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Twisted Signs for Finite General Linear Groups

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Twisted Signs for Finite General Linear Groups

A DISSERTATION APPROVED FOR THE

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Abstract

For an irreducible representation of a finite group G, a twisted sign is a generalization of the Frobenius-Schur indicator, which indicates whether a bilinear form preserving the G-action is symmetric or skew-symmetric. We will look at such signs for the finite general linear groups twisted by the transpose-inverse map. In particular, we will look at the relationship between these twisted signs and Harish-Chandra's philosophy of cusp forms. From this relationship, we show that these signs must all be one, recovering a result by Gow.

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Chapter 1

Introduction

Let $G = \mathrm{GL}_n(\mathbb{F}_q)$, where $\mathbb{F}_q$ is the finite field of cardinality q, and let θ send $g \in G$ to its transpose inverse. A complex representation (ρ, V) of G will be sent to its contragredient under the map $\rho \mapsto \rho^\theta := \rho \circ \theta$, giving us a bilinear form on V that is G-invariant under the twisted diagonal action

$$g.(v, v') := (\rho(g)v, \rho^\theta(g), v).$$

When (ρ, V) is irreducible, such a bilinear form is symmetric or skew-symmetric, which is indicated by a sign: $\epsilon_\theta(\rho) = 1$ in the symmetric case; $\epsilon_\theta(\rho) = -1$ in the skew-symmetric case.

For any finite group of Lie type, there is a special class of subgroups called Levi subgroups. In the case of our G, the Levi subgroups are the conjugates of the subgroups of block diagonal matrices. These latter subgroups are called standard Levi subgroups. Harish-Chandra showed that an irreducible representation ρ determines, up to conjugacy, a pair (L, σ) consisting of a Levi subgroup L and an irreducible representation σ of L with a special property called cuspidality. The representation ρ can then be recovered (up to equivalence) from σ and an associ-

ated object called a Hecke algebra module. Due to this, the study of the irreducible representations of a finite group of Lie type reduces to two main activities:

- studying or constructing the irreducible cuspidal representations of the Levi subgroups of the given group and
- determining the associated Hecke algebra modules.

This is Harish-Chandra's philosophy of cusp forms. Through the work of Deligne-Lusztig, Lusztig, Howlett-Lehrer and many others, Harish-Chandra's program has been carried out for a general finite group of Lie type.

Now let ρ be an irreducible representation of G and let σ be an associated irreducible cuspidal representation of a standard Levi subgroup L of G (that is, a subgroup of block diagonal matrices). Note that θ preserves L and so the sign $\epsilon_\theta(\sigma)$ is defined.

We show that the problem of determining the twisted sign $\epsilon_\theta(\rho)$ fits well with Harish-Chandra's philosophy. More precisely, one of our main results is that

$$\epsilon_\theta(\rho) = \epsilon_\theta(\sigma).$$

The proof is long and constitutes the technical heart of the thesis. It involves studying the Hecke algebra modules alluded to above.

In [Gow83], Gow considered the group $G^+ := G \rtimes \langle\theta\rangle$, showing that all of the irreducible representations are self-dual, and further, that all of the irreducible representations can be realized over the reals. This is equivalent to showing that $\epsilon_\theta(\rho) = 1$ for any irreducible G-representation (ρ, V).

He shows this to be the case for odd characteristic, mentioning that this should also hold for the characterstic two case, though his methods, depending heavily on the structure of the Sylow 2-subgroups in G, did not encompass this case. Vinroot in section 3 of [Vin06] gave an alternate proof of Gow's theorem, valid for both

odd and even characteristic. Again, however, Vinroot's method relies on structure specific to GL_n.

In the final chapter, we will show that the sign (σ, E) must be one, due to E being irreducible and cuspidal, hence reproving Gow's result. The novelty comes not from the result itself—for obvious reasons—but from the approach, what we later call the bridge. While the final few chapters are dependent on our specific setting, $G = \mathrm{GL}_n(\mathbb{F}_q)$, most of the techniques used in building the bridge use only the general structure theory of finite groups of Lie type and functorial constructions in representation theory. Further, they depend only on the existence of a map θ with certain nice properties. Much of this part of the paper should generalize to other finite groups of Lie type.

The author is hopeful, though it is less certain, that some of these techniques should also generalize to the p-adic case. New tools such as these would be beneficial in an area where much is still not known about the representation theory—for example, in the case of symplectic groups over local fields.

Chapter 2

Background

We first want to introduce some of the key components of our specific context and terminology pertaining to this context. Afterward, we will present some standard results and definitions that we will be using throughout the remainder of the paper.

2.1 The Context in Which We Find Ourselves

Let $G = \mathrm{GL}_n(\mathbb{F}_q)$ for some finite field $\mathbb{F}_q$ of cardinality q. We will call those subgroups P of G of block upper triangular matrices the **standard parabolic subgroups** of G. There is a decomposition of P into L, the **standard parabolic subgroup** of P, which are the block diagonal matrices within P, and U, the so-called **unipotent radical** of P. This latter group, U, will take the form of block identity matrices, in accordance to the block structure of L, and arbitrary entries elsewhere above the diagonal. We get the split exact sequence

$$1 \to U \to P \to L \to 1.$$

To give a better picture, we give an example in $\mathrm{GL}_3(\mathbb{F}_q)$:

$$1 \to \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} \to \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix} \to 1.$$

2.2 Adjoint Functors

This should be a pretty brief section, but an important one nevertheless. These results are almost entirely from Emily Riehl's [Rie16].

Let $\mathcal{C}$ and $\mathcal{D}$ be locally small categories—it works more generally, but the definition is simpler in this case and, as it is the only case we will need, it does no harm to make our lives a little easier—and let

$$\mathsf{F} : \mathcal{C} \rightleftharpoons \mathcal{D} : \mathsf{G}$$

be (covariant) functors between the categories such that for any $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, we have

$$\mathrm{Hom}_{\mathcal{D}}(\mathsf{F}X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, \mathsf{G}Y)$$

and this isomorphism is natural in both X and Y. We call F and G an **adjoint functor pair**, denoted $\mathsf{F} \dashv \mathsf{G}$, and say that F is **left-adjoint** to G or G is **right-adjoint** to F.

Associated to an adjoint functor pair $\mathsf{F} \dashv \mathsf{G}$ are the natural **counit** $\mathsf{FG}Y \to Y$ and **unit** $X \to \mathsf{GF}X$ morphisms, which are adjoint to the identity morphisms $\mathsf{G}Y \to \mathsf{G}Y$ and $\mathsf{F}X \to \mathsf{F}X$, respectively. These morphisms often given us familiar and useful parts of mathematics; for example, the evaluation morphisms of vector spaces are the counits of the tensor-hom adjunctions; the embeddings of vector spaces into their respective double dual are the units of that same adjunction. In fact, the unit

and counit give us a useful way to determine the adjoints to morphisms: Take the morphism in question, send it through the appropriate functor in the pair, and attach either the unit or counit, again appropriately, and this will give you the adjoint to that morphism.

Given a morphism $f \in \mathrm{Hom}_{\mathcal{D}}(\mathsf{F}X, Y)$, we will often denote the adjoint by $f^\sharp$; for $g \in \mathrm{Hom}_{\mathcal{C}}(X, \mathsf{G}Y)$, we will likewise denote the adjoint by $g^\flat$. It should be mentioned that this notation is cleverly used in the [Rie16], from which we have borrowed it.

2.3 Representation Theory

For representation theory results, we will refer to the classic by Fulton and Harris [FH91].

Before we get into the more heavy duty results pertaining to functors, we want to recall that the irreducible representations of S_n can be obtained from the group algebra $\mathbb{C}[S_n]$ by multiplication against special idempotent elements called Young symmetrizers. The Young symmetrizers are defined over $\mathbb{Q}$—in fact, over $\mathbb{Z}$—meaning the irreducible representations of S_n can be realized over $\mathbb{Q}$.

The Inflation and Coinvariant Functors

Let P be a parabolic subgroup of G and

$$1 \to U \to P \xrightarrow{\pi} L \to 1 \tag{2.1}$$

be the Levi decomposition of P; let $W = W_L$ be the Weyl group relative to L and $N = N_L$ be the normalizer of L.

Given a complex representation (σ, E) of L, we want to construct a representation of G; the process is similar to that of coinduction, however it will be mitigated by the parabolic group P. Because of this, it is called *parabolic* coinduction. (It should also be noted that this process, which will be shown below to be a right adjoint to a restriction functor, is often called parabolic induction, though we will reserve this name for the corresponding left adjoint.)

From the projection map $\pi : P \to L$, we have a functor π^* that takes L-representations to P-representations, often called the inflation functor: To the representation morphism $\sigma : L \to \mathrm{GL}(W)$, we precompose π and get the P-representation $(\sigma^\pi, E^\pi) := (\sigma\pi, E)$. Notice that this implies that the unipotent radical U of P acts trivially: Precomposing the inclusion of $U \to P$ is exact and must get sent to the identity in L. Given another L-representation (σ', E') with an L-equivariant morphism $f : E \to E'$, notice that $\pi(p) \in L$ for any $p \in P$, so we still have

$$f(\sigma^\pi_p w) = \sigma'^\pi_p f(w),$$

so that L-equivariant morphisms between the modules are already P-equivariant under this inflation process. Hence, π^* is a functor between representation categories.

This functor has a left and right adjoint: Let (ρ, V) be any P-representation with a morphism $f : V \to \pi^* E$. Then, for any element $v \in V$ and any unipotent element $u \in U$, it follows that

$$f(\rho_u v) = \sigma^\pi_u f(v) = f(v),$$

or $f(\rho(u)v - v) = 0$. We then get an L-representation $V/V(\pi)$, where $V(\pi) := \{\rho(u)v - v \mid u \in U, v \in V\}$ and a morphism

$$f^\flat : V/V(\pi) \to W$$

that is L-equivariant. The space $V_\pi := V/V(\pi)$ is called either the coinvariant space of V with respect to L or the Jacquet module, though the latter is most often used in the p-adic case, not in our finite group case. Notice that if we had $\varphi : V \to V'$, where V and V' were both P-representations, then $f(V(\pi)) \leqslant V'(\pi)$, so that this process works nicely on morphisms and indeed gives a functor. In fact, we've just shown that this gives us a left adjoint to the inflation functor, denoted $\pi^\flat \dashv \pi^*$. There is also a right adjoint, the invariant space, but we will not have a need for this.

The Parabolic Coinduction Functor

From the inclusion $P \xrightarrow{\iota} G$, we get the restriction functor ι^* from G-representations to P-representations—this is again a precomposition functor like π^* is above, however the nature of the functor is very different, reflecting the stark contrast between a monomorphism and an epimorphism.

For any group G, let $\mathsf{Rep}_G(\mathbb{C})$ denote the category of G-representations and G-equivariant linear morphisms between representation spaces. Let $\iota^\flat : \mathsf{Rep}_P(\mathbb{C}) \to \mathsf{Rep}_G(\mathbb{C})$ denote the left adjoint to the restriction functor—that is, the induction functor—and $\iota^\sharp : \mathsf{Rep}_P(\mathbb{C}) \to \mathsf{Rep}_G(\mathbb{C})$ denote the right adjoint, the coinduction functor. We will be mostly concerned with the composite functor

$$\mathrm{c}_L^G := \iota^\sharp \pi^*,$$

which we will call the **parabolic coniduction** functor. If we define **parabolic restriction** to be the composite functor

$$\mathrm{r}_L^G := \pi^\flat \iota^*,$$

then without too much work, we can show that this is left adjoint to the parabolic coinduction functor.

Proposition 1. $\mathrm{r}_L^G \dashv \mathrm{c}_L^G$

Proof. It is little more than a definition chase: Notice that for any G-representation E and L-representation F, we have

$$\begin{aligned}\mathrm{Hom}_L(\mathrm{r}_L^G E, F) &= \mathrm{Hom}_L(\pi^\flat \iota^* E, F) \\ &\simeq \mathrm{Hom}_p(\iota^* E, \pi^* F) \\ &\simeq \mathrm{Hom}_G(E, \iota^\sharp \pi^* F) \\ &= \mathrm{Hom}_G(E, \mathrm{c}_L^G F).\end{aligned}$$

This is exactly the condition that says they are indeed adjoint functors. □

For ease of notation, we will usually denote these functors c and r if the underlying groups are clear. But it is important to remember that the *parabolic* restriction and coinduction functors always go between the representation cagetories of a Levi component L, and the ambient group G.

Remark 1. It is also maybe of passing interest to mention that as we've defined parabolic restriction it does not have a left adjoint, though the composite functor $\pi^\sharp \iota^*$ would have a left adjoint. This composite functor could give an alternate restriction functor, and in our case where the groups are finite, this would actually be equivalent to our parabolic restriction functor, though in the case of p-adic groups, they would no longer be equivalent functors.

One final detail that we need to mention to make our calculations later more manageable: The more common way to construct the parabolic coinduction functor is as follows: For an L-representation, we inflate it to a P-representation, and then we coinduce up to a G-representation, which is as we have described above—this is the composite process that defines our parabolic coinduction functor—so a

functor that is isomorphic is to take an L-representation (σ, E) and send it to the P-equivariant maps $\mathrm{Hom}_P(G, E)$, where G is considered as a left P-set. The G-action on this new space is right multiplication, so given $f(x) \in \mathrm{Hom}_P(G, E)$ and $g \in G$, we have

$$g.f(x) = f(xg).$$

Notice that

$$g.(h.f(x)) = g.f(xh) = f(xgh),$$

so this indeed defines a left action by G. All this is to say $\mathrm{c}_L^G(E) \simeq \mathrm{Hom}_P(G, E)$, so we will often use these interchangeably in what follows.

