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P-HARMONIC THEORY ON AN ELLIPSOID AND GEOMETRIC APPLICATIONS

A DISSERTATION APPROVED FOR THE DEPARTMENT OF MATHEMATICS

BY

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DEDICATION

to

My parents

Guizhi Xiong and Shunzhong Wu

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# Contents

# Chapter 1

# Introduction and Statements of Main Results

## 1.1 Background

Let $M^n$ be a compact Riemannian manifold with possibly nonempty boundary, and let $N^k$ be isometrically immersed in $R^q$. $L_1^p(M, N)$ denotes the set of maps $u : M \to R^q$ whose component functions have first weak derivatives in $L^p$ and $u(x) \in N$ a.e. on $M$.

**Definition 0.1:** The $p$-energy for $u \in L_1^p(M, N)$ is given by

$$E_p(u) = \frac{1}{p} \int_M |du|^p dv$$

where $du$ denotes the differential of $u$, and $dv$ is the volume element of $M$.

**Definition 0.2:** A map $u \in L_1^p(M, N)$ is said to be weakly $p$-harmonic if it is a weak solution to the following Euler-Lagrange equation for $E_p$ on $L_1^p(M, N)$:

$$div(|du|^{p-2}du) = 0$$

When $p = 2$, a $p$-harmonic map is reduced to a harmonic map. $u$ is called $p$-stable(or $p$-minimizing) if $u$ is a local(resp. global)minimum of $p$-energy functional $E_p$ within a homotopy class of $L_1^p(M, N)$ having the same trace on $\partial M$.

In the first part of this paper, we are interested in representing homotopy groups by $p$-harmonic maps with applications to minimal submanifolds of ellipsoids. Just as harmonic forms represent cohomology classes, stable currents represent homology classes, so do $p$-harmonic maps represent homotopy classes.

It is well known that on one hand, each class in nontrivial fundamental group $\pi_1$ can be represented by a closed geodesic of minimum length on a compact Riemannian manifold by the existence theorem due to Cartan. On the other hand, there does not exist a closed length-minimizing geodesic on an orientable and even-dimensional manifold with positive sectional curvature by the nonexistence theorem due to Synge. Combining the existence theorem and the nonexistence theorem together, Synge obtains the following conclusion:

$$\pi_1(N) = 0 \text{ if } N \text{ is a compact orientable and even-dimensional manifold with positive sectional curvature.}$$

We also use an idea that curvature implies topology via p-harmonic maps. The technique of striking a delicate balance between the existence and the nonexistence, leads from a hypothesis on curvature to a topological conclusion.

Along this direction, we discuss the existence and the nonexistence for the higher homotopy classes $\pi_i, i \geq 1$.

**Existence Theorem(S.W.Wei)([48])**: If $N$ is a compact Riemannian manifold, then each class in $\pi_i(N) \neq 0$ can be represented by a $C^{1,\alpha}$ $p$-harmonic maps from $S^i$ into $N$ minimizing $p$-energy in its homotopy class for any $p > i$.

**Nonexistence Theorem(S.W.Wei)([48])**: Let $N$ be a minimal $k$-submanifold of a unit Euclidean sphere such that the Ricci curvature $Ric^N$ of $N$ satisfies

$$Ric^N > k(1 - \frac{1}{p})$$

where $2 \leq p < k$, then $N$ is $p$-superstrongly unstable ($p$-SSU) which implies that there are no nonconstant $p$-stable maps on $N$.

Combining the above two Wei's theorems, we obtain the following conclusions:

1. Ricci curvature implies topological vanishing properties,

2. **Topological Vanishing Theorem([48])**: Every compact $p$-SSU manifold $N$ is $[p]$-connected, i.e.

$$\pi_1(N) = \pi_2(N) = \cdots = \pi_{[p]}(N) = 0$$

Let's recall what's the meaning of $p$-SSU manifolds.

**Definition 0.3**: A Riemannian manifold is $p$-strongly unstable ($p$-SU) if it is neither the domain nor the target of any nonconstant smooth $p$-stable map.

**Definition 0.4**: A Riemannian manifold $N$ with a Riemannian metric $\langle , \rangle_N$ is said to be $p$-superstrongly unstable ($p$-SSU) for $p \geq 2$, if there exists an isometric immersion in $R^q$ such that, for all unit tangent vectors $X$ to $N$ at every point $y \in N$, the following functional is always *negative*:

$$F_{p,y}(X) = (p-2)\langle h(X,X), h(X,X)\rangle_{R^q} + \langle Q_y^N(X), X\rangle_N$$

where

$$\langle Q_y^N(X), X\rangle_N = \sum_{i=1}^{k}(2\langle h(X,\alpha_i), h(X,\alpha_i)\rangle_{R^q} - \langle h(X,X), h(\alpha_i,\alpha_i)\rangle_{R^q}),$$

h is the second fundamental form of $N$ in $R^q$, and $\{\alpha_1, \cdots, \alpha_k\}$ is a local orthonormal frame on $N$ near $y$.

In fact, every compact $p$-SSU manifold is $p$-SU.(cf. Theorem 3.1 ([51])).

**Definition 0.5**: A p-SSU index $\omega_p$ on a p-SSU manifold $N$ is defined to be:

$$\omega_p = \inf_{y \in N} \frac{\phi_p(y)}{(k+p-2)S(y)}$$

where $\phi_p(y) = \inf_{\|X\|=1}(-F_{p,y}(X))$ and $S(y) > 0$ is a positive upper bound of the sectional curvature of $N$ at $y$.

We obtain the main result in the first part which is a topological vanishing theorem for minimal submanifolds in ellipsoids by a Ricci curvature assumption. Furthermore, on ellipsoids we explore geometric $k_p$-connectedness by applying Topological Vanishing Theorem, and solve the Dirichlet problems for $p$-harmonic maps with any given boundary data.

Here and throughout this paper, unless stated otherwise, we denote an ellipsoid $E^n = \{(x_1, \cdots, x_{n+1}) \in R^{n+1} : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1, a_i > 0, \forall 1 \leq i \leq n+1\}$, a closed upper-half ellipsoid $\bar{E}^n_+ = \{(x_1, \cdots, x_{n+1}) \in E^n : x_{k+1} \geq 0\}$, $\max(a_i) = \max_{1 \leq i \leq n+1}\{a_i\}$, $\min(a_i) = \min_{1 \leq i \leq n+1}\{a_i\}$, and $\mu = \frac{\max(a_i)}{\min(a_i)}$.

In the second part of this paper, we discuss Liouville-type theorems for $p$-harmonic or $p$-stable maps into either a closed half-ellipsoid $\bar{E}^n_+$ or an ellipsoid $E^n$ with the $p$-superstrongly unstable indice $\omega_p$. We derive an $L^p$ gradient estimate and establish Liouville for $p$-minimizing tangent maps into a closed half-ellipsoid $\bar{E}^n_+$ by a Bochner formula with related estimates and the first, second variational formulas for $p$-energy functional on maps between Riemannian manifolds. We establish Liouville-type theorems on an ellipsoid $E^n$ with a $p$-SSU index $\omega_p$ by developing fundamental tools in the study of $\omega_p$, the second fundamental form of an ellipsoid $E^n$ isometrically imbedded in $R^{n+1}$, and an $L^p$ gradient estimate in a $p$-SSU manifold. In fact, most of geometric, analytic and topological properties of $E^n$ are characterized by $\omega_p$ depending only on $n$, $p$, and the ration $\mu$. For example, $\omega_p$ serves as an indication of the regularity theory. Naively speaking, the closer $\omega_p$ is 1, the easier it is to establish a Liouville theorem in terms of $p$-th volume growth condition in the domain. Furthermore, we study the Liouville-type theorems when the domain is a $p$-parabolic manifold. Here it should be mentioned that there is a "gap phenomenon" between a $p$-parabolic manifold and a manifold with $p$-th volume growth condition. Actually, we can furnish an example of gap phenomena which is a $p$-parabolic manifold with exponential volume growth instead of $p$-th volume growth.

In the third part of this paper, we are interested in the existence and non-existence of stable rectifiable currents (or submanifolds) on an ellipsoid $E^n$ which extends Lawson and Simons' statement about nonexistence of stable varifolds in a unit sphere.

Our basic method is to apply for Trace Formulas due to R. Howard and S.W.Wei ([33]) and a way of estimating the second fundamental form of $E^n$ in $R^{n+1}$ (cf. Lemma 2.1). The study of nonexistence of stable submanifolds has received wide attention among mathematicians. The earliest result of this type known to us is the result of Synge that a compact, orientable, even dimensional Riemannian manifold of positive sectional curvature has no stable closed geodesics. The first results about the nonexistence of stable submanifolds other than closed geodesics seem to be in the celebrated paper of Simons ([42]). Lawson and Simons show that a Euclidean sphere $S^n$ contains no stable submanifolds. They ([24]) made a conjecture: there are no closed stable submanifolds in any compact, simply connected, strictly $\frac{1}{4}$-pinched Riemannian manifold. Ralph Howard and S.Walter Wei ([33]) verify this conjecture for several classes of positively curved manifolds. They show how the geometry of a Riemannian manifold $M^n$ affects the existence, or the nonexistence of stable submanifolds by giving conditions on the second fundamental form of $M^n$ immersed in a higher dimensional Euclidean space. On the basis of these results made by Ralph Howard and S.Walter Wei, we obtain the nonexistence theorem of stable currents on $E^n$.

In the fourth part, we study $E^n$ as a geometric application of Yang-Mills instability of convex hypersurfaces. Let's recall what's the instability of Yang-Mills connections.

**Definition 0.6:** Let $M$ be a compact Riemannian manifold and a principal $G$-bundle $P$ over $M$, where $G$ is a compact Lie group. On the space $C$ of the connections in $P$, the Yang-Mills functional $J : C \to R$ is defined by

$$J(\omega) = \frac{1}{2}\int_M ||\Omega||^2, \omega \in C$$

where $\Omega$ is the curvature of the connection.

**Definition 0.7:**A critical point of $J$ is called a Yang-Mills connection and its curvature a Yang-Mills field. A Yang-Mills connection $\omega$ is said to be weakly stable if the second variation of $J$ at $\omega$ is non-negative, i.e.,

$$\frac{d^2}{dt^2}J(\omega_t)|_{t=0} \geq 0$$

for every smooth family of connections $\omega_t, -\delta < t < \delta$, with $\omega_0 = \omega$. We say that a compact Riemannian manifold $M$ is Yang-Mills instable if, for every choice of $G$ and every principle $G$-bundle $P$ over $M$, none of the nonflat Yang-Mills connections in $P$ is weakly stable.

At the Tokyo Symposium on "Minimal Submanifolds and Geodesics" in September of 1997, J. Simons announced the following theorem in his talk entitled "Gauge Fields". (His lecture has never been published).

**Theorem:**For $n \geq 5$, the n-sphere $S^n$ with the natural metric is Yang-Mills instable.

In our paper, we obtain Yang-Mills instability of an ellipsoid based on the work of Y.B Shen and Y.L.Pan ([43]) and the works of Shoshichi Kobayashi, Yoshihiro Ohnita, and Masaru Takeuchi ([23]).

In the fifth part of this paper, we verify that all of conclusions in the above topological, analytic and geometric theorems on $E^n$ are still valid on a compact convex hypersurface $N^n$ in $R^{n+1}$ if the assumptions on the ratio $\mu$ on an ellipsoid $E^n$ are replaced by the assumptions on its maximum principal curvature $\max(\lambda_i)$ and minimum principal curvature $\min(\lambda_i)$ on $N^n$.

**Definition 0.8**: Suppose $N^n$ is a compact hypersurface in a Euclidean space $R^{n+1}$, let $\nu$ be a unit normal vector filed to $N^n$, and $\{e_1, \cdots, e_n\}$ be a local orthonormal frame filed on $N$ s.t. the second fundamental form $h_{ij}(x) = \langle \nabla_{e_i} \nu, e_j \rangle = \lambda_i(x)\delta_{ij}$

$$\lambda_1 = \min_{x \in N^n} \inf_{\|X\|=1, X \in T_x(N^n)} \langle \nabla_X \nu, X \rangle$$

$$\lambda_n = \max_{x \in N^n} \sup_{\|X\|=1, X \in T_x(N^n)} \langle \nabla_X \nu, X \rangle$$

$$\mu_0 = \frac{\lambda_n}{\lambda_1}$$

In the sixth part of this paper ([52]), we make sharp global integral estimates by a unified method, and find a dichotomy between constancy and ”infinity” of weak sub- and supersolutions of a large class of degenerate and singular nonlinear partial differential equations on complete noncompact Riemannian manifolds, by introducing the concepts of their corresponding ”p-finite, p-mild, p-obtuse, p-moderate”, and ”p-small” growth, and their counter-parts ”p-infinite, p-severe, p-acute, p-immoderate,” and ”p-large” growth. These lead naturally to a Generalized Uniformization Theorem, a Generalized Bochner’s Method, and an interative method, by which we approach various geometric and variational problems in complete noncompact manifolds of general dimensions. In particular, we give applications to minimal surfaces and to biharmonic maps between complete noncompact manifolds.

## 1.2 Main Results

Let’s describe the following important Lemmas before we look at the main theorems.

**Lemma 2.1(Estimate on the second fundamental form of an ellipsoid $E^n$ embedded in $R^{n+1}$)**: Suppose $\nu$ is a unit normal vector field on an ellipsoid $E^n$ imbedded in $R^{n+1}$ and $Y$ is any unit tangent vector on $E^n$ at a point x, then the second fundamental form

$$h_{YY} = \langle \nabla_Y \nu, Y \rangle$$

satisfies:

$$\frac{\min(a_i)}{(\max(a_i))^2} \leq h_{YY} \leq \frac{\max(a_i)}{(\min(a_i))^2}$$

where $\nabla$ is the Riemannian connection on $R^{n+1}$ and $\langle , \rangle$ is the Riemannian metric on $R^{n+1}$.

**Lemma 2.2(Estimates on principle curvatures and sectional curvatures of $E^n$ embedded in $R^{n+1}$)(Y.B.Shen and Y.L.Pan ([43])**: Suppose $\{\lambda_i\}_{i=1}^n$ is a family of principle curvatures on $E^n$ embedded in $R^{n+1}$ satisfying $0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$ and $Sec^{E^n}$ denotes any sectional curvature on $E^n$ embedded in $R^{n+1}$. Then:

$$\lambda_1 = \frac{\min(a_i)}{(\max(a_i))^2}$$
$$\lambda_n = \frac{\max(a_i)}{(\min(a_i))^2}$$
$$\frac{(\min(a_i))^2}{(\max(a_i))^4} \leq Sec^{E^n} \leq \frac{(\max(a_i))^2}{(\min(a_i))^4}$$

**Lemma 2.3(Criteria for $p$-SSU ellipsoids ):** An ellipsoid $E^n$ is p-SSU if $\mu < \sqrt[3]{\frac{n}{p}}$.

**Lemma 2.4(An Estimate on $p$-Superstrongly Unstable Index $\omega_p$ on p-SSU ellipsoids):** Suppose an ellipsoid $E^n$ is a $p$-SSU manifold, then:

$$\omega_p \geq \frac{\frac{n}{\mu^6} - p}{n + p - 2}$$

The estimate is sharp on sphere $S^n(r)$.

In this section we describe the main results presented in this thesis into the following categories:

**A. Representing homotopy groups and spaces of maps by p-harmonic maps into ellipsoids**

**Theorem 2.1(Topological Vanishing Theorem on Minimal Submanifolds into Ellipsoids)**: Let $N$ be a minimal $k$-submanifold of an ellipsoid $E^n$ such that the Ricci curvature $Ric^N$ of $N$ satisfies:

$$Ric^N > k(1 - \frac{1}{p})\frac{(\max(a_i))^2}{(\min(a_i))^4}$$

where $2 \leq p < k$. Then $N$ is $p$-SSU and

$$\pi_1(N) = \pi_2(N) = \cdots = \pi_{[p]}(N) = 0$$

When $a_1 = \cdots = a_{n+1}$, we recapture S.W.Wei's nonexistence theorem which is sharp (cf. Remark 2.1).

**Theorem 2.2(Geometric $k_p$ connectedness and Dirichlet problems on ellipsoids**: Let an ellipsoid $E^n$ be a compact $p$-SSU n-manifold with an appropriate $p$-SSU index $\omega_p$. Then $E^n$ is geometrically $(d(\omega_p, \delta) - 1)_p$-connected. Hence:

1. $E^n$ is $(d(\omega_p, \delta) - 1)$-connected.

2. $C^1(N, E^n)$ is dense in $L_1^{d(\omega_p,\delta)-1}(N, E^n)$ for any compact manifold $N$ with possibly non-empty boundary.

3. Each class in $\pi_{d(\omega_p,\delta)}(E^n)$ can be represented by a sum of $C^{1,\alpha}$ $p$-harmonic map of $S^{d(\omega_p,\delta)}$ into $E^n$. Furthermore, if the class is nontrivial, these nonconstant $p$-harmonic representatives are $p$-unstable.

4. Each class in $\pi_{d(\omega_p,\delta)}(E^n)$ can be represented by a sum of $C^{1,\alpha}d(\omega_p,\delta)$-harmonic map of $S^{d(\omega_p,\delta)}$ into $E^n$.

5. For any $p^{'}$ <dim$N$ and $2 \leq p^{'} \leq d(\omega_p,\delta)$, and any given boundary data $\eta \in L^{p^{'}}_{1-\frac{1}{p^{'}}}(\partial N, E^n)$, the Dirichlet problem for $p^{'}$-harmonic maps into $E^n$ has a solution, which is locally Hölder continuous up to $\partial N$ and is $C^{1,\alpha}$ in the interior of N off the singular set of Hausdorff dimension $\leq \dim N - p^{'} - 1$.

6. In particular, the Dirichlet problem for $p$-harmonic maps into $E^n$ has a solution with singular set of Hausdorff dimension $\leq \dim N - d(\omega_p,\delta) - 1$.

7. For any $2 \leq p^{'} \leq d(\omega_p,\delta) - 1$, the homotopy class of any map to or from $E^n$ contains a map of arbitrarily small $p^{'}$-energy.

8. If $n \leq 2d(\omega_p,\delta) - 1$, then $E^n$ is homeomorphic to a sphere.

where for each constant $0 \leq \sigma \leq 1$, and each $p$-SSU index $\omega_p$ on a $p$-SSU manifold $M^n$, we set an integer:

$$d(\omega_p,\delta) = 2 + [[p + 2\sqrt{p-\delta}]]$$

if $\omega_p \geq \frac{p+1-\delta+2\sqrt{p-\delta}}{p+1+2\sqrt{p-\delta}}$;

$$d(\omega_p,\delta) = max\{1 + [\frac{\delta}{1-\omega_p}], [p]\}$$

otherwise. Here $[[t]]$, is the greatest integer $[t]$of t, if t is not an integer; otherwise $[[t]] = t - 1$.

**B. Regularity of $p$-minimizers into a closed half-ellipsoid $\bar{E}^n_+$**

**Theorem 3.1 ( Regularity of $p$-Minimizers into Closed Half-ELLIPSOIDS ):** Every $p$-stable or $p$-minimizing tangent map ($p > 1$) from $R^l(l > 2)$ into $\bar{E}^n_+$ is constant for $p + \frac{2}{p-1} - \frac{2\sqrt{p^2-3p+3}}{p-1} \leq l \leq p + \frac{2}{p-1} + \frac{2\sqrt{p^2-3p+3}}{p-1}$ and $\mu < \sqrt[4]{\frac{p-1}{2(2n+1)}}$. Moreover, for any $p$-minimizing map $u : M \to \bar{E}^n_+$, it has a closed singular set $\Sigma$ of Hausdorff dimension $dim(\Sigma) \leq dim(M) - \{p + \frac{2}{p-1} + \frac{2\sqrt{p^2-3p+3}}{p-1}\} - 1$.

