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VALUATIVE TREES IN ARBITRARY CHARACTERISTIC AND APPLICATIONS TO LOG CANONICAL THRESHOLDS

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VALUATIVE TREES IN ARBITRARY CHARACTERISTIC AND APPLICATIONS TO LOG CANONICAL THRESHOLDS

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BY THE COMMITTEE CONSISTING OF

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[1]Now called SLMath.

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Abstract

Valuations naturally encode birational invariants of algebraic varieties. This thesis investigates the valuative tree—the valuative space centered at the affine plane over a field—via two complementary approaches. The first analyzes its tree structure through the observer's topology and introduces two parameterizations, skewness and thinness, defined using the sequence of key polynomials (SKP). Although obtained from distinct constructions, thinness is shown to coincide with log-discrepancy functions through the universal dual graph. The second approach employs tools from non-Archimedean geometry. We review the Berkovich affine line over a complete valued field, with particular attention to Hahn series and completed Puiseux series; in these cases, the valuative tree embeds naturally into the Berkovich line, revealing characteristic-dependent phenomena.

As an application, we use valuative trees to compute the log canonical threshold of analytically irreducible plane curves. We conclude by discussing an alternative approach to defining log discrepancy functions in positive characteristic, along with the counterpart to the log canonical threshold—the F-pure threshold.

Chapter 1

Introduction

Valuative spaces play an important role in algebraic geometry. For example, given a field extension $K/\Bbbk$, the *Riemann-Zariski space* is a locally ringed space whose underlying set consists of all valuation rings of K containing $\Bbbk$. In 1962, Nagata used the Riemann-Zariski space to prove that any algebraic $\Bbbk$-variety can be embedded into a proper $\Bbbk$-variety (see [Nag62]). Alternatively, the Riemann-Zariski space can be realized as follows: let X be a projective $\Bbbk$-variety with function field isomorphic to K. The collection of all proper birational morphisms $\pi\colon Y \to X$ forms an inverse system, whose limit recovers the same Riemann-Zariski space. Therefore, even if X is not projective, the Riemann-Zariski space encodes information about all possible algebraic compactifications of X.

In non-Archimedean geometry, valuative spaces also provide crucial foundations. Let (K, v_K) be a real-valued field. Equipped with the ultrametric topology, K is totally disconnected; hence, it lacks a natural notion of analytic continuation as in complex analysis, making it difficult to develop a theory of analytic functions. In 1971, Tate addressed this by introducing Grothendieck topologies, though they are not always intuitive to describe explicitly (see [Tat71]). An alternative approach, due to Berkovich, is to enlarge the space K as follows: observe that K can be seen as the set of K-valued points of $X = \operatorname{Spec} K[x]$, and Berkovich defines X^{an} to be the set of all real semivaluations on $K[x]$ extending v_K (see [Ber90]). The resulting space enjoys desirable topological properties: it is Hausdorff, locally compact, and path-connected.

This construction extends to general K-varieties, thus embedding any such variety into a locally compact Hausdorff space X^{an}.

Rather than considering all real semivaluations on a $\Bbbk$-variety X, one may focus on normalized real semivaluations centered at a closed point $x \in X$. This space, denoted $\mathcal{V}_{X,x}$, was studied by Boucksom, Favre, and Jonsson (see [BFJ08]). In particular, when X is a surface and x is a regular point, $\mathcal{V}_{X,\mathfrak{m}}$ forms a topological tree, known as the *valuative tree*: the case $\Bbbk = \mathbb{C}$ is thoroughly treated in [FJ04], and [Gra07] generalizes this result to arbitrary two-dimensional regular Noetherian local rings. On the other hand, the space of real valuations has been extensively studied by Jonsson and Mustaţă [JM12], who developed a theory of discrepancy functions for quasi-monomial valuations, thereby offering a valuative perspective on the log canonical threshold in birational geometry.

In this thesis, we achieve the following.

- We generalize the valuative tree introduced in [FJ04] into the setting without any restriction on the characteristic of the ground field.

- We further develop the theory of Berkovich affine line by studying the field of Hahn series, generalizing the development using the field of Puiseux series in characteristic 0.

- We provide a method to compute the log canonical threshold of an analytically irreducible plane curve using the valuative tree. Again, this works in arbitrary characteristic.

- We speculate that F-pure thresholds can be computed from the information of valuative spaces.

Structure of the thesis

The thesis is organized as follows.

In Chapter 2, we review the foundational notions introduced in [FJ04], such as sequences of key polynomials and parametrizations including skewness and thinness, and reformulate them in a general setting without assuming $\Bbbk = \mathbb{C}$. We also review necessary background from birational geometry and conclude the chapter with a discussion of the universal dual graph.

Chapter 3 focuses on the field of Hahn series and Puiseux series, along with basic constructions in affine analytic geometry. We then apply these tools to the case where the base field is the field of formal Laurent series, yielding an alternative construction of the valuative tree. At the end of the chapter, we compare the two versions of valuative trees and explore the relationships between their respective parametrizations.

In Chapter 4, we apply our framework to derive a formula for the log canonical threshold of an irreducible formal plane curve, valid in arbitrary characteristic.

Finally, Chapter 5 explores several topics in positive characteristic. In particular, we investigate the computation of the F-pure threshold—the positive characteristic counterpart to the log canonical threshold, and we speculate that F-pure thresholds can be computed via valuative spaces.

Notations and Conventions

- $(\Gamma, \leq)$ always denotes a (totally) ordered abelian group (with binary operation $+$), and the order $\leq$ is omitted when no confusion arises. $\overline{\Gamma}$ is defined as $\Gamma \cup \{\infty\}$, which is a totally ordered set by declaring that $\gamma < \infty$ for all $\gamma \in \Gamma$. We extend the arithmetic operation $+$ to $\overline{\Gamma}$ via the rule: $\gamma + \infty = \infty$ for any $\gamma \in \overline{\Gamma}$.

In particular, $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ are endowed with the usual total orders.

- For an ordered abelian group $\overline{\Gamma}$, we set

$$\begin{aligned}
\Gamma_{>0} &= \{\gamma \in \Gamma \mid \gamma > 0\}, \\
\Gamma_{\geq 0} &= \{\gamma \in \Gamma \mid \gamma \geq 0\}, \\
\overline{\Gamma}_{>0} &= \left\{\gamma \in \overline{\Gamma} \mid \gamma > 0\right\}, \text{ and} \\
\overline{\Gamma}_{\geq 0} &= \left\{\gamma \in \overline{\Gamma} \mid \gamma \geq 0\right\}.
\end{aligned}$$

- A ring in this thesis is always commutative and contains 1, and a ring homomorphism sends 1 to 1. A ring is written as A or B, and a ring homomorphism is usually denoted by $\phi\colon A \to B$ or $\pi^{\#}\colon A \to B$, where $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is the corresponding morphism of spectra. When A is an integral domain, $\operatorname{char} A$ denotes the characteristic of A. In particular, we always let $R = \Bbbk[[x_1, x_2, \ldots, x_n]]$ or more specifically $R = \Bbbk[[x, y]]$.

  We use $\mathfrak{a}$, $\mathfrak{b}$, I, J to denote an ideal. The notations $\mathfrak{p}$ and $\mathfrak{m}$ mean a prime ideal and a maximal ideal, respectively.

- We write F to mean an arbitrary field, and $\Bbbk$ for an algebraically closed field. The symbols K and L usually mean real-valued fields (with the valuation omitted). Note that $\Bbbk$ can be regarded as a real-valued field with the trivial valuation. We reserve the notation $\mathbb{K}$ to mean the complete algebraic closure $\widehat{\Bbbk((x))^{\mathrm{alg}}}$ of the formal Laurent series field $\Bbbk((x))$ (see Definition 3.3.9).

- A variety in this thesis is an integral, separated, finite type $\Bbbk$-scheme.

- There are three symbols relating to valuative spaces of $R = \Bbbk[[x, y]]$ (see Definition 2.3.7):

$$\begin{aligned}
\mathcal{V}_R &= \{\text{centered, nonnegative real semivaluations on } R\}\,, \\
\mathcal{V}_{\mathfrak{m}} &= \{v \in \mathcal{V}_R \mid v(\mathfrak{m}) = 1\}\,, \text{ and} \\
\mathcal{V}_x &= \{v \in \mathcal{V}_R \mid v(x) = 1\}\,.
\end{aligned}$$

  There is also a valuative tree as a subset of the Berkovich affine line $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{K}} = (\operatorname{Spec} \mathbb{K}[y])^{\mathrm{an}}$ from Definition 3.6.3:

$$\widehat{\mathcal{V}}_x = \left\{v \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{K}} \mid v(y) > 0\right\} \cup \{\hat{v}_*\}.$$

- There are several symbols regarding dual graphs from Definition 2.10.4, Definition 2.10.6, and Definition 2.10.8:

$$\begin{aligned}
\Gamma_\pi &= \text{the combinatorial graph induced from } \pi\colon Y \to X, \\
\Gamma^*_\pi &= \text{the vertices of } \Gamma_\pi, \\
\Gamma^* &= \operatorname*{colim}_{\pi \in \mathfrak{B}} \Gamma^*_\pi, \\
\Gamma^*_{\mathbb{R}} &= \text{the nonmetric } \mathbb{R}\text{-tree induced by } \Gamma^*, \text{ and} \\
\Gamma &= \text{the completion of } \Gamma^*_{\mathbb{R}}.
\end{aligned}$$

- In a valued field (K, v), a closed ball is denoted by $B(f, \gamma)$ (see Definition 3.1.6), which should not be confused with an open ball.

Chapter 2

Valuative Trees via Key Polynomials

In this section, we introduce the valuative tree following [FJ04]. The notions of key polynomials and the language of non-metric trees will serve as the main tools for describing valuative trees, which will be reviewed in Section 2.4 and Section 2.5, respectively.

2.1 Semivaluations and Multiplicative Seminorms

We begin by recalling several notions of valuations used in mathematics.

Definition 2.1.1 (Krull Semivaluations)**.** Let A be a field, and Γ a totally ordered abelian group. Denote the set $\Gamma \cup \{\infty\}$ by $\overline{\Gamma}$ equipped with the extended total order and the binary operation (see Notations and Conventions).

- A *(Krull) semivaluation* is a non-constant function $v \colon A \to \overline{\Gamma}$ such that, for $f, g \in A$,

  (V1) $v(fg) = v(f) + v(g)$,

  (V2) $v(f+g) \geq \min\{v(f), v(g)\}$, and

  (V3) $v(0) = \infty$.

- A semivaluation is *real* if $\Gamma = \mathbb{R}$, and is *nonnegative* if its image is in $\overline{\Gamma}_{\geq 0}$.

- The *home* of a semivaluation v is the prime ideal $h_A(v) = v^{-1}(\infty)$. If $h_A(v) = \{0\}$, we call v a *valuation*. A semivaluation $v\colon A \to \overline{\Gamma}$ always induces a valuation on the quotient $\bar{v}\colon A/h_A(v) \to \overline{\Gamma}$.

- The *valuation ring* of a valuation v is the subring $\mathcal{O}_v = v^{-1}(\overline{\Gamma}_{\geq 0})$ of A, and v restricts to a nonnegative valuation on $\mathcal{O}_v$.

- Let v be a nonnegative valuation on A. The *center* is the prime ideal $c_A(v) = v^{-1}(\overline{\Gamma}_{>0})$. We clearly have the inclusions $h_A(v) \subset c_A(v) \subset A$. A nonnegative valuation v is *centered* on A if $c_A(v) \neq \{0\}$, and is *centered at* $\mathfrak{p}$ if $c_A(v) = \mathfrak{p} \neq \{0\}$.

- A *valued field* is a pair (K, v), where K is a field and v is a valuation on K. (K, v) is a *real-valued field* if v is real.

Remark 2.1.2 (Multiplicative Seminorms)**.** Let A be a ring. A (nonarchimedean) *multiplicative seminorm* is a function $||\cdot||\colon A \to \mathbb{R}_{\geq 0}$ such that, for $f, g \in A$,

- $||f \cdot g|| = ||f|| \cdot ||g||$,

- $||f + g|| \leq \max\{||f||, ||g||\}$, and

- $||0|| = 0$.

A multiplicative seminorm $||\cdot||$ induces a real valuation v via $v(f) = -\log(||f||)$, and vice versa.

Remark 2.1.3. Another way to define a real valuation is via the notion of a semiring. Define the *tropical addition* and the *tropical multiplication* on $\overline{\mathbb{R}}$ by the rules: for $x, y \in \overline{\mathbb{R}}$,

$$x \oplus y = \min\{x, y\}, \text{ and}$$

$$x \otimes y = x + y.$$

With these operations, $(\overline{\mathbb{R}}, \oplus, \otimes)$ becomes an idempotent semiring with additive identity ∞ and multiplicative identity 0, and is called the *tropical semiring*.

For an ideal $I \subset A$ and a real semivaluation, define $v(I) = \inf\{v(f) \mid f \in I\}$. Then, it is not hard to see that, for two ideals I and J of A, we have $v(I \cdot J) = v(I) + v(J)$ and $v(I + J) = \min\{v(I), v(J)\}$. If we define $\mathcal{I}$ to be the collection of ideals in A, then $(\mathcal{I}, +, \cdot)$ is an idempotent semiring with additive identity $\{0\}$ and multiplicative identity A. Therefore, a real valuation v always induces a homomorphism of semirings $v\colon (\mathcal{I}, +, \cdot) \to (\overline{\mathbb{R}}, \oplus, \otimes)$. Conversely, given a semiring homomorphism $v\colon (\mathcal{I}, +, \cdot) \to (\overline{\mathbb{R}}, \oplus, \otimes)$, we can define a real semivaluation on A via $v(f) = v(fA)$ for $f \in A$. However, these two constructions are mutually inverse only when v is centered, see [JM12, Section 1.2].

Theorem 2.1.4 ([HS06, Proposition 6.5.4])**.** *Let $A \hookrightarrow B$ be a faithfully flat inclusion of domains, where A is Noetherian. Then, a nonnegative Krull valuation on A extends to a nonnegative Krull valuation on B.*

Remark 2.1.5. We will only use Theorem 2.1.4 in the special case that $(A, \mathfrak{m})$ is a regular local ring, and B is the $\mathfrak{m}$-adic completion of A. Also, see [HS06, Discussion 6.5.5] for topics about the uniqueness of extensions in Theorem 2.1.4.

Theorem 2.1.4 applies to nonnegative real semivaluations on a domain: such a real semivaluation can be lifted to a Krull valuation in the field of fractions.

Theorem 2.1.6. *If (v_j) is a (point-wise) increasing sequence of real semivaluations on A indexed by a totally ordered set J, then $v \triangleq \sup_{j \in J} v_j$ is a real semivaluation on A.*

Proof. It is clear that v is a well-defined function. For $i, j \in I$ with $i \leq j$ and $f, g \in A$, we observe that

$$v_j(f \cdot g) = v_j(f) + v_j(g) \leq v(f) + v(g), \text{ and}$$
$$v_i(f) + v_j(g) \leq v_j(f) + v_j(g) = v_j(f \cdot g) \leq v(f \cdot g).$$

Taking supremum yields $v(f \cdot g) = v(f) + v(g)$. For the sub-additivity of v, we notice that

$$\min\{v_i(f),\ v_j(g)\} \leq \min\{v_j(f),\ v_j(g)\} \leq v_j(f+g) \leq v(f+g).$$

Taking supremum twice and noting that the min function is 'continuous', we obtain the desired inequality $\min\{v(f),\ v(g)\} \leq v(f+g)$. □

2.2 Arithmetic Examples

Most semivaluations arise either from the order of vanishing along a prime divisor or from the pullback along a ring homomorphism. Specifically, if there is a ring map $\phi\colon A \to B$ and a semivaluation $v\colon B \to \overline{\mathbb{R}}$, the composition $v \circ \phi\colon A \to \overline{\mathbb{R}}$ is again a semivaluation. These two constructions account for the majority of semivaluations encountered in practice.

Example 2.2.1 (The Trivial Valuation)**.** For any ring A, we define the *trivial valuation* v_{triv} by the rule that $v_{\mathrm{triv}}(f) = 0$ if and only if $f \neq 0$.

Example 2.2.2 ($\mathfrak{p}$-adic Valuations)**.** Let A be a principal ideal domain. Pick a prime ideal $\mathfrak{p}$ and a number $c \in [0, \infty]$. Then the function

$$v_{\mathfrak{p},c} = c \cdot \mathrm{ord}_{\mathfrak{p}}$$

is a semivaluation,[1] where $\operatorname{ord}_{\mathfrak{p}}(a) = q \min\{e \in \mathbb{Z}_{\geq 0} \mid a \in \mathfrak{p}^e\}$ for $a \in A$ is the order of $\mathfrak{p}$. Note that when $c = \infty$, the valuation $v_{\mathfrak{p},\infty}$ is equal to the composition

$$A \to A/\mathfrak{p} \xrightarrow{v_{\mathrm{triv}}} \overline{\mathbb{R}}$$

of the natural quotient map and the trivial valuation on the quotient.

When $A = \mathbb{Z}$, this construction gives all possible semivaluations. This is part of Ostrowski's Theorem.

When $A = F[x]$ where F is a field, we can consider the *degree valuation*[2] $v_{\infty,c}$, for $c \in [0, \infty)$, by

$$v_{\infty,c}(f) = -c \cdot \deg(f).$$

In this case, every semivaluation arises from one of the constructions given above. See Figure 2.1 for a visualization of the collections of semivaluations for $A = \mathbb{Z}$ and $A = F[x]$.

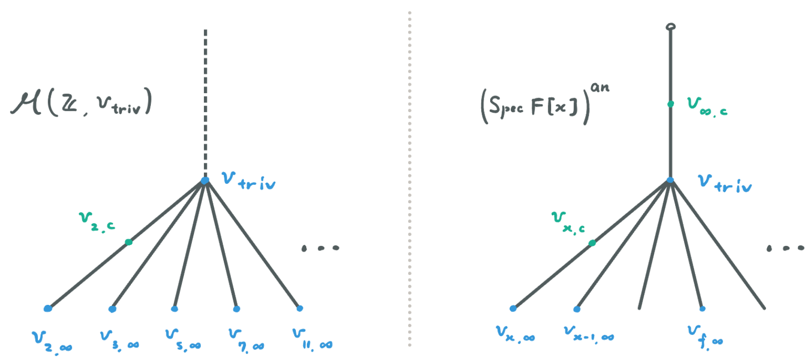


Figure 2.1: The picture on the left (resp. right) is the collection of semivaluations on $\mathbb{Z}$ (resp. $F[x]$).

[1] If $c = 0$, we just recover the trivial valuation on A.
[2] One way to think of the degree valuation is $-\deg = \operatorname{ord}_{x^{-1}}$.

Remark 2.2.3 (I-adic valuations, [HS06, Theorem 6.7.8])**.** In the previous example, we used the order $\operatorname{ord}_{\mathfrak{p}}$ to define valuations. More generally, let I be an ideal of a Noetherian ring A so that $\cap_n I^n = 0$. The order function ord_I is a valuation if and only if the associated graded ring $\operatorname{gr}_I A = \bigoplus_{n\geq 0} I^n/I^{n+1}$ is an integral domain.

2.3 Geometric Examples

Now we look at examples from the geometric side. If X is a variety over an algebraically closed field $\Bbbk$ and $x \in X$ is a regular point, then the stalk $\mathcal{O}_{X,x}$ is a Noetherian regular local ring. We assume that $\mathcal{O}_{X,x}$ is equicharacteristic (this holds, for example, when $\operatorname{char} \Bbbk = 0$). Then the Cohen structure theorem says that the ring completion $\widehat{\mathcal{O}_{X,x}}$ is isomorphic to $\Bbbk[[x_1, \ldots, x_n]]$, where n is the codimension of $x \in X$. By Theorem 2.1.4, the semivaluations on $\mathcal{O}_{X,x}$ centered at the maximal ideal $\mathfrak{m}_x\mathcal{O}_{X,x}$ and those on $\Bbbk[[x_1, \ldots, x_n]]$ centered at $\mathfrak{m} = (x_1, \ldots, x_n)$ are identical.[3]

In this section, we write $R = \Bbbk[[x_1, \ldots, x_n]]$. A monomial $x_1^{u_1} \cdots x_n^{u_n}$ in R will be written as x^u for $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n_{\geq 0}$.

Example 2.3.1 (Monomial Valuations)**.** Pick $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$.[4] Define the *monomial valuation* (or *Gauss valuation*) $v_{\mathbf{w}}$ on R as

$$v_{\mathbf{w}}\left(\sum_{\mathrm{u}} c_{\mathrm{u}} x^{\mathrm{u}}\right) = \min\left\{\mathbf{w} \cdot \mathrm{u} \mid c_{\mathrm{u}} \neq 0\right\}.$$

If $\mathbf{w} = (1, 1, \ldots, 1)$, then $v_{\mathbf{w}}$ is nothing but the $\mathfrak{m}$-adic valuation $\operatorname{ord}_{\mathfrak{m}}$.

[3]However, the extended semivaluation on $\widehat{\mathcal{O}_{X,x}}$ of a valuation on $\mathcal{O}_{X,x}$ may not be a valuation.

[4]In general, we can pick arbitrary $\mathbf{w} \in \mathbb{R}^n$, but those $\mathbf{w} \in \mathbb{R}^n \setminus \mathbb{R}^n_{\geq 0}$ give non-centered valuations on R.

Example 2.3.2 (Divisorial Valuations)**.** Let $\pi\colon Y \to X = \operatorname{Spec} R$ be a (proper) birational morphism from a normal variety Y, and let $E \subset Y$ be a prime divisor which gives the pullback of functions $\pi^{\#}\colon R \to \mathcal{O}_{Y,E}$. Then the composition

$$\operatorname{div}_E \triangleq \pi_* \operatorname{ord}_E = \operatorname{ord}_E \circ \pi^{\#}$$

defines a valuation, called a *divisorial valuation*.

Note that $c_R(\operatorname{div}_E) = \mathfrak{m}$ if and only if $E \subset \pi^{-1}(\mathfrak{m})$ is an exceptional component. Also, we will write $c_R(E) = c_R(\operatorname{div}_E)$.

Remark 2.3.3. A divisorial valuation can also be intrinsically defined (see [HS06, Definition 9.3.1]). Let A be a Noetherian integral domain, and v a valuation centered on A. Then v has its valuation ring V in $\operatorname{Frac} A$ and the residue field κ_v. Write $\kappa = \operatorname{Frac}(A/c_v(A))$. If the transcendence degree of the field extension κ_v/κ, denoted by $\operatorname{tr.deg}(v)$, is equal to $\operatorname{ht}(c_A(v)) - 1$, then v is a *divisorial valuation*. When A is a finite-type $\Bbbk$-algebra domain, we can reach $\operatorname{Spec} V$ by a sequence of blow-ups (see [KM98, Lemma 2.45]).

Example 2.3.4 (Quasimonimial Valuations)**.** Again, let $\pi\colon Y \to \operatorname{Spec} R$ be a (proper) birational morphism for Y normal. Pick a regular closed point[5] y and a monomial valuation $v_{\mathbf{w}}$ with respect to some local system of parameters at y. Then the composition

$$\pi_* v_{\mathbf{w}} = v_{\mathbf{w}} \circ \pi^{\#}$$

is a valuation, named a *quasimonimial valuation*.

Every divisorial valuation is quasimonomial. Let $E \subset Y$ be a divisor over $\operatorname{Spec} R$ for some normal variety Y. The normality of Y implies that the singular locus of Y must

[5]Because R has a unique closed point, we must have $y \in \pi^{-1}(\mathfrak{m})$.

have codimension at least 2, so we can pick a regular closed point y in E. On the local ring $\mathcal{O}_{Y,y}$, the divisor E is cut out by an equation f_1. Extending f_1 to a local system of parameters $f_1, \ldots, f_n$, we get the completion map $\mathcal{O}_{Y,y} \to \Bbbk[[y_1, \ldots, y_n]]$ which sends f_j to y_j. The divisorial valuation div_E is the quasi-monomial valuation $v_{(1,0,\ldots,0)}$ at y with respect to the indeterminates $y_1, \ldots, y_n$.

Example 2.3.5 (Curve Semivaluations)**.** In this example, let $n = 2$ for simplicity. Let $f \in R$ be an irreducible element; geometrically, it defines an analytically irreducible curve $\mathrm{V}(f)$ on $\operatorname{Spec} R$. The intersection multiplicity defined by

$$v_f \colon g \mapsto \dim_{\Bbbk} \frac{\Bbbk[[x_1, x_2]]}{(f, g)}$$

is a semivaluation, called a *curve semivaluation*. Note that $v_f(g) = \infty$ if and only if g is a multiple of f, and thus if $v_f = v_g$ then $f = gu$ for some $u \in R^\times$.

By abuse of notation, we will denote the curve semivaluation in $\mathcal{V}_{\mathfrak{m}}$ or $\mathcal{V}_x$ also by v_f.

Notation 2.3.6 (The Ring R)**.** From now on, until the end of the thesis, we focus on the semivaluations on $\Bbbk[[x, y]]$ centered at the origin, where $\Bbbk$ is an algebraically closed field. We write this ring $R = \Bbbk[[x, y]]$ and the unique maximal ideal $\mathfrak{m} = (x, y)$, and x, y will always be the regular system of parameters when we talk about monomial valuations. However, we may choose an alternative regular system of parameters if needed, and still refer to them as x and y.

Definition 2.3.7 (Valuative Spaces)**.** We define $\mathcal{V}_R$ to be the collection of all centered, nonnegative real semivaluations v on R.[6]

[6] Our definition $\mathcal{V}_R$ is bigger than the space $\widetilde{\mathcal{V}}$ defined in [FJ04] because we include the trivial valuation.

Further, define $\mathcal{V}_\mathfrak{m}$ (resp. $\mathcal{V}_x$) to be the collection of $v \in \mathcal{V}_R$ such that $v(\mathfrak{m}) = 1$ (resp. $v(x) = 1$).

Definition 2.3.8 (Basic Structures on Valuative Spaces)**.**

1. Each $f \in R$ gives a function on $\mathcal{V}_R$ to $\overline{\mathbb{R}}_{\geq 0}$ by the evaluation map

$$\mathrm{eval}_f \colon v \mapsto v(f).$$

   Hence, the valuative space $\mathcal{V}_R$ is naturally equipped with the *weak topology* such that eval_f is continuous for all $f \in R$. At the same time, $\mathcal{V}_\mathfrak{m}$ and $\mathcal{V}_x$ are equipped with the subspace topologies from $\mathcal{V}_R$, called the weak topologies as well.

2. For $v, w \in \mathcal{V}_\mathfrak{m}$ (resp. $\mathcal{V}_x$), define $v \leq w$ if $v(f) \leq w(f)$ for all $f \in R$. This makes $\mathcal{V}_\mathfrak{m}$ (resp. $\mathcal{V}_x$) a poset with a unique minimum $\mathrm{ord}_\mathfrak{m}$ (resp. ord_x).

We will see later that $(\mathcal{V}_\mathfrak{m}, \leq)$ has a non-metric tree structure which induces a topology equal to the weak topology. The discussion of the space $(\mathcal{V}_x, \leq)$ will be in Section 2.11.

Remark 2.3.9. Examples 2.3.1 to 2.3.5 give elements in $\mathcal{V}$, but normalizations are needed. For example, if $v_{(a,b)} \in \mathcal{V}_\mathfrak{m}$ is a monomial valuation, then $\min\{a, b\} = 1$.

A quasimonomial valuation that is not divisorial is called an *irrational valuation.*

Note that the above examples do not exhaust the space $\mathcal{V}$; the rest are valuations, called *infinitely singular valuations.*

2.4 Sequences of Key Polynomials (SKPs)

Recall that $R = \Bbbk[[x, y]]$. To analyze the valuative space $\mathcal{V}_\mathfrak{m}$, we introduce the notion of a sequence of key polynomials, which Mac Lane initially develops in [Mac36].

Definition 2.4.1 (SKPs)**.** Let $k \in \overline{\mathbb{Z}}_{>0}$. A *sequence of key polynomials* (SKP) of R is a sequence of polynomials $(U_j)_{j=0}^k$ and a sequence of positive real numbers $(\tilde{\beta}_j)_{j=0}^k$ that satisfy the following conditions:

(S1) $U_0 = x$ and $U_1 = y$.

(S2) For $1 \leq j < k$, there are relations

$$\tilde{\beta}_{j+1} > n_j \tilde{\beta}_j = \sum_{l=0}^{j-1} m_{j,l} \tilde{\beta}_l,$$

where $n_j \in \mathbb{Z}_{>0}$ and $m_{j,l} \in \mathbb{Z}_{\geq 0}$ satisfy the conditions: for all $1 \leq j < k$ and $0 < l < j$,

$$n_j = \min \left\{ n \in \mathbb{Z}_{>0} \,\middle|\, n\tilde{\beta}_j \in \mathbb{Z}\tilde{\beta}_1 + \cdots + \mathbb{Z}\tilde{\beta}_{j-1} \right\}$$

and $0 \leq m_{j,l} < n_j$.

(S3) For $j \geq 1$, there is $\theta_j \in \Bbbk^*$ such that

$$U_{j+1} = U_j^{n_j} - \theta_j U_0^{m_{j,0}} U_1^{m_{j,1}} \cdots U_{j-1}^{m_{j,j-1}}.$$

An SKP can be denoted by a pair $[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$. If $k < \infty$, we say that the SKP is *finite*, and *infinite* otherwise.

Remark 2.4.2. In Definition 2.4.1 (S3), the exponents n_j, $m_{j,0}, \ldots, m_{j,j-1}$ are always uniquely determined. In other words, the relations in (S2) are uniquely satisfied. See [FJ04, Lemma 2.7] for a complete statement.

Lemma 2.4.3 (First Properties of SKPs)**.** *Let $[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ be an SKP.*

- *All U_j are irreducible (in R) and in Weierstrass form.*

- *The sequence* $(\tilde{\beta}_j/\deg_y U_j)_0^k$ *is strictly increasing.*[7]

Proof. See [FJ04, Lemma 2.4]. $\square$

Theorem 2.4.4 (From SKPs to Semivaluations)**.** *An SKP* $[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ *determines a smallest semivaluation* $v = \mathrm{val}[(U_j)_0^k; (\tilde{\beta}_j)_0^k] \in \mathcal{V}_R$ *so that* $v(U_j) = \tilde{\beta}_j$ *for all* $0 \leq j \leq k$.

Proof. Let us sketch the idea of how an SKP determines a semivaluation by induction on the length k of the SKP. Assume $k < \infty$ for a moment. First, define the monomial valuation v_1 with the weight $(\tilde{\beta}_0, \tilde{\beta}_1)$. Assume that $v_1, \ldots, v_{k-1}$ are defined from $[(U_j)_0^{k-1}; (\tilde{\beta}_j)_0^{k-1}]$. Note that for $f \in \Bbbk[x, y]$, we can write

$$f = \sum_i g_i U_k^i,$$

where $g_i \in k[x, y]$ with $\deg_y g_i < \deg_y U_k$. Define

$$v_k(f) = \min_i \left\{ v_{k-1}(g_i) + i\tilde{\beta}_k \right\}.$$

Then, v_k is a semivaluation on $k[x, y]$ (see [FJ04, Section 2.1.3]). By Theorem 2.1.4, v_k extends uniquely to a semivaluation on R. We then let $v = v_k$.

Now, assume $k = \infty$. By the above construction we have $v_1, v_2, \ldots, v_j, \ldots$, and then Lemma 2.4.5, stated below, says that v_j converges (weakly) to a semivaluation v_∞ as $j \to \infty$. We let $v = v_\infty$. $\square$

Lemma 2.4.5 (Infinite SKPs)**.** *Let* $[(U_j)_0^\infty; (\tilde{\beta}_j)_0^\infty]$ *be an infinite SKP, and set* $v_k = \mathrm{val}[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ *for all* $k \geq 1$. *Recall the* n_j *from Definition 2.4.1.*

- v_k *converges (weakly) to a semivaluation* v_∞ *as* $k \to \infty$.

---

[7] We may define $\deg_y U_0 = \deg_y x = -\infty$, so $\tilde{\beta}_0/\deg_y U_0 = 0$.