2.4 Philosophy of Cusp Forms

According to Daniel Bump in [Bum98], "...the essential paradigm in the representation theory of [reductive groups over] finite fields is that the representations are built up from basic building blocks, which are the cuspidal representations." All of this theory is laid out in [Bum98], among various other places, so I am going to state the philosophy without proofs, though enough details will be included to give the reader an idea of the rudiments of the theory.

A representation (ρ, V) of $G = \mathrm{GL}_n(\mathbb{F}_q)$ is called **cuspidal** if parabolically restricting to any of the Levi components gives the zero module. It turns out it is enough to show that this restriction condition holds with respect to the standard Levi components since the other Levi components will be conjugate to these. Given a standard Levi component L, which we know to be block diagonal matrices, we can define a cuspidal representation analogously. Let

$$L = \mathrm{GL}_{n_1}(\mathbb{F}_q) \times \mathrm{GL}_{n_2}(\mathbb{F}_q) \times \ldots \times \mathrm{GL}_{n_d}(\mathbb{F}_q) \tag{2.2}$$

be the decomposition that comes from our understanding of L as the product of the blocks within the block diagonal, where $n_1+n_2+\ldots+n_d = n$. Then a representation (σ, E) similarly decomposes into a tensor product of (σ_i, E_i) with $\sigma_i : \mathrm{GL}_{n_i}(\mathbb{F}_q) \to \mathrm{GL}(E_i)$. We call (σ, E) **a cuspidal representation of** L exactly when it comprises a set of cuspidal representations on the components $\mathrm{GL}_{n_i}(\mathbb{F}_q)$. Within the context of L being a subgroup of G, this happens whenever it is parabolically restricts to the zero module, as is the case for G representations.

The philosophy of cusp forms as it pertains to the representation theory of finite groups of Lie type, and $\mathrm{GL}_n(\mathbb{F}_q)$ in particular, is that the representation theory can be broken down into finding cuspidal representations and determining how the coinduced representations for cuspidal representations of Levi components L decompose into irreducible representations.

For example, if (ρ, V) is an irreducible representation of G, but *not* cuspidal, then parabolically restricting to some Levi L gives a non-zero module (σ, E), meaning there is some nonzero L-equivariant map $\mathbf{r}V \to E$. But our adjunction, this tells us that

$$\mathrm{Hom}_G(V, \mathbf{c}E) \simeq \mathrm{Hom}_L(\mathbf{r}V, E) \neq 0,$$

which is to say that V is indeed an irreducible submodule of some irreducible L-representation. Nothing about this tells us that E itself need be cuspidal, but if it is not, then we can in turn say that it parabolically restricts to the Levi component of a parabolic subgroup further down the poset of parabolic subgroups (in our context, recall that these are block upper triangular matrices, so it is not hard to see that they form a poset by inclusion; conjugating this poset will give us the other connected components of the poset; for other such groups, this will happen as well). This process must stop since there are finitely many parabolic subgroups

in any connected component of the poset. That is to say, we can always move our considerations to an irreducible cuspidal (σ, E) by changing the Levi component in question, and so the irreducible representations of G will either themselves be cuspidal or be components of modules coinduced from a Levi component cuspidal.

A Word of Warning

To be clear about something that may not be obvious: Throughout the paper, functions are most often referred to as morphisms irrespective of any properties they may have. It is assumed that the reader can glean the requisite properties from the categorical context in which we are working at any given moment: This is most often a representation category, so the morphisms will usually be linear functions between vector spaces with some of kind of group-action equivariance that should be clear from the context. Occasionally, we will be talking about the category of groups, in which case the term morphism refers to group homomorphisms. Seldom will mention be made of this expressly.

Chapter 3

The Morphism θ

One of the main characters in our story is the morphism

$$\theta : \mathrm{GL}_n(\mathbb{F}_q) \to \mathrm{GL}_n(\mathbb{F}_q) : g \mapsto {}^T g^{-1},$$

for a fixed positive integer n, where $\mathbb{F}_q$ is the finite field of cardinality q. In this, ${}^T g$ denotes the transpose of the matrix g, in the sense of sending $(\alpha_{ij})_{i,j}$ to $(\alpha_{ji})_{i,j}$, and g^{-1} is the inverse element of g. The morphisms $g \mapsto {}^T g$ and $g \mapsto g^{-1}$ are both invertible and reverse multiplication. Hence, the composite morphism $\theta : g \mapsto {}^T g^{-1}$ is an isomorphism of the group $G := \mathrm{GL}_n(\mathbb{F}_q)$.

Properties of θ

Throughout the paper, we will need certain properties of the map θ, which we record here.

Proposition 2. $\theta^2 = \mathbb{1}_G$.

Proof. Notice that for $g \in G$, we have

$$\begin{aligned} {}^T(g^{-1})\,{}^T g &= {}^T(gg^{-1}) \\ {}^T(g^{-1})\,{}^T g &= \mathbb{1}_G \\ {}^T(g^{-1}) &= ({}^T g)^{-1}, \end{aligned}$$

so that transposition and inversion commute with one another. From this, it follows that

$$\theta^2(g) = \theta({}^T g^{-1}) = {}^T({}^T g^{-1})^{-1}(g^{-1})^{-1} = g.$$

Thus, $\theta^2 = \mathbb{1}_G$. □

Proposition 3. *Let $w \in G$ be a permutation matrix—a matrix wherein each row and column has a single nonzero value of one. Then ${}^T w = w^{-1}$ and $\theta(w) = w$.*

Proof. It follows directly from ${}^T w = w^{-1}$ that $\theta(w) = w$, so we need only show the former property. We will prove this property inductively based on length of element in terms of the generating set of adjacent transpositions $(i\ i+1)$ for $0 < i < n$.

For the base case of adjacent transpositions themselves, it is clear by their being invariant under both transposition *and* inversion: ${}^T(i\ i+1) = (i\ i+1) = (i\ i+1)^{-1}$.

Now, suppose this is true for elements of length k; that is, suppose that it holds for any permutation that can be written as the product of k many adjacent transpositions. Let $w \in G$ be of length $k+1$. Then $w = w'(i\ i+1)$ for some permutation w' of length k and adjacent transposition $(i\ i+1)$. Then, we have

$${}^T w = {}^T(w'(i\ i+1)) = {}^T(i\ i+1)\,{}^T w' = (i\ i+1)^{-1} w'^{-1} = (w'(i\ i+1))^{-1} = w^{-1}.$$

□

The Twisted Bilinear Form and the θ-sign on G

For any representation (ρ, V) of G, we can define a new representation $(\rho^\theta, V^\theta) := (\rho\theta, V)$ of G. If we denote by $({}^*\rho, V^*)$ the contragredient representation of G, there exists an G-equivariant isomorphism

$$V^\theta \xrightarrow{\beta} V^*. \tag{3.1}$$

That is, we have the following result.

Proposition 4. ${}^*\rho \simeq \rho^\theta$.

Proof. This can be seen using characters: Let χ_{ρ^θ} and $\chi_{{}^*\rho}$ be the characters of ρ^θ and ${}^*\rho$, respectively. One the one hand, for $g \in G$, we have

$$\begin{aligned} \chi_{\rho^\theta}(g) &= \operatorname{tr}(\rho^\theta(g)) \\ &= \operatorname{tr}(\rho(\theta(g))) \\ &= \operatorname{tr}(\rho({}^T g^{-1})). \end{aligned} \tag{3.2}$$

Over any field, the transpose of a square matrix is always conjugate to the matrix itself, meaning we have some $h \in G$ with ${}^T g = hgh^{-1}$. This lets us continue from (3.2) to get

$$\begin{aligned} \operatorname{tr}(\rho({}^T g^{-1})) &= \operatorname{tr}(\rho(hg^{-1}h^{-1})) \\ &= \operatorname{tr}(\rho(h)\rho(g^{-1})\rho(h)^{-1}) \\ &= \operatorname{tr}(\rho(g^{-1})). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\chi_{{}^*\rho}(g) &= \mathrm{tr}({}^*\rho(g)) \\ &= \mathrm{tr}(\rho(g^{-1})^*) \\ &= \mathrm{tr}(\rho(g^{-1})) \\ &= \mathrm{tr}(\rho(g)^{-1}).\end{aligned}$$

This shows us that the characters are equal. Hence, the corresponding representations are isomorphic. □

Remark 2. To be clear about a notational decision, the natural action on V^* we inherit from ρ is ρ^*, where g will act by pre-composition of $\rho_g := \rho(g)$, denoted ρ_g^*. However, this is naturally a right action, so to acknowledge that discrepancy and to emphasize the contragredient as a left action, we will use the non-standard notation ${}^*\rho$ to denote the contragredient representation, though we still denote the underlying vector space by V^*.

Through the tensor-hom adjunction, the adjoint of this isomorphism gives us a linear map

$$V \otimes V^\theta \to \mathbb{C}$$

where G acts on $V \otimes V^\theta$ by $g.(v, w) = (\rho_g v,\ {}^\theta\rho_g w)$ and G acts on $\mathbb{C}$ trivially. This is our non-degenerate bilinear form, which we will denote by $\langle -, - \rangle_\beta : V \otimes V^\theta \to \mathbb{C}$, where the β subscript is to remind us of its relation to (3.1).

Let $V \otimes V^\theta \xrightarrow{\tau} V \otimes V^\theta : v \otimes v' \mapsto v' \otimes v$ be the tensor twist map. Then we have another non-degenerate bilinear form $\langle -, - \rangle'_\beta := \langle -, - \rangle_\beta \circ \tau$. It might be worth mentioning that, at first blush, this doesn't look like a bilinear form on $V \otimes V^\theta$ since τ doesn't preserve the G-action on $V \otimes V^\theta$. However,

$$\langle v, \rho_g^\theta w \rangle'_\beta = \langle \rho_g^\theta w, v \rangle_\beta = \langle w, \rho_g v \rangle_\beta = \langle \rho_g v, w \rangle'_\beta,$$

which is equivalent to saying that this map is G-equivariant on $V \otimes V^\theta$, or more accurately G-invariant since the action on $\mathbb{C}$ is the trivial action.

If we suppose that V is an irreducible G-representation, then so are V^* and V^θ. From this, Schur's lemma tells us that there is a unique such non-degenerate bilinear form, up to scaling, which is to say there is some $\alpha \in \mathbb{C}$ such that

$$\langle v, w \rangle'_\beta = \alpha \langle v, w \rangle_\beta.$$

It follows from $\tau^2 = \mathbb{1}_V$ that $\alpha^2 = 1$, or $\alpha = \pm 1$. This number α we call the θ-**sign** and denote by $\epsilon_\theta(\rho)$.

Let P be a standard parabolic subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ with standard Levi component L. Then, since L comprises block diagonal matrices, it is clear that $\theta(L) = L$. That means that we can restrict θ to a function on L and play the same game that we have above with representations of L: Let (σ, E) be a representation of L, so that $(\sigma^\theta, E^\theta) := (\sigma\theta, E)$ is another representation of L. Mutatis mutandis, if E is an irreducible representation, we can define the θ-sign of (σ, E), which we will likewise denote by $\mathcal{E}_\theta(\sigma)$. To future-proof this *mutatis mutandis*, notice that we will get an isomorphism $E^\theta \xrightarrow{\gamma} E^*$ and the corresponding twisted bilinear form $\langle -, - \rangle_\gamma : E \otimes E^\theta \to \mathbb{C}$.

Chapter 4

Isotypy Component Decomposition

To build the bridge that we will need to relate the signs of ρ and σ, we will need to use a decomposition of the isotypy component of ρ in $\mathrm{Hom}_P(G, V)$. This chapter should be fairly short and self-contained, but it will be crucial once we've talked about the bilinear forms in the next couple chapters.

4.1 The Commuting Algebra

Let (σ, E) be a representation of L, where L is the Levi component of some standard parabolic subgroup P in G. Suppose that for an irreducible G-representation (ρ, V), we have $V \hookrightarrow \mathrm{Hom}_P(G, E)$ as G-representations. Let

$$C(\sigma) := \mathrm{End}_G(\mathrm{Hom}_P(G, E)),$$

that is, the G-equivariant linear maps from $\mathrm{Hom}_P(G, E)$ to itself. We will call this algebra the commuting algebra of σ, though it is a slight misnomer.

We also have a representable contravariant functor from the left representations of G to the left modules of $C(\sigma)$ in the morphism that sends any G-representation (ν, W) to $[W \Rightarrow E] := \mathrm{Hom}_G(W, \mathrm{Hom}_P(G, E))$: Any element

$f \in [W \Rightarrow E]$ is acted on by $A \in C(\sigma)$ by post-composition: $A.f := Af$. From this, we can see that this is a left $C(\sigma)$-module and that, if (ν, W) is *not* a subrepresentation of $\mathrm{Hom}_P(G, E)$, then this functor sends it to the zero module. To be clear, this is a functor because morphisms between representations of G get sent to the pre-composition morphism as is standard with representable functors.

For our irreducible subrepresentation (ρ, V), let us denote by V^ρ the isotypy component of (ρ, V) in $\mathrm{Hom}_P(G, E)$, that is, the subrepresentation of $\mathrm{Hom}_P(G, E)$ comprising all the copies of (ρ, V). We can see that this isotypy component has both a G-action defined on it and a $C(\sigma)$-action defined on it since any element of $C(\sigma)$ will have to send a copy of (ρ, V) to another isomorphic copy, so all of $C(\sigma)$ sends V^ρ to itself. In fact, the isotypy component is a $C(\sigma) \otimes \mathbb{C}G$ irreducible subrepresentation of $\mathrm{Hom}_P(G, V)$, as becomes evident in the following section.

