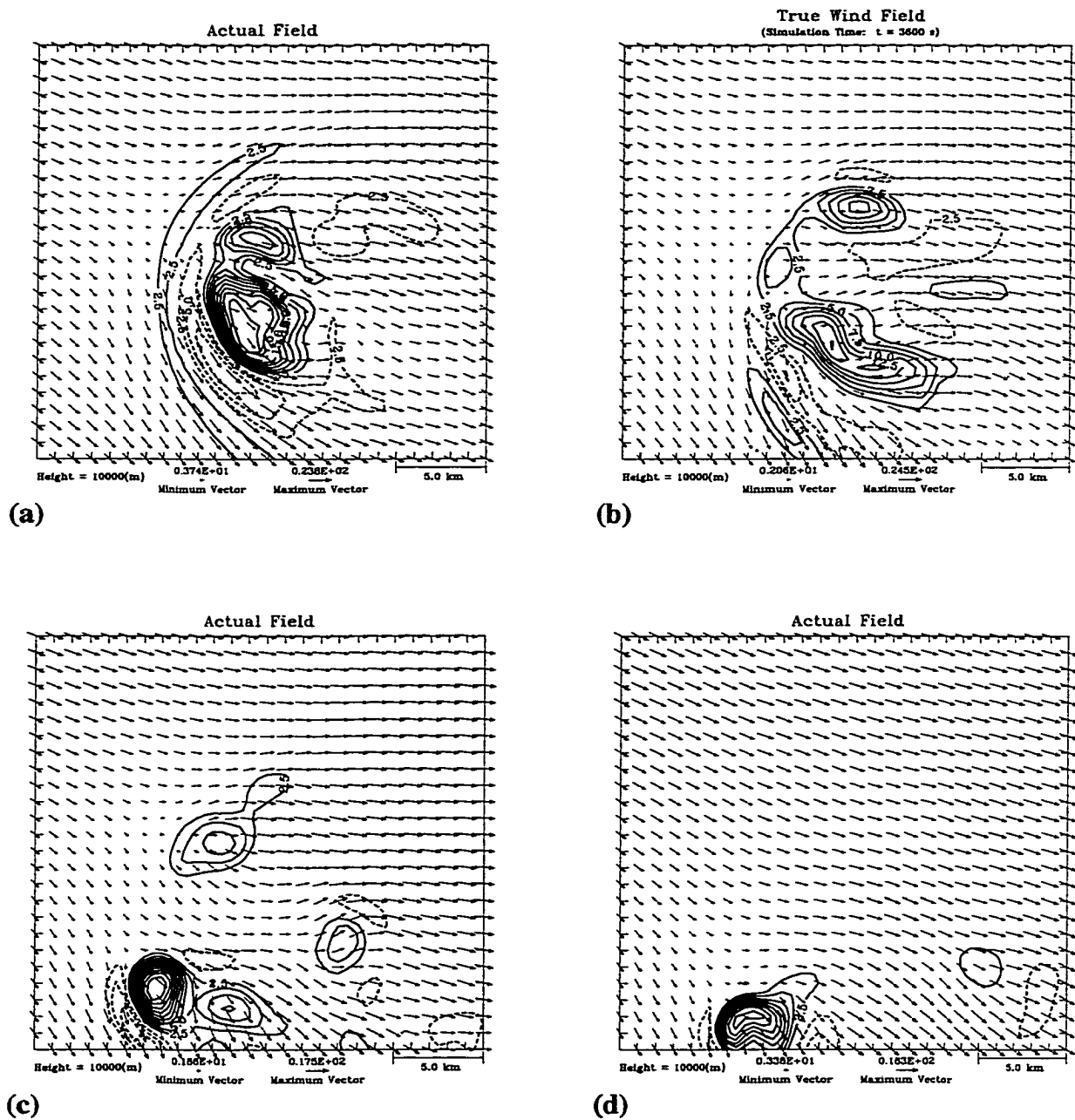


Figure 5.32: Horizontal wind vectors and contours of w at $z = 10.0$ km in the microburst storm, taken from the true field at (a) $t = 1800$ s, (b) $t = 3600$ s, (c) $t = 5400$ s, and (d) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 2° and 20° elevation for each radar (Case III).

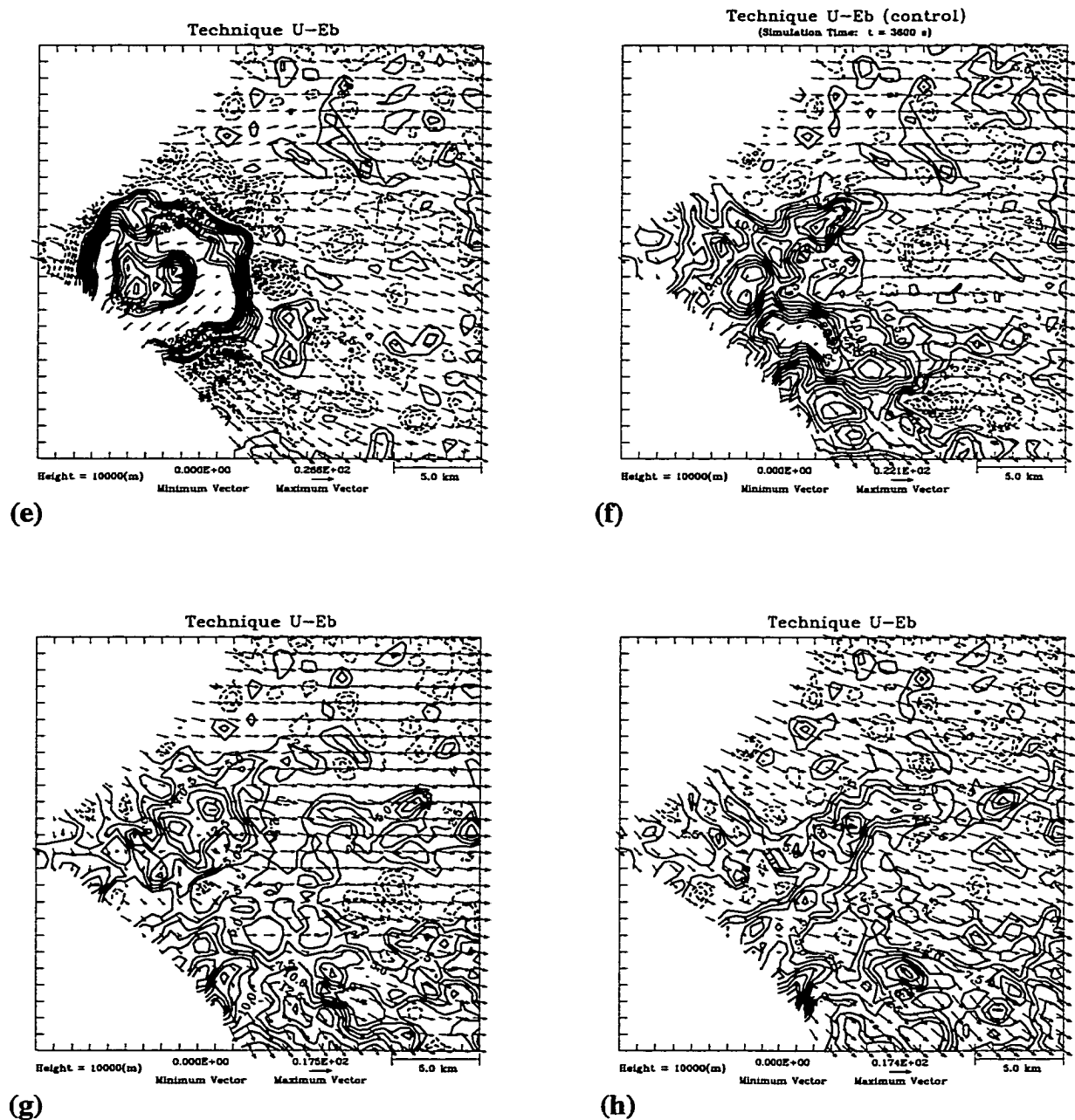


Figure 5.32 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the microburst storm, taken from control technique U-Eb at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 2° and 20° elevation for each radar (Case III).

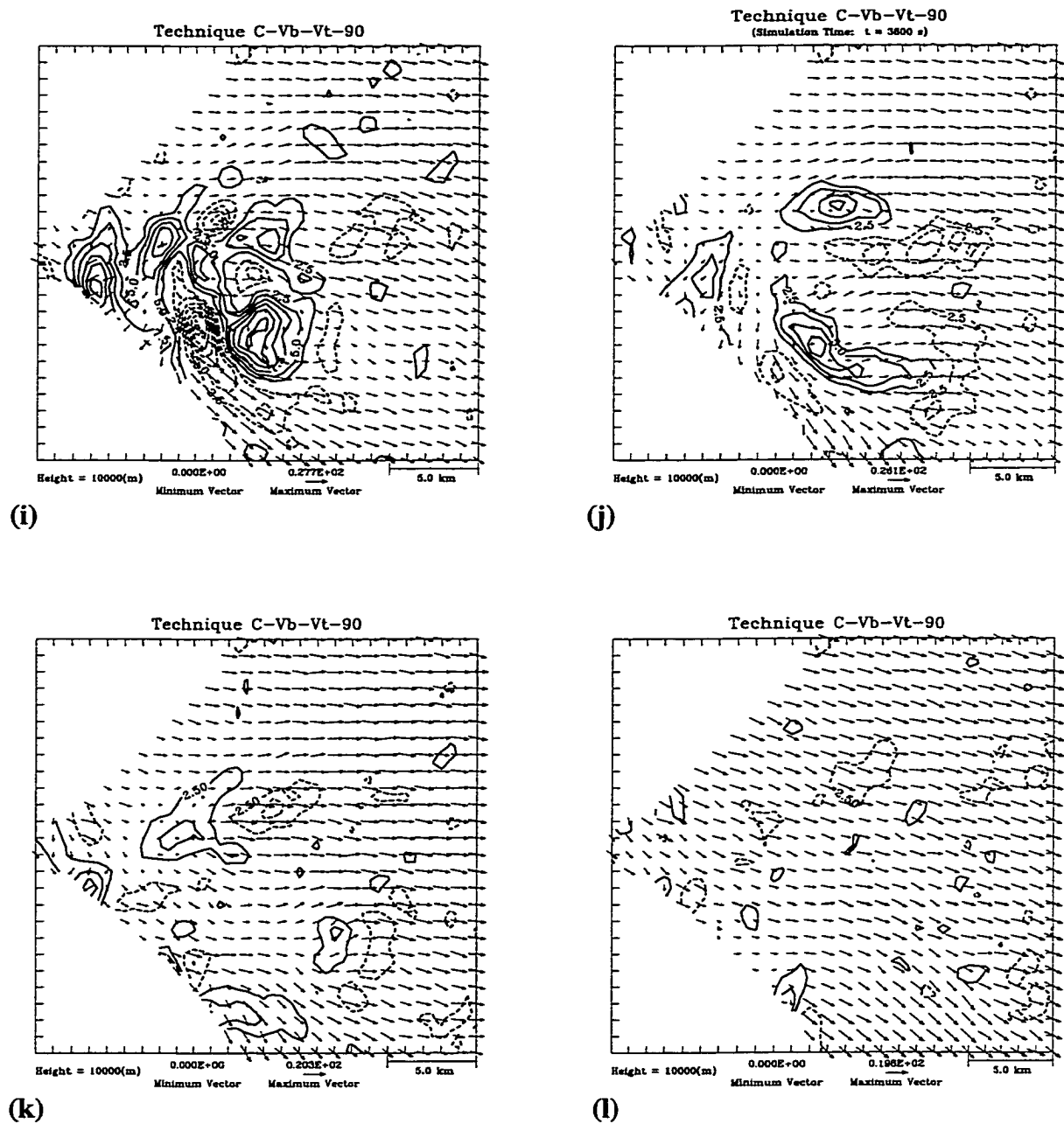


Figure 5.32 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the microburst storm, taken from the 'best' vorticity technique C-Vb-Vt-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is 2.5 m s^{-1} . Scan geometry is limited to regions between 2° and 20° elevation for each radar (Case III).

this case of downward integration, comparison of Figs. 5.34a and 5.34b shows that a number of small-scale spurious features still appear in the vertical velocity field from D-It. On the other hand, the best technique for this case, C-Vb-It-90 (Fig. 5.34c), removes nearly all of these features completely, and exhibits a very good overall representation of the true vertical velocity field. One could argue that, in terms of spatial interpretability of the fields (at least at $t = 3600$ s and $z = 1.0$ km), technique C-Vb-It-90 may be more advantageous over technique D-It than the modest RMS error reductions of Fig. 5.33 alone would seem to indicate. Technique C-Vb-It-300 (Fig. 5.34d) also shows good control of small-scale noise, although the bulk features are not represented as well. Technique C-Eb-It (Fig. 5.34e), a very common traditional technique, is clearly the least desirable of the techniques shown for this case.

Figures 5.35 and 5.36 explore the wind fields at $t = 1800$ s, $t = 3600$ s, $t = 5400$ s, and $t = 7200$ s from the control technique (which is now downward technique D-It) and the ‘best’ (as determined from Fig. 5.33) vorticity based technique (C-Vb-It-90). Similar to the Del City storm, the fields are much improved for the control technique here as compared to the earlier cases. Aside from the small scale noise with technique D-It, few significant spatial differences can be found between the fields from techniques D-It and C-Vb-It-90.

