UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

VOLUMES AND TOPOLOGY OF HYPERBOLIC 4-MANIFOLDS

A Dissertation

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

Doctor of Philosophy

By

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Norman, Oklahoma
1998

UMI Number: 9905631

VOLUMES AND TOPOLOGY OF HYPERBOLIC 4-MANIFOLDS

A Dissertation APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

BY

Dedicated to my parents

Ivan and Zlatka Ivanšić

ACKNOWLEDGEMENTS

I would like to thank my advisor Professor Boris Apanasov for the many helpful conversations we had concerning this dissertation and other matters. This work would not have been possible without his knowledge, direction, support and encouragement.

Additionally, many thanks are due to Professor Darryl McCullough, who taught me many useful things in topology and always had time to answer my questions. I thank the other committee members, Professors Ronald Kantowski, Andy Miller, Gerard Walschap and former committee member Robert Miner for their willingness to serve in that capacity.

Many people whom I talked to helped streamline proofs, gave very useful pointers and provided additional encouragement. For this I am indebted to Professors Ed Cline, Robert Daverman, Kyung-Bai Lee and Andy Miller.

Very special thanks go to Professor Leonard Rubin and Mrs. Diane Rubin. Their kindness and hospitality during my stay in Norman have made the years here even more memorable.

Most of all, I would like to thank my father and my late mother for their generous and unswerving support of all my endeavors. They set a wonderful example for me to follow.

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1. INTRODUCTION

In the late seventies, W. Thurston proved or conjectured a number of results that initiated a revolution in 3-manifolds and brought hyperbolic geometry into the spotlight. The most famous of these is the conjecture that the interior of every compact 3-manifold may be decomposed into pieces, each of which admits one of eight geometries on it. Thurston proved the conjecture for a large number of special cases (Haken manifolds) and showed that many manifolds (for example, Haken, homotopically atoroidal) automatically have a hyperbolic geometry on it. This and the fact that 3-manifolds with all geometries except the hyperbolic one are well understood (classified) spawned a great deal of interest in the hyperbolic case. Thus, in the years that ensued, the field of 3-dimensional hyperbolic manifolds bloomed and a lot of research is still being done on them.

The direction that we took in our doctoral work was to contribute to research of higher-dimensional hyperbolic manifolds. While there are several important results about these manifolds, such as Mostow's rigidity theorem or Wang's theorem about their volumes, this field has remained largely uninvestigated, especially from the topological point of view. Our studies center on dimension 4 for actual examples, while we also obtain some results in higher dimensions.

Most problems we look at are inspired by similar considerations in the 3-

dimensional case. What is interesting is that the results are usually not direct generalizations of the 3-dimensional case. In several instances we come up with results starkly different from the 3-dimensional case.

First, we construct a sequence of noncompact finite-volume hyperbolic 4-manifolds by way of side-pairings of a polyhedron. While examples of such manifolds were available before, almost all were constructed using arithmetic methods, which shed little light on the topology or volume of the manifolds. Before the constructions that we carry out here, there was only one (compact) example ([D]) of a hyperbolic 4-manifold known that came from side-pairings of a polyhedron. In the meantime, several authors ([R-T] or [N]) have independently come up with noncompact examples by the method of pairing sides of a polyhedron, and they used them to prove interesting results.

One of the major themes of our study is the volume of hyperbolic 4-manifolds. Wang's theorem immediately implies that there exist only finitely many hyperbolic n-manifolds with the same volume ($n \geq 4$). Although not a consequence of Wang's theorem, this is true also in dimension three. That there is no bound on how many may have the same volume was proved in dimension three by Wielenberg for the compact case and Apanasov and Gutsul in the noncompact case. Both results were obtained by using polyhedra that had their sides paired in different ways. Using our examples we extend this theorem into dimension 4 and prove that there exist at least $n+1$ noncompact hyperbolic manifolds that have volume $2n \cdot 4\pi^2/3$. All of them share the same fundamental polyhedron.

The Gauss-Bonnet theorem for a hyperbolic 4-manifold M says that the volume of M is proportional to the Euler characteristic of M, i.e. $\mathrm{Vol}(M) = \chi(M) \cdot 4\pi^2/3$. Our examples give manifolds with Euler characteristic $2n$, thus showing that at least "half" the possibilities for volumes are attained. Recently, Ratcliffe and Tschantz independently showed that for every n there exists a hyperbolic 4-manifold with volume $n \cdot 4\pi^2/3$. However, their proof does not give an explicit polyhedron with side-pairings on it for each n.