- *If $n_j = 1$ for all k large enough, then U_k converges to an irreducible formal power series $U_\infty \in R$, and $v_\infty = v_{U_\infty}$.*

Proof. See [FJ04, Theorem 2.22]. $\square$

Theorem 2.4.6 (SKPs Determine All Semivaluations)**.** *For any $v \in \mathcal{V}_R$, there exists a unique SKP $[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ such that $v = \mathrm{val}[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$. We say that k is the* length *of v, denoted by* SKPLength v.[8]

Proof. See [FJ04, Theorem 2.29]. $\square$

Remark 2.4.7. From Theorem 2.4.6, we should be able to associate an SKP with $v \in \mathcal{V}_R$. The procedure is as follows: first, let $U_0 = x$, $U_1 = y$, $\tilde{\beta}_0 = v(x)$, and $\tilde{\beta}_1 = v(y)$. If v is monomial, then we are done; otherwise, there must be a (unique) binomial of the form $U_2 = y^{n_1} - \theta_1 x^{m_{1,0}}$, where $\theta_1 \in \Bbbk^*$ and $n_1, m_{1,0} \in \mathbb{Z}_{\geq 0}$, satisfying $n_1\tilde{\beta}_1 = m_{1,0}\tilde{\beta}_0$, such that $\tilde{\beta}_2 \triangleq v(U_2) > n_1\tilde{\beta}_1$.

From Remark 2.4.2, there exist $n_2, m_{2,1}, m_{2,0} \in \mathbb{R}$, satisfying the conditions in $(S2)$, so that the polynomial

$$U_3 = U_2^{n_2} - \theta_2 x^{m_{2,0}} y^{m_{2,1}}$$

is defined, where $\theta_2 \in \Bbbk^*$. If $\tilde{\beta}_3 \triangleq v(U_3) = n_2\tilde{\beta}_2$ for all $\theta_2 \in \Bbbk^*$, then we are done. Otherwise, there exists a (unique) θ_2 such that $\tilde{\beta}_3 > n_2\tilde{\beta}_2$, and we continue similarly.

Theorem 2.4.8 (Classification of Semivaluations via SKPs)**.** *Assume that $v \in \mathcal{V}_R$ is obtained by an SKP $[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$.*

1. *v is monomial if and only if $k = 1$ and $\tilde{\beta}_0, \tilde{\beta}_1 < \infty$.*

2. *v is quasimonomial if and only if $k < \infty$, $\tilde{\beta}_0, \tilde{\beta}_k < \infty$.*

3. *v is divisorial if and only if v is quasimonomial and $\tilde{\beta}_k \in \mathbb{Q}$.*

[8] In [FJ04], they use length v to denote the length of the corresponding SKP.

4. *v is irrational if and only if v is quasimonomial and $\tilde{\beta}_k \notin \mathbb{Q}$.*

5. *v is a curve semivaluation if and only if either $k = \infty$ and $\deg_y U_j \nrightarrow \infty$, or $k < \infty$ and $\max\{\tilde{\beta}_0, \tilde{\beta}_k\} = \infty$.*

6. *v is infinitely singular if and only if $k = \infty$ and $\deg_y U_j \to \infty$ as $j \to \infty$.*

Proof. See [FJ04, Section 2.2] for a more detailed discussion.

Item 1 is straightforward. For item 2, see [FJ04, Proposition 6.41].

To prove item 3, use [FJ04, Theorem 2.28] to show that $\operatorname{tr.deg}(v) = 1$ if and only if the corresponding SKP $[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ has the properties that $k < \infty$, $\tilde{\beta}_0 < \infty$, and $\tilde{\beta}_k \in \mathbb{Q}$. When item 2 and item 3 are already proved, item 4 immediately follows.

To prove item 5, suppose that the SKP satisfies the assumption. If $k < \infty$ then either $v(U_0) = \infty$ or $v(U_k) = \infty$, meaning that either $v = v_x$ or $v = v_{U_k}$. The case when $k = \infty$ is from Lemma 2.4.5.

We know that infinitely singular valuations are those that do not belong to any of the other five types. The remaining SKP condition is exactly what characterizes this case, so item 6 follows. $\square$

Remark 2.4.9. Note that in item 5 of Theorem 2.4.8, if v_f is a curve semivaluation given by $f \in R$ such that f is monic in y, then f need not be equal to U_k ($k \leq \infty$). In this case, we point out that $U_\infty \in \Bbbk[[x]][y]$ must be the corresponding Weierstrass polynomial of f in y. The key idea is that in the proof of [FJ04, Theorem 2.22], it is shown that when k is large enough so that $\deg_y U_k$ stabilizes,

$$U_{k+1} - U_k \in x^{e_k} k[[x]][y]$$

for some power e_k, and $e_k \to \infty$ as $k \to \infty$.

Although this connection between SKPs and the Weierstrass form is not explicitly stated in [FJ04], we should emphasize that the SKP method provides an efficient way to compute the Weierstrass form of an irreducible polynomial in R.

Example 2.4.10. Here is an example illustrating how to use SKPs to approximate the Weierstrass form of a polynomial. Assume $\operatorname{char} \Bbbk = 2$ and consider

$$f = y^4 + (1 + x^2)y^2 + x^4 y + x^3 \in \Bbbk[[x]][y].$$

We then run the SKP computation for the normalized curve valuation v_f and obtain $[(U_0, U_1, \ldots, U_9); (\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_9)]$:

U_j	$\hat{\beta}_j$
x	1
y	3/2
$y^2 + x^3$	7/2
$y^2 + x^2y + x^3$	5
$y^2 + x^2y + x^3 + x^5$	11/2
$y^2 + (x^2 + x^4)y + x^3 + x^5$	6
$y^2 + (x^2 + x^4)y + x^3 + x^5 + x^6$	8
$y^2 + (x^2 + x^4)y + x^3 + x^5 + x^6 + x^8$	9
$y^2 + (x^2 + x^4)y + x^3 + x^5 + x^6 + x^8 + x^9$	19/2
$y^2 + (x^2 + x^4 + x^8)y + x^3 + x^5 + x^6 + x^8 + x^9$	10

One can show that this computation never ends by analyzing the (S2) condition, and see that U_9 is a truncation of the Weierstrass form of f.

2.5 Non-metric Trees, $\mathbb{R}$-trees, and Topologies

Definition 2.5.1 (Convex Subsets of Posets)**.** A totally ordered subset $\mathcal{S}$ of a poset $(\mathcal{T}, \leq)$ is *convex* if $\sigma, \sigma' \in \mathcal{S}$ and $\tau \in \mathcal{T}$ with $\sigma \leq \tau \leq \sigma'$ implies $\tau \in \mathcal{T}$.

Definition 2.5.2 (Non-metric Trees)**.** Let Λ be a totally ordered set. A *non-metric Λ-tree*[9] is a poset $(\mathcal{T}, \leq)$ with the following properties.

(T1) Every nonempty subset $\mathcal{S}$ of $\mathcal{T}$ admits an infimum $\wedge_{\tau \in \mathcal{S}} \tau$. The unique minimal element τ_0 of $\mathcal{T}$ is called the *root* of $\mathcal{T}$.

(T2) If $\tau \in \mathcal{T}$, the set $\{\sigma \,|\, \sigma \leq \tau\}$ is isomorphic to a Λ-interval.

(T3) Every convex subset of $\mathcal{T}$ is isomorphic to a Λ-interval.

We will use the term *non-metric tree* to mean a non-metric $\mathbb{R}$-tree. A non-metric tree is *complete* if every totally ordered subset of $\mathcal{T}$ has an upper bound. Also, a *non-rooted non-metric tree* is obtained from a non-metric tree by forgetting the order.

Remark 2.5.3. In the Definition 2.5.2, the purpose of (T1) is to avoid the following Figure 2.2, and the condition (T3) is to avoid the 'long half-line' (see [FJ04, Remark 3.2]).

Remark 2.5.4. A non-metric tree $(\mathcal{T}, \leq)$ is a 'tree' because it doesn't contain any 'cycle'. Suppose $\tau, \tau', \sigma, \sigma' \in \mathcal{T}$ so that $\sigma \not\leq \sigma'$, $\sigma \not\geq \sigma'$, $\tau < \sigma < \tau'$, and $\tau < \sigma' < \tau'$. Then the set $\{\sigma'' \,|\, \sigma'' \leq \tau'\}$ is not totally ordered, so it is not isomorphic to a real interval, which contradicts (T2).

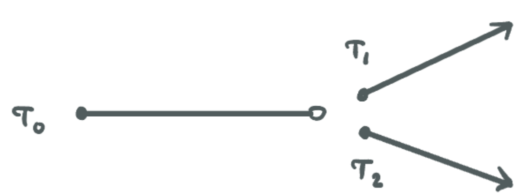


Figure 2.2: A poset with root τ_0 but $\tau_1 \wedge \tau_2$ is not defined. We intend to avoid such 'disconnection of tripod'.

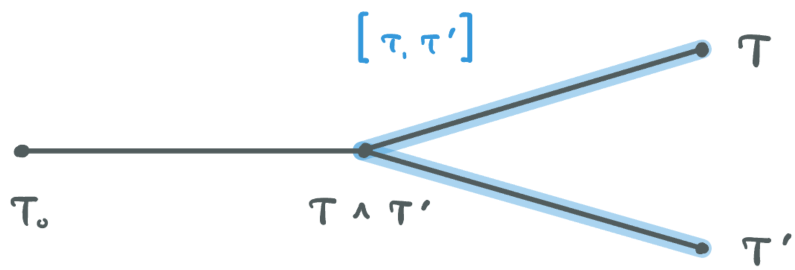
[9]Our definition is called a *rooted nonmetric tree* in [FJ04], and by 'forgetting the root' we get their version of nonmetric tree.

Definition 2.5.5 (Points in Non-metric Trees)**.** Let $(\mathcal{T}, \leq)$ be a non-metric tree.

- If $\tau, \tau' \in \mathcal{T}$ then we define a *segment* as

$$[\tau, \tau'] \triangleq \{\sigma \mid \tau \wedge \tau' \leq \sigma \leq \tau \text{ or } \tau \wedge \tau' \leq \sigma \leq \tau'\}.$$

  See the following picture for a segment.



  We define an open-closed interval $(\tau, \tau']$ or an open interval (τ, τ') in the same way without the corresponding end points.

- Let $\tau \in \mathcal{T}$. Define a relation on $\mathcal{T} \setminus \{\tau\}$ by claiming that σ and σ' are equivalent if the segments $(\tau, \sigma]$ and $(\tau, \sigma']$ intersect. This is an equivalence relation, and an equivalence class is called a *tangent vector* at τ. Thus, the collection of all equivalence classes is called the *tangent space* at τ, denoted by T_τ. If $\sigma \in \mathcal{T}$ so that $[\sigma] = \mathbf{v} \in T_\tau$, we say that σ *represents* $\mathbf{v}$.

- $\tau \in \mathcal{T}$ is an *end* if T_τ is a singleton, is a *regular* point if T_τ is a doubleton, and is a *regular point* otherwise.

Definition 2.5.6 (Parametrizations)**.** Given a non-metric Λ-tree $(\mathcal{T}, \leq)$, a *parameterization* is an increasing function $\alpha \colon \mathcal{T} \to \Lambda \cup \{\pm\infty\}$ such that it gives a bijection from any convex subset of $\mathcal{T}$ to an interval in Λ.

In the case that $\Lambda = \mathbb{R}$, we also allow the codomain of α to be $\overline{\mathbb{R}}_{\geq 0}$ or $[0, 1]$. However, all candidates are (order-preserving) isomorphic as ordered sets. In particular, they are all homeomorphic.

Remark 2.5.7. Intuitively, it seems that any non-metric tree admits a parametrization, but this is false. See the errata of [FJ04] for the construction of an example.

Definition 2.5.8 ($\mathbb{R}$-trees)**.** Let X be a topological (resp. metric) space.

- For $x, y \in X$, an *arc* (resp. *geodesic*) jointing x and y is a topological (resp. isometric) embedding $f\colon [a, b] \to X$ with $f(a) = x$ and $f(b) = y$.
- X is a *topological (resp. metric)* $\mathbb{R}$*-tree* if every two different points in X are jointed by a unique arc (resp. geodesic).

Theorem 2.5.9 ([FJ04, Proposition 3.10])**.** *Let $(\mathcal{T}, \leq)$ be a non-metric tree with a parametrization $\alpha\colon \mathcal{T} \to [0, 1]$. Define a function on $\mathcal{T} \times \mathcal{T}$ by*

$$d\colon (\tau, \sigma) \mapsto (\alpha(\tau) - \alpha(\tau \wedge \sigma)) + (\alpha(\sigma) - \alpha(\tau \wedge \sigma)).$$

Then d is a metric on $\mathcal{T}$, making $(\mathcal{T}, d)$ a metric $\mathbb{R}$*-tree.*

Remark 2.5.10. A metric $\mathbb{R}$-tree $(\mathcal{T}, d)$ induces a (non-canonical) parameterized tree structure as follows. Pick a point $\tau_0 \in \mathcal{T}$ and define a partial order $\leq$ by declaring that $\sigma \leq \tau$ whenever σ lies on the geodesic from τ_0 to τ, which turns $(\mathcal{T}, \leq)$ into a non-metric tree. Then define a parametrization α by the rule $\alpha(\tau) \triangleq d(\tau_0, \tau)$.

Definition 2.5.11 (Morphisms of Trees)**.** Given non-metric trees $\mathcal{S}, \mathcal{T}$ with roots σ_0, τ_0 and a function $\Phi\colon \mathcal{S} \to \mathcal{T}$,

- Φ is a *morphism* of non-metric trees if for any $\sigma \in \mathcal{S}$, the map Φ is an order-preserving bijection on segments $[\sigma_0, \sigma]$ and $[\tau_0, \Phi(\sigma)]$;

- if Φ is also bijective, then it is an *isomorphism* of non-metric trees.

Definition 2.5.12 (Observer's Topology)**.** Let $(\mathcal{T}, \leq)$ be a non-metric tree. Given a point $\tau \in \mathcal{T}$ and a tangent vector $\mathbf{v} \in T_\tau$, define

$$U_\tau(\mathbf{v}) = \{\sigma \in \mathcal{T} \setminus \{\tau\} \mid \sigma \text{ represents } \mathbf{v}\}.$$

The *observer's topology*[10] is a topology generated by the subbase of the collection of all possible $U_\tau(\mathbf{v})$.

Note that by the (T2) axiom of non-metric trees (see Definition 2.5.2), if τ is not the root τ_0 then any $\tau' \in [\tau_0, \tau)$ represents the same tangent vector at τ. In particular, we can always choose a representative of that tangent vector to be τ_0.

Lemma 2.5.13. *The observer's topology is Hausdorff.*

Proof. Pick two different points τ and τ' in $\mathcal{T}$. We set $\sigma = \tau \wedge \tau'$ if $\tau \wedge \tau'$ is different from τ and τ'. Otherwise, we may assume $\tau < \tau'$, and we can pick $\sigma \in \mathcal{T}$ so that $\tau < \sigma < \tau'$ in view of (T2) in Definition 2.5.2.

Then, $U_\sigma([\tau])$ and $U_\sigma([\tau'])$ are disjoint open sets containing τ and τ' respectively. □

Proposition 2.5.14 ([FJ04, Proposition 3.8])**.** *Any parametrization α on a non-metric tree $(\mathcal{T}, \leq)$ is lower semi-continuous for the observer's topology.*

Proof. To prove that α is lower semi-continuous, we should show that, for all $t \in \mathbb{R}$, $\alpha^{-1}((t, \infty])$ is open for the observer's topology.

Let $t \in \mathbb{R}$ and set $\mathcal{U} = \alpha^{-1}((t, \infty])$. Assume that $\mathcal{U}$ is nonempty and pick an element $\tau \in \mathcal{U}$. Suppose that we can pick $\sigma \in \mathcal{T}$ so that $\sigma < \tau$ and $\alpha(\sigma) \geq t$. Then τ represents a tangent vector $\mathbf{v}$ at σ, so $\tau \in U_\sigma(\mathbf{v})$. If $\tau' \in \mathcal{T}$ also represents $\mathbf{v}$, then we

[10]The terminology observer's topology is from the paper [CHL07]. It is called *the weak tree topology* in [FJ04].

must have $\tau' > \sigma$ because $\mathbf{v}$ is not the vector represented by the root τ_0. This implies that $\alpha(\tau') > \alpha(\sigma)$, so $U_\sigma(\mathbf{v}) \subset \mathcal{U}$ as desired.

Now we explain why we can pick such σ. By (T2) axiom of non-metric trees, the segment $[\tau_0, \tau]$ is isomorphic to a real interval, so it is convex. Since α is a bijection from the segment $[\tau_0, \tau]$ to the real interval $[\alpha(\tau_0), \alpha(\tau)]$, we can safely pick such σ we want. □

Remark 2.5.15. If a non-metric tree $(\mathcal{T}, \leq)$ admits a parametrization, it has the observer's topology and the metric topology from the metric induced by the parametrization. Those two topologies may not coincide (see Theorem 2.6.5 for an example).

Before we end this section, let me put two theorems that will be used later.

Definition 2.5.16 (Completion of Non-metric Trees)**.** The *completion* of a non-metric tree $(\mathcal{T}, \leq)$ is a non-metric tree $(\overline{\mathcal{T}}, \leq)$ obtained by adding 'missing ends' to $\mathcal{T}$: for any unbounded increasing sequence in $\mathcal{T}$, we add an end dominating the sequence, and the order $\leq$ of $\mathcal{T}$ naturally extends to $\overline{\mathcal{T}}$.

Theorem 2.5.17 (Completion of Non-metric $\mathbb{Q}$-Trees, [FJ04, Proposition 3.12])**.** *Let $(\mathcal{T}, \leq)$ be a non-metric $\mathbb{Q}$-tree. Then there exist a non-metric $\mathbb{R}$-tree $\mathcal{T}_{\mathbb{R}}$ and an order-preserving injection $\iota\colon \mathcal{T} \to \mathcal{T}_{\mathbb{R}}$ satisfying the following conditions.*

- *The image of ι is dense for the observer's topology.*
- *Any point in $\mathcal{T}_{\mathbb{R}} \setminus \iota(\mathcal{T})$ is a regular point of $\mathcal{T}_{\mathbb{R}}$.*
- *If there is another order-preserving injection $\iota'\colon \mathcal{T} \to \mathcal{T}'_{\mathbb{R}}$ into a non-metric $\mathbb{R}$-tree, then there exists a morphism $\Phi : \mathcal{T}_{\mathbb{R}} \to \mathcal{T}'_{\mathbb{R}}$ of non-metric trees such that $\iota' = \Phi \circ \iota$, and that Φ extends to an isomorphism of non-metric trees from $\overline{\mathcal{T}_{\mathbb{R}}}$ to $\overline{\mathcal{T}'_{\mathbb{R}}}$.*

Furthermore, any parametrization α on $\mathcal{T}$ extends uniquely to a parametrization $\alpha_{\mathbb{R}}$ on $\mathcal{T}_{\mathbb{R}}$ (and on $\overline{\mathcal{T}_{\mathbb{R}}}$).

2.6 The Valuative Tree $\mathcal{V}_{\mathfrak{m}}$

In this section, we will see that the valuative space $\mathcal{V}_{\mathfrak{m}}$ is a non-metric tree. The key is to verify that (T1) of Definition 2.5.2 holds. That is, every nonempty subset admits an infimum.

Lemma 2.6.1 (The Infimum of Semivaluations via SKPs, [FJ04, Corollary 3.19])**.** *Let $\{v^i \mid i \in I\}$ be a subset of $\mathcal{V}_{\mathfrak{m}}$. Write $v^i = \mathrm{val}[(U^i_j)_0^{k_i}; (\tilde{\beta}^i_j)_0^{k_i}]$ and assume $1 = \tilde{\beta}^i_0 < \tilde{\beta}^i_1$ for all $i \in I$. Let $k \in \mathbb{Z}_{>0}$ be maximal such that, for all $i, i' \in I$, we have $U^i_j = U^{i'}_j \triangleq U_j$ for $0 \leq j \leq k$ and $\tilde{\beta}^i_j = \tilde{\beta}^{i'}_j$ for $0 \leq j < k$. Then the infimum of $\{v^i\}$ exists and is given by*

$$\bigwedge_{i\in I} v^i = \mathrm{val}\left[(U^i_j)_0^k; \left(\tilde{\beta}_0, \tilde{\beta}_1, \ldots, \tilde{\beta}_{k-1}, \inf_{i\in I} \tilde{\beta}^i_k\right)\right].$$

Remark 2.6.2. The assumption in the Lemma 2.6.1 that $1 = \tilde{\beta}^i_0 < \tilde{\beta}^i_1$ for all $i \in I$ is natural: if there is one v_i has the opposite property that $1 = \tilde{\beta}^i_1 < \tilde{\beta}^i_0$, then the infimum is $\mathrm{ord}_{\mathfrak{m}}$.

Theorem 2.6.3. *The valuative space $\mathcal{V}_{\mathfrak{m}}$ is a complete, non-metric tree rooted at* $\mathrm{ord}_{\mathfrak{m}}$*.*

Proof. Let me give a rough sketch. See [FJ04, Theorem 3.14] for a complete proof.

Definition 2.3.8 already equips $\mathcal{V}_{\mathfrak{m}}$ with a partial order $\leq$, so (T1) follows from Lemma 2.6.1.

To prove (T2), let $v \in \mathcal{V}_{\mathfrak{m}}$ and, without loss of generality, assume that $1 = v(x) \leq v(y)$ (if $v(y) < v(x)$, we can swap the symbols x and y). Write $v = \mathrm{val}[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$. For the case that $k < \infty$, we can show that the segment $[\mathrm{ord}_{\mathfrak{m}}, v]$ is isomorphic to the real interval $[1, \tilde{\beta}_k / \deg_y U_k]$. The case that $k = \infty$ is similar: by Lemma 2.4.3

the sequence $(\tilde{\beta}_j/\deg_y U_j)_0^\infty$ has a limit, denoted by M. Then we can show that the segment $[\mathrm{ord}_\mathfrak{m}, v]$ is isomorphic to the real interval $[1, M]$.

Notice that (T3) follows if we can show the completeness condition for $\mathcal{V}_\mathfrak{m}$ (by using [FJ04, Remark 3.3]). Let $\mathcal{S} \subset \mathcal{V}_\mathfrak{m}$ be a totally ordered subset. We may assume $1 = v(x) < v(y)$ for all $v \in \mathcal{S}$. By Lemma 2.6.1, SKPLength: $\mathcal{S} \to \bar{\mathbb{Z}}_{>0}$ is an increasing function. If SKPLength $v \to \infty$ as $v \in \mathcal{S}$ increases, due to Theorem 2.4.8, v must tend to a curve semivaluation or an infinitely singular valuation, which dominates all $v \in \mathcal{S}$. If there exist $v_0 \in \mathcal{S}$ such that $n = \mathrm{SKPLength}\, v < \infty$ for all $v \geq v_0$, then, for $v \geq v_0$, we must have

$$v = \mathrm{val}[(U_j)_0^n; (\tilde{\beta}_1, \ldots, \tilde{\beta}_{n-1}, \tilde{\beta}_n^v)]$$

for some $\tilde{\beta}_n^v \in \mathbb{R}$. Take $\tilde{\beta} = \sup_{v \geq v_0} \tilde{\beta}_n^v \in \overline{\mathbb{R}}_{\geq 0}$, and the semivaluation

$$\mathrm{val}[(U_j)_0^n; (\tilde{\beta}_1, \ldots, \tilde{\beta}_{n-1}, \tilde{\beta})]$$

dominates all $v \in \mathcal{S}$. $\square$

Proposition 2.6.4 (Dendrology of $\mathcal{V}_\mathfrak{m}$, [FJ04, Proposition 3.20])**.** *The points in the non-metric tree $\mathcal{V}_\mathfrak{m}$ can be classified as follows.*

- *The root of $\mathcal{V}_\mathfrak{m}$ is* $\mathrm{ord}_\mathfrak{m}$.
- *The ends of $\mathcal{V}_\mathfrak{m}$ are the curve semivaluations and the infinitely singular valuations.*
- *Any tangent vector in $\mathcal{V}_\mathfrak{m}$ can be represented by a curve semivaluation or an infinitely singular valuation.*
- *The regular points in $\mathcal{V}_\mathfrak{m}$ are the irrational valuations.*
- *The branch points in $\mathcal{V}_\mathfrak{m}$ are the divisorial valuations, and the tangent space at a divisorial valuation is bijective to $\mathbb{P}^1_\Bbbk(\Bbbk)$.*

See Figure 2.3 to visualize a part of the space $\mathcal{V}_\mathfrak{m}$.

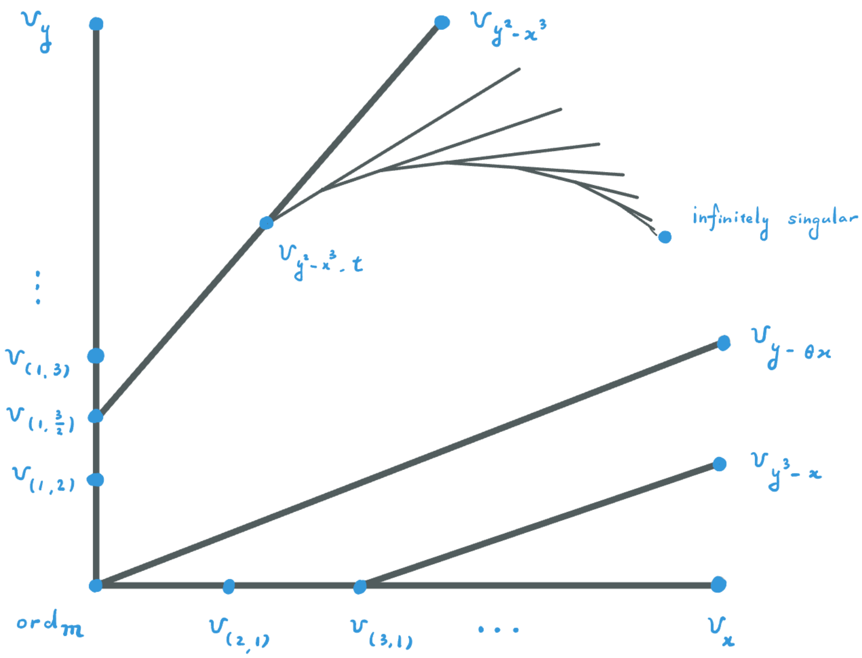


Figure 2.3: Some monomial valuations and curve semivaluations are labeled. The notation $v_{y^2-x^3,t}$ will be defined in the next section.

Theorem 2.6.5 (Properties of the Weak Topology)**.** *The observer's topology and the weak topology on $\mathcal{V}_\mathfrak{m}$ are the same, and they satisfy the following properties.*

- *$\mathcal{V}_\mathfrak{m}$ is weakly compact.*
- *The weak topology is not metrizable if $\Bbbk$ is uncountable.*
- *The collection of all divisorial (resp. irrational, infinitely singular, curve) valuations is weakly dense in $\mathcal{V}_\mathfrak{m}$.*

Proof. It is obvious to show that the two topologies are identical (see [FJ04, Theorem 5.1]). For the second statement, see [FJ04, Proposition 5.2].[11] The last property is from [FJ04, Proposition 5.3]. $\square$

[11] It is natural to ask if the converse is true. The author does not know the answer.

2.7 Skewness

Now, we introduce the first parametrization on $\mathcal{V}_\mathfrak{m}$, called the skewness, which measures how much a semivaluation deviates from $\operatorname{ord}_\mathfrak{m}$. We follow [FJ04, Subsection 3.3.1].

Definition 2.7.1 (Skewness). The skewness of a semivaluation $v \in \mathcal{V}_\mathfrak{m}$ is

$$\alpha(v) \triangleq \sup \left\{ \frac{v(f)}{\operatorname{ord}_\mathfrak{m}(f)} \,\middle|\, f \in \mathfrak{m} \right\}.$$

Proposition 2.7.2. *Given $v \in \mathcal{V}_\mathfrak{m}$ and any irreducible $f \in \mathfrak{m}$, we have*

$$v(f) = \alpha(v \wedge v_f) \operatorname{ord}_\mathfrak{m}(f).$$

This quickly implies that $\alpha(v) = v(f)/\operatorname{ord}_\mathfrak{m}(f)$ if and only if $v_f \geq v$.

Proof. See [FJ04, Proposition 3.25]. □

Lemma 2.7.3 (Skewness via SKPs, [FJ04, Lemma 3.32]). *Let $v \in \mathcal{V}_\mathfrak{m}$ with its associated SKP $[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$.*

- *If v is quasimonimial, then $\alpha(v) = \tilde{\beta}_0 \tilde{\beta}_k / \deg_y U_j$.*
- *If v is a curve semivaluation, then $\alpha(v) = \infty$.*
- *If v is infinitely singular, then $\alpha(v) = \lim_{j \to \infty} \tilde{\beta}_0 \tilde{\beta}_j / \deg_y U_j \in (1, \infty]$.*

Theorem 2.7.4 (Classification of Semivaluations via Skewness). *The skewness function $\alpha \colon \mathcal{V}_\mathfrak{m} \to [1, \infty]$ is a parametrization. Furthermore, it has specific values on different types of semivaluations.*

- *If v is divisorial, then $\alpha(v) \in \mathbb{Q}$.*
- *If v is irrational, then $\alpha(v) \notin \mathbb{Q}$.*

- *If v is a curve semivaluation, then $\alpha(v) = \infty$.*
- *If v is infinitely singular, then $\alpha(v) \in (1, \infty]$. Moreover, for any $t \in (1, \infty]$, there is an infinitely singular valuation v such that $\alpha(v) = t$.*

Proof. See [FJ04, Theorem 3.26]. Also see [FJ04, Appendix A.2] for a construction of prescribed infinitely singular valuations. □

Notation 2.7.5 ($v_{f,t}$)**.** For irreducible $f \in \mathfrak{m}$ and $t \in [1, \infty)$, we write $v_{f,t}$ to encode the unique valuation on the segment $[\mathrm{ord}_{\mathfrak{m}}, v_f]$ with skewness t. Note that for $v \in \mathcal{V}_{\mathfrak{m}}$ we have $v = v_{f,t}$ for any $v_f > v$.

Proposition 2.7.6. *The skewness α is weakly lower semicontinuous, but not weakly continuous.*

Proof. The lower semicontinuity is just a corollary from Proposition 2.5.14 because the skewness is a parametrization.

Let's show that the skewness is not continuous. Since $\Bbbk$ is infinite, consider a sequence (θ_n) of $\Bbbk^*$ and thus the sequence (v_n) of $\mathcal{V}_{\mathfrak{m}}$, where $v_n \triangleq v_{y-\theta_n x,2}$. We see that $\alpha(v_n) = 2$ and $v_n \to \mathrm{ord}_{\mathfrak{m}}$ as $n \to \infty$, but $\alpha(\mathrm{ord}_{\mathfrak{m}}) = 1$. □

2.8 Multiplicity and Thinness

Definition 2.8.1 (Multiplicity)**.** We define two kinds of multiplicity:

- The *multiplicity*[12] of $v \in \mathcal{V}_{\mathfrak{m}}$ is

$$m(v) = \min \{\mathrm{ord}_{\mathfrak{m}}(f) \mid v_f \geq v\}.$$

[12] In [FJ04], this definition only applies for quasimonomial valuations, but it can be extended to any semivaluation.

- For a tangent vector $\mathbf{v} \in T_v$, define the *multiplicity* of $\mathbf{v}$ by the following rule: if $\mathbf{v}$ is represented by $\mathrm{ord}_{\mathfrak{m}}$ then set $m(\mathbf{v}) = m(v)$; otherwise,

$$m(\mathbf{v}) = \min \{\mathrm{ord}_{\mathfrak{m}}(f) \mid v_f \text{ represents } \mathbf{v}\}.$$

Theorem 2.8.2 (Computing Multiplicities via SKPs)**.** *Let $v = \mathrm{val}[(U_j)_0^k; (\tilde{\beta}_j)_0^k] \in \mathcal{V}_{\mathfrak{m}}$, and assume that $1 = v(x) \leq v(y)$. Then, $m(v) = \sup\left\{\deg_y U_j \mid 1 \leq j \leq k\right\}$. In particular, we have the following immediate corollaries.*

- *If v is a quasimonimial valuation or a curve semivaluation, then $m(v) = \deg_y U_k$;*
- *v is infinitely singular if and only if $m(v) = \infty$;*
- *moreover, if $v > w$ and v is not infinitely singular, then $m(w)$ divides $m(v)$.*

Proof. The theorem is from [FJ04, Lemma 3.42 and Proposition 3.37], but let me go through the proof.