**C. Liouville theorems for $p$-stable maps from $p$-parabolic manifolds into $p$-SSU ellipsoids $E^n$**

**Theorem 4.1($L^p$ and $L^q$ Gradient Estimates)**: Let $u$ be a $p$-stable map from a complete Riemannian manifold $M$ into an ellipsoid $E^n$ with $\mu < \sqrt[3]{\frac{n}{p}}$. Then for any $\xi \in C^\infty_0(M)$, we have:

1. $p$-Stability Inequality:

$$\int_M \frac{\frac{n}{\mu^4} - p\mu^2}{(n + p - 2)(\min(a_i)^2)} \xi^2 |du|^p dv \leq \int_M |\nabla\xi|^2 |du|^{p-2} dv$$

2. $L^p$ Gradient Estimate:

$$\int_M \xi^p |du|^p dv \leq C \int_M |\nabla\xi|^p dv$$

where $C$ is constant depending on $n, p, \mu$.

3. $L^q$ Gradient Estimate: Let $-K$ be the lower bound of the Ricci curvature of $M$, and $q \in [p+2, p+2\frac{n(\frac{n}{\mu^6}-p)}{(n-1)(n+p-2)}]$, then

$$\int_M \xi^q |du|^q dv \leq C_1 \int_M K\xi^q |du|^{q-2} dv + C_2 \int_M |\nabla \xi|^q dv$$

where $C_1, C_2$ are constants depending on $n, p, \mu$ and $K$.

**Theorem 4.2(Liouville-type Theorem for $p$-Parabolic or $q$-Parabolic Manifolds)**:

1. Every $p$-stable map $u$ from a $p$-parabolic manifold $M$ into an ellipsoid $E^n$ with $\mu < \sqrt[3]{\frac{n}{p}}$ is constant for $p \geq 2$.

2. Every $p$-stable map $u$ from $q$-parabolic manifold $M$ with $Ric^M \geq 0$ into an ellipsoid $E^n$ with $\mu < \sqrt[3]{\frac{n}{p}}$ is constant, where

$$q \in [p+2, p+2\frac{n(\frac{n}{\mu^6}-p)}{(n-1)(n+p-2)}].$$

**D. On the non-existence of stable currents and submanifolds on ellipsoids**

**Theorem 5.1 (Nonexistence of stable submanifolds in an ellipsoid)**: An ellipsoid $E^n$ has no closed stable $k$-dimensional submanifolds if $\mu^4 \leq \frac{n-k+2}{2}$. In particular, a unit sphere $S^n$ has no closed $k$-dimensional submanifolds for any $1 \leq k < n$.

## E. Applications to Yang-Mills Instability

**Theorem 6.1(Yang-Mills instability of an ellipsoid):** An ellipsoid $E^n$ with $\mu < \sqrt[3]{\frac{n-2}{2}}$ is Yang-Mills instable.

The case $a_1 = a, a_2 = \cdots = a_{n+1} = 1$ with $a < n-3$ is due to Shoshichi Kobayashi, Yoshihiro Ohnita, and Masaru Takeuchi ([23]).

## F. The main theorems on a compact convex hypersurface in a Euclidean space

**Theorem 7.1**: Suppose $N^n$ is a compact convex hypersurface in $R^{n+1}$, then $N^n$ is a $p$-SSU Riemannian manifold if $\mu_0 \leq \frac{n}{p}$ where $\mu_0$ is defined as in Definition 0.8.

**Theorem 7.2**: Let $N^n$ be a compact convex hypersurface in $R^{n+1}$. Suppose $N_1$ is a minimal $k$-submanifold of $N^n$ such that the Ricci curvature $Ric^{N_1}$ satisfies:

$$Ric^{N_1} > k(1-\frac{1}{p})\lambda_n^2$$

Then $N_1$ is $p$-SSU where $\lambda_n$ is defined as in Definition 0.8.

**Theorem 7.6**: Suppose $N^n$ is a compact convex hypersurface in $R^{n+1}$. Then if $\mu_0 \leq \frac{n}{p}$, we have the following properties:

1. Any $p$-harmonic stable map $\varphi$ from $N^n$ into a compact manifold is constant.
2. Any $p$-harmonic stable map $\varphi$ from a compact manifold into $N^n$ is constant.
3. Any $p$-harmonic map from $N^n$ into a compact manifold with nonpositive sectional curvature is constant.
4. The identity map $Id|_{N^n}$ is $p$-unstable.

5. $N^n$ is $[p]$-connected. i.e. $\pi_1(N^n) = \cdots = \pi_{[p]}(N^n) = 0$

6. $\inf\{E_p(\eta); \eta$ is homotopic to a map $\xi : M \to N^n\} = \inf\{E_p(\eta) : \eta$ is homotopic to $\xi : N^n \to K\} = 0$ for any compact manifold $M$ and any map $\xi : M \to N^n$; for any complete manifold $K$ and any map $\xi : N^n \to K$.

7. $C^1(M, N^n)$ is dense in $L_1^p(M, N^n)$ for any compact manifold $M$ with possibly nonempty boundary.

8. For any $p' < \dim M$ and $2 \le p' \le p + 1$, and given boundary data $\eta \in L^{p'}_{1-\frac{1}{p'}}(\partial M, N^n)$, the Dirichlet problem for $p'$-harmonic maps has a solution $u \in L_1^{p'}(M, N^n)$ minimizing $p'$-energy in the class of maps $u \in L_1^{p'}(M, N^n)$ with the trace $u|_{\partial N^n} = \eta$ which is locally Holder continuous up to $\partial M$, and is $C^{1,\alpha}$ in the interior of $M$ off the singular set off Hausdorff dimension $\le \dim M - p' - 1$.

# Chapter 2

# Representing homotopy groups and spaces of maps by $p$-harmonic maps into ellipsoids

## 2.1 Preliminaries

Now we begin with some concepts of the second fundamental form:

Let $M^n$ be an n-dimensional Riemannian manifold isometrically immersed in a Euclidean space $R^{n+m}$. The normal bundle of $M$ in $R^{n+m}$ will be denoted by $TM^\perp$, covariant derivative on $M$ by $\nabla$ and the standard covariant derivative on $R^{n+m}$ by $\bar{\nabla}$. Let $h$ be the second fundamental form of $M$ in $R^{n+m}$. Then for each $x \in M, h_x$ is a symmetric bilinear map from $TM_x \times TM_x$ to $TM_x^\perp$. If $X, Y$ are vector field on $M$, then $\nabla, \tilde{\nabla}$, and $h$ are related by the Gauss equation:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

where $X, Y$ are smooth vector fields on $M$. For each $x \in M$ and $\eta \in TM_x^\perp$, let $A^\eta$ be the Weingarten map for the direction $\eta$. It is a self-adjoint linear map $A^\eta : TM_x \to TM_x$ and is related to the second fundamental form by

$$\langle h(X, Y), \eta \rangle = \langle A^\eta X, Y \rangle$$

for $X, Y \in TM_x$ and $\eta \in TM_x^\perp$. Denote by $\nabla^\perp$ the induced connection on the normal bundle, that is if $\eta$ is a section of $TM^\perp$ and $X$ is a vector field on $M$ then $\nabla_X \eta^\perp$ is the orthogonal projection of $\bar{\nabla}_X Y$ onto $TM^{\perp}$. It is related to $A$ and $\bar{\nabla}$ by the Weingarten equation:

$$A^{\eta}X = \bar{\nabla}_X\eta - (\bar{\nabla}_X\eta)^{\perp} = (\bar{\nabla}_X\eta)^{\top}$$

where $(.)^{\perp}$ is an orthogonal projection onto $TM^{\perp}$ and $(.)^{\top}$ is a tangential projection onto $TM$. The connections $\nabla$ and $\nabla^{\perp}$ induce a connection $\bar{\nabla}$ on all the tensor bundles constructed from $TM$ and $TM^{\perp}$. In particular $A$ can be viewed as a smooth section of $\mathrm{hom}(TM^{\perp}, \mathrm{hom}(TM, TM))$.

**Lemma 2.1(Estimate of the second fundamental form of an ellipsoid $E^n$ embedded in $R^{n+1}$)**: Suppose $\nu$ is a unit normal vector field on an ellipsoid $E^n$ imbedded in $R^{n+1}$ and $Y$ is any unit tangent vector on $E^n$ at a point x, then the second fundamental form

$$h_{YY} = \langle \nabla_Y \nu, Y\rangle$$

satisfies:

$$\frac{\min(a_i)}{(\max(a_i))^2} \leq h_{YY} \leq \frac{\max(a_i)}{(\min(a_i))^2}$$

where $\nabla$ is the Riemannian connection on $R^{n+1}$ and $\langle,\rangle$ is the Riemannian metric on $R^{n+1}$.

**Proof:** Suppose $\{U_1, U_2, \cdots, U_{n+1}\}$ is a standard o.n.b.in $R^{n+1}$. Since $E^n$ is a level set of $\frac{x_1^2}{a_1^2} + \cdots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1$, we know a normal vector $Z = \sum_{i=1}^{n+1} \frac{x_i}{a_i^2} U_i$, then a unit normal vector $\nu = \frac{Z}{||Z||} = \frac{1}{(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4})^{\frac{1}{2}}} \sum_{i=1}^{n+1} \frac{x_i}{a_i^2} U_i$. Suppose $Y = \sum_{i=1}^{n+1} \beta_i U_i$ is a unit tangent vector field on $E^n$ with $\sum_{i=1}^{n+1} \beta_i^2 = 1$, then we compute

$$\bar{\nabla}_Y \nu = \frac{1}{||Z||}\bar{\nabla}_Y Z + Y(\frac{1}{||Z||})Z$$

$$h_{YY} = \langle \overline{\nabla}_Y \nu, Y \rangle = \langle \frac{1}{\|Z\|} \overline{\nabla}_Y Z, Y \rangle$$

$$\frac{1}{\|Z\|} \overline{\nabla}_Y Z = \frac{1}{(\sum_{i=1}^{n+1} (\frac{x_i^2}{a_i^4}))^{\frac{1}{2}}} \sum_{i=1}^{n+1} \frac{Y[x_i]}{a_i^2} U_i$$

$$= \frac{1}{(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4})^{\frac{1}{2}}} \sum_{i=1}^{n+1} \frac{\beta_i}{a_i^2} U_i$$

where $Y[x_i] = dx_i(Y) = \beta_i$.

$$h_{YY} = \frac{1}{(\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4})^{\frac{1}{2}}} \sum_{i=1}^{n+1} \frac{\beta_i^2}{a_i^2}$$

(Recall: $\sum_{i=1}^{n+1} \beta_i^2 = 1$ because of a unit tangent vector $Y$ and $\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^2} = 1$ on $E^n$))

$$\frac{1}{(\max(a_i))^2} = (\sum_{i=1}^{n+1} \beta_i^2) \frac{1}{(\max(a_i))^2} \leq \sum_{i=1}^{n+1} \frac{\beta_i^2}{a_i^2} \leq \sum_{i=1}^{n+1} \frac{\beta_i^2}{(\min(a_i))^2} = \frac{1}{(\min(a_i))^2}$$

$$\frac{1}{\|Z\|} \frac{1}{(\max(a_i))^2} \leq h_{YY} \leq \frac{1}{\|Z\|} \frac{1}{(\min(a_i))^2}$$

$$\frac{1}{(\max(a_i))^2} = (\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^2}) \frac{1}{(\max(a_i))^2} \leq \sum_{i=1}^{n+1} \frac{x_i^2}{a_i^4} \leq (\sum_{i=1}^{n+1} \frac{x_i^2}{(a_i^2)}) \frac{1}{(\min(a_i))^2} = \frac{1}{(\min(a_i))^2}$$

$$\sum_{i=1}^{n+1} \frac{\beta_i^2}{a_i^2} \leq \sum_{i=1}^{n+1} \frac{\beta_i^2}{min(a_i)^2} = \frac{1}{min(a_i)^2}$$

$$h_{YY} \leq \frac{\max(a_i)}{(\min(a_i))^2}$$

In fact, we also prove the fact: $\frac{1}{(\max(a_i))^2} \leq |\nabla_Y Z| \leq \frac{1}{(\min(a_i))^2}$ for any unit tangent vector $Y$.

Furthermore, if we rewrite an ellipsoid $E^k : x_{k+1} = f(x_1, \cdots, x_k)$ where $f = a_{k+1}(1 - \sum_{i=1}^{k} \frac{x_i^2}{a_i^2})^{\frac{1}{2}}$, we can estimate the principle curvature $\{\lambda_1, \cdots, \lambda_k\}$ on $E^k$ based on the calculation of $h_{ij}^{\nu} = \langle \overline{\nabla}_{e_i} \nu, e_j \rangle, g_{ij} = \delta_{ij} - \frac{f_i f_j}{\omega^2}$ where $f_i = \frac{\partial f}{\partial x_i}$, $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$, and $\omega =$

$(1+\sum_{i=1}^{k} f_i^2)^{\frac{1}{2}}, i, j = 1, 2, \cdots, k$ because the principle curvatures $\{\lambda_1, \cdots, \lambda_k\}$ are solutions of a system of equations $det(h_{ij} - \lambda g_{ij}) = 0$. So we get:

$$\max_{1\le i\le k}(\lambda_i) = \frac{\max_{1\le i\le k+1}(a_i)}{[\min_{1\le i\le k+1}(a_i)]^2}$$
$$\min_{1\le i\le k} = \frac{\min_{1\le i\le k+1}(a_i)}{[\max_{1\le i\le k+1}(a_i)]^2}$$

Since the sectional curvature $Riem$ on $E^k$ is a product of two principle curvatures, we can estimate the sectional curvature $Riem^{E^k}$:

$$\sup Riem^{E^k} = \left(\frac{\max_{1\le i\le k+1}(a_i)}{[\min_{1\le i\le k+1}(a_i)]^2}\right)^2$$
$$\inf Riem^{E^k} = \left(\frac{\min_{1\le i\le k+1}(a_i)}{[\max_{1\le i\le k+1}(a_i)]^2}\right)^2$$

The most details can be found in Y.B.Chen and Y.L.Pan's work ([43]) as follows:

**Lemma 2.2(Estimates of principle curvatures and sectional curvatures of $E^n$ embedded in $R^{n+1}$)(Y.B.Shen and Y.L.Pan ([43]))**: Suppose $\{\lambda_i\}_{i=1}^n$ is a family of principle curvatures on $E^n$ embedded in $R^{n+1}$ satisfying $0 < \lambda_1 \le \lambda_2 \cdots \le \lambda_n$ and $Sec^{E^n}$ denotes any sectional curvature on $E^n$ embedded in $R^{n+1}$. Then:

$$\lambda_1 = \frac{\min(a_i)}{(\max(a_i))^2}$$
$$\lambda_n = \frac{\max(a_i)}{(\min(a_i))^2}$$
$$\frac{(\min(a_i))^2}{(\max(a_i))^4} \le Sec^{E^n} \le \frac{(\max(a_i))^2}{(\min(a_i))^4}$$

**Proof:**(cf. Lemma 2, Lemma 3, and Corollary 1 in ([43]).

Next we talk about a p-SSU manifold with p-SSU index $\omega_p$. In 1992, S.W.Wei ([49]) for the first time identified and discussed a large class of manifolds with positive Ricci curvature called "superstrongly unstable manifolds" and their associated

”superstrongly unstable indices”. In 1994, Wei and Yau ([51]) found to find a large class of ”distinguished” manifolds equipped with ”$p$-superstrongly unstable manifolds”(that is, $p$-SSU) together with ”p-superstrongly unstable indices”(that is, $\omega_p$) which included his previous work in 1992 as a special case for treating $p = 2$. Aided by the work of Wei, we can identify $E^n$ with a $p$-SSU manifold together with $p$-SSU index $\omega_p$.

**Example 2.1**: The simplest example of an SSU manifold is $S^k$ where $k > 2$ and the simplest example of a p-SSU manifold is $S^k$ where $k > p$. The reason is the following: On a unit sphere $N = S^k$, the covariant derivative is a position operator. that is: $\nabla_X \nu = X$ for any unit tangent vector $X$ where $\nu$ is a unit normal vector to $S^k$. Then the second fundamental form $h$:

$$h_{X,\alpha_i} = A^\nu(X, \alpha_i) = \langle \nabla_X \nu, \alpha_i \rangle \nu = \langle X, \alpha_i \rangle \nu$$

$$h_{X,X} = A^\nu(X, X) = \langle \nabla_X \nu, X \rangle \nu = \langle X, X \rangle \nu = \nu$$

$$h_{\alpha_i,\alpha_i} = A^\nu(\alpha_i, \alpha_i) = \langle \nabla_{\alpha_i} \nu, \alpha_i \rangle \nu = \langle \alpha_i, \alpha_i \rangle \nu = \nu$$

So, we have:

$$\begin{aligned}
\langle Q_y^N(X), X \rangle_N &= \sum_{i=1}^{k} \Big( 2|h_{X,\alpha_i}|^2 - h_{X,X} \cdot h_{\alpha_i,\alpha_i} \Big) && (2.1)\\
&= 2 \sum_{i=1}^{k} |\langle X, \alpha_i \rangle|^2 - k && (2.2)\\
&= 2|X|^2 - k && (2.3)\\
&= 2 - k && (2.4)
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}F_{p,y}(X) &= (p-2)|h_{X,X}|^2 + \langle Q_y^N(X), X\rangle_N && (2.5)\\ &= p-2+2-k && (2.6)\\ &= p-k && (2.7)\end{aligned}$$

**Example 2.2**: The simplest SSU index $\omega$ on an SSU manifold $N$ is $\omega = \frac{k-2}{k}$ on $S^k$ for $k > 2$ and the simplest p-SSU index $\omega_p$ on a p-SSU manifold $N$ is $\omega_p = \frac{k-p}{k+p-2}$ on $S^k$ for $k > p$. The reason is following: A unit sphere $S^k$ is a Riemannian manifold with the constant sectional curvature equal to 1. i.e. $S(y) = 1$. $\omega = \inf_{y\in N} \frac{\varphi(y)}{kS(y)} = \frac{k-2}{k}$ where $\varphi(y) = k-2$ satisfying $\langle Q_y^N(X), X\rangle_N \le -\varphi(y)\langle X, X\rangle_N$ and $\omega_p = \inf_{y\in N} \frac{\varphi_p(y)}{(k+p-2)S(y)} = \frac{k-p}{k+p-2}$ where $\varphi_p(y) = \inf_{|X|=1}(-F_{p,y}(X)) = -(p-k) = k-p$.

Every complete hypersurface in $R^{k+1}$ is SSU, if and only if all principal curvatures are positive (with respect to a normal vector), $0 \le \lambda_i \le \lambda_2 \cdots \le \lambda_k$, and the largest curvature is less than the sum of the others. $\lambda_k < \lambda_1 + \cdots + \lambda_{k-1} \Longleftrightarrow \lambda_k < \frac{1}{2}\sum_{i=1}^k \lambda_i$. Every complete hypersurface in $R^{k+1}$ is p-SSU, if and only if all principal curvatures are positive(with respect to a normal vector), $0 < \lambda_1 \le \lambda_2 \cdots \lambda_k$, and the largest principal curvature is less that one $p$-th of the sum of all principal curvatures, $\lambda_k < \frac{1}{p}\sum_{i=1}^k \lambda_i$. Let's look at an ellipsoid with a p-SSU manifold.