Another major direction of this thesis is investigating possible hyperbolic structures on complements of submanifolds inside closed n-manifolds. This direction draws on the fact that a lot of research in 3-manifolds is based on the phenomenon that there exist many knot complements in S^3 that allow a hyperbolic structure on them. Furthermore, Thurston has proved that almost every Dehn surgery on such a link results in a closed hyperbolic manifold and Ouyang has shown that different Dehn surgeries yield infinitely many different manifolds. Also, due to the results of Lickorish and Wallace and Adams, every 3-manifold may be obtained by Dehn surgery on a hyperbolic link in S^3.

With the 3-dimensional case in mind, we ask the following general question: when may we think of a noncompact finite-volume hyperbolic manifold M as the complement of a codimension-k closed submanifold A inside a closed manifold N ($\dim N = \dim M$)? If we may do so, we say that M is a codimension-k complement. Since M is always the union of a compact piece and finitely many noncompact pieces (ends), each of which has the form $E \times [0, \infty)$, where E is a compact flat (Euclidean) n-manifold, the answer to this question will depend only on the ends. By investigating these ends, we find the following: if

$M = N - A$, as above, then every manifold E corresponding to an end $E \times [0, \infty)$ must either be an S^0- or an S^1-bundle over a component of A, i.e. $k = 1$ or $k = 2$. Furthermore, the components of A must be flat manifolds.

We consider first the case $k = 2$, which generalizes the much-investigated 3-dimensional situation (a link inside a 3-dimensional manifold is a codimension-2 submanifold). The torus and the Klein bottle are the only flat 2-manifolds and they are both S^1-bundles over the circle, so every hyperbolic 3-manifold is a codimension-2 complement. In contrast to flat 2-manifolds we find that there exist flat 3-manifolds E that are not S^1-bundles; consequently, any hyperbolic 4-manifold having $E \times [0, \infty)$ as an end is not a codimension-2 complement. In general, we prove a criterion in terms of the fundamental group of E that detects when E is an S^1-bundle. We use it to determine which, of the 10 possible, flat 3-manifolds are S^1-bundles and which are not, and to construct examples of flat manifolds in every dimension that are not S^1-bundles, thereby showing that for $\dim M \geq 4$ there exists an obstruction to M being a codimension-2 complement.

If M has dimension 4, it can be a codimension-2 complement only of tori and Klein bottles. The natural question, in light of the fact that so many link complements in S^3 have hyperbolic structure, is whether there exist hyperbolic 4-manifolds M that are torus or Klein bottle complements inside the four-sphere S^4? We show that there can be at most finitely many such manifolds.

As far as this author knows, hyperbolic manifolds have not been studied as codimension-1 complements (this is the case $k = 1$). It turns out that this way of looking at them differs a great deal from looking at them as codimension-2 complements. For example, filling in ends of $M = S^3 - \{\text{link}\}$ in various (codim-2) ways corresponds to Dehn surgery on the link. It is known (see [Ou]) that all the different surgeries result in infinitely many nonhomeomorphic 3-manifolds. It is also known that almost all of those surgeries yield closed hyperbolic manifolds (see [T] or [R]).

However, if every manifold E corresponding to an end of a hyperbolic M is an S^0-bundle, and we represent M as a codimension-1 complement, i.e. $M = N^{n+1} - A^n$, then we get the following facts. First, the universal cover of N is $\mathbb{R}^n$. This is very different from the 3-dimensional codimension-2 case, where we often got that $N = S^3$ (so, not even contractible). Second, the manifold N can never be hyperbolic. Third, if every E is an S^0-bundle over a Euclidean manifold B that has the same holonomy group as E, then there are only finitely many choices for N, where components of A are Euclidean manifolds with the same holonomy groups as the E's corresponding to them. (This result is restricted to a 3- or 4-dimensional M and there are indications it does not hold in higher dimensions.)

Finally, we give criteria in terms of the fundamental group of a flat manifold that allow us to detect whether a flat manifold is an S^0-bundle over a flat manifold B with the same holonomy group. We use them to show that all but two flat 3-manifolds are S^0-bundles over flat 3-manifolds with the same holonomy group. For the two exceptions, we show that they are not S^0-bundles at all.

2. Euclidean and hyperbolic geometries

In this section we briefly review the basic facts about Euclidean and hyperbolic spaces that we will need later.