A summary of the errors from the various techniques applied on the microburst storm is available in Table 5.2. Comparison of Table 5.2 with Table 5.1 reveals several notable features. First, the overall RMS errors are much lower with the microburst storm, most likely because the vertical velocity magnitudes are smaller. However, the techniques applying the vorticity equation tended to exhibit more improvement with the

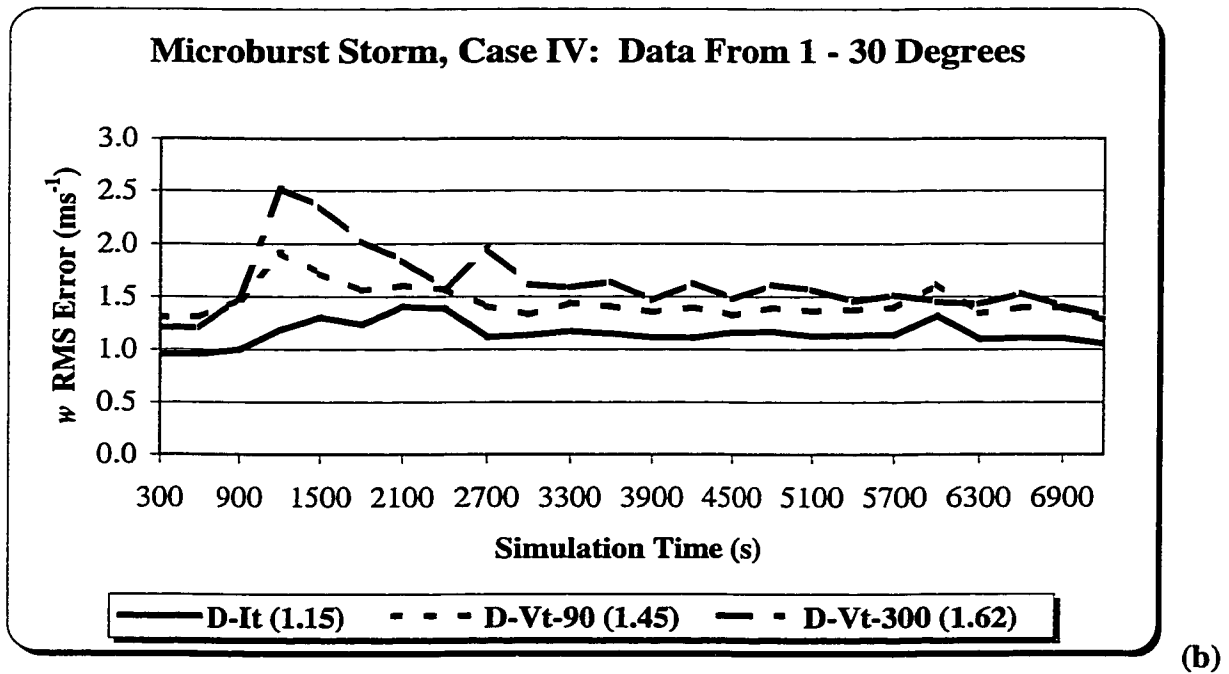
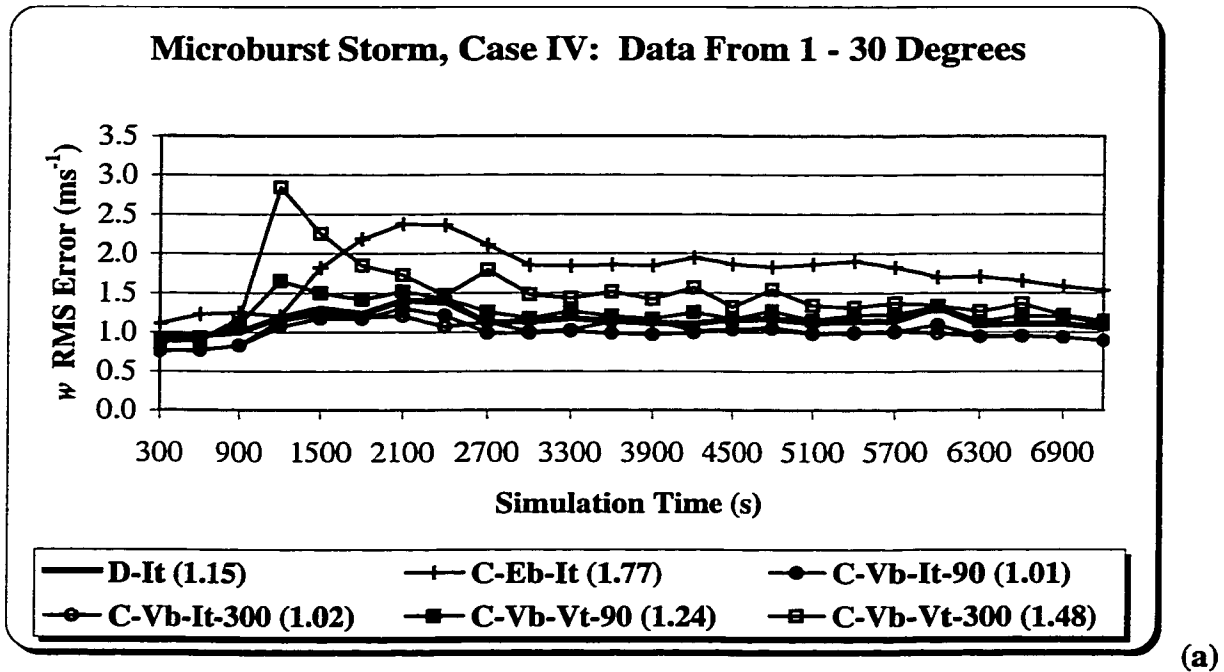


Figure 5.33: Root mean square vertical velocity retrieval errors from experiments on the microburst storm simulation using volume scan geometries providing data between 1° and 30° (Case IV).

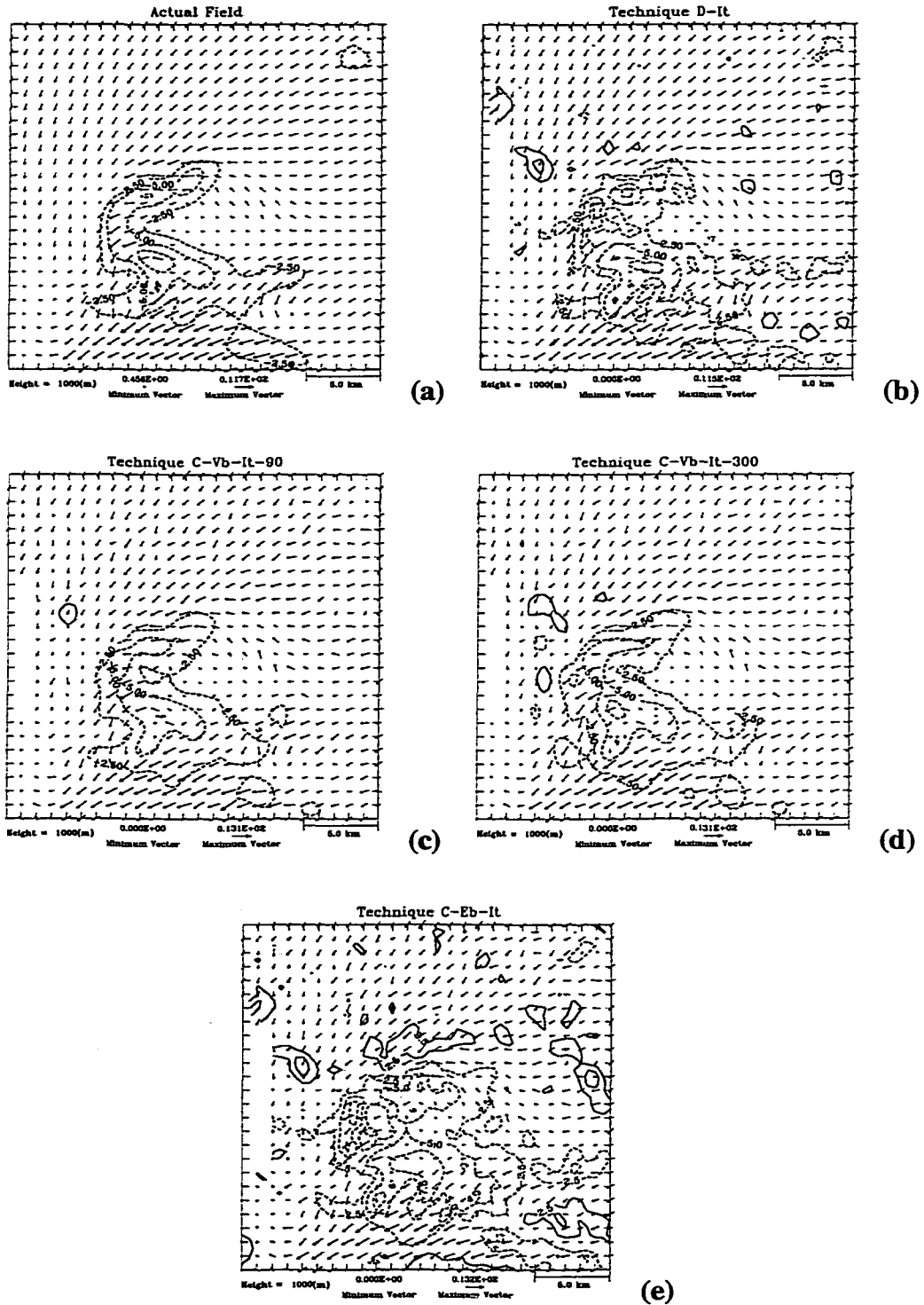


Figure 5.34: Horizontal wind vectors and contours of w at $z = 1.0$ km in the microburst storm, taken from (a) the true field, (b) control technique D-It, (c) technique C-Vb-It-90, (d) technique C-Vb-It-300, and (e) technique C-Vb-Vt-90. Contour interval is 2.5 m/s⁻¹. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).

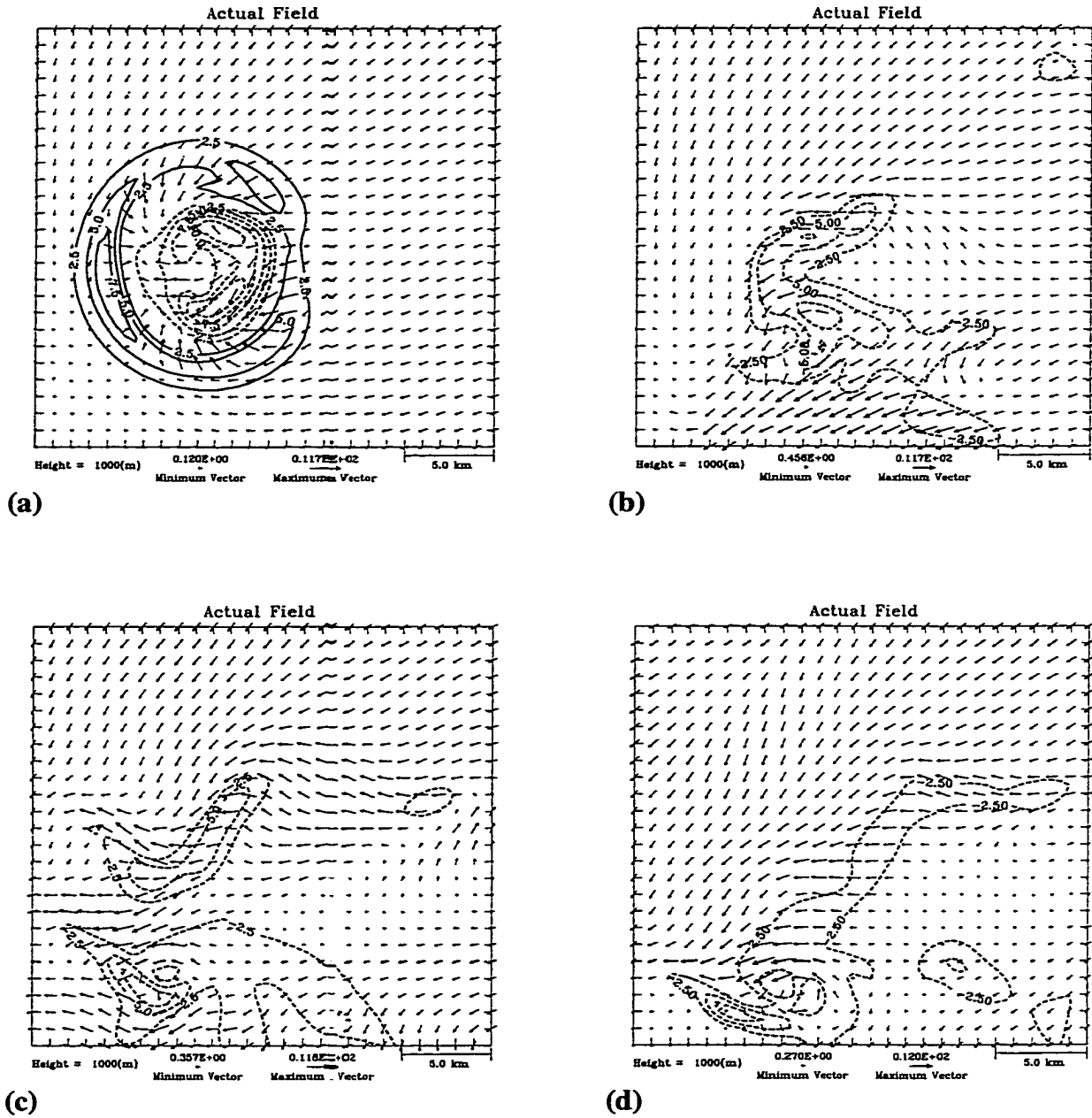
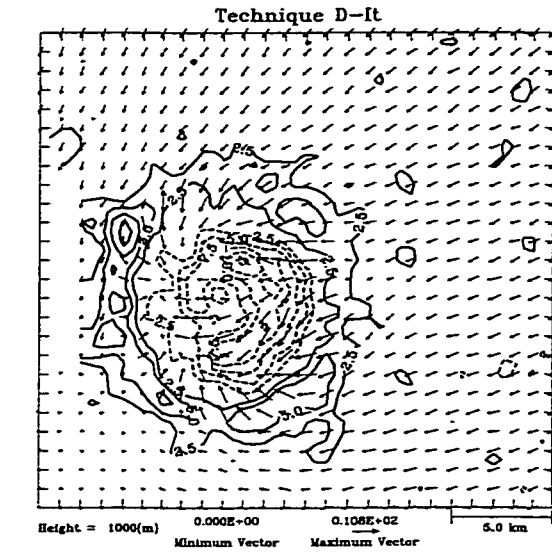
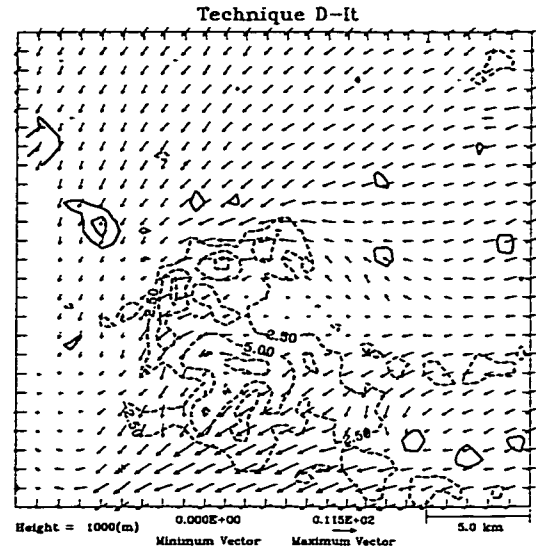