4.2 The Evaluation Isomorphism

Let

$$V \otimes [V \Rightarrow E] \to V^\rho \leqslant \mathrm{Hom}_P(G, E) : v \otimes \varphi \mapsto \varphi(v).$$

This is an isomorphism. To show this, first let us mention that, by definition of V^ρ, $\varphi(v) \leqslant V^\rho$ since v has to get sent to an isomorphic copy of V as G-representations by definition of $[V \Rightarrow E] = \mathrm{Hom}_G(V, \mathrm{Hom}_P(G, E)$; V^ρ is all the isomorphic copies of V, so φ must send v into V^ρ. This ensures that the codomain can indeed be restricted to the isotypy component.

To show that the evaluation map is surjective onto V^ρ, let $v \in V^\rho$ be any element. By definition of V^ρ, v is an element of some irreducible G-representation (ρ', V') that is a subrepresentation of $\mathrm{Hom}_P(G, E)$, meaning we have $V' \overset{a}{\hookrightarrow} \mathrm{Hom}_P(G, E)$; moreover, by definition of V^ρ, there is a G-equivariant isomorphism $V \overset{b}{\to} V'$,

meaning such that $b(\rho_g x) = \rho'_g b(x)$ for any $x \in V$. Then $ab \in \mathrm{Hom}_G(V, \mathrm{Hom}_P(G, E))$ such that there is some $v' \in V$ with $\alpha\beta(v') = v$.

Instead of worrying about this map being injective, since we are dealing with finite dimensional vector spaces, it is enough to show that they have the same dimension; surjectivity and the rank-nullity theorem will then tell us this is an isomorphism.

We can decompose V^ρ into a direct some of submodules V_i, all of which are isomorphic to V by definition of V^ρ. Let

$$V^\rho = V_1 \oplus V_2 \oplus \ldots \oplus V_k \tag{4.1}$$

be such a decomposition for some positive integer k, and let φ_i be an isomorphism from V to V_i, so that $\varphi_i \in [V \Rightarrow E]$. Schur's lemma tells us that any other isomorphism $V \to V_i$ will be a scalar multiple of φ_i, meaning $\{\varphi_i\}_i$ gives us a basis of $[V \Rightarrow E]$. Then, the dimension of $V \otimes [V \to E]$ is $\dim(V) \cdot k$. The dimension of V^ρ will also be $\dim(V) \cdot k$ by our decomposition (4.1). Hence, they have the same dimension, so they must be isomorphic by our surjective map.

Chapter 5

The Inherited Bilinear Form

In this chapter and the next, we will be using the bilinear form on (σ, E) corresponding to the isomorphism $\gamma : E^\theta \to E^*$ to construct bilinear forms on related G-modules. With these tools, we will be able to elucidate the relationship between the θ-sign on the L-representation (σ, E) and that on the G-representation (ρ, V). In this chapter, we will address the θ-twisted bilinear form on $\mathrm{Hom}_P(G, E)$ that is inherited from (σ, E).

5.1 The Inherited Bilinear Form

Let L again be the standard Levi component of some standard parabolic subgroup P, so that L is a subgroup of block diagonal matrices of G and let (σ, E) be an irreducible representation of L. We first want to show that there is a naturally defined non-degenerate bilinear form on the G-representation $(\mathrm{c}\sigma, \mathrm{c}E) = (c\sigma, \mathrm{Hom}_P(G, E))$.

Let $\langle -, - \rangle_\gamma = \mathrm{ev}_E(-, \gamma(-))$ be the bilinear on E, with which we've defined the θ-sign. This corresponds to an isomorphism $E^\theta \xrightarrow{\gamma} E^*$, which we can parabolically coinduce to an isomorphism $\mathrm{c}(E^\theta) \to \mathrm{c}(E^*)$, and it is this isomorphism that forms

the heart of the parabolically coinduced bilinear form. In more familiar notation, this gives us an isomorphism

$$\mathrm{Hom}_P(G, E^\theta) \xrightarrow{\gamma_*} \mathrm{Hom}_P(G, E^*),$$

which is the post-composition map γ_*.

The Non-Parabolically Coinduced Form

Notice that we can *non*-parabolically coinduce γ as well to get the isomorphism

$$\mathrm{Hom}_L(G, E^\theta) \xrightarrow{\gamma_*} \mathrm{Hom}_L(G, E^*),$$

and the γ_* between P-equivariant maps is just the restriction of the more broadly defined γ_* between L-equivariant maps. It is this L-equivariant form that we will use to define the coinduced bilinear form.

Because $\theta(L) = L$, we have an isomorphism

$$\mathrm{Hom}_L(G, E)^\theta \xrightarrow{\theta^*} \mathrm{Hom}_L(G, E^\theta) : f \mapsto f\theta.$$

To flesh that claim out a bit more, let $f \in \mathrm{Hom}_L(G, E)^\theta$, so that $g \in G$ acts on f by the rule $g.f(x) = f(x\theta(g))$. Letting $f_\theta(x) := f(\theta(x))$, we see that g acts on $f_\theta(x)$ by

$$g.f_\theta(x) = f_\theta(x\theta(g)) = f(\theta(x)\theta^2(g)) = f(\theta(x)g),$$

which is to say that the g-action on f_θ is *not* twisted by θ. However, if $\ell \in L$, we have

$$f_\theta(\ell g) = f(\theta(\ell g)) = {}^\theta\sigma(\ell) f_\theta(g).$$

Because the G-action is not twisted, but the element f_θ itself is equivariant with respect to the L-action that *is* twisted, we can see that $f_\theta \in \mathrm{Hom}_L(G, E^\theta)$, and

indeed $f \mapsto f_\theta$ gives us a G-equivariant map from $\mathrm{Hom}_L(G, E)^\theta$ to $\mathrm{Hom}_L(G, E^\theta)$. It is quick work to see that this map is in fact a complex linear map as well.

To see that it is an isomorphism, recall that $\theta^2 = \mathbb{1}_G$; it follows that

$$(\theta^*)^2 = \mathbb{1}_{\mathrm{Hom}_L(G,E^\theta)} \qquad \& \qquad (\theta^*)^2 = \mathbb{1}_{\mathrm{Hom}_L(G,E)^\theta}.$$

The other isomorphism we need to finish building our new bilinear form is

$$\mathrm{Hom}_L(G, E^*) \to \mathrm{Hom}_L(G, E)^*;$$

this isomorphism takes a bit more work.

Let us first consider the counit morphism $\mathrm{ev}_E(-,-) : E \otimes E^* \to \mathbb{C} : f \otimes \varphi \mapsto \varphi(f)$. If $\Delta : G \to G \times G$ is the diagonal map, then composing with the counit gives us a morphism

$$\begin{aligned}\mathrm{Hom}_L(G, E) \otimes \mathrm{Hom}_L(G, E^*) \to \mathrm{Hom}_L(G, E \otimes E^*) \to \mathrm{Hom}_L(G, \mathbb{C}) :& \\ f \otimes \varphi \mapsto (f \otimes \varphi) \circ \Delta \mapsto \mathrm{ev}_V(-,-) \circ (f \otimes \varphi) \circ \Delta,& \quad (5.1)\end{aligned}$$

which is a hideous looking thing. But if we look at what it does to elements $g \in G$, we will see it isn't too ghastly:

$$\begin{aligned}(\mathrm{ev}_E(-,-) \circ (f \otimes \varphi) \circ \Delta)(g) &= (\mathrm{ev}_E(-,-) \circ (f \otimes \varphi))(g, g) \\ &= \mathrm{ev}_E(-,-)(f(g) \otimes \varphi(g)) \\ &= \mathrm{ev}_E(f(g), \varphi(g)).\end{aligned}$$

Ultimately, we want a bilinear form. Integrating over G will gives us the correct codomain and our morphism becomes

$$\mathrm{Hom}_L(G, E) \otimes \mathrm{Hom}_L(G, E^*) \to \mathbb{C} : f \otimes \varphi \mapsto \int_G \mathrm{ev}_E(f(g), \varphi(g))\, dg, \qquad (5.2)$$

which is, on the whole, a little less unsightly than (5.1). This of course corresponds to a morphism

$$\mathrm{Hom}_L(G, E^*) \to \mathrm{Hom}_L(G, E)^* : \varphi \mapsto (f \mapsto \int_G \mathrm{ev}_E(f(g), \varphi(g))\, dg. \tag{5.3}$$

Proposition 5. *The bilinear form in* (5.2) *is non-degenerate. That is to say, the corresponding G-equivariant linear map in* (5.3) *is an isomorphism.*

Proof. Now, for $v \in E^*, v \neq 0$, we can define an element of $\mathrm{Hom}_L(G, E^*)$ as follows: For $\ell \in L$, let $\varphi_{h,v}$ send Lh to $\sigma(\ell)v$; for $g \notin Lh$, we will have $\varphi_{h,v}(g) = 0$. Suppose that $f \otimes \varphi \mapsto 0$ for every $\varphi \in \mathrm{Hom}_L(G, E^*)$. For any of our $\varphi_{h,v}$, we have

$$\begin{aligned} f \otimes \varphi_{h,v} &\mapsto \int_G \mathrm{ev}_E(f(g), \varphi_{h,v}(g))\, dg \\ &= \int_L \mathrm{ev}_E(f(\ell h), \varphi_{h,v}(\ell h))\, dg \\ &= \big|L\big| \cdot \mathrm{ev}_E(f(h), v) \\ &= \big|L\big| v(f(h)). \end{aligned}$$

But saying this is zero for all $\varphi_{h,v}$ is saying that $v(f(h)) = 0$ for all $v \in E^*$ and $h \in G$: To say $v(f(h)) = 0$ for all $v \in E^*$ is to say that $f(h) = 0$; to say this holds for all $h \in G$, $f(G) = 0$, is to say that f must be the zero map. Summing up, this tells us that if there is an $f \in \mathrm{Hom}_L(G, E)$ such that $f \otimes \varphi \mapsto 0$ for all $\varphi \in \mathrm{Hom}_L(G, E^*)$, it follows that $f \equiv 0$, in which case $f \otimes \varphi$ is the zero element in the tensor.

We can ask what about if there is some $\varphi \in \mathrm{Hom}_L(G, E^*)$ such that $f \otimes \varphi \mapsto 0$ for all $f \in \mathrm{Hom}_L(G, E)$? Running the same game as above, with $v \in E$ instead, we can show that it follows that $\varphi \equiv 0$ and $f \otimes \varphi$ is again the zero element. □

Putting these three isomorphisms together, we get a final isomorphism

$$\mathrm{Hom}_L(G, E)^\theta \xrightarrow{\Gamma} \mathrm{Hom}_L(G, E)^* : f \mapsto \Gamma_f,$$

where Γ_f acts on $F \in \mathrm{Hom}_L(G, E)$ by

$$\Gamma_f(F) = \int_G \mathrm{ev}_E(F(g), \gamma f_\theta(g))\, dg = \int_G \langle F(g), f_\theta(g) \rangle_\gamma \, dg.$$

This, of course, has an adjoint that gives us the **inherited bilinear form**:

$$\mathrm{Hom}_L(G, E) \otimes \mathrm{Hom}_L(G, E)^\theta \to \mathbb{C} : F \otimes f \mapsto \langle F, f \rangle_\Gamma := \Gamma_f(F), \tag{5.4}$$

which by construction is non-degenerate and G-invariant.

Remark 3. Though we are using integral notation, in our present case, this is over the counting measure on a finite group, so it is a finite sum—we use integral notation with an eye on eventually generalizing this to p-adic groups, which will no longer be using counting measure.

What is not nice is that we've constructed a θ-twisted, non-degenerate, bilinear form on $\mathrm{Hom}_L(G, E)$, not on the P-equivariant morphisms, $\mathrm{Hom}_P(G, E)$ with no obvious way to ensure that, when restricted to $\mathrm{Hom}_P(G, E) \otimes \mathrm{Hom}_P(G, E)^\theta$, it doesn't become the zero bilinear form. How do we know that we aren't restricting it to a subspace that is in the kernel of the form?

The most straightforward way to deal with this question is to consider the parabolically coinduced version.