If $[(U'_j)_0^k; (\tilde{\beta}'_j)_0^k]$ is an SKP with the normalization that $1 = \tilde{\beta}'_0 \leq \tilde{\beta}'_1$, then an immediate corollary of Theorem 4.1.7 is that $\mathrm{ord}_{\mathfrak{m}}(U'_j) = \deg_y U'_j$ for all j.

If v is infinitely singular, then $\deg_y U_j \to \infty$ due to Theorem 2.4.8. If $v = v_f$ is a curve semivaluation, then $m(v) = \mathrm{ord}_{\mathfrak{m}}(f) = \deg_y U_k = \sup\left\{\deg_y U_j \,|\, 1 \leq j \leq k\right\}$.

Now, assume v is quasimonomial. If $v_f > v$ with the SKP $[(U'_j)_0^{k'}; (\tilde{\beta}'_j)_0^{k'}]$, then $k' \geq k$ and $\mathrm{ord}_{\mathfrak{m}}(f) = \deg_y U_{k'} \geq \deg_y U_k$. This indicates that $m(v) \geq \deg_y U_k$. But $v_{U_k} > v$, which implies that $m(v) = \deg_y U_k = \sup\left\{\deg_y U_j \,|\, 1 \leq j \leq k\right\}$. □

Proposition 2.8.3. *The multiplicity $m\colon \mathcal{V}_{\mathfrak{m}} \to \overline{\mathbb{Z}}_{>0}$ is weakly lower semicontinuous, but not weakly continuous.*

Proof. The lower semicontinuity of m is from Theorem 2.8.2. To show that m is not continuous, we follow the similar method when we show that the skewness α is not continuous: let (θ_n) be a sequence in $\mathbb{k}^*$ and consider the sequence (v_n) of curve semivaluations in $\mathcal{V}_\mathfrak{m}$, where $v_n = v_{(y+\theta_n x)^2+x^3}$. Then, $v_n \to \mathrm{ord}_\mathfrak{m}$ as $n \to \infty$, but $m(v_n) = 2$ and $m(\mathrm{ord}_\mathfrak{m}) = 1$. □

Theorem 2.8.4 (Generic Multiplicities, [FJ04, Proposition 3.39])**.** *Let $v \in \mathcal{V}_\mathfrak{m}$ be given. Then there is an integer $b(v) > 0$, called the* generic multiplicity *of v, such that one of the following holds.*

- *$m(\mathbf{v}) = b(v)$ for all tangent vector $\mathbf{v} \in T_v$, and thus $m(v) = b(v)$.*
- *$m(\mathbf{v}) = b(v)$ for all $\mathbf{v} \in T_v$ except two tangent vectors, and one of the two is represented by $\mathrm{ord}_\mathfrak{m}$. In this case, $m(v)$ divides $b(v)$ non-trivially.*

Proposition 2.8.5. *Recall that for a divisor E over $\mathrm{Spec}\, R$, we have $\pi_* \mathrm{ord}_E \in \mathcal{V}_R$. The normalized divisorial valuation on E is*

$$v_E \triangleq b^{-1}\pi_* \mathrm{ord}_E \in \mathcal{V}_\mathfrak{m},$$

where $b = \mathrm{ord}_E(\mathfrak{m})$. Then, b is the generic multiplicity $b(v_E)$ of v_E.

Proof. This is a corollary of Lemma 2.10.18 and Remark 2.10.22. □

Definition 2.8.6 (Approximating Sequences)**.** Let $v \in \mathcal{V}_\mathfrak{m}$ be given.

- Assume $m(v) < \infty$. Then there exists a finite sequence of divisorial valuations $v_1, v_2, \ldots, v_g$ so that

  $$\mathrm{ord}_\mathfrak{m} = v_0 < v_1 < v_2 < \cdots < v_g < v_{g+1} = v,$$

  and a strictly increasing sequence of positive integers $m_0, m_1, m_2, \ldots, m_g, m_{g+1}$ such that $m(w) = m_j$ for all $w \in (v_{j-1}, v_j]$ and $1 \le j \le g+1$.

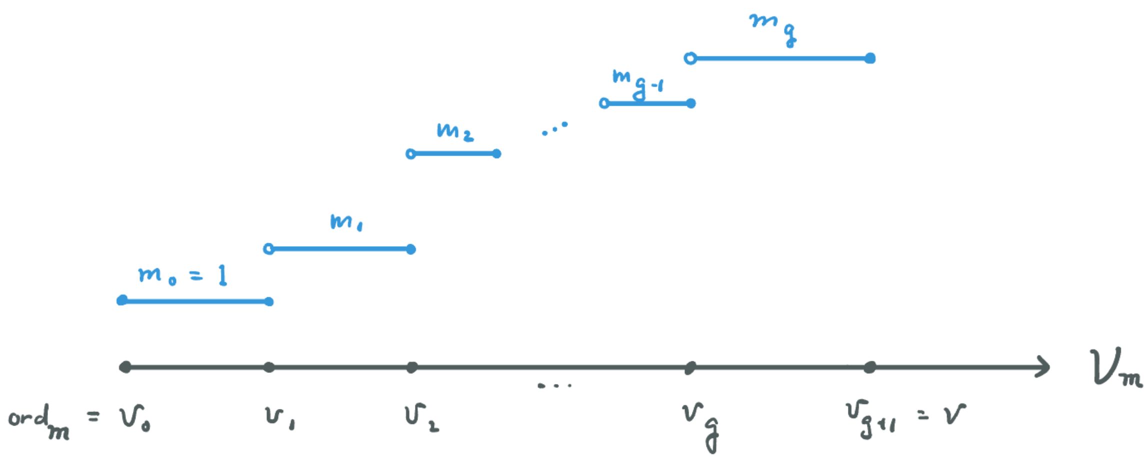


Figure 2.4: Given an approximating sequence, $(m_j)_0^g$ is a piecewise increasing function on the segment $[\mathrm{ord}_\mathfrak{m}, v]$.

- If v is infinitely singular, we can extend the previous definition naturally to this case.

The obtained pair of sequences $[(v_j)_0^g; (m_j)_0^g]$, $g \leq \infty$, is called an *approximating sequence.*[13] See Figure 2.4 to visualize an approximating sequence.

Proof. Let me quickly explain why such a pair of sequences exists (the uniqueness of the sequence is clear).

Let $v \in \mathcal{V}_\mathfrak{m}$ and now we focus on the segment $[\mathrm{ord}_\mathfrak{m}, v]$. We start from $\mathrm{ord}_\mathfrak{m}$ and go toward v. If the multiplicity jumps up to another integer, then we must be at a divisorial valuation v_1 (because $b(v) > m(v)$ only happens at divisorial valuations). Then, we continue until we reach v. $\square$

Now, we are ready to define another parametrization: the thinness. Although the skewness gives us a nice parametrization (see Theorem 2.7.4), thinness gives much finer birational geometry information: the log discrepancy of a valuation.

[13]In [FJ04], they denote an approximating sequence only using $(v_j)_0^g$. Here, we also include the data $(m_j)_0^g$.

Definition 2.8.7 (Thinness)**.** The *thinness*[14] of a valuation $v \in \mathcal{V}_{\mathfrak{m}}$ is

$$A^{\text{thin}}(v) \triangleq 2 + \int_{\text{ord}_{\mathfrak{m}}}^{v} m(w) d\alpha(w).$$

Equivalently, if $[(v_j)_0^g; (m_j)_0^g]$ is the approximating sequence of v, then

$$A^{\text{thin}}(v) = 2 + \sum_{j=0}^{g} m_j(\alpha(v_{j+1}) - \alpha(v_j)).$$

Remark 2.8.8. The integer 2 looks strange at the beginning. However, we will see in Section 2.10 that to make the thinness correspond to the log discrepancy function, we must have

$$A_{\operatorname{Spec} R}(E_0) = 2 = A^{\text{thin}}(\text{ord}_{\mathfrak{m}}),$$

where the notation of discrepancy on the left is from Definition 2.9.5, and E_0 is the exceptional prime divisor of the blow-up at the origin $\mathfrak{m} \subset \operatorname{Spec} R$.

Theorem 2.8.9 (Classification of Semivaluations via Thinness)**.** *The thinness function $A^{\text{thin}} \colon \mathcal{V}_{\mathfrak{m}} \to [2, \infty]$ gives a parametrization of the non-metric tree $(\mathcal{V}_{\mathfrak{m}}, \leq)$. Also, the thinness enjoys the following properties.*

- *If v is divisorial, then $A^{\text{thin}}(v) \in \mathbb{Q}$.*
- *If v is irrational, then $A^{\text{thin}}(v) \notin \mathbb{Q}$.*
- *If v is a curve semivaluation, then $A^{\text{thin}}(v) = \infty$.*
- *If v is infinitely singular, then $A^{\text{thin}}(v) \in (2, \infty]$. Moreover, for any $t \in (2, \infty]$, there is an infinitely singular valuation v such that $A^{\text{thin}}(v) = t$.*

Proof. This is a direct application of Theorem 2.7.4.

[14] For a motivation of the name 'thinness', see [FJ04, Remark 3.49].

See [FJ04, Appendix A.2] for a construction of prescribed infinitely singular valuations. □

Proposition 2.8.10. *The thinness* $A^{\text{thin}}\colon \mathcal{V}_{\mathfrak{m}} \to [2,\infty]$ *is weakly lower semicontinuous, but not weakly continuous.*

Proof. Since A^{thin} gives a parametrization, lower semicontinuity comes from Proposition 2.5.14. The same example (i.e. $v_n = v_{y-\theta_n x,2}$) we used in the proof of Theorem 2.7.4 also works to show that A^{thin} is not continuous. □

2.9 Discrepancies in Birational Geometry

In this section, we review some constructions in birational geometry, even thought they have already appeared before. A variety will always be a $\Bbbk$-variety.

Definition 2.9.1 (Divisors)**.** Let X be a Noetherian integral (separated) scheme, which is regular in codimension 1 (for example, X is normal).

- A *prime divisor* is a codimension 1 integral closed subscheme of X.
- A $\mathbb{Q}$-*Weil divisor* (or $\mathbb{Q}$-divisor) is an element of $\bigoplus_{D_i} \mathbb{Q}D_i$, where the sum runs through all prime divisors D_i of X. A *Weil divisor* (or *integral divisor*) is a $\mathbb{Q}$-divisor with all coefficients in $\mathbb{Z}$. The *zero divisor* is the divisor with all coefficients zero, denoted by 0. When writing $D = \sum a_i D_i$, we always assume that the D_i are distinct. If $D = \sum a_i D_i$, then its *round down* is the divisor $\lfloor D \rfloor = \sum \lfloor a_i \rfloor D_i$. We use the notation $\operatorname{ord}_{D_i}(D) \triangleq a_i$.
- A $\mathbb{Q}$-divisor is *effective* if its coefficients are all nonnegative.
- A $\mathbb{Q}$-divisor D is $\mathbb{Q}$-*Cartier* if mD is Cartier for some $m \in \mathbb{Z}$. In this case, the *Cartier index* is the smallest positive m.

Definition 2.9.2 (SNC Divisors)**.** We follow the setup from [JM12] and [Can20].

Let X be a regular scheme of dimension n. An effective $\mathbb{Q}$-divisor $D = \sum_{i=1}^{N} a_i D_i$ on X is *simple normal crossing* (SNC) if for any point $x \in X$, the reduced divisor $D = \sum_{i=1}^{N} D_i$ is cut out at x by an equation $x_1 x_2 \cdots x_r$, where $0 \leq r \leq n$ and $x_1, x_2, \ldots, x_r$ form a part of a regular system of parameters of $\mathcal{O}_{X,x}$.

Definition 2.9.3 (Pairs)**.**

- A *(log) pair* is a pair (X, Δ), where X is a normal integral excellent scheme (of dimension $n \geq 2$), and Δ is an (effective) $\mathbb{Q}$-divisor on X.[15]

- A pair (X, Δ) is *log* $\mathbb{Q}$*-Gorenstein* if $K_X + \Delta$ is $\mathbb{Q}$-Cartier for some (and thus any) canonical divisor K_X of X. In particular, X is $\mathbb{Q}$*-Gorenstien* if $(X, 0)$ is log $\mathbb{Q}$-Gorenstein.

Remark 2.9.4. Let me make a remark about the existence of a canonical divisor K_X. If X is a smooth variety or a normal variety, the existence is well-known by taking $\omega_X = \left(\bigwedge^{\dim X} \Omega_{X/\Bbbk}\right)^{\vee\vee}$.

Now, assume that X is a normal integral scheme with a dualizing complex $\omega_X^{\bullet}$.[16] If $\omega_X^{\bullet}$ is concentrated in a single degree, then that sheaf is called a *dualizing sheaf* of X; this happens, for instance, when X is Cohen-Macaulay. Then, the *canonical sheaf* is a rank 1 reflexive sheaf ω_X which agrees with the dualizing sheaf on the Cohen-Macaulay locus of X (see [ST14, Section 2.4]). An alternative approach, leveraging the normality of X, is to consider the regular locus U of X with the open embedding $j \colon U \to X$. There is a canonical sheaf ω_U on U which is invertible, and we can let $\omega_X = j_*(\omega_U)$. In any case, ω_X is a rank 1 reflexive sheaf on X, so we pick any divisor K_X such that $\omega_X \cong \mathcal{O}_X(K_X)$.

[15] This notion is not uniform among the community. For example, in [Kol97] the effectiveness is not necessary, but it is assumed in [Can20], for instance.

[16] $\omega_X^{\bullet}$ exists, for example, if X is excellent (see [Sta25, Tag 0BFR]).

Definition 2.9.5 (Discrepancies)**.** Let (X, Δ) be a log $\mathbb{Q}$-Gorenstein pair with a canonical divisor K_X. Let $\pi\colon Y \to X$ be a proper birational morphism from a normal scheme Y, and $E \subset Y$ be a prime divisor. Let K_Y be a canonical divisor on Y with $\pi_* K_Y = K_X$. Then, we obtain a $\mathbb{Q}$-divisor Δ_Y on Y by the formula[17]

$$K_Y + \Delta_Y = \pi^*(K_X + \Delta_X).$$

The *discrepancy* of (X, Δ) at E is

$$\begin{aligned} a(E; X, \Delta) &\triangleq -\operatorname{ord}_E(\Delta_Y) \\ &= \operatorname{ord}_E(K_Y - \pi^*(K_X + \Delta)). \end{aligned}$$

The *log discrepancy* of (X, Δ) at E is

$$\begin{aligned} A_{(X,\Delta)}(E) &\triangleq 1 + a(E; X, \Delta) \\ &= 1 + \operatorname{ord}_E(K_Y - \pi^*(K_X + \Delta)). \end{aligned}$$

Note that this definition is independent of the choice of the model Y where the divisor E lives because ord_E is a valuation on the function field of X. If $\Delta = 0$, we may write $A_X(E)$ instead of $A_{(X,0)}(E)$.

Remark 2.9.6. If X is $\mathbb{Q}$-Gorenstien, then Δ is already $\mathbb{Q}$-Cartier and we can rewrite the log discrepancy as

$$A_{(X,\Delta)}(E) = 1 + \operatorname{ord}_E(K_{Y/X}) + \operatorname{ord}_E(\Delta),$$

[17]Whenever $\pi_* K_Y = K_X$ holds, Δ_Y is a well-defined $\mathbb{Q}$-divisor and is independent of the choice of K_X and K_Y.

where $K_{Y/X} \triangleq K_Y - \pi^* K_X$ is the *relative canonical divisor* of π. Since $A_{(X,\Delta)}(E)$ is independent of the choice of the model Y where E lies, we may write $\operatorname{ord}_E(K_{-/X}) \triangleq \operatorname{ord}_E(K_{Y/X})$ for some choice of model Y.

Definition 2.9.7 (Global Discrepancies)**.** Let (X, Δ) be a log $\mathbb{Q}$-Gorenstein pair.

- The *discrepancy* of (X, Δ) is

$$\operatorname{discrep}(X, \Delta) \triangleq \inf\{a(E; X, \Delta) \mid E \text{ is an exceptional divisor over } X\}.$$

- The *minimal log discrepancy* of (X, Δ) is[18]

$$\operatorname{mld}(X, \Delta) \triangleq \inf\{A_{(X,\Delta)}(E) \mid E \text{ is a divisor over } X\}.$$

Remark 2.9.8. In Definition 2.9.7, the condition that E is a divisor over X is equivalent to $c_X(E) \neq \emptyset$ (i.e., E is centered on X). Similarly, the condition that E is an exceptional divisor over X is equivalent to saying that $c_X(E) \neq \emptyset$ with codimension ≥ 2.

Definition 2.9.9 ([Kol97, Definition 3.5] and [KM98, Definition 2.34])**.** Let (X, Δ) be a log $\mathbb{Q}$-Gorenstein pair.

- (X, Δ) is *terminal* if $\operatorname{discrep}(X, \Delta) > 0$.
- (X, Δ) is *canonical* if $\operatorname{discrep}(X, \Delta) \geq 0$.
- (X, Δ) is *purely log terminal* (plt) if $\operatorname{discrep}(X, \Delta) > -1$.
- (X, Δ) is *Kawamata log terminal* (klt) if it is plt and $\lfloor \Delta \rfloor \leq 0$.[19]
- (X, Δ) is *log canonical* (lc) if $\operatorname{discrep}(X, \Delta) \geq -1$.

[18] The definition and the notation is from [Mus04], where our $\operatorname{mld}(X, \Delta)$ is the same as $\operatorname{mld}(X; X, \Delta)$ in his notation. Also, in [KM98] they have a similar definition: $\operatorname{totaldiscrep}(X, \Delta) \triangleq \operatorname{mld}(X, \Delta) - 1$.

[19] As $\Delta \geq 0$ in our setting, $\lfloor \Delta \rfloor \leq 0$ simply says that all coefficients of Δ are in the interval $[0, 1)$.

We omit the divisor Δ when $\Delta = 0$; for example, we say that X is terminal if $(X, 0)$ is.

Remark 2.9.10. Terminal singularities are those that appear on minimal models, in the sense of the Minimal Model Program (MMP), for smooth varieties. Canonical singularities arise as the singularities of canonical models of projective varieties. The class of klt singularities is the natural setting in which the MMP is carried out, and vanishing theorems (e.g., Kawamata–Viehweg vanishing) hold in this class. Log canonical (lc) singularities are precisely those for which the notion of a discrepancy is well-defined.

Example 2.9.11 ([Kol97, Theorem 3.6])**.** Let X be a germ of a $\mathbb{Q}$-Gorenstein (analytic) surface over $\mathbb{C}$ with an isolated singularity, then its singularity type can be described explicitly:

- X is terminal if and only if X is smooth over $\mathbb{C}$. This follows from the fact that if X is terminal, then its singularities have codimension ≥ 3.[20]
- X is canonical if and only if X is analytically isomorphic to the germ of a quotient of $\mathbb{A}^2_{\mathbb{C}}$ by a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.[21]
- X is klt if and only if X is isomorphic to the germ of a quotient of $\mathbb{A}^2_{\mathbb{C}}$ by a finite subgroup of $\mathrm{GL}_2(\mathbb{C})$.
- When X is lc, it is classified in [Kaw88].

Lemma 2.9.12. *The pair* (X, Δ) *is lc if and only if* $\mathrm{mld}(X, \Delta) \geq 0$.

Proof. This is actually an immediate corollary of [KM98, Corollary 2.31.(1)], but let me give a proof. Since there is an obvious inequality $\mathrm{mld}(X, \Delta) \leq \mathrm{discrep}(X, \Delta) + 1$, the 'if' direction of the statement follows.

[20] Therefore, it is natural to look at terminal 3-folds, and these are classified in [Mor85].

[21] A 2-dimensional canonical singularity is also called a *du Val singularity.*

For the 'only if' direction, assume that there is a divisor E over X such that $A_{(X,\Delta)}(E) < 0$, which means that $a(E; X, \Delta) < -1$. If E is exceptional, then we have $\operatorname{discrep}(X, \Delta) < -1$ and we are good; otherwise, we have $E \subset \pi_*^{-1}(\Delta)$ for some proper birational morphism of normal varieties $\pi\colon Y \to X$. Define the $\mathbb{Q}$-divisor $\Delta_Y = \pi^*(K_X + \Delta) - K_Y$ on Y. Pick a closed subscheme $Z \subset Y$ of codimension 2 so that $Z \subset E$ and Z is not in other exceptional divisors, and consider the blow-up, $\pi'\colon \operatorname{Bl}_Z Y \to Y$, along Z with the exceptional divisor E' dominating Z. Since $\Delta_Y = a(E, X, \Delta)E + \cdots$, we obtain

$$a(E', X, \Delta) = a(E', Y, \Delta_Y) = \operatorname{codim}_Y Z - 1 + a(E, X, \Delta) = 1 + a(E, X, \Delta) \triangleq c < 0.$$

Then, pick $Z' = E' \cap (\pi'_*)^{-1}E$ and consider the blow-up along Z' with the exceptional divisor E''. A similar computation gives $a(E'', X, \Delta) = 2c$. Repeating this process n times for n large enough, we will get an exceptional divisor $E^{(n)}$ such that $a(E^{(n)}, X, \Delta) < -1$. Therefore, $\operatorname{discrep}(X, \Delta) < -1$, completing the proof. $\square$

Definition 2.9.13 (Log Canonical Thresholds, LCTs)**.** Let (X, Δ) be a $\mathbb{Q}$-Gorenstein pair so that X is plt and $\Delta \neq 0$. Then the *log canonical threshold* (LCT) of (X, Δ) is[22]

$$\operatorname{lct}(X, \Delta) \triangleq \sup\{t > 0 \mid \text{the pair } (X, t\Delta) \text{ is lc}\}.$$

When Δ is $\mathbb{Q}$-Cartier, then

$$\operatorname{lct}(X, \Delta) = \inf\left\{ \frac{1 + \operatorname{ord}_E(K_{-/X})}{\operatorname{ord}_E(\Delta)} \,\middle|\, E \text{ is a divisor over } X \right\}.$$

[22]In the modern language, the LCT is defined as follows: let (X, Δ) be a klt pair, and Γ is an effective divisor on X. Then, we define the *log canonical threshold* of (X, Δ) with respect to Γ as

$$\operatorname{lct}(X, \Delta; \Gamma) \triangleq \sup\{t > 0 \mid \text{the pair } (X, \Delta + t\Gamma) \text{ is lc}\}.$$

Therefore, our definition of $\operatorname{lct}(X, \Delta)$ is nothing but $\operatorname{lct}(X, 0; \Delta)$ in their setting.

Remark 2.9.14. Assume that (X, Δ) admits log resolutions. Then we can use a single log resolution to compute the LCT.

Proof. The following quick argument, under the assumption that Δ is $\mathbb{Q}$-Cartier, is from the proof of [Mus12, Theorem 1.1]. Let me add more details based on it.

First, note that computing an LCT is independent of the choice of a log resolution. See [Laz04, Theorem 9.2.18] for a proof of this claim.[23]

Since (X, Δ) admits a log resolution, we pick such a one. Given a divisor E over X, we can find a common log resolution containing E and dominating the original resolution. Because these two log resolutions compute the LCT, the quantity

$$\frac{1 + \mathrm{ord}_E(K_{-/X})}{\mathrm{ord}_E(\Delta)}$$

must be non-less than $\mathrm{lct}(X, \Delta)$. The remark now follows. $\square$

Remark 2.9.15 (Local LCTs)**.** It is also common to consider the LCT at a point: if $\mathfrak{p} \in X$ then define

$$\mathrm{lct}_{\mathfrak{p}}(X, \Delta) = \inf \left\{ \frac{1 + \mathrm{ord}_E(K_{-/X})}{\mathrm{ord}_E(\Delta)} \;\middle|\; E \text{ is a divisor over } X \text{ with } c_X(E) = \mathfrak{p} \right\}.$$

The same argument in Remark 2.9.14 indicates that if (X, Δ) admits log resolutions, $\mathrm{lct}_{\mathfrak{p}}(X, \Delta)$ does not depend on the resolution chosen.

2.10 Universal Dual Graph

In this section, we give another approach to the valuative tree, which is from the aspect of birational geometry and is more combinatorial.

[23] In [Laz04], they only work on varieties over $\mathbb{C}$, but the proof goes without restriction on the base field.

Theorem 2.10.1. *Let $\pi\colon Y \to X$ be a proper birational morphism between integral Noetherian regular schemes of dimension 2. Then π is a sequence of blow-ups at closed points.*

Proof. See [Sta25, Tag 0C5R]. □

Theorem 2.10.2. *If $\pi\colon Y \to X$ and $\pi'\colon Y' \to X$ are two proper birational morphisms between integral Noetherian schemes of dimension 2, and if Y and Y' are regular, then there is a scheme Z with morphisms $\varphi\colon Z \to Y$ and $\varphi'\colon Z \to Y'$ so that*

- *φ and φ' are sequences of blow-ups at closed points.*

- *the diagram commutes: $\pi \circ \varphi = \pi' \circ \varphi'$.*

Proof. This is a corollary of [Sta25, Tag 0C5S]. □

Notation 2.10.3 ($\mathfrak{B}$)**.** Recall $R = \Bbbk[[x, y]]$. We define $\mathfrak{B}$ to be the collection of all proper birational maps $\pi\colon Y \to \operatorname{Spec} R$ with Y a regular scheme.

Definition 2.10.4 (Dual Graph of π)**.** Let $\pi\colon Y \to \operatorname{Spec} R$ be a proper birational morphism. Then π is associated with a (combinatorial) graph Γ_π, called the *dual graph* of π, constructed as follows: Vertices of Γ_π are irreducible components of $\pi^{-1}(\mathfrak{m})$, and if two irreducible components intersect, there is an edge between them.

- The collection of vertices of Γ_π is denoted by Γ_π^*, and is also called the *dual graph* of π if no confusion arises.

- If $\pi_0 : \operatorname{Bl}_\mathfrak{m} R \to \operatorname{Spec} R$ is the single blow-up at $\mathfrak{m}$, we write E_0 the exceptional divisor of π_0. By the universal property of the blow-up, $E_0 \in \Gamma_\pi$ for all $\pi \in \mathfrak{B}$.

- For $\pi \in \mathfrak{B}$, define a partial order on Γ_π^* as follows: $E' \leq_\pi E$ if there is a containment of segments $[E_0, E'] \subset [E_0, E]$ in Γ_π. This construction makes $(\Gamma_\pi^*, \leq_\pi)$ a poset.

Remark 2.10.5. Since π is a sequence of point blow-ups (see Theorem 2.10.1), the dual graph Γ_π can be constructed inductively. See Figure 2.5 for an illustration, and from Figure 2.5 it is easy to see that a dual graph Γ_π is always a trivalent tree.

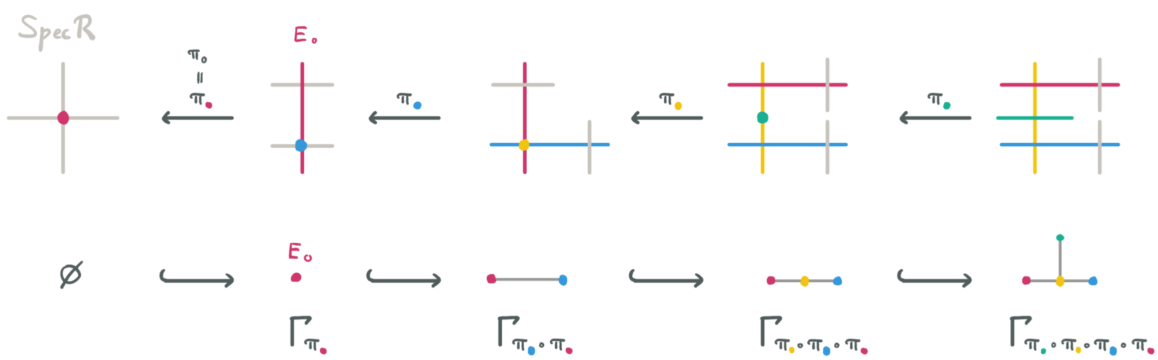


Figure 2.5: The top is a diagram of four blow-ups, while the bottom is the corresponding dual graphs. The yellow vertex is obtained from blowing up at the intersection of exceptional divisors, and is called a *satellite point* in [FJ04]. The other vertices, obtained from blowing up a point in a single exceptional divisor, are called *free points*.

Definition 2.10.6 (Universal Dual Graph)**.** Define a partial order on $\mathfrak{B}$ by declaring that $\pi \geq \pi'$ if π factors through π'. By Theorem 2.10.2, $(\mathfrak{B}, \leq)$ is a directed poset.

Note that if $\pi' \leq \pi$ then there is an order-preserving inclusion map between dual graphs $\iota_{\pi',\pi} \colon \Gamma^*_{\pi'} \to \Gamma^*_\pi$, which indicates that $((\Gamma^*_\pi, \leq_\pi)_{\pi\in\mathfrak{B}}, (\iota_{\pi',\pi})_{\pi'\leq\pi\in\mathfrak{B}})$ is a direct system of posets. The colimit (direct limit) of this direct system

$$(\Gamma^*, \leq) \triangleq \underset{\pi\in\mathfrak{B}}{\operatorname{colimit}}(\Gamma^*_\pi, \leq_\pi)$$

is called the *universal dual graph*. Intuitively, Γ^* is the union of all dual graphs Γ^*_π.

Note that $E' \leq E$ if and only if $E' \leq_\pi E$ for some $\pi \in \mathfrak{B}$.

Theorem 2.10.7 ([FJ04, Proposition 6.2])**.** *The universal dual graph* $(\Gamma^*, \leq)$ *is a non-metric* $\mathbb{Q}$*-tree, rooted at* E_0*. All elements of* Γ^* *are branch points.*

Definition 2.10.8 (Universal Dual Graph, Non-metric $\mathbb{R}$-Tree Version)**.** By Theorem 2.5.17 and Definition 2.5.16, the non-metric $\mathbb{Q}$-tree $(\Gamma^*, \leq)$ induces a non-metric $\mathbb{R}$-tree $\Gamma^*_{\mathbb{R}}$,[24] and its completion is also called the *universal dual graph*, denoted by Γ.

Remark 2.10.9. In Theorem 2.5.17, it is possible that some points in a non-metric $\mathbb{Q}$-tree $(\mathcal{T}, \leq)$ are regular points in $(\mathcal{T}_{\mathbb{R}}, \leq)$, while in the case of the universal dual graph, every point in Γ^* is a branch point of Γ because of Theorem 2.10.7.

To build the connection of Γ and the valuative tree $\mathcal{V}_{\mathfrak{m}}$, we introduce the following quantity: Farey weights.

Definition 2.10.10 (Farey Weights and Farey Parameters)**.** The *Farey Weight* is a function $\Gamma^* \to \mathbb{Z}^2_{\geq 1}$ sending E to a pair of positive integers $(a(E), b(E))$, constructed as follows. Recall that every $E \in \Gamma^*$ can be reached by a sequence of point blow-ups (see [HS06, Exercise 9.9]), so we can define the Farey Weight inductively.

First, define $(a(E_0), b(E_0)) = (2, 1)$. Now let $E \in \Gamma^*$; then, there is $\pi\colon Y \to \operatorname{Spec} R$ such that E is the exceptional divisor of a point blow-up $\beta : \mathrm{Bl}_p Y \to Y$, where p is some point in $\pi^{-1}(\mathfrak{m})$. If p is only in a single divisor E' of Y, define $(a(E), b(E)) = (a(E') + 1, b(E'))$; otherwise, if p lies in the intersection of two exceptional divisors E' and E'' of Y, define $(a(E), b(E)) = (a(E'), b(E')) + (a(E''), b(E''))$. Figure 2.6 illustrates the Farey weights of the first exceptional divisors.

The *Farey parameter* of $E \in \Gamma^*$ is the number $A^{\mathrm{Farey}}(E) \triangleq a(E)/b(E) \in \mathbb{Q}_{\geq 0}$.

Figure 2.6: The Farey weights are assigned for the vertices shown in Figure 2.5.