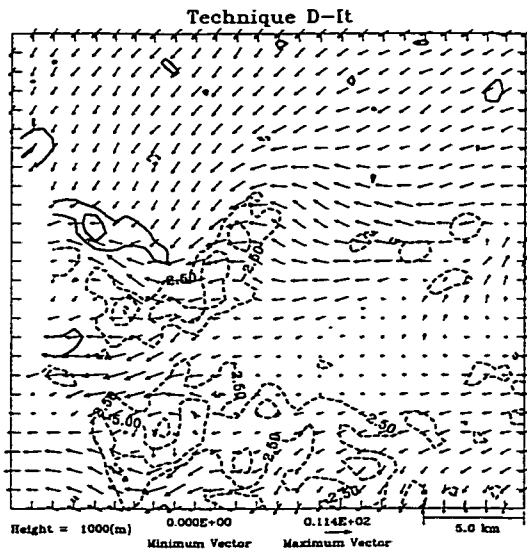
Figure 5.35: Horizontal wind vectors and contours of w at $z = 1.0$ km in the microburst storm, taken from the true field at (a) $t = 1800$ s, (b) $t = 3600$ s, (c) $t = 5400$ s, and (d) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).



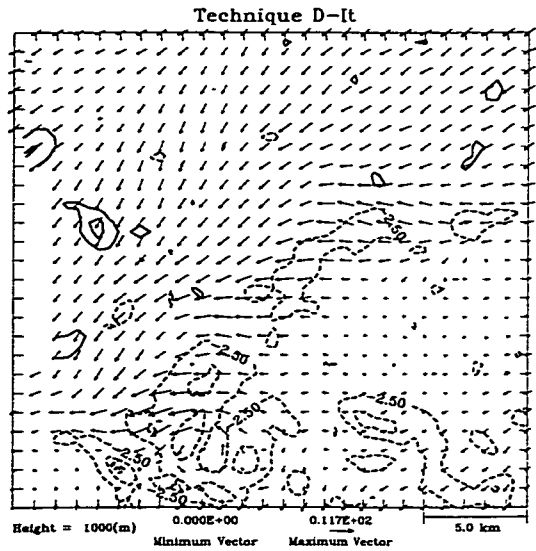
(e)



(f)



(g)



(h)

Figure 5.35 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the microburst storm, taken from control technique D-It at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).

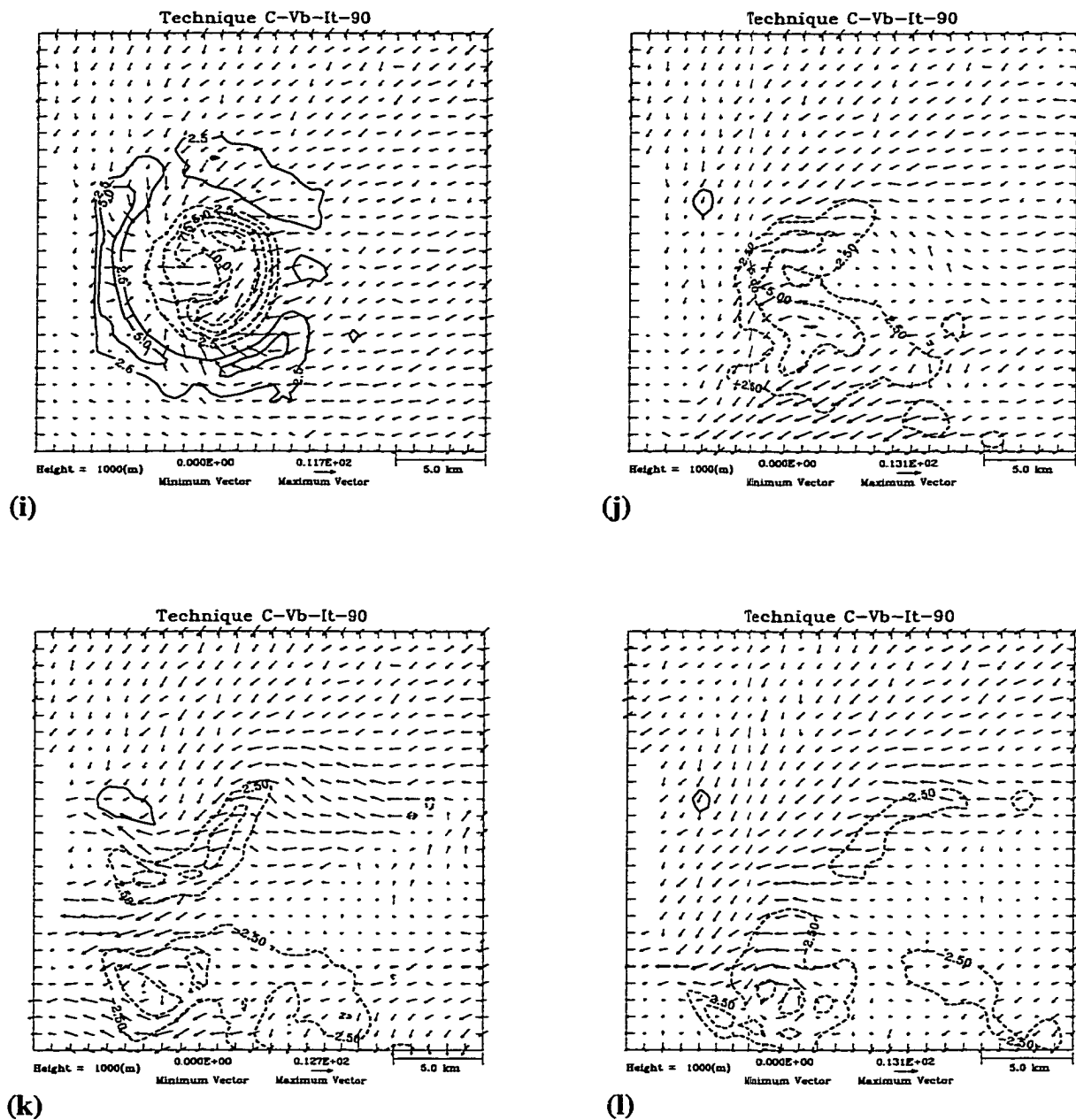


Figure 5.35 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ km in the microburst storm, taken from the 'best' vorticity technique C-Vb-It-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is $2.5 \text{ m} \cdot \text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).

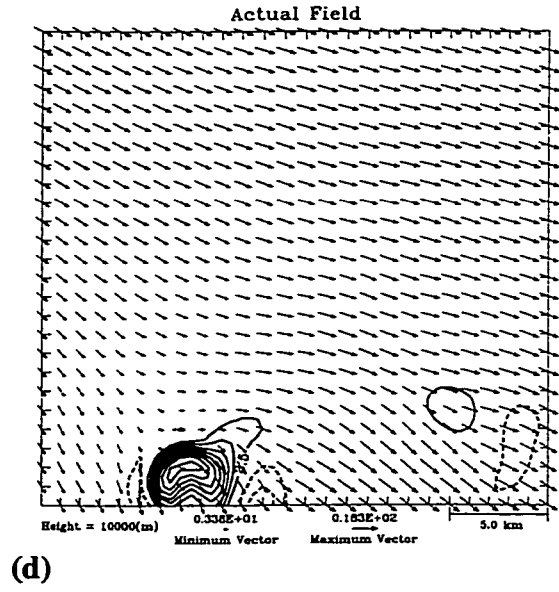
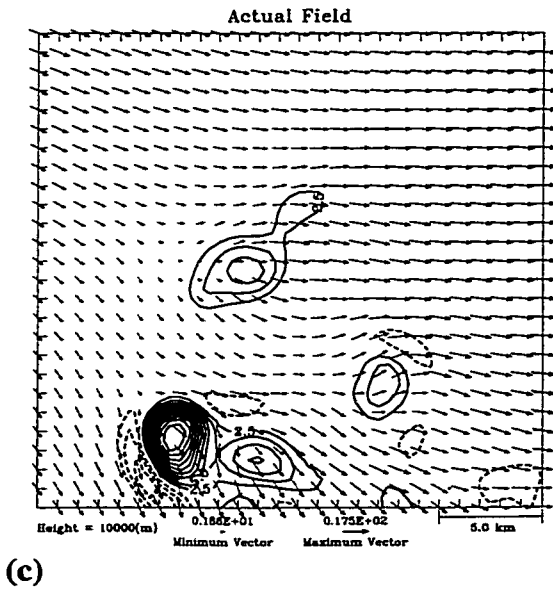
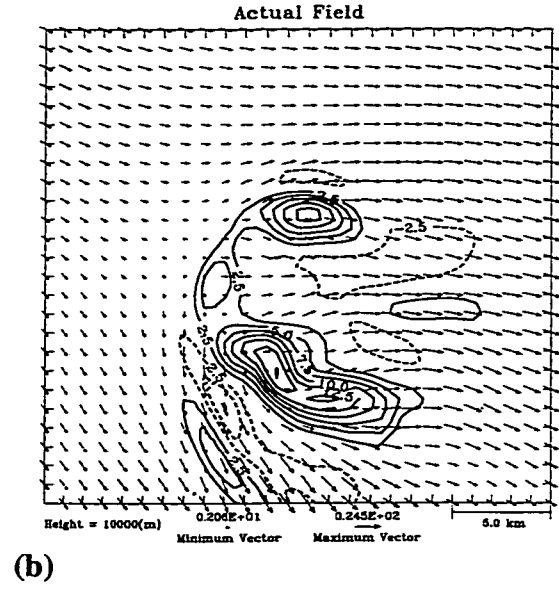
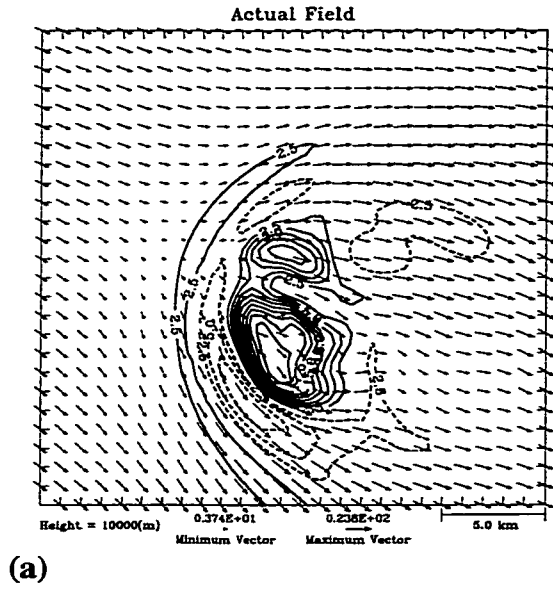


Figure 5.36: Horizontal wind vectors and contours of w at $z = 10.0$ km in the microburst storm, taken from the true field at (a) $t = 1800$ s, (b) $t = 3600$ s, (c) $t = 5400$ s, and (d) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).