**Lemma 2.3(Criteria for $p$-SSU ellipsoids):** An ellipsoid $E^n$ is p-SSU if $\mu < \sqrt[3]{\frac{n}{p}}$.

**Proof:** By Lemma 2.2, we have an estimate for its principal curvatures $\lambda_1, \lambda_2, ...\lambda_n$ as follows:

$$\frac{min(a_1,...a_{(n+1)})}{max(a_1,...a_{(n+1)})^2} \le \lambda_i \le \frac{max(a_1,...a_{(n+1)})}{min(a_1,...a_{(n+1)})^2}$$

A complete hypersuface in $R^{n+1}$ is p-SSU if and only if its principle curvatures satisfy

$$0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_n < \frac{H}{p},$$

where $H = (\lambda_1 + ...\lambda_n)$(cf.Theorem 3.3 ([51])) We only need to prove that any principal curvature is less than one $p$-th of the sum of all principal curvatures. In fact, we have:

$$H \ge n\frac{min(a_i)}{max(a_i)^2}$$

and

$$\lambda_j \le \frac{max(a_i)}{min(a_i)^2}$$

then

$$\lambda_j - \frac{H}{p} < \frac{max(a_i)}{(min(a_i))^2} - \frac{n}{p}\frac{min(a_i)}{(max(a_i))^2}$$

Hence we know that $E^n$ is $p$-SSU if

$$\mu = \frac{max(a_i)}{min(a_i)} < \sqrt[3]{\frac{n}{p}}.$$

**Lemma 2.4(An Estimate on $p$-Superstrongly Unstable Index $\omega_p$ on $p$-SSU Ellipsoids):** Suppose an ellipsoid $E^n$ is a $p$-SSU manifold, then:

$$\omega_p \ge \frac{\frac{n}{\mu^6} - p}{n + p - 2}$$

The estimate is sharp on sphere $S^n(r)$.

**Proof:** Suppose

$$E^n : g = \frac{x_1^2}{a_1^2} + \cdots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1$$

then

$$\omega_p = inf_{y\in E^n} \frac{\phi_p(y)}{(n+p-2)S(y)}$$

where

$$S(y) = sup_{y\in E^n} Sec^{E^n}(y) = \frac{(max(a_i))^2}{(min(a_i))^4}$$

and

$$\phi_p(y) = inf_{||x||=1}(-F_{p,y}(X)), E^n \hookrightarrow R^{n+1}$$

$$F_{p,y}(X) = (p-2)|h_{XX}|^2 + \textstyle\sum_{j=1}^n 2|h_{X\alpha_j}|^2 - h_{XX}H$$

where $H$ is the mean curvature(i.e. the trace of the second fundamental forms from $E^n$ into $R^{n+1}$) and $\{\alpha_1, \cdots, \alpha_n\}$ is an o.n.b. in $E^n$, $\nu$ is an unit normal vector to $E^n$. Here $\{\alpha_1, \cdots, \alpha_n, \nu\}$ is an o.n.b in $R^{n+1}$. By Lemma 2.1 we have

$$\frac{\min(a_i)}{(\max(a_i))^2} \leq h_{XX} \leq \frac{\max(a_i)}{(\min(a_i))^2}$$

$$-F_{p,y}(X) = -(p-2)|h_{XX}|^2 + h_{XX}H - 2\sum_{j=1}^n |h_{X\alpha_j}|^2$$
$$\geq -(p-2)\frac{(max(a_i))^2}{(min(a_i))^4} + \frac{min(a_i)}{(max(a_i))^2} n \frac{min(a_i)}{(max(a_i))^2} - 2||\frac{1}{||Z||}\nabla_X Z||^2$$

where we can estimate the mean curvature: $n\frac{min(a_i)}{max(a_i)^2} \leq H \leq n\frac{max(a_i)}{min(a_i)^2}$ and we also know the following fact: $\{\alpha_1, \alpha_2, \cdots, \alpha_n, \nu\}$ is also an o.n.b. in $R^{n+1}$, then

$$\frac{1}{||Z||}\nabla_X Z = \sum_{j=1}^n \langle \frac{1}{||Z||}\nabla_X Z, \alpha_j \alpha\rangle \alpha_j + \langle \frac{1}{||Z||}\nabla_X Z, \nu\rangle \nu$$

$$
\begin{aligned}
\|\frac{1}{\|Z\|}\nabla_X Z\|^2 &\geq \sum_{j=1}^{n} |\langle \frac{1}{\|Z\|}\nabla_X Z, \alpha_j\rangle|^2 \\
&= \sum_{j=1}^{n} |h_{X\alpha_j}|^2 \\
\|\frac{1}{\|Z\|}\nabla_X Z\|^2 &\leq \frac{(\max(a_i))^2}{(\min(a_i))^4}
\end{aligned}
$$

(cf. Lemma 2.1)

$$
\begin{aligned}
-F_{p,y}(X) &\geq -(p-2)\frac{(max(a_i))^2}{(min(a_i))^4} + n\frac{(min(a_i))^2}{(max(a_i))^4} - 2\frac{(max(a_i))^2}{(min(a_i))^4} \\
&= -p\frac{(max(a_i))^2}{(min(a_i))^4} + n\frac{(min(a_i))^2}{(max(a_i))^4} \\
\omega_p &\geq \frac{-p\frac{(max(a_i))^2}{(min(a_i))^4} + n\frac{(min(a_i))^2}{(max(a_i))^4}}{(n+p-2)\frac{(max(a_i))^2}{(min(a_i))^4}} \\
&= -\frac{p}{(n+p-2)} + \frac{n}{(n+p-2)}\frac{(min(a_i))^6}{(max(a_i))^6} \\
\omega_p &\geq \frac{n\frac{1}{\mu^6} - p}{(n+p-2)}
\end{aligned}
$$

The estimate is sharp on sphere $S^n(r)$ because of $\langle \frac{1}{\|Z\|}\nabla_X Z, \nu\rangle = 0$.

## 2.2 Topological Vanishing Theorem on Minimal Submanifolds in Ellipsoids

**Theorem 2.1(Topological Vanishing Theorem on Minimal Submanifolds in Ellipsoids)**: Let $N$ be a minimal $k$-dimensional submanifold of an ellipsoid $E^n$ such that the Ricci curvature $Ric^N$ of $N$ satisfies:

$$
Ric^N > k(1 - \frac{1}{p})\frac{(\max(a_i))^2}{(\min(a_i))^4}
$$

where $2 \leq p < k$. Then $N$ is $p$-SSU and

$$\pi_1(N) = \pi_2(N) = \cdots = \pi_{[p]}(N) = 0$$

**Proof:** For any unit vector $X \in T_x(N)$, we choose an orthonormal frame $\{\alpha_1, \cdots, \alpha_k\}$ on N such that at x, $X = \alpha_k$. For$1 \leq i, j \leq k$, let$R_{ij}(resp.\overline{R}_{ij})$ denote the sectional curvature of N(resp. an ellipsoid $E^n$) for the section $\alpha_i \wedge \alpha_j$. For $1 \leq i, j \leq k$, let $h_{ij} = (\nabla_{\alpha_i}\alpha_j)^{\perp}$ where $\nabla$ is the Riemannian connection on $E^n$ and $()^{\perp}$ is the projection onto the normal space $T_x(N)^{\perp}$. The fact that N is minimal says:

$$\sum_{i=1}^{k} h_{ii} = 0 \cdots (1)$$

Let $\overline{h}(Y, Z) = \langle \overline{\nabla}_Y Z, \nu \rangle$ where $\overline{\nabla}$ is the Riemannian connection on $R^{n+1}$, Y and Z are local vector fields on $E^n$, and $\nu$ is the unit normal vector to $E^n$. And the second fundamental form $\widetilde{h}$ of N in $R^{n+1}$ splits into:

$$|\widetilde{h}(X, X)|^2 = h_{kk}^2 + |\overline{h}(X, X)|^2 \cdots (2)$$

Applying the Gauss-curvature equation, we have:

$$R_{ik} = \overline{R}_{ik} + \langle h_{ii}, h_{kk} \rangle_{E^n} - h_{ik}^2 \text{ where } 1 \leq i \leq k \cdots (3)$$

Summing (3) over $1 \leq i \leq k$ and applying (1), we have:

$$Ric(X) = Ric(\alpha_k) = \sum_{i=1}^{k-1} \overline{R}_{ik} - \sum_{i=1}^{k} h_{ik}^2 \cdots (4)$$

In view of (4), the assumption $Ric^N > k(1 - \frac{1}{p})\frac{\max(a_i)^2}{\min(a_i)^4}$ implies:

$$\sum_{i=1}^{k-1} \overline{R}_{ik} - \sum_{i=1}^{k} h_{ik}^2 = Ric(X) > k(1 - \frac{1}{p})\frac{max(a_i)^2}{min(a_i)^4}$$

$$\sum_{i=1}^{k} h_{ik}^2 < \sum_{i=1}^{k-1} \overline{R}_{ik} - k(1-\tfrac{1}{p})\tfrac{max(a_i)^2}{min(a_i)^4}$$

And by Lemma 2.2, we have an estimate for the sectional curvature on $E^n$, i.e.

$$\tfrac{min(a_i)^2}{max(a_i)^4} \leq \overline{R}_{ij} \leq \tfrac{max(a_i)^2}{min(a_i)^4}$$

Hence

$$\begin{aligned} h_{kk}^2 &\leq (k-1)\frac{max(a_i)^2}{min(a_i)^4} - k(1-\frac{1}{p})\frac{max(a_i)^2}{min(a_i)^4} \\ &= (\frac{k}{p}-1)\frac{(\max(a_i))^2}{(\min(a_i))^4} \cdots (5) \end{aligned}$$

On the other hand, by the assumption that $N$ is a minimal submanifold of $E^n$ and the formula (cf. (2.12) ([34]) p.322), we have:

$$\begin{aligned} \langle Q_y^N(X), X\rangle_N &= \langle -2Ric(X), X\rangle_N + (\sum_{i=1}^{k}\langle\overline{\nabla}_{\alpha_i}\nu, \alpha_i\rangle_{R^{n+1}})\langle\overline{\nabla}_X\nu, X\rangle_{R^{n+1}} \\ &= \langle -2RicX, X\rangle_N + (\sum_{i=1}^{k}\overline{h}_{ii})\overline{h}_{XX} \end{aligned}$$

By Lemma 2.1, we can estimate $\overline{h}_{XX}$ as follows:

$$\overline{h}_{XX} \leq \tfrac{max(a_i)}{min(a_i)^2} \cdots (6)$$

Therefore, we can get:

$$F_{p,y}(X) = (p-2)|\tilde{h}_{kk}|^2 + \langle Q_y^N(X), X\rangle$$

where $X = \alpha_k$

$$= (p-2)(h_{kk}^2 + \overline{h}_{kk}^2) + \langle -2RicX, X\rangle + (\sum_{i=1}^{k}\overline{h}_{ii})\overline{h}_{kk}$$

(by (5) and (6))

$$< (p-2)\{(\frac{k}{p}-1)\frac{(max(a_i))^2}{(min(a_i))^4} + \frac{(max(a_i))^2}{(min(a_i))^4}\} - 2k(1-\frac{1}{p})\frac{(max(a_i))^2}{(min(a_i))^4} + k\frac{(max(a_i))^2}{(min(a_i))^4}$$

$$= 0$$

Then, N is p-SSU by the definition. Furthermore, by Topological Vanishing Theorem, we know that $N$ is $[p]$-connected, i.e. $\pi_1(N) = \pi_2(N) = \cdots = \pi_{[p]}(N) = 0$.

In contrast to the above theorem, S.W.Wei obtained Sphere Theorem as follows:

**Sphere Theorem(S.W.Wei)([48]):** Let $N$ be a minimal $k$-submanifold of a unit Euclidean sphere $S^{q-1}$ such that the Ricci curvature $Ric^N$ of $N$ satisfies:

$$Ric^N > k(1 - \frac{1}{p})$$

where $2 \leq p < k$. Then $N$ is $p$-SSU. As a consequence, $\pi_1(N) = \cdots = \pi_{[p]}(N) = 0$.

**Remark 2.1:**

1. We recapture Sphere Theorem when the lower bound for Ricci curvature in an ellipsoid is equal to that in a unit sphere.

2. Sphere Theorem is sharp; that is: Let

$$N^k = S^j(\sqrt{\frac{j}{2p}}) \times S^{2p-j}(\sqrt{\frac{2p-j}{2p}}) \subset S^{2p+1}(1)$$

for any $1 \leq j \leq p$. Then $N$ is a minimal $k$-dimensional submanifold of a unit Euclidean sphere such that for each tangent vector to its first factor $S^j(\sqrt{\frac{j}{2p}})$,

$$Ric^N = \frac{2p}{j}(j-1) \leq 2p(1 - \frac{1}{p}) = k - \frac{k}{p},$$

where $k = 2p$. But $\pi_j(N) \neq 0, \pi_{2p-j}(N) \neq 0$. In particular, if $j = p$, the Clifford embeddings

$$N^k = S^p(\frac{1}{\sqrt{2}}) \times S^p(\frac{1}{\sqrt{2}}) \subset S^{2p+1}(1)$$

with

$$Ric^N = k(1 - \frac{1}{p})$$

but

$$\pi_p(N) \neq 0$$

for all $p$. There also exist totally geodesic embeddings $N^k = S^p(1)$ in the unit sphere with $Ric^N = k(1 - \frac{1}{p})$ where $p = k$ but $\pi_p(N) \neq 0$.

## 2.3 Geometric connectedness and Dirichlet problems on $p$-SSU ellipsoids

In this section, we can explore geometric $k_p$-connectedness (hence $k$-connectedness) for some $k \geq\geq [p]$ on an p-SSU ellipsoid $E^n$ where $k$ depends on some appropriate p-SSU index $\omega_p$. Moreover, we obtain a nonexistence theorem which asserts that no p-SSU ellipsoid can be either the domain or the target of any nonconstant p-stable maps (into a complete manifold or from a compact manifold). We also solve the Dirichlet problem for p-harmonic maps from a compact manifold $M$ into a p-SSU ellipsoid.

**Theorem 2.2(Geometric $k_p$ connectedness and Dirichlet problems on ellipsoids**: Let an ellipsoid $E^n$ be a compact p-SSU n-manifold with an appropriate p-SSU index $\omega_p$. Then $E^n$ is geometrically $(d(\omega_p, \delta) - 1)_p$-connected. Hence:

1. $E^n$ is $(d(\omega_p, \delta) - 1)$-connected.

2. $C^1(N, E^n)$ is dense in $L_1^{d(\omega_p,\delta)-1}(N, E^n)$ for any compact manifold N with possibly non-empty boundary.

3. Each class in $\pi_{d(\omega_p,\delta)}(E^n)$ can be represented by a sum of $C^{1,\alpha}$ p-harmonic map of $S^{d(\omega_p,\delta)}$ into $E^n$. Furthermore, if the class is nontrivial, these nonconstant p-harmonic representatives are p-unstable.

4. Each class in $\pi_{d(\omega_p,\delta)}(E^n)$ can be represented by a sum of $C^{1,\alpha} d(\omega_p, \delta)$-harmonic map of $S^{d(\omega_p,\delta)}$ into $E^n$.

5. For any $p^{'}$ <dim$N$ and $2 \leq p^{'} \leq d(\omega_p, \delta)$, and any given boundary data $\eta \in L^{p^{'}}_{1-\frac{1}{p^{'}}}(\partial N, E^n)$, the Dirichlet problem for $p^{'}$-harmonic maps into $E^n$ has a solution, which is locally Hölder continuous up to $\partial N$ and is $C^{1,\alpha}$ in the interior of N off the singular set of Hausdorff dimension $\leq \dim N - p^{'} - 1$.

6. In particular, the Dirichlet problem for p-harmonic maps into $E^n$ has a solution with singular set of Hausdorff dimension $\leq \dim N - d(\omega_p, \delta) - 1$.

7. For any $2 \leq p^{'} \leq d(\omega_p, \delta) - 1$, the homotopy class of any map to or from $E^n$ contains a map of arbitrarily small $p^{'}$-energy.

8. If $n \leq 2d(\omega_p, \delta) - 1$, then $E^n$ is homeomorphic to a sphere.

where for each constant $0 \leq \sigma \leq 1$, and each p-SSU index $\omega_p$ on a p-SSU manifold $M^n$, we set an integer:

$$d(\omega_p, \delta) = 2 + [[p + 2\sqrt{p - \delta}]]$$

if $\omega_p \geq \frac{p+1-\delta+2\sqrt{p-\delta}}{p+1+2\sqrt{p-\delta}}$;

$$d(\omega_p, \delta) = max\{1 + [\frac{\delta}{1-\omega_p}], [p]\}$$

otherwise. Here $[[t]]$, is the greatest integer $[t]$of t, if t is not an integer; otherwise $[[t]] = t - 1$.

# Chapter 3

# Regularity of $p$-energy minimizing maps into a closed upper-half ellipsoid $\bar{E}^n_+$

Mappings between Riemannian manifolds $u : M \to N$ that are critical points or minimizers for p-energy functional $E_p$ have been investigated extensively. Regularity estimates for elliptic systems, in particular the Euler-Lagrange equation for p-energy, were first obtained by Uhlenbeck ([45]) for $p \geq 2$, and later by Tolksdorff ([44]) for $p > 1$. Regularity theory of p-energy minimizing maps between Riemannian manifolds has been established by Hardt-Lin ([14]) and Luckhaus ([26]), for case $p > 1$. the question of minimizing the p-energy in appropriate homotopy classes has been studied by White ([53]). S.Walter Wei and Chi-Ming Yau ([51]) derives the first and second variational formulas for the p-energy functional on maps between Riemannian manifolds and obtain a Bochner formula and related estimates as a fundamental tool in the study of p-harmonic maps.

In this chapter, our work is built on the regularity of local p-minimizers on an ellipsoid $E^n$, via the first and second variational formula for p-energy functional on maps, together with a technique of Bochner on $E^n$.

## 3.1 Preliminaries

Consider a smooth map $u : M \to N$, where $M$ is compact. Denote the pull-back tangent bundle of $N$ by $u$ as $u^{-1}TN$, and denote the pull-back connection by $\nabla^u$. Let $v, w$ be smooth sections in $u^{-1}TN$, vanishing identically on a neighborhood of $\partial M$.

Choose a one-parameter family of smooth maps $u_t$ such that $u_0 = u$ and $\frac{du_t}{dt}|_{t=0} = v$ and a two-parameter variations $F(., s.t) = u_{s,t}$ such that $v = \frac{\partial u_{s,t}}{\partial s}|_{s,t=0}$ and $w = \frac{\partial u_{s,t}}{\partial t}|_{s,t=0}$. We shall also denote by $F : M \times R \to N$ the smooth map defined by $F(x,t) = u_t(x)$ for one-parameter variations.

**First variational formula for p-energy**

$$\frac{d}{dt}E_p(u_t)|_{t=0} = \int_M |du|^{p-2} \sum_i \langle \nabla^u_{e_i} v, \tilde{e}_i \rangle dv$$

where $\{e_1, \cdots, e_n\}$ is a local orthonormal frame field on $M$, and $\tilde{e}_i$ means $du(e_i)$.