Euclidean space in dimension n is defined as the set $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}$ with the length differential $ds^2 = dx_1^2 + \cdots + dx_n^2$. The geodesics (curves whose length realizes distance between two points) in Euclidean space are standard lines. A *hyperplane* is, in general, a codimension-1 subspace that is totally geodesic, that is, whenever two points are in the subspace, so is the geodesic connecting them. In Euclidean space, hyperplanes are the standard hyperplanes: sets satisfying the equation $a_1x_1 + \cdots + a_nx_n = d$ for fixed real numbers $a_1, \ldots, a_n, d$. If a plane is given in this form, the vector $(a_1, \ldots, a_n)$ is a normal vector for the plane. The angle between two hyperplanes is the acute angle between any two of their normal vectors. With the above differential metric, the Euclidean space has sectional curvature 0.

We may define a metric on $\mathbb{R}^n$ induced by the length differential: the distance between two points is the infimum of lengths of all paths connecting them. It turns out that this metric is the same as the standard Euclidean metric on $\mathbb{R}^n$, under which $\mathbb{R}^n$ is a complete metric space.

Hyperbolic n-space $\mathbb{H}^n$ has several models — for our purposes the most convenient one will be the "upper half-space" model. In this model

$$\mathbb{H}^n = \{(x_1, \ldots, x_{n-1}, t) \in \mathbb{R}^n \mid t > 0\}$$

with the length differential $ds^2 = \frac{dx_1^2 + \cdots + dx_{n-1}^2 + dt^2}{t^2}$. Similarly as above, the length differential turns $\mathbb{H}^n$ into a complete metric space of constant sectional curvature -1. Its geodesics are either Euclidean lines or Euclidean half-circles perpendicular to $\partial\mathbb{H}^n = \mathbb{R}^{n-1} \times \{0\} \cup \{\infty\}$. Hyperplanes in this model are either Euclidean hyperplanes or half-spheres that are perpendicular to $\partial\mathbb{H}^n$. The *boundary at infinity* of a set $S \subset \mathbb{H}^n$ is the set of all points in $\partial\mathbb{H}^n$ that are in the Euclidean closure of S. A hyperplane is uniquely determined by its own boundary at infinity which can be either a Euclidean $(n-2)$-plane (which we will assume to contain the point ∞) or an $(n-2)$-sphere in $\partial\mathbb{H}^n$. We will say that the hyperplane is *based*, respectively, on a plane or a sphere. Using the length differential, the angle between hyperplanes is well-defined and it turns out to be the same as the angle between normal vectors to their boundaries at infinity.

The set of isometries of $\mathbb{R}^n$ and $\mathbb{H}^n$ can be given the compact-open topology. A Euclidean manifold E is a space of form $\mathbb{R}^n/G$ where G is a discrete, freely-acting subgroup of $\operatorname{Isom}\mathbb{R}^n$. Euclidean manifolds are often called *flat*. Similarly, a hyperbolic manifold M is a space of form $\mathbb{H}^n/G$, with G a discrete and freely-acting subgroup of $\operatorname{Isom}\mathbb{H}^n$. The freeness-of-action condition is necessary to ensure that E and M are spaces that are locally homeomorphic to a unit ball in $\mathbb{R}^n$. It can be seen that an element $g \in \operatorname{Isom}\mathbb{R}^n$ or $\operatorname{Isom}\mathbb{H}^n$ that generates a discrete subgroup acts freely if and only if it has infinite order, so the condition of freeness of action can be replaced by requiring torsion-freeness of G.

We will give a few facts about Euclidean manifolds in §4, while we finish this section with some facts about hyperbolic ones. The fact about topology of hyperbolic manifolds that we most frequently use is the following one (see [A1], [A2] or [B-P]).

Theorem 2.1. *If M is a noncompact finite-volume hyperbolic n-manifold, then it is the union of a compact part and finitely many disjoint noncompact pieces, called ends, each of which is of the form $E \times [0, \infty)$, where E is a compact Euclidean $(n-1)$- manifold.*

The compact part of a manifold M with ends $E_1 \times [0, \infty), \ldots, E_m \times [0, \infty)$ is the manifold with boundary $\overline{M} = M - (E_1 \times (0, \infty) \cup \cdots \cup E_m \times [0, \infty))$. The boundary components are, of course, $E_1, \ldots, E_m$. Since the interior of $\overline{M}$ is homeomorphic to M we will abuse language and say that each E bounds M if $E \times [0, \infty)$ is one of its ends.