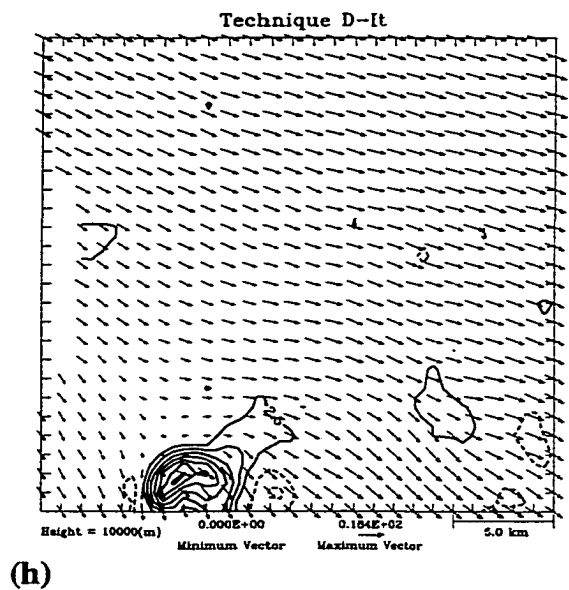
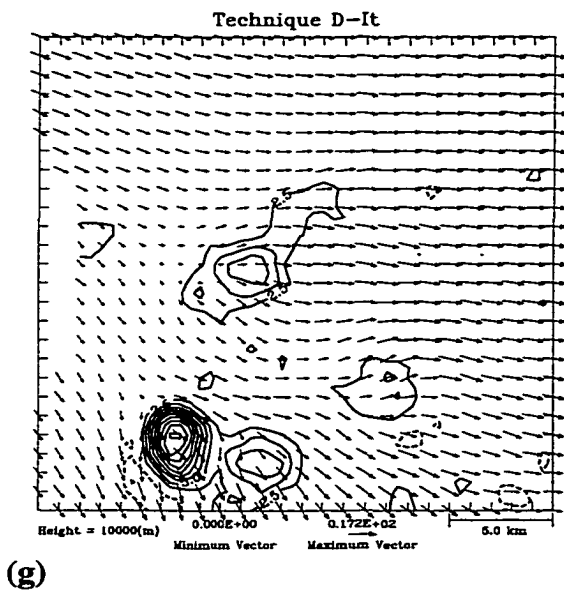
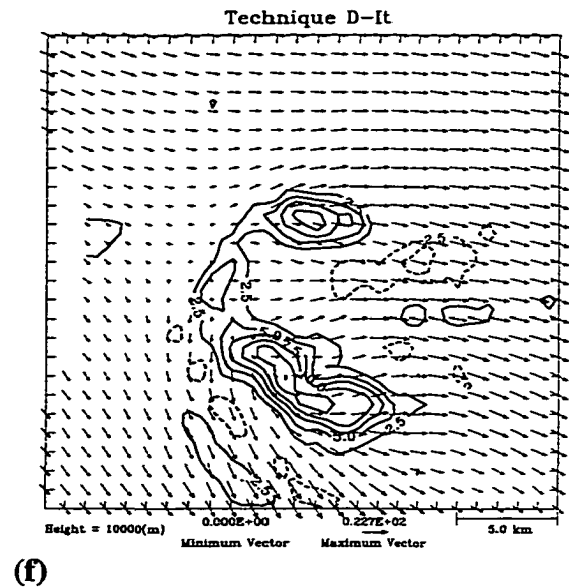
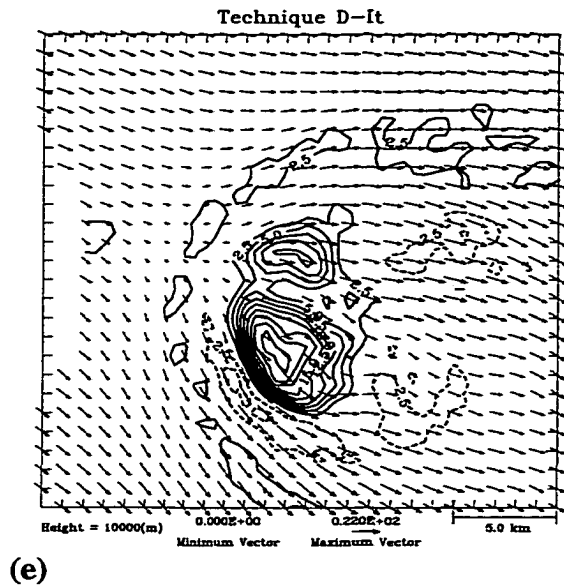


Figure 5.36 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the microburst storm, taken from control technique D-It at (e) $t = 1800$ s, (f) $t = 3600$ s, (g) $t = 5400$ s, and (h) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).

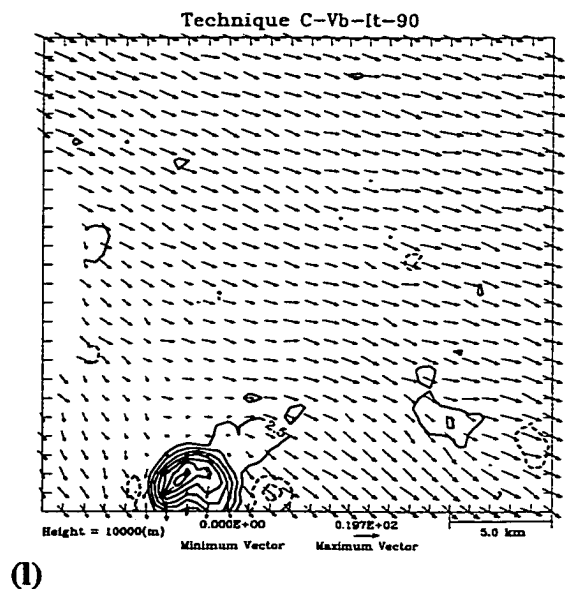
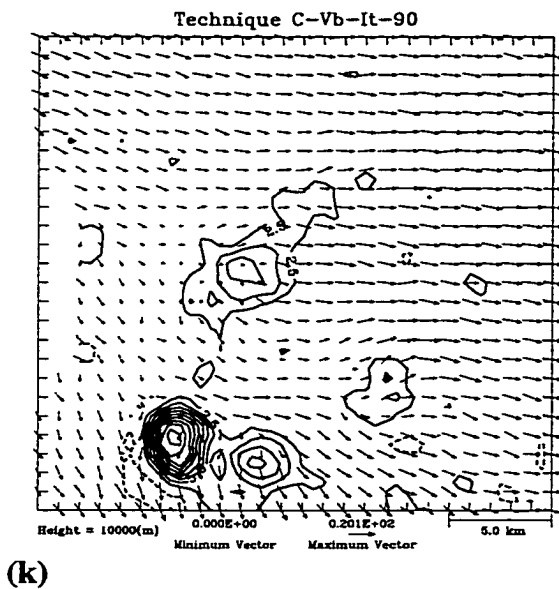
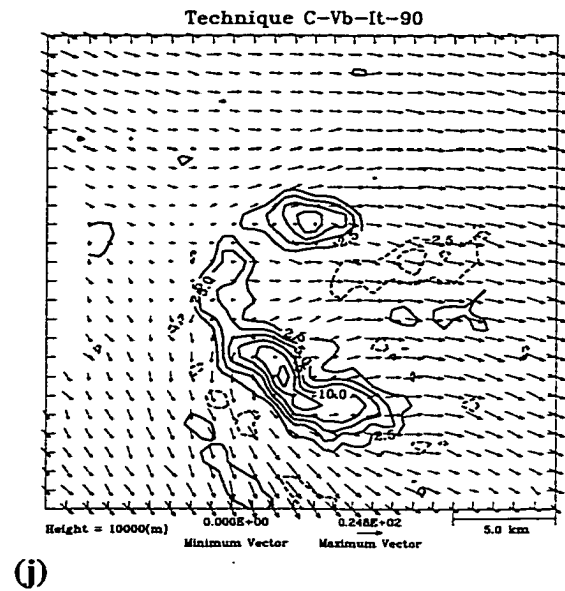
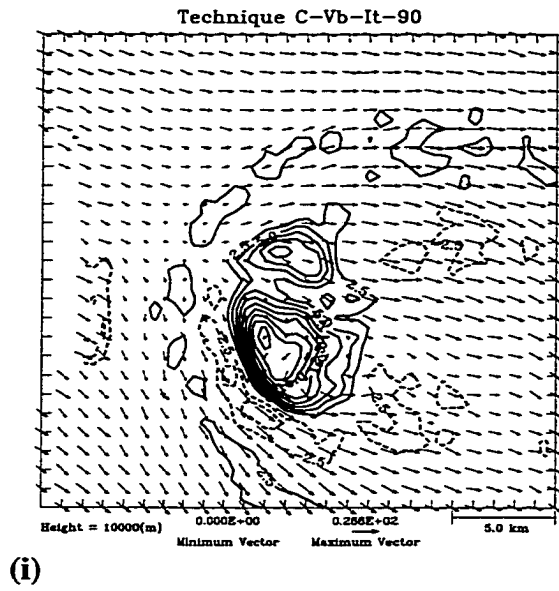


Figure 5.36 (continued): Horizontal wind vectors and contours of w at $z = 10.0$ km in the microburst storm, taken from the 'best' vorticity technique C-Vb-It-90 at (i) $t = 1800$ s, (j) $t = 3600$ s, (k) $t = 5400$ s, and (l) $t = 7200$ s. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV).

microburst storm than the control techniques. This is encouraging given the weaker shear environment of the microburst storm. Second, there is much less difference between application of 90 and 300 s volume scan periods with the vorticity techniques for this storm. From the discussion in Chapter 3 this is expected when ζ exhibits are more linear temporal evolution. The “loaded gun”, high-shear environment of the Del City storm apparently supports a much more rapid and non-linear evolution than the environment of the microburst storm. Third, the ‘best’ technique for most of the scan geometry cases is unchanged from the Del City storm, with the exception of Case II. In Case II, technique C-Vb-Vt-90 is now the preferred technique (as opposed to technique C-Eb-Vt-90 for the Del City storm). Thus, application of the vorticity equation to retrieve a lower boundary condition in this storm appears to be preferred over extrapolating divergence regardless of whether data are available below 1° or 2° .

Section D: Limiting the data region to the storm volume

In all of the experiments discussed thus far, data have been limited only by radar elevation angle, leaving data available over a significant region where no hydrometeors would be available to provide radar observations. These experiments were useful because they provide a clear picture of the results from each technique that is not dependent upon storm shape, presence of precipitation, or other logistical factors complicating the analyses. However, for real data applications it is important to show that the methods proposed here continue to work well under more realistic data coverage scenarios. To investigate this situation it is useful to revisit Cases I and IV of the microburst storm but

	Case I (0° to 20°)	Case II (1° to 20°)	Case III (2° to 20°)	Case IV (1° to 30°)
U-Ib	2.17 (Control)	-	-	-
U-Eb	-	2.97 (Control)	4.43 (Control)	-
U-Vb-90	2.50	2.71	2.67	-
U-Vb-300	2.64	2.82	2.79	-
D-It	-	-	-	1.15 (Control)
D-Vt-90	-	-	-	1.45
D-Vt-300	-	-	-	1.62
C-Eb-It	-	-	-	1.77
C-Ib-Vt-90	<u>1.20</u>	-	-	-
C-Ib-Vt-300	1.24	-	-	-
C-Eb-Vt-90	-	2.03	3.44	-
C-Eb-Vt-300	-	2.05	3.44	-
C-Vb-Vt-90	1.40	<u>1.67</u>	<u>1.73</u>	1.24
C-Vb-Vt-300	1.64	1.92	1.98	1.48
C-Vb-It-90	-	-	-	<u>1.01</u>
C-Vb-It-300	-	-	-	1.02

Table 5.2: RMS vertical velocity errors for each of the techniques and data coverages discussed in Section C (the microburst storm), averaged across all 24 retrieval times. Hyphens are shown in place of RMS errors for techniques that were not applied in each case. The control technique for each case is labeled as such, and the 'best' technique for each case is underlined.

limiting data to regions containing hydrometeors (as defined by a $0.1 \text{ g}\cdot\text{kg}^{-1}$ total mixing ratio of rain water, ice, snow, and hail). From Srivastava (1985) we may deduce that this region should correspond roughly to a 20 to 25 dBZ reflectivity threshold. The east-west cross sectional area of the new restricted analysis region is slightly smaller than the region contained within the $0.1 \text{ g}\cdot\text{kg}^{-1}$ mixing ratio contour of Figs. 5.21b, 5.21d, 5.21f, and 5.21h, which includes cloud water mixing ratio in addition to those previously mentioned.

Technique U-Ib was again chosen as the control technique for Case I. However, unlike the experiments of Section C of this chapter, data were not available to ground level in all areas since hydrometeors were not present down to ground level in every column. Given this fact, technique U-Eb would appear to have been a logical control technique. However, the RMS errors from the two techniques favored technique U-Ib. This result was likely a combination of the shortcomings of extrapolating divergence in microburst situations and also the need to extrapolate divergence from varying heights dependent upon the hydrometeor distribution, which can create considerable noise in the vertical velocity field. Since we are not restricted to either of these modes of “guessing” a boundary condition with technique C-Vb-Vt, it was chosen as an ideal vorticity technique for comparison. The results of this experiment (Fig. 5.37) continue to be encouraging. Even when faced with a much more limited data region, technique C-Vb-Vt, with either 90 or 300 s volume scan periods, is still preferable over the control technique. Techniques C-Vb-Vt-90 and C-Vb-Vt-300 show error reductions of as much as 27% over the control technique, comparable to the 36% reduction in error noted for the same case and technique (but with a much larger data region) in Section C. Although this same experiment was not carried out for Cases II and III, we can likely infer from the