[24] In [FJ04], $\Gamma^*_{\mathbb{R}}$ is denoted by Γ°.

Theorem 2.10.11 ([FJ04, Lemma 6.15]). *The Farey weight $A^{\mathrm{Farey}}\colon \Gamma^* \to \mathbb{Q}_{\geq 2}$ is a strictly increasing function.*

Definition 2.10.12 (Farey Multiplicity). For $E \in \Gamma^*_{\mathbb{R}}$, we define

$$m^{\mathrm{Farey}}(E) = \min\{b(E') \mid E' \geq E\},$$

called the *Farey multiplicity* of E.[25]

Now, we are ready to state the key fact in this section.

Theorem 2.10.13 (The Isomorphism between Γ and $\mathcal{V}_{\mathfrak{m}}$). *There is an isomorphism*

$$\Phi\colon (\Gamma, A^{\mathrm{Farey}}) \to (\mathcal{V}_{\mathfrak{m}}, A^{\mathrm{thin}})$$

of parametrized trees preserving the multiplicities (i.e., $m^{\mathrm{Farey}} = m \circ \Phi$).

Remark 2.10.14. We may refer the reader to [FJ04, Section 6.4] for a proof of Theorem 2.10.13. However, in that book after Lemma 6.27, they prove that $A^{\mathrm{Farey}} = A^{\mathrm{thin}} \circ \Phi$ by using the statement that $A^{\mathrm{Farey}} = 1 + \hat{\beta}$, which is true in characteristic 0 but false in positive characteristic (we will discuss $\hat{\beta}$ in the next chapter).

In the rest of this section, we establish an alternative proof of Theorem 2.10.13 that works in arbitrary characteristic. The strategy is first to prove it on the restriction Γ^*; for this purpose, we list the properties needed.

Definition 2.10.15 ($\mathcal{V}_{\mathrm{div}}$). We define $\mathcal{V}_{\mathrm{div}}$ to be the collection of all divisorial valuations in $\mathcal{V}_{\mathfrak{m}}$.

Lemma 2.10.16 ([FJ04, Subsection 6.5.1]). *The natural map $\Phi\colon \Gamma^* \to \mathcal{V}_{\mathrm{div}}$ sending E to its normalized divisorial valuation v_E is a bijection.*

[25] In [FJ04], they simply call m^{Farey} multiplicity and use the notation m.

Proof. By the definition of v_E, the map Φ is clearly surjective.

For the injectivity, let $E, E' \in \Gamma^*$ and $E \neq E'$. Then, there exists a proper birational morphism $\pi\colon Y \to \operatorname{Spec} R$ so that $E, E' \in \pi^{-1}(\mathfrak{m})$. Since $c_Y(v_E) = \eta_E \neq \eta_{E'} = c_Y(v_{E'})$, we must have $v_E \neq v'_E$. (Here, the notation η_Z denotes the generic point of an irreducible closed subscheme Z.) $\square$

Lemma 2.10.17 ([FJ04, Subsection 6.5.3])**.** *The natural map $\Phi\colon \Gamma^* \to \mathcal{V}_{\text{div}}$ and its inverse are order-preserving.*

Lemma 2.10.18. *Given $\pi \in \mathfrak{B}$ and $E \in \Gamma^*$, we have equalities $a(E) = A_X(E)$ and $b(E) = \operatorname{ord}_E(\pi^*\mathfrak{m})$, where $X = \operatorname{Spec} R$.*

Proof. Write $\pi\colon Y \to X$ as a sequence of $n+1$ blow-ups of closed points from Theorem 2.10.1. The only relevant case is that E is the exceptional divisor in the last blow-up, so we assume it and write $\pi = \pi' \circ \varphi$, where $\pi'\colon Y' \to X$ is the first n blow-ups and $\varphi\colon Y \to Y'$ is the last blow-up at a closed point $p \in (\pi')^{-1}(\mathfrak{m})$. We prove by induction on n; the base case $n = 0$ is clear.

To show that $a(E) = A_X(E)$, note from [dFEM11, Appendix A] that the relative canonical divisors can be defined via *the sheaf of special differentials*, and we have the transformation rule

$$K_{Y/X} = K_{Y/Y'} + \varphi^*\left(K_{Y'/X}\right)$$

from [dFEM11, Lemma A.13]. If p is in a single exceptional divisor E' of Y', we have

$$\begin{aligned}
A_X(E) &= 1 + \mathrm{ord}_E\left(K_{Y/X}\right) \\
&= 1 + \mathrm{ord}_E(K_{Y/Y'}) + \mathrm{ord}_E(\varphi^*(K_{Y'/X})) \\
&= 1 + 1 + \mathrm{ord}_{E'}(K_{Y'/X}) \\
&= 1 + a(E') \\
&= a(E).
\end{aligned}$$

Similarly, if p is in the intersection of exceptional components E' and E'' of Y', we obtain

$$\begin{aligned}
A_X(E) &= 1 + \mathrm{ord}_E(K_{Y/Y'}) + \mathrm{ord}_E(\varphi^*(K_{Y'/X})) \\
&= 1 + 1 + \mathrm{ord}_{E'}(K_{Y'/X}) + \mathrm{ord}_{E''}(K_{Y'/X}) \\
&= a(E') + a(E'') \\
&= a(E).
\end{aligned}$$

Checking $b(E) = \mathrm{ord}_E(\pi^*\mathfrak{m})$ is straightforward. If p is in a single divisor E' locally cut out by a function f and if E is locally cut by a function g, then

$$\pi^*\mathfrak{m} = \varphi^*\pi'^*\mathfrak{m} = \varphi^*\left(g^{b(E')}u'\right) = f^{b(E')}u\varphi^*(u'),$$

where u' and u are local invertible functions around E' and E respectively. Similarly, if p is in the intersection of two exceptional divisors E' and E'', locally cut out by functions g_1 and g_2, then

$$\pi^*\mathfrak{m} = \varphi^*\pi'^*\mathfrak{m} = \varphi^*\left(g_1^{b(E')}g_2^{b(E'')}u'\right) = f^{b(E')+b(E'')}u\varphi^*(u'),$$

where again u' and u are local invertible functions around E' and E respectively. $\square$

Lemma 2.10.19 ([FJ04, Lemma 6.16])**.** *Let $E, E' \in \Gamma^*$ so that E' is the exceptional divisor of a blow-up at a point lying only on a single exceptional divisor E. We then have $m^{\text{Farey}}(E') = b(E') = b(E)$, and the Farey multiplicity is constant on the interval $(E, E']$ with the value $b(E')$.*

Lemma 2.10.20. *The map $\Phi\colon \Gamma^* \to \mathcal{V}_{\text{div}}$ satisfies $m^{\text{Farey}} = m \circ \Phi$.*

Proof. The details of the proof can be found in [FJ04]. Let me go through the rough idea.

The first step is to identify all analytically irreducible curves C as ends of Γ via [FJ04, Proposition 6.12]. The Farey multiplicity m^{Farey} naturally extends onto those curves by taking the limit.[26] From [FJ04, Corollary 6.21], we obtain

$$m^{\text{Farey}}(E) = \left\{m^{\text{Farey}}(C) \mid C > E\right\}.$$

Then, [FJ04, Lemma 6.30] says that $m^{\text{Farey}}(C) = m(v_f)$ whenever $C = V(f)$. For $E \in \Gamma^*$, we can conclude that

$$\begin{aligned} m(\Phi(E)) &= \min\left\{m(v_f) \mid v_f > \Phi(E)\right\} \\ &= \min\left\{m^{\text{Farey}}(C) \mid C > E\right\} \\ &= m^{\text{Farey}}(E) \end{aligned}$$

as desired. $\square$

Theorem 2.10.21. *The natural map $\Phi\colon \Gamma^* \to \mathcal{V}_{\text{div}}$ satisfies $A^{\text{Farey}} = A^{\text{thin}} \circ \Phi$.*

Proof. Similar to the proof of Lemma 2.10.18, we prove by induction on the number of blow-ups. That is, pick $\pi \in \mathfrak{B}$ so that $\pi = \pi' \circ \varphi$, where $\pi'\colon Y \to \operatorname{Spec} R$ is a

[26] In [FJ04], $m^{\text{Farey}}(C)$ is denoted by $m_\Gamma(C)$.

composition of n blow-ups on closed points, and $\varphi\colon Y \to Y'$ is a single blow-up with the exceptional divisor E at a closed point $p \in (\pi')^{-1}(\mathfrak{m})$. The base case $n = 0$ is clear. There are two cases in the inductive step, and we will frequently invoke Lemma 2.10.19 and Lemma 2.10.20 without further mention.

Assume that p only belongs to a single exceptional divisor E' in Y', so $b(E) = b(E')$. Because of Lemma 2.10.19, the Farey multiplicity is constant on the segment $(E', E]$ with the value $m^{\mathrm{Farey}}(E) = b(E)$. Pick $f \in \mathfrak{m}$ so that $V(f) > E$ and $m(v_E) = \mathrm{ord}_{\mathfrak{m}}(f)$. Hence, $\pi_*^{-1}(f)$ must intersect E transversely, and we can see that $\mathrm{ord}_E(f) = \mathrm{ord}_{E'}(f)+1$ by a local computation. Using Proposition 2.7.2, we then compute

$$\begin{aligned}
A^{\mathrm{thin}}(v_E) - A^{\mathrm{thin}}(v_{E'}) &= m(v_E)\left(\alpha(v_E) - \alpha(v_{E'})\right) \\
&= m(v_E)\left(\frac{v_E(f)}{\mathrm{ord}_{\mathfrak{m}}(f)} - \frac{v_{E'}(f)}{\mathrm{ord}_{\mathfrak{m}}(f)}\right) \\
&= \frac{\mathrm{ord}_E(f)}{b(E)} - \frac{\mathrm{ord}_{E'}(f)}{b(E')} \\
&= \frac{1}{b(E')} \\
&= \frac{a(E')+1}{b(E')} - \frac{a(E')}{b(E')} \\
&= \frac{a(E)}{b(E)} - \frac{a(E')}{b(E')} \\
&= A^{\mathrm{Farey}}(E) - A^{\mathrm{Farey}}(E').
\end{aligned}$$

As $A^{\mathrm{thin}}(v_{E'}) = A^{\mathrm{Farey}}(E')$ from the induction hypothesis, we see that $A^{\mathrm{thin}}(v_E) = A^{\mathrm{Farey}}(E)$.

Now, assume that p lies in the intersection of two exceptional components E' and E'' in Y', where $E' < E''$ in Γ^*. As a corollary of Lemma 2.10.19, the Farey multiplicity is constant on the segment $(E', E'']$ with the value $m^{\mathrm{Farey}}(E'') = m^{\mathrm{Farey}}(E)$. Let $f \in \mathfrak{m}$

such that $m(v_{E''}) = \mathrm{ord}_{\mathfrak{m}}(f)$. We write (a,b) (resp. (a',b') and (a'',b'')) for the Farey parameter of E (resp. E' and E''). Leveraging Proposition 2.7.2 again, we obtain

$$\begin{aligned}
A^{\mathrm{thin}}(v_E) - A^{\mathrm{thin}}(v_{E'}) &= m(v_E)(\alpha(v_E) - \alpha(v_{E'})) \\
&= m(v_E)\left(\frac{v_E(f)}{\mathrm{ord}_{\mathfrak{m}}(f)} - \frac{v_{E'}(f)}{\mathrm{ord}_{\mathfrak{m}}(f)}\right) \\
&= \frac{\mathrm{ord}_E(f)}{b} - \frac{\mathrm{ord}_{E'}(f)}{b'} \\
&= \frac{\mathrm{ord}_{E'}(f) + \mathrm{ord}_{E''}(f)}{b' + b''} - \frac{\mathrm{ord}_{E'}(f)}{b'} \\
&= \frac{b' \,\mathrm{ord}_{E''}(f) - b'' \,\mathrm{ord}_{E'}(f)}{b'(b' + b'')} \\
&= \frac{b''}{b' + b''}\left(\frac{\mathrm{ord}_{E''}(f)}{b''} - \frac{\mathrm{ord}_{E'}(f)}{b'}\right) \\
&= \frac{b''}{b' + b''}\left(A^{\mathrm{thin}}(v_{E''}) - A^{\mathrm{thin}}(v_{E'})\right).
\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}
A^{\mathrm{Farey}}(E) - A^{\mathrm{Farey}}(E') &= \frac{a}{b} - \frac{a'}{b'} \\
&= \frac{a' + a''}{b' + b''} - \frac{a'}{b'} \\
&= \frac{a''b' - a'b''}{b'(b' + b'')} \\
&= \frac{b''}{b' + b''}\left(\frac{a''}{b''} - \frac{a'}{b'}\right) \\
&= \frac{b''}{b' + b''}\left(A^{\mathrm{Farey}}(E'') - A^{\mathrm{Farey}}(E')\right).
\end{aligned}$$

The induction hypothesis on the number of blow-ups says that $A^{\mathrm{thin}}(v_{E'}) = A^{\mathrm{Farey}}(E')$ and $A^{\mathrm{thin}}(v_{E''}) = A^{\mathrm{Farey}}(E'')$, so we get $A^{\mathrm{thin}}(v_E) = A^{\mathrm{Farey}}(E)$ as desired. $\square$

Proof of Theorem 2.10.13. Lemma 2.10.16, Lemma 2.10.17, and Theorem 2.10.21 imply that the natural map $\Phi\colon (\Gamma^*, A^{\text{Farey}}) \to (\mathcal{V}_{\text{div}}, A^{\text{thin}})$ is an isomorphism of parametrized non-metric $\mathbb{Q}$-trees and preserves multiplicities. Therefore, Φ extends to an isomorphism $\Phi\colon (\Gamma, A^{\text{Farey}}) \to (\mathcal{V}_{\mathfrak{m}}, A^{\text{thin}})$ of parametrized non-metric $\mathbb{R}$-trees (we still write the extension of Farey parameter as A^{Farey}).

Note that m^{Farey} and m are integral-valued, and lower-semicontinuous on Γ and on $\mathcal{V}_{\mathfrak{m}}$ respectively, and that $m^{\text{Farey}} = m \circ \Phi$ on Γ^*. It follows that $m^{\text{Farey}} = m \circ \Phi$ on Γ as desired. □

Remark 2.10.22. From our isomorphism of Theorem 2.10.13 between the two trees, we can conclude that, for an exceptional divisor E, the number $b(E)$ is equal to the generic multiplicity $b(v_E)$ defined in Definition 2.8.1. See [FJ04, Subsection 6.3.2] for more details.

2.11 The Relative Valuative Tree $\mathcal{V}_x$

Recall that $R = \Bbbk[[x, y]]$ with unique maximal ideal $\mathfrak{m}$. To connect the valuative tree with that in the next chapter, we normalize $\mathcal{V}_R$ not by assigning $v(\mathfrak{m}) = 1$, but by choosing an element that is part of a regular system of parameters. This is a reason why we want to consider the space $(\mathcal{V}_x, \leq)$ defined in Definition 2.3.7. To avoid confusion, in the space $\mathcal{V}_x$ we may denote the partial order by $\leq_x$, and the minimum element operation by $\wedge_x$.

Theorem 2.11.1 ([FJ04, Lemma 3.59 and Proposition 3.61])**.** *The re-normalization map $N\colon \mathcal{V}_{\mathfrak{m}} \to \mathcal{V}_x$ defined by $N(v) = v/v(x)$ is a homeomorphism with respect to weak topologies, making $\mathcal{V}_x$ a non-rooted non-metric tree, and is an isomorphism of non-rooted non-metric trees. We call $\mathcal{V}_x$ the* relative valuative tree.

Remark 2.11.2 ([FJ04, Lemma 3.60]). Let $v, w \in \mathcal{V}_x \setminus \{\mathrm{ord}_x\}$. Then, $w \leq_x w$ if and only if $[v_x, N^{-1}v] \subset [v_x, N^{-1}w]$ in $\mathcal{V}_\mathfrak{m}$. Roughly speaking, the order $\leq_x$ is obtained by simply reversing the order $\leq$ on the segment $[\mathrm{ord}_\mathfrak{m}, v_x] \subset \mathcal{V}_\mathfrak{m}$.

Definition 2.11.3 (Relative Parametrizations). We define the parametrizations similarly to what we have done previously. Let $v \in \mathcal{V}_x$.

- The *relative skewness* of v is

$$\alpha_x(v) = \sup\left\{ \frac{v(f)}{v_x(f)} \,\middle|\, f \in \mathfrak{m} \right\} \in \overline{\mathbb{R}}_{\geq 0}.$$

  In particular, $\alpha_x(\mathrm{ord}_x) = 0$.

- The *relative multiplicity* of v is

$$m_x(v) = \min\{v_x(f) \mid N(v_f) \geq_x v\} \in \overline{\mathbb{Z}}_{\geq 1}.$$

- The *relative thinness* of v is

$$A_x^{\mathrm{thin}}(v) = 1 + \int_{\mathrm{ord}_x}^{v} m_x(w) d\alpha_x(w) \in \overline{\mathbb{R}}_{\geq 1}.$$

Theorem 2.11.4 ([FJ04, Proposition 3.65]). *For $v \in \mathcal{V}_\mathfrak{m} \setminus \{v_x\}$, we have $\alpha_x(N(v)) = \alpha(v)/v(x)^2$ and $A_x^{\mathrm{thin}}(N(v)) = A^{\mathrm{thin}}(v)/v(x)$. For the relative multiplicity, we have $m_x(N(v)) = 1$ if $v \in [\mathrm{ord}_\mathfrak{m}, v_x]$ and $m_x(N(v)) = m(v)v(x)$ otherwise.*

The next theorem, similar to Proposition 2.7.2, is not in [FJ04], but it is needed in the next chapter. The proof of that theorem requires the following simple lemma.

Lemma 2.11.5. *For any $v \in \mathcal{V}_\mathfrak{m}$, we have $\alpha_x(N(v \wedge v_x)) = v(x)^{-1}$.*

Proof. Applying Proposition 2.7.2 and Theorem 2.11.4 yields

$$v(x) = \alpha(v \wedge v_x) \operatorname{ord}_{\mathfrak{m}}(x) = \alpha_x(N(v \wedge v_x))v(x)^2.$$

The lemma thus follows. $\square$

Theorem 2.11.6. *For $N(v) \in \mathcal{V}_x$ and an irreducible element $f \in \mathfrak{m}$, we have*

$$N(v)(f) = \alpha_x(N(v) \wedge_x N(v_f))v_x(f).$$

In particular, $\alpha_x(N(v)) = N(v)(f)/v_x(f)$ for any f such that $N(v_f) \geq_x N(v)$.

Proof. Fix $N(v) \in \mathcal{V}_x$ and an irreducible element $f \in \mathfrak{m}$.

By Proposition 2.7.2, we see that

$$\frac{N(v)(f)}{v_x(f)} = \frac{v(f)}{v(x)v_x(f)} = \frac{\alpha(v \wedge v_f) \operatorname{ord}_{\mathfrak{m}}(f)}{v(x)\alpha(v_x \wedge v_f) \operatorname{ord}_{\mathfrak{m}}(f)} = \frac{\alpha(v \wedge v_f)}{v(x)\alpha(v_x \wedge v_f)}. \tag{2.1}$$

We process by discussing 3 possible cases. To visualize these cases, see Figure 2.7.

Case 1: $v \wedge v_f = v \wedge v_x \leq v_f \wedge v_x$. Applying Lemma 2.11.5, we see that

$$\frac{\alpha(v \wedge v_f)}{v(x)\alpha(v_x \wedge v_f)} = \frac{v(x)}{v(x)\alpha(v_x \wedge v_f)} = \frac{1}{\alpha(v_x \wedge v_f)} = \alpha_x(N(v) \wedge_x N(v_f))$$

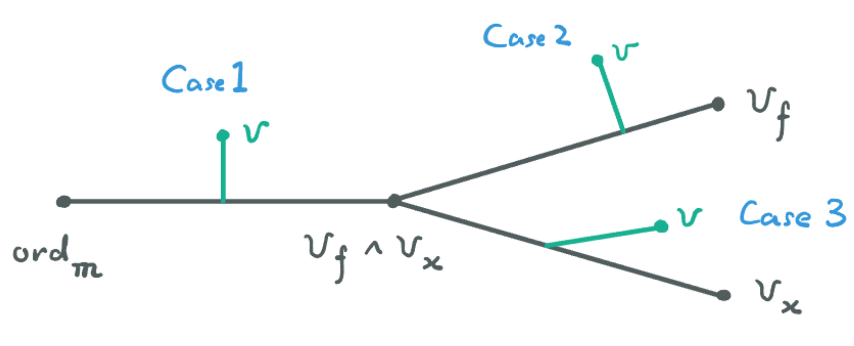


Figure 2.7: Three different cases in the proof of Theorem 2.11.6.

Case 2: $v \wedge v_x = v_f \wedge v_x < v \wedge v_f$. In this case, it follows that

$$\frac{\alpha(v \wedge v_f)}{v(x)\alpha(v_x \wedge v_f)} = \frac{\alpha(v \wedge v_f)}{v \wedge v_f(x)^2} = \alpha_x(N(v \wedge v_f)) = \alpha_x(N(v) \wedge_x N(v_f))$$

Case 3: $v \wedge v_f = v_f \wedge v_x < v \wedge v_x$. In view of Lemma 2.11.5, the computation continues as follows.

$$\frac{\alpha(v \wedge v_f)}{v(x)\alpha(v_x \wedge v_f)} = \frac{1}{v(x)} = \frac{1}{v \wedge v_x(x)} = \alpha_x(N(v \wedge v_x)) = \alpha_x(N(v) \wedge_x N(v_f)).$$

$\square$

Chapter 3

Valuative Trees via Puiseux and Hahn Series

The previous chapter presents an 'analytic' perspective on valuative trees, whereas in this chapter we approach them from an algebraic viewpoint. The key construction will be the Berkovich (affine) analytic spaces, which we will discuss in Section 3.4. At the beginning of this chapter, we develop the theory of Puiseux and Hahn series and review certain properties of real-valued field extensions.

3.1 Hahn Series

Definition 3.1.1 (Hahn Series)**.** Let F be a field and Γ a totally ordered abelian group. A *Hahn series* (or *generalized power series*) is a series of the form

$$f = \sum_{j \in \Gamma} c_j x^j,$$

where $c_j \in F$ and x is an indeterminate, such that the support of f, written as $\operatorname{supp}(f) \triangleq \{j \in \Gamma \mid c_j \neq 0\}$, is well-ordered. The collection of Hahn series is denoted by $F((x^\Gamma))$.

Proposition 3.1.2 (Operations on Hahn series)**.** *Define the addition and the multiplication on $F((x^\Gamma))$ by the following roles: for $f = \sum_{j\in\Gamma} c_j x^j$ and $g = \sum_{j\in\Gamma} d_j x^j$,*

$$f + g \triangleq \sum_{j\in\Gamma} (c_j + d_j) x^j, \text{ and}$$
$$f \cdot g \triangleq \sum_{k\in\Gamma} \left(\sum_{i+j=k} c_i d_j \right) x^k.$$

This makes $F((x^\Gamma))$ a field, call the field of Hahn series.[1]

To prove Proposition 3.1.2, we need some lemmas regarding well-ordered subsets.

Lemma 3.1.3 ([Pas77, 13.2 Lemma 2.9])**.** *Let Γ be a totally ordered abelian group, and A, B two well-ordered subsets of Γ.*

- *For $\gamma \in \Gamma$, the subset $A \cap (\gamma - B)$ of Γ is finite (here, $\gamma - B \triangleq \{\gamma - b \mid b \in B\}$).*
- *The sum $A + B \triangleq \{a + b \mid a \in A, b \in B\}$ and the union $A \cup B$ are well-ordered.*

Proof. Assume on the contrary that $A \cap (\gamma - B) = \{a \in A \mid \gamma - a \in B\}$ is infinite. Since A is well-ordered, we can pick an infinite strictly increasing sequence (a_n) of $A \cap (\gamma - B)$. This gives an infinite strictly decreasing sequence $(\gamma - a_n)$ of B, contradicting the well-ordered-ness of B. This shows the finiteness of $A \cap (\gamma - B)$.

Suppose that $A + B$ is not well-ordered, so it contains an infinite strictly decreasing sequence (c_n), where $c_n = a_n + b_n \in A + B$. We pick a subsequence of (a_n) in A inductively by choosing

$$a_{n_1} = \min\{a_n\}, \quad a_{n_2} = \min\{a_n \mid n > n_1\}, \dots, a_{n_{k+1}} = \min\{a_n \mid n > n_k\}, \dots.$$

[1] This construction also works if F is replaced by a general ring A and Γ is non-abelian. Such a collection, written as $A((\Gamma))$ in literature, is called the *Mal'cev-Neumann ring of generalized power series*, see [Poo93] for related topics. However, $A((\Gamma))$ may not be a field or a division ring.

As subsequence (a_{n_k}) is increasing by construction and (c_n) is strictly decreasing, the subsequence (b_{n_k}) of (b_n) is strictly decreasing, contradicting the well-ordered assumption of B. Hence, $A+B$ is well-ordered.

Finally, if $C \subset A \cup B$, then $\min C = \min\{\min(C \cap A), \min(C \cap B)\}$, so $A \cup B$ is well-ordered. $\square$

Lemma 3.1.4 ([Pas77, 13.2 Lemma 2.10])**.** *Let Γ be a totally ordered abelian group, and let A be a well-ordered subset of $\Gamma_{>0}$.*

- *The sub-semigroup $\widetilde{A} \triangleq \bigcup_{n=1}^{\infty} nA$ of $\Gamma_{>0}$ is well-ordered.*
- *We have $\widetilde{A} \supset 2\widetilde{A} \supset \cdots \supset n\widetilde{A} \supset \cdots$, and then $\bigcap_{n=1}^{\infty} n\widetilde{A} = \emptyset$.*

Proof. See the proof in [Pas77] for details about the well-ordered-ness of $\widetilde{A}$.

Clearly, we have the descending chain $\widetilde{A} \supset 2\widetilde{A} \supset \cdots \supset n\widetilde{A} \supset \cdots$. Assume that the intersection is nonempty. Since $\widetilde{A}$ is well-ordered, we can pick a minimum $a \in \bigcap_{n=1}^{\infty} n\widetilde{A}$. For all $n > 1$, write $a = b_n + c_n \in (n-1)\widetilde{A} + \widetilde{A}$, and then set $B = \{b_n\}$ and $C = \{c_n\}$. As $b_n = a - c_n$ for all $n > 1$, it shows that $B \subset \widetilde{A} \cap (a - \widetilde{A})$. By Lemma 3.1.3, B is a finite set, so there exists $b \in B$ so that $b = b_n \in (n-1)\widetilde{A}$ for infinitely many $n > 1$. It follows that $b \in \bigcap_{n=1}^{\infty} n\widetilde{A}$. However, picking one such n we obtain $b = a - c_n < a$, contradicting the minimality of $a \in \bigcap_{n=1}^{\infty} n\widetilde{A}$. $\square$

Proof of Proposition 3.1.2. Write $A = \mathrm{Supp}(\mathrm{f})$ and $B = \mathrm{Supp}(g)$. By Lemma 3.1.3, $\mathrm{Supp}(f+g) \subset A \cup B$ is well-ordered, so the addition is well-defined.

Let's see why the multiplication is also well-defined. First, for $k \in \Gamma$, notice that the sum $\sum_{i+j=k} c_i d_j$ is supported on $A \cap (k - B)$, which is finite by Lemma 3.1.3, so sum is well-defined. Next, $\mathrm{Supp}(f \cdot g) \subset A + B$ is well-ordered by Lemma 3.1.3, which implies that the multiplication is well-defined.

With the defined operations, it is not hard to see that $F((x^\Gamma))$ forms a commutative ring with the additive identity $0 = 0x^0$ and multiplicative identity $1 = 1x^0$. Let's show that $F((x^\Gamma))$ is a field.

Consider the special case that $f = 1 - h \in F((x^\Gamma))$, where $h \in F((x^\Gamma))$ with $\mathrm{Supp}(h) \subset \Gamma_{>0}$. Our goal is to show that the geometric series $g \triangleq \sum_{n=1}^{\infty} h^n$ is a well-defined element and is the inverse of $1 - h$. We write $A = \mathrm{Supp}(h)$ and $\widetilde{A} = \bigcup_{n=1}^{\infty} nA$. Lemma 3.1.3 says that $\bigcap_{n=1}^{\infty} n\widetilde{A} = \emptyset$ and that $\widetilde{A}$ is well-ordered. Hence, for any $k \in \Gamma$ it only lies in finitely many supports $\mathrm{Supp}(h^n)$ (because $\mathrm{Supp}(h^n) \subset n\widetilde{A}$), and thus all the coefficient sums of g are finite. Hence, g is a well-defined series; furthermore, g is supported inside the well-ordered set $\{1\} \cup \widetilde{A}$, showing that g is a Hahn series. Finally, we can check that $fg = gf = 1$ by calculating the coefficients at each $k \in \Gamma$.

For general $f = \sum_j c_j x^j \in F((x^\Gamma)) \setminus \{0\}$, let $j_0 = \min \mathrm{Supp(f)}$. This shows that $f = c_{j_0} x^{j_0}(1 - h)$, where

$$h \triangleq (c_{j_0}^{-1} x^{-j_0}) f - 1$$

has the support $\mathrm{Supp}(h) \subset \Gamma_{>0}$. We know that $1 - h$ is invertible, so f is invertible as desired.[2] □

Definition 3.1.5 (Valuation on Hahn Series)**.** Define the 'x-adic' Krull valuation $v_* \colon F((x^\Gamma)) \to \Gamma$ by

$$v_* \colon f \mapsto \min \mathrm{Supp}(f),$$

making $(F((x^\Gamma)), v_*)$ a valued field with the valued group Γ and the residue field F.

The field of Hahn series $F((x^\Gamma))$ enjoys many nice properties which indicates that $F((x^\Gamma))$ is 'the largest field in many senses'.

[2] In [Pas77], the same proof works even if Γ is not abelian. In that case, we conclude that $F((x^\Gamma))$ is a division F-algebra.

Definition 3.1.6 (Spherical Completeness)**.** Let (K, v) be a valued field with the value group Γ.

- A *closed ball* in K is a set of the form $B(f, \gamma) \triangleq \{g \in K \mid v(f - g) \geq \gamma\}$, where $f \in K$.
- (K, v) is *spherically complete* if any descending sequence of closed balls, indexed by a totally-ordered set, has a nonempty intersection.

Remark 3.1.7. A valued field K is called *maximally complete* if it does not have any nontrivial immediate extension. In [Kap42, Theorem 4], Kaplansky shows that maximal completeness is equivalent to the condition that every descending sequence of closed balls, indexed by a well-ordered set, has a nonempty intersection. (His original language is that every *pseudo convergence* sequence in K has a *pseudo limit*.) Although not every totally ordered set is well-ordered, every unbounded totally ordered set admits a cofinal well-ordered subset whose ordinal type is a limit ordinal. Therefore, Kaplansky's condition is the same as the modern setup of spherical completeness.

Remark 3.1.8. An (ultra)metric space is called *spherically complete* if any countable descending sequence of closed balls has nonempty intersection (and thus it is a complete metric space).

If the value group of (K, v) is of rank 1 so that $\Gamma \subset \mathbb{R}$, the valuation v induces a norm on K by letting $|f|_v = \exp(-v(f))$ for $f \in K$; in particular, K is a metric space. In this case, if K is spherically complete as a valued field, then it is so as a metric space.

Proposition 3.1.9 (Topological Properties of Hahn Series)**.** *The field of Hahn series $F((x^\Gamma))$ is spherically complete. Moreover, if* $\operatorname{char} F = 0$*, then $F((x^\Gamma))$ is (up to a non-unique isomorphism) the only spherically complete field with the value group Γ and the residue field F.*

Proof. See [Kru32] for the spherical completeness of $F((x^\Gamma))$, and see [Kap42, Theorem 6 and Theorem 8] for the uniqueness of spherical completeness. $\square$

Remark 3.1.10. In [Kap42], the following question is investigated: if (K, v) is spherically complete and has value group Γ and residue field F, is there an isomorphism $K \cong F((x^\Gamma))$ of value fields? Kaplansky gives a partial answer. He defines that (K, v) satisfies 'Hypothesis A' if either $\operatorname{char} K = 0$, or $\operatorname{char} K = p > 0$ with the following conditions on residue field F and value group Γ:

- Any polynomial of the form

$$x^{p^n} + a_1 x^{p^{n-1}} + \cdots + a_{n-1} x^p + a_n x + a_{n+1} \in F[x]$$

  has a root in F.