One of the items we consider with regard to hyperbolic manifolds is their volume. The following theorem is a very important result about volumes of higher-dimensional hyperbolic manifolds.

Theorem 2.2. *(Wang, [Wg]) Let $n \geq 4$. Given a real number $c > 0$ there exist only finitely many nonisometric hyperbolic n-manifolds with volume less than c.*

In dimension 3 this theorem is not valid, but it is still true that only finitely many hyperbolic 3-manifolds have the same volume.

More can be said about volumes of hyperbolic manifolds.

Theorem 2.3. *(Gauss-Bonnet formula) The volume of a finite-volume hyperbolic 4-manifold is* $\operatorname{Vol} M = \chi(M) \cdot 4\pi^2/3$, *where* $\chi(M)$ *is the Euler characteristic of* M. □

This formula for the compact case comes from [Ho], while to get the noncompact version [G, page 84] should be applied. Notice that the Euler characteristic of the compact part $\overline{M}$ of M is the same as $\chi(M)$, since $\overline{M}$ is a retract of M.

3. Poincaré's polyhedron theorem and torsion-freeness

In this section, we formulate Poincaré's polyhedron theorem: it holds in all constant curvature geometries and it is going to be one of our essential tools.

Constructing discrete, torsion-free subgroups of $\operatorname{Isom}\mathbb{R}^n$ and $\operatorname{Isom}\mathbb{H}^n$ (i.e. constructing Euclidean and hyperbolic manifolds) is in general a nontrivial task. One very useful method that also offers some insight into the topology of the resulting manifold is to pair sides of a polyhedron in $\mathbb{R}^n$ or $\mathbb{H}^n$. The conditions that such a pairing has to satisfy in order that the resulting quotient space be a Euclidean or hyperbolic manifold are described by Poincaré's polyhedron theorem. The approach we use is from [E-P] and [R] — some other sources are [A2] and [M].

First, we review some basic definitions. Let $X = \mathbb{R}^n$ or $\mathbb{H}^n$. (We could also allow X to be the n-sphere since the notion of hyperplanes is also defined there, but there will be no need for spherical geometry here.) A hyperplane in X divides X into two closed pieces, called half-spaces, whose interiors are disjoint and whose intersection is the hyperplane. In $\mathbb{H}^n$, the $(n-2)$-plane or -sphere on which the hyperplane is based divides $\mathbb{R}^{n-1} \cup \{\infty\}$ into two closed sets, each of which is the boundary at infinity of one of the half-spaces that the hyperplane determines.

A *polyhedron* in X is a connected closed subset P of X with a nonempty interior whose (topological) boundary is contained in a locally finite collection $\mathcal{C}$ of hyperplanes. For example, an intersection of finitely many half-spaces is going to be a polyhedron — we work only with such examples of polyhedra. A *codimension-one side* S of P is a subset of ∂P such that $S = P \cap H$ and $S = \mathrm{cl}_H(\mathrm{int}_H\, S)$, where H is a hyperplane from the collection $\mathcal{C}$. Then S is a union of $(n-1)$-dimensional polyhedra in H. Proceeding inductively, we may define a *codimension-i side* of P to be a codimension-one side of a codimension-$(i-1)$ side of P. More details on polyhedra may be found in [A2] or [E-P].)

Since every codimension-i side is a polyhedron in dimension $n-i$, we also call it an $(n-i)$-side. Codimension-one sides we will simply call *sides*, codimension-two sides we call *edges*, and we will use the term *vertex* for a 0-side of P. When we talk about a polyhedron in $\mathbb{H}^n$ its vertices are also called *finite vertices* or *real vertices* as opposed to *vertices at infinity* or *ideal vertices* which are the isolated boundary points of P in $\partial\mathbb{H}^n$.

Often, to simplify notation, a hyperplane, the side of P lying on the hyperplane and, in the case of a hyperbolic polyhedron, the boundary at infinity of the hyperplane, will be denoted by the same letter. (No confusion should arise here because in our polyhedra, each hyperplane supports only one side.)

To get a *side-pairing* Φ of P, for every side S of P we choose another side S' and an isometry $g \in \mathrm{Isom}\, X$ so that $g(S) = S'$. We say that g pairs S and S' and that S' is the pair of S. A side-pairing must also satisfy the following requirements (for more details, see [E-P] or [R]):

(1) if S' is the pair of S then the pair of S' is S and g^{-1} is the isometry that pairs S' and S.