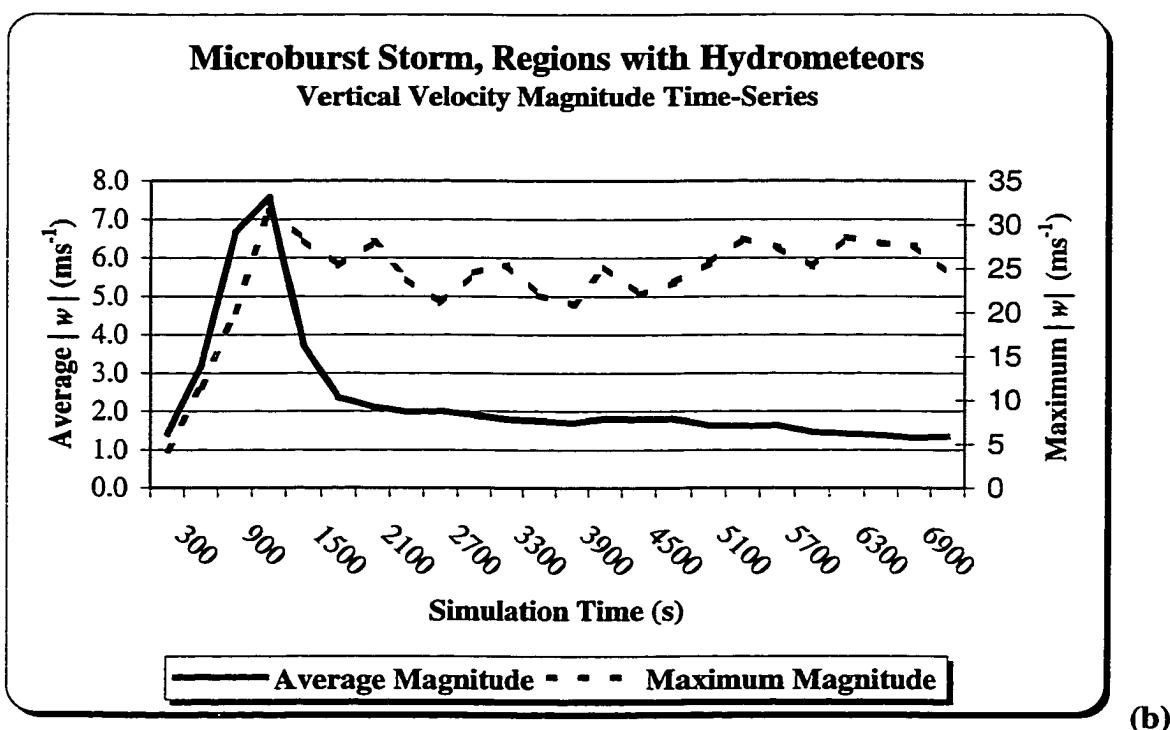
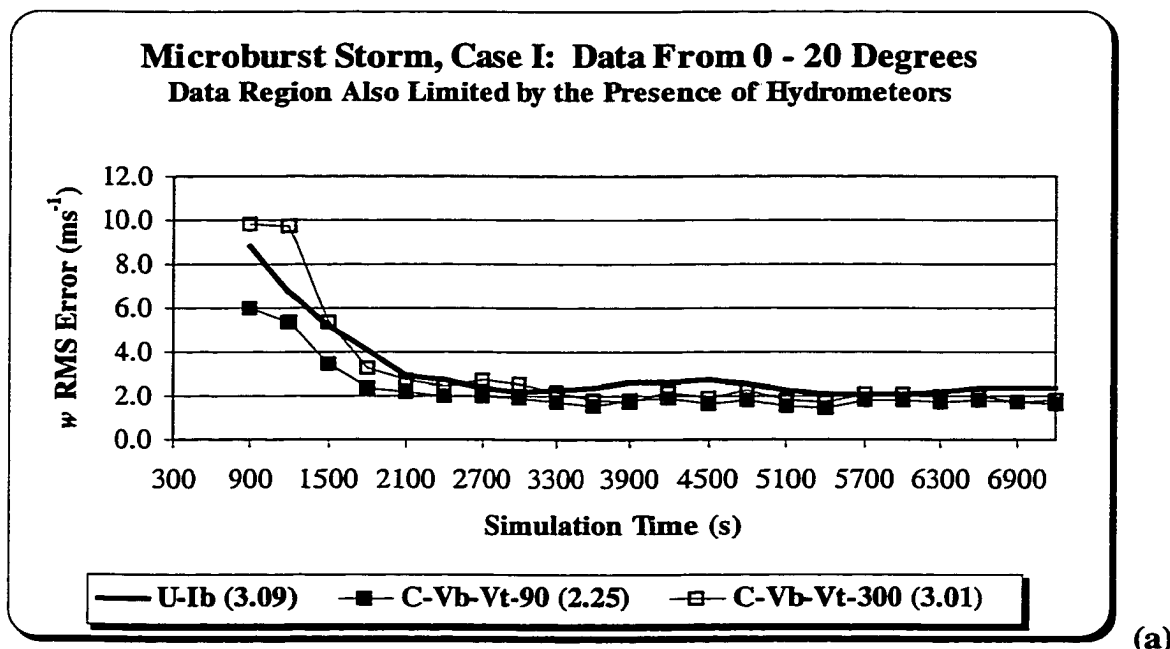


Figure 5.37: Root mean square vertical velocity retrieval errors (a) from experiments on the microburst storm simulation using volume scan geometries providing data between 0° and 20° (Case I) and further limiting the data region by the presence of hydrometeors. For comparison, average and maximum vertical velocity magnitudes over the portions of the domain in which hydrometeors are present are depicted in (b), which is also applicable in interpretation of Fig. 5.39.

results of Section C that the vorticity based techniques would become even more advantageous when data at low elevation angles are lacking.

The spatial structures of the fields retrieved from each of the techniques for which results are shown in Fig. 5.37 are explored in Fig. 5.38. The fields at $z = 1.0$ and 10.0 km at $t = 3600$ s are again shown to permit visual comparison of results with earlier retrievals (in Section C) for which data were not limited to regions with hydrometeors. Visually, techniques U-Ib and C-Vb-Vt-90 appear to have equally good retrievals of the low level vertical velocity (see Figs. 5.38c and 5.38e), with slightly more significant errors arising with technique C-Vb-Vt-300 (Fig. 5.38g). However, at $z = 10.0$ km, techniques C-Vb-Vt-90 and C-Vb-Vt-300 each show reasonably good representations of the bulk features of the vertical velocity field (compare Figs. 5.38h and 5.38f with Fig. 5.38b). However, the field from control technique U-Ib is dramatically degraded by the presence of divergence related noise (see Fig. 5.38d). Note also that the vertical velocity field at $z = 10.0$ from the best technique in this case (C-Vb-Vt-90; Fig. 5.38f) compares very favorably to the field obtained with the same scan geometry limitations but without the extra hydrometeor limitation shown in Fig. 5.23e of Section C. Thus, the considerably smaller data region does not appear to significantly degrade the results for this vorticity based technique.

Case IV continues to be more problematic for the vorticity techniques given this limited data region. Only technique C-Vb-It-90 was able to show any improvement over technique D-It, with a small 3% error reduction from 1.85 to $1.78 \text{ m}\cdot\text{s}^{-1}$ (see Fig. 5.39). Technique C-Vb-Vt-90, the best technique for Case I, showed slightly larger errors than technique D-It. It should be noted that the true vertical velocity at storm top as defined by

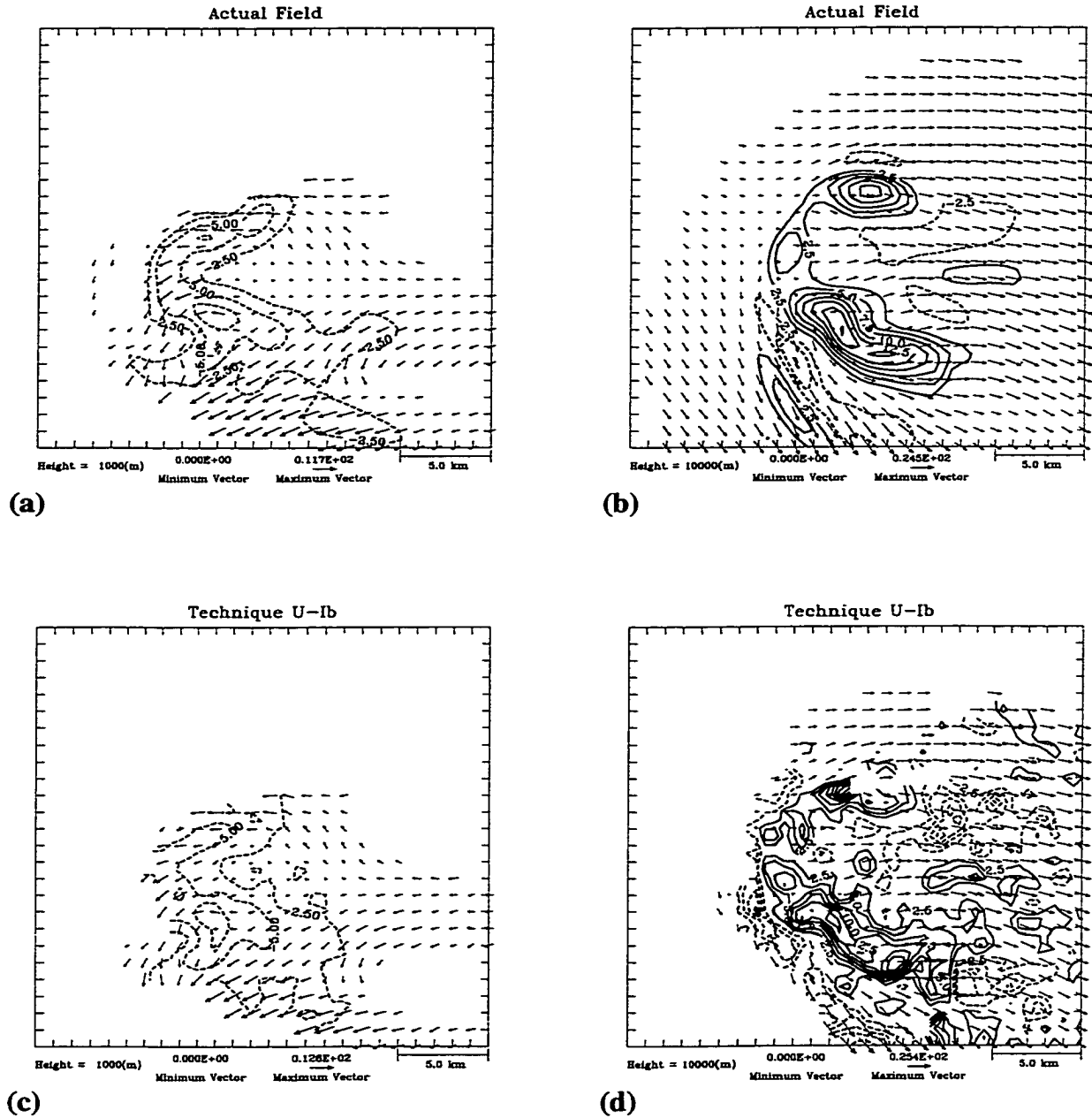
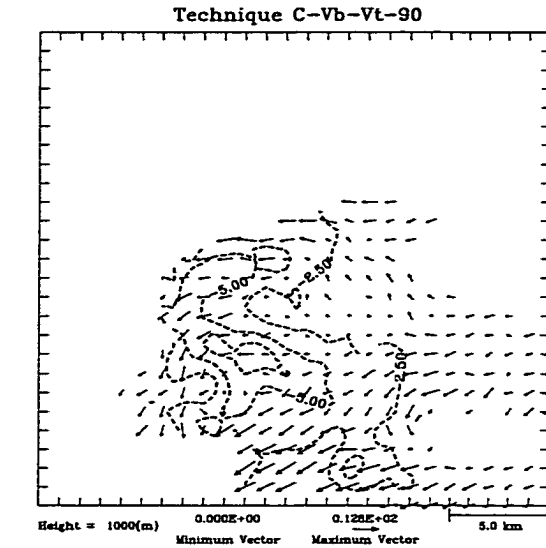
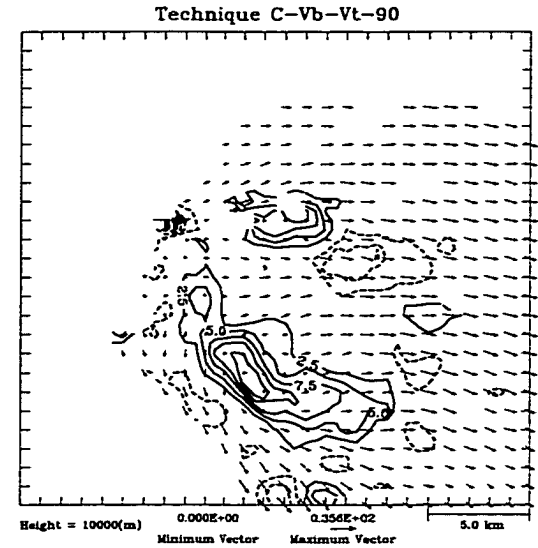


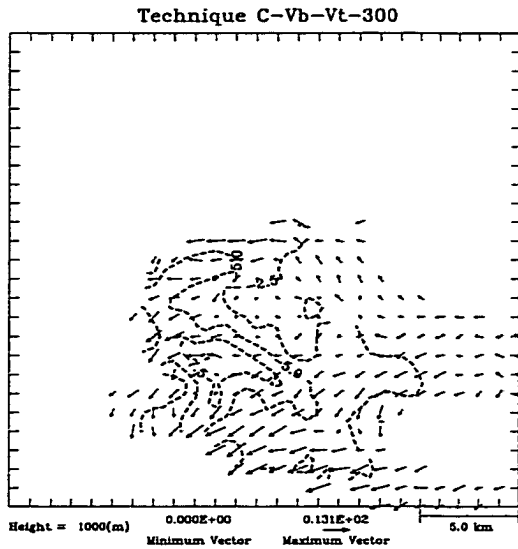
Figure 5.38: Horizontal wind vectors and contours of w at $z = 1.0$ and 10.0 km in the microburst storm, taken from (a) the true field at $z = 1.0$ km, (b) the true field at $z = 10.0$ km, (c) control technique U-Ib at $z = 1.0$ km, and (d) control technique U-Ib at $z = 10.0$ km. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I), and further limited to regions with a $0.1 \text{ g}\cdot\text{kg}^{-1}$ concentration of precipitable hydrometeors.