**Second variational formula of two parameters for p-energy** ($u$ is not a necessary p-harmonic map)

$$\begin{aligned} \frac{\partial^2}{\partial t \partial s} E_p(u_{s,t})|_{s,t=0} &= \int_M (p-2)|du|^{p-4} \Big( \sum_i \langle \nabla^u_{e_i} v, \tilde{e}_i \rangle \Big) \Big( \sum_j \langle \nabla^u_{e_j} w, \tilde{e}_j \rangle \Big) \\ &+ |du|^{p-2} \Big( \sum_{i=1}^n \langle \nabla^u_{e_i} \nabla^u_{\frac{\partial}{\partial t}} \frac{\partial u_{s,t}}{\partial s}|_{s,t=0} + R^N(w, \tilde{e}_i)v, \tilde{e}_i \rangle \\ &+ \sum_{i=1}^n \langle \nabla^u_{e_i} v, \nabla^u_{e_i} w \rangle \Big) dv \end{aligned} \tag{3.1}$$

**The second variational formula of one-parameter for p-harmonic map**

$$
\begin{aligned}
\frac{d^2}{dt^2}E_p(u_t)|_{t=0} &= \int_M (p-2)|du|^{p-4}\Big(\sum_i \langle \nabla^u_{e_i} v, \tilde{e}_i\rangle\Big)^2 \\
&+ |du|^{p-2}\Big(\sum_i \langle R^N(v,\tilde{e}_i)v, \tilde{e}_i\rangle \\
&- \sum_i \langle \nabla^u_{e_i}\nabla^u_{e_i} v - \nabla^u_{\nabla_{e_i}e_i} v, v\rangle + div\theta^v\Big)dv \qquad (3.2)
\end{aligned}
$$

where $\theta^v$ is a differential 1-form given by $\theta^v(x) = \langle \nabla^u_x v, v\rangle$, for a local vector field on $M$.

We now turn to the following pointwise estimate on the Bochner Formula based on Wei and Yau's work in 1994 ([51]).

**Lemma 3.1:( Bochner Formula):**Let $u : M \to N$ be a p-harmonic map when M is compact, then:

$$\int_M -K|du|^{p-2+2\beta}dv \le -(p-1)(2\beta-1)\int_M |du|^{p-4+2\beta}|\nabla|du||^2 dv + \frac{n-1}{n}\int_M S|du|^{p+2\beta}dv$$

where $n = dimM$ and $-K$ =lower bound of $Ric^M$ and $S$ =upper bound for the sectional curvature on N for every positive constant $\beta \ge 1$.

**Proof:**(cf. Lemma 1.3 ([47]))

As an application of the first variational formula, we can look at the Liouville theorem for tangent maps that serves as a basis for regularity theorems as follows: Let $u : S^{n-1} \to N$ be a smooth, p-harmonic map and let $\phi$ be a strictly convex function such that $u(S^{n-1})$ lies in the domain of $\phi$.(If $N$ has a nonpositive sectional curvature, $\phi$ is the distance function on $N$) Choose $\nabla\phi$ to be the variational vector field. If $u$ were not constant, then the first variational formula and the strict convexity of $\phi$ would lead to a contradiction:

$$0 = \int_{S^{n-1}} |du|^{p-2} \sum_{i=1}^{n} \langle \tilde{e}_i, \nabla_{\tilde{e}_i}(\nabla\phi)\rangle > 0$$

Based on this reason, any p-minimizing tangent map $R^n \to N$ is trivial for each $n = 1, 2, \cdots$. Therefore, S.W.Wei ([51]) proves the regularity for p-minimizing map into a domain of a strictly convex function as below:

**Theorem**: Every p-minimizing $L_1^p$ map into a manifold with nonpositive sectional curvature, or into the domain of a strictly convex function is $C^{1,\alpha}$. In particular, every p-minimizing $L_1^p$ maps into either a complete noncompact manifold with positive sectional curvature or an open hemisphere is $C^{1,\alpha}$.

By the same story of a contradiction between the first variational formula and the strictly convexity, S.W.Wei ([48]) also obtain theorems about the behavior of p-harmonic maps and p-stable maps.

**Theorem**: Every smooth p-harmonic map $u$ defined on a compact Riemannian manifold $M$ into a manifold $N$ that supports a strictly convex function $\phi$ is constant.

**Theorem**: Let $M$ be compact with possibly nonempty boundary $\partial M$ and $N$ be compact with nonpositive sectional curvature. Then every p-harmonic map $u : M \to N$ minimizes p-energy in its homotopy class ( that agrees with $\partial M$ if $\partial M$ is nonempty). In particular, every p-harmonic map is stable.

On an ellipsoid $E_a^n = \{(x_1, \cdots, x_{n+1}) \in R^{n+1} : ax_1^2 + x_1^2 + \cdots + x_{n+1}^2 = 1, a > 0\}$ with one parameter as the target space, S.W.Wei ([47]) study Liouville-type theorem for $p$-minimizing tangent maps into half-ellipsoid $\overline{E}_{a,+}^n = \{(x_1, \cdots, x_{n+1}) \in E_a^n; x_{n+1} \geq 0\}$ as follows:

**Theorem**: Every p-stable or p-minimizing tangent map from $R^\ell$ into $\bar{E}^n_{a,+}$ is constant for $\ell < d(a, \sigma, p)$ where $d(a, \sigma, p)$ is a constant depending on $a, p$ and $\sigma \in (0, 1)$.

## 3.2 Regularity for $p$-minimizer into closed half-ellipsoids

In this section, we discuss the Liouville's problem for tangent maps based on the first and second variational formulas for p-energy and a Bochner formula.

**Theorem 3.1 ( Regularity of p-Minimizers into Closed Half-ELLIPSOIDS ):** Every p-stable or p-minimizing tangent map($p > 1$) from $R^l(l > 2)$ into $\bar{E}^n_+$ is constant for $p + \frac{2}{p-1} - \frac{2\sqrt{p^2-3p+3}}{p-1} \leq l \leq p + \frac{2}{p-1} + \frac{2\sqrt{p^2-3p+3}}{p-1}$ and $\mu < \sqrt[4]{\frac{p-1}{2(2n+1)}}$. Moreover, for any $p$-minimizing map $u : M \to \bar{E}^n_+$, it has a closed singular set $\Sigma$ of Hausdorff dimension $dim(\Sigma) \leq dim(M) - \{p + \frac{2}{p-1} + \frac{2\sqrt{p^2-3p+3}}{p-1}\} - 1$.

**Proof**: By the first variational formula for p-energy functionals, any p-harmonic map $u \in L^p_1(M, N)$ is a weak solution of

$$div(|du|^{p-2} \bigtriangledown u) + A(\bigtriangledown u, \bigtriangledown u)|du|^{p-2} = 0$$

where A is the second fundamental form of $N$ into $R^q$. From the elementary calculus,$E^n$ is the level set of $\frac{u_1^2}{a_1^2} + \cdots + \frac{u_{n+1}^2}{a_{n+1}^2} - 1$, whose gradient vector field $(\frac{u_1}{a_1^2}, \cdots, \frac{u_{n+1}}{a_{n+1}^2})$ is normal to the convex hypersurface $E^n$ in $R^{n+1}$. It follows that any p-harmonic map $u = (u_1, \cdots, u_{n+1}) : S^{l-1} \to \bar{E}^n_+ \subset R^{n+1}$ of class $C^1$ is a weak solution to

$$div(|du|^{p-2}\nabla u_i) + |du|^p u_i \varrho_i = 0 \cdots (.1)$$

for some function $\varrho_i > 0$ and $1 \leq i \leq (n+1)$. Taking $i = n+1$ in this equation, integrating it over $S^{l-1}$ and using Stokes' theorem, we have

$$\int_{S^{l-1}} |du|^p u_{n+1}\varrho_{n+1} = 0 \cdots (.2)$$

Define $B = \{x \in S^{l-1}; |du|(x) > 0\}$. Suppose $|du| \neq 0$; then B is open and nonempty. Hence, from (.2), $u_{n+1} \geq 0$ implies $u_{n+1}|_{\bar{B}} \equiv 0$. On the other hand, $|du_{n+1}| \leq |du| = 0$ in $S^{l-1}\backslash\bar{B}$. Thus $u_{n+1} \equiv 0$ in $S^{l-1}$. Combining these two facts we have either $u_{n+1} \equiv 0$ (if B is nonempty) or u constant ( if B is empty). If u is constant, we are done. So we assume $u_{n+1} \equiv 0$ so that $u(S^{l-1}) \subset \partial(\bar{E}^n_+)$. Now we deform its p-minimizing homogeneous extension $\bar{u} = (\bar{u}_1, \cdots, \bar{u}_n, 0) : R^l \to \partial(\bar{E}^n_+)$ along $(0, 0, \cdots, 0, \varphi)$ where $\varphi$ is a positive function and $\varphi \in C^1(R^l\backslash\{0\})$ and $\partial(\bar{E}^n_+) = \{(x_1, \cdots, x_{n+1} \in E^n_+, x_{n+1} = 0\}$. Since we have

$$\frac{(\bar{u}_1)^2}{a_1^2} + \cdots + \frac{(\bar{u}_n)^2}{a_n^2} = 1$$

this generates a variation $\bar{u}_t(x)$ in $\bar{E}^n_+$ given by

$$\bar{u}_t(x) = \frac{1}{\sqrt{1+t^2\varphi^2}}(\bar{u}_1(x), \cdots, \bar{u}_n(x), ta_{n+1}\varphi) : R^l \to \bar{E}^n_+$$

where t is a nonnegative number. Let $\dot{\bar{u}}$ and $\ddot{\bar{u}}$ denote the first and second derivative of $\bar{u}_t$ with respect to t evaluated at $t = 0$. Then clearly we have:

$$\dot{\bar{u}} = (0, \cdots, 0, a_{n+1}\varphi)$$

$$\ddot{\bar{u}} = -\varphi^2(\bar{u}_1, \bar{u}_2, \cdots, 0)$$

Actually we have the following identity:

$$\frac{d^2}{dt^2}(\frac{1}{p}|d\bar{u}_t|^p) = |d\bar{u}_t|^{p-2}\{\langle d\ddot{\bar{u}}_t, d\bar{u}_t\rangle + |d\dot{\bar{u}}_t|^2\} + (p-2)|d\bar{u}_t|^{p-4}\langle d\dot{\bar{u}}_t, d\bar{u}_t\rangle^2$$

$$\frac{d^2}{dt^2}(\frac{1}{p}|d\bar{u}_t|^p)|_{t=0} = |d\bar{u}|^{p-2}(|d\dot{\bar{u}}|^2 + \langle d\bar{u}, d\ddot{\bar{u}}\rangle) + (p-2)|d\bar{u}|^{p-4}\langle d\bar{u}, d\dot{\bar{u}}\rangle^2 \cdots (*)$$

$$\langle d\bar{u}, d\ddot{\bar{u}}\rangle + \varphi^2|d\bar{u}|^2 = -\frac{1}{2}d(\varphi^2)d(\bar{u}^2) = -2\bar{u}\varphi d(\varphi)d\bar{u}$$

and the Cauchy-Schwarz inequality implies:

$$\langle d\bar{u}, d\ddot{\bar{u}}\rangle + \varphi^2|d\bar{u}|^2 \le 2|\frac{\varphi d\bar{u}}{\sqrt{2}}||\sqrt{2}(\bar{u})d\varphi| \le \frac{1}{2}\varphi^2|d\bar{u}|^2 + 2|d\varphi|^2|\bar{u}|^2$$

where

$$|\bar{u}|^2 = (\bar{u}_1)^2 + \cdots + (\bar{u}_k)^2 \le a_1^2 + \cdots + a_n^2 = \sum_{i=1}^{n} a_i^2 \cdots (.3)$$

$$\langle d\bar{u}, d\ddot{\bar{u}}\rangle \le \frac{-1}{2}\varphi^2|d\bar{u}|^2 + 2(\sum_{i=1}^{n} a_i^2)|d\varphi|^2 \cdots (.4)$$

Since $\bar{u}$ is minimizing,

$$0 \le \frac{d^2}{dt^2}E_p(\bar{u}_t)|_{t=0} = \int_{R^l}\frac{d^2}{dt^2}(\frac{1}{p}|d(\bar{u}_t)|^p)|_{t=0}dv$$

$$= \int_{R^l}|d\bar{u}|^{p-2}(|d\dot{\bar{u}}|^2 + \langle d\bar{u}, d\ddot{\bar{u}}\rangle) + (p-2)|d\bar{u}|^{p-4}\langle d\bar{u}, d\dot{\bar{u}}\rangle^2$$

$$\le \int_{R^l}|d\bar{u}|^{p-2}a_{n+1}^2|d\varphi|^2 + |d\bar{u}|^{p-2}\{\frac{-1}{2}\varphi^2|d\bar{u}|^2 + 2(\sum_{i=1}^{n} a_i^2)|d\varphi|^2\}$$

$$= \int_{R^l}|d\bar{u}|^{p-2}|d\varphi|^2(a_{n+1}^2 + 2\sum_{i=1}^{n} a_i^2) - \frac{1}{2}|d\bar{u}|^p\varphi^2$$

So we can get:

$$\frac{1}{2}\int_{R^l}|d\bar{u}|^p\varphi^2 \le \int_{R^l}(a_{n+1}^2 + 2\sum_{i=1}^{n} a_i^2)|d\bar{u}|^{p-2}|d\varphi|^2$$

$$\int_{R^l} C_a \varphi^2 |d\bar{u}|^p dx \le \int_{R^l} |d\varphi|^2 |d\bar{u}|^{p-2} dx \cdots (.5)$$

where

$$C_a = \frac{1}{2} \frac{1}{(a_{n+1}^2 + 2\sum_{i=1}^n a_i^2)}$$

Splitting the integrals over $R^l$ into spherical and radial directions as follows: if we choose $\varphi = f(r)g(\nu)$ where $r = |x|$ and $\nu = \frac{x}{|x|} \in S^{l-1}$, we substitute $\varphi$ into $fg$ in (.5) and we obtain:

$$\frac{\int_{S^{l-1}} (-|\bigtriangledown g|^2 |du|^{(p-2)} + C_a g^2 |du|^p) d\nu}{\int_{S^{l-1}} g^2 |du|^{(p-2)} d\nu} \le \frac{\int_0^\infty f'^2 r^{l-p+1} dr}{\int_0^\infty f^2 r^{l-p-1} dr}$$

Since the inf of the right-hand side is $\frac{1}{4}(l-p)^2$, and we know the basic following fact:

$$ds^2 = dr^2 + \frac{1}{r^2} d\Theta^2$$

$$|\nabla f(r) g(\Theta)|^2 = |\frac{df}{dr} g(\Theta)|^2 + |\frac{1}{r} f(r) \frac{dg}{d\Theta}|^2$$

then, we can get:

$$\int_{S^{l-1}} (-|dg|^2 |du|^{p-2} + C_a g^2 |du|^p) d\nu \le \frac{1}{4}(l-p)^2 \int_{S^{l-1}} g^2 |du|^{p-2} d\nu$$

for every nonzero $g \in C^1(S^{l-1})$. Setting $g = |du|$, this gives:

$$-\int_{S^{l-1}} |\bigtriangledown |du||^2 |du|^{p-2} d\nu + \int_{S^{l-1}} C_a |du|^{p+2} d\nu \le \frac{1}{4}(l-p)^2 \int_{S^{l-1}} |du|^p d\nu \cdots (.6)$$

On the other hand, the Bochner formula for a p-harmonic map from $S^{l-1}$ into $\bar{E}_+^k$ gives:

$$\int_{S^{l-1}} (l-2)|du|^p d\nu$$
$$\le -(p-1) \int_{S^{l-1}} |du|^{p-2} |\bigtriangledown |du||^2 d\nu + \frac{l-2}{l-1} \int_{S^{l-1}} \frac{(max(a_i))^2}{(min(a_i))^4} |du|^{p+2} d\nu \cdots (.7)$$

Combining (.6) and (.7), we can get the following inequality:

$$\frac{l-2}{p-1}\int_{S^{l-1}}|du|^p dv-\frac{l-2}{(p-1)(l-1)}\int_{S^{l-1}}\frac{(max(a_i))2}{(min(a_i))^4}|du|^{p+2}dv$$
$$\leq\frac{1}{4}(l-p)^2\int_{S^{l-1}}|du|^p dv-\int_{S^{l-1}}C_a|du|^{p+2}dv$$
$$[\frac{l-2}{p-1}-\frac{1}{4}(l-p)^2]\int_{S^{l-1}}|du|^p dv\leq[\frac{l-2}{(p-1)(l-1)}\frac{(max(a_i))^2}{(min(a_i))^4}-C_a]\int_{S^{l-1}}|du|^{p+2}dv$$

if $\frac{l-2}{p-1}-\frac{1}{4}(l-p)^2\geq 0$ and $\frac{l-2}{(p-1)(l-1)}\frac{(max(a_i))^2}{(min(a_i))^4}-C_a<0$, then u is constant. So we get our conclusion:

$$\text{If } p+\frac{2}{p-1}-\frac{2\sqrt{p^2-3p+3}}{p-1}\leq l\leq p+\frac{2}{p-1}+\frac{2\sqrt{p^2-3p+3}}{p-1} \text{ and}$$
$$\frac{(min(a_i))^4}{(max(a_i))^2}>\frac{1}{p-1}2(2n+1)(\max(a_i))^2\geq\frac{1}{(p-1)C_a}$$

then u is constant. Via Hardt and Lin's work on regularity in 1987 ([14]), we can get the estimates of Hausdorff dimension of the singular set.

## 3.3 A stability inequality, $L^p$ and $L^q$ gradient estimates on upper-half closed ellipsoids

In this section, we derive a stability inequality and $L^p, L^q$ gradient estimates which plays a dominating role in our Liouville theorems for p-harmonic, or p-stable maps developed in next section. We begin with S.W.Wei's statement about a stability inequality and $L^p, L^q$ gradient estimates on $E^n_a$. Briefly speaking, his technique is built on the argument that any p-harmonic map $u=(u_1,u_2,\cdots,u_{n+1}):M\to\bar{E}^n_{a,+}\subset R^{n+1}$ of class $C^1$ is a weak solution to

$$div(|du|^{p-2}\nabla u_i)+|du|^p u_i\rho_i=0$$

for some positive function $\rho_i, \forall 1 \leq i \leq n+1$ which are associated with the second fundamental form $A^\nu(\nabla u_i, \nabla u_i)$ and are estimated by the principal curvatures (eigenvalues of the second fundamental form A)$\lambda_1, \cdots, \lambda_{n+1}$ on $E_a^n$ characterized by $a$. Choosing an appropriate test function and applying integration by parts and the Schwarz inequality yield $L^p, L^q$ gradient estimates on $E_a^n$. Now, we improve the results of Wei's work in $E^n$ as long as we investigate that any harmonic map $u : M \to \bar{E}_+^n$ is a weak solution of the same type equation : $div(|du|^{p-2}\nabla u_i) + |du|^p u_i \rho_i = 0$ where the positive function $\rho_i$ is determined by the principle curvatures on $E^n$ characterized by $\{a_1, \cdots, a_{n+1}\}$. To be more precise, we prove that $\rho_i$ is depending on $\mu = \frac{\max(a_i)}{\min(a_i)}$ based on the estimation about the principle curvatures given by Y.B.Shen and Y.L.Pan ([43]).