- $\Gamma = p\Gamma$.

With the assumption of Hypothesis A, Theorem 6 of [Kap42] says that if $\operatorname{char} K = \operatorname{char} F$, then K is analytically isomorphic to a field of 'twist' Hahn series $F((x^\Gamma, c_{\alpha,\beta}))$, where the *factor set* $\{c_{\alpha,\beta}\}$ is part of the data of the multiplication in the sense that

$$x^\alpha \cdot x^\beta = c_{\alpha,\beta} x^{\alpha+\beta}$$

for all $\alpha, \beta \in \Gamma$; and Theorem 8 of [Kap42] says that if F contains all roots of unity, then K is uniquely determined by F, Γ, and the additional $v(p)$ when K has mixed characteristic $(0, p)$. That is, if $\operatorname{char} K = \operatorname{char} F$, then the data of the factor set in Theorem 6 can be determined from F and Γ.

Therefore, we can observe the following. Since $F((x^\Gamma))$ is spherically complete, Theorem 6 and Theorem 8 imply that $F((x^\Gamma))$ is the only spherically complete valued field (up to isomorphism) with residue field F and value group Γ if

- either $\operatorname{char} F = 0$, or
- F is algebraically closed with $\operatorname{char} F = p > 0$ and $\Gamma = p\Gamma$.

In particular, this observation always holds when F is algebraically closed and Γ is divisible.

With Hypothesis A assumed, Theorem 8 says that a spherically complete K is still uniquely determined in the mixed-characteristic case, but clearly K is not isomorphic to a field of Hahn series. One possibility is to use the 'p-adic Mal'cev-Neumann field', introduced in [Poo93], to describe K.

Proposition 3.1.11 (Valuative Properties of Hahn Series)**.** *Let (K, v) be a valued field that satisfies the following properties.*

- *The valuation v is trivial on the prime field of K.*[3]
- *The value group Γ of K is divisible, and the residue field F of K is algebraically closed.*

Then (K, v) is isomorphic to a subfield of $F((x^\Gamma))$ such that $v_*|_K = v$.

Proof. See [Poo93, Corollary 5]. □

Proposition 3.1.12 (An Algebraic Property of Hahn Series)**.** *If F is algebraically closed and Γ is divisible, then $F((x^\Gamma))$ is algebraically closed.*

Proof. See [Poo93, Corollary 4]. □

Remark 3.1.13. Let (K, v) be a valued field with the value group Γ and the residue field F so that v is trivial on the prime field of K. By propositions 3.1.11 and 3.1.12, we conclude that $F^{\mathrm{alg}}((x^{\overline{\Gamma}}))$ contains an algebraic closure K^{alg} of K, where $\overline{\Gamma}$ denotes

[3]This assumption is essential. See [Poo93] for a generalization of the proposition when this assumption is false.

the divisible closure of Γ. However, K^{alg} is usually much smaller than $F^{\mathrm{alg}}((x^{\overline{\Gamma}}))$. See Section 3.2 for one such example.

Example 3.1.14. Fix a field F. The 'smallest' Hahn series is $F((x^{\mathbb{Z}}))$, which is nothing but the field of formal Laurent series $F((x))$. It is fundamental that $F((x))$ is a complete field, which is a special case of Proposition 3.1.9.

3.2 Puiseux Series

Definition 3.2.1 (Puiseux Series)**.** Let F be a field. Then *the field of Puiseux series*, denoted by $F\{\{x\}\}$, is defined by[4]

$$F\{\{x\}\} = \bigcup_{n=0}^{\infty} F((x^{1/n})).$$

$F\{\{x\}\}$ is clearly a field because it is a colimit of fields.

Remark 3.2.2. Since $F((x^{1/n})) \subset F((x^{\mathbb{Q}}))$ for all n, the colimit in Definition 3.2.1 is nothing but a union inside $F((x^{\mathbb{Q}}))$. Hence, $F\{\{x\}\} \subset F((x^{\mathbb{Q}}))$, and the valuation v_* on $F((x^{\mathbb{Q}}))$ naturally restricts onto $F\{\{x\}\}$. We always think of a Puiseux series as a series with a common denominator for all the exponents.

Now, assume that $F = \Bbbk$ is algebraically closed. In view of Remark 3.1.13, we have the inclusions of valued fields

$$\Bbbk\{\{x\}\} \xhookrightarrow{\varphi} \Bbbk\{\{x\}\}^{\mathrm{alg}} \hookrightarrow \Bbbk((x^{\mathbb{Q}})).$$

Therefore, we will always fix such an algebraic closure of $\Bbbk\{\{x\}\}$ inside $\Bbbk((x^{\mathbb{Q}}))$. On the other hand, Since $\Bbbk\{\{x\}\}$ is algebraic over $\Bbbk((x))$, it follows that $\Bbbk\{\{x\}\}^{\mathrm{alg}}$ is also an algebraic closure of $\Bbbk((x))$.

[4]This notation is borrowed from [MS15].

Theorem 3.2.3 (Newton-Puiseux). *If* $\operatorname{char} \Bbbk = 0$*, the Puiseux series* $\Bbbk\{\{x\}\}$ *is algebraically closed. That is, the inclusion map* $\varphi : \Bbbk\{\{x\}\} \to \Bbbk\{\{x\}\}^{\mathrm{alg}}$ *is an isomorphism.*

Proof. See [MS15, Theorem 2.1.5]. The idea is that given a non-constant polynomial $f(y) \in \Bbbk\{\{x\}\}[y]$, we can construct a solution of f by iteratively building the terms. The assumption that $\operatorname{char} \Bbbk = 0$ is heavily used in this process. □

Remark 3.2.4. If $\operatorname{char} \Bbbk = p > 0$, the inclusion map $\varphi : \Bbbk\{\{x\}\} \to \Bbbk\{\{x\}\}^{\mathrm{alg}}$ is *never* surjective: consider the following *Artin-Schreier polynomial*

$$y^p - y - x^{-1} \in \Bbbk((x))[y].$$

This polynomial has p different roots in $\Bbbk((x^{\mathbb{Q}}))$:

$$\sum_{e=1}^{\infty} x^{-\frac{1}{p^e}} + c = \left(x^{-\frac{1}{p}} + x^{-\frac{1}{p^2}} + \cdots + x^{-\frac{1}{p^e}} + \cdots \right) + c,$$

where $c \in \mathbb{F}_p$, but neither of them is a Puiseux series.

In this case, the algebraic closure $\Bbbk\{\{x\}\}^{\mathrm{alg}}$ is not easy to describe (see [Ked01, Theorem 8] for a full description). In spite of this, let me put some lemmas below regarding the algebraic closure.

Lemma 3.2.5 ([Ked01, Theorem 1]). *Assume that* $\operatorname{char} \Bbbk = p > 0$*. Let* L *be the collection of Hahn series* $f \in \Bbbk((x^{\mathbb{Q}}))$ *satisfying the following property: there exists* $m \in \mathbb{Z}_{>0}$ *so that every element of* $m \cdot \operatorname{Supp}(f)$ *has a denominator which is a power of* p*. In other words,*

$$m \cdot \operatorname{Supp}(f) \subset \bigcup_{e=1}^{\infty} \frac{1}{p^e}\mathbb{Z}.$$

Then L *is an algebraically closed field. (Therefore,* $\Bbbk\{\{x\}\}^{\mathrm{alg}}$ *is a subfield of* L*.)*

Lemma 3.2.6 ([Ked01, Lemma 3]). *Assume that* $\operatorname{char} \Bbbk = p > 0$. *Then any finite normal extension of* $\Bbbk((x))$ *is a tower of Artin-Schreier extensions over* $\Bbbk((x^{1/n}))$ *for some* $n \in \mathbb{Z}_{>0}$.

3.3 Complete Algebraic Closure of Laurent Series

In this section, we review the classical construction in the valuation theory to the following question: given a real-valued field (K, v), what is the smallest complete, algebraically closed field where (K, v) embeds? In particular, we care most about the case that $K = \Bbbk(x)$ is the field of fractions of $\Bbbk[x]$, and v is the x-adic valuation.

We will mostly follow [BGR84, 3.4.1].

Definition 3.3.1 (Completion of real-valued fields). Let (K, v) be a real-valued field. Since v is a valuation, it induces a norm by $|f|_v \triangleq \exp(-v(f))$ for $f \in K$, and makes K an ultra-metric space.

- (K, v) is *complete* if it is a complete metric space.
- A real-valued field (L, w) is a *completion* of (K, v) if (L, w) is complete and there is an isometric embedding of real-valued fields $(K, v) \hookrightarrow (L, w)$ such that the image is dense.

Proposition 3.3.2 ([BGR84, 3.2.3 Theorem 2]). *Let* (K, v) *be a complete real-valued field, and* L *an algebraic extension of* K. *Then* v *can be uniquely extended to a valuation* w *on* L, *called the* spectral valuation, *by the rule: for* $f \in L$,

$$v_{\mathrm{sp}}(f) = \frac{1}{n} \mathrm{N}_{L/K}(f),$$

where n is the degree of the minimal polynomial of f over K, and $\mathrm{N}_{L/K}$ denotes the field norm of the extension L/K.[5] *If additionally $[L:K]<\infty$, then (L, v_{sp}) is also complete.*

Remark 3.3.3. If a real-valued field (K, v) is not necessarily complete, and if L is an algebraic extension of K, we can still define the *spectral norm* on L as follows: if $f \in L$ has the minimal polynomial $y^n + a_1 y^{n-1} + \cdots + a_{n-1}y + a_n \in K[y]$, then

$$|f|_{\mathrm{sp}} \triangleq \max\left\{ |a_j|_v^{1/j} \,\middle|\, 1 \leq j \leq n \right\},$$

where $|\cdot|_v \triangleq \exp(-v(\cdot))$ is the norm induced by v. The spectral norm is a power-multiplicative K-algebra norm extending $|\cdot|_v$ (see [BGR84, 3.2.2 Theorem 2]).

We care about the extension of valuations. When K is complete, the spectral norm becomes multiplicative and thus defines a valuation: if $f \in L$ has minimal polynomial $y^n + a_1 y^{n-1} + \cdots + a_{n-1}y + a_n \in K[y]$, then

$$v_{\mathrm{sp}} \triangleq \min\left\{ \frac{v(a_j)}{j} \,\middle|\, 1 \leq j \leq n \right\}.$$

Moreover, one can show that $v(a_j) \geq \frac{j}{n} v(a_n)$ for all $1 \leq j \leq n$, so we recover the classical valuation theoretic expression in Proposition 3.3.2 (see [BGR84, 3.2.4 Proposition 3]).

Proposition 3.3.4 ([BGR84, 3.4.1 Proposition 3])**.** *If a real-valued field (K, v) is algebraically closed, its completion $\widehat{K}$ is algebraically closed.*

Remark 3.3.5. Now we can answer the question we asked at the beginning of the section. Let (K, v) be a real-valued field. In view of Proposition 3.3.2, the extension

[5] A real-valued field is called *Henselian* if its valuation can be uniquely extended to an algebraic extension. Hence, this proposition says that a complete real-valued field is Henselian.

of valuations behaves well on complete fields, so first we pass to the completion $(\widehat{K}, v)$. Then, we further extend the completion $(\widehat{K}, v)$ to the algebraic closure $(\widehat{K}^{\mathrm{alg}}, v_{\mathrm{sp}})$. Usually $\widehat{K}^{\mathrm{alg}}$ loses the completeness, so we should complete again and obtain $(\widehat{\widehat{K}^{\mathrm{alg}}}, v_{\mathrm{sp}})$, which is algebraically closed by Proposition 3.3.4. By this explicit construction, it is clear that $\widehat{\widehat{K}^{\mathrm{alg}}}$ is the smallest complete, algebraically closed real-valued field into which (K, v) is embedded.

Lemma 3.3.6. *If (K, v) is a real-valued field, the algebraic closure K^{alg} of K inside $\widehat{\widehat{K}^{\mathrm{alg}}}$ is dense. That is, the subset*

$$K^{\mathrm{alg}} \triangleq \left\{ f \in \widehat{\widehat{K}^{\mathrm{alg}}} \mid f \text{ is algebraic over } K \right\}$$

is dense.

Proof. See Figure 3.1 for the relation of all the fields appearing in the proof.

Although there may be more than one valuation on K^{alg} extending (K, v), the valuation v_{sp} on $\widehat{\widehat{K}^{\mathrm{alg}}}$ always restricts onto K^{alg}. Hence, $(K^{\mathrm{alg}}, v_{\mathrm{sp}})$ is an algebraically closed real-valued field. Its completion $\widehat{K^{\mathrm{alg}}}$ is inside $\widehat{\widehat{K}^{\mathrm{alg}}}$, and is algebraically closed due to Proposition 3.3.4. From the Remark 3.3.5, we must have $\widehat{K^{\mathrm{alg}}} = \widehat{\widehat{K}^{\mathrm{alg}}}$, and the lemma follows. $\square$

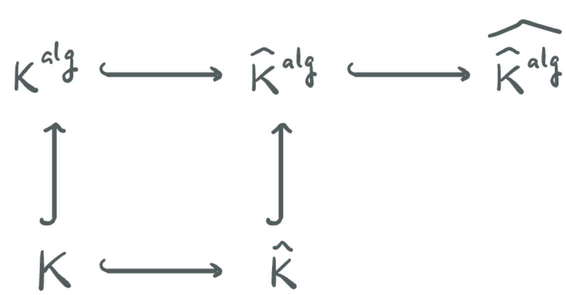


Figure 3.1: The relation between all completions and algebraic closures.

Example 3.3.7 (Arithmetic Cases)**.** Let $K = \mathbb{Q}$ with the p-adic valuation $v_p \triangleq v_{p,1} = \mathrm{ord}_p$ (see Example 2.2.2). The completion of $(\mathbb{Q}, v_p)$ is the field of p-adic numbers $(\mathbb{Q}_p, v_p)$. Its algebraic closure $\mathbb{Q}_p^{\mathrm{alg}}$ is not complete because it is not a Baire space and thus is not a complete metric space (see [Rob00, Chapter 3, Section 1.4, Theorem, Page 129] for details). We then take the completion again and get the field $\mathbb{C}_p$, which is complete but not spherically complete (See [Rob00, Chapter 3, Section 3.4, Proposition, Page 143] for a proof). In that book, the spherical completion is denoted by Ω_p.

Note that the algebraic closure $\mathbb{Q}^{\mathrm{alg}}$ in $\mathbb{C}$ naturally embeds into $\mathbb{C}_p$ (and its image is dense in $\mathbb{C}_p$ because of Lemma 3.3.6). Using this embedding, one may construct a non-canonical isomorphism of fields $\mathbb{C} \to \mathbb{C}_p$ (see [Rob00, Chapter 3, Section 3.5, Theorem, Page 145] for a proof).

Example 3.3.8 (Geometric Cases)**.** We care about this example very much. Let $K = \Bbbk(x)$ with the x-adic valuation ord_x (see Example 2.2.2). First, the completion is the field of formal Laurent series $\Bbbk((x))$. Then, we have seen in Remark 3.2.2 that its algebraic closure $\Bbbk((x))^{\mathrm{alg}}$ is noting but the algebraic closure of Puiseux series $\Bbbk\{\{x\}\}^{\mathrm{alg}}$.

Assuming $\mathrm{char}\,\Bbbk = 0$ for the rest of the discussion, we know that $\Bbbk((x))^{\mathrm{alg}} = \Bbbk\{\{x\}\}$. The field of Puiseux series is not complete, and we can obtain the completion $\widehat{\Bbbk\{\{x\}\}}$. We can explicitly describe the field $\widehat{\Bbbk\{\{x\}\}}$ as

$$\widehat{\Bbbk\{\{x\}\}} = \left\{ f = \sum_{j \geq 0} c_j x^{\hat{\beta}_j} \in \Bbbk((x^{\Gamma})) \;\middle|\; \lim_{j \to \infty} \hat{\beta}_j \to \infty \text{ if } \mathrm{Supp}(f) \text{ is an infinite set} \right\}.$$

Notation 3.3.9. (The field $\mathbb{K}$) Regardless of the characteristic of $\Bbbk$, we will write

$$\mathbb{K} = \widehat{\Bbbk((x))^{\mathrm{alg}}}.$$

3.4 Affine Analytic Spaces

In this section, we review Berkovich's construction of analytic spaces. Berkovich's theory is fruitful, but we will only see those essential parts we need. For example, I won't introduce the notion of the Berkovich spectrum of a Banach ring.

In this section, a real-valued field (K, v_K) will always be complete and algebraically closed. We write $\Bbbk$ for an algebraically closed field with the trivial valuation.

Definition 3.4.1 (Affine Analytic Spaces)**.** Let A be a finite-type K-algebra. We define the *affine analytic space* $(\operatorname{Spec} A)^{\mathrm{an}}$, in the sense of Berkovich, as follows. As a set, it is the collection of all real semivaluations v on A extending the valuation v_K on K. We endow $(\operatorname{Spec} A)^{\mathrm{an}}$ with the weak topology, the weakest topology making the evaluation maps $\mathrm{eval}_f \colon v \to v(f)$ continuous for all $f \in A$.

If $A = K[y_1, y_2, \cdots, y_n]$ is a polynomial ring, then the induced space $(\operatorname{Spec} A)^{\mathrm{an}}$ is denoted by $\mathbb{A}_K^{n,\mathrm{an}}$ and called the *Berkovich affine n-space*.

Remark 3.4.2. In algebraic geometry, we can glue affine schemes and get general schemes. Similarly, we can glue affine analytic spaces and obtain more general analytic spaces. We will not go in this direction further if unnecessary.

Remark 3.4.3. Let A be a finite-type K-algebra.

- In Berkovich's original construction, $v \colon A \to \overline{\mathbb{R}}$ is defined by requiring that the restriction of $|\cdot|_v$ to $(K, |\cdot|_{v_K})$ is *bounded*, instead of using the now-common phrasing that v extends v_K. However, one can use the definition of a real-valued field to easily show that these two statements are the same.

- If the completeness assumption on K is dropped, then any real semivaluation on A is also a real semivaluation on $A \otimes_K \widehat{K}$, which means that $(\operatorname{Spec} A)^{\mathrm{an}} = (\operatorname{Spec} A \otimes_K \widehat{K})^{\mathrm{an}}$. The algebraically-closed assumption on K is not necessary;

however, without this assumption, we will encounter closed points in $\operatorname{Spec} A$ which are not rational, and this makes things more difficult to deal with.

Definition 3.4.4 (Val_X and $X^\beth$)**.** Let $X = \operatorname{Spec} A$, where A is a finite-type $\Bbbk$-algebra. Recall home and center from Definition 2.1.1.

- The affine *Thuillier's $\beth$-space* (read 'beth' space) of X is

$$X^\beth = \{v \in X^{\mathrm{an}} \mid v \text{ is a nonnegative with } c_A(v) \neq \{0\}\}$$

with the subspace topology from X^{an} (cf. [Thu07]).

- The Val-*space* Val_X of X is defined as[6]

$$\begin{aligned}\mathrm{Val}_X &= \left\{v \in X^\beth \mid h_A(v) = \{0\}\right\} \\ &= \{v \text{ is a centered on } A\},\end{aligned}$$

with the subspace topology from $X^\beth$.

Remark 3.4.5. We assumed that $K = \Bbbk$ (i.e., the valuation is trivial) when discussing the center map, and this was the necessary situation that we can talk about the center. Given $v \in X^{\mathrm{an}}$, if there exists $c \in K^*$ such that $v(c) \neq 0$, then we may assume that $v(c) < 0$. We see that if $f \in c_A(v)$ then $v(c^n f) < 0$ if n is sufficiently large; It follows that $c^n f \notin c_A(v)$, so $c_A(v)$ is not an ideal of A.

One possibility to proceed with this theory when v_K is non-trivial is to consider algebras over the valuation ring $\mathcal{O}_{v_K}$. We will not focus on this direction.

Proposition 3.4.6 (Home is Continuous)**.** *Let A be a finite-type K-algebra. The home $h_A\colon (\operatorname{Spec} A)^{\mathrm{an}} \to \operatorname{Spec} A$ is continuous (so the home map $h\colon X^{\mathrm{an}} \to X$ is also continuous for general locally of finite-type K-scheme X).*

[6]The notation Val_X is from [JM12].

Proof. Let $Z = V(I) \subset \operatorname{Spec} A$ be a closed subset for some ideal $I \subset A$. If $v \in h_A^{-1}(Z)$, then $h_A(v) \supset I$, and we see that $v(f) = \infty$ for all $f \in I$. We conclude that $h_A^{-1}(Z) = \bigcap_{f \in I} \operatorname{eval}_f^{-1}(\infty)$ is a closed subset. $\square$

Proposition 3.4.7 (Center is Anti-Continuous)**.** *Let A be a finite-type $\mathbb{k}$-algebra. The center $c_A \colon (\operatorname{Spec} A)^{\beth} \to \operatorname{Spec} A$ is anti-continuous (i.e., the pullbacks of open sets are closed).*

When no confusion can arise, we adopt the notations $h_X \triangleq h_A$ or $c_X \triangleq c_A$.

Proof. Let $Z = V(I) \subset \operatorname{Spec} A$ be a closed subset for some ideal $I \subset A$. Since A is Noetherian, we can write $I = (f_1, f_2, \cdots, f_n)$. If $v \in c_A^{-1}(Z)$, then $c_A(v) \supset I$, meaning that $v(f_j) > 0$ for all $1 \leq j \leq n$. Hence, it follows that $c_A^{-1}(Z) = \bigcap_{j=1}^{n} \operatorname{eval}_f^{-1}(\overline{\mathbb{R}}_{>0})$ is an open subset.[7] $\square$

Theorem 3.4.8. *Let $X = \operatorname{Spec} A$, where A is a finite-type K-algebra.*

- *X^{an} is Hausdorff.*
- *If $K = \mathbb{k}$, the space $X^{\beth}$ is compact, and thus a closed subset of X^{an}.*

Proof. To show the Hausdorff-ness, let $v, w \in (\operatorname{Spec} A)^{\mathrm{an}}$ with $v \neq w$. Therefore, there exists $f \in A$ such that $v(f) \neq w(f)$, so may assume $v(f) < w(f)$. Choosing a real number $r \in (v(f), w(f))$, we get $v \in f^{-1}([0, r))$ and $w \in f^{-1}((r, \infty])$.

By the definition of weak topology, there is a topological embedding

$$(\operatorname{Spec} A)^{\beth} \longrightarrow \prod_{f \in A} \overline{\mathbb{R}}_{\geq 0}.$$

into the product space $\prod \overline{\mathbb{R}}_{\geq 0}$, which is compact by Tychonoff's theorem. The valuative conditions make $(\operatorname{Spec} A)^{\beth}$ a closed subset, so $(\operatorname{Spec} A)^{\beth}$ is compact. (The same proof works for proving the compactness in Lemma 2.6.5.) $\square$

[7] In the proof of [Can20, Lemma 2.24], the author uses the union from all eval_f, which is not true.

Remark 3.4.9 (Morphisms between Affine Analytic Spaces)**.** Let $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism of finite-type K-schemes from the ring map $\pi^{\#}\colon A \to B$. We obtain a map $\pi^{\mathrm{an}}\colon (\operatorname{Spec} B)^{\mathrm{an}} \to (\operatorname{Spec} A)^{\mathrm{an}}$ defined by the pushforward: $\pi^{\mathrm{an}}(w) \triangleq w \circ \pi^{\#}$.

- It is clear that π^{an} is continuous, and that $(\cdot)^{\mathrm{an}}$ is a functor from the category of (affine) K-schemes to the category of (affine) analytic spaces. The functor $(\cdot)^{\mathrm{an}}$ 'preserves' many properties of morphisms of schemes. For instance, π is a closed embedding only if π^{an} is a closed embedding.[8]

- When $K = \Bbbk$, the map π^{an} restricts to $\beth$-spaces $\pi^{\beth}\colon (\operatorname{Spec} B)^{\beth} \to (\operatorname{Spec} A)^{\beth}$ because of the equality $c_A(\pi^{\mathrm{an}}(v)) = \pi(c_B(v))$ for all $v \in (\operatorname{Spec} B)^{\beth}$. In this particular affine case, $\pi^{\beth}$ is automatically a closed map in view of Theorem 3.4.8. Similarly, $h_A(\pi^{\mathrm{an}}(v)) = \pi(h_B(v))$ for all $v \in (\operatorname{Spec} B)^{\mathrm{an}}$, so $\pi^{\beth}$ restricts to a continuous map of Val-spaces and is denoted by π^{Val}.

Remark 3.4.10. Let $X = \operatorname{Spec} A$, and $\mathfrak{R}$ denote the nil-radical of A. Since $\mathfrak{R} \subset h_X(v)$ for any $v \in X^{\mathrm{an}}$, v induces a real semivaluation on $A/\mathfrak{R}$ by composition:

$$A/\mathfrak{R} \to A/h_X(v) \xrightarrow{\bar{v}} \overline{\mathbb{R}}.$$

It follows that the reduction morphism $\pi_{\mathrm{red}}\colon X_{\mathrm{red}} \to X$ induces a continuous bijection $\pi_{\mathrm{red}}^{\mathrm{an}}\colon (X_{\mathrm{red}})^{\mathrm{an}} \to X^{\mathrm{an}}$. Because $\pi_{\mathrm{red}}^{\mathrm{an}}$ is closed by Remark 3.4.9, it is a homeomorphism. Similarly, $\pi_{\mathrm{red}}^{\beth}$ is also a homeomorphism when $K = \Bbbk$. Hence, it is harmless to assume that A is reduced.

Theorem 3.4.11. *Let X and Y be two integral finite-type $\Bbbk$-schemes, and let $\pi\colon Y \to X$ be a proper birational morphism. Then $\pi^{\mathrm{Val}}\colon \mathrm{Val}_Y \to \mathrm{Val}_X$ is a homeomorphism.*

[8] In fact, this is an 'if and only if' in this case.

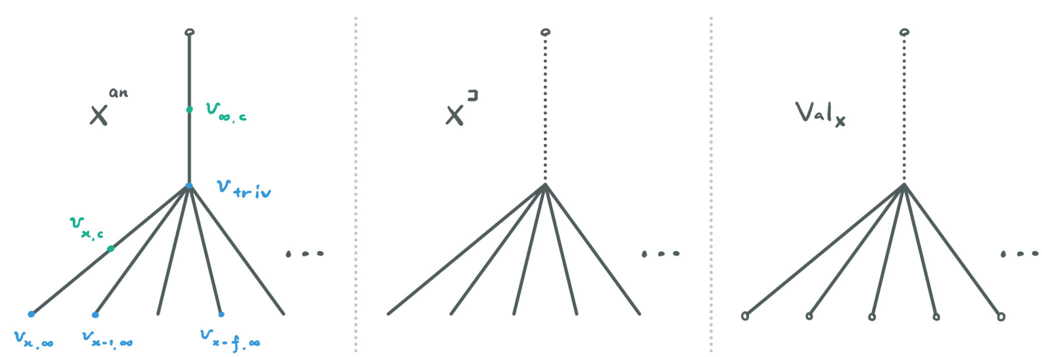
Proof. First, we claim that the continuous map π is bijective. Let $v \in \mathrm{Val}_X$. Since π is birational, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Val}_Y \arrow[r, hook] \arrow[d, "\pi^{\mathrm{Val}}"] & Y^{\beth} \arrow[d, "\pi^{\beth}"] \\
\mathrm{Val}_X \arrow[r, hook] & X^{\beth}
\end{tikzcd}$$

By the valuative criterion of properness, there exists a unique morphism $\operatorname{Spec} O_v \to Y$ through which the morphism $\operatorname{Spec} O_v \to X$ factors. The bijectivity of π^{Val} now follows.

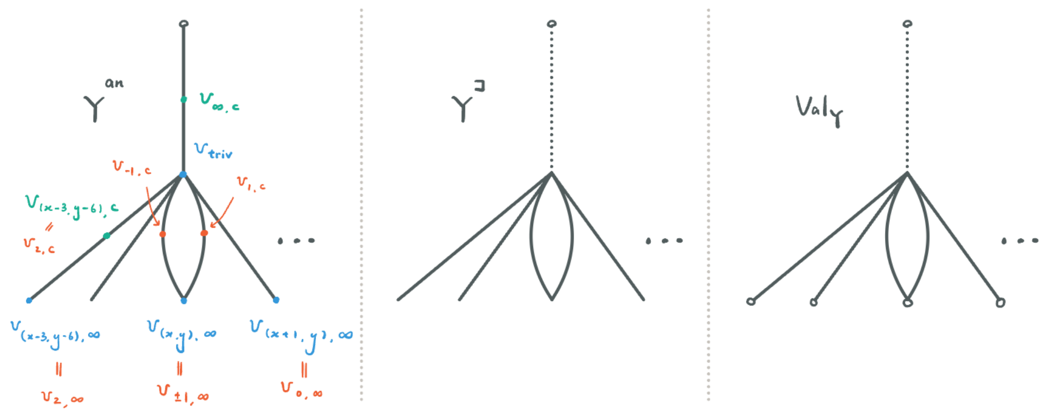
It remains to show that π^{Val} is a closed map. Let $Z \subset \mathrm{Val}_Y$ be a closed set, which means that $Z = \overline{Z} \cap \mathrm{Val}_Y$ for some closed set $\overline{Z} \subset Y^{\beth}$. Clearly, we know that $\pi^{\beth}(\overline{Z}) \cap \mathrm{Val}_X$ is a closed subset of Val_X containing π^{Val}(Z), so it suffices to show that the containment is an equality. Let $v \in \pi^{\beth}(\overline{Z}) \cap \mathrm{Val}_X$; this indicates that $v = \pi^{\beth}(w)$ for some $w \in \overline{Z} \cap (\pi^{\beth})^{-1}(\mathrm{Val}_X)$. However, from the above diagram we see that if $h_X(v) = \eta_X$ then $h_Y\left((\pi^{\beth})^{-1}(v)\right) = \eta_Y$ (where η_W denotes the generic point of an irreducible scheme W), which implies that $(\pi^{\beth})^{-1}(\mathrm{Val}_X) = \mathrm{Val}_Y$. We therefore deduce that $w \in \overline{Z} \cap \mathrm{Val}_Y = Z$, and thus $v = \pi^{\beth}(w) \in \pi^{\mathrm{Val}}(Z)$ as desired. $\square$

Example 3.4.12. Let $X = \operatorname{Spec} \Bbbk[x]$. The following pictures show the spaces X^{an}, $X^{\beth}$, and $\mathrm{Val}_{\mathrm{X}}$.



We already have the full description of X^{an} in Example 2.2.2. The space $X^{\beth}$ is obtained from X^{an} by removing all $v_{\infty,c}$ for $c > 0$. To get the space Val_X, we further remove the semivaluations $v_{f,\infty}$ for all $f \in \Bbbk$.

Let $A = \Bbbk[x,y]/(y^2 - x^2 - x^3)$ and let $Y = \operatorname{Spec} A$ be a node on the plane. The space Y^{an} is illustrated below on the left.



Using the coordinates from $\mathbb{A}^2_{\Bbbk} = \operatorname{Spec} \Bbbk[x,y]$, we can describe semivaluations such as $v_{(x+1,y),\infty}$. Consider also the normalization map $A \to \Bbbk[t]$ defined by $x \mapsto t^2 - 1$ and $y \mapsto t(t^2-1)$. This shows that A can be identified with the subring $\Bbbk[t^2-1, t(t^2-1)] \subset \Bbbk[t]$, and the semivaluations can be described using the new coordinates t, as illustrated in orange in the figure. We observe that $v_{1,c}$ and $v_{-1,c}$ are distinct valuations unless $c = 0$ or $c = \infty$. Removing all $v_{\infty,c}$ for $c > 0$ yields the space $Y^{\beth}$, and discarding the remaining boundary semivaluations $v_{\mathfrak{p},\infty}$, where $\mathfrak{p} \in Y \setminus \{0\}$, gives the space $\mathrm{Val}_{\mathrm{Y}}$. We find that Val_Y is homeomorphic to Val_X, which is an example of Theorem 3.4.11.