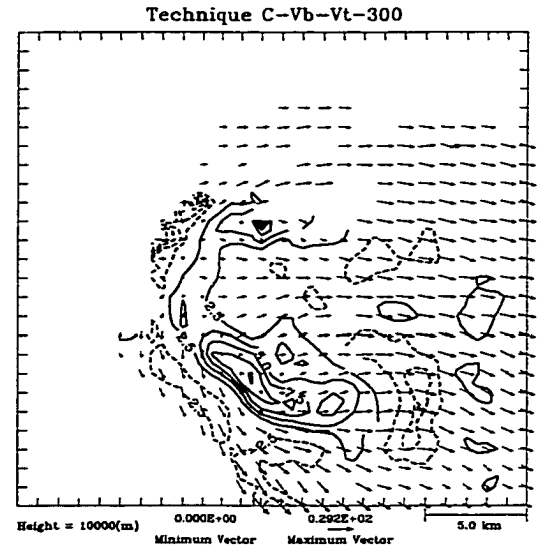
(e)



(f)



(g)



(h)

Figure 5.38 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ and 10.0 km in the microburst storm, taken from (e) technique C-Vb-Vt-90 at $z = 1.0$ km, (f) technique C-Vb-Vt-90 at $z = 10.0$ km, (g) technique C-Vb-Vt-300 at $z = 1.0$ km, and (h) technique C-Vb-Vt-300 at $z = 10.0$ km. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 0° and 20° elevation for each radar (Case I), and further limited to regions with a $0.1 \text{ g}\cdot\text{kg}^{-1}$ concentration of precipitable hydrometeors.

the presence of hydrometeors in this simulation is very nearly zero. However, it has been noted in the literature that this is not always the case with the “storm top” as observed by radar (Protat and Zawadzki, 1999). When the vertical velocity at storm top is suspected of differing significantly from zero, the vorticity based techniques may hold more promise for improvement. Also, all previous discussion pertaining to the potential for more significant improvements over the control technique when systematic errors are present is still valid with this case.

The spatial structure of the vertical velocity fields from this final experiment are explored in Fig. 5.40. Fields at $z = 1.0$ and 10.0 km and $t = 3600$ s are once again used in the figures to facilitate section to section comparisons within this chapter. Examination of the vertical velocity fields at $z = 10.0$ km shows that application of the impermeability condition at storm top appears to be preferable to use of the vorticity equation to retrieve an upper boundary condition. The poorest representations of the true field (Fig. 5.40b) are realized with technique C-Vb-Vt (Figs. 5.40j and 5.40l), although the fields from this technique are still quite reasonable. Each of the remaining techniques shown in Fig. 5.40 apply impermeability at storm top.

Examination of the low-level vertical velocity fields does not show any technique as being a clear favorite. The field from technique D-It again shows the most small-scale noise (Fig. 5.40c), but does a reasonably good job of retaining the bulk features of the true field (shown in Fig. 5.40a). Surprisingly, technique C-Vb-Vt-300 actually appears to show the best representation of the true vertical velocity field at $z = 1.0$ km. Although it seems odd at first that 300 s volume scan periods would produce better results than 90 s

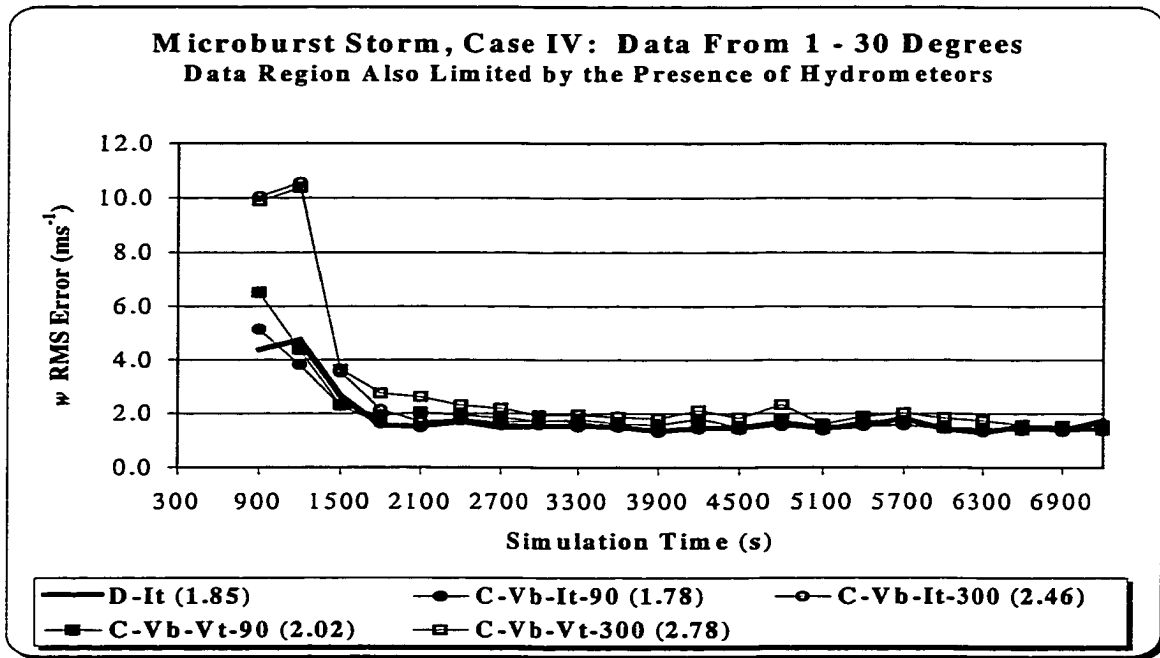


Figure 5.39: Root mean square vertical velocity retrieval errors from experiments on the microburst storm simulation using volume scan geometries providing data between 1° and 30° (Case IV) and further limiting the data region by the presence of hydrometeors.

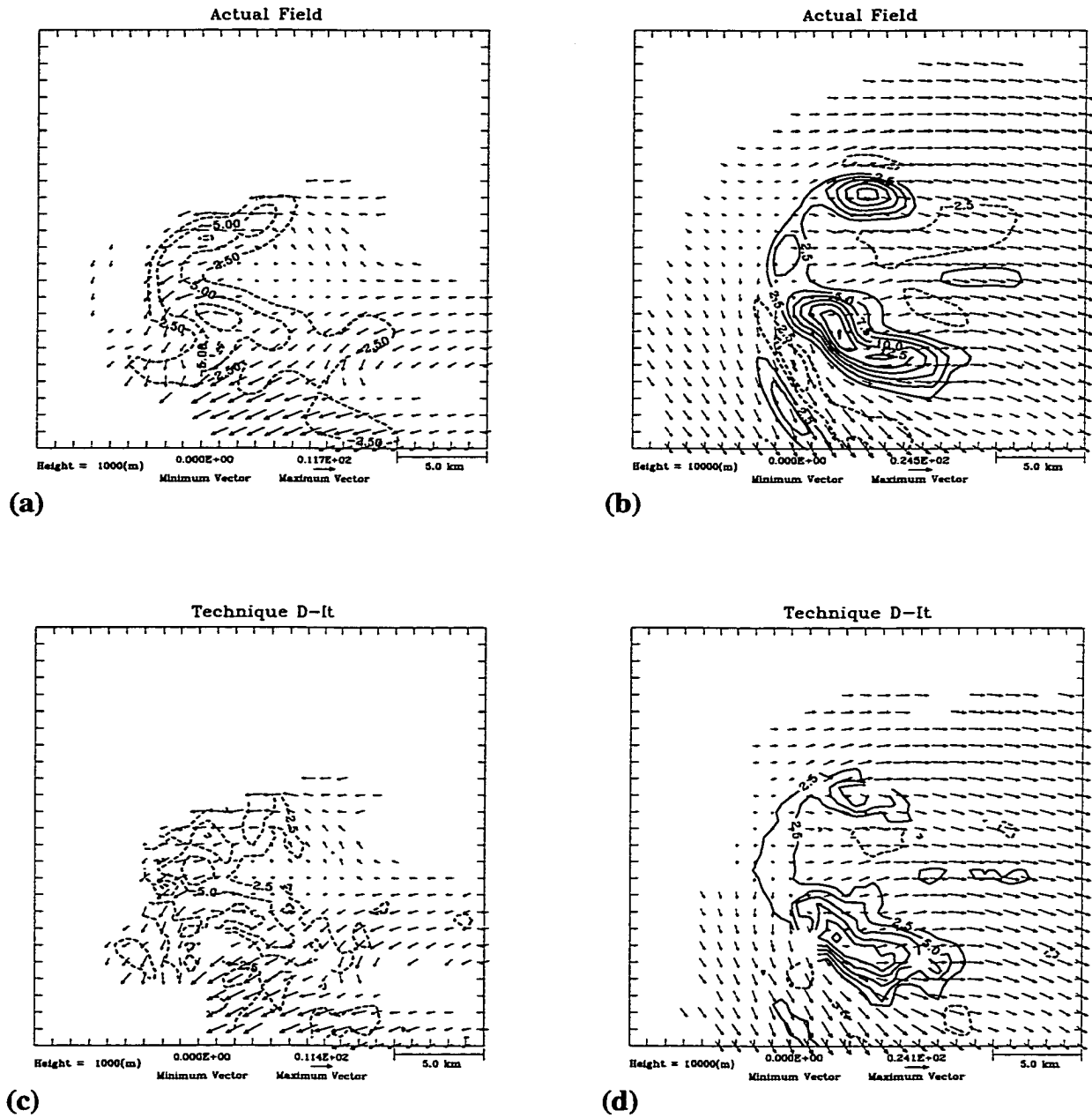


Figure 5.40: Horizontal wind vectors and contours of w at $z = 1.0$ and 10.0 km in the microburst storm, taken from (a) the true field at $z = 1.0$ km, (b) the true field at $z = 10.0$ km, (c) control technique D-It at $z = 1.0$ km, and (d) control technique D-It at $z = 10.0$ km. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV), and further limited to regions with a $0.1 \text{ g}\cdot\text{kg}^{-1}$ concentration of precipitable hydrometeors.

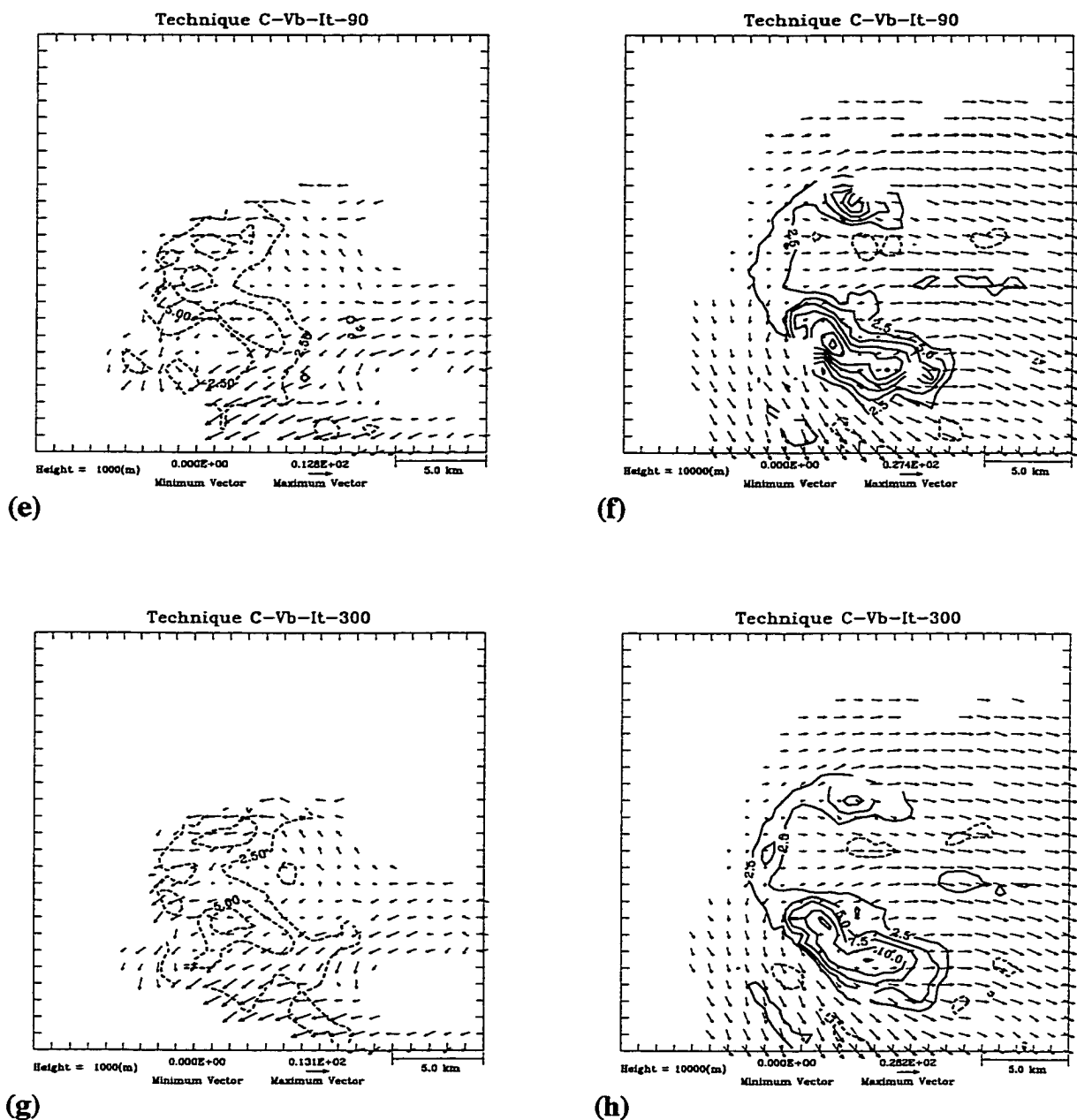


Figure 5.40 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ and 10.0 km in the microburst storm, taken from (e) technique C-Vb-It-90 at $z = 1.0$ km, (f) technique C-Vb-It-90 at $z = 10.0$ km, (g) technique C-Vb-It-300 at $z = 1.0$ km, and (h) technique C-Vb-It-300 at $z = 10.0$ km. Contour interval is $2.5 \text{ m}\cdot\text{s}^{-1}$. Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV), and further limited to regions with a $0.1 \text{ g}\cdot\text{kg}^{-1}$ concentration of precipitable hydrometeors.