The Parabolically Coinduced Form

As mentioned above, we can parabolically coinduce γ_* to be an isomorphism between $\mathrm{Hom}_P(G, E^\theta)$ and $\mathrm{Hom}_P(G, E^*)$, the P-equivariant morphisms instead of the L-equivariant ones. This is simply the restriction of the γ_* between L-equivariant morphisms, so this is not where the issue lies. Nor is the issue with the isomorphism $\mathrm{Hom}_P(G, E^*) \to \mathrm{Hom}_P(G, E)^*$, which is the same as above: We can see that

$$\mathrm{Hom}_P(G, E) \otimes \mathrm{Hom}_P(G, E^*) \to \mathbb{C} : f \otimes \varphi \mapsto \int_G \mathrm{ev}_E(f(g), \varphi(g))\, dg$$

still gives the requisite isomorphism. What is at issue is that $f \mapsto f_\theta$ no longer gives us an isomorphism $\mathrm{Hom}_P(G,E)^\theta \to \mathrm{Hom}_P(G,E^\theta)$. This is because, while f_θ is equivariant with E^θ, it is no longer P-equivariant (though it still is L-equivariant). If we let $\overline{P} := \theta(P)$, then $f_\theta \in \mathrm{Hom}_{\overline{P}}(G,E^\theta)$. We need to account for this by making it U-invariant, instead of $\overline{U}$-invariant, where $\overline{U} := \theta(U)$ is the unipotent radical of $\overline{P}$. We instead get the morphism

$$\mathrm{Hom}_P(G,E)^\theta \to \mathrm{Hom}_P(G,E^\theta) : f(x) \mapsto f'(x) := \int_U f\theta(ux)\,du.$$

For any $u' \in U$, we get

$$f'(u'x) = \int_U f\theta(uu'x)\,du = \int_U f\theta(ux)\,du = f'(x),$$

which shows us that this is indeed now a U-invariant morphism again. To show that it takes $f \in \mathrm{Hom}_P(G,E)^\theta$ to $\mathrm{Hom}_P(G,E^\theta)$, it remains to show that it is L-equivariant (though this is the same calculation as above): For $\ell \in L$, we have

$$\begin{aligned}
f'(\ell x) &= \int_U f\theta(u\ell x)\,du \\
&= \int_U f\theta(\ell u^\ell x)\,du, \quad u^\ell := \ell^{-1}u\ell \\
&= \int_U f\big(\theta(\ell)\theta(u^\ell x)\big)\,du \\
&= \int_U \sigma(\theta(\ell))f\theta(u^\ell x)\,du \\
&= \int_U \sigma^\theta(\ell)f\theta(u^\ell x)\,du \\
&= \sigma^\theta(\ell)\int_U f\theta(u^\ell x)\,du \\
&= \sigma^\theta(\ell)f'(x).
\end{aligned}$$

There is the notable issue that there isn't an obvious reason why this integral transformation would be an isomorphism. However, Howlett and Lehrer tell us, in [HL94], that this indeed the case.

Putting these three isomorphisms together gives us the new bilinear form

$$\mathrm{Hom}_P(G,E)\otimes\mathrm{Hom}_P(G,E)^\theta \xrightarrow{\Gamma'} \mathbb{C} : F\otimes f \mapsto \int_G \mathrm{ev}_E(f(g),\gamma(F'(g)))\,dg,$$

which can also be written

$$f\otimes F \mapsto \int_G \langle f(g), F'(g)\rangle_\gamma\,dg.$$

This is in distinction from the restricted L-equivariant form of

$$f\otimes F \mapsto \int_G \langle f(g), F\theta(g)\rangle_\gamma\,dg.$$

But we will show that because of the integration over G, these end up being equivalent bilinear forms.

The Equivalence of These Forms

Let $\langle -,-\rangle_{\Gamma'}$ denote the bilinear form we get from parabolically coinducing γ; that is, for $f\in\mathrm{Hom}_P(G,E)$ and $F\in\mathrm{Hom}_P(G,E)^\theta$,

$$\langle f,F\rangle_{\Gamma'} := \int_G \langle f(g),F'(g)\rangle_\gamma\,dg,$$

and let $\langle -,-\rangle_\Gamma$ be the original *non*-parabolically coinduced form; that is,

$$\langle f,F\rangle_{\Gamma} := \int_G \langle f(g),F\theta(g)\rangle_\gamma\,dg.$$

We want to show that these are equivalent bilinear forms in the sense that one is a scalar multiple of the other. In subsequent chapters, we are going to want to use $\langle -,-\rangle_\Gamma$, as the calculations are simplified by not having to concern ourselves with U-invariance. However, by construction we know that $\langle -,-\rangle_{\Gamma'}$ is non-degenerate, something we cannot easily say about $\langle -,-\rangle_\Gamma$; we need to show equivalence of these forms lest we make much ado about a degenerate form.

Let $f \in \mathrm{Hom}_P(G, E)$ and $F \in \mathrm{Hom}_P(G, E)^\theta$. The trick, as mentioned above, is that integration over G renders the U-invariance component of the bilinear form superfluous, which we can show by a change of variables in the integration process. Again, it is helpful to remember that we can be cavalier about integration since, in our case, they are truly finite sums. We have

$$\begin{aligned}
\langle f, F\rangle_{\Gamma'} &= \int_G \langle f(g), F'(g)\rangle_\gamma \, dg \\
&= \int_G \langle f(g), \int_U F\theta(ug)\, du\rangle_\gamma \, dg \\
&= \int_G \int_U \langle f(g), F\theta(ug)\rangle_\gamma \, du dg \\
&= \int_U \int_G \langle f(g), F\theta(ug)\rangle_\gamma \, dg du \\
&= \int_U \int_G \langle f(u^{-1}h), F\theta(h)\rangle_\gamma \, dh du, \quad h = ug,\ dh = dg \\
&= \int_U \int_G \langle f(h), F\theta(h)\rangle_\gamma \, dh du \\
&= \int_U du \int_G \langle f(h), F\theta(h)\rangle_\gamma \, dh \\
&= \int_U du \int_G \langle f(h), F\theta(h)\rangle_\gamma \, dh \\
&= \mu\langle f, F\rangle_\Gamma,
\end{aligned}$$

where μ is the measure of U; if the measure is on-the-nose counting measure, this will be the cardinality of U; if, however, we have normalized the measure so that U has measure one, these forms are exactly equal. In any case, they differ by a scalar, as needed. This tells us that restricting $\langle -, -\rangle_\Gamma$ to a θ-twisted form on $\mathrm{Hom}_P(G, E)$ will still give us a non-degenerate form, so we need only concern ourselves with this form in what follows.

5.2 A θ-Sign Nevertheless

One thing that is interesting about this bilinear form is that, despite $\mathrm{Hom}_P(G, E)$ *not* in general being an irreducible representation of G, there is nevertheless a θ-sign associated to this form.

Proposition 6. *Let* $\langle -, - \rangle_\Gamma : \mathrm{Hom}_P(G, E) \otimes \mathrm{Hom}_P(G, E)^\theta \to \mathbb{C}$ *be the bilinear form from* (5.4)*. For any* $f_0, f_1 \in \mathrm{Hom}_P(G, E)$*, we have*

$$\langle f_0, f_1 \rangle_\Gamma = \epsilon_\theta(\sigma) \langle f_1, f_0 \rangle_\Gamma.$$

Proof. We've put in all the legwork already: For $F, f \in \mathrm{Hom}_L(G, E)$ we have

$$\begin{aligned}
\langle F, f \rangle_\Gamma &= \int_G \langle F(g), f\theta(g) \rangle_\gamma \, dg \\
&= \int_G \langle F\theta(h), f(h) \rangle_\gamma \, dh, \quad h = \theta(g) \\
&= \int_G \epsilon_\theta(\sigma) \langle f(h), F\theta(h) \rangle_\gamma \, dh \\
&= \epsilon_\theta(\sigma) \int_G \langle f(h), F\theta(h) \rangle_\gamma \, dh \\
&= \epsilon_\theta(\sigma) \langle f, F \rangle_\Gamma.
\end{aligned}$$

□

Chapter 6

Building A Bridge

We have one of the bilinear forms that we will need. But there are still a couple others that we will need to complete the bridge that we are building. Here, briefly, is big picture: We will put a bilinear form on $V \otimes [V \Rightarrow E]$ by putting one on each component of the tensor product; the inherited bilinear form on $\mathrm{Hom}_P(G, E)$ restricts to a non-degenerate form on the isotypy component V^ρ of V; finally, we will show that the evaluation isomorphism between $V \otimes [V \Rightarrow E]$ and V^ρ preserves these bilinear forms.

Once we have shown this, it is an immediate corollary that the θ-sign of the L-module E and the θ-sign of the irreducible component V of $\mathrm{Hom}_P(G, E)$ will be related by a sign associated to the necessary $C(\sigma)$-module. This is our bridge.

6.1 The Bilinear Form on C(σ)-Modules

A Pairing Using Schur's Lemma

Toward that end, we need a new bilinear form on $[V \Rightarrow E]$—recall this is our notation for the $C(\sigma)$-module $\mathrm{Hom}_G(V, \mathrm{Hom}_P(G, E))$.

Schur assures us that $\mathrm{Hom}_G(V, V) \simeq \mathbb{C}$. Thus, for any

$$F \in [E \Rightarrow V] := \mathrm{Hom}_G(\mathrm{Hom}_P(G, E), V)$$

and $\varphi \in [V \Rightarrow E]$, we have $F\varphi \in \mathbb{C}$, meaning that $F\varphi$ is a scaling map on V, which we will denote by $\mu_{F\varphi}$. This gives us a morphism $[E \Rightarrow V] \to [V \Rightarrow E]^*$ and, equivalently, a pairing form

$$[V \Rightarrow E] \otimes [E \Rightarrow V] \to \mathbb{C}.$$

Let us first show that this pairing is non-degenerate by showing that the morphism $[E \Rightarrow V] \to [V \Rightarrow E]^*$ is an isomorphism.

Suppose that $F\varphi = 0$ for arbitrary $\varphi \in [V \Rightarrow E]$. If $\varphi \neq 0$, then $\varphi(V)$ is a nonzero factor of $\mathrm{Hom}_P(G, E)$, so $F\varphi = 0$ means that F must send the entire factor $\varphi(V)$ to zero. But this holds for arbitrary φ, so $F = 0$ meaning $[E \Rightarrow V] \to [V \Rightarrow E]^*$ is injective.

Now, let $A \in [V \Rightarrow E]^*$ be some morphism in the dual. Then for any $\varphi \in [V \Rightarrow E]$, we have $A\varphi = \alpha \mathbb{1}_V$ for some $\alpha \in \mathbb{C}$, meaning for $v \in V$, we have $A\varphi(v) = \alpha v$. But $V \simeq \varphi(V)$ via φ; we can extend the map $F(\varphi(V)) = \alpha V$ by zero to all of $\mathrm{Hom}_P(G, E)$, so that $F \in [E \Rightarrow V]$ and $F\varphi = \alpha \mathbb{1}_V$. This shows that $[E \Rightarrow V] \to [V \Rightarrow E]^*$ is surjective, hence it is an isomorphism.

A Bilinear Form From the Pairing

This isn't quite what we want since $[V \Rightarrow E] \otimes [E \Rightarrow V] \to \mathbb{C}$ isn't yet a bilinear form. However, we can correct this: Let $\varphi : V \to \mathrm{Hom}_P(G, E)$, so that $\varphi^* : \mathrm{Hom}_P(G, E)^* \to V^*$. Precomposing by the G-equivariant isomorphism $\Gamma : \mathrm{Hom}_P(G, E)^\theta \to \mathrm{Hom}_P(G, E)^*$ and postcomposing by the G-equivariant isomor-

phism $\gamma^{-1} : V^* \to V^\theta$ gives us an isomorphism

$$\varphi^\circ := \gamma^{-1}\varphi^*\Gamma : \mathrm{Hom}_P(G, E)^\theta \to V^\theta,$$

which is G-equivariant exactly when φ is. We need a lemma to show that this turns our pairing into a bilinear form.

This gives us the required isomorphism

$$[V \Rightarrow E] \to [V \Rightarrow E]^*.$$

Let us record all that has been discussed above in the following proposition.

Proposition 7. *Let*

$$\langle -, - \rangle_\circ : [V \Rightarrow E] \otimes [V \Rightarrow E] \to \mathbb{C} : \varphi \otimes \psi \mapsto \mu_{\varphi^\circ\psi},$$

where $\mu_{\varphi^\circ\psi}$ denotes the scalar that the composite morphism $\varphi^\circ\psi$ acts by according to Schur's lemma. Then this defines a non-degenerate, bilinear form.

Proof. The only detail that hasn't been discussed above is that the assignment $\varphi \mapsto \varphi^\circ$ is a $\mathbb{C}$-linear assignment. Notice that for $\varphi, \psi \in [V \Rightarrow E]$, we have

$$(\varphi + \psi)^\circ = \gamma^{-1}(\varphi + \psi)^*\Gamma = \gamma^{-1}\varphi^*\Gamma + \gamma^{-1}\psi^*\Gamma = \varphi^\circ + \psi^\circ,$$

and, for $\alpha \in \mathbb{C}$,

$$(\alpha\varphi)^\circ = \gamma^{-1}(\alpha\varphi)^*\Gamma = \gamma^{-1}\alpha\varphi^*\Gamma = \alpha\gamma^{-1}\varphi^*\Gamma = \alpha\varphi^\circ,$$

so $\varphi \mapsto \varphi^\circ$ is indeed a $\mathbb{C}$-linear isomorphism, making the morphism $\langle\rangle_\circ$ a G-invariant, non-degenerate bilinear form. □

In fact, this non-degenerate bilinear form has a sign as well. We first need to show that $[V \Rightarrow E]$ is simple before we can show this to be the case.

Lemma 1. *If (ρ, V) is an irreducible representation of G, then $[V \Rightarrow E]$ is a simple $C(\sigma)$-module.*

Proof. Let $\varphi, \psi \in [V \Rightarrow E]$ be any two nonzero elements. We want to show that there is some $A \in C(\sigma)$ such that $A\varphi = \psi$. Let us denote $\psi(V) := V_\psi \leqslant \mathrm{Hom}_P(G, E)$ and, likewise, $\varphi(V) := V_\varphi \leqslant \mathrm{Hom}_P(G, E)$. Since φ, ψ are nonzero, $V_\varphi \simeq V \simeq V_\psi$, and we can extend this isomorphism by zero to an element of $C(\sigma)$. That is, letting this isomorphism be A, we have $A(v) \in V_\psi$ whenever $v \in V_\varphi$ and $A(v) = 0$ whenever $v \notin V_\varphi$. It follows directly that this A satisfies our condition, so that $A\varphi = \psi$. □

Using a familiar trick, define a bilinear form

$$\langle -, - \rangle'_\circ : [V \Rightarrow E] \otimes [V \Rightarrow E] \to \mathbb{C} : \varphi \otimes \psi \mapsto \langle \psi, \varphi \rangle_\circ,$$

so that $\langle \varphi, \psi \rangle'_\circ = \langle \psi, \varphi \rangle_\circ$. The isomorphisms $[V \Rightarrow E] \mapsto [V \Rightarrow E]^*$ that define both of these bilinear forms must differ by a scalar, so it follows that

$$\langle \varphi, \psi \rangle_\circ = \epsilon(\rho/\sigma) \langle \varphi, \psi \rangle'_\circ,$$

with $\epsilon(\rho/\sigma)^2 = 1$.