(2) For any g that pairs some two sides, $g(\operatorname{int} P) \cap \operatorname{int} P = \emptyset$.

In order that the side-pairing isometries generate a discrete subgroup of $\operatorname{Isom} X$, sides cannot be paired in any arbitrary way: the so-called *edge cycle condition* must be satisfied. (This condition is called Cyclic in [E-P].) In general, a side-pairing on P induces an equivalence relation on P that is generated by the relation $x \sim s(x)$, where $x \in \partial P \cap S$, S is a side of P and s its side-pairing. The equivalence class $[x]$ of x under this equivalence relation is called the *cycle* of x. The cycle of an i-side is defined analogously so that it contains all the i-sides of P that are identified by a string of side-pairings. Cycles of ideal vertices are defined in exactly the same way. The number of cycles of ideal vertices is equal to the number of ends (see Theorem 2.1) of a manifold generated by a side-pairing.

Every edge (codimension-2 side) of P is the intersection of two uniquely determined sides of P. The *dihedral angle* at an edge is the angle in the interior of P that the two sides subtend. A cycle of edges can be obtained in the following way. Start with an edge E_1, which is the intersection of sides S_1 and R_1, and let g_1 be the isometry pairing R_1 and some side S_2 of P. We get that $g_1(E_1) = E_2$, where E_2 is an edge determined by S_2 and some other side R_2. Now let g_2 be the isometry pairing R_2 and some side S_3. Continuing in the same way we get a sequence of edges, sides and isometries $\{\sigma_i = (E_i, S_i, R_i, g_i)\}_{i=1,2,\ldots}$. This procedure is commonly called "edge-chasing". We require that the above sequence have a period q (called *first cycle length* in [E-P]), that is $\sigma_{q+1} = \sigma_1$ for some q. The cycle of edges will then consist of exactly $E_1, \ldots, E_q$. This is the first part of the edge cycle condition. If the polyhedron is finite-sided, this condition will automatically be satisfied.

It is clear that $g_q \circ \cdots \circ g_1(E_1) = E_1$, but it may happen that the restriction of $g_q \circ \cdots \circ g_1$ on E_1 is not the identity. The second part of the edge cycle condition is that there must be a k so that $(g_q \circ \cdots \circ g_1)^k \mid_{E_1} = id$. The number kq is called the *second cycle length* in [E-P].

The third and final part of the edge-cycle condition is that if θ_i, $i = 1, \ldots, q$ are the dihedral angles of edges E_i, where $E_1, \ldots, E_q$ is a cycle of edges, then there is a nonzero integer m so that $k(\theta_1 + \cdots + \theta_q) = 2\pi/m$.

The edge-cycle condition is enough to guarantee the conclusion of the polyhedron theorem if $X = \mathbb{R}^n$. When $X = \mathbb{H}^n$ and we are dealing with a hyperbolic polyhedron, we need another condition. Before we state it, recall that a *horosphere in* $\mathbb{H}^n$ is either a Euclidean sphere tangent to $\partial\mathbb{H}^n$ or a Euclidean hyperplane $\{t = c\}$ parallel to $\partial\mathbb{H}^n$. The former are said to be *centered* at the point of tangency with $\partial\mathbb{H}^n$, the latter at infinity.

Consistent horosphere condition: there exists a set T of disjoint horospheres centered at ideal vertices of P, each intersecting only those sides of P whose boundary contains center of the horosphere, so that if g is a side-pairing of a side that contains the center of a horosphere $H \in T$ in its boundary, then $g(H)$ is again a horosphere from T.

We may now formulate Poincaré's polyhedron theorem as follows.

Theorem 3.1. *(Poincaré's polyhedron theorem) Let Φ be a side-pairing on a polyhedron $P \subset X$ that satisfies the edge cycle condition If $X = \mathbb{H}^n$, let the side-pairing additionally satisfy the consistent horosphere condition. Then the side-pairings of Φ generate a discrete group $G \subset \operatorname{Isom}\mathbb{H}^n$ whose fundamental polyhedron is P.* $\square$

Poincaré's polyhedron theorem gives only the discreteness of G - in order to get manifolds, we will also need to show torsion-freeness of G. (Otherwise, X/G is only going to be a Euclidean or hyperbolic orbifold.) Torsion-freeness is checked in the following way.