**Theorem 3.2**: Let u be a p-harmonic map from a complete n-manifold $M$ into an open half-ellipsoid $E_k^+ = \{(x_1, \cdots, x_{k+1}) : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_{k+1}^2}{a_{k+1}^2} = 1 and x_{k+1} > 0\}$, or a p-stable map from $M$ into a closed half-ellipsoid $\bar{E}_+^k$. Then for each $\varphi \in C_0^\infty(M)$, we have:

1. A $p$-stability inequality

$$\int_M \rho\varphi^2 |du|^p dv \leq \int_M |\nabla \varphi|^2 |du|^{p-2} dv \cdots (4.1)$$

   where $\rho$ is a positive function.

2. An $L^p$ gradient estimate

$$\int_M \varphi^p |du|^p dv \leq C \int_M |\nabla \varphi|^p dv \cdots (4.2)$$

   where C is a constant depending only on p,k,and $\mu = \frac{max(a_i)}{min(a_i)}$.

3. An $L^q$ gradient estimate

$$\int_M \varphi^q |du|^q dv \le C_1 \int_M K\varphi^q |du|^{q-2} dv + C_2 \int_M |\nabla \varphi|^q dv \cdots (4.3)$$

where $K \geq -Ric^{M_1}$, $C_1$ and $C_2$ are constants depending only on k,p, K,$\mu$,$n =$ $dimM$, and $q > p$.

.

**Proof** By the argument given before, any p-harmonic map $u = (u_1, \cdots, u_{k+1}) :$ $M \to \overline{E}^k_+ \subset R^{k+1}$ of class $C^1$ is a weak solution to

$$div(|du|^{p-2} \nabla u_i) + |du|^p u_i \rho_i = 0 \cdots (4.4)$$

for some function $\rho_i > 0$ and $\{1 \leq i \leq (k+1)\}$. Let $A = \{x \in M; |du|(x) > 0$ and $u_{k+1}(x) > 0\}$. Applying integration by parts and Schwarz inequality, we have:

$$- \int_A \varphi^2 div(|du|^{p-2} \nabla log(u_{k+1}))dv = 2\int_A \varphi \langle \nabla\varphi, |du|^{p-2} \nabla log(u_{k+1})\rangle dv$$
$$\leq \int_A |\nabla \varphi|^2 |du|^{p-2} dv + \int_A \varphi^2 |du|^{p-2} |\nabla log(u_{k+1})|^2 dv$$

for every $\varphi \in C_0^\infty(A)$. on the other hand, in the set A we have:

$$div(|du|^{p-2} \nabla log(u_{k+1})) = \langle \nabla |du|^{p-2}, \frac{\nabla u_{k+1}}{u_{k+1}} \rangle + |du|^{p-2} \triangle log(u_{k+1})$$
$$= \frac{1}{u_{k+1}} (\langle \nabla |du|^{p-2}, \nabla u_{k+1}\rangle + |du|^{p-2} \triangle u_{k+1}) - |du|^{p-2} \frac{|\nabla u_{k+1}|^2}{u_{k+1}^2}$$
$$= \frac{1}{u_{k+1}} div(|du|^{p-2} \nabla u_{k+1}) - |du|^{p-2} |\nabla log(u_{k+1})|^2$$
$$= -\rho |du|^p - |du|^{p-2} |\nabla log(u_{k+1})|^2$$

where the last step follows from (4.4) and we have set $\rho = \rho_{k+1}$. Combining this with the preceding equation we have (4.1).

Replacing $\varphi$ by $\varphi^{\frac{p}{2}}$ in (4.1) and applying Young's inequality, we have

$$\int_A \rho\varphi^p |du|^p \leq \int_A \frac{p^2}{4}\varphi^{p-2}|\bigtriangledown\varphi|^2|du|^{p-2}dv$$

$$\leq \int_A \frac{1}{4}(p(p-2))\epsilon^{\frac{p}{p-2}}\varphi^p|du|^p + \int_A \frac{1}{2}p\epsilon^{-\frac{p}{2}}|\bigtriangledown\varphi|^p dv$$

Since the eigenvalues of the second fundamental form A are the principal curvatures $\lambda_i$, for $1 \leq i \leq k$, (4.4) and elementary linear algebra imply:

$$\rho_i a_i \geq min(\lambda_1, \cdots, \lambda_k)$$

$$\rho \geq \frac{min(\lambda_j)}{max(a_i)}$$

$$\rho \geq \frac{min(a_i)}{max(a_i)^3} \cdots (4.5)$$

Choosing $\epsilon$ sufficiently small yields the $L^p$ estimate (4.2), in which C is a constant depending on $\mu$ and p and k.

To prove Theorem 3.2(3), replace $\varphi$ by $|du|^\beta\varphi$in (4.1) to get:

$$\int_A \rho\varphi^2|du|^{p+2\beta}dv \leq \int_A [\beta|du|^{\beta-1}(\bigtriangledown|du|)\varphi + |du|^\beta\bigtriangledown\varphi]^2|du|^{p-2}dv$$

$$= \int_A |du|^{p-2}\{|\bigtriangledown\varphi|^2|du|^{2\beta} + \frac{1}{2}\bigtriangledown(\varphi^2)\bigtriangledown(|du|^{2\beta}) + \varphi^2|\bigtriangledown(|du|^\beta)|^2\}dv \cdots (4.6)$$

Integrating the third term by parts, we have:

$$\int_A |du|^{p-2}\frac{1}{2}\bigtriangledown\varphi^2\bigtriangledown(|du|^{2\beta}) = -\int_A |du|^{p-2}\varphi^2|\bigtriangledown(|du|^\beta)|^2 - \int_A |du|^{p-2+\beta}\bigtriangleup(|du|^\beta)\varphi^2$$

$$\int_A \rho\varphi^2|du|^{p+2\beta} + \int_A |du|^{p-2}|du|^\beta\bigtriangleup(|du|^\beta)\varphi^2 \leq \int_A |du|^{p-2+2\beta}|\bigtriangledown\varphi|^2 \cdots (4.7)$$

We have the following pointwise estimate on the Bochner Formula in [15]. If $u : M \to N$ p-harmonic map and S is an upper bound for the sectional curvature on

N and $-K \leq Ric^M$, then:

$$|du|^\beta \triangle (|du|^\beta) \geq \beta |du|^{2\beta-2}(\langle \triangle du, du\rangle + (\beta-1)|\nabla|du||^2 - K|du|^2 - \frac{n-1}{n} S\,|du|^4)$$

at all points where $|du| \neq 0$ and $n = dimM$. So:

$$\int_A |du|^{p-2}|du|^\beta \triangle (|du|^\beta)\varphi^2$$
$$\geq \{\beta(\beta-2)\int_A \varphi^2 |du|^{p+2\beta-4}|\nabla|du||^2$$
$$+\int_A -K\beta\varphi^2|du|^{p+2\beta-2}$$
$$-\int_A \beta\frac{n-1}{n}\frac{max(a_i)^2}{min(a_i)^4}\varphi^2|du|^{p+2\beta}\}$$

At the same time, we know:

$$\int_A \rho\varphi^2|du|^{p+2\beta} \geq \int_A \frac{min(a_i)}{max(a_i)^3}\varphi^2|du|^{p+2\beta}$$
$$\{\int_A (\frac{min(a_i)}{max(a_i)^3} - \beta\frac{n-1}{n}\frac{\mu^2}{min(a_i)^2})\varphi^2|du|^{p+2\beta}$$
$$+\int_A \beta(\beta-2)\varphi^2|du|^{p+2\beta-4}|\nabla|du||^2$$
$$+\int_A -K\beta\varphi^2|du|^{p+2\beta-2}\}$$
$$\leq \int_A \rho\varphi^2|du|^{p+2\beta} + \int_A |du|^{p-2}|du|^\beta \triangle (|du|^\beta)\varphi^2$$
$$\leq \int_A |du|^{p-2+2\beta}|\nabla\varphi|^2$$

Suppose we choose $\beta$ satisfying:

$$\frac{min(a_i)}{max(a_i)^3} - \beta\frac{n-1}{n}\frac{\mu^2}{min(a_i)^2} \geq 0$$

that is:

$$\beta \leq \frac{n}{n-1}\frac{1}{\mu^5} \cdots (4.8)$$

then we get:

$$\int_A \beta(\beta-2)\varphi^2|du|^{p+2\beta-4}|\nabla|du||^2 \le \int_A K\beta\varphi^2|du|^{p+2\beta-2} + \int_A |du|^{p-2+2\beta}|\nabla\varphi|^2 \cdots (4.9)$$

On the other hand, from the Cauchy-Schwarz inequality, (4.6) implies:

$$\int_A \rho\varphi^2|du|^{p+2\beta}dv$$

$$\le \int_A |du|^{p-2}\{|\nabla\varphi|^2|du|^{2\beta} + \frac{1}{2}(\nabla(\varphi^2))(\nabla(|du|^{2\beta})) + \varphi^2|\nabla(|du|^\beta)|^2\}$$

$$\le \int_A |\nabla\varphi|^2|du|^{p-2+2\beta} + 2\beta|\varphi||\nabla\varphi||du|^{p+2\beta-3}|\nabla|du|| + \varphi^2\beta^2|du|^{p+2\beta-4}|\nabla|du||^2$$

$$= \int_A |\nabla\varphi|^2|du|^{p-2+2\beta} + 2(|\nabla\varphi||du|^{\frac{p}{2}-1+\beta})(|\varphi|\beta|du|^{\frac{p}{2}+\beta-2}\nabla|du|) + \varphi^2\beta^2|du|^{p+2\beta-4}|\nabla|du||^2$$

$$\le 2\int_A |\nabla\varphi|^2|du|^{p-2+2\beta} + 2\beta^2\int_A \varphi^2|du|^{p+2\beta-4}|\nabla|du||^2$$

Substituting (4.9) into this above inequality, we have:

$$\int_A \rho\varphi^2|du|^{p+2\beta} \le \int_A 4\frac{\beta-1}{\beta-2}|\nabla\varphi|^2|du|^{p-2+2\beta} + \int_A \frac{2\beta^2}{\beta-2}K\varphi^2|du|^{p+2\beta-2} \cdots (4.10)$$

Young's inequality implies that:

$$\varphi^{p-2+2\beta}|\nabla\varphi|^2|du|^{p-2+2\beta}$$

$$\le \epsilon^{\frac{p+2\beta}{p-2+2\beta}}\frac{p-2+2\beta}{p+2\beta}\varphi^{p+2\beta}|du|^{p+2\beta} + \frac{1}{\epsilon^{\frac{p+2\beta}{2}}}\frac{2}{p+2\beta}|\nabla\varphi|^{p+2\beta} \cdots (4.11)$$

$$\varphi^{p+2\beta}|du|^{p-2+2\beta}$$

$$\le \epsilon^{\frac{p+2\beta}{p-2+2\beta}}\frac{p-2+2\beta}{p+2\beta}\varphi^{p+2\beta}|du|^{p+2\beta} + \frac{1}{\epsilon^{\frac{p+2\beta}{2}}}\frac{2}{p+2\beta}\varphi^{p+2\beta} \cdots (4.12)$$

Choosing $\beta$ satisfying (4.8) and replacing $\varphi$ by $\varphi^{\frac{p+2\beta}{2}}$ in (4.10), using (4.5), applying (4.11) and (4.12) for sufficiently small $\epsilon$, and setting $q = p+2\beta$, we then obtain the $L^q$ gradient estimate (4.3), proving Theorem 3.2(3).

## 3.4 Liouville theorems for p-harmonic or p-stable maps from a p-parabolic manifold $M$ into $\bar{E}^k_+$

We study the applications to p-harmonic maps from a p-parabolic manifold into $\bar{E}^k_+$. First let's recall the definitions of p-parabolic manifolds, manifolds with p-moderate volume growth, and manifolds with p-th volume growth conditions.

**Definition 3.1:**A $C^2$ function $f : M \to R$ is said to be $p$-harmonic (resp. $p$-superharmonic, $p$-subharmonic) if its $p$-Laplacian $\triangle_p f = div(|\nabla f|^{p-2}\nabla f$ = (resp. $\leq, \geq$)0.

**Definition 3.2:**A complete noncompact Riemannian manifold $M$ is said to be $p$-parabolic if it admits no nonconstant positive $p$-superharmonic function, and $p$-hyperbolic otherwise.

**Definition 3.3:**A complete noncompact Riemannian manifold $M$ is said to have $p$-moderate volume growth if there exist $x_0 \in M$ and $F(r) \in \mathcal{F}$, such that

$$\limsup_{r\to\infty} \frac{1}{r^p F^{p-1}(r)} vol(B(x_0; r)) < \infty$$

where

$$\mathcal{F} = \{F : [a, \infty) \to (0, \infty) | \int_a^\infty \frac{dr}{rF(r)} = \infty, a > 0\}$$

It should be mentioned that there is a "gap phenomenon" between a p-parabolic manifold and a manifold with the p-th or q-th volume growth condition. Actually, in this section, we can furnish an example of gap phenomena which is a p-parabolic manifold with exponential volume growth instead of p-th volume growth. Because the essence of a successful approach of Liouville type problems has its roots in $L^p$ gradient estimate which help us to control an integral of $|du|^p$ by an integral of $|\nabla\varphi|^p$ where $\varphi$ is a test function good enough, we can successfully widen the Liouville theorems from a manifold with volume growth conditions to a p-parabolic manifold.

**Proposition 3.1(A Maximum Principle for p-harmonic maps into an ellipsoid)**: Let $\Omega$ be any bounded domain in $M$ with nonempty smooth boundary $\partial\Omega$. Let $u = (u_1, \cdots, u_{k+1}) : M \to \overline{E}^k_+ \subset R^{k+1}$ of class $C^1$ be a p-harmonic map such that $u_{k+1}|_{\partial\Omega} \geq C$ where $C = \inf_{\partial\Omega} u_{k+1} \geq 0$ and $M$ be a complete noncompact Riemannian manifold and $\overline{E}^k_+$ denote a closed upper half-ellipsoid. Then, $u_{k+1} \geq C$ on $\Omega$.

**Proof**: Let $\phi = min\{u_{k+1} - C, 0\}$. If $u = (u_1, \cdots, u_{k+1}) : M \to \overline{E}^k_+ \subset R^{k+1}$ of class $C^1$ is a p-harmonic map, then it is a weak solution to

$$div(|du|^{p-2}\nabla u_i) + |du|^p u_i \rho_i = 0$$

for some function $\rho_i > 0$ and $1 \leq i \leq k+1$. Applying integration by parts, we get:

$$\begin{aligned} 0 &\leq \int_M div(|\nabla u|^{p-2}\nabla u_{k+1})\phi dv \\ &= -\int_M |\nabla u|^{p-2}\nabla u_{k+1} \cdot \nabla\phi dv \\ &= -\int_{\{u_{k+1}<C\}} |\nabla u|^{p-2}|\nabla u_{k+1}|^2 \\ &\leq 0 \end{aligned}$$

Hence, if there exists a point $x_0$ at which $u_{k+1} < C$, then $|\nabla u_{k+1}| \equiv 0$ in a neighborhood $\Omega$ of $x_0$, and $u_{k+1} \equiv$ constant$< C$ on $\Omega$, contradicting $C = \inf_{\partial\Omega} u_{k+1}$. This completes the proof.

**Theorem 3.3**: Every p-harmonic map $u : M \to \bar{E}^k_+$, whose image either lies inside an open upper half-ellipsoid $E^k_+$ or on the equator. If the image $u(M)$ lies inside $E^k_+$, then $u$ is a constant provided $M$ is a p-parabolic manifold.

**Proof**: In view of Proposition 3.1 in which $C = 0$, if there exists a point $x_0$ at which $u_{k+1} = 0$, then $u_{k+1} \equiv 0$ on any domain $\Omega$ with $x_0 \in \partial\Omega$. Hence, $u(M)$ lies on the equator.

It follows from Theorem 3.2(2) that we have an $L^p$ gradient estimate for every $\varphi \in C_0^\infty(M)$,

$$\int_M \varphi^p |du|^p dv \le C \int_M |\nabla\varphi|^p dv \cdots (*)$$

where C is a positive constant depending only on p,k,and $\mu = \frac{max(a_i)}{min(a_i)}$. For any fixed compact set K in $M$, we choose a sufficiently large $r > 0$ so that the geodesic ball $B_r$ of radius r centered at a fixed point $x_0$ covers K. Let $A(r) = B_r \backslash K$, where $r > 1$. Then there exists a unique function $u_r \equiv 1$ on the compact K and $u_r$ is vanishing on outside $B_r$ and $u_r$ is a p-harmonic function between K and $B_r$ such that $u_r$ is the p-minimizer of the p-energy functional $E_p(u)$ among all functions in the space

$$S_r = \{u \in L_1^p(A(r)) : u|_{\partial K} = 1 \text{ and } u|_{\partial B_r} = 0\}$$

Indeed, given a p-minimizing sequence $\{u_{r_i}\}$ in $S_r$, by the Poincare Inequality

$$\int_{A(r)} |\varphi|^p dv \leq C(r,p) \int_{A(r)} |\nabla\varphi|^p dv,$$

for any $\varphi \in C_0^\infty(A(r))$, the lower-semicontinuity of $E_p$, and the closeness of $S_r$ in $L_1^p$, one can extract a subsequence $\{u_{r_j}\}$ converges to $u_r \in L_1^p$ that realizes the infimium as the minimum:

$$E_p(u_r) = \inf_{u\in S_r} E_p(u)$$

Since R is the domain of a convex function, it follows that $u_r$ is $C^{1,\alpha}$ in the interior and $C^\alpha$ up to the boundary. Then by the maximum principle of p-harmonic functions we get $0 \leq u_r \leq 1$. For any $r < r'$ and $x \in A(r), u_r(x) \leq u_{r'}(x)$. By the Monotone sequence theorem, $\lim_{r\to\infty} u_r = u_\infty$ in $M\backslash\partial K$. It follows from Harnack inequality for p-harmonic functions that in a small ball we can have uniformly Hölder estimate and hence $C^{1,\alpha}$ estimate on $u_r$ by Hardt-Lin. By a diagonal subsequence argument a subsequence of $\{u_r\}$ converges in $C^{1,\alpha}$ norm to $u_\infty$ on every compact subset of $M_1\backslash K$, and $u_\infty$ is a p-harmonic function. By the assumption $M$ being p-parabolic, $u_\infty$ is a constant which implies $u_\infty \equiv 1$ by the maximum principle. Hence on every compact subset of $M\backslash\partial K$

$$\nabla u_r \to 0 \cdots (**)$$

uniformly as $r \to \infty$; and we substitute $\varphi = u_r$ into (*) and using the divergence theorem and (**), we have:

$$\int_K |du|^p dv \leq C_1 \int_{B_r\backslash K} |\nabla u_r|^p dv$$

$$= C_1 \int_{B_r \setminus K} div(u_r |\nabla u_r|^{p-2} \nabla u_r) dv$$

$$= C_1 \int_{\partial K} |\nabla u_r|^{p-2} \frac{\partial u_r}{\partial \nu} ds$$

$$= C_1 \int_{\sum} |\nabla u_r|^{p-2} \frac{\partial u_r}{\partial \nu} ds \to 0$$

as $r \to \infty$ where $\sum$ is a hypersurface between $\partial K$ and $\partial B_r$ and $\nu$ is its unit outer normal vector. The last step follows from the p-harmonicity of $u_r$ between $\partial K$ and $\sum$. It follows that $|du|^p \equiv 0$ on any compact set K and hence $u \equiv 0$ on $M$. (cf. ([52]))

**Theorem 3.4**: Every $p$-stable map $u$ from a $p$-parabolic manifold $M$ into a closed upper half-ellipsoid $\overline{E}^k_+$ is a constant. Every stable $p$-harmonic

Proof: Proceeding in the similarly way, we can get our desired from an $L^p$ gradient estimate.