Example 3.4.13 (Berkovich Affine Lines)**.** Let $A = K[y]$ and $X = \operatorname{Spec} A$. Berkovich classifies the points in $X^{\mathrm{an}} = \mathbb{A}^{1,\mathrm{an}}_K$. We have seen in the above example that when $K = \Bbbk$, the space $\mathbb{A}^{1,\mathrm{an}}_{\Bbbk}$ is simple.

For $\hat{f} \in K$ and $\hat{\beta} \in \mathbb{R}$, recall that we can define the corresponding *Gauss valuation* $v_{K,\hat{f},\hat{\beta}}$ as follows: any element of $K[y]$ is of the form $f(y) = \sum_{j=0}^{n} c_j(y - \hat{f})^j$, and we let

$$\hat{v}_{K,\hat{f},\hat{\beta}}\left(\sum_{j=0}^{n} c_j(y - \hat{f})^j\right) = \min_{0 \le j \le n}\left\{v_K(c_j) + i\hat{\beta}\right\},$$

meaning that $\hat{v}_{K,\hat{f},\hat{\beta}}$ is the smallest real valuation so that the value of $y - \hat{f}$ is $\hat{\beta}$. When there is no confusion of the field K, we shorten the notation as $\hat{v}_{\hat{f},\hat{\beta}}$.

We have three observations:

- Since K is algebraically closed, there is a factorization $g(y) = a_n \prod_{j=1}^{n}(y - \hat{g}_j)$ for any $g(y) \in K[y]$, so to describe $\hat{v}_{\hat{f},\hat{\beta}}$ it suffices to describe the values on all possible linear factors. Actually we have, for any $\hat{g} \in K$,

$$\hat{v}_{\hat{f},\hat{\beta}}(y - \hat{g}) = \min\left\{\hat{\beta},\ v_K(\hat{f} - \hat{g})\right\}.$$

- The above construction works even if $\hat{\beta} = \infty$. That is, for any $\hat{g} \in K$, we have

$$\hat{v}_{\hat{f},\infty}(y - \hat{g}) = v_K(\hat{f} - \hat{g}).$$

  Therefore, we adopt the terminology *Gauss semivaluation* to mention $\hat{v}_{\hat{f},\hat{\beta}}$ with the possibility that $\hat{\beta} = \infty$.

- If $\hat{\beta} = 0$, then any $\hat{v}_{\hat{f},0}$, where $\hat{f} \in B(0,0) \subset K$ with $v_K(\hat{f}) \ge 0$, defines the same valuation which sends any $y - \hat{g}$, where $\hat{g} \in B(0,0)$, to zero. Therefore, it is natural to regard this valuation as an extension of v_K, written as $\hat{v}_K \in \mathbb{A}_K^{1,\mathrm{an}}$.

Definition 3.4.14 (Types of Points in $\mathbb{A}_K^{1,\mathrm{an}}$)**.** Berkovich names real semivaluations in $\mathbb{A}_K^{1,\mathrm{an}}$. A Gauss semivaluation $\hat{v}_{K,\hat{f},\hat{\beta}}$ is *Type 1* if $\hat{\beta} = \infty$, is *Type 2* if $\hat{\beta} \in v_K(K^*)$, and is *Type 3* if $\hat{\beta} \in \mathbb{R} \setminus v_K(K^*)$. The rest of elements in $\mathbb{A}_K^{1,\mathrm{an}}$ is called *Type 4*.

Remark 3.4.15. The natural map sending $\hat{f} \in K$ to $\hat{v}_{\hat{f},\infty} \in \mathbb{A}_K^{1,\mathrm{an}}$ is injective, so we may think of K as a subset of $\mathbb{A}_K^{1,\mathrm{an}}$.

3.5 Berkovich Affine Lines

We concluded the previous section by introducing the Berkovich affine line as an example. In this section, we focus on this example more deeply. Again, we fix a complete and algebraically closed real-valued field (K, v_K).

Lemma 3.5.1. *Recall from Definition 3.1.6 the notion of closed balls in K: we define $B(\hat{f}, \hat{\beta}) = \{\hat{g} \in K \mid v_K(\hat{f} - \hat{g}) \geq \hat{\beta}\}$ for $\hat{f} \in K$ and $\hat{\beta} \in \overline{\mathbb{R}}$. Then, a ball $B(\hat{f}, \hat{\beta})$ uniquely determines a Gauss semivaluation $\hat{v}_{\hat{f},\hat{\beta}}$ (and vice versa).*

Proof. It suffices to show that if we change the 'center' $\hat{f}$ of the ball $B(\hat{f}, \hat{\beta})$, the resulting Gauss semivaluation remains the same. Assume that $\hat{g} \in B(\hat{f}, \hat{\beta})$, which means $v_K(\hat{g} - \hat{f}) \geq \hat{\beta}$. Therefore, for $\hat{h} \in K$, either $v_K(\hat{g} - \hat{h}) = v_K(\hat{f} - \hat{h}) < \hat{\beta}$, or $v_K(\hat{f} - \hat{h}) \geq \hat{\beta}$ and $v_K(\hat{g} - \hat{h}) \geq \hat{\beta}$ simultaneously. It indicates that

$$\hat{v}_{\hat{f},\hat{\beta}}(y - \hat{h}) = \min\left\{\hat{\beta},\ v_K(\hat{f} - \hat{h})\right\} = \min\left\{\hat{\beta},\ v_K(\hat{g} - \hat{h})\right\} = \hat{v}_{\hat{g},\hat{\beta}}(y - \hat{h}),$$

which implies $\hat{v}_{\hat{f},\hat{\beta}} = \hat{v}_{\hat{g},\hat{\beta}}$. □

Theorem 3.5.2. *A descending sequence of closed balls in K, indexed by a totally-ordered set, determines a unique element in $\mathbb{A}_K^{1,\mathrm{an}}$. Furthermore, every element in $\mathbb{A}_K^{1,\mathrm{an}}$ is of this form.*

Proof. Let $\mathcal{B} = (B(\hat{f}_j, \hat{\beta}_j))_{j \in J}$ be a descending sequence of closed balls in K, and write $\hat{v}_j = \hat{v}_{\hat{f}_j,\hat{\beta}_j}$ for $j \in J$. If $i, j \in J$ with $i \leq j$, then $\hat{\beta}_i \leq \hat{\beta}_j$ and therefore $\hat{v}_i \leq \hat{v}_j$ by Lemma 3.5.1. Hence, $\hat{v}_{\mathcal{B}} \triangleq \sup_{j \in J} \hat{v}_j$ is a real semivaluation because of Theorem 2.1.6.

Conversely, let $\hat{v} \in \mathbb{A}_K^{1,\mathrm{an}}$ and consider the collection of balls

$$\mathcal{B} = \left\{ B(\hat{f}, \hat{v}(y - \hat{f})) \mid \hat{f} \in K \right\}.$$

We claim that if $\hat{v}(y - \hat{f}) < \hat{v}(y - \hat{g})$ then $B(\hat{f}, \hat{v}(y - \hat{f})) \supset B(\hat{g}, \hat{v}(y - \hat{g}))$. The assumption $\hat{v}(y - \hat{f}) < \hat{v}(y - \hat{g})$ implies that $v_K(\hat{f} - \hat{g}) \geq \min \left\{ \hat{v}(y - \hat{f}),\ \hat{v}(y - \hat{g}) \right\} = \hat{v}(y - \hat{f})$. Hence, if $\hat{h} \in B(\hat{g}, \hat{v}(y - \hat{g}))$ then we deduce that

$$\begin{aligned} v_K(\hat{h} - \hat{f}) &\geq \min \left\{ v_K(\hat{h} - \hat{g}),\ v_K(\hat{f} - \hat{g}) \right\} \\ &\geq \min \left\{ \hat{v}(y - \hat{g}),\ \hat{v}(y - \hat{f}) \right\} = \hat{v}(y - \hat{f}), \end{aligned}$$

which means that $\hat{h} \in B(\hat{f}, \hat{v}(y - \hat{f}))$. The claim thus follows.

Therefore, $\mathcal{B}$ is a collection of decreasing sequences of balls, and they uniquely determine a real semivaluation $\hat{v}_{\mathcal{B}}$. Now for a fixed $\hat{h} \in K$, we obtain

$$\begin{aligned} \hat{v}(y - \hat{h}) &= \hat{v}\left((y - \hat{f}) + (\hat{f} - \hat{h}) \right) \\ &\geq \min \left\{ \hat{v}(y - \hat{f}),\ v_K(\hat{f} - \hat{h}) \right\} \\ &= \hat{v}_{\hat{f}, \hat{v}(y - \hat{f})}(y - \hat{h}) \end{aligned}$$

for each $\hat{f} \in K$. Thus, $\hat{v}(y - \hat{h}) \geq \sup \hat{v}_{\hat{f}, \hat{v}(y - \hat{f})}(y - \hat{h}) = \hat{v}_{\mathcal{B}}(y - \hat{h})$. Also, notice that if $\hat{f} = \hat{h}$ in the above equations, we get an equality $\hat{v}(y - \hat{h}) = \hat{v}_{\hat{h}, \hat{v}(y - \hat{h})}(y - \hat{h})$. Therefore, $\hat{v}(y - \hat{h}) = \hat{v}_{\mathcal{B}}(y - \hat{h})$ for any fixed $\hat{h} \in K$, which proves that $\hat{v} = \hat{v}_{\mathcal{B}}$. $\square$

Remark 3.5.3. Let $\mathcal{B} = (B(\hat{f}_j, \hat{\beta}_j))_{j \in J}$ be a descending sequence of closed balls in K, and assume that $\hat{g} \in \bigcap B(\hat{f}_j, \hat{\beta}_j)$. Defining $\hat{\beta} = \sup \hat{\beta}_j$, we see that $B(\hat{g}, \hat{\beta}) = \bigcap B(\hat{f}_j, \hat{\beta}_j)$. Therefore, we conclude that $\hat{v}_{\mathcal{B}} \in \mathbb{A}_K^{1,\mathrm{an}}$ is not Type 4 if and only if

$\bigcap B(\hat{f}_j, \hat{\beta}_j) \neq \emptyset$. In particular, if K is spherically complete, then there are no Type 4 points.

Remark 3.5.4. We write K^{sph} as the spherical completion of K, so the inclusion map $\pi^{\#} \colon K[y] \to K^{\mathrm{sph}}[y]$ induces an analytic morphism $\pi^{\mathrm{an}} \colon \mathbb{A}^{1,\mathrm{an}}_{K^{\mathrm{sph}}} \to \mathbb{A}^{1,\mathrm{an}}_K$ (see Remark 3.4.9). Theorem 3.5.2 says that every $\hat{v} \in \mathbb{A}^{1,\mathrm{an}}_K$ comes from a descending sequence of balls, so π^{an} is subjective. Therefore, any Type 4 point in $\mathbb{A}^{1,\mathrm{an}}_K$ is a restriction of a Gauss semivaluation $\hat{v}_{\hat{f},\hat{\beta}}|_{K[y]}$ for some $\hat{f} \in K^{\mathrm{sph}} \setminus K$. We may write $\hat{v}_{\hat{f},\hat{\beta},K}$ for that restriction.

Next, we shall see that $\mathbb{A}^{1,\mathrm{an}}_K$ is naturally an $\mathbb{R}$-tree. To prove this, we need to introduce new notions.

Definition 3.5.5 (Discs in $\mathbb{A}^{1,\mathrm{an}}_K$)**.** For $\hat{f} \in K$ and $\hat{\beta} \in \overline{\mathbb{R}}$, they define a *closed disc*

$$\overline{\mathbb{D}}_K(\hat{f}, \hat{\beta}) = \mathrm{eval}^{-1}_{y-\hat{f}}\left(\overline{\mathbb{R}}_{\geq \hat{\beta}}\right) = \left\{\hat{v} \in \mathbb{A}^{1,\mathrm{an}}_K \,\middle|\, \hat{v}(y - \hat{f}) \geq \hat{\beta}\right\},$$

and an *open disc* $\mathbb{D}^{\circ}_K(\hat{f}, \hat{\beta}) = \mathrm{eval}^{-1}_{y-\hat{f}}\left(\overline{\mathbb{R}}_{>\hat{\beta}}\right)$.[9] Thus, it is obvious that an open disc is open, and a closed disc is closed and nonempty (because $\hat{v}_{\hat{f},\hat{\beta}} \in \overline{\mathbb{D}}_K(\hat{f}, \hat{\beta})$).

If no confusion can arise, we drop the K in the notation of discs.

Remark 3.5.6. Using similar analysis in the proof of Theorem 3.5.2, we can see that a closed disc in $\mathbb{A}^{1,\mathrm{an}}_K$ uniquely determines a closed ball in K, and vice versa. Combining Lemma 3.5.1, we find that there are one-to-one correspondences between three data:[10]

$$\begin{aligned} \left\{\text{closed discs } \overline{\mathbb{D}}(\hat{f}, \hat{\beta}) \subset \mathbb{A}^{1,\mathrm{an}}_K\right\} &\longleftrightarrow \left\{\text{closed balls } B(\hat{f}, \hat{\beta}) \subset K\right\} \\ &\longleftrightarrow \left\{\text{Gauss semivaluations } \hat{v}_{\hat{f},\hat{\beta}} \in \mathbb{A}^{1,\mathrm{an}}_K\right\}. \end{aligned}$$

[9] In the literature, open discs are typically denoted by $\mathbb{D}(\hat{f}, \hat{\beta})$. For clarity of notation, we adopt the slightly modified symbol $\mathbb{D}^{\circ}(\hat{f}, \hat{\beta})$ to emphasize openness.

[10] For this reason, closed discs are used in the literature to prove Theorem 3.5.2.

A corollary is that $\overline{\mathbb{D}}(\hat{f},\hat{\beta}) = \overline{\mathbb{D}}(\hat{g},\hat{\beta})$ if and only if $v_K(\hat{f}-\hat{g}) \geq \hat{\beta}$.

Lemma 3.5.7 ($\hat{v}_{\hat{f},\hat{\beta}}$ is the 'Boundary' of $\overline{\mathbb{D}}(\hat{f},\hat{\beta})$)**.** *For $\hat{f} \in K$ and $\hat{\beta} \in \overline{\mathbb{R}}$, we have*

$$\overline{\mathbb{D}}(\hat{f},\hat{\beta}) \setminus \{\hat{v}_{\hat{f},\hat{\beta}}\} = \bigcup_{\hat{g}\in B(\hat{f},\hat{\beta})} \mathbb{D}^{\circ}(\hat{g},\hat{\beta}).$$

In particular, $\overline{\mathbb{D}}(\hat{f},\hat{\beta}) \setminus \{\hat{v}_{\hat{f},\hat{\beta}}\}$ is open.

Proof. For the containment ($\supset$), if $\hat{g} \in B(\hat{f},\hat{\beta})$, we have $\mathbb{D}^{\circ}(\hat{g},\hat{\beta}) \subset \overline{\mathbb{D}}(\hat{g},\hat{\beta}) = \overline{\mathbb{D}}(\hat{f},\hat{\beta})$; furthermore, since $\hat{v}_{\hat{f},\hat{\beta}}(y-\hat{g}) = \min\left\{\hat{\beta},\ v_K(\hat{f}-\hat{g})\right\} = \hat{\beta}$, we see that $\hat{v}_{\hat{f},\hat{\beta}} \notin \mathbb{D}^{\circ}(\hat{g},\hat{\beta})$.

For the other containment ($\subset$), we assume $\hat{v} \in \overline{\mathbb{D}}(\hat{f},\hat{\beta}) \setminus \bigcup_{\hat{g}\in B(\hat{f},\hat{\beta})} \mathbb{D}^{\circ}(\hat{g},\hat{\beta})$, and we will show that $\hat{v} = \hat{v}_{\hat{f},\hat{\beta}}$. Pick an element $\hat{h} \in K$. If $\hat{h} \in B(\hat{f},\hat{\beta})$, we have

$$\hat{\beta} \geq \hat{v}(y-\hat{h}) \geq \min\left\{\hat{v}(y-\hat{f}),\ v_K(\hat{f}-\hat{h}))\right\} \geq \hat{\beta},$$

where the first inequality is from the assumption that $\hat{v} \notin \mathbb{D}^{\circ}(\hat{h},\hat{\beta})$. These inequalities force $\hat{v}(y-\hat{h}) = \hat{\beta} = \hat{v}_{\hat{f},\hat{\beta}}(y-\hat{h})$. Instead, if $\hat{h} \notin B(\hat{f},\hat{\beta})$, then $v_K(\hat{f}-\hat{h}) < \hat{\beta} \leq \hat{v}(y-\hat{f})$, which implies that

$$\hat{v}(y-\hat{h}) = \min\left\{\hat{v}(y-\hat{f}),\ v_K(\hat{f}-\hat{h})\right\} = v_K(\hat{f}-\hat{h}) = \hat{v}_{\hat{f},\hat{\beta}}(y-\hat{h}).$$

Therefore, $\hat{v} = \hat{v}_{\hat{f},\hat{\beta}}$, which completes the proof. $\square$

Remark 3.5.8. It can be shown that the closure of $\mathbb{D}^{\circ}(\hat{f},\hat{\beta})$ in $\mathbb{A}^{1,\mathrm{an}}_K$ is $\mathbb{D}^{\circ}(\hat{f},\hat{\beta}) \cup \{\hat{v}_{\hat{f},\hat{\beta}}\}$, but in general it is just a proper subset of $\overline{\mathbb{D}}(\hat{f},\hat{\beta})$ when $\hat{\beta} \in \Gamma_K \triangleq v_K(K)$.

For example, let $K = \Bbbk$ and consider $\mathbb{D}^{\circ}(0,0) = \{\hat{v}_{\hat{f},t} \mid v_K(\hat{f}) > 0 \text{ and } t \in \overline{\mathbb{R}}_{>0}\}$. We see that its closure in $\mathbb{A}^{1,\mathrm{an}}_{\Bbbk}$ is $\mathbb{D}^{\circ}(0,0) \cup \{v_{\mathrm{triv}}\}$, which is not $\overline{\mathbb{D}}(0,0) = \mathbb{A}^{1,\beth}_{\Bbbk}$.

Theorem 3.5.9. *The Berkovich affine line $\mathbb{A}^{1,\mathrm{an}}_K$ is a topological $\mathbb{R}$-tree.*

Proof. First, we show that, for any $\hat{f} \in K$, the injective map $\Phi\colon \overline{\mathbb{R}}_{\geq 0} \to \mathbb{A}_K^{1,\mathrm{an}}$ defined by $\hat{\beta} \mapsto \hat{v}_{\hat{f},\hat{\beta}}$ is a homeomorphism onto the image. Since $\mathbb{A}_K^{1,\mathrm{an}}$ is equipped with the weakest topology making all eval_f continuous for $f \in K[y]$, the continuity of Φ follows if the compositions $\overline{\mathbb{R}}_{\geq 0} \xrightarrow{\Phi} \mathbb{A}_K^{1,\mathrm{an}} \xrightarrow{\mathrm{eval}_f} \overline{\mathbb{R}}$, defined by $\hat{\beta} \to \hat{v}_{\hat{f},\hat{\beta}}(f)$, is continuous for all $f \in K[y]$; however, we see that $\hat{v}_{\hat{f},\hat{\beta}}\left(\sum a_j(y-\hat{f})^j\right) = \min\{v_K(a_j) + j\hat{\beta}\}$ is clearly continuous in $\hat{\beta}$. Moreover, $\overline{\mathbb{R}}_{\geq 0}$ is compact and $\mathbb{A}_K^{1,\mathrm{an}}$ is Hausdorff (see Theorem 3.4.8), so Φ is an open map and thus is a homeomorphism onto the image.

Next, we show that $\Phi([\hat{\alpha},\hat{\gamma}])$, where $\hat{\alpha} < \hat{\gamma}$, is a unique arc connecting $u \triangleq \hat{v}_{\hat{f},\hat{\alpha}}$ and $w \triangleq \hat{v}_{\hat{f},\hat{\gamma}}$. Picking an element $v = \hat{v}_{\hat{f},\hat{\beta}}$ for some $\hat{\beta} \in (\hat{\alpha},\hat{\gamma})$, we discover that $\mathbb{A}_K^{1,\mathrm{an}} \setminus \{v\} = U \dot{\cup} V$ is a disjoint union of open sets, where

$$U = \mathrm{eval}_{y-\hat{f}}^{-1}\left((-\infty,\hat{\beta})\right)$$

and

$$V = \overline{\mathbb{D}}(\hat{f},\hat{\beta}) \setminus \{\hat{v}_{\hat{f},\hat{\beta}}\}.$$

This proves our assertion. (The openness of V follows from Lemma 3.5.7.)

Third, for general $\hat{v}_{\hat{f},\hat{\alpha}}$ and $\hat{v}_{\hat{g},\hat{\gamma}} \in \mathbb{A}_K^{1,\mathrm{an}}$ with $v_K(\hat{f}-\hat{g}) < \min\{\hat{\beta},\hat{\gamma}\}$, we define $\hat{\beta} = v_K(\hat{f}-\hat{g})$ and therefore $\hat{v}_{\hat{f},\hat{\beta}} = \hat{v}_{\hat{g},\hat{\beta}}$. There are unique arcs from $\hat{v}_{\hat{f},\hat{\alpha}}$ to $\hat{v}_{\hat{f},\hat{\beta}}$ and from $\hat{v}_{\hat{g},\hat{\beta}}$ to $\hat{v}_{\hat{g},\hat{\gamma}}$, which gives an arc from $\hat{v}_{\hat{f},\hat{\alpha}}$ to $\hat{v}_{\hat{g},\hat{\gamma}}$. The uniqueness of this arc follows from the previous argument that $\hat{v}_{\hat{f},\hat{\alpha}} \in V$ and $\hat{v}_{\hat{g},\hat{\gamma}} \in U$.

So far we have proved that $\mathbb{A}_K^{1,\mathrm{an}}$, excluding all type 4 points, is a topological $\mathbb{R}$-tree. By Lemma 3.5.1, we can define a partial ordering on $\overline{\mathbb{D}}(0,\hat{\alpha}) \setminus \{\text{type 4 points}\}$, for any $\hat{\alpha} \in \mathbb{R}$, by declaring $\hat{v}_{\hat{f},\hat{\beta}} \leq \hat{v}_{\hat{g},\hat{\gamma}}$ if $B(\hat{f},\hat{\beta}) \supset B(\hat{g},\hat{\gamma})$. This makes $\overline{\mathbb{D}}(0,\hat{\alpha}) \setminus \{\text{type 4 points}\}$ a non-metric tree rooted at $v_{0,\hat{\alpha}}$. In view of Theorem 3.5.2, a type 4 point should live in the completion of $\overline{\mathbb{D}}(0,\hat{\alpha})$ as an end, so we can extend the topological $\mathbb{R}$-tree structure onto whole $\overline{\mathbb{D}}(0,\hat{\alpha})$. Since, $\mathbb{A}_K^{1,\mathrm{an}} = \bigcup_{\hat{\alpha}\in R} \overline{\mathbb{D}}(0,\hat{\alpha})$, we eventually conclude that $\mathbb{A}_K^{1,\mathrm{an}}$ is a topological $\mathbb{R}$-tree. $\square$

Lemma 3.5.10 ([PT21, Corollary I.3.6])**.** $\mathbb{A}_K^{1,\mathrm{an}}$ *is metrizable if and only if K is a separable space (i.e., K contains a countable dense subset).*

Definition 3.5.11 (Parametrization on $\mathbb{A}_K^{1,\mathrm{an}}$)**.** A corollary of Lemma 3.5.1 is that if $\hat{v}_{\hat{f},\hat{\beta}} = \hat{v}_{\hat{g},\hat{\gamma}}$ then $\hat{\beta} = \hat{\gamma}$. Therefore, if $\hat{v} \in \mathbb{A}_K^{1,\mathrm{an}}$ is a Gauss semivaluation, we define the notation $\hat{\beta}(\hat{v}) \in \overline{\mathbb{R}}$ to mean a unique number such that $\hat{v} = \hat{v}_{\hat{f},\hat{\beta}(v)}$ for some $\hat{f} \in K$. We immediately have two observations.

- We can alternatively write $\hat{\beta}(\hat{v}) = \sup_{\hat{f}\in K} \hat{v}(y-\hat{f})$, and the supremum is achieved if $\hat{v}$ is not Type 4.
- The function $\hat{\beta}\colon \mathbb{A}_K^{1,\mathrm{an}} \to \overline{\mathbb{R}}$ is a parametrization on any closed balls $\overline{\mathbb{D}}(0,\hat{\beta})$ (see Definition 2.5.6). This is from the first part of the proof of Theorem 3.5.9. Thus, $\mathbb{A}_K^{1,\mathrm{an}}$ has a metric $\mathbb{R}$-tree structure induced by α (see Theorem 2.5.9).

Remark 3.5.12. When the value group Γ_K of K is dense in $\mathbb{R}$, [FJ04, Section 3.1.6] provides another prospective to see the metric $\mathbb{R}$-tree structure of $\mathbb{A}_K^{1,\mathrm{an}}$. Let (X,d) be an ultra-metric space, and define an equivalence relation on $X \times \mathbb{R}_{>0}$ by setting $(x,s) \sim (y,t)$ if the conditions $d(x,y) \leq s = t$ hold.[11] Then, the quotient space $(X \times \mathbb{R}_{>0})/\sim$ is a metric $\mathbb{R}$-tree. Moreover, if the diameter of any ball of radius r is actually r, then an element $[x,s] \in (X \times \mathbb{R}_{>0})/\sim$ can be regarded as a closed ball of radius s centered at x.

Combining Lemma 3.5.1 and noting that Γ_K is dense in $\mathbb{R}$, we now see that $\mathbb{A}_K^{1,\mathrm{an}}$, excluding Type 4 points, is a metric $\mathbb{R}$-tree.

[11] In [FJ04, Section 3.1.6], they assume that X has diameter 1. This is not a necessary assumption.

Theorem 3.5.13 (Dendrology of $\mathbb{A}_K^{1,\mathrm{an}}$)**.** *Let $\hat{v} = \hat{v}_{\hat{f},\hat{\beta}} \in \mathbb{A}_K^{1,\mathrm{an}}$. Recall Definition 2.5.5 for the terminologies.*

- *$\hat{v}$ is a branch point if and only if $\hat{v}$ is Type 2.*
- *$\hat{v}$ is a regular point if and only if $\hat{v}$ is Type 3.*
- *$\hat{v}$ is an end if and only if $\hat{v}$ is Type 1 or Type 4.*

Theorem 3.5.14 (Tangent Vectors at Type 2 points)**.** *Any tangent vector at a Type 2 point $\hat{v}_{\hat{f},\hat{\beta}} \in \mathbb{A}_K^{1,\mathrm{an}}$ can be represented by a Type 1 point (i.e., by an element of K).*

- *Any element $\hat{h} \in K$ such that $v_K(\hat{f} - \hat{h}) > \hat{\beta}$ represents the same vector.*
- *Any element $\hat{h} \in K$ such that $v_K(\hat{f} - \hat{h}) < \hat{\beta}$ represents the same vector.*
- *For a tangent vector that does not belong to the above, it is represented by a unique $\hat{h} \in K$ such that $v_K(\hat{f} - \hat{h}) = \hat{\beta}$.*

Example 3.5.15. Let $K = \mathbb{C}_3$ with the 3-adic valuation ord_3 extended from the one on $\mathbb{Q}$. Figure 3.2 shows the space $\mathbb{A}_{\mathbb{C}_3}^{1,\mathrm{an}}$. Since the residue field is $\mathbb{F}_3$, each branch point only has $3+1$ distinct tangent vectors.

Next, we talk about how to relate two Berkovich affine lines by extension of scalars. We have seen a special case of $K[y] \to K^{\mathrm{sph}}[y]$, where K^{sph} is the spherical completion of K. In general, consider a real-valued field extension L/K and we obtain the analytic map $\pi_{L/K} \colon \mathbb{A}_L^{1,\mathrm{an}} \to \mathbb{A}_K^{1,\mathrm{an}}$. We almost follow [PT21, Section 1.5] for the discussion.

Definition 3.5.16 (Assumption on the field extension L/K)**.** In the rest of this section, we only assume that L/K is a field extension of complete real-valued fields (i.e., we drop the assumption that K is algebraically closed, which will help us to analyze $\mathbb{A}_K^{1,\mathrm{an}}$ in general).

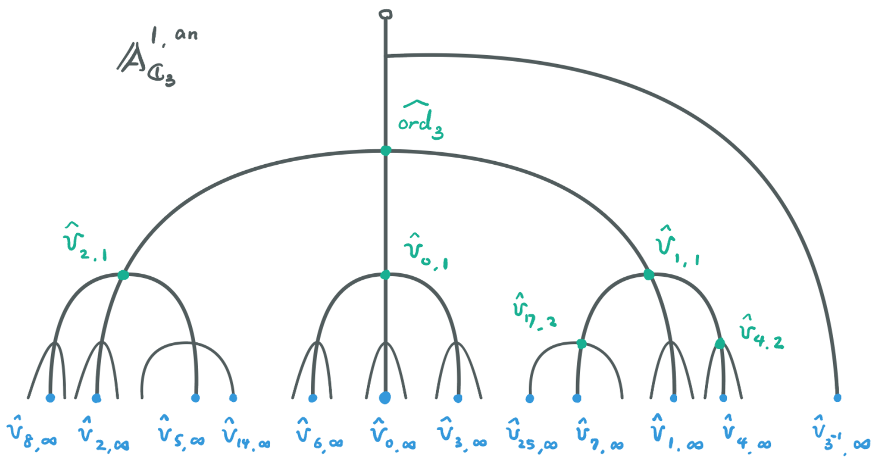


Figure 3.2: Some Gauss semivaluations are labeled in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_3}$. For example, the fact $v_3(7-1) = 1$ means that $v_{1,1} = v_{7,1} = v_{1,\infty} \wedge v_{7,\infty}$, which indicates the locations of those semivaluations.

The following theorem generalizes our observation for the case K^{sph}/K.

Theorem 3.5.17 ([PT21, Proposition I.5.1])**.** *The continuous map $\pi_{L/K}$ is proper and surjective.*

Theorem 3.5.18 ([PT21, Lemma I.5.2])**.** *Assume that both K and L are algebraically closed and that L is spherically complete. If $\hat{v} \in \mathbb{A}^{1,\mathrm{an}}_K$ is a Type 4 point, then $\pi^{-1}_{L/K}(\hat{v}) = \overline{\mathbb{D}}(\hat{f}, \hat{\beta}(\hat{v}))$ for some $\hat{\beta} \in L$.*

Definition 3.5.19. Assume that L/K is algebraic with the automorphism group $G = \mathrm{Aut}(L/K)$. The group action on L by G naturally extends to an action on $L[y]$ by the rule: if $\sigma \in G$ and $\sum a_j y^j \in L[y]$, then $\sigma \cdot \sum a_j y^j = \sum a_j^\sigma y^j$. Dually, we obtain an action on the Berkovich affine line: if $\sigma \in G$ and $\hat{v} \in \mathbb{A}^{1,\mathrm{an}}_L$ then $\sigma(\hat{v})(f) = \hat{v}(\sigma(f))$.

Theorem 3.5.20 ([PT21, Proposition I.5.5])**.** *Assume that L/K is a finite Galois extension.*

- *The action map $G \times \mathbb{A}^{1,\mathrm{an}}_L \to \mathbb{A}^{1,\mathrm{an}}_L$ is continuous and proper.*
- *$\pi_{L/K}$ is G-invariant, and therefore induces a homeomorphism $\mathbb{A}^{1,\mathrm{an}}_L/G \to \mathbb{A}^{1,\mathrm{an}}_K$. As a result, $\pi_{L/K}$ is an open map.*

We want to extend Theorem 3.5.20 to arbitrary algebraic extensions, in particular to the case that $L = K^{\mathrm{alg}}$. Because the natural restriction $\mathrm{Aut(K^{alg}/K)} \to \mathrm{Gal(K^{sep}/K)}$ is a bijection, it suffices to consider the case of $L = K^{\mathrm{sep}}$. Here, K^{sep} denotes the subfield of the algebraic closure K^{alg} collecting all elements that are separable over K.