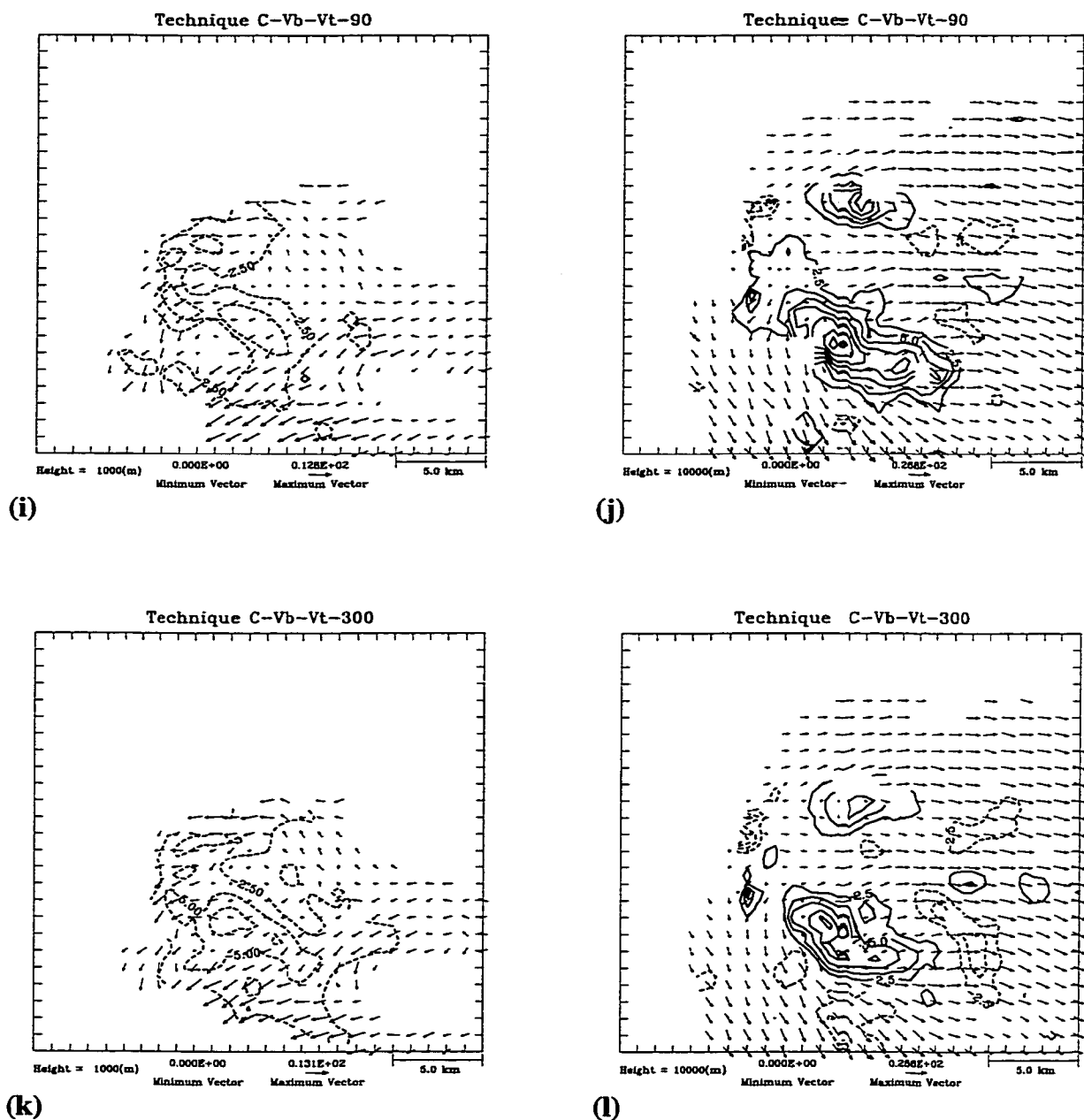


Figure 5.40 (continued): Horizontal wind vectors and contours of w at $z = 1.0$ and 10.0 km in the microburst storm, taken from (i) technique C-Vb-Vt-90 at $z = 1.0$ km, (j) technique C-Vb-Vt-90 at $z = 10.0$ km, (k) technique C-Vb-Vt-300 at $z = 1.0$ km, and (l) technique C-Vb-Vt-300 at $z = 10.0$ km. Contour interval is 2.5 m s^{-1} . Scan geometry is limited to regions between 1° and 30° elevation for each radar (Case IV), and further limited to regions with a 0.1 g kg^{-1} concentration of precipitable hydrometeors.

volume scan periods, it can be explained in terms of the two different sources for errors in the temporal derivative.

Referring to equation (A1.2) of Appendix 1, we see that the error variance of a spatial derivative is actually inversely proportional to the grid spacing. A similar equation can be derived for the temporal derivative, in which case we find that a longer volume scan period can actually be beneficial to proper estimation of the derivative. This seems to contradict the findings in Chapter 3, which showed that rapid volume scans were very beneficial to estimation of the temporal derivative. However, the discussion in Chapter 3 was focused upon controlling the potentially disastrous effects of non-linear local evolution on the temporal derivative through diminishing the advection of small-scale features. In many cases, this is the dominant type of error in the temporal derivative. However, as the local evolution becomes more linear, the random errors in the vorticity fields at each of the two times used in the finite difference estimate can become the dominant source of errors in the temporal derivative. In this case, as the time period between volume scans is lengthened the significance of the random errors relative to the true signal is diminished. Given the much slower evolution of this storm relative to the Del City storm, it would not be surprising for these random errors to be dominant so that 300 s volume scans are at least occasionally preferable.

Chapter 6: Conclusions

This work revisits the longstanding problem of dual-Doppler analysis of the vertical velocity field. The approach taken here focuses on the use of the anelastic vertical vorticity equation as a new analysis constraint. Several new multi-step dual-Doppler analysis methods are proposed, each building upon traditional analysis frameworks but incorporating the anelastic vertical vorticity equation as a weak constraint to aid in the retrieval of appropriate boundary condition(s). One technique is designed to retrieve the single boundary condition required when mass continuity is applied as a strong constraint in dual-Doppler analyses (such as in AR69 or BR77). The Euler-Lagrange equation and associated boundary conditions are the same regardless of whether the boundary condition is applied at the top, bottom, or somewhere in the middle of the data region. In a second technique, the vorticity equation is applied to retrieve the set of two boundary conditions required when the vertical velocity is broken down into its ‘weak constraint form’. The Euler-Lagrange equations for this approach consist of a set of two linear elliptic PDEs which must be solved as a system. However, if one boundary condition can be estimated reliably using other means (e.g., applying impermeability), this technique can be easily adapted to retrieve only one opposing boundary condition. This can be very useful in limiting accumulating divergence errors in traditional analyses. With this technique, an extra adjustment step to the horizontal winds is required in order to ensure mass continuity is satisfied exactly in the final analysis field. A new correction step specifically designed for application with this vorticity technique was presented.

The techniques were tested using a series of four data coverage scenarios. Each scenario was designed around the scanning capabilities of current operational and research radars. These scenarios were implemented in two different simulations, one of a supercell thunderstorm and another of a microburst producing thunderstorm. A special case where data were limited by the presence of precipitable hydrometeors was also investigated.

Based on the results of retrievals using these simulated datasets the vorticity equation appears to hold significant potential for improving vertical velocity retrievals in datasets where data are not available to storm top. When data are available to storm top, the vorticity equation was only capable of lending modest improvements to the traditional downward integration technique in the experiments shown. However, the errors applied to in these experiments were strictly random. Techniques relying upon direct integration of mass continuity (such as techniques U and D) have been shown by MB98 to be very susceptible to bias errors, while those constraining the vertical velocity field with two boundary conditions (such as technique C of this work and MB98's technique B) were shown to be much more robust in the presence of these systematic errors. This suggests that the vorticity equation and technique C might provide more substantial benefits over downward integration from storm top in other datasets which exhibit non-random errors, although this has yet to be verified experimentally.

Overall, the results of this work indicate that the vorticity equation will not be preferable to application of the impermeability condition as a means of determining an appropriate boundary condition, at least in cases where the impermeability condition

can be safely applied. However, when use of the impermeability condition is not appropriate, it does appear that the vorticity equation may be preferable to extrapolating divergence for a lower boundary condition in many cases. The wide adaptability of the vorticity techniques is highlighted when applied to retrieve a second opposing boundary condition when one boundary condition is already known through other means, as it can help limit accumulating divergence errors in this case. This may prove to be one of the most widely applicable scenarios for using the vorticity equation, since datasets frequently fail to span from ground level to storm top (so that impermeability cannot be applied safely at both boundaries of the data region). In cases where data fail to extend to any level where impermeability could be applied, the vorticity equation still provides significant potential for reliable vertical velocity retrievals. This appears to be true even in situations that traditionally would have prohibited such analyses.

A few important findings have come out of this work. First, it has been shown that retrievals performed with 90 s volume scan periods provide widely superior results to retrievals performed with 300 s volume scan periods. However, in cases of slow evolution, shorter volume scan periods can occasionally be detrimental. At least one such case was discussed in Chapter 5. In general, the high temporal resolution (90 s) data allows the vorticity techniques to outperform traditional methods of boundary condition specification in most cases. This fact argues strongly for the deployment of more rapid scanning research radars in meteorological field experiments, and (eventually) in operational radar networks. Another significant finding is that although the vorticity equation is sensitive to shear magnitude, the vertically

integrated nature in which it has been applied here seems to negate the need for strong vertical wind shear, and additionally provides an error filtering effect. Results were actually superior with the lower shear environment of the microburst storm.

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Appendix I: Parameterization of Errors in the Shear Vector Magnitude and Direction

As discussed in Chapter 3, the magnitude of the vertical wind shear plays a key role in determining the accuracy of the vertical velocity fields retrieved using the vorticity equation. Assessing the magnitude of the expected errors in the calculated magnitude and direction of the shear vector is therefore a logical first step in estimating errors that may appear in solutions for w . Since the shear is calculated using the horizontal wind fields at two vertical levels and a finite-difference approximation, the distribution of the errors in the calculated vertical wind shear can be estimated using the distributions of the errors in the horizontal wind fields.

For example, assume that errors in the u -component of the horizontal wind field are random and exhibit little or no bias. These errors could then be expected to exhibit an approximately Gaussian distribution with an expected value (μ_u) and a standard deviation (σ_u) calculated from error analysis. If there is no bias in the estimates of u , then the expected value of the error is zero ($\mu_u=0$). If the errors in u at two nearby grid points draw from similar unbiased distributions of errors, it can be shown that their sums and/or differences will exhibit the following properties:

$$\begin{aligned} \mu_{(u_{k+1}-u_{k-1})} &= \mu_{u_{k+1}} - \mu_{u_{k-1}} = 0 \\ \text{var}(u_{e_{k+1}} - u_{e_{k-1}}) &= \text{var}(u_{e_{k+1}}) + \text{var}(u_{e_{k-1}}) = \sigma_u^2 + \sigma_u^2 = 2\sigma_u^2 \end{aligned} \quad (\text{A1.1})$$

The expected value of the error in the calculated vertical wind shear using a typical 2nd order centered difference is then zero while the standard deviation is given by:

$$(A1.2) \quad \sigma_{\partial u / \partial z} = \sigma_{u_z} = \frac{\sqrt{2} \sigma_u}{2 \Delta z}.$$

From (A1.2) we can deduce that if the standard deviation of the errors in a horizontal wind component is given by σ_u , then the standard deviation of the errors in this component of the vertical wind shear is $\sqrt{2}/2\Delta z$ times that value. For example, if $\sigma_u=1.0$ m/s and $\Delta z=500$ meters, then approximately 68% of the calculated shear values would be expected to be in error by less than 0.00141 s^{-1} (or one standard deviation) while approximately 96% of the calculated shear values would be expected to be in error by less than 0.00283 s^{-1} (or two standard deviations). These figures exclude the error introduced by approximating continuous derivatives with finite differences.