A Duality on $\mathbf{C}(\sigma)$

Recall that $[V \Rightarrow E]$ is a $C(\sigma)$-module; we can use $\langle -, - \rangle_\circ$ to define a duality on the operators of $C(\sigma)$ and, using this duality, a new action of $C(\sigma)$ on $[V \Rightarrow E]$. Let

$\varphi, \psi \in [V \Rightarrow E]$. Then,

$$\begin{aligned}
\langle A\varphi, \psi\rangle_\circ &= (A\varphi)^\circ\psi \\
&= \gamma^{-1}(A\varphi)^*\Gamma\psi \\
&= \gamma^{-1}\varphi^* A^*\Gamma\psi \\
&= \gamma^{-1}\varphi^*\Gamma(\Gamma^{-1}A^*\Gamma)\psi \\
&= \varphi^\circ(\Gamma^{-1}A^*\Gamma)\psi \\
&= \langle\varphi, (\Gamma^{-1}A^*\Gamma)\psi\rangle_\circ,
\end{aligned}$$

so that we have the process

$$A \mapsto A^\vee := (\Gamma^{-1}A^*\Gamma). \tag{6.1}$$

Remark 4. To be clear, to be able to pre- and post-compose by Γ^{-1} and Γ, respectively, we have to remember that elements of $C(\sigma)$ also intertwine the θ-action.

There is no reason so far to believe that this process is a duality on our operators in $C(\sigma)$, so we need to verify this before moving forward. That is, we want to show that $A^{\vee\vee} = A$. To do this, we need the following lemma.

Lemma 2. *For $A \in C(\sigma)$, we have $(\Gamma A\Gamma^{-1})^* = \Gamma^{-1}A^*\Gamma$.*

Proof. Let $f_0, f_1 \in \mathrm{Hom}_P(G, E)$ and $A \in C(\sigma)$. Then, we have

$$\begin{aligned}
\langle f_0, (\Gamma^{-1}A^*\Gamma)f_1\rangle_\Gamma &= \mathrm{ev}_{\mathrm{Hom}_P(G,E)}(f_0, \Gamma(\Gamma^{-1}A^*\Gamma)f_1) \\
&= \mathrm{ev}_{\mathrm{Hom}_P(G,E)}(f_0, A^*\Gamma f_1) \\
&= \mathrm{ev}_{\mathrm{Hom}_P(G,E)}(Af_0, \Gamma f_1) \\
&= \langle Af_0, f_1\rangle_\Gamma.
\end{aligned}$$

We also have

$$\begin{aligned}
\langle Af_0, f_1\rangle_\Gamma &= \mathcal{E}_\theta(\sigma)\langle f_1, Af_0\rangle_\Gamma \\
&= \mathcal{E}_\theta(\sigma)\mathrm{ev}_{\mathrm{Hom}_P(G,E)}(f_1, \Gamma A f_0) \\
&= \mathcal{E}_\theta(\sigma)\mathrm{ev}_{\mathrm{Hom}_P(G,E)}(f_1, (\Gamma A\Gamma^{-1})\Gamma f_0) \\
&= \mathcal{E}_\theta(\sigma)\mathrm{ev}_{\mathrm{Hom}_P(G,E)}((\Gamma A\Gamma^{-1})^* f_1, \Gamma f_0) \\
&= \mathcal{E}_\theta(\sigma)\langle(\Gamma A\Gamma^{-1})^* f_1, f_0\rangle_\Gamma \\
&= \langle f_0, (\Gamma A\Gamma^{-1})^* f_1\rangle_\Gamma.
\end{aligned}$$

Together, this gives us

$$\langle f_0, (\Gamma^{-1}A^*\Gamma)f_1\rangle_\Gamma = \langle f_0, (\Gamma A\Gamma^{-1})^* f_1\rangle_\Gamma. \qquad (*)$$

Because $(*)$ holds for all f_0 and f_1, non-degeneracy of $\langle -, -\rangle_\Gamma$ gives us our result. □

Corollary 1. *For any $A \in C(\sigma)$, we have $A^{\vee\vee} = A$.*

Proof.

$$A^{\vee\vee} = (\Gamma(\Gamma A\Gamma^{-1})^*\Gamma^{-1})^* = (\Gamma(\Gamma^{-1}A^*\Gamma)\Gamma^{-1})^* = A^{**} = A.$$

□

6.2 The Final Bilinear Form

We need one last bilinear form to build the bridge we are constructing; it will be built from other bilinear forms and will help us show that the two θ-signs of the G-representation (ρ, V) and the underlying L-representation (σ, E) are related. Making that relation precise is the point of building the bridge.

The Bilinear Form Itself

Because V is an irreducible G-representation, we have the twisted bilinear form that comes from V^θ and V^* being isomorphic. Let the isomorphism in question be

$$V^\theta \xrightarrow{\gamma} V^*,$$

so that the bilinear form we have is

$$\langle -, - \rangle_\gamma : V \otimes V^\theta \to \mathbb{C} : v \otimes v' \mapsto \mathrm{ev}_V(v, \gamma(v')).$$

We also have the isomorphism

$$[V \Rightarrow E] \to [V \Rightarrow E]^* : \varphi \mapsto \varphi^\circ.$$

Taking the tensor of these gives us an isomorphism

$$V^\theta \otimes [V \Rightarrow E] \mapsto V^* \otimes [V \Rightarrow E]^* \simeq (V \otimes [V \Rightarrow E])^*,$$

which, by adjunction, gives us a non-degenerate bilinear form

$$(V \otimes [V \Rightarrow E]) \otimes (V^\theta \otimes [V \Rightarrow E]) \to \mathbb{C} : (v \otimes \varphi) \otimes (v' \otimes \varphi') \mapsto \langle v, v' \rangle_\beta \langle \varphi, \varphi' \rangle_\circ.$$

We will denote this bilinear form simply by $\langle -, - \rangle$, unadorned.

Building a Bridge

We can now return to the evaluation isomorphism and the decomposition that it gives us of the isotypy component, $V \otimes [V \Rightarrow E] \simeq V^\rho$. We want to show that the bilinear form is preserved by this isomorphism. That is to say, we want to show that

Proposition 8. $\langle v \otimes \varphi, v' \otimes \varphi' \rangle = \langle \varphi(v), \varphi'(v') \rangle_\Gamma$.

Proof. For ease of notation, the morphism $\varphi(v) \in \mathrm{Hom}_P(G, E)$, we will denote by φ_v. Then, for $v, v' \in V$ and $\varphi, \varphi' \in [V \Rightarrow E]$, we have

$$\begin{aligned}
\langle v, v' \rangle_\beta \langle \varphi, \varphi' \rangle_\circ &= \mu_{\varphi^\circ \varphi'} \langle v, v' \rangle_\beta \\
&= \langle v, \mu_{\varphi^\circ \varphi'} v' \rangle_\beta \\
&= \langle v, \varphi^\circ \varphi' v' \rangle_\beta \\
&= \left\langle v, (\gamma^{-1} \varphi^* \Gamma \varphi') v' \right\rangle_\beta \\
&= \mathrm{ev}_V(v, \gamma(\gamma^{-1} \varphi^* \Gamma \varphi') v') \\
&= \mathrm{ev}_V(v, \varphi^* (\Gamma \varphi') v') \\
&= \mathrm{ev}_{\mathrm{Hom}_P(G,E)}(\varphi v, \Gamma \varphi' v') \\
&= \langle \varphi v, \varphi' v' \rangle_\Gamma
\end{aligned}$$

□

The proposition is important to us because of the following.

Corollary 2. $\epsilon_\theta(\rho)\epsilon(\rho/\sigma) = \epsilon_\theta(\sigma)$

And this corollary is the bridge between the θ-signs that we have been looking for.

Chapter 7

A Brief Detour: Projective Representations

We have discussed bilinear forms, isotypy decomposition, the equality between θ-signs; we want to push a little further and show that $\epsilon(\rho/\sigma) = 1$ in our case of $\mathrm{GL}_n(\mathbb{F}_q)$. Toward this end, we need some tools from projective representation theory and, in particular, a way to extend a linear representation of L.

We don't need all the standard results of projective representations, so this will by no means be a comprehensive introduction to them; rather, we will cherry-pick just the bits we need and elaborate on them as needed. For a more comprehensive introduction, from which most of this has come, the reader may want to refer to Karpilovsky [Kar85].

7.1 Two Different Vantage Points with the Same View

Let H be a normal subgroup of a group G—in this chapter, G will not necessarily be $\mathrm{GL}_n(\mathbb{F}_q)$; in the following chapters, we will return to our convention that

$G = \mathrm{GL}_n(\mathbb{F}_q)$—and let (σ, E) be an irreducible representation of H. Let K be the cokernel of $H \to G$, so that

$$1 \to H \xrightarrow{\iota} G \xrightarrow{\pi} K \to 1$$

is a short exact sequence. There are two ways to think about the projective extension of this linear representation of H.

We can take the conjugate action of G on H, which gives a natural action of G on the representations of H: For $g \in G$ and $h \in H$, if we let $c_g(h) := ghg^{-1}$, then we have the action of $\sigma \mapsto \sigma c_g$. Properly speaking, this would be a right action, but we are presently only concerned with when this stabilizes σ, so it is of little consequence in this context whether we "fix" this by making it a left action or leave it as is. In either case, let G_σ be the stabilizer of σ.

Alternately, given a section $s : K \to G$ of the projection above, we can define the conjugation action of K on H by way of s, that is for $k \in K$ and $h \in H$, we let $d_k(h) := s(k)hs(k)^{-1}$. Similar to above, this K-action on H gives a K-action on the representations of H: σc_k is again another representation of H. Let K_σ denote the stabilizer of this action.

Notice that $d_k = c_{s(k)}$. Let $g \in G_\sigma$. Then $\pi(g) \in K$ and we can see that $s(\pi(g))$ differs by an element of H since $\pi(s(\pi(g))) = \pi(g)$ by definition of s. In particular, precomposing σ by the conjugation of $s(\pi(g))$ will differ by precomposing by the conjugation of g by a conjugation of h—that is, if $s(\pi(g)) = gh$ for some $h \in H$, then

$$s(\pi(g))h's(\pi(g))^{-1} = ghh'h^{-1}g^{-1}$$

—and conjugation by an element of H is an isomorphism of the vector space, so that $\pi(g) \in K_\sigma$ whenever $g \in G_\sigma$. Conversely, let $k \in K_\sigma$. Then it is clear by definition that $s(k) \in G_\sigma$. So these two perspectives on the projective extension of

(σ, E) give the same results, but one accentuates the importance of K whereas the other accentuates the importance of G.

7.2 The Projective Extension

In what follows, we will take the vantage point of extending to G_σ.

Saying that $g \in G_\sigma$ stabilizes the representation (σ, E) is to say that $(\sigma c_g, {}^g E)$ is isomorphic to (σ, E) as H-representations. In particular, there is some linear isomorphism

$$\widehat{\sigma}(g) : E \to {}^g E$$

that is H-equivariant, meaning $\widehat{\sigma}(g)\sigma(h) = {}^g\sigma(h)\widehat{\sigma}(g)$, where ${}^g\sigma := \sigma c_g$; this can also be written more suggestively as

$$\widehat{\sigma}(g)\sigma(h)\widehat{\sigma}(g)^{-1} = \sigma(ghg^{-1}).$$

But because we have picked these operators $\widehat{\sigma}(g)$ more or less at random, there is no reason to believe that they will multiply nicely: For $g, g' \in G_\sigma$, there is no reason $\widehat{\sigma}(g)\widehat{\sigma}(g') = \widehat{\sigma}(gg')$ will hold. And *this* is where Schur's lemma comes in handy: Because (σ, E) is an irreducible representation of H and both $\widehat{\sigma}(g)\widehat{\sigma}(g')$ and $\widehat{\sigma}(gg')$ are H-isomorphisms from (σ, E) to $({}^{gg'}\sigma, {}^{gg'}E)$, they will differ by a nonzero complex scalar, and by the usual following the implications of associativity of multiplication of G_σ, we can see that this scalar will define a 2-cocycle of G_σ with values in $\mathbb{C}^\times$.

A nice detail that we have not yet mentioned is that for any element $h \in H$, we can define $\widehat{\sigma}(h) := \sigma(h)$ so that this map $\widehat{\sigma} : G_\sigma \to \mathrm{GL}(E)$ restricts to our original $\sigma : H \to \mathrm{GL}(E)$. However, since multiplication is only preserved up to something in $\mathbb{C}^\times$, this only gives us a group homomorphism if we consider

it as a map $\widehat{\sigma} : G_\sigma \to \mathrm{PGL}(E)$. Hence, this gives us an extension of the linear representationt σ to a projective representation $\widehat{\sigma}$.