Lemma 3.5.21 ([BGR84, Section 3.4.1, Proposition 6])**.** *For a complete real valued field (K, v_K), the separable closure K^{sep} of K is dense in K^{alg}.*

To be precise, if $\hat{f} \in K^{\mathrm{alg}}$ is of degree n over K, then for any $\hat{\beta} \in \mathbb{R}$ there exists $\hat{h} \in K^{\mathrm{sep}}$ of degree n over K such that $v_{\mathrm{sp}}(\hat{f} - \hat{h}) > \hat{\beta}$, where v_{sp} is the spectral valuation.

Remark 3.5.22. As an immediate corollary, we see that under the same setup as in Lemma 3.5.21, K^{sep} is also dense in $\widehat{K^{\mathrm{alg}}}$. Furthermore, for each $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$, the K-algebra morphism $\sigma \colon K^{\mathrm{sep}} \to K^{\mathrm{sep}}$ preserves the values and therefore is continuous. By continuity, we get a K-algebra morphism $\sigma \colon \widehat{K^{\mathrm{sep}}} \to \widehat{K^{\mathrm{sep}}}$, which yields a group action $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $\widehat{K^{\mathrm{sep}}} = \widehat{K^{\mathrm{alg}}}$.

Definition 3.5.23 (Profinite Topology on $\mathrm{Gal}(K^{\mathrm{sep}}/K)$)**.** The collection of all finite Galois extensions of K is a direct system under the inclusion, so dually the finite Galois groups over K is an inverse system under the restriction. Thus, $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ is an (inverse) limit of all finite Galois groups over K, and can be topologized by the profinite topology. At the same time, $\mathrm{Aut}(K^{\mathrm{alg}}/K)$ is equipped with the profinite topology.

Theorem 3.5.24 ([PT21, Corollary I.5.6 and Lemma I.5.7])**.** *Write $G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ and $L = \widehat{K^{\mathrm{alg}}}$.*

- *The action map $G \times \mathbb{A}_L^{1,\mathrm{an}} \to \mathbb{A}_L^{1,\mathrm{an}}$ is continuous and proper.*
- *$\pi_{L/K}$ is G-invariant, and therefore induces a homeomorphism $\mathbb{A}_L^{1,\mathrm{an}}/G \to \mathbb{A}_K^{1,\mathrm{an}}$. As a result, $\pi_{L/K}$ is an open map.*
- *If $\pi_{L/K}(\hat{v}) = \pi_{L/K}(\hat{w})$, then $\hat{v}$ and $\hat{w}$ are of the same type, and $\hat{\beta}(\hat{v}) = \hat{\beta}(\hat{w})$.*

3.6 Valuative Trees Via Series

For the rest of the chapter, we apply the theories from the previous sections to the special case that $(K, v_K) = (\Bbbk((x)), v_*)$. Via this way, we obtain another perspective to look at the valuative tree.

Notation 3.6.1 (Notations $\mathbb{K}$ and $\mathbb{G}$)**.** Recall the notation $\mathbb{K} = \widehat{\Bbbk((x))^{\mathrm{alg}}}$ from Notation 3.3.9, and we denote $\mathbb{G} = \mathrm{Gal}\left(\Bbbk((x))^{\mathrm{sep}}/\Bbbk((x))\right) = \mathrm{Aut}\left(\Bbbk((x))^{\mathrm{alg}}/\Bbbk((x))\right)$.

Theorem 3.6.2 (Separable Representatives of Gauss Valuations)**.** *If $\hat{v} \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{K}}$ is of Type 2 or Type 3, then $\hat{v} = \hat{v}_{\hat{f},\hat{\beta}}$ for some $\hat{f} \in \Bbbk((x))^{\mathrm{sep}}$.*

Proof. Write $\hat{v} = \hat{v}_{\hat{f},\hat{\beta}}$ for some $\hat{f} \in \mathbb{K}$ and $\hat{\beta} < \infty$. Since $\Bbbk((x))^{\mathrm{sep}} \subset \mathbb{K}$ is dense (see Remark 3.5.22), there is a separable $\hat{h} \in \Bbbk((x))^{\mathrm{sep}}$ such that $v_*(\hat{f} - \hat{h}) > \hat{\beta}$, which implies $\hat{v} = \hat{v}_{\hat{h},\hat{\beta}}$. $\square$

Definition 3.6.3 (Valuative Space $\widehat{\mathcal{V}}_x$)**.** We define

$$\widehat{\mathcal{V}}_x = \left\{v \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{K}} \mid v(y) > 0\right\} \cup \{\hat{v}_*\},$$

which is nothing but the closure of the open disc $\mathbb{D}^{\circ}_{\mathbb{K}}(0,0)$ in $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{K}}$. It inherits the partial order $\leq$ from $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{K}}$, and is a non-metric tree rooted at $\hat{v}_*$ for the existence of a parameterization $\hat{\beta}$.

Theorem 3.6.4. *The natural map $\pi_{\mathbb{K}/\Bbbk((x))} \colon \mathbb{A}^{1,\mathrm{an}}_{\mathbb{K}} \to \mathbb{A}^{1,\mathrm{an}}_{\Bbbk((x))}$ restricts to a $\mathbb{G}$-invariant map $\pi_{\mathbb{K}/\Bbbk((x))} \colon \widehat{\mathcal{V}}_x \to \mathcal{V}_x$ and induces a homeomorphism*

$$\widehat{\mathcal{V}}_x/G \to \mathcal{V}_x.$$

Proof. First, we show that $\mathcal{V}_x$ can be identified as the subset

$$\mathrm{cl}\left(\mathbb{D}^{\circ}_{\Bbbk((x))}(0,0)\right) \triangleq \{\hat{v} \in \mathbb{A}^{1,\mathrm{an}}_{\Bbbk((x))} \mid \hat{v}(y) > 0\} \cup \{\hat{v}_*\}$$

of $\mathbb{A}^{1,\mathrm{an}}_{\Bbbk((x))}$. The natural inclusion $\Bbbk[[x]][y] \to \Bbbk((x))[y]$ induces the continuous map between valuative spaces $\mathbb{A}^{1,\mathrm{an}}_{\Bbbk((x))} \to \mathcal{V}_x$ via the restriction. Conversely, since $\Bbbk((x))[y] = \Bbbk[[x]][y]_x$ is the localization at x, any $v \in \mathcal{V}_x$ extends to a semivaluation on $\Bbbk((x))[y]$ by setting $v(f/x^n) = v(f) - n$, where $f \in \Bbbk[[x]][y]$. We therefore obtain a map $\mathcal{V}_x \to \mathbb{A}^{1,\mathrm{an}}_{\Bbbk((x))}$ whose image is nothing but $\mathrm{cl}\left(\mathbb{D}^{\circ}_{\Bbbk((x))}(0,0)\right)$. Therefore, the restriction

$$\mathrm{cl}\left(\mathbb{D}^{\circ}_{\Bbbk((x))}(0,0)\right) \to \mathcal{V}_x$$

is a homeomorphism because $\mathrm{cl}\left(\mathbb{D}^{\circ}_{\Bbbk((x))}(0,0)\right)$ is compact and $\mathcal{V}_x$ is Hausdorff.

It is clear that the preimage of $\mathrm{cl}\left(\mathbb{D}^{\circ}_{\Bbbk((x))}(0,0)\right)$ by $\pi_{\mathbb{K}/\Bbbk((x))}$ is $\widehat{\mathcal{V}}_x$, which is closed under the $\mathbb{G}$ action, so the proof is completed by Theorem 3.5.24. □

3.7 Skewness and Thinness via Series

In this section, we show that the space $\widehat{\mathcal{V}}_x$ also has the notion of skewness and multiplicity, and they make the parametrization $\hat{\beta}$ similar to the relative thinness.

Definition 3.7.1 (Notations $\sim_{\mathbb{G}}$ and $\deg_{\Bbbk((x))} \hat{f}$)**.** Let $\hat{f}, \hat{h} \in \Bbbk((x))^{\mathrm{alg}}$ be given.

- We write $\hat{h} \sim_{\mathbb{G}} \hat{f}$ if there is an automorphism $\sigma \in \mathbb{G}$ such that $\hat{h} = \sigma(\hat{f})$.
- Also, define $\operatorname{alg.deg} \hat{f}$ (resp. $\operatorname{sep.deg} \hat{f}$) to be the degree (resp. separable degree) of the field extension $\Bbbk((x))(\hat{f})$ over $\Bbbk((x))$.
- We always write $f \in \Bbbk((x))[y]$ to mean the minimal polynomial of $\hat{f}$ over $\Bbbk((x))$.

Definition 3.7.2 (Hahn Skewness)**.** For $\hat{v} \in \widehat{\mathcal{V}}_x$, we define the *Hahn skewness* to be

$$\hat{\alpha}_x(\hat{v}) = \sup\left\{ \frac{\hat{v}(f)}{\operatorname{alg.deg} \hat{f}} \;\middle|\; \hat{f} \in \Bbbk((x))^{\mathrm{alg}} \right\}.$$

Remark 3.7.3. For $\hat{f} \in \Bbbk((x))^{\mathrm{alg}}$, it has the minimal polynomial of the form

$$f = \prod_{i=1}^{n} (y - \hat{f}_i)^{p^m},$$

where $p = \operatorname{char} \Bbbk$, $n = \operatorname{sep.deg} \hat{f}$, and $np^m = \operatorname{alg.deg} \hat{f} = \deg_y f$.[12] Therefore, we see that

$$\frac{\hat{v}(f)}{\operatorname{alg.deg} \hat{f}} = \frac{p^m \sum_{i=1}^{n} \hat{v}(y - \hat{f}_i)}{np^m} = \frac{\sum_{i=1}^{n} \hat{v}(y - \hat{f}_i)}{n},$$

and thus the definition of $\hat{\alpha}$ can be re-defined as

$$\hat{\alpha}_x(\hat{v}) = \sup\left\{ \frac{\sum_{\hat{h} \sim_{\mathbb{G}} \hat{f}} \hat{v}(y - \hat{h})}{\operatorname{sep.deg} \hat{f}} \;\middle|\; \hat{f} \in \Bbbk((x))^{\mathrm{alg}} \right\}.$$

Here, the summand $\sum_{\hat{h} \sim_{\mathbb{G}} \hat{f}} \hat{v}(y - \hat{h})$ is a finite sum: since when $\Bbbk((x))^{\mathrm{alg}} \subset \mathbb{K}$ is fixed, there are only finitely many conjugates of $\hat{f} \in \Bbbk((x))^{\mathrm{alg}}$.

Theorem 3.7.4. *For any $\hat{v} \in \widehat{\mathcal{V}}_x$, we have $\alpha_x(\pi_{\mathbb{K}/\Bbbk((x))}\hat{v}) = \hat{\alpha}_x(\hat{v})$.*

Proof. It follows from the identities $\hat{v}(f) = (\pi_{\mathbb{K}/\Bbbk((x))}\hat{v})(f)$ and $\operatorname{alg.deg} \hat{f} = \deg_y f = v_x(f)$. $\square$

Theorem 3.7.5. *Given a Type 2 or Type 3 point $\hat{v} \in \widehat{\mathcal{V}}_x$, we have*

$$\hat{\alpha}_x(\hat{v}) = \frac{\sum_{\hat{h} \sim_{\mathbb{G}} \hat{f}} \hat{v}(y - \hat{h})}{\operatorname{sep.deg} \hat{f}}$$

for some $\hat{f} \in \Bbbk((x))^{\mathrm{alg}}$ if and only if $\hat{v} \leq \hat{v}_{\hat{f},\infty}$.

[12] Here, we make a convention that $0^m = 1$.

Proof. It is a corollary of Theorem 2.11.6 and Theorem 3.7.4. $\square$

Definition 3.7.6 (Hahn Multiplicity)**.** For $\hat{v} \in \widehat{\mathcal{V}}_x$, define the *Hahn multiplicity* of $\hat{v}$ to be

$$\hat{m}_x(\hat{v}) = \min\left\{\operatorname{sep.deg} \hat{f} \;\middle|\; \hat{f} \in \Bbbk((x))^{\mathrm{alg}} \text{ with } \hat{v} \le \hat{v}_{\hat{f},\infty}\right\}.$$

Proposition 3.7.7. *The Hahn multiplicity $\hat{m}_x$ enjoys the following properties.*

- *$\hat{m}_x$ is an increasing function, which meaning that $\hat{v} \le \hat{w}$ in $\widehat{\mathcal{V}}_x$ implies that $\hat{m}_x(\hat{v}) \le \hat{m}_x(\hat{w})$.*
- *$\hat{m}_x(\hat{v}) = \operatorname{alg.deg} \hat{f}$ for some $\hat{f} \in \Bbbk((x))^{\mathrm{sep}}$ with $\hat{v} \le \hat{v}_{\hat{f},\infty}$.*
- *If $\hat{f} \in \Bbbk((x))^{\mathrm{alg}}$, then $\hat{m}_x(\hat{v}_{\hat{f},\hat{\beta}}) = \operatorname{sep.deg} \hat{f}$ whenever $\hat{\beta} > \sup \operatorname{Supp}(\hat{f})$.*

Proof. The first and the third statements are straightforward. The second statement is directly from Lemma 3.5.21. $\square$

Conjecture 3.7.8. *Let $\hat{v} = \hat{v}_{\hat{f},\hat{\beta}} \in \widehat{\mathcal{V}}_x$ for some $\hat{f} \in \Bbbk((x))^{\mathrm{sep}}$ such that $\hat{m}_x(\hat{v}) = \operatorname{sep.deg} \hat{f}$, and let $\sigma \in \mathbb{G}$ with the property $\hat{f} \ne \hat{f}^{\sigma}$. Then, we have $v_*(\hat{f} - \hat{f}^{\sigma}) \le \hat{\beta}(\hat{v})$.*

Definition 3.7.9 (Approximating Sequences on $\widehat{\mathcal{V}}_x$)**.** For $\hat{v} \in \widehat{\mathcal{V}}_x$, the *approximating sequence* of $\hat{v}$ consists of a strictly increasing sequence $(\hat{v}_j)_0^{g+1}$ in $\widehat{\mathcal{V}}_x$ and a strictly increasing sequence $(\hat{m}_j)_0^g$ in $\mathbb{Z}_{>0}$, such that $\hat{v}_0 = \hat{v}_*$, $\hat{v}_{g+1} = \hat{v}$, $\hat{m}_0 = 1$, and for $0 \le j \le g$ the Hahn multiplicity $\hat{m}_x$ has the constant value $\hat{m}_j$ on the open-closed segment $(\hat{v}_j, \hat{v}_{j+1}]$. An approximating sequence is written as $[(\hat{v}_j)_0^{g+1}; (\hat{m}_j)_0^g]$.

Remark 3.7.10. Note that $\hat{m}_0 = 1$ can be deduced from the definition. Assume we have such a sequence but $\hat{m}_0 > 1$. Write $\hat{v}_1 = \hat{v}_{\hat{f},\hat{\beta}}$ with $\hat{\beta} > v_*(\hat{f})$, and let $\hat{\gamma} = v_*(\hat{f})/2$ Since $\hat{m} = 1$ on the segment $[\hat{v}_*, \hat{v}_{0,\infty}]$, we see that $\hat{m}_x(\hat{v}_{\hat{f},\hat{\gamma}}) = \hat{m}_x(\hat{v}_{0,\hat{\gamma}}) = 1$, which contradicts the assumption that $\hat{m}_0 > 1$.

Theorem 3.7.11. *Suppose that Conjecture 3.7.8 is true. For $\hat{v} \in \widehat{\mathcal{V}}_x$, we have the identity*

$$\hat{\beta}(\hat{v}) = \int_{\hat{v}_*}^{\hat{v}} \hat{m}_x(\hat{w}) d\hat{\alpha}_x(\hat{w}).$$

Proof. Fix $\hat{v} \in \widehat{\mathcal{V}}_x$. Let $[(\hat{v}_j)_0^{g+1}; (\hat{m}_j)_0^g]$ be the corresponding approximating sequence. We may assume $g < \infty$, and prove by induction on g. In this case, the integral formula reduces to a sum

$$\hat{\beta}(\hat{v}) = \sum_{j=0}^{g} \hat{m}_j(\hat{\alpha}_x(\hat{v}_{j+1}) - \hat{\alpha}_x(\hat{v}_j)).$$

Assume $g = 0$ and so $\hat{m}_0 = 1$. By Proposition 3.7.7, there is $\hat{f} \in \Bbbk((x))$ such that $\hat{v} = \hat{v}_{\hat{f},\hat{\beta}}$ for some $\hat{\beta} \in \overline{\mathbb{R}}$. Theorem 3.7.5 tells us that $\hat{\alpha}_x(\hat{v}) = \hat{v}(y - \hat{f}) \cdot 1 = \hat{\beta}$, and thus $\hat{m}_0(\hat{\alpha}_x(\hat{v}) - \hat{\alpha}_x(\hat{v}_*)) = 1 \cdot (\hat{\beta} - 0) = \hat{\beta}$.

Now assume $g > 0$; our goal is to show that $\hat{m}_g(\hat{\alpha}_x(\hat{v}) - \hat{\alpha}_x(\hat{v}_g)) = \hat{\beta}(\hat{v}) - \hat{\beta}(\hat{v}_g)$. By Theorem 3.7.5, say $\hat{v} = \hat{v}_{\hat{f},\hat{\beta}}$ for some $\hat{f} \in \Bbbk((x))^{\mathrm{sep}}$ such that $\hat{m}_g = \operatorname{alg.deg} \hat{f}$. By Conjecture 3.7.8, $v_*(\hat{f} - \hat{f}^\sigma) \leq \hat{\beta}_g$ for any $\sigma \in \mathbb{G}$ with $\hat{f}^\sigma \neq \hat{f}$, which shows that $\hat{v}_{\hat{f},\hat{\beta}}(y - \hat{f}^\sigma) = \hat{v}_{\hat{f},\hat{\beta}_g}(y - \hat{f}^\sigma)$. We then compute

$$\begin{aligned}
\hat{m}_g(\hat{\alpha}_x(\hat{v}) - \hat{\alpha}_x(\hat{v}_g)) &= \sum_{\sigma \in \mathbb{G}} \hat{v}_{\hat{f},\hat{\beta}}(y - \hat{f}^\sigma) - \sum_{\sigma \in \mathbb{G}} \hat{v}_{\hat{f},\hat{\beta}_g}(y - \hat{f}^\sigma) \\
&= \sum_{\sigma \in \mathbb{G}} \Big(\hat{v}_{\hat{f},\hat{\beta}}(y - \hat{f}^\sigma) - \hat{v}_{\hat{f},\hat{\beta}_g}(y - \hat{f}^\sigma) \Big) \\
&= \hat{\beta} - \hat{\beta}_g
\end{aligned}$$

as desired. □

Remark 3.7.12. Conjecture 3.7.8 is true when $\operatorname{char} \Bbbk = 0$, which is the situation in [FJ04, Theorem 4.17]. With this assumption, every extension of $\Bbbk((x))$ is separable,

$\Bbbk((x))^{\mathrm{alg}}$ is the field of Puiseux series $\Bbbk\{\{x\}\}$ (see Theorem 3.2.3), and the Galois group is isomorphic to the profinite integer:

$$\mathbb{G} = \varprojlim \mathrm{Gal}\left(\Bbbk((x^{1/n}))/\Bbbk((x))\right) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}}.$$

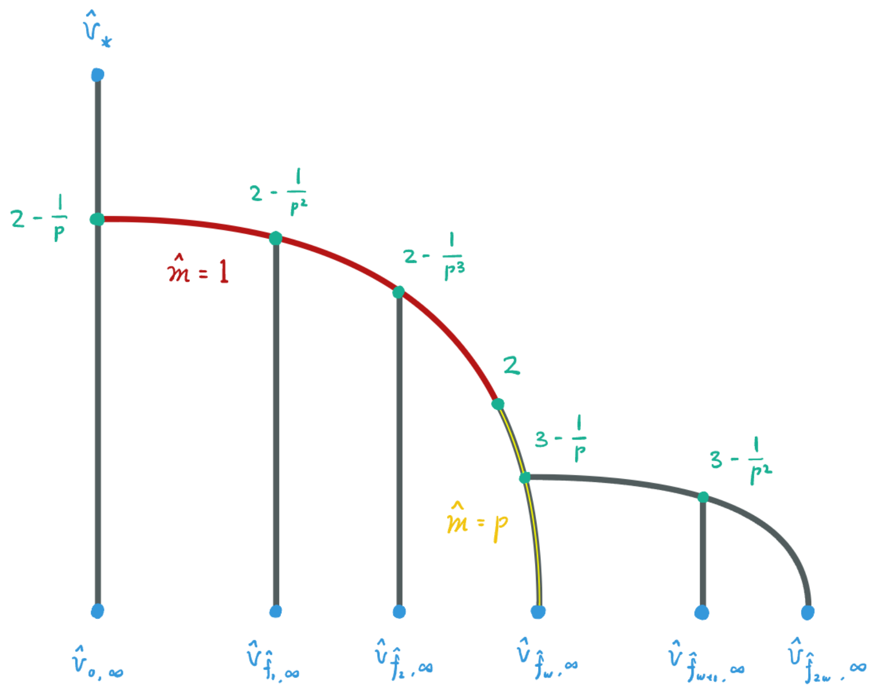
Since $\hat{m}_x = m_x \circ \pi_{\mathbb{K}/\Bbbk((x))}$ in this case, we conclude that $A_x^{\mathrm{thin}} = 1 + \hat{\beta}$.

Example 3.7.13. Let $p > 2$,[13] and consider the following elements in $\mathbb{K}$:

$$\begin{aligned}
\hat{f}_1 &= x^{2-\frac{1}{p}}, \\
\hat{f}_2 &= x^{2-\frac{1}{p}} + x^{2-\frac{1}{p^2}}, \\
\hat{f}_\omega &= \sum_{k=1}^{\infty} x^{2-\frac{1}{p^k}}, \\
\hat{f}_{\omega+1} &= \sum_{k=1}^{\infty} x^{2-\frac{1}{p^k}} + x^{3-\frac{1}{p}}, \text{ and} \\
\hat{f}_{2\omega} &= \sum_{k=1}^{\infty} x^{2-\frac{1}{p^k}} + \sum_{k=1}^{\infty} x^{3-\frac{1}{p^k}}.
\end{aligned}$$

We can label the related Gauss semivaluations of those elements in the following picture. Any Gauss valuation lying on the red curve has Hahn multiplicity $\hat{m} = 1$ (they are $\mathbb{G}$-invariant), and the one lying on the yellow curve has $\hat{m} = p$. The numbers in green mark the parameterizations $\hat{\beta}$.

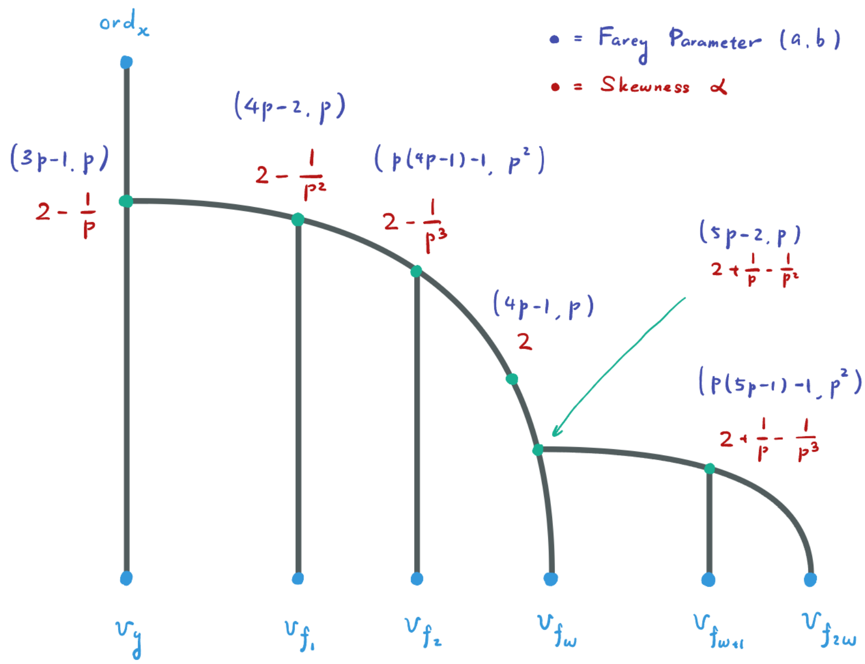
[13] The picture of $\widehat{\mathcal{V}}_x$ when $p = 2$ is slightly different.



We then compute the minimal polynomials of those five elements and get

$$
\begin{aligned}
f_1 &= y^p - x^{2p-1}, \\
f_2 &= y^{p^2} - x^{p(2p-1)} - x^{2p^2-1}, \\
f_\omega &= y^p - x^{2(p-1)}y - x^{2p-1}, \\
f_{\omega+1} &= y^{p^2} - x^{2p(p-1)}y^p - x^{p(3p-1)} + x^{(2p-1)(p+1)} - x^{p(2p-1)}, \text{ and} \\
f_{2\omega} &= y^p + (x^2(x+1))^{p-1}y - x^{3p-1} - x^{2p-1}.
\end{aligned}
$$

In the following picture, we label the corresponding curve semivaluations in the relative valuative tree $\widehat{\mathcal{V}}_x$ with (relative) Farey parameters and (relative) skewness for the middle branch points. Theorem 2.11.4 ensures that the skewness is equal to the relative skewness for those branch points (and also the absolute/relative Farey parameters).



As a result, we see that $A_x^{\text{thin}} = A^{\text{thin}} = A^{\text{Farey}}$ from Theorem 2.10.21, and conclude that the relation $A_x^{\text{thin}} = 1 + \hat{\beta}$, which is true in characteristic 0, does not hold in positive characteristic.

3.8 A Dictionary between Valuative Trees

We have introduced many notions and numerical invariants. Let's compare them.

Recall from Theorem 3.6.4, there is a continuous map $\widehat{\mathcal{V}}_x \to \mathcal{V}_x$. With this map, we can connect the types of points and label them in the following table.

Non-rooted Non-Metric Trees	$\mathcal{V}_x/\mathcal{V}_\mathfrak{m}$	$\hat{v}_{\hat{f},\hat{\beta}} \in \widehat{\mathcal{V}}_x$
branch points	divisorial	Type 2
regular points	irrational	Type 3
ends	curve semivaluations	Type 1 with $\hat{f} \in \Bbbk((x))^{\mathrm{alg}}$
	infinitely singular	Type 1 with $\hat{f} \notin \Bbbk((x))^{\mathrm{alg}}$ or Type 4

If $\operatorname{char} \Bbbk = 0$, the correspondence is more precise (see [FJ04, Section 4.7]).

Chapter 4

Computations of Log Canonical Thresholds

In general, computing the log canonical threshold (LCT) of a pair is a complicated problem; for example, see [Kol97, Chapter 8] and [Mus12] for more details.

Because we only work on the affine plane $\mathbb{A}^2_{\Bbbk}$, we will simply write $\operatorname{lct}_{\mathfrak{m}}(f)$ to mean $\operatorname{lct}_{\mathfrak{m}}(\mathbb{A}^2_{\Bbbk}, V(f))$. The goal of this chapter is to prove Theorem 4.2.1.

4.1 Preliminaries

Definition 4.1.1 (Newton Polyhedra)**.** Let $A = \Bbbk[x_1, x_2, \dots, x_n]$. Recall from Section 2.3 the notation $x^{\mathbf{u}} = x_1^{u_1} x_2^{u_2} \dots x_n^{u_n}$, where $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{Z}^n_{\geq 0}$.

- For $f = \sum_{\mathbf{u}} c_{\mathbf{u}} x^{\mathbf{u}} \in A$, the *support* of f is the set $\operatorname{Supp}(f) = \left\{u \in \mathbb{Z}^n_{\geq 0} \,|\, c_{\mathbf{u}} \neq 0\right\}$.
- For $f \in A$, the *monomial ideal of* f is the ideal

$$\operatorname{Monom}(f) = (x^{\mathbf{u}} \mid \mathbf{u} \in \operatorname{Supp}(f))$$

  of A. That is, it is the ideal generated by the monomials appearing in f.
- If $\mathfrak{a}$ is a monomial ideal, which means that $\mathfrak{a}$ can be generated by monomials, then the *Newton polyhedron of* $\mathfrak{a}$, denoted by $\operatorname{Newt}(\mathfrak{a})$, is defined to be the convex hull of the subset $\left\{\mathbf{u} \in \mathbb{Z}^n_{\geq 0} \mid x^{\mathbf{u}} \in \mathfrak{a}\right\}$ of $\mathbb{R}^n_{\geq 0}$.
- For $f \in A$, we define the *Newton polyhedron of* f to be $\operatorname{Newt}(\operatorname{Monom}(f))$.

The above definitions remain valid if $A = \Bbbk[[x_1, x_2, \dots, x_n]]$.

Remark 4.1.2. In order to state the next theorem, note that Definition 2.9.13 of LCT can be defined for arbitrary ideals rather than only divisors. The new definition is the verbatim translation of the original one, with the divisor Δ replaced by the ideal $\mathfrak{a}$.

Theorem 4.1.3. *If $\mathfrak{a} \subseteq \Bbbk[x_1, x_2, \cdots, x_n]$ is a monomial ideal, then*

$$\mathrm{lct}_{\mathfrak{m}}(\mathfrak{a}) = \max\{\lambda \in \mathbb{R}_{\geq 0} \mid (1, 1, \dots, 1) \in \lambda \cdot \mathrm{Newt}(\mathfrak{a})\}.$$

The same result holds if $\mathfrak{a} \subseteq \Bbbk[[x_1, x_2, \cdots, x_n]]$ is a monomial ideal.[1]

Example 4.1.4. If $\mathfrak{a} = (x_1^{u_1}, x_2^{u_2}, \dots, x_n^{u_n}) \in \mathbb{C}[x_1, x_2, \cdots, x_n]$ then

$$\mathrm{lct}_{\mathfrak{m}}(\mathfrak{a}) = \frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_n}.$$

Remark 4.1.5 (Normalized SKPs)**.** Recall the sequence of key polynomials (SKP) from Definition 2.4.1. Given an SKP $[(U_j)_0^k; (\tilde{\beta})_0^k]$, if $\tilde{\beta}_0 > \tilde{\beta}_1$ then we can exchange x and y and assume that $\tilde{\beta}_0 \leq \tilde{\beta}_1$. Next, if U_2 is of the form $y - \theta_1 x$ for some $\theta_1 \in \Bbbk^*$, then $\Bbbk[[x, y]] = \Bbbk[[x, U_2]]$ and thus we can replace y by U_2. After having done so, the new SKP will have U_2 of the form $y^{n_1} - \theta_1 x^{m_{1,0}}$ for some $\theta_1 \in \Bbbk^*$ and some $m_{1,0} > 1$.

Therefore, we say that an SKP $[(U_j)_0^k; (\tilde{\beta})_0^k]$ is *normalized* if $1 = \tilde{\beta}_0 < \tilde{\beta}_1$. We have seen in Chapter 2 that the normalization assumption simplifies statements.