The *normalized solid angle at point* x of a polyhedron P is defined as $\omega(x) = \operatorname{Vol}(B(x,r) \cap P)/\operatorname{Vol} B(x,r)$ (see [R]). Here $B(x,r)$ is a ball about x or radius r, Vol is volume and r is taken small enough so that $B(x,r)$ intersects only those sides of P on which x lies. Let $[x] = \{x_1, \dots, x_n\}$ be the cycle of x for some side-pairing of P. We define the *normalized solid angle sum of* $[x]$ as $\omega[x] = \sum_{y\in[x]} \omega(y)$. The next theorem is Theorem 11.1.1 from [R].

Theorem 3.2. *If $\omega[x] = 1$ for every $x \in P$ then the group G generated by the side-pairings of P is torsion-free.* $\square$

Remark 3.3. Knowing that P is a fundamental polyhedron for a discrete group G already implies $\omega[x] \leq 1$ for every point of P. Really, for every $x_i \in [x]$, choose an isometry $g_i \in G$ taking x_i to x. (In general, there may be many ways to make the choices.) We now have an injective map from $\{x_1, \dots, x_n\}$ to $\{$translates of P under G containing $x\}$ given by $x_i \mapsto g_i(P)$. Since $\{g_1(P) \cap B(x,r), \dots, g_n(P) \cap$

$B(x,r)\}$ fill out maybe only a portion of $B(x,r)$, we get $\omega[x] \leq 1$. (Note that the strict inequality will occur if and only if x is the fixed point of an element in G.)

By Remark 3.3, to check torsion-freeness it will be enough to see that $\omega[x] \geq 1$ for every $x \in P$.

4. Fundamental flat manifolds facts

In this section we recall some basic facts about flat (Euclidean) manifolds and their fundamental groups that we are going to use later. Our source was mostly [C], but some information can also be found in [A2] and [Wo].

As we have said before, a flat manifold is a manifold of form $\mathbb{R}^n/G$, where G is a discrete, torsion-free subgroup of Euclidean isometries of $\mathbb{R}^n$. An element $g \in \operatorname{Isom}\mathbb{R}^n$ may be written as $g(x) = Ax + a$, where $A \in O(n)$ and $x, a \in \mathbb{R}^n$. The orthogonal transformation A is called the *rotational part* of g — notice that $g \mapsto A$ defines a homomorphism $\operatorname{Isom}\mathbb{R}^n \xrightarrow{p} O(n)$. Clearly g is a translation if and only if $A = I$. When an element $g \in \operatorname{Isom}\mathbb{R}^n$ has no fixed points (which is equivalent to it having infinite order if the subgroup it generates is discrete) it can be seen that $\ker(A - I) \neq 0$ and that a has a nonzero component in the first factor of the decomposition $\mathbb{R}^n = \ker(A - I) \oplus \operatorname{im}(A - I)$. Furthermore, after a change of coordinates by a translation we may assume that $a \in \ker(A - I)$. Then we call a the *translational* part of g.

The following well-known theorems reveal the structure of discrete subgroups of $\operatorname{Isom}\mathbb{R}^n$ whose quotient $\mathbb{R}^n/G$ is compact. Such groups are called *crystallographic*. The torsion-free among them are the groups that yield flat manifolds and are called *Bieberbach groups*.

Theorem 4.1. *(Bieberbach's theorems.) Let G be a crystallographic subgroup of* $\operatorname{Isom}\mathbb{R}^n$.

First. The subgroup K of translations of G is a rank n free abelian group whose vectors span $\mathbb{R}^n$. Furthermore, K has finite index in G.

Second. If G' is another group as in the assumption of the theorem, and $f: G \to G'$ is an isomorphism, then $f(g) = dgd^{-1}$, where d is an affine transformation ($d(x) = Ax + a$, $A \in GL(n, \mathbb{R})$, $a \in \mathbb{R}^n$).

Third. Up to an affine change of coordinates, there exist only finitely many groups G.

An algebraic characterization of crystallographic groups is: G is crystallographic if and only if it is finitely generated and contains a maximal, under inclusion, subgroup K among all its abelian subgroups of finite index and K is torsion-free. The subgroup K can also be characterized as the set of all elements in G with finitely many conjugates. Its elements are called *translations.* Every crystallographic group G fits into the exact sequence $0 \to K \to G \xrightarrow{p} \Phi \to 1$, where Φ is finite. If we think of G as a subgroup of $\operatorname{Isom}\mathbb{R}^n$ then K is the set of elements of G that are translations (in the usual sense) and p may be viewed as the restriction to G of the above map $\operatorname{Isom}\mathbb{R}^n \to O(n)$. The group Φ is called the *holonomy group* of G.