7.3 Projective Equivalences

Let (σ_1, E_1) and (σ_2, E_2) be complex projective representations of G. We say that these representations are **projectively equivalent** if there exists a 1-cocycle $\mu : G \to \mathbb{C}^\times$ and an isomorphism $\alpha : E_1 \to E_2$ such that for every $v \in E_1$ and $g \in G$, we have

$$\alpha(\sigma_1(g)v) = \mu(g)\sigma_2(g)\alpha(v).$$

If $\mu \equiv 1$, then we say they are **linearly equivalent**.

The Ambiguity of the Contragredient

Let (σ, E) be a complex linear representation of H and $(\widehat{\sigma}, E)$ be a projective extension to G_σ. We can consider the contragredient representation of H, which we have been designating by $({}^*\sigma, E^*)$, where for $f \in E^*$ and $h \in H$, we have

$${}^*\sigma(h)f(x) = f(\sigma(h^{-1})x).$$

There is a fundamental ambiguity in the definition of how we extend the contragredient representation to a projective representation of G_σ. It arises from the equality $\sigma(h^{-1}) = \sigma(h)^{-1}$ for the original representation of H. As with σ, there is a natural definition of a *right* action of G_σ on E^*, but unlike in the case of the contragredient representation $({}^*\sigma, E^*)$ of H, $\widehat{\sigma}(g^{-1}) = \widehat{\sigma}(g)^{-1}$ need no longer hold, so we can define the corresponding left action as

$${}^*\widehat{\sigma}(g)f(x) := f(\widehat{\sigma}(h^{-1})x)$$

or as

$$^*\widehat{\sigma}(g)f(x) := f(\widehat{\sigma}(h)^{-1}x).$$

While these are projectively equivalent definitions, they are distinct definitions unlike in the case of the contragredient action of H—each of these gives a different map $G_\sigma \to \mathrm{GL}(E^*)$, whereas they would give the same map $H \to \mathrm{GL}(E^*)$. This means that the projective extension of the contragredient from H to G_σ is only ever defined up to cohomology.

The reason this matters to us is that it offers a small space of ambiguity that we can exploit to get a nice property from our projective operators associated to G_σ.

Moreover, notice that for $g \in G_\sigma$, we have an action on E^* as well: For $f \in E^*$, we can still precompose, $\widehat{\sigma}(g)^* f(x) = f(\widehat{\sigma}(g)x)$, which will be a right action, so to make it a left action in keeping with the contragredient action we can define the action

$$^*\widehat{\sigma}(g)f(x) := f(\widehat{\sigma}(g^{-1})x),$$

so that elements of G_σ act by their inverse elements, just as elements of H do in the contragredient. In fact, for $g, g' \in G_\sigma$, this will give us

$$\begin{aligned}
^*\widehat{\sigma}(g)\,^*\widehat{\sigma}(g')f(x) &= {}^*\widehat{\sigma}(g)f(\widehat{\sigma}(g'^{-1})x) \\
&= f(\widehat{\sigma}(g'^{-1})\widehat{\sigma}(g^{-1})x) \\
&= f(c(g'^{-1}, g^{-1})\widehat{\sigma}(g'^{-1}g^{-1})x) \\
&= c(g'^{-1}, g^{-1})f(\widehat{\sigma}((gg')^{-1})x) \\
&= c(g'^{-1}, g^{-1})\,^*\widehat{\sigma}((gg'))f(x),
\end{aligned}$$

where $c : G_\sigma \times G_\sigma \to \mathbb{C}^\times$ is the 2-cocycle associated to $(\widehat{\sigma}, E)$. But, it can be shown that $d(g, g') := c(g'^{-1}, g^{-1})$ is also a 2-cocycle, so this gives us a projective extension

of the contragredient to a projective representation of G_σ, albeit with a different 2-cocycle associated to it.

7.4 Back To Our Context

So, why does this all matter to us? Consider the twisted bilinear form $\langle -, - \rangle_\gamma$. Let N_σ be the stabilizer of the N action on the L-representation (σ, E) and let W_σ be the corresponding stabilizer in the relative Weyl group. But let us be even more specific—because L is a standard Levi component, it is a subgroup of block diagonal matrices in G, so the normalizer in G, namely N, is the subgroup of matrices in G that send those blocks to one another. This means that if we let $L = \mathrm{GL}_{n_1}(\mathbb{F}_q) \times \mathrm{GL}_{n_2}(\mathbb{F}_q) \times \ldots \times \mathrm{GL}_{n_d}(\mathbb{F}_q)$, where $\sum_{i=1}^{d} n_i = n$, then N can send $\mathrm{GL}_{n_i}(\mathbb{F}_q)$ to $\mathrm{GL}_{n_j}(\mathbb{F}_q)$ if and only if $n_i = n_j$; every element $n \in N$ can be decomposed into a direct sum of isomorphisms between the component blocks of L. Of course L itself normalizes itself, which is to say that the blocks within L act on themselves by conjugation; if we then consider $W = N/L$, we see that we have a map of W back into N as the pure permutation maps: If $n_i = n_j$ for $i \neq j$, then we have the isomorphism

$$\mathrm{GL}_{n_i}(\mathbb{F}_q) \to \mathrm{GL}_{n_j}(\mathbb{F}_q) : A \mapsto A,$$

which we *want* to call an identity map, but we must remember that within the framework of L, these are *not* the same general linear group—they act on different parts of $\mathbb{F}_q^n$. If we also let $\mathrm{GL}_{n_j}(\mathbb{F}_q)$ map to $\mathrm{GL}_{n_i}(\mathbb{F}_q)$ via this abstract identity map, and extend this to all of L via the actual identity on all the rest of the blocks, this gives us an element of W considered as an element of N—in fact, a generating set of elements of W—so we have a group morphism $W \to N$ and the short exact

sequence

$$1 \to L \to N \to W \to 1$$

is split exact.

This means something very nice for our projective extensions—something that will not be true in the general case: We can decompose (σ, E) into representations on the blocks $\mathrm{GL}_{n_i}(\mathbb{F}_q)$, so that $(\sigma, E) = (\bigotimes_i \sigma_i, \bigotimes_i E_i)$, and an element of W fixes the representation (σ, E) exactly when it exchanges (σ_i, E_i) and (σ_j, E_j) that are isomorphic to one another. Without loss of generality, we may assume that when (σ_i, E_i) and (σ_j, E_j) are isomorphic, they are in fact equal representations: If this is not the case, we can move our concerns to an isomorphic representation where this is the case. This will allow us to find a particularly nice form for the operators in question.

We have a map $W \to \mathrm{GL}(E)$ generated by the basic transpositions (of blocks) described above: If $w_{ij} \in W$ is the element that exchanges the blocks $\mathrm{GL}_{n_i}(\mathbb{F}_q)$ and $\mathrm{GL}_{n_j}(\mathbb{F}_q)$, then w_{ij} maps to the element in $\mathrm{GL}(E)$ that exchanges the spaces E_i and E_j. In *particular* particular (as Dr Kujawa says), the 2-cocycle associated to this mapping is the trivial cocycle. This is to say that, for $w, w' \in W_\sigma$, we have

$$\widehat{\sigma}(w)\widehat{\sigma}(w') = \widehat{\sigma}(ww'),$$

and the projective extension is in fact a linear extension.

Chapter 8

The Hecke Algebra Trick

From Corollary 2 in Chapter 7, we have

$$\epsilon_\theta(\rho)\epsilon(\rho/\sigma) = \epsilon_\theta(\sigma).$$

But in our case, we can do better than this: We can show that all $C(\sigma)$-signs are one, $\epsilon(\rho/\sigma) = 1$. Thus, the θ-sign of an irreducible G-representation V depends only on the θ-sign of the underlying irreducible L-representation E.

8.1 Duality on the Standard Intertwining Operators

Recall that we defined a process on the operators of $C(\sigma)$ by way of the bilinear form on $[V \Rightarrow E]$ that turned out to be a duality, namely for $A \in C(\sigma)$

$$A \mapsto A^\vee := \Gamma^{-1} A^* \Gamma.$$

In this section, we will be interested in how this process works on a particular basis of $C(\sigma)$. Given $w \in W_\sigma$, we define **a standard intertwining operator** T_w as follows: For $f \in \mathrm{Hom}_P(G, E)$,

$$T_w f(x) := \widehat{\sigma}_w \int_{U_w} f(w^{-1}ux)\, du,$$

where $U_w := U \cap w\overline{U}w^{-1}$, $\overline{U} := \theta(U)$, and $\widehat{\sigma}_w := \widehat{\sigma}(w)$ is a linear operator as in 7.2. In fact, in chapter 10 of [Car85], Carter shows us that these operators form a basis of $C(\sigma)$ whenever the L-representation (σ, E) is cuspidal, in addition to irreducible—going forward, we will assume that (σ, E) is both irreducible and cuspidal. We are interested in how these operators are affected by the duality process $T_w \mapsto T_w^\vee$.

The Dual of $\mathbf{T_w}$

Let $f \in \mathrm{Hom}_P(G, E)^\theta$ and consider the action of $T_w^\vee$ on f. This is the composite action $\Gamma^{-1}T_w^*\Gamma f$. However, we want to be able to say something more specific about them: We want to say that the duality process permutes these operators, and in fact, more specifically, for a given $w \in W_\sigma$, the duality sends T_w to $T_{w^{-1}}$. Our approach to this will be to show that, given $f_0 \in \mathrm{Hom}_P(G, E)$, we have $\langle f_0, T_w^\vee f\rangle_\Gamma = \langle f_0, T_{w^{-1}} f\rangle_\Gamma$, after which non-degeneracy of the bilinear form implies that $T_w^\vee = T_{w^{-1}}$. We first know, by definition of $T_w^\vee$, that

$$\langle f_0, T_w^\vee f\rangle_\Gamma = \langle T_w f_0, f\rangle_\Gamma,$$

from the short calculation

$$\begin{aligned}
\langle f_0, T_w^\vee f\rangle_\Gamma &= \mathrm{ev}_{\mathrm{Hom}_P(G,E)}(f_0, \Gamma(T_w^\vee f)) \\
&= \mathrm{ev}_{\mathrm{Hom}_P(G,E)}(f_0, \Gamma(\Gamma^{-1}T_w^*\Gamma f)) \\
&= \mathrm{ev}_{\mathrm{Hom}_P(G,E)}(f_0, T_w^*\Gamma f) \\
&= \mathrm{ev}_{\mathrm{Hom}_P(G,E)}(T_w f_0, \Gamma f) \\
&= \langle T_w f_0, f\rangle_\Gamma.
\end{aligned}$$

We now want to show that

$$\langle T_w f_0, f\rangle_\Gamma = \langle f_0, T_{w^{-1}} f\rangle_\Gamma$$

and this is where the work comes in. From the previous chapter, we know that for $w \in W_\sigma$ we have

$$\langle \widehat{\sigma}_w v, v' \rangle_\gamma = \langle v, \widehat{\sigma}_{w^{-1}} v' \rangle_\gamma$$

for any two $v, v' \in E$, which comes into play in the proof of the following lemma.

Lemma 3. $\langle T_w f_0, f \rangle_\Gamma = \langle f_0, T_{w^{-1}} f \rangle_\Gamma$

Proof. We have the equality

$$\langle T_w f_0, f \rangle_\Gamma = \int_G \left\langle \widehat{\sigma}_w \int_{U_w} f_0(w^{-1}ug)\, du, f\theta(g) \right\rangle_\gamma dg$$

by definition of bilinear form $\langle -, - \rangle_\Gamma$ and the operator T_w. Recall that $\widehat{\sigma}_w$ is a linear operator on E, so it commutes with the integral (which, as mentioned earlier, is only a finite sum in our case). As mentioned above, we also know that it can be brought to the other side of the bilinear form at the expense of becoming $\widehat{\sigma}_{w^{-1}}$. This gives us

$$\int_G \left\langle \widehat{\sigma}_w \int_{U_w} f_0(w^{-1}ug)\, du, f\theta(g) \right\rangle_\gamma dg = \int_{U_w} \int_G \left\langle f_0(w^{-1}ug), \widehat{\sigma}_{w^{-1}} f\theta(g) \right\rangle_\gamma dg\, du.$$

Now, by the change of variable $h = w^{-1}ug$, the latter half of the above equality becomes $\displaystyle\int_{U_w} \int_G \left\langle f_0(h), \widehat{\sigma}_{w^{-1}} f\theta(u^{-1}wh) \right\rangle_\gamma dh du$; bringing the integral over U_w back into the bilinear form gives us the equality

$$\int_{U_w} \int_G \left\langle f_0(w^{-1}ug), \widehat{\sigma}_{w^{-1}} f\theta(g) \right\rangle_\gamma dg du = \int_G \left\langle f_0(h), \widehat{\sigma}_{w^{-1}} \int_{U_w} f\theta(u^{-1}wh)\, du \right\rangle_\gamma dh$$

so it only remains to show that $T_{w^{-1}} f\theta(x) = \widehat{\sigma}_{w^{-1}} \displaystyle\int_{U_w} f\theta(u^{-1}wh)\, du$, which comes down to showing that the integrals $\displaystyle\int_{U_w} f\theta(u^{-1}wh)\, du$ and $\displaystyle\int_{U_{w^{-1}}} f(wu\theta(h))\, du$ are equal, which we will show to be true below. This completes the proof. □

Lemma 4. $\displaystyle\int_{U_w} f\theta(u^{-1}wh)\, du = \int_{U_{w^{-1}}} f(wu\theta(h))\, du$

Proof. Let $u \in U_w$ for $w \in W_\sigma$, and $h \in G$. Notice that $\theta(w) = w$ as discussed in 3, so that $\theta(u^{-1}wh) = \theta(u^{-1}w)\theta(h)$. Because $u \in U_w = U \cap w\overline{U}w^{-1}$, we know that $u = w\overline{u}w^{-1}$ for some $\overline{u} \in \overline{U}$, or $w^{-1}u = \overline{u}w^{-1}$. Negating both sides, we get $u^{-1}w = w\overline{u}^{-1}$. This gives us

$$\theta(u^{-1}wh) = \theta(u^{-1}w)\theta(h) = \theta(w\overline{u}^{-1})\theta(h) = w\,{}^T\overline{u}\theta(h).$$

The transpose of something in $\overline{U}$ is something in U, so ${}^T\overline{u} \in U$, but by definition of u,

$$\begin{aligned} {}^Tu &= {}^T(w\overline{u}w^{-1}) \\ {}^Tu &= w\,{}^T\overline{u}w^{-1} \\ w^{-1}\,{}^Tuw &= {}^T\overline{u}, \end{aligned}$$

so that ${}^T\overline{u} \in w^{-1}\overline{U}w$ as well. Thus, ${}^T\overline{u} \in U \cap w^{-1}\overline{U}w = U_{w^{-1}}$.