Lemma 4.1.6. *Let $[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ be a normalized SKP with $k \geq 2$, and write $v_{1,\tilde{\beta}_1}$ the monomial valuation sending x to 1 and y to $\tilde{\beta}_1$ (see Example 2.3.1). For $j \geq 2$, we have*

$$v_{1,\tilde{\beta}_1}(U_0^{m_{j,0}} \cdots U_{j-1}^{m_{j,j-1}}) > \tilde{\beta}_1 d_{j+1},$$

[1] In the original paper [How01], the theorem was proven only for the case $\mathfrak{a} \subseteq \mathbb{C}[x_1, x_2, \cdots, x_n]$. In [Mus12], a proof involving toric resolution is given, and this method works for any base field $\Bbbk$. On the other hand, the monomial ideals in $\Bbbk[x_1, x_2, \cdots, x_n]$ and the ones in $\Bbbk[[x_1, x_2, \cdots, x_n]]$ are in canonical bijection, so the result extends to the setting of formal power series.

where $d_{j+1} \triangleq \deg_y U_{j+1} = n_1 n_2 \cdots n_{j-1} n_j$.[2]

Proof. Recall the skewness α from Definition 2.7.1. By Proposition 2.7.2, for $l > 0$ we see that

$$v_{1,\tilde{\beta}_1}(U_l) = \alpha(v_{1,\tilde{\beta}_1} \wedge v_{U_l}) v_{\mathfrak{m}}(U_l) = \alpha(v_{1,\tilde{\beta}_1}) \deg_y U_l = \tilde{\beta}_1 d_l.$$

Now for $j \geq 2$, we can compute

$$\begin{aligned} v_{1,\tilde{\beta}_1}(U_0^{m_{j,0}} \cdots U_{j-1}^{m_{j,j-1}}) &= \sum_{l=0}^{j-1} m_{j,l}\, v_{1,\tilde{\beta}_1}(U_l) \\ &= m_{j,0} + m_{j,1}\tilde{\beta}_1 d_1 + m_{j,2}\tilde{\beta}_1 d_2 + \cdots + m_{j,j-1}\tilde{\beta}_1 d_{j-1}. \end{aligned}$$

Therefore, to prove the lemma, it suffices to solve for $(m_{j,0}, m_{j,1}, \ldots, m_{j,j-1}) \in \mathbb{R}^j_{\geq 0}$ in the following optimization problem

$$\begin{aligned} \text{min} \qquad & m_{j_0} + \tilde{\beta}_1 \sum_{l=1}^{j-1} m_{j,l} d_l \\ \text{subject to} \quad & n_j \tilde{\beta}_j = \sum_{l=0}^{j-1} m_{j,l} \tilde{\beta}_l \\ & m_{j,l} \geq 0 \ \text{ for } \ l = 0, \ldots, j-1, \end{aligned}$$

where the constraint is from Definition 2.4.1 (S2), and to show that the minimum is still greater than $\tilde{\beta}_1 d_{j+1}$.

This is a linear programming problem, so we know that the minimum is on a vertex of the simplex determined by the simplex. Each vertex is of the form

$$(m_{j,0}, m_{j,1}, \ldots, m_{j,j-1}) = n_j \tilde{\beta}_j \tilde{\beta}_l^{-1} \mathbf{e}_l,$$

[2]If $k < \infty$ and $j = k$, then even thought U_{k+1} has not been determined yet, its degree $\deg_y U_{k+1}$ must be $n_1 n_2 \cdots n_{k-1} n_k$. As a result, the notation d_{k+1} still makes sense.

where $\mathbf{e}_l \in \mathbb{R}^j$ is the l-th standard basis vector. In view of Lemma 2.4.3, the sequence $(\tilde{\beta}_l/d_l)$ is strictly increasing, so the vertex $(0, 0, \ldots, 0, n_j\tilde{\beta}_j\tilde{\beta}_{j-1}^{-1})$ gives the minimum $\tilde{\beta}_1 n_j\tilde{\beta}_j\tilde{\beta}_{j-1}^{-1}d_{j-1}$. Using the relation $\tilde{\beta}_j > n_{j-1}\tilde{\beta}_{j-1}$, we see the desired inequality

$$\tilde{\beta}_1 n_j\tilde{\beta}_j\tilde{\beta}_{j-1}^{-1}d_{j-1} > \tilde{\beta}_1 n_j n_{j-1} d_{j-1} = \tilde{\beta}_1 d_{j+1}.$$

□

Theorem 4.1.7. *Let $[(U_j)_0^k; (\tilde{\beta}_j)_0^k]$ be an SKP with $2 \le k \le \infty$ such that $\tilde{\beta}_0 < \tilde{\beta}_1$. For all $2 \le j \le k$, we have the following equality*

$$\mathrm{Newt}(U_j) = \frac{\deg_y U_j}{n_1} \cdot \mathrm{Newt}(U_2).$$

Here, $\deg_y U_\infty$ only makes sense when U_∞ is defined (see Lemma 2.4.5).

Proof. Without loss of generality, we may assume $\tilde{\beta}_0 = 1$ to make the SKP normalized. We write $d_j = \deg_y U_j$, and prove the theorem by induction on $j < \infty$; it is true for $j = 2$ because $d_2 = n_1$, so we assume that $j > 2$ and that the theorem is true for smaller j's.

Write $U_0^{m_{j,0}} \cdots U_{j-1}^{m_{j,j-1}} = \sum c_{ij}x^i y^j$. Lemma 4.1.6 indicates that

$$i + \tilde{\beta}_1 j = v_{1,\tilde{\beta}_1}(x^i y^j) \ge v_{1,\tilde{\beta}_1}\left(\sum c_{ij}x^i y^j\right) > \tilde{\beta}_1 d_{j+1},$$

which is equivalent to

$$\frac{n_1}{d_{j+1}}\left(\frac{i}{m_{1,0}} + \frac{j}{n_1}\right) > 1$$

due to the relation $n_1\tilde{\beta}_1 = m_{1,0}\tilde{\beta}_0 = m_{1,0}$. Since

$$\mathrm{Newt}(U_2) = \left\{(u_1, u_2) \in \mathbb{R}^2_{\ge 0} \,\middle|\, \frac{u_1}{m_{1,0}} + \frac{u_2}{n_1} \ge 1\right\}, \tag{4.1}$$

the indicated inequality implies that the point $(i, j) \in \mathbb{R}^2_{\geq 0}$ lies in the interior of the set $(d_{j+1}/n_1) \cdot \mathrm{Newt}(U_2)$. By the induction hypothesis, we also have

$$\mathrm{Newt}(U_j^{n_j}) = n_j \cdot \mathrm{Newt}(U_j) = n_j \cdot \frac{d_j}{n_1} \cdot \mathrm{Newt}(\mathrm{U}_2) = \frac{d_{j+1}}{n_1} \cdot \mathrm{Newt}(U_2),$$

so we can conclude that

$$\begin{aligned}
\mathrm{Newt}(U_{j+1}) &= \mathrm{Newt}(U_j^{n_j} - \theta_j U_0^{m_{j,0}} \cdots U_{j-1}^{m_{j,j-1}}) \\
&\subseteq \frac{d_{j+1}}{n_1} \cdot \mathrm{Newt}(U_2) \\
&= \mathrm{Newt}(U_j^{n_j}) \\
&\subseteq \mathrm{Newt}(U_{j+1}).
\end{aligned}$$

Finally, assume $j = \infty$ in the case that U_∞ is defined. Since $U_j \to U_\infty$ and $d_\infty = d_{j_0}$ for some j_0 large enough, we see that

$$\mathrm{Newt}(U_\infty) = \mathrm{Newt}(U_{j_0}) = \frac{d_{j_{j_0}}}{n_1} \cdot \mathrm{Newt}(U_2) = \frac{d_\infty}{n_1} \cdot \mathrm{Newt}(U_2)$$

as desired. □

Remark 4.1.8. We have seen that (4.1) gives an explicit description of $\mathrm{Newt}(U_2)$, so combining Theorem 4.1.7 we deduce that, for $j \geq 2$,

$$\mathrm{Newt}(U_j) = \left\{ (u_1, u_2) \in \mathbb{R}^2_{\geq 0} \,\middle|\, \frac{n_1}{\deg_y U_j} \left(\frac{u_1}{m_{1,0}} + \frac{u_2}{n_1} \right) \geq 1 \right\}.$$

In view of the relation $n_1 \tilde{\beta}_1 = m_{1,0} \tilde{\beta}_0$ and Theorem 4.1.3, we can conclude that

$$\mathrm{lct}_{\mathfrak{m}} \left(\mathrm{Monom}(U_k) \right) = \frac{n_1}{\deg_y U_k} \left(\frac{1}{n_1} + \frac{1}{m_{1,0}} \right) = \frac{1}{\deg_y U_k} \left(1 + \frac{\tilde{\beta}_0}{\tilde{\beta}_1} \right).$$

4.2 The Main Result

Theorem 4.2.1. *Let $f \in \Bbbk[[x,y]]$ be irreducible, and assume that the tangent cone of $V(f)$ is defined by the equation $y^{\operatorname{ord}_{\mathfrak{m}}(f)} = 0$. Write v_f the corresponding non-normalized curve semivaluation. Then, we have*

$$\operatorname{lct}_{\mathfrak{m}}(f) = \frac{1}{v_f(x)} + \frac{1}{v_f(y)}.$$

Proof. Under the assumption of the tangent cone of f, we can write $v_f = \operatorname{val}[(U_j)_0^k; (\tilde{\beta})_j^k]$ so that $f = U_k$, $1 \le k \le \infty$, and $\tilde{\beta}_0 < \tilde{\beta}_1$. We denote $\deg_y U_j$ by d_j as usual. To prove the desired equality, we verify the ($\le$) and ($\ge$) directions.

For ($\le$) direction, we apply Theorem 4.1.7 to get

$$\begin{aligned}
\operatorname{lct}_{\mathfrak{m}}(f) &= \operatorname{lct}_{\mathfrak{m}}(U_k) \\
&\le \operatorname{lct}_{\mathfrak{m}}(\operatorname{Monom}(U_k)) \\
&= \frac{1}{d_k}\left(1 + \frac{\tilde{\beta}_0}{\tilde{\beta}_1}\right) \\
&= \frac{1}{v_f(x)} + \frac{1}{v_f(y)},
\end{aligned}$$

where the inequality is from the fact that $\mathfrak{a} \subseteq \mathfrak{b}$ implies $\operatorname{lct}_{\mathfrak{m}}(\mathfrak{a}) \le \operatorname{lct}_{\mathfrak{m}}(\mathfrak{b})$, and the last equality is because $\tilde{\beta}_0 = v_f(x) = v_x(f) = \alpha(v_x \wedge v_f) \deg_y f = d_k$.

For ($\ge$) direction, let $\pi : Y \to X \triangleq \operatorname{Spec} \Bbbk[[x,y]]$ be a log resolution of $V(f)$, and pick an exceptional component $E \subseteq \pi^{-1}(\mathfrak{m})$. Our goal is to show that

$$\frac{A_X(E)}{\operatorname{ord}_E(\pi^* f)} \ge \frac{1}{v_f(x)} + \frac{1}{v_f(y)}.$$

We have the following estimate:

$$\begin{aligned}\frac{A_X(E)}{\mathrm{ord}_E(\pi^* f)} &= \frac{\mathrm{ord}_E(\pi^*\mathfrak{m})}{\mathrm{ord}_E(\pi^* f)} \cdot \frac{A_X(E)}{\mathrm{ord}_E(\pi^*\mathfrak{m})} \\ &= \frac{1}{v_E(f)} \cdot A^{\mathrm{thin}}(E) \\ &= \frac{1}{v_{\mathfrak{m}}(f)\alpha(v_E \wedge v_f)} \cdot A^{\mathrm{thin}}(v_E) \\ &\geq \frac{1}{d_k} \cdot \frac{1}{\alpha(v_E \wedge v_f)} \cdot A^{\mathrm{thin}}(v_E \wedge v_f),\end{aligned}$$

where the second equality is from Theorem 2.10.21. Hence, it remains to show the following inequality

$$\frac{A^{\mathrm{thin}}(v_E \wedge v_f)}{\alpha(v_E \wedge v_f)} \geq 1 + \frac{\tilde{\beta}_0}{\tilde{\beta}_1}.$$

Notice that A^{thin}/α defines a real-valued function on the valuative tree $\mathcal{V}_{\mathfrak{m}}$. We claim that A^{thin}/α satisfies the following lemma and postpone its proof to the end of this section.

Lemma 4.2.2. *Let $v \in \mathcal{V}_{\mathfrak{m}}$ with $m(v) < \infty$ (so v is not infinitely singular), and v_1 be the first divisorial valuation after $v_{\mathfrak{m}}$ in the approximating sequence of v. Then A^{thin}/α is strictly decreasing on $[v_{\mathfrak{m}}, v_1]$ and strictly increasing on $[v_1, v]$. Moreover, the minimum is $A^{\mathrm{thin}}(v_1)/\alpha(v_1) = 1 + 1/\alpha(v_1)$.*

Setting $v = v_f$ in Lemma 4.2.2, we now show that $v_1 = v_y \wedge v_f$. On $[v_{\mathfrak{m}}, v_y \wedge v_f]$, the multiplicity function m is 1, so $v_1 \geq v_y \wedge v_f$. Conversely, consider a valuation $w \in (v_y \wedge v_f, v_f)$. Because w is not monomial, we must have $m(w) > 1$; therefore, $v_1 \leq v_y \wedge v_f$.

Since $v_E \wedge v_f \in [v_{\mathfrak{m}}, v_f)$, by Lemma 4.2.2 we obtain

$$\frac{A^{\mathrm{thin}}(v_E \wedge v_f)}{\alpha(v_E \wedge v_f)} \geq \frac{A^{\mathrm{thin}}(v_y \wedge v_f)}{\alpha(v_y \wedge v_f)} = 1 + \frac{1}{\alpha(v_y \wedge v_f)} = 1 + \frac{\tilde{\beta}_0}{\tilde{\beta}_1}.$$

This completes the proof. $\square$

Proof of Lemma 4.2.2. Let $[(v_j)_0^g; (m_j)_0^g]$, $0 \le g < \infty$, be the approximating sequence. Recall from Definition 2.8.6 that $[(v_j)_0^g; (m_j)_0^g]$ consists of an increasing sequence of semivaluations

$$v_{\mathfrak{m}} = v_0 < v_1 < v_2 < \cdots < v_g < v_{g+1} = v$$

and an increasing sequence of positive integers $1 = m_1 < m_2 < \cdots < m_g < m_{g+1}$ such that $m(w) = m_j$ for $w \in (v_{j-1}, v_j]$. Put $t_j = \alpha(v_j)$ for $j = 0, \ldots, g+1$.

For $w \in [v_{\mathfrak{m}}, v_1]$, we see that the function

$$\frac{A^{\text{thin}}(w)}{\alpha(w)} = \frac{2 + 1 \cdot (\alpha(w) - t_0)}{\alpha(w)} = 1 + \frac{1}{\alpha(w)}$$

is strictly decreasing as w increases. Furthermore, $A(v_1)/\alpha(v_1) = 1 + 1/\alpha(v_1)$.

On the other hand, let $w \in (v_{j-1}, v_j]$ for $j \ge 2$. Because $m_l - m_{l+1} \le -1$ (from Theorem 2.8.2, m_l divides m_{l+1} non-trivially) and $t_l > 1$ for any $l > 0$, the following computation

$$\begin{aligned} \frac{A^{\text{thin}}(w)}{\alpha(w)} &= \frac{1}{\alpha(w)} \left(2 + \sum_{l=1}^{j-1} m_l(t_l - t_{l-1}) + m_j(\alpha(w) - t_{j-1}) \right) \\ &= m_j + \frac{1}{\alpha(w)} \left(1 + \sum_{l=1}^{j-1} t_l(m_l - m_{l+1}) \right) \end{aligned}$$

implies that A^{thin}/α is strictly increasing on $(v_{j-1}, v_j]$. Since both α and A^{thin} are parameterizations, they are continuous (for the observer's topology) on $[v_{\mathfrak{m}}, v]$, and thus A^{thin}/α is continuous on $[v_{\mathfrak{m}}, v]$ too, which shows that A^{thin}/α is strictly increasing on $[v_1, v]$. $\square$

Remark 4.2.3. The assumption in Theorem 4.2.1 that the tangent cone of f has the equation $y^{\text{ord}_m(f)} = 0$ is unnecessary. The purpose of this assumption is to let the description of the theorem be neat and simple, and to simplify the argument in the proof. If an analytically irreducible curve has a different tangent cone, we can always perform a suitable change of coordinates to make the assumption happen.

Remark 4.2.4. Our result generalizes the statement in [Kol97, Example 8.9], which says that if $f \in \mathbb{C}[[x,y]]$ is irreducible, then

$$\mathrm{lct}_{\mathfrak{m}}(\mathbb{A}^2_{\mathbb{C}}, f) = \frac{1}{m} + \frac{1}{n},$$

where $\mathfrak{m}$ denotes the maximal ideal of $\mathbb{C}[[x,y]]$, $m = \mathrm{ord}_{\mathfrak{m}}(f)$, and n/m is the first Puiseux exponent of f.

This statement is proven in [Igu77]. Originally, Igusa deals with different problems and computes invariants related to Bernstein-Sato polynomials, which turn out to be related to the LCT. However, the computation is complex, and the connection to the LCT is indirect.

Chapter 5

Characteristic p Phenomena

In Chapter 4, we showed how to compute LCTs via the valuative tree. Although it works for any characteristic, the LCT is more important in characteristic 0 than in characteristic $p > 0$. In the latter situation, an analogy, called the F-pure threshold, plays an important role. It is therefore natural to ask whether a similar strategy can be applied to the F-pure threshold.

5.1 Characteristic p Algebraic Geometry

Definition 5.1.1 (Frobenius Morphisms)**.** Let $p > 0$ be a prime number.

- Let A be an $\mathbb{F}_p$-algebra. The *Frobenius homomorphism* $F_A \colon A \to A$ defined by $F(a) = a^p$ is a ring homomorphism. It is injective if and only if A is reduced.

- Let X be a $\Bbbk$-scheme with $\operatorname{char} \Bbbk = p$. On each affine chart of X, there is a morphism induced from the Frobenius homomorphism, so those morphisms glue and give a *Frobenius morphism* $F_X \colon X \to X$. A space X is *F-finite* if the Frobenius morphism is a finite morphism.

When no confusion can arise, we drop the subscript from F_A or F_X and just write F, and write F^e to mean e times iteration of Frobenius.

Remark 5.1.2 (The ring $A^{1/p}$)**.** Recall that if A is reduced, then A can be embedded into a product of fields, and we can define a ring $A^{1/p}$ as the collection of all p-th roots of elements of A from such embedding. In this case, the Frobenius homomorphism F_A can be regarded as the inclusion map $A \hookrightarrow A^{1/p}$.

Remark 5.1.3 (Assumptions on $\Bbbk$-schemes X and rings A)**.** Let $p > 0$ be a prime number.

- We make a convention that, in the rest of the chapter, every scheme and every ring is of characteristic p.

- Let X be a $\Bbbk$-scheme. Note that the Frobenius morphism is a morphism of $\Bbbk$-schemes if and only if $\Bbbk$ is perfect. Also, the F-finiteness plays an important role throughout the theory; for example, any F-finite affine scheme admits a dualizing complex (see [ST14]), and if an F-finite scheme X has a dualizing complex $\omega_X^\bullet$, then $F^!\omega_X^\bullet$ is a dualizing complex. Therefore, we will always assume that $\Bbbk$ is a perfect field of characteristic p and X is F-finite; the latter assumption holds when X is essentially finite-type over $\Bbbk$. The same assumption applies when we talk about a $\Bbbk$-algebra A.

Theorem 5.1.4 ([Kun69, Theorem 2.1])**.** *A locally Noetherian $\Bbbk$-scheme X is regular if and only if the Frobenius morphism is flat (under the F-finiteness assumption, it is equivalent to say that $F_*\mathcal{O}_X$ is locally free).*

Remark 5.1.5. Let (X, Δ) be a pair. Since Δ is effective, so is the divisor $\lceil (p^e - 1)\Delta \rceil$ for any $e > 0$, which induces a natural map

$$0 \to F_*^e\mathcal{O}_X \to F_*^e\mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil).$$

With the e-iterated Frobenius map $\mathcal{O}_X \to F_*^e\mathcal{O}_X$, we get the following composition

$$0 \to \mathcal{O}_X \to F_*^e\mathcal{O}_X \to F_*^e\mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil). \tag{5.1}$$

Definition 5.1.6 (F-Singularities of Pair, [PST17, Definition 4.1 and Definition 4.2])**.** Let (X, Δ) be a pair.

- (X, Δ) is *globally F-split* if (5.1) splits for some $e > 0$;[1] it is *sharply F-pure* if it has an affine open cover by globally F-split pairs.
- (X, Δ) is *globally F-regular* if, for any effective integral divisor D, there is $e > 0$ such that the composition

  $$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$$

  splits; it is *strongly F-regular* if it has an affine open cover by globally F-regular pairs.

Remark 5.1.7. Let me explain the terminology 'pure'. Recall that a B-module map $\varphi\colon M \to N$, where B is an arbitrary ring, is *pure* if $M \otimes_B Q \to N \otimes_B Q$ is injective for any B-module Q, and φ is a *split monomorphism* if the following sequence

$$0 \to M \to N \to N/\varphi(M) \to 0$$

is exact and split exact. When φ is pure, φ is a split monomorphism if and only if $N/\varphi(M)$ is finitely presented. Assuming that B is Noetherian and N is a finite A-module, we see that $N/\varphi(M)$ is always finitely presented.

We say that A is *F-pure* if the Frobenius homomorphism F_A is pure. If A is Noetherian, the F-finiteness assumption of A implies that F_A is pure if and only if F_A is a split monomorphism.

[1] In general, the term 'splitting' is used in the context of split exact sequences, or one might speak of a 'split monomorphism.' However, for the Frobenius morphism, it is customary to simply say 'split' instead of 'split monomorphism.'

Remark 5.1.8. In [PST17], a connection between singularities in characteristic 0 and in characteristic p is provided:

characteristic p	characteristic 0
globally F-regular	log Fano
strongly F-regular	Kawamata log terminal
globally F-split	log Calabi-Yau
sharply F-pure	log canonical

Definition 5.1.9 (F-pure Thresholds, [PST17, Definition 7.1])**.** Let $X = \operatorname{Spec} R$ be affine, and assume that (X, Δ) is sharply F-pure. Let H be a Cartier divisor. The *F-pure threshold of* (X, Δ) *along* H is

$$\operatorname{fpt}((X, \Delta), H) = \sup \{t \in \mathbb{R} \mid (X, \Delta + tH) \text{ is sharply } F\text{-pure}\}.$$

When $\Delta = 0$, we write $\operatorname{fpt}(X, H) = \operatorname{fpt}((X, 0), H)$.

Definition 5.1.10 (F-Thresholds, [MTW05])**.** Let $(A, \mathfrak{m})$ be a regular local ring.[2]

- For $e > 0$ and an ideal $\mathfrak{a} \subset A$, define an ideal $\mathfrak{a}^{[p^e]} = (f^{p^e} \mid f \in \mathfrak{a})$.
- For ideals $\mathfrak{a}$ and J of A with $\mathfrak{a} \subset \sqrt{J} \subset \mathfrak{m}$, we define, for any $e > 0$,

$$\nu_{\mathfrak{a}}^J(p^e) = \max \left\{ r \in \mathbb{Z}_{\geq 0} \mid \mathfrak{a}^r \not\subset J^{[p^e]} \right\}.$$

  The assumption $\mathfrak{a} \subset \sqrt{J}$ makes sure that this number is well-defined. If $\mathfrak{a} = J = \mathfrak{m}$, we write $\nu_{\mathfrak{m}}(p^e) = \nu_{\mathfrak{a}}^J(p^e)$.

[2]In [MTW05], A is not necessarily F-finite.

- Since $v_{\mathfrak{a}}^J(p^{e+1}) \geq p \cdot \nu_{\mathfrak{a}}^J(p^e)$, the following limit

$$\lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e} = \sup_{e>0} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$$

is defined, called the F-*threshold* of the pair $(R, \mathfrak{a})$ with respect to J, and denoted by $c^J(\mathfrak{a})$.

Remark 5.1.11. The beginning definition of F-pure threshold was defined in [TW04] as follows. Let A be a reduced F-finite F-pure ring, and write

$$\begin{aligned} A^\circ &= \{f \in A \mid f \text{ is not in any minimal prime ideal of } A\} \\ &= \{f \in A \mid f \text{ is not a zero divisor}\}. \end{aligned}$$

Let $\mathfrak{a} \subset A$ be an ideal such that $\mathfrak{a} \cap A^\circ \neq \emptyset$.

- Given $t \in \mathbb{R}_{\geq 0}$, the pair $(A, \mathfrak{a}^t)$ is F-*pure* if for all $e > 0$ large enough, there is $d \in \mathfrak{a}^{\lfloor t(p^e-1)\rfloor}$ such that $d^{1/p^e}A \hookrightarrow A^{1/p^e}$ is a split monomorphism.

- The F-*pure threshold* of $(A, \mathfrak{a})$, in the sense of Takagi and Watanabe, is defined to be

$$c(A, \mathfrak{a}) = \sup\left\{t \geq 0 \mid (A, \mathfrak{a}^t) \text{ is } F\text{-pure}\right\}.$$

  We can translate between two definitions of F-pure threshold: if $X = \operatorname{Spec} A$ and H is an effective Cartier divisor, then we see that $\operatorname{fpt}(X, H) = c(A, \mathcal{O}_X(-H))$.

In [MTW05, Remark 1.5], it says that under the assumption that $(A, \mathfrak{m})$ is a regular local ring with an ideal $\mathfrak{a} \subset \mathfrak{m}$, the F-threshold and the F-pure threshold (in the sense of Takagi and Watanabe) are the same: $c^{\mathfrak{m}}(\mathfrak{a}) = c(A, \mathfrak{a})$. In practice, we therefore may use the definition of an F-threshold to compute the F-pure threshold.

Example 5.1.12 (The F-pure Threshold of the Cusp)**.** Let $f = y^2 - x^3 \in \Bbbk[x, y]$ (or in $\Bbbk[x, y]_{\mathfrak{m}}$, where $\mathfrak{m} = (x, y)$), and let $X = \operatorname{Spec} \Bbbk[x, y]$. We have

$$\operatorname{fpt}(X, V(f)) = \begin{cases} \dfrac{1}{2} & \text{if } p = 2 \\ \dfrac{2}{3} & \text{if } p = 3 \\ \dfrac{5}{6} & \text{if } p \equiv 1 \pmod 3 \\ \dfrac{5}{6} - \dfrac{1}{6p} & \text{if } p \equiv 1 \pmod 3. \end{cases}$$

It is possible to compute those numbers via the definition of an F-threshold. For example, $\nu_{\mathfrak{m}}(p^e) = 5/6 \cdot (p^e - 1)$ when $p \equiv 1 \pmod 3$, and

$$\nu_{\mathfrak{m}}(p^e) = \begin{cases} \dfrac{5}{6}p - \dfrac{7}{6} & \text{if } e = 1 \\ \dfrac{5}{6}p^e - \dfrac{1}{6}p^{e-1} - 1 & \text{if } e > 1 \end{cases}$$

when $p \equiv 2 \pmod 3$. See [MTW05] from a D-module prospective to F-thresholds.

5.2 Questions and Speculations

Remark 5.2.1. In the end of Chapter 3, we developed the theory of Hahn multiplicity (although the full result remains conjectural). It has been shown in [FJ04] that $\hat{\beta}+1 = A_x^{\mathrm{thin}} \circ \pi_{\mathbb{K}/\Bbbk(x))}$ if $\operatorname{char} \Bbbk = 0$, while it is false when $\operatorname{char} \Bbbk > 0$. This positive characteristic feature hints that the numerical invariants of singularities may be elicited from these valuative spaces.

For instance, when computing the LCT, we consider the function of the form

$$1 + \operatorname{ord}_E(K_{-/X}) / \operatorname{ord}_E(\Delta)$$

on the space Val_X.

Question 5.2.2. *In characteristic p, is there a function defined on a subspace of X^{an} that can be used to compute the F-pure threshold?*

Remark 5.2.3. Let (X, Δ) be a log $\mathbb{Q}$-Gorenstien pair. The *multiplier ideal* is defined by

$$J(X; \Delta) = \bigcap_{\pi\colon Y \to X} \pi_* \mathcal{O}_Y \left(\lceil K_Y - \pi^*(K_X + \Delta) \rceil \right),$$

where π ranges all proper birational morphisms with Y normal and K_Y is chosen so that $\pi_* K_Y = K_X$. If (X, Δ) admits log resolutions, such a resolution computes $J(X; \Delta)$. See [Laz04, Chapter 9] or [BST15, Section 2.4] for more details. By Definition 2.9.13, it is obvious that the LCT is the smallest positive number λ with the following property: for any $\varepsilon > 0$, $J(X; \lambda(1-\varepsilon)\Delta) \supsetneq J(X; \lambda\Delta)$. Any positive number with this property is called a *jumping number*.

We have an analogy in characteristic p. Let (X, Δ) be a pair in positive characteristic. The *(big) test ideal*, denoted by $\tau(X; \Delta)$, is a unique smallest nonzero ideal $\mathcal{J} \subset \mathcal{O}_X$ with the property that the containment $\phi(F^e_* \mathcal{J}) \subset \mathcal{J}$ holds for any $e > 0$ and any local section $\phi \in \mathcal{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil), \mathcal{O}_X)$. An *$F$-jumping number* is a positive number λ such that for any $\varepsilon > 0$, $J(X; \lambda(1-\varepsilon)\Delta) \supsetneq J(X; \lambda\Delta)$; the fact is that the F-pure threshold is the first F-jumping number.

Therefore, if we can compute the test ideal using birational geometry, that might be a solution to Question 5.2.2. In [BST15, Theorem 4.6], if (X, Δ) is log $\mathbb{Q}$-Gorenstein, then there exists a finite separable map (or a regular separable alteration) $\pi\colon Y \to X$, where Y is normal, such that

$$\tau(X; \Delta) = \mathrm{Image}\left(\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) \rceil) \xrightarrow{\mathrm{Tr}_\pi} K(X) \right),$$

where $K_Y = \pi^* K_X + \mathrm{Ram}_\pi$ and Ram_π denotes the ramification divisor of π, and Tr_π is the Grothendieck trace of π. This result is further strengthened in [STZ12] as follows: let (X, Δ) be the same as above, and let D be an effective Cartier divisor on X. Then, there is a finite separable morphism (or a regular separable alteration) $\pi\colon Y \to X$, where Y is normal, such that for any $\lambda \geq 0$,

$$\tau(X; \Delta + \lambda D) = \mathrm{Image}\left(\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta + \lambda D)\rceil) \xrightarrow{\mathrm{Tr}_\pi} K(X)\right).$$

The good news from this result is that a single finite separable map (or a regular separable alteration) computes the F-pure threshold, which is similar to the fact that we can compute the LCT via a single log resolution. Although this result provides a theoretical approach for constructing such a map, it does not offer a practical means of carrying it out.[3]

One may wonder, instead of using all alterations, whether we can only use proper birational maps. However, [ST14, Example 7.12] gives an counterexample in characteristic 2: let

$$R = \frac{\mathbb{F}_2[[x, y, z]]}{(z^2 + xyz + xy^2 + x^2 y)}$$

and let $X = \operatorname{Spec} R$. We can consider a ring extension

$$S = \frac{R[u, v]}{u^2 + xu + x, v^2 + yv + y, z + xv + yu} \cong \mathbb{F}_2[[u, v]],$$

which gives a resolution $\pi\colon Y \triangleq \operatorname{Spec} S \to X$. The test ideal is $(x, y, z) \subset R$, but $\mathrm{Tr}_\pi(\mathcal{J}(Y)) = R$.

[3]The author verified this technical difficulty with Kevin Tucker during the 2025 Summer Research Institute in Algebraic Geometry.

Example 5.2.4. Let $f = y^2 - x^3 \in \mathbb{F}_p[x, y]$. The author tried to find one such finite separable map using Macaulay 2. If $p = 2$, there are many candidates, and one of them is the morphism induced by the finite ring homomorphism

$$\mathbb{F}_p[x, y] \to \frac{\mathbb{F}[x, y, z]}{(z^2 - fz - x)}.$$

If $p = 3$, there are still several candidates, and one of them is given by

$$\mathbb{F}_p[x, y] \to \frac{\mathbb{F}[x, y, z]}{(z^3 - fz - y)}.$$

If $p = 5$, only the following candidate is found:

$$\mathbb{F}_p[x, y] \to \frac{\mathbb{F}[x, y, z]}{(z^5 - fz - xy)}.$$

For the case that $p \geq 7$, the author's Macaulay 2 code never terminates, and the reason is that computing Weil divisors can be time-consuming. There is a strong evidence that each resulting finite map above computes the F-pure threshold.

Remark 5.2.5. In [Can20], Canton proposes a discrepancy function defined on the whole $\daleth$-space $X^{\daleth}$ using p^{-e}-linear maps, and shows that if $(X, 0)$ admits a log resolution, then Canton's discrepancy agrees with the discrepancy function defined in [JM12]. Furthermore, his theory can detect whether a given pair is sharply F-pure or not (see [Can20, Theorem 5.3] for a precise statement). However, it is not clear if the coefficient of the boundary changes continuously, how Canton's discrepancy changes accordingly.

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