For a given finite group Φ and free abelian group K on which Φ acts, there is a one-to-one correspondence between equivalent exact sequences $0 \to K \to G \xrightarrow{p} \Phi \to 1$ and elements of the group $H^2(\Phi; K)$. (K need not be the maximal abelian subgroup in G.) Elements of $H^2(\Phi; K)$ are certain classes represented by maps $c : \Phi \times \Phi \to K$ (called *cocycles*) which satisfy $\delta c = 0$. (For the definition of δ, see [C]). It can be seen that every element of $H^2(\Phi; K)$ can be represented by a *normalized* cocycle, that is, one which satisfies $c(\sigma, 1) = c(1, \sigma) = 0$. Then the group G corresponding to the element of $H^2(\Phi; K)$ that is represented by c may be viewed as the set $\Phi \times K$ with the multiplication

$$(\sigma, k)(\sigma', k') = (\sigma\sigma', \sigma \cdot k' + k + c(\sigma, \sigma')),$$

where $\sigma, \sigma' \in \Phi$ and $k, k' \in K$. Here $\sigma \cdot k'$ denotes the action of Φ on K: $\sigma \cdot k = gkg^{-1}$, where $g \in G$ is any element so that $p(g) = \sigma$.

If $f : G \to G$ is an isomorphism such that $f(K) = K$ (for example, if K is the maximal free abelian subgroup) then it is easily seen that it must have the form $f(\sigma, m) = (f \cdot \sigma, f(m))$ and must satisfy $f(c(\sigma, \sigma')) = c(f \cdot \sigma, f \cdot \sigma')$. (We use the same letter f to denote both $f : G \to G$ and the restriction of f to K.) The expression $f \cdot \sigma$ denotes the action of f on Φ by conjugation, which is defined since we may think of both f and Φ as subgroups of $\operatorname{Aut} K$.

We are going to make much use of the following theorem that characterizes flat manifolds. It is a special case of the Borel conjecture, which says that any two aspherical manifolds that have isomorphic fundamental groups are homeomorphic.

Theorem 4.2. *(Farrell-Hsiang) Let F^n, $n \neq 3, 4$ be a closed aspherical manifold such that $\pi_1 F^n$ is isomorphic to $\pi_1 E^n$, where E^n is a closed connected flat manifold. If $n = 3$ assume, in addition, that F^n is irreducible. Then F^n is homeomorphic to E^n.*

Proof. The theorem for $n \neq 3, 4$ is a result of Farrell and Hsiang (see [F-H]). The claim in dimension 3 is a simple application of Waldhausen's theory of sufficiently large manifolds. We will need the following theorem from [H]:

Theorem 4.3. *(Heil) Let E, F, be closed 3-manifolds that are P^2-irreducible, i.e, they are irreducible and do not contain any 2-sided projective planes. If E is orientable, let it be sufficiently large. Suppose $\sigma : \pi_1 F \to \pi_1 E$ is an isomorphism. Then there exists a homeomorphism $g : F \to E$ which induces σ.* □

Thus, in order to extend Farrell and Hsiang's result to dimension 3 we just need to make sure that F does not contain a 2-sided P^2 and that every flat manifold E is P^2-irreducible and sufficiently large when orientable. If F contained a 2-sided P^2, $\pi_1 F$ would have an element of order two, which is impossible, since $\pi_1 F \cong \pi_1 E$, a torsion-free group. This also shows that E doesn't contain a 2-sided P^2. Furthermore, E is irreducible since its universal covering is $\mathbb{R}^3$.