This gives us the calculation

$$\begin{aligned} \int_{U_w} f\theta(u^{-1}wh)\,du &= \int_{U_w} f({}^Tuw\theta(h))\,du \\ &= \int_{U_w} f(w\,{}^T\overline{u}\theta(h))\,du \\ &= \int_{U_{w^{-1}}} f(w^{-1}u\theta(h))\,du, \end{aligned}$$

and we are done. □

Once we have these two equalities, $\langle f_0, T_w^\vee f\rangle_\Gamma = \langle T_w f_0, f\rangle_\Gamma$ and $\langle f_0, T_{w^{-1}} f\rangle_\Gamma = \langle T_w f_0, f\rangle_\Gamma$, we can put them together to get the following.

Proposition 9. $T_w^\vee = T_{w^{-1}}$

The Trace of The Duality

Let us now turn to the trace of the duality operation, which will be denoted $\mathrm{tr}(-^\vee)$. Roughly, we will calculate the trace of the duality using two different bases to

attain a formula that we can use to compare the signs of simple $C(\sigma)$-modules under the duality $A \mapsto A^\vee$ to the signs of irreducible W_σ representations under the duality provided by the contragredient.

On the one hand,

Lemma 5. *Let* $\mathrm{Irr}(W_\sigma)$ *denote the set of characters of irreducible representations of* W_σ*. Then,*

$$\mathrm{tr}(-^\vee) = \sum_{\chi \in \mathrm{Irr}(W_\sigma)} \chi(1).$$

Proof. In the basis $\{T_w\}_{w \in W_\sigma}$, the trace will count the number of T_w fixed by duality. By Proposition 9, this will be all $w \in W_\sigma$ such that $T_w = T_w^\vee = T_{w^{-1}}$, or

$$\mathrm{tr}(-^\vee) = \big|w \in W_\sigma \mid w^2 = 1\big|.$$

From [Isa76], we know that

$$\big|\{w \in W_\sigma \mid w^2 = 1\}\big| = \sum_{\chi \in \mathrm{Irr}(W_\sigma)} \epsilon(\chi)\chi(1).$$

As mentioned in 2.3, the irreducible representations of S_n are realizable over the rational numbers, meaning over the reals. This last statement is equivalent to saying that the Frobenius-Schur indicators are all one. Being that W_σ is a product of symmetric groups, it follows that the signs of W_σ are also all one, giving us

$$\mathrm{tr}(-^\vee) = \big|\{w \in W_\sigma \mid w^2 = 1\}\big| = \sum_{\chi \in \mathrm{Irr}(W_\sigma)} \chi(1).$$

□

On the other hand, we know from [Car85], that $C(\sigma) \simeq \mathbb{C}[W_\sigma]$, so $C(\sigma)$ is a semi-simple algebra. Hence, we have a decomposition

$$C(\sigma) = C_1 \oplus C_2 \oplus \ldots \oplus C_r$$

into the minimal two-sided ideals C_i for $1 \leqslant i \leqslant r$. These ideals correspond to the irreducible representations of $C(\sigma)$. Let M_i be the irreducible $C(\sigma)$-representation corresponding to the ideal C_i, so that $C_i \simeq \mathrm{End}_{C_i}(M_i)$. We can, roughly speaking, find a basis of $C(\sigma)$ that respects this decomposition and calculate the trace using this basis.

If $C_i^\vee \neq C_i$, then the trace on this submodule of $C(\sigma)$ will be zero, so we have that $\mathrm{tr}(-^\vee) = \sum_{C_i^\vee = C_i} \mathrm{tr}\left(-^\vee\big|_{C_i}\right)$ and our calculation reduces to finding what the trace is on each of these components of $C(\sigma)$. Toward that end, we want to consider these traces restricted to the C_i such that $C_i^\vee = C_i$.

The dual to M_i, $M_i^* = \mathrm{Hom}_{\mathbb{C}}(M_i, \mathbb{C})$, has a natural right $C(\sigma)$-action by precomposition. To make this a left $C(\sigma)$-action, we use the duality on $C(\sigma)$:

$$A.f := fA^\vee.$$

Observe that this allows us, for $A, B \in C(\sigma)$, and $f \in M_i^*$,

$$A.(B.f)(x) = A.f(B^\vee x) = f(B^\vee A^\vee x) = f((AB)^\vee x) = (AB).f(x),$$

showing that it is, indeed, left action now.

We now want to show that the condition $C_i = C_i^\vee$ is equivalent to the condition $M_i \simeq M_i^\vee$.

Lemma 6. *In the decomposition of $C(\sigma)$ above, let C_i be one of the component ideals and let M_i be the corresponding irreducible module. Let $M_i^\vee$ denote the dual space of M_i viewed as a left $C(\sigma)$-module via $A.f(x) := f(A^\vee x)$. Then, $M_i \simeq M_i^\vee$ if and only if $C_i^\vee = C_i$.*

Proof. Suppose $\alpha : M_i \to M_i^\vee$ is a (left) $C(\sigma)$-module isomorphism, so that for $A \in C(\sigma)$ and $m \in M_i$, we have $\alpha(A.m) = A.\alpha(m)$. Then, we have a bilinear form

$$\langle -, - \rangle_\alpha : M_i \otimes M_i \to \mathbb{C} : m \otimes m' \mapsto \mathrm{ev}_{M_i}(m, \alpha(m')),$$

such that $\langle Am, m' \rangle_\alpha = \langle m, A^\vee m' \rangle_\alpha$ for $m, m' \in M_i$. In particular, for $m, m' \in M_i$ such that $\langle Am, m' \rangle_\alpha \neq 0$, then $\langle m, A^\vee m' \rangle_\alpha \neq 0$; this implies that $A^\vee \in C_i$, so $C_i^\vee \subseteq C_i$. As it is a duality, we get the inclusion $C_i \subseteq C_i^\vee$ for free, so that $C_i^\vee = C_i$.

Conversely, suppose $C_i^\vee = C_i$. This already means that the left modules corresponding to each of these ideals are isomorphic. The module corresponding to C_i is of course M_i; C_i naturally acts on M_i^*, the dual of M_i as a right module, meaning $C_i^\vee$ acts on M_i^* as a left module; from this, it follows that M_i and $M_i^\vee$ are indeed isomorphic as $C(\sigma)$-modules. □

Taking stock of the situation: We know that we can calculate the trace by concerning ourselves with only the C_i in the decomposition that are fixed by the duality because the C_i that are not fixed will not contribute to the trace. We also know that the C_i with $C_i = C_i^\vee$ correspond exactly to the simple modules M_i with $M_i \simeq M_i^\vee$. We have a further lemma that helps us calculate the trace with reference to these C_i.

Lemma 7. *For $C_i = C_i^\vee$, we have $\mathrm{tr}(-^\vee\big|_{C_i}) = \epsilon(M_i)\dim(M_i)$, where M_i is the simple module corresponding to C_i.*

Proof. Because $C_i = C_i^\vee$, the previous lemma tells us that $M_i \simeq M_i^\vee$; let $\alpha : M_i \to M_i^\vee$ be a C_i-module isomorphism. This gives us a non-degenerate bilinear form

$$\langle -, - \rangle_\alpha : M_i \otimes M_i \to \mathbb{C} : m \otimes m' \mapsto \mathrm{ev}_{M_i}(m, \alpha(m')).$$

We can use the isomorphism $C_i \simeq \mathrm{End}_{\mathbb{C}}(M_i) \simeq M_i \otimes M_i$, where $M_i \otimes M_i \xrightarrow{\sim} C_i : m \otimes n \mapsto A_{m \otimes n}$, where

$$A_{m \otimes n}(x) = \langle x, n \rangle_\alpha m, \ x \in M_i$$

We can see that for $m, n, x, y \in M_i$, this gives us

$$\begin{aligned}
\langle x, A^{\vee}_{m\otimes n}(y)\rangle_\alpha &= \langle A_{m\otimes n}(x), y\rangle_\alpha \\
&= \langle \langle x, n\rangle_\alpha m, y\rangle_\alpha \\
&= \langle x, n\rangle_\alpha \langle m, y\rangle_\alpha \\
&= \langle x, n\rangle_\alpha \epsilon_{\rho/\sigma} \langle y, m\rangle_\alpha \\
&= \epsilon_{\rho/\sigma} \langle x, \langle y, m\rangle_\alpha n\rangle_\alpha \\
&= \epsilon_{\rho/\sigma} \langle x, A_{n\otimes m} y\rangle_\alpha \\
&= \langle x, \epsilon_{\rho/\sigma} A_{n\otimes m} y\rangle_\alpha,
\end{aligned}$$

which tells us that $A^{\vee}_{m\otimes n} = \epsilon(\rho/\sigma) A_{n\otimes m}$. Picking a basis $\{m_j\}$ for M_i, we can see that the duality will fix, up to a sign $\epsilon(\rho/\sigma)$, the basis elements $A_{m_j\otimes m_j}$, and there will be exactly $\dim(M_i)$ of these basis elements. Hence,

$$\operatorname{tr}(-^{\vee}\big|_{C_i}) = \epsilon(\rho/\sigma)\dim(M_i).$$

□

By the lemma, we get

$$\operatorname{tr}(-^{\vee}) = \sum_{C_i^{\vee}=C_i} \operatorname{tr}\left(-^{\vee}\big|_{C_i}\right) = \sum_{C_i^{\vee}=C_i} \epsilon(M_i)\dim(M_i).$$

Comparing this to our calculation using the basis of standard intertwining operators T_w, which was

$$\operatorname{tr}(-^{\vee}) = \left|w \in W_\sigma \mid w^2 = 1\right| = \sum_{\chi\in \mathrm{Irr}(W_\sigma)} \chi(1),$$

we have

$$\sum_{\chi\in \mathrm{Irr}(W_\sigma)} \chi(1) = \sum_{C_i^{\vee}=C_i} \epsilon(M_i)\dim(M_i).$$

This brings us to our final result of the chapter.

Proposition 10. $\epsilon(M_i) = 1$ *for all irreducible* $C(\sigma)$*-modules* M_i*.*

Proof. From basic character theory, we know that $\chi(1) = \dim(M_\chi)$, where M_χ is a module in the isomorphism class corresponding to the character χ. But because $C(\sigma) \simeq \mathbb{C}[W_\sigma]$, there is a one-to-one correspondence between the the simple $C(\sigma)$-modules M_i and the irreducible W_σ representation M_χ, so that we get

$$\sum_{\substack{\chi=\overline{\chi} \\ \chi \in \mathrm{Irr}(W_\sigma)}} \dim(M_\chi) = \sum_{C_i^\vee = C_i} \epsilon(M_i) \dim(M_i).$$

It follows that $M_i \simeq M_i^\vee$ and $\epsilon(M_i) = 1$, $0 < i \leqslant r$. □

Finally,

Corollary 3. $\epsilon_\theta(\sigma) = \epsilon_\theta(\rho)$.

Proof. We have made note previously that $[V \Rightarrow E]$ is a simple $C(\sigma)$-module, so by Proposition 2, $[V \Rightarrow E]$ is self-dual—a fact we were already made aware of—and that $\epsilon(\rho/\sigma) = 1$. Applying this to Chapter 6, Corollary 2, this tells us $\epsilon_\theta(\rho) = \epsilon_\theta(\sigma)$. □

Chapter 9

Even Better Than Equal

We want to show that the θ-sign, $\epsilon_\theta(\sigma)$, corresponding to $\langle -, - \rangle_\gamma$, is always equal to one whenever (σ, E) is both irreducible and cuspidal.

9.1 Generic Representations

Putting our specific context on hold for the following section, let $G = \mathrm{GL}_m(\mathbb{F}_q)$. To be clear, this will indeed hold in our context when $n = m$, but we are talking more broadly in what follows.

Let U be the strict upper triangular matrices and let $\lambda : U \to \mathbb{C}^\times$ be a linear character. Because $\mathbb{C}^\times$ is abelian, this descends to a map $U/U' \mapsto \mathbb{C}^\times$, where U' is the derived subgroup of U. We have the projection map $U \to \mathbb{F}_q^{m-1}$ defined off U' by

$$\begin{bmatrix} 1 & x_1 & & & & \\ & 1 & x_2 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & x_{m-2} & \\ & & & & 1 & x_{m-1} \end{bmatrix} \mapsto (x_1, x_2, \ldots, x_{m-1}).$$

A quick calculation shows us that multiplication in U is, up to something in U', addition along the superdiagonal, making this is a surjective group homomorphism. Moreover, we can see that U' will not only be contained in the kernel of this map, but will in fact be the kernel of this map. Hence, this gives an isomorphism $U/U' \simeq \mathbb{F}_q^{m-1}$.