**Corollary 3.1**: Every smooth $p$-harmonic map, where $p \geq 2$, on a complete n-manifold $M$ into $\overline{E}^k_+$ either lies on the equator or is a constant, provided $vol(B_r) = O(r^p)$ as $r \to \infty$.

**Corollary 3.2**: Every smooth p-stable map from a complete n-manifold $M_1$ into a closed half-ellipsoid $\overline{E}^k_+$ is constant, provided that one of the following conditions is satisfied:

1. $M$ has p-th volume growth, or

2. $M$ has a nonnegative Ricci curvature tensor and q-th volume growth where q is the same as q in Theorem 3.2(3).

**Remark 3.1:** Every Riemannian manifold M with $vol(B_r) = O(r^p)$ as $r \to \infty$ is a manifold with p-moderate volume growth. Every complete noncompact Riemannian manifold M with p-moderate volume growth is p-parabolic for $1 < p < \infty$. The p-moderate volume growth of $M$ is a sufficient condition for a p-parabolic manifold, but it is by far not a necessary one. The case of Liouville-type theorem from the domain with p-th volume growth into $E^n$ with $\{a_2 = \cdots = a_{n+1} = 1\}$ is due to S.W.Wei ([47]). The case $E^n$ with $a_1 = \cdots = a_{n+1} = 1$ in Theorem 3.3 is due to Wei-Li-Wu ([52]).

The following is a good example of a complete manifold that does not support nonconstant positive p-superharmonic function with volume growth of geodesic balls in $M$ increasing exponentially:

**Example 3.1:** (**p-parabolic manifolds with exponential volume growth**)(cf. Example 3.1 ([52]))

Let $M$ be a Euclidean space $R^n$ endowed with a metric $ds^2 = f(x)|dx|^2, x = (x_1, \cdots, x_n)$, where $f > 0$ is smooth on $R^n$, such that $f(x) = 1$ for $x_n \leq 0$ and $f(x) = \frac{1}{(x_n+1)^2 \log^2(x_n+1)}$ for $x_n \geq \frac{1}{4}$. Then $M$ is a complete Riemannian manifold that does not support a nonconstant nonnegative p-superharmonic function. In particular, $M$ is an n-parabolic manifold. But volume growth of geodesic balls in $M$ increases exponentially. Indeed, $Sec^M \leq \inf\{-(1+\log(x_n+1))^2, -(1+\log(x_n+1)+\log^2(x_n+1))\} < -1$ for $x_n > \frac{1}{4}$. Thus by Volume Comparison Theorem, there exists a sequence of points $\{y_i\} \subset M$, and numbers $\{r_i > 0\}$ such that $B(y_i; r_i)$ lies in the half space $x_n > \frac{1}{4}$, with $Vol(B(y_i; r_i)) = O(e^{r_i})$ as $r_i \to \infty$.

# Chapter 4

# Liouville theorems for $p$-stable maps into an $E^n$ with $p$-superstrongly unstable index $\omega_p$

In this chapter, we establish Liouville theorems on a $p$-SSU ellipsoid with a $p$-SSU index $\omega_p$. In the previous chapter, we also discuss Liouville problems in a closed upper-half-ellipsoid $\bar{E}^n_+$ with the boundary of an equator $A = \{(x_1, \cdots, x_{n+1}) \in E^n, x_{n+1} = 0\}$. As one of valuable techniques, this boundary $A$ generates a variation associated with the test function. After that, we can derive $L^p$ gradient estimate which is followed by Liouville type theorems. In light of Liouville theorems on an ellipsoid, it is of interest to relate this problem to the intrinsic geometry of an ellipsoid. In view of a $p$-SSU manifold, we can recapture Liouville problems in $E^n$ instead of $\bar{E}^n_+$ via $\omega_p$ which marks the birth of the importance of $\mu = \frac{\max(a_i)}{\min(a_i)}$ in $E^n$.

Aided by the work of Wei, we can identify $E^n$ with a $p$-SSU manifold together with $p$-SSU index $\omega_p$. The purpose of this chapter is to show how the $p$-SSU index $\omega_p$ affects the geometry of $E^n$.More precisely, most of geometric, analytic and topological properties of $E^n$ are characterized by $\omega_p$ depending only on $n, p$ and the ratio $\mu = \frac{\max(a_i)}{\min(a_i)}$.

In the precious chapter, via the extrinsic variational method as well as an expanded technique of Bochner, we establish $L^p$ gradient estimate leading to a Liouville theorem in terms of the volume growth condition. Compared with the method in the previous chapter, we realize that this principle of a $p$-SSU ellipsoid $E^n$ with $\omega_p$ gives a new role in Riemannian geometry to $p$-harmonic maps, $p$-stable maps, resulting in simple new proofs of Liouville-type theorems. $\omega_p$ serves as an indication of the regularity theory. Naively speaking, the closer $\omega_p$ is to 1, the easier it is to establish a Liouville theorem in terms of the volume growth condition.

## 4.1 $p$-Stability inequality, $L^p$ and $L^q$ Gradient Estimates on $p$-SSU Ellipsoids

**Theorem 4.1($p$-Stability inequality, $L^p$ and $L^q$ Gradient Estimates)**: Let $u$ be a $p$-stable map from a complete Riemannian manifold $M$ into an ellipsoid $E^n$ with $\mu < \sqrt[3]{\frac{n}{p}}$. Then for any $\xi \in C_0^\infty(M)$, we have:

1. $p$-Stability inequality:

$$\int_M \frac{\frac{n}{\mu^4} - p\mu^2}{(n+p-2)(\min(a_i))^2} \xi^2 |du|^p dv \leq \int_M |\nabla \xi|^2 |du|^{p-2} dv$$

2. $L^p$ Gradient estimate:

$$\int_M \xi^p |du|^p dv \leq C \int_M |\nabla \xi|^p dv$$

where $C$ is constant depending on $n, p, \mu$.

3. $L^q$ Gradient estimate: Let $-K$ be the lower bound of the Ricci curvature of $M$, and $q \in [p+2, p+2\frac{n(\frac{n}{\mu^6}-p)}{(n-1)(n+p-2)}]$, then

$$\int_M \xi^q |du|^q dv \leq C_1 \int_M K \xi^q |du|^{q-2} dv + C_2 \int_M |\nabla \xi|^q dv$$

where $C_1, C_2$ are constants depending on $n, p, \mu$ and $K$.

**Proof:** Combining Lemma 2.3 and Lemma 2.4 and the following theorem, we can get the desired.

**Theorem(S.W.Wei)([51]):** Let $u$ be a $p$-stable map from a complete Riemannian manifold $M$ into a $p$-SSU manifold $N$. Suppose $\xi \in C_0^\infty(M)$.

1. $p$-Stability Inequality:

$$\int_M \frac{\phi_p(u)}{(k+p-2)}\xi^2|du|^p dv \leq \int_M |\nabla\xi|^2|du|^{p-2}dv$$

2. $L^p$ Gradient Estimate: Assume that $u(M)$ lies in a compact subset $N_0$ of the $p$-SSU manifold $N$, then

$$\int_M \xi^p|du|^p dv \leq C\int_M |\nabla\xi|^p dv$$

   where $C$ is constant depending on $\phi_p, k, p$ and $N_0$.

3. $L^q$ Gradient Estimate: Let $-K$ be the lower bound of the Ricci curvature of $M$, and $q \in [p+2, p+\frac{2n\omega_p}{n-1}]$, then

$$\int_M \xi^q|du|^q dv \leq C_1\int_M K\xi^q|du|^{q-2}dv + C_2\int_M |\nabla\xi|^q dv$$

   where $C_1, C_2$ are constants depending on $k, p, \omega_p$ and $K$.

The following series of results indicate for appropriate maps into $p$-SSU manifolds how quantitatively, the closer this $p$-SSU index $\omega_p$ is to 1, the easier it is to establish the Liouville theorem in terms of the volume growth condition in the domain, and moreover, the ”smoother” the maps is in terms of the Hausdorff dimension of the singular set in the domain.

**Liouville Theorem** Every $p$-stable map from a complete Riemannian manifold $M$ into a $p$-SSU manifold $N$ is constant provided one of the following is true:

1. $M$ has $p$-th volume growth(i.e. volume $(B_r) = O(r^p)$ as $r \to \infty$), and $u(M) \subset N_0$, a compact subset of $N$.

2. $M$ has nonnegative Ricci curvature tensor and $q$-th volume growth, where $q = p + \frac{2\omega_p n}{n-1}$.

**Liouville theorem for a tangent map** Every $p$-minimizing tangent map $u$ defined on $R^n$ into a $p$-SSU manifold is constant for $n \le d(\omega_p)$ where $d(\omega_p) = [[p + 2 + 2\sqrt{p-1}]]$ if $\omega_p \ge \frac{p+2\sqrt{p-1}}{p+1+2\sqrt{p-1}}$, otherwise, $d(\omega_p) = [1 + \frac{1}{1-\omega_p}]$. Here $[t]$ denotes the greatest integer $\le t$ and $[[t]] = [t]$ if $t$ is not an integer and $[[t]] = t - 1$ if $t$ is an integer. The details of the proofs can be found in Wei’s work ([51]).

## 4.2 Liouville theorems for $p$-stbale maps into $p$-SSU ellipsoids

**Theorem 4.2(Liouville-type Theorem for the $p$-Parabolic or $q$-Parabolic Manifolds)**:

1. Every $p$-stable map $u$ from a $p$-parabolic manifold $M$ into an ellipsoid $E^n$ with $\mu < \sqrt[3]{\frac{n}{p}}$ is constant for $p \ge 2$.

2. Every $p$-stable map $u$ from a $q$-parabolic manifold $M$ with $Ric^M \geq 0$ into an ellipsoid $E^n$ with $\mu < \sqrt[3]{\frac{n}{p}}$ is constant, where

$$q \in [p+2, p+2\frac{n(\frac{n}{\mu^6} - p)}{(n-1)(n+p-2)}].$$

**Proof**: It follows from $L^p$ and $L^q$ gradient estimates in the similarly way in previous chapter.

This result arguments the work of Wei-Li-Wu for ellipsoids with one parameter $a_2 = \cdots = a_{n+1}$ ([52]).

**Corollary 4.1(Liouville-type Theorem for the Domain with a $p$-moderate Volume Growth)**: Every $p$-stable map from a complete Riemannian manifold $M$ into an ellipsoid $E^n$ with $\mu < \sqrt[3]{\frac{n}{p}}$ is constant provided one of the following is true:

1. $M$ has $p$-moderate volume growth

2. $M$ has nonnegative Ricci curvature tensor and $q$-moderate volume growth, where

$$q = p + 2\frac{n(\frac{n}{\mu^6} - p)}{(n-1)(n+p-2)}$$

**Proof:** This follows at once from Theorem 4.2 by Remark 3.1.

# Chapter 5

# Non-existence of stable currents on ellipsoids

The first results about the nonexistence of stable submanifolds seems to be in J.Simons's work that there are no oriented close stable hypersurfaces in an oriented manifold with positive Ricci tensor([42]). J.Simons and Lawson prove that the standard sphere $S^n$ contains no stable submanifolds. Along this way, Lawson and Simons made an conjecture ([24]):

**Conjecture**(L-S) :There are no closed stable submanifolds in any compact, simply connected, strictly $\frac{1}{4}$-pinched Riemannian manifold.

The work of Howard and S.W.Wei ([33]) links the relation between the nonexistence of stable currents and the curvature of ambient manifolds. They study how the geometry of a Riemannian manifold $M$ affects the nonexistence of stable currents in $M$ by giving conditions on the second fundamental form of $M$ isometrically imbedded in a Euclidean space. Based on their work, we can obtain the nonexistence results in $E^n$.

## 5.1 Preliminaries

In the following section, we will consider how the geometry of a Riemannian manifold $M^n$ effects the non-existence of the stable currents in $M^n$. In particular, we will give conditions on the second fundamental form of an immersed manifold $M^n$ of a Euclidean space which implies that $M^n$ supports no stable currents in dimension $k$. First, we talk about the above problem in a general manifold. Second, we restrict it to an ellipsoid. Our basic method is to isometrically immerse the manifold $M^n$ in a Euclidean space $R^{n+m}$. Then for any parallel vector field V on $R^{n+m}$ and compact $k$-dimensional currents $N^k$ of $M^n$ we deform N along the flow $\phi_t^{V^\perp}$ of the vector field $V^\top$ obtained by taking the orthogonal projection of V onto tangent spaces of $M^n$. And the flow $\phi_t^{V^\top}$ is the flow generated by $V^\top$, i.e. $\frac{d}{dt}(\phi_t^{V^\top}(x)) = V^\top(\phi_t(x))$. In this case the formula for the second variation

$$\frac{d^2}{dt^2}|_{t=0} = vol(\phi_t^{V^\top} N) \cdots (*)$$

can be greatly simplified. The resulting formula still is hard to compute. But if (*) is averaged over an orthonormal basis $V_1, \cdots, V_{n+m}$ of $R^{n+m}$ ), it is easy to compute. The result is a pointwise algebraic condition on the second fundamental form of $M^n$(corresponding to the average of (*) over $V_1, \cdots, V_{n+m}$) which implies $M^n$ has no stable currents in dimension $k$. By applying the above method, we have:

**Lemma 5.1 (Trace Formulas by R.Howard and S.W.Wei)([33]):** Suppose we have $N^k \subset M^n \hookrightarrow R^{n+m}$.

$$trace(Q_g) = \int_{M^n} trace(Q_\xi) d||g||(x)$$

and if g is stable then $trace(Q_\xi) \geq 0$. If $\{e_1, \cdots, e_{n+m}\}$ is an orthonormal basis of $R^{n+m}$ such that $\{e_1, \cdots, e_n\}$ is an o.n.b. of $TM_x, \{e_{n+1}, \cdots, e_{n+m}\}$ is a basis of the normal space of $TM_x$ in $R^{n+m}$, and $\xi = e_1 \wedge \cdots \wedge e_k$, then

$$trace(Q_\xi) = \sum_{i=1}^{k} \sum_{r=k+1}^{n} (2\|h(e_i, e_r)\|^2 - h(e_i, e_i)h(e_r, e_r))$$

And if for some k, $trace(Q_\xi) < 0$ holds for all $\xi = e_1 \wedge \cdots \wedge e_k$ tangent to $M^n$, thus $M^n$ has no closed stable submanifolds of dimension $k$.

## 5.2 Nonexistence of stable currents or submanifolds in ellipsoids

In this section, by applying Trace Formulas, we consider how the geometry of an ellipsoid effects the nonexistence of the stable submanifolds on $E^n$.

**Theorem 5.1 (Nonexistence of stable submanifolds in an ellipsoid**:) An ellipsoid $E^n$ has no closed stable $k$-dimensional submanifold if $\mu^4 \leq \frac{n-k+2}{2}$, In particular, there are no closed stable $k$-dimensional submanifold in a sphere $S^n(r)$ for any $1 \leq k < n$.

**Proof:** Let $N$ be a k-dimensional submanifold in $E^n$. Suppose $\{e_1, \cdots, e_k\}$ is an o.n.b. in $TN_x$, $\{e_1, \cdots, e_n\}$ is an o.n.b. in $TE^n_x$, and $\{e_1, \cdots, e_n, \nu\}$ is an o.n.b. in $R^{n+1}$. For any fixed $i$, $(1 \leq i \leq k)$, we have:

$$\frac{1}{\|Z\|}\nabla_{e_i}Z = \sum_{j=1}^{n}\langle\frac{1}{\|Z\|}\nabla_{e_i}Z, e_j\rangle e_j + \langle\frac{1}{\|Z\|}\nabla_{e_i}Z, \nu\rangle\nu$$

$$= \sum_{j=1}^{n} h_{e_ie_j}e_j + \langle\frac{1}{\|Z\|}\nabla_{e_i}Z, \nu\rangle\nu$$

$$|\frac{1}{\|Z\|}\nabla_{e_i}Z|^2 \geq \sum_{j=1}^{n}|h_{e_ie_j}|^2$$

$$\sum_{r=k+1}^{n} |h_{e_ie_r}|^2| \leq |\frac{1}{||Z||}\nabla_{e_i} Z|^2 - \sum_{j=1}^{k} |h_{e_ie_j}|^2$$

Applying Lemma 5.1, we get:

$$trace(Q_\xi) = 2\sum_{i=1}^{k}[\sum_{r=k+1}^{n} |h_{e_ie_r}|^2] - \sum_{i=1}^{k}\sum_{r=k+1}^{n} h_{e_ie_i}h_{e_re_r}$$

$$\leq 2\sum_{i=1}^{k}[\frac{1}{||Z||^2}\frac{1}{(min_{1\leq sn+1}\{a_s\})^4} - \sum_{j=1}^{k} |h_{e_ie_j}|^2] - \frac{1}{||Z||^2}\frac{1}{(max_{1\leq s\leq n+1}\{a_s\})^4}k(n-k)$$

$$= \frac{1}{||Z||^2}[\frac{2k}{(min_{1\leq sn+1}\{a_s\})^4} - \frac{k(n-k)}{(max_{1\leq s\leq n+1}\{a_s\})^4}] - 2\sum_{i,j=1}^{k} |h_{e_ie_j}|^2$$

$$= \frac{1}{||Z||^2}[\frac{2k}{(min_{1\leq sn+1}\{a_s\})^4} - \frac{k(n-k)}{(max_{1\leq s\leq n+1}\{a_s\})^4}] - 2\sum_{i=j=1}^{k} |h_{e_ie_j}|^2 - 2\sum_{i\neq j} |h_{e_ie_j}|^2$$

$$\leq \frac{1}{||Z||^2}[\frac{2k}{(min_{1\leq sn+1}\{a_s\})^4} - \frac{k(n-k)}{(max_{1\leq s\leq n+1}\{a_s\})^4}] - 2k\frac{1}{||Z||^2}\frac{1}{(\max_{1\leq s\leq n+1}\{a_s\})^4}$$

$$= \frac{k}{||Z||^2}\{\frac{2}{(min_{1\leq sn+1}\{a_s\})^4} - \frac{n+2-k}{(max_{1\leq s\leq n+1}\{a_s\})^4}\}$$

($\frac{1}{(max_{1\leq s\leq n+1}\{a_s\})^2} \leq |\nabla_Y z| \leq \frac{1}{(min_{1\leq sn+1}\{a_s\})^2}$ and $\frac{1}{||Z||(max_{1\leq s\leq n+1}\{a_s\})^2} \leq h_{YY} \leq \frac{1}{||Z||(min_{1\leq sn+1}\{a_s\})^2}$

(cf. the proof of Lemma 2.1)

Hence, $\mu^4 \leq \frac{n+2-k}{2}$, then $trace(Q_\xi) < 0$.

**Remark 5.1:** The above argument can be generalized to a stable currents or rectifiable varifolds (cf. ([33])).

# Chapter 6

# Applications to Yang-Mills Instability

**Theorem 6.1(Yang-Mills instability of an ellipsoid):** An ellipsoid $E^n$ with $\mu < \sqrt[3]{\frac{n-2}{2}}$ is Yang-Mills instable.