Up to homeomorphism, there are only ten closed 3-dimensional flat manifolds, 6 orientable and 4 nonorientable. A list may be found in [Wo] or the original paper [H-W]. Both of these sources use $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, and $\mathcal{G}_6$ to denote the subgroups of $\operatorname{Isom}\mathbb{R}^n$ that generate the orientable flat manifolds and $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$ to denote subgroups generating nonorientable manifolds. Of the 6 orientable ones, 5 have infinite first homology groups, which implies that they are sufficiently large. The 6th one, $\mathbb{R}^3/\mathcal{G}_6$, can directly be found to contain an incompressible 2-sided surface. The paper [H-W] shows we may choose as fundamental polyhedron for $\mathcal{G}_6$ the cube $C = [-1,1] \times [-1,1] \times [-1,1]$ with side pairings

$$f_1 : \{x_2 = 1\} \to \{x_1 = 1\} = (\text{translation by } (1,-1,0)) \circ$$
$$(\text{rotation by } \pi \text{ about line passing through } (1,0,0) \text{ and } (0,1,0))$$
$$f_2 : \{x_1 = -1\} \to \{x_2 = -1\} = (\text{translation by } (1,-1,0)) \circ$$
$$(\text{rotation by } \pi \text{ about line passing through } (-1,0,0) \text{ and } (0,-1,0))$$
$$f_3 : \{x_3 = -1\} \to \{x_3 = 1\} = (\text{translation by } (2,0,0)) \circ$$
$$(\text{rotation by } \pi \text{ about the } z\text{-axis}).$$

The square $[-1,1] \times [-1,1] \times \{0\}$ projects down to a totally geodesic embedded Klein bottle $K \subset \mathbb{R}^3/\mathcal{G}_6$, as can be seen by applying Theorem 7.4. While incompressible, this surface is not 2-sided. A regular neighborhood W of K is the image of $[-1,1] \times [-1,1] \times [-\epsilon, \epsilon]$ under identification. Its boundary is the torus ∂W which is 2-sided. It is also incompressible, because the composite of the double covering $T^2 \to K$ with the embedding $K \hookrightarrow \mathbb{R}^3/\mathcal{G}_6$ is injective on fundamental groups and is homotopic to the embedding $T^2 \xrightarrow{\cong} \partial W \hookrightarrow \mathbb{R}^3/\mathcal{G}_6$. $\square$

5. Two noncompact finite-volume hyperbolic 4-manifolds

In this section we construct two noncompact hyperbolic 4-manifolds by using Poincaré's polyhedron theorem.

We start with a polyhedron P in $\mathbb{H}^4$ that is the intersection of finitely many half-spaces. Recall that a hyperplane in $\mathbb{H}^4$ is determined by its boundary in $\mathbb{R}^3$ — these will be either 2-planes or 2-spheres in $\mathbb{R}^3$.

Consider the planes that bound the rectangular box $R \subset \mathbb{R}^3$, $R = [-2,2] \times [-2,2] \times [-2\sqrt{2}, 2\sqrt{2}]$. Add to them the 12 spheres of radius $\sqrt{2}$ with centers $(\pm 1, \pm 1, j2\sqrt{2})$ for $j = -1, 0, 1$ and the 18 spheres of the same radius with centers $(j, k, \pm\sqrt{2})$ for $j, k = -2, 0, 2$. The upper part of figure 2, going from left to right, depicts intersections of these spheres with planes with constant z-coordinates $-2\sqrt{2}$, $-\sqrt{2}$, 0, $\sqrt{2}$, $2\sqrt{2}$. Label the spheres by the letters A_i, A'_i, B_i, B'_i, C_i, C'_i, D_i, D'_i in either of the ways suggested by figure 2. Let X_1, X'_1, Y_1, Y'_1, Z_1, Z'_1 be respectively the planes $\{x = -2\}$, $\{x = 2\}$, $\{y = -2\}$, $\{y = 2\}$, $\{z = -2\sqrt{2}\}$, $\{z = 2\sqrt{2}\}$.

Each of the planes that comprise the boundary of R and each of the above spheres determine a hyperplane in $\mathbb{H}^4 = \{(x, y, z, t) \in \mathbb{R}^4 \mid t > 0\}$ that divides $\mathbb{H}^4$ into two half-spaces. These hyperplanes will be denoted by the same letters as their boundaries at infinity. For the spheres we choose the half-spaces whose boundary at infinity is unbounded in $\mathbb{R}^3$, for the planes the half-spaces so that the intersection of their boundaries at infinity is the rectangular box R. The polyhedron P is defined as the intersection of those half-spaces. P has only one side on each of the defining hyperplanes, which we denote by the same letter as the hyperplane. For later convenience, we set $P_- = \{(x,y,z,t) \in P \mid z \leq 0\}$, $P_+ = \{(x,y,z,t) \in P \mid z \geq 0\}$.

The following observations about the spheres and planes that we just defined are easy to check.