Our linear character, then, gives us a character $\mathbf{F}_q^{m-1} \simeq U/U' \to \mathbb{C}^\times$, and a character of $\mathbb{F}_q^{m-1}$ is exactly a choice of characters for each component $\mathbb{F}_q$. We call λ **a generic character** of U if none of these choices are the trivial character.

In fact, letting $\lambda_0 : \mathbb{F}_q \to \mathbb{C}^\times$ be any nonzero character of $\mathbb{F}_q$, any other character μ can be shown to satisfy $\mu(x) = \lambda(ax)$ for some $a \in \mathbb{F}_q^\times$. This means that, by conjugating U by some element of T, we can insist on $\lambda_i = \lambda_j$ for all $i, j < m$, so that every element along the superdiagonal gets sent through the same character on $\mathbb{F}_q$.

Now, let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. We say ρ is **a generic representation** of G exactly when it contains a generic character of U. That is, for the some generic character $\lambda : U \to \mathbb{C}^\times$ of U, there is an nonzero element $v \in V$ such that

$$\sigma(u)v = \lambda(u)v.$$

This is equivalent to saying that

$$\mathrm{Hom}_U(\mathrm{res}V, \mathbb{C}_\lambda) \neq 0,$$

where $\mathbb{C}_\lambda$ is the one-dimensional space affording the U-action corresponding to the linear character $\lambda : U \to \mathbb{C}^\times$. And, as mentioned above, we can insist on the character being $(x_i)_i \mapsto \prod_i \lambda(x_i)$ for some character $\lambda : \mathbb{F}_q \to \mathbb{C}^\times$.

Remark 5. If (ρ, V) is an irreducible representation, this gives us an G-equivariant embedding $V \hookrightarrow \mathrm{Hom}_U(G, \mathbb{C}_\lambda)$ called the Whittaker model, though we will have no use for these in the current paper.

There is a theorem originally due to Gelfand and Graev [GfG62], and generalized by Steinberg [Ste16] that we will need.

Theorem 1. *If (ρ, V) is irreducible and generic, containing the generic character $\lambda : U \to \mathbb{C}^\times$, then λ occurs in $\rho\big|_U$ with multiplicity one.*

This tells us that not only do we know that

$$\mathrm{Hom}_U(\mathrm{res}V, \mathbb{C}_\lambda) \neq 0,$$

but we can say more precisely that

$$\mathrm{Hom}_U(\mathrm{res}V, \mathbb{C}_\lambda) \simeq \mathbb{C}.$$

9.2 The Multiplicity One Condition

How do generic representations pertain to our concerns? Proposition 6.3 of [Spr74] tells us the following:

Proposition 11. *Let (ρ, V) be an irreducible, cuspidal representation of G whose degree is relatively prime to the characteristic of $\mathbb{F}_q$. Then V is generic.*

Further, in [Gre55], Green shows that the degrees of the irreducible, cuspidal representations of G will all be $\prod_{1\leqslant k\leqslant n-1}(q^k-1)$. Hence, they are not even divisible by q, let alone p.

Thus, taken together, these results tell us that the irreducible, cuspidal representations of G are generic. This result coupled with the Gelfand-Graev-Steinberg result tells us that an irreducible, cuspidal representation of $GL_m(\mathbb{F}_q)$ will contain a copy of a generic character with multiplicity one.

9.3 A Dual to the Fixed Vector

Piecing Together the Blocks

Recall that L is a subgroup of block diagonal matrices, so

$$L = \mathrm{GL}_{n_1}(\mathbb{F}_q) \times \mathrm{GL}_{n_2}(\mathbb{F}_q) \times \ldots \times \mathrm{GL}_{n_r}(\mathbb{F}_q)$$

for some integer $r > 0$. We will write L_i for the i^{th} diagonal block, so that $L_i = \mathrm{GL}_{n_i}(\mathbb{F}_q)$, $1 \leqslant i \leqslant r$. The representation space E decomposes along this decomposition into a tensor product of irreducible representations E_i of the L_i; denote the corresponding representation of L_i by σ_i so that $\sigma(\ell)v = \bigotimes_i \sigma_i(\ell_i)v_i$ for $\ell = (\ell_1, \ell_2, \ldots, \ell_r)$ and $v = \bigotimes_i v_i$. Within each L_i, let the strictly upper-triangular matrices be denoted U_i, and let the product of these be denoted U_L—that is, U_L is the subgroup of strict upper triangular matrices of G *contained in* L.

Fix a character $\mu : \mathbb{F}_q \to \mathbb{C}^\times$. For a given L_i, we can apply the results from chapter 9 to find a v_{λ_i} such that $\sigma_i(u)v_{\lambda_i} = \lambda_i(u)v_{\lambda_i},\ u \in U_i$ for the character defined by

$$\lambda : U_i/U_i' \simeq \mathbb{F}_q^{n_i-1} \to \mathbb{C}^\times : (x_j) \mapsto \prod_j \mu(x_j).$$

Having found v_{λ_i} for all $0 < i \leqslant r$, we let $v_0 = \bigotimes_i v_{\lambda_i}$, so that

$$\sigma(u)v_0 = \lambda(u)v_0,\ u \in U.$$

Here $\lambda : U \to \mathbb{C}^\times$ is the obvious character defined by $\lambda : U/U' \simeq \mathbb{F}_q^{n-1} \to \mathbb{C}^\times : (x_i) \mapsto \prod_i \mu(x_i)$ for our chosen $\mu : F_q \to \mathbb{C}^\times$.

In each L_i, we have an element

$$w_i := \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}.$$

Also, notice that θ fixes L_i: Let $\overline{U}_i := \theta(U_i)$, so that $\overline{U}_i = w_i U_i w_i$—conjugation by w_i sends strict upper triangular matrices to strict lower triangular matrices; it also sends the lower to the upper. Now, let w_0 be the product of all the w_i—that is, w_0 will be a block diagonal matrix with each block being the cross-diagonal matrix.

Almost the Correct Bilinear Form

The second component in our current bilinear form, $\langle -, - \rangle_\gamma$, is acted on by σ^θ. However, $\theta(U_L) = \overline{U}_L \neq U_L$, so we cannot say much about how σ^θ acts on E when restricted to U_L. To fix this, we have to slightly change the bilinear form we are considering to one defined on $E \otimes E^{\theta'}$, where $\theta'(\ell) = \theta(w_0 \ell w_0^{-1})$ for $\ell \in L$. A direct calculation shows that

$$\sigma^{\theta'}(u)v_\lambda = \lambda(u)^{-1}v_\lambda,$$

by our choice of λ and w_0.

Let $\gamma' := \gamma \circ \sigma(w_0) : E^{\theta'} \to E^*$. This is an isomorphism of the underlying vector spaces, but we want to show that this is G-equivariant as well. Let $\ell \in L$. Then, for

$v \in V^{\theta'}$, we have

$$\begin{aligned}\gamma'(\sigma^{\theta'}(\ell)v) &= \gamma(\sigma(w_0)\sigma^{\theta}(w_0\ell w_0)v)\\ &= \gamma(\sigma^{\theta}(\ell w_0)v)\\ &= {}^*\sigma(\ell)\gamma(\sigma^{\theta}(w_0)v)\\ &= {}^*\sigma(\ell)\gamma'(v).\end{aligned}$$

This, in turn, gives us a new bilinear form

$$\langle -, - \rangle'_\gamma : E \otimes E^{\theta'} \to \mathbb{C} : v \otimes v' \mapsto \mathrm{ev}_E(v, \gamma'(v')),$$

which is non-degenerate and L-invariant.

Lemma 8. *The linear functional $\gamma'(v_\lambda) \in E^*$ evaluates to a nonzero value at v_λ. In particular $\langle v_\lambda, v_\lambda \rangle'_\gamma \neq 0$*

Proof. Because $v_\lambda \in E^{\theta'}$ is nonzero and is an eigenvector for the λ^{-1} action of U on $E^{\theta'}$, γ' sends v_λ to a nonzero element in $\mathbb{C}_{\lambda^{-1}} = \mathbb{C}^*_\lambda \leqslant E^*$ that acts as an eigenvector for the λ^{-1} action of U. By the multiplicity one of $\mathbb{C}_\lambda$ in E, it follows that $\mathbb{C}_{\lambda^{-1}}$ has multiplicity one in E^* as well, so that $\gamma'(v_\lambda) \in \mathbb{C}^*_\lambda$ and $v_\lambda \in \mathbb{C}_\lambda$ both being nonzero elements implies that $[\gamma'(v_\lambda)](v_\lambda) \neq 0$. This tells us immediately that

$$\langle v_\lambda, v_\lambda \rangle'_\gamma = \mathrm{ev}_V(v_\lambda, \gamma'(v_\lambda)) = [\gamma'(v_\lambda)](v_\lambda) \neq 0.$$

□

By a standard argument (that has been made previously in this paper), V being irreducible means that, for any $v, v' \in V$, $\langle v, v' \rangle'_\gamma = \epsilon_{\theta'}(\sigma)\langle v', v \rangle'_\gamma$ with $\epsilon_{\theta'}(\sigma)^2 = 1$; we will refer to this as the θ'-sign if need be.

Lemma 9. *With notation as above, $\epsilon_\theta(\sigma) = \epsilon_{\theta'}(\sigma)$.*

Proof. On the one hand, for any $v, v' \in V$, we have

$$\langle v, v' \rangle'_\gamma = \epsilon_{\theta'}(\sigma) \langle v', v \rangle'_\gamma$$

by definition of $\epsilon_{\theta'}(\sigma)$. On the other hand, we have

$$\begin{aligned}
\langle v, v' \rangle'_\gamma &= \mathrm{ev}_V(v, \gamma'(v')) \\
&= \mathrm{ev}_V(v, \gamma(\sigma(w_0)v')) \\
&= \mathrm{ev}_V(v, {}^*\sigma(w_0)\gamma(v')) \\
&= \mathrm{ev}_V(\sigma(w_0)v, \gamma(v')) \\
&= \langle \sigma(w_0)v, v' \rangle_\gamma \\
&= \epsilon_\theta(\sigma) \langle v', \sigma(w_0)v \rangle_\gamma \\
&= \epsilon_\theta(\sigma) \mathrm{ev}_V(v', \gamma(\sigma(w_0)v)) \\
&= \epsilon_\theta(\sigma) \mathrm{ev}_V(v', \gamma'(v)) \\
&= \epsilon_\theta(\sigma) \langle v, v' \rangle'_\gamma.
\end{aligned}$$

From these two equalities, we have

$$\epsilon_{\theta'}(\sigma) \langle v', v \rangle'_\gamma = (\sigma) \langle v, v' \rangle'_\gamma = \epsilon_\theta(\sigma) \langle v', v \rangle'_\gamma,$$

for any $v, v' \in V$. The result follows directly. □

The main result shows that these signs are all equal to one.

Theorem 2. $\epsilon_\theta(\sigma) = 1$.

Proof. By Lemma 2, it is enough to show that $\epsilon_{\theta'}(\sigma) = 1$. Letting $v = v' = v\lambda$ in the calculations in the proof of Lemma 2 shows us that

$$\langle v_\lambda, v_\lambda \rangle'_\gamma = \epsilon_{\theta'}(\sigma) \langle v_\lambda, v_\lambda \rangle'_\gamma. \qquad (*)$$

By Lemma 1, we have $\langle v_\lambda, v_\lambda \rangle'_\gamma \neq 0$. It follows that $\epsilon_{\theta'}(\sigma) = 1$. □

Bibliography

[Bum98] Daniel Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1998.

[Car85] Roger W. Carter. *Finite groups of Lie type*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1985. Conjugacy classes and complex characters, A Wiley-Interscience Publication.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[GfG62] I. M. Gel fand and M. I. Graev. Construction of irreducible representations of simple algebraic groups over a finite field. *Dokl. Akad. Nauk SSSR*, 147:529–532, 1962.

[Gow83] R. Gow. Properties of the characters of the finite general linear group related to the transpose-inverse involution. *Proc. London Math. Soc. (3)*, 47(3):493–506, 1983.

[Gre55] J. A. Green. The characters of the finite general linear groups. *Trans. Amer. Math. Soc.*, 80:402–447, 1955.

[HL94] R. B. Howlett and G. I. Lehrer. On Harish-Chandra induction and restriction for modules of Levi subgroups. *J. Algebra*, 165(1):172–183, 1994.

[Isa76] I. Martin Isaacs. *Character theory of finite groups*, volume No. 69 of *Pure and Applied Mathematics*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976.

[Kar85] Gregory Karpilovsky. *Projective representations of finite groups*, volume 94 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1985.

[Rie16] Emily Riehl. *Category theory in context*. Aurora Dover Modern Math Originals. Dover Publications, Inc., Mineola, NY, 2016.

[Spr74] T. A. Springer. Caractères de groupes de Chevalley finis. In *Séminaire Bourbaki, 25ème année (1972/1973)*, volume Vol. 383 of *Lecture Notes in Math.*, pages Exp. No. 429, pp. 210–233. Springer, Berlin-New York, 1974.

[Ste16] Robert Steinberg. *Lectures on Chevalley groups*, volume 66 of *University Lecture Series*. American Mathematical Society, Providence, RI, corrected edition, 2016. Notes prepared by John Faulkner and Robert Wilson, With a foreword by Robert R. Snapp.

[Vin06] C. Ryan Vinroot. Involutions acting on representations. *J. Algebra*, 297(1):50–61, 2006.