**Proof:**Suppose $\{\lambda_i\}$ is a family of principal curvatures on an ellipsoid $E^n$.By Lemma 2.2, we have:

$$\lambda_i + \lambda_j \leq 2\frac{\max(a_i)}{(\min(a_i))^2}$$

$$(n-2)\frac{\min(a_i)}{(\max(a_i))^2} \leq \sum_{s=1,s\neq i,j}^{n} \lambda_s$$

for all pair $i \neq j$. Hence, if

$$2\frac{\max(a_i)}{(\min(a_i))^2} < (n-2)\frac{\min(a_i)}{(\max(a_i))^2}$$

we know that $E^n$ is Yang-Mills instable (cf. Theorem (5.3) ([23]))

**Remark 6.1**: On $E^n = E^n_a$,(i.e. $E^n_a = \{(x_1, \cdots, x_{n+1}) \in R^{n+1}; ax_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1, a > 0\}$), the estimate becomes $1 \leq \mu = a < \sqrt[3]{\frac{n-2}{2}}$ for Yang-Mills instability. This implies the estimate $a < n-3$ due to Shoshichi Kobayashi, Yoshihiro Ohnita and Masaru Takeuchi (cf. Example (2) ([23])).

# Chapter 7

# A compact convex hypersurface in a Euclidean space

In this section, we will discuss the regularity theory, existence and nonexistence theorems, and representing homotopy groups and spaces of p-harmonic maps in a compact convex hypersurface in a Euclidean space, generalizing our previous results in $E^n$.

**Theorem 7.1**: Suppose $N^m$ is a compact convex hypersurface in $R^{m+1}$. $\mu_0$ is defined as in Definition 0.8 then:

1. If $p \leq \frac{m}{\mu_0}$, then $N^m$ is $p$-SSU.

2. the index form $\omega_p \geq \frac{\frac{m}{\mu_0^2} - p}{m+p-2}$.

**Proof**: $\forall x \in N^m$, suppose the principle curvatures satisfy: $0 \leq \lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_m(x)$ Well,

$$\frac{(\lambda_1(x) + \cdots + \lambda_m(x))}{p} \geq \frac{m\lambda_1}{p} \geq \lambda_m \geq \lambda_m(x).$$

So, we know that $N^m$ is p-SSU from the definition. Next, we compute the p-SSU index $\omega_p$ as follows:

$$\omega_p = \inf_{y \in N^m} \frac{\phi_p(y)}{(m + p - 2)S(y)}$$

where $S(y) = \sup Riem^{N^m}(y) \leq \lambda_m^2$

$$\phi_p(y) = \inf_{\|X\|=1, X \in T_y(N^m)} (-F_{p,y}(X))$$

$$-F_{p,y}(X) = -(p-2)|h_{XX}|^2 + h_{XX}H - 2\sum_{j=1}^{m} |h_{X,e_j}|^2$$

$$\geq -(p-2)\lambda_m^2 + m\lambda_1^2 - 2\lambda_m^2$$

$$= m\lambda_1^2 - p\lambda_m^2$$

where let $X = \sum_{i=1}^{m} \beta_i e_i$ with $\sum_{i=1}^{m} \beta_i^2 = 1$,

$$h_{X,e_j} = \sum_{i=1}^{m} \beta_i h_{ij} = \sum_{i=1}^{m} \beta_i \lambda_i \delta_{ij} = \beta_j \lambda_j,$$

$$\sum_{i=1}^{m} |h_{X,e_j}|^2 = \sum_{j=1}^{m} \beta_j^2 \lambda_j^2 \leq (\sum_{j=1}^{m} \beta_j^2)\lambda_m^2 = \lambda_m^2.$$

$$\omega_p \geq \frac{m\lambda_1^2 - p\lambda_m^2}{(m+p-2)\lambda_m^2} = \frac{\frac{m}{\mu_0^2} - p}{m+p-2}.$$

**Theorem 7.2**: Let $N^m$ be a compact convex hypersurface in $R^{m+1}$. Suppose $N_1$ is a minimal $k$-submanifold of $N^m$ such that the Ricci curvature $Ric^{N_1}$ satisfies:

$$Ric^{N_1} > k(1 - \frac{1}{p})\lambda_m^2$$

where $\lambda_m$ is defined as Definition 0.8. Then $N_1$ is $p$-SSU.

**Proof**: For any unit vector $X \in T_x(N_1)$, choose an orthonormal frame field $\{\alpha_1, \cdots, \alpha_k\}$ on $N_1$ such that at point x, $X = \alpha_k$. For $1 \leq i, j \leq k$, let $R_{ij}(resp.\bar{R}_{ij})$ denote the sectional curvature of $N_1(resp.N^m)$ for the section $\alpha_i \wedge \alpha_j$. Let $h_{ij} = (\nabla_{\alpha_i}\alpha_j)^{\perp}$ where $\nabla$ is the Riemannian connection on $N^m$ and $()^{\perp}$ is the projection onto the normal space $T(N_1)^{\perp}$. The fact that $N_1$ is minimal says:

$$\sum_{i=1}^{k} h_{ii} = 0 \cdots (1)$$

Let $\bar{h}(Y,Z) = \langle \bar{\nabla}_Y Z, \nu \rangle$ where $\bar{\nabla}$ is the Riemannian connection on $R^{m+1}$, Y and Z are local vector fields on $N^m$, and $\nu$ is the unit normal vector to $N^m$. And the second fundamental form $\widetilde{h}$ of $N_1$ in $R^{m+1}$ splits into:

$$|\widetilde{h}(X,X)|^2 = |h(X,X)|^2 + |\bar{h}(X,X)|^2 = h_{kk}^2 + |\bar{h}(X,X)|^2 \cdots (2)$$

Applying the Gauss-curvature equation, we have:

$$R_{ik} = \bar{R}_{ik} + \langle h_{ii}, h_{kk} \rangle - h_{ik}^2, 1 \le i \le k \cdots (3)$$

Summing (3) over $1 \le i \le k$ and applying (1), we have:

$$Ric(X) = \sum_{i=1}^{k-1} \bar{R}_{ik} - \sum_{i=1}^{k} h_{ik}^2 \cdots (4)$$

In view of (4), our assumption of $Ricci^{N_1}$ implies:

$$\sum_{i=1}^{k-1} \bar{R}_{ik} - \sum_{i=1}^{k} h_{ik}^2 = Ric(X) > k(1 - \frac{1}{p})\lambda_m^2$$

On the other hand, we have:

$$\sum_{i=1}^{k-1} \bar{R}_{ik} \le (k-1)Riem^{N^m} \le (k-1)\lambda_m^2$$

$$h_{kk}^2 \le \sum_{i=1}^{k} h_{ik}^2 \le (k-1)\lambda_m^2 - k(1 - \frac{1}{p})\lambda_m^2 = (\frac{k}{p} - 1)\lambda_m^2 \cdots (5)$$

At the same time, we have:

$$\begin{aligned}\langle Q_x^{N_1}(X), X \rangle &= \langle -2Ric(X), X \rangle + (\sum_{i=1}^{k} \langle \bar{\nabla}_{\alpha_i} \nu, \alpha_i \rangle) \langle \bar{\nabla}_X \nu, X \rangle \\ &< -2k(1 - \frac{1}{p})\lambda_m^2 + k\lambda_m^2 \\ &= k(\frac{2}{p} - 1)\lambda_m^2\end{aligned}$$

$$F_{p,x}(X) = (p-2)|\tilde{h}_{kk}|^2 + \langle Q_x^{N_1}(X), X\rangle$$

$$= (p-2)(|h_{kk}|^2 + |\bar{h}_{XX}|^2) + \langle Q_x^{N_1}(X), X\rangle$$

$$< (p-2)[(\frac{k}{p} - 1)\lambda_m^2 + \lambda_m^2] + k(\frac{2}{p} - 1)\lambda_m^2 = 0$$

Hence, $N_1$ is p-SSU.

**Theorem 7.3**: Let $u$ be a $p$-stable map from a complete Riemannian manifold $M$ into a compact convex hypersurface $N^m$ in $R^{m+1}$ with $p \leq \frac{m}{\mu_0}$. Then for any $\xi \in C_0^\infty(M)$, we have:

1. $p$-stability inequality: $\int_M \frac{m\lambda_1^2 - p\lambda_m^2}{m+p-2}\xi^2|du|^p dv \leq \int_M |\nabla\xi|^2|du|^{p-2}dv$
2. $L^p$ gradient estimate: $\int_M \xi^p|du|^p dv \leq C\int_M |\nabla\xi|^p dv$ where $C$ is constant depending on $\lambda_1, \lambda_m, m$, and $p$.
3. $L^q$ gradient estimate: Let $-K$ be the lower bound of the Ricci curvature of $M$, and $q \in [p+2, p+\frac{2m(\frac{m}{\mu_0^2}-p)}{(m-1)(m+p-2)}]$, then $\int_M \xi^q|du|^q dv \leq C_1\int_M K\xi^q dv + C_2\int_M |\nabla\xi|^q dv$ where $C_1, C_2$ are constants depending on $m, p, \mu_0$, and $K$.

**Liouville Theorem 7.4**:

1. Every $p$-stable map $u$ from a $p$-parabolic manifold $M$ into a compact convex hypersurface $N^m$ in $R^{m+1}$ with $p \leq \frac{m}{\mu_0}$ is a constant where $p \geq 2$.
2. Every $p$-stable map $u$ from a $q$-parabolic manifold $M$ with $Ric^M \geq 0$ into a complete compact hypersurface $N^m$ in $R^{m+1}$ with $p \leq \frac{m}{\mu_0}$ is a constant, where $q \in [p+2, p+\frac{2m(\frac{m}{\mu_0^2}-p)}{(m-1)(m+p-2)}]$.

**Theorem 7.5**: Suppose $N^m$ is a complete compact convex hypersurface in $R^{m+1}$. If $u$ is a $p$-harmonic map from a compact Riemannian manifold $M$ into $N^m$ and $u(M) \subset B_{(\frac{\pi}{2\lambda_m})}$ where $\lambda_m$ is defined in Definition 0.8, then $u$ is constant.

**Theorem 7.6**: Suppose $N^m$ is a compact convex hypersurface in $R^{m+1}$, if $p \leq \frac{m}{\mu_0}$ where $\mu_0$ is defined as in Definition 0.8, we have the following properties:

1. Any $p$-harmonic stable map $\varphi$ from $N^m$ into a compact Riemannian manifold is constant.

2. Any $p$-harmonic stable map $\varphi$ from a compact Riemannian manifold into $N^m$ is constant.

3. Any $p$-harmonic map from $N^m$ into a compact Riemannian manifold with nonpositive sectional curvature is constant.

4. The identity map $Id|_{N^m}$ is p-unstable.

5. $N^m$ is [p]-connected. i.e. $\pi_1(N^m) = \pi_2(N^m) = \cdots = \pi_{[p]}(N^m) = 0$

6. $\inf\{E_p(\eta) : \eta$ is homotopic to a map $\xi : M \to N^m\} = \inf\{E_p(\eta) : \eta$ is homotopic to $\xi : N^m \to K\} = 0$ for any compact manifold M and any map $\xi : M \to N^m$; for any complete manifold K and any map $\xi : N^m \to K$ where $E_p(\eta)$ denotes p-energy for a map $\eta$.

7. $C^1(M, N^m)$ is dense in $L_1^p(M, N^m)$ for any compact manifold $M$ with possibly nonempty boundary.

8. For any $p' < dimM$ and $2 \leq p' \leq (p+1)$, and any given boundary data $\eta \in L^{p'}_{1-\frac{1}{p'}}(\partial M, N^m)$, the Dirichlet problem for $p'$-harmonic maps has a solution $u_0 \in L_1^{p'}(M, N^m)$ minimizing $p'$-energy in the class of maps $u \in L_1^{p'}(M, N^m)$ with the trace $u|_{\partial M} = \eta$ which is locally Hölder continuous up to $\partial M$, and is $C^{1,\alpha}$ in the interior of M off the singular set of Hausdorff dimension $\leq dimM - p' - 1$.

.

**Theorem 7.7**: Let $N^m$ be a compact convex hypersurface in $R^{m+1}$ and $N^m$ be a $p$-SSU m-manifold with an appropriate $p$-SSU index $\omega_p$. Then $N^m$ is geometrically $(d(\omega_p, \delta) - 1) - connected$. Hence:

1. $N^m$ is $(d(\omega_p, \delta) - 1) - connected$.

2. $C^1(M, N^m)$ is dense in $L_1^{d(\omega_p,\delta)-1}(M, N^m)$ for any compact manifold M with possibly nonempty boundary.

3. Each class in $\pi_{d(\omega_p,\delta)}(N^m)$ can be represented by a sum of $C^{1,\alpha}$ p-harmonic map of $S^{d(\omega_p,\delta)}$ into $N^m$. Furthermore, if the class is nontrivial, these nonconstant p-harmonic representatives are p-unstable.

4. Each class in $\pi_{d(\omega_p,\delta)}(N^m)$ can be represented by a sum of $C^{1,\alpha}$ $d(\omega_p,\delta)$-harmonic map of $S^{d(\omega_p,\delta)}$ into $N^m$.

5. For any $p^{'} < dimM$ and $2 \leq p^{'} \leq d(\omega_p,\delta)$, and any given boundary data $\eta \in L^{p^{'}}_{1-\frac{1}{p^{'}}}(\partial M, N^m)$, the Dirichlet problem for $p^{'}$-harmonic maps into $N^m$ has a solution, which is locally Hölder continuous up to $\partial M$ and is $C^{1,\alpha}$ in the interior of M off the singular set of Hausdorff dimension $\leq dimM - p^{'} - 1$.

6. In particular, the Dirichlet problem for p-harmonic maps into $N^m$ has a solution with singular set of Hausdorff dimension $\leq dimM - d(\omega_p,\delta) - 1$.

7. For any $2 \leq p^{'} \leq d(\omega_p,\delta) - 1$, the homotopy class of any map to or from $N^m$ contains a map of arbitrarily small $p^{'}$-energy.

8. If $m \leq 2d(\omega_p,\delta) - 1$, then $N^m$ is homeomorphic to a sphere.

where for each constant $0 \leq \delta < 1$, and each $p$-SSU index $\omega_p$ on a $p$-SSU manifold $M$, we can get an integer:

$$d(\omega_p,\delta) = 2 + [[p + 2\sqrt{p-\delta}]]$$

if $\omega_p \geq \frac{p+1-\delta+2\sqrt{p-\delta}}{p+1+2\sqrt{p-\delta}}$;

$$d(\omega_p,\delta) = max\{1 + [\frac{\delta}{1-\omega_p}], [p]\}$$

otherwise. Here $[[t]]$ is the greatest integer $[t]$of t, if t is not an integer; otherwise $[[t]] = t - 1$.

# Chapter 8

# $p$-Harmonic Generalizations of the Uniformization theorem and Bochner's Method, and Geometric Applications ([52])

It is well known that on a complete Riemannian manifold, any $\mathbf{L^2}$ harmonic function is **constant**. A proof of this result can be based on essentially the arguments of Andreotti-Vesentini ([2]), or Stampacchia inequality ( ([2]) Proposition 3 on page 90). R. Greene and H. Wu prove the following:

**Theorem (Greene-Wu ([12]))** *If for some $p \ge 1$, $f$ is a continuous nonnegative $\mathbf{L^p}$ subharmonic function on a complete Riemannian manifold $M$ of nonnegative sectional curvature outside a compact set , then* $\mathbf{f \equiv 0}$ .

A similar result was obtained by Yau ( ([54]) Theorem 3, and Appendix ) for $p > 1$, and $f \in C^\infty$ without any curvature condition. More precisely, he showed that if $f$ is a smooth nonnegative subharmonic function on $M$ with $\liminf_{\mathbf{r}\to\infty} \frac{\mathbf{1}}{\mathbf{r}} \int_{\mathbf{B(x_0;r)}} \mathbf{f^p dv} = \mathbf{0}$ for some $p > 1$ and some $x_0$ in $M$ , then $f$ is **constant.** In particular, it follows that if $f$ is in $L^p$ then $f$ is constant. Here and throughout this paper, unless stated otherwise, we assume $M$ is a complete manifold with volume element $dv$ , and denote the geodesic ball of radius $r$ centered at $x_0$ in $M$ by $B(x_0;r)$ , its volume by $\text{vol}(B(x_0;r))$ , and its boundary by $\partial B(x_0;r)$ (later we suppress $x_0$ and denote them by $B(r)$ , $\text{vol}(B(r))$ , and $\partial B(r)$ respectively.)

Ahlfors ([1]) and Nevanlinna ([28]) using the conformal mapping argument prove that every positive superharmonic function on a Riemann surface $M$ is **constant**, or equivalently $M$ is **parabolic**, if for some $x_0 \in M$, and $a > 0$,

$$\int_a^\infty \frac{1}{\mathrm{vol}(\partial B(x_0; r))} dr = \infty \, . \tag{8.1}$$

J. Milnor ([27]), R. Osserman ([29])([30]), Blanc-Fiala-A. Huber ([17]), Greene-Wu ([13]), P. Yang ([57]), J. Jenkins ([21]), H. Huber ([18]), and R. Rüedy ([35]) have made further studies on deciding whether or not a surface is parabolic or hyperbolic under a curvature or analytic assumption. In general dimensions, Cheng-Yau ([6]) prove that $M$ is **parabolic** if $\limsup_{r\to\infty} \frac{1}{r^2}\mathrm{vol}(B(x_0; r)) < \infty$, based on the following Cheng-Yau estimate: Let $f$ be a positive $C^2$ function on $B(x_0; b)$ and suppose $\int_{B(x_0;r)} \frac{\Delta f}{f} dv$ is decreasing in $r \geq a$. Then for any sequence $a = r_0 < r_1 < \cdots < r_n = b$,

$$\int_{B(x_0;a)} |\nabla \log f|^2 dv \leq \int_{B(x_0;a)} \frac{\Delta f}{f} dv + \left(\sum_{j=0}^{n-1}\left(\frac{(r_{j+1} - r_j)^2}{\mathrm{vol}(B(x_0; r_{j+1})) - \mathrm{vol}(B(x_0; r_j))}\right)\right)^{-1} . \tag{8.2}$$

Karp then proves the following:

**Theorem (Karp([22]))** *Every continuous nonnegative subharmonic function $f$ on $M$ is either* **constant or** *both* $\liminf_{r\to\infty} \frac{1}{r^2} \int_{B(x_0;r)} f^p dv = \infty$ *and*

$$\limsup_{r\to\infty} \frac{1}{r^2 F(r)} \int_{B(x_0;r)} f^p dv = \infty \tag{8.3}$$

*hold for every $p > 1$, every $x_0 \in M$, and every positive* **nondecreasing** *function $F$ satisfying $\int_a^\infty \frac{dr}{rF(r)} = \infty$ for some $a > 0$.*

Karp also refines the results of Greene-Wu, Blanc-Fiala-A. Huber and others for surfaces, and of Cheng-Yau for Riemannian manifolds, by showing that $M$ is parabolic if $M$ has **moderate volume growth**, i.e. there exist $x_0 \in M$, and a function $F(r)$ as above, such that

$$\limsup_{r\to\infty} \frac{1}{r^2 F(r)} \mathrm{vol}(B(x_0;r)) < \infty\,.$$

This result is further refined to state that $M$ is parabolic, if for some $x_0 \in M$, and $a > 0$,

$$\int_a^\infty \frac{r}{\mathrm{vol}(B(x_0;r))} dr = \infty\,,$$

by Grigor'yan ([10]) ([11]) and Varopoulos ([46]), or if (8.1) holds, for some $x_0 \in M$, and $a > 0$, by Lyons-Sullivan ([25]), Varopoulos ([46]), Grigor'yan ([10]) ([11]), Sturm ([37]), and Carron ([3]) for Riemannian manifolds. Sturm ([36]) generalizes and further refines the works of Greene-Wu, Yau and Karp by showing that if either $p > 1$ and $f$ is a nonnegative subharmonic function , or $p < 1$ and $f$ is a nonnegative superharmonic function, then $f$ is **constant or**

$$\int_a^\infty \frac{r}{\int_{B(x_0;r)} f^p dv} dr < \infty \tag{8.4}$$

holds for every $x_0 \in M$, and $a > 0$.

The celebrated uniformization theorem of F. Klein, P. Keobe and H. Poincaré is a classification theorem that sharply divides complete noncompact surfaces into parabolic and hyperbolic ones and enables us to solve many geometric variational problems on surfaces(e.g.([9]),([8]),([31]),([15]),([50])). However, in dealing with higher dimensional geometric problems, the scope of the uniformization theorem and its related Laplace operator needs to be widened. For example, the manifolds $R^3$ and $R^4$ are both hyperbolic, and therefore can not be distinguished from each other in this way, while manifolds with Ricci curvature bounded below by a nonpositive constant behave like the Euclidean space $R^n$ from the viewpoint of harmonic functions(cf. ([55])) This motivates us to study the geometric significance and applications of the $p$-Laplace operator $\Delta_p$ $\Big($ defined by $\Delta_p f = \operatorname{div}(|\nabla f|^{p-2}\nabla f)$, where $\nabla f$ denotes the gradient of $f$ $\Big)$, as well as its perturbation, the $\mathcal{A}$-operator, which does not arise from the ordinary Laplace operator $\Delta$. This also motivates us, for our subsequent study, to state

**A Generalized Uniformization Theorem:** *A complete noncompact Riemannian manifold is either* $p$-**parabolic** *or* $p$-**hyperbolic**, *for* $1 < p < \infty$ .

Recall a $C^2$ function $f : M \to R$ is said to be $p$-**harmonic** ( resp. $p$-**superharmonic,** $p$-**subharmonic** ) if its $p$-Laplacian $\Delta_p f =$ ( resp. $\le$, $\ge$) $0$, and a complete noncompact Riemannian manifold $M^n$ is said to be $p$-**parabolic** if it admits no nonconstant positive $p$-superharmonic function, and $p$-**hyperbolic** otherwise. Accordingly, $R^3$ is 3-parabolic, but $R^4$ is not; and the hyperbolic space $H^n$ is $n$-hyperbolic, whereas $n$-manifolds with $Ric^M \ge 0$ are $n$-parabolic.

We initiate this paper by observing that while Karp optimally refines Caccioppoli inequalities and manipulates them beautifully in an algebraic and analytic manner, his assumption in (8.3) that the auxiliary function $F$ is nondecreasing can be removed, and the new result becomes optimal and is equivalent to the Sturm's result (8.4). Distilling further the basic ideas of Karp inequalities, we develop a unified method to refine and generalize the Cheng-Yau estimate and all other results displayed above, as well as their $L^p$ versions due to I.Holopainen ([16]) T. Coulhon, I.Holopainen, and L. Saloff-Coste ([7]). In fact, we find a dichotomy between constancy and "infinity" of weak sub- and supersolutions of a large class of degenerate and singular nonlinear partial differential equations on complete noncompact Riemannian manifolds, by introducing the concepts of their corresponding *"p-finite,* $p$-mild, $p$-obtuse, $p$-moderate, *and* $p$-small*" growth*, and their counter-parts ***"p-infinite, p-severe, p-acute, p-immoderate,* and *p-large*" growth**:

**Definition 8.1**A function $f$ has *p-finite growth* (or, simply, *is p-finite*) if there exists $x_0 \in M$ such that

$$\liminf_{r\to\infty} \frac{1}{r^p} \int_{B(x_0;r)} |f|^q dv < \infty\,, \tag{8.5}$$

and has *p-infinite growth* (or, simply, *is p-infinite*) otherwise.

**Definition 8.2**A function $f$ has *p-mild growth* (or, simply, *is p-mild*) if there exists $x_0 \in M$, and a strictly increasing sequence of $\{r_j\}_0^\infty$ going to infinity, such that for every $l_0 > 0$, we have

$$\sum_{j=\ell_0}^{\infty} \left(\frac{(r_{j+1}-r_j)^p}{\int_{B(x_0;r_{j+1})\setminus B(x_0;r_j)} |f|^q dv}\right)^{\frac{1}{p-1}} = \infty\,, \tag{8.6}$$

and has *p-severe growth* (or, simply, *is p-severe*) otherwise.

**Definition 8.3** A function $f$ has *p-obtuse growth* (or, simply, *is p-obtuse*) if there exists $x_0 \in M$ such that for every $a > 0$, we have

$$\int_a^\infty \left(\frac{1}{\int_{\partial B(x_0;r)} |f|^q dv}\right)^{\frac{1}{p-1}} dr = \infty\,, \tag{8.7}$$

and has *p-acute growth* (or, simply, *is p-acute*) otherwise.

**Definition 8.4** A function $f$ has *p-moderate growth* (or, simply, *is p-moderate*) if there exist $x_0 \in M$, and $F(r) \in \mathcal{F}$,such that

$$\limsup_{r\to\infty} \frac{1}{r^p F^{p-1}(r)} \int_{B(x_0;r)} |f|^q dv < \infty\,, \tag{8.8}$$

and has *p-immoderate growth* (or, simply, *is p-immoderate*) otherwise, where

$$\mathcal{F} = \{F : [a,\infty) \longrightarrow (0,\infty) | \int_a^\infty \frac{dr}{rF(r)} = \infty \ \ for \ \ some \ \ a \geq 0\}\,. \tag{8.9}$$

(Notice that the function in $\mathcal{F}$ are not necessarily monotone.)

**Definition 8.5**A function $f$ has *p-small growth* (or, simply, *is p-small*) if there exists $x_0 \in M$ , such that for every $a > 0$ ,we have

$$\int_a^\infty \left(\frac{r}{\int\limits_{B(x_0,r)} |f|^q dv}\right)^{\frac{1}{p-1}} dr = \infty\,, \tag{8.10}$$

and has *p-large growth* (or, simply, *is p-large*) otherwise.

The above definition of "*p-finite, p-mild, p-obtuse, p-moderate, p-small*" and "*p-infinite, p-severe, p-acute, p-immoderate, p-large*" growth depends on $q$, and $q$ will be specified in the context in which the definition is used. This has extended the scope of previous $L^2$ or $L^p$ function growth due to Andreotti-Vesentini ([2]), Greene-Wu ([12]), Yau ([54]) and others, via moderate growth due to Karp ([22]), to 2-*moderate* growth and more generally "*p-finite, p-mild, p-obtuse, p-moderate,* and *p-small*" growth. Such a dichotomy between constancy and "infinity" also occurs for $\mathcal{A}-$sub- and $\mathcal{A}-$ superharmonic functions, which are central to the theory of quasiconformal and quasiregular maps, and for a global version of a Reverse Cheng-Yau inequality on complete noncompact Riemannian manifolds. Here and throughout this paper, unless stated otherwise, we denote by $\mathcal{A}$ a measurable cross section in the bundle whose fiber at a.e. $x$ in $M$ is a continuous map $\mathcal{A}_x$ on the tangent space $T_x(M)$ into $T_x(M)$. We assume further that there are constants $1 < p < \infty$ and $0 < \alpha \leq \beta < \infty$ such that for a.e. $x$ in $M$ and all $h \in T_x(M)$,

$$\langle \mathcal{A}_x(h), h \rangle_M \geq \alpha |h|^p \tag{8.11}$$

$$|\mathcal{A}_x(h)| \leq \beta |h|^{p-1} \tag{8.12}$$

$$\langle \mathcal{A}_x(h_1) - \mathcal{A}_x(h_2), h_1 - h_2 \rangle_M > 0, \quad h_1 \neq h_2, \tag{8.13}$$

and

$$\mathcal{A}_x(\lambda h) \equiv |\lambda|^{p-2} \lambda \mathcal{A}_x(h), \quad \lambda \in R \backslash \{0\}. \tag{8.14}$$

A function $f \in L^{1,p}_{loc}(M)$ is a weak solution ( resp. supersolution, subsolution ) of the equation

$$\mathrm{div} \mathcal{A}_x(\nabla f) = 0 (resp. \quad \leq 0, \quad \geq 0) \tag{8.15}$$

if for all nonnegative $\varphi \in C_0^\infty(M)$

$$\int_M \langle \mathcal{A}_x(\nabla f), \nabla \varphi \rangle dv = 0 (resp. \quad \geq 0, \quad \leq 0) \tag{8.16}$$

Here $L^{1,p}_{loc}(M)$ is the Sobolev space whose functions are locally $p$-integrable and have locally $p$-integrable partial distributional first derivatives. Continuous solutions of (8.15) are called $\mathcal{A}-$harmonic. In the case $\mathcal{A}_x(h) \equiv |h|^{p-2}h$, $\mathcal{A}-$harmonic functions are $p$-harmonic. A lower ( resp. upper ) semicontinuous function $f : M \to R \cup \{\infty\}$ ( resp. $\{-\infty\} \cup R$ ) is $\mathcal{A}$-superharmonic ( resp. $\mathcal{A}$-subharmonic ) if it is not identically infinite, and satisfies the $\mathcal{A}$-comparison principle: i.e., for each domain $D \subset\subset M$ and for each function $g \in C(\overline{D})$ which is $\mathcal{A}$-harmonic in $D$, $g \leq f$ ( resp. $g \geq f$) in $\partial D$ implies $g \leq f$ ( resp. $g \geq f$) in $D$. An $\mathcal{A}$-superharmonic ( resp. $\mathcal{A}$-subharmonic ) function $f$ is called $p$-superharmonic ( resp. $p$-subharmonic ) if $\mathcal{A}_x(h) \equiv |h|^{p-2}h$ . $\mathcal{A}$-superharmonic and $\mathcal{A}$-subharmonic functions are closely related to subsolutions and supersolutions of (8.15). For a discussion of the $\mathcal{A}$-harmonic equation, we refer the reader to I. Holopainen, T. Kipeläinen and O. Martio's book ([19]).

We define function $f \in L^{1,p+1}_{loc}(M)$ to be a weak solution of the differential inequality

$$f\,\mathrm{div}\mathcal{A}_x(\nabla f) \geq 0 \tag{8.17}$$

if for all nonnegative $\varphi \in C^\infty_0(M)$

$$\int_M \langle \mathcal{A}_x(\nabla f), \nabla(\varphi f)\rangle dv \quad \leq 0 \tag{8.18}$$

As an example, we prove the following:

**Theorem A**: Every locally bounded weak solution $f : M \to (-\infty, \infty)$ of the differential inequality $f div(A_x(\nabla f)) \geq 0$ , with $1 < p < \infty$ , is constant a.e. provided $f$ has one of the following: *p-finite, p-mild, p-obtuse, p-moderate,* or *p-small* growth, for some $q > p - 1$.Or equivalently, for every $q > p - 1$ , any nonconstant a.e. locally bounded weak solution $f : M \to (-\infty, \infty)$ of $f div(A_x(\nabla f)) \geq 0$ has *p-infinite, p-severe, p-acute, p-immoderate,* and *p-large* growth.

**The Generalized Bochner's Method :** *On a complete manifold, every nonnegative p-subharmonic ( resp. p-superharmonic ) function f is constant provided f has one of the following:* $p$-finite*,* $p$-mild*,* $p$-obtuse*,* $p$-moderate*, or* $p$-small *growth. for* $q > p - 1$ *( resp.* $q < p - 1$ *).*

We derive composition formulas and find the **first** set of nontrivial **geometric quantities** that are $p$-subharmonic (resp. $p$-superharmonic) functions, i.e. the composition of $p$-harmonic morphisms with convex (resp. concave) functions are $p$-subharmonic (resp. $p$-superharmonic) for $p \geq 2$. As an application of The Generalized Bochner's Method, we prove

**Theorem B** *Let* $u : M \to S^k_+ \subset R^{k+1}$ *be a smooth (a) harmonic map (where* $p = 2$*) or (b) p-harmonic morphism with* $p > 2$*, where* $S^k_+$ *is an open hemisphere centered at pole* $y_0$*. If for some* $q < p - 1$ *, the height function, defined by* $f(x) = \langle u(x), y_0 \rangle_{R^{k+1}}$ *for* $x \in M$ *, has one of the following:* $p$-finite*,* $p$-mild*,* $p$-obtuse*,* $p$-moderate*, or* $p$-small *growth, then u is constant.*

In an attempt to extend Sacks-Uhlenbeck's work ([38]) beyond the conformal dimension, FangHua Lin recently has made a breakthrough. In particular, he proves that if there is no smooth nonconstant harmonic map from $S^2$ into a compact manifold $N$, then the the Hausdorff dimension of the singular set of any stationary harmonic map from an $n$-manifold into $N$ does not exceed $n - 4$. The composition formulas lead to the **existence of smooth nonconstant harmonic maps from $S^3$** into such manifolds $N$ or into $K(\pi, 1)$ manifolds of dimension no less than 3. In particular, this solves **a problem** of FangHua Lin and ChangYou Wang in dimension 3

We develop an $L^p$ version of the method employed in ([50]) (cf. ([50]) p. 152-153), so that one can use $p$-(super)harmonic functions to study minimal surfaces. In ([39]), Schoen-Simon-Yau prove that if $M$ is a stable minimal hypersurface in a manifold $N^{n+1}$ with constant sectional curvature $K_2 \geq 0$, and if $M$ satisfies the volume growth condition

$$\lim_{R\to\infty} R^{-p}\mathrm{vol}(B_R) = 0 \tag{8.19}$$

for some $p \in (0, 4 + \sqrt{8/n})$, then $M$ is totally geodesic. This solves a generalized Bernstein Problem in Euclidean space under the volume growth condition (8.19) on $M$ for $n \leq 5$. In fact, Yau raised **the question** ( ([56]), p.692, Problem 102) whether one can prove that a complete minimal hypersurface $M$ in $R^{n+1}$ is a hyperplane for $n \leq 7$. For the case $n = 2$, it is proved by Colbrie-Fisher - Schoen ([9]), do Carmo - Peng ([8]), and Pogorelov ([31]) that $M$ is a plane . The condition (8.19), which is stronger than a *p-moderate* volume growth condition, implies that $M$ is $p$-parabolic. On the other hand, a $p$-parabolic manifold can have exponential volume growth by the example we constructed (cf. Example 3.1). Just as we use parabolicity and the uniformization theorem to study variational problems on surfaces in $R^3$ and their related differential equations (cf. ([50]) p.153), so do we use $p$-parabolicity and the generalized uniformization theorem to study stable minimal hypersurfaces in $R^{n+1}$. Thus, extending the work of Schoen -Simon-Yau, we solve the **generalized Bernstein Problem** under the $p$-parabolicity condition on $M$ for $n \le 5$ :

**Theorem C** *Every p-parabolic stable minimal hypersurface in $R^{n+1}$ is a hyperplane for $n \le 5$, where $p \in [4, 4 + \sqrt{8/n})$.*

The estimates we make on weak solutions not only work for functions but also for differential forms, vector fields, sections of bundles of various tensor types, fundamental groups of isometric covering, Riemannian manifolds and their geodesic balls, mappings between Riemannian manifolds,submanifolds, and solutions of fourth order strongly elliptic semilinear PDEs. As another example, applying our estimates, and the Generalized Bochner's Method, and our **iterative method**, we reduce partial differential system of the fourth-order, via the third-order and the second-order, to the zero-order and obtain the uniqueness of constant solutions and information on geometry. In fact, Jiang ([20]) proves that a smooth map $u$ from a compact manifold $M$ into a compact manifold of nonpositve sectional curvature is a harmonic map if and only if $u$ is a biharmonic map. We extend Jiang's work to **biharmonic maps** between complete Riemannian manifolds.

**Theorem D** *Let u be a smooth biharmonic map from a complete manifold M to* *a complete manifold $N$ of nonpositve sectional curvature. If for some $q > 2$, its* **tension field** $\tau(u)$ *has one of the following:* 2-finite, 2-mild, 2-obtuse, 2-moderate, *or* 2-small *growth, then $\tau(u)$ is parallel; and if $q = 2$, its* **differential** *$du$ has one of the following:* 2-finite, 2-mild, 2-obtuse, 2-moderate, *and* 2-small *growth, then $u$ is an energy minimizing harmonic map in its homotopy class; and if for some $q > 2$, and $y_0$ in a simply-connected $N$, its* **distance function**, *defined by $f(x) = dist(u(x), y_0)$ for $x \in M$, has one of the following:* 2-finite, 2−mild, 2-obtuse, 2-moderate, *or* 2-small *growth, then $u$ is constant.*

**Corollary A** *Every smooth biharmonic isometric immersion of a complete manifold $M$ into $R^k$ is a minimal submanifold of $R^k$, if its* **mean curvature vector field** *has one of the following:* 2-finite, 2-mild, 2-obtuse, 2-moderate, *or* 2-small *growth.*

This Corollary provides some interesting partial answers to Chen **Conjecture** on biharmonic immersions, namely, the only biharmonic isometric immersions of a Riemannian manifold into a Euclidean space are the minimal ones (cf. B. Y. Chen's survey article ([5])).

Similarly, as a special case of our results, we have via a variational approach and the $L^p$ method, extending our previous work of $p$-harmonic maps and $p$-stable maps under volume growth condition in ([47]) (cf. Theorem 5.1, p. 196):

**Theorem E** **1.** *Every $p$-harmonic map $u$ from a $p$-parabolic manifold $M$ into an open upper half-ellipsoid $E^k_{a,+}$ is constant.* **2.** *Every nonconstant $p$-stable map $u$ into a closed upper half-ellipsoid $\overline{E^k_{a,+}}$ is supported by a $p$-hyperbolic manifold*

*$M$, with* $p$-infinite, $p$-severe, $p$-acute, $p$-immoderate, *and* $p$-large *volume growth. Furthermore, if $M$ has nonnegative Ricci curvature, then $M$ is $q$-hyperbolic, with* $q$-infinite, $q$-severe, $q$-acute, $q$-immoderate, *and* $p$-large *volume growth. where for some $q > p + 2$ .*

Extending our previous work ([51]), we also obtain a Liouville-type theorem for $p$-stable maps into $p$-SSU manifolds. This theorem is new even for $p = 2$ and generalizes a theorem of Wei ([49]), and further refines a theorem of Schoen-Uhlenbeck ([40]) stating that every stable harmonic map from $R^n$ to $S^k$ is constant for $k \geq 3$ and $2 \leq n \leq \overline{d}(k)$, where $\overline{d}(3) = 2$ and $\overline{d}(k) = \min([[\frac{k}{2}]], 4)$ if $k \geq 4$ .

The $p$-harmonic theory with sharp estimates, the generalized uniformation theorem, the generalized Bochner's method, the iterative method, and the $L^p$- method, together with the geometric applications listed below seem to provide a new perspective in approaching geometric and variational problems occurring in complete noncompact manifolds of all dimensions.

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