An expression for the approximate error distribution for the estimates of the two individual components of the vertical wind shear is now available (A1.2). However, the distribution of errors in the magnitude and direction of the total vertical wind shear vector is dependent upon both of these distributions. Following Doviak and Ray (1976) note that the magnitude and direction of the shear vector are given by:

$$(A1.3) \quad \begin{aligned} |\bar{S}| &= \left(\frac{\partial u^2}{\partial z} + \frac{\partial v^2}{\partial z} \right)^{1/2} = (u_z^2 + v_z^2)^{1/2} \\ \psi &= \tan^{-1} \left(\frac{\partial v / \partial z}{\partial u / \partial z} \right) = \tan^{-1} \left(\frac{v_z}{u_z} \right), \end{aligned}$$

where u_z and v_z represent the partial derivatives of u and v with respect to the vertical coordinate z . Note that the direction ψ as defined by (A1.3) does not take on the traditional form for meteorology of $0^\circ=360^\circ\equiv\text{North}$. Assuming errors concentrated around the expected values of u_z and v_z , the error variance of the magnitude of the vertical wind shear vector can be shown to be approximately:

$$(A1.4) \quad \sigma_{|\bar{S}|}^2 \approx \left(\frac{\partial |\bar{S}|}{\partial u_z} \right)^2 \sigma_{u_z}^2 + \left(\frac{\partial |\bar{S}|}{\partial v_z} \right)^2 \sigma_{v_z}^2 + 2 \left(\frac{\partial |\bar{S}|}{\partial u_z} \right) \left(\frac{\partial |\bar{S}|}{\partial v_z} \right) \sigma_{u_z} \sigma_{v_z} r(u_z, v_z),$$

where $r(u_z, v_z)$ is the correlation coefficient between errors in the estimates of u_z and v_z . Although the form of this correlation coefficient could, in principle, be determined, it is beyond the scope of this analysis. Instead, if it is assumed that the errors in the individual shear components are uncorrelated we find:

$$(A1.5) \quad \sigma_{|\bar{S}|}^2 \approx \frac{u_z^2 \sigma_{u_z}^2 + v_z^2 \sigma_{v_z}^2}{u_z^2 + v_z^2}.$$

Further, if $\sigma_{u_z} \approx \sigma_{v_z}$ the standard deviations can be factored out of the right-hand side of (A1.5) leaving:

$$(A1.6) \quad \sigma_{|\vec{s}|} \approx \sigma_{u_z} \approx \sigma_{v_z} .$$

Equation (A1.6) states that if the errors in the individual components of the shear vector are uncorrelated, then the standard deviation of the error in the magnitude of the shear vector is approximately equal to the standard deviation of the error in the individual components of the wind shear vector. How are these results changed if the errors in the individual components of the shear vector are correlated (i.e., $r(u_z, v_z) \neq 0.0$)? In this case it is useful to note that:

$$(A1.7) \quad \begin{aligned} \frac{\partial |\vec{s}|}{\partial u_z} &= \frac{u_z}{|\vec{s}|} = \cos \psi \\ \frac{\partial |\vec{s}|}{\partial v_z} &= \frac{v_z}{|\vec{s}|} = \sin \psi \end{aligned} .$$

This fact allows simplification of (A1.7) as:

$$(A1.8) \quad \sigma_{|\vec{s}|}^2 \approx \sigma_{u_z}^2 \cos^2 \bar{\psi} + \sigma_{v_z}^2 \sin^2 \bar{\psi} + 2\sigma_{u_z} \sigma_{v_z} r(u_z, v_z) \cos \bar{\psi} \sin \bar{\psi} ,$$

where $\bar{\psi}$ is the expected value of the shear vector direction. If it is once again assumed that $\sigma_{u_z} \approx \sigma_{v_z}$, application of trigonometric identities leads to:

$$(A1.9) \quad \sigma_{|\bar{s}|}^2 \leq \sigma_{u_z}^2 + \sigma_{u_z}^2 r(u_z, v_z).$$

The error variance in the magnitude of the shear can therefore range from zero up to twice the error variance for the individual components of the shear, depending on the value of the correlation coefficient $r(u_z, v_z)$. A worst-case scenario is that the standard deviation of the magnitude of the shear would be $\sqrt{2}$ times the standard deviation for the individual components. The expected standard deviation of the errors in the shear magnitude should then be:

$$(A1.10) \quad \sigma_{|\bar{s}|} \leq \sqrt{2} \sigma_{u_z} = \frac{\sigma_u}{\Delta z} = \frac{\sigma_v}{\Delta z}.$$

In a similar manner a formula can be derived to predict the standard deviation of the errors in the shear vector's direction. Using the definition for ψ from (A1.3) and following a procedure similar to that used to deduce the error variance of the wind shear magnitude estimates it can be shown that:

$$(A1.11) \quad \left[\bar{s} \right]^2 \sigma_{\psi}^2 \approx \sigma_{u_z}^2 \sin^2 \bar{\psi} + \sigma_{v_z}^2 \cos^2 \bar{\psi} - 2 \sigma_{u_z} \sigma_{v_z} r(u_z, v_z) \sin \bar{\psi} \cos \bar{\psi},$$

where $\overline{S}$ is the expected value of the magnitude of the shear vector. Making the simplification again that $\sigma_{u_z} \approx \sigma_{v_z}$ leads to:

$$(A1.12) \quad \overline{S}^2 \sigma_\psi^2 \leq \sigma_{u_z}^2 - \sigma_{u_z}^2 r(u_z, v_z),$$

or:

$$(A1.13) \quad \sigma_\psi^2 \leq \left(\frac{\sigma_{u_z}^2 - \sigma_{u_z}^2 r(u_z, v_z)}{\overline{S}^2} \right).$$

Assuming complete anti-correlation between the errors in u_z and v_z the standard deviation of the wind direction (in radians) is bounded by:

$$(A1.14) \quad \sigma_\psi \leq \left(\frac{\sqrt{2}\sigma_{u_z}}{\overline{S}} \right) = \frac{\sqrt{2}\sigma_u}{\overline{S}\Delta z} = \frac{\sqrt{2}\sigma_v}{\overline{S}\Delta z}.$$

Equations (A1.10) and (A1.14) give upper bounds on the standard deviations of the errors in the wind shear vector magnitude and direction given the standard deviations of the errors in the individual horizontal wind components and the expected values of the wind shear magnitude. For example, if values of $1.0 \text{ m}\cdot\text{s}^{-1}$ are prescribed for σ_u and σ_v along with a 500 meter vertical grid spacing (Δz), (A1.10) shows that the

standard deviation of the errors in the magnitude of the vertical wind shear vector would be bounded by 0.002 s^{-1} . If the errors are normally distributed, 68% of the errors in the shear magnitude would be less than this value, while 95% of the errors would be less than 0.004 s^{-1} . Under this same situation, (A1.14) shows that the standard deviation of the errors in the shear vector direction should be less than 2.83 radians (162°) for a true shear magnitude of 0.001 s^{-1} , less than 0.57 radians (33°) for a true shear magnitude of 0.005 s^{-1} , and less than 0.28 radians (16°) for a true shear magnitude of 0.010 s^{-1} . Since (A1.10) and (A1.14) were derived assuming worst-case scenarios in several instances, it is likely that these equations at least slightly overestimate the magnitude of errors that will be found in practice.

Appendix II: Determination of Advection Velocities

In order to apply the vertical vorticity equation as a means of determining the vertical velocity in dual-Doppler analyses it is essential that we be able to accurately evaluate the local derivative ($\partial/\partial t$) of the vertical component of relative vorticity. In order to improve estimates of this derivative it is useful to define the observations in a reference frame within which the phenomena of interest is as stationary as possible. This minimizes the effects of advection of small-scale features in the analysis, a prime source of non-linear local evolution in ζ (as discussed in Chapter 3). Knowledge of the advection speed is also useful when attempting to interpolate radar data to common times.

The most practical technique for application in dual-Doppler analyses is that of Gal-Chen (1982). Gal-Chen utilized the non-Cartesian radial wind velocity vectors from two Doppler radars at (at least) several different times to determine the advection velocity of observed features. This technique is described in the following paragraphs. Although Gal-Chen also proposed other techniques using scalars such as radar reflectivity, this technique is ideal for the current application since it utilizes the observations most relevant to dual-Doppler analyses. Gal-Chen also states that analysis of Doppler radar data reveals that while the radial velocities often display quasi-stationary behavior, the reflectivity field does not, further supporting use of this method.

To begin, assume that the radial winds are observed in a four-dimensional Cartesian domain (x, y, z, t) . Define a Galilean transformation of variables by:

$$\begin{aligned}
(a2.1) \quad & x' = x - U(t - t_0), \quad u' = u - U, \\
& y' = y - V(t - t_0), \quad v' = v - V, \\
& t' = t, \quad z' = z, \quad w' = w, \\
& f(x, y, z, t) = f(x', y', z', t')
\end{aligned}$$

where U and V are the constant velocities of a moving Galilean frame of reference (the advection velocities we want to solve for). Using the chain rule one can verify that for the transformation defined by (a2.1):

$$(a2.2) \quad \left. \frac{\partial}{\partial t'} \right)_{x', y', z'} = \left. \frac{\partial}{\partial t} \right)_{x, y, z} + U \left. \frac{\partial}{\partial x} \right)_{y, z, t} + V \left. \frac{\partial}{\partial y} \right)_{x, z, t}.$$

Also, the following notation conventions will be adopted for convenience:

$$(a2.3) \quad dxdydzdt = d^4R, \quad dx'dy'dz'dt' = d^4R'$$

The remainder of this derivation proceeds under the assumption that the phenomena of interest is in a quasi-steady state, or in other words, advection is dominating over evolution in causing temporal variations in the velocity field. In this derivation we are looking for a quantity that is stationary in the moving reference frame. The quantity $v_r r$, (where v_r is the observed radial wind velocity at a point and r is the distance from that point to the radar) turns out to be one such quantity. To show this, first note that:

$$(a2.4) \quad \alpha = v_r r = u(x-a) + v(y-b) + w(z-c),$$

where (a, b, c) are the coordinates of the radar in the fixed frame of reference. To express (a2.4) in the moving frame of reference assume that the radar observes the same point in the phenomena of interest (or equivalently, the same point in the moving reference frame) at a time $t=0$ and again at a time $t=\Delta t$. Application of (a2.1) yields:

$$(a2.5) \quad \begin{aligned} x_1 &= x', & x_2 &= x' + U\Delta t \\ y_1 &= y', & y_2 &= y' + V\Delta t \end{aligned}$$

Since we've assumed that the phenomena is quasi-steady in the moving frame of reference:

$$(a2.6) \quad \begin{aligned} u(x', y', z, t) &= u(x', y', z, t + \Delta t) = u(x_1, y_1, z, t) = u(x_2, y_2, z, t + \Delta t) \\ v(x', y', z, t) &= v(x', y', z, t + \Delta t) = v(x_1, y_1, z, t) = v(x_2, y_2, z, t + \Delta t) \end{aligned}$$

If we write (a2.4) separately at time t and $t+\Delta t$ and subtract (utilizing (a2.1), (a2.5) and (a2.6)) we get the relation:

$$(a2.7) \quad \frac{\alpha(x', y', z, t + \Delta t) - \alpha(x', y', z, t)}{\Delta t} = uU + vV = (u' + U)U + (v' + v)V$$

or:

$$(a2.8) \quad \frac{\partial(v_r r)}{\partial t'} - U^2 - V^2 = u'U + v'V$$

Further, if we take another derivative of (a2.8) with respect to t' we find (recalling that we've assumed the phenomena is steady in the moving frame of reference):

$$(a2.9) \quad \frac{\partial^2}{\partial t'^2}(v_r r) = 0 \Rightarrow \frac{\partial}{\partial t'}(v_r r) = \text{constant}.$$

Now, if two Doppler radars observe the same point at multiple times we find that in general:

$$(a2.10) \quad \frac{\partial}{\partial t}(v_r^{(1)} r^{(1)}) \neq \frac{\partial}{\partial t}(v_r^{(2)} r^{(2)}).$$

However, inspection of (a2.8) reveals that if there is no evolution then there is a frame of reference moving with velocity (U,V) in which:

$$(a2.11) \quad \frac{\partial}{\partial t'}(v_r^{(1)} r^{(1)}) = \frac{\partial}{\partial t'}(v_r^{(2)} r^{(2)}) = \text{constant}.$$

If we define a quantity W , where W is given by:

$$(a2.12) \quad W = v_r^{(1)} r^{(1)} - v_r^{(2)} r^{(2)},$$

then in the moving frame of reference:

$$(a2.13) \quad \frac{\partial W}{\partial t'} = 0.$$

Expressed in the fixed frame of reference (using (a2.2)) this requires:

$$(a2.14) \quad \frac{\partial W}{\partial t} + U \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial y} = 0$$

Equation (a2.14) is valid only if advection dominates over evolution and is only valid for a single observation point. In general, we have to integrate over all observations, in which case we require that:

$$(a2.15) \quad \int \left(\frac{\partial W}{\partial t} + U \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial y} \right)^2 d^4 R = Min$$

Taking the first variation of (a2.15) with respect to U and V yields a system of two equations in two unknowns:

$$\begin{aligned}
 (a2.16) \quad & AU + BV = D \\
 & BU + CV = E
 \end{aligned}$$

where:

$$\begin{aligned}
 (a2.17) \quad & A = \int \left(\frac{\partial W}{\partial x} \right)^2 d^4 R \\
 & B = \int \left(\frac{\partial W}{\partial x} \right) \left(\frac{\partial W}{\partial y} \right) d^4 R \\
 & C = \int \left(\frac{\partial W}{\partial y} \right)^2 d^4 R \quad . \\
 & D = \int \left(\frac{\partial W}{\partial t} \right) \left(\frac{\partial W}{\partial x} \right) d^4 R \\
 & E = \int \left(\frac{\partial W}{\partial t} \right) \left(\frac{\partial W}{\partial y} \right) d^4 R
 \end{aligned}$$

Once A, B, C, D, and E have been calculated for a particular domain it is straightforward to solve for the advection velocities U and V.

Recalling the reasoning for determining the advection velocities given in the first few paragraphs of this appendix, it is immediately evident that the terms D and E in (a2.17) may be subject to large errors due to inaccuracies in the estimates of the local temporal derivatives involved in these terms. Errors in terms D and E will in turn lead to imprecise estimates of U and V. To counter this problem an iterative approach is employed. The iterative approach is as follows. Using the first guess estimates $U^{(0)}$ and $V^{(0)}$ define the observations in a new frame:

$$(a2.18) \quad \begin{aligned} x' &= x - U^{(0)}(t - t_0) \\ y' &= y - V^{(0)}(t - t_0) \end{aligned}$$

Note that the radial wind velocities must also be corrected to account for the motion of the new reference frame by subtracting the scalar products of v_r with the advection velocity (U,V). The algorithm for estimating advection velocities is then performed in this new reference frame. If the new frame is optimal, the estimate of the advection velocities should be zero. If not, residual advection velocities $\Delta U^{(1)}$ and $\Delta V^{(1)}$ are obtained. From these residuals a second guess of the advection velocities is obtained:

$$(a2.19) \quad \begin{aligned} U^{(1)} &= U^{(0)} + \Delta U^{(1)} \\ V^{(1)} &= V^{(0)} + \Delta V^{(1)} \end{aligned}$$

and the observations are once again redefined in a new reference frame. The iterations are continued until $\Delta U^{(k)}$ and $\Delta V^{(k)}$ are sufficiently small. Experiments with real data in Gal-Chen indicate that sufficient convergence is usually obtained in 3-4 iterations. Once convergence is achieved the radial wind observations can be simply redefined in the 'stationary' reference frame. The dual-Doppler analysis can then proceed using the redefined data, and the vorticity equation can be directly applied on the wind field of the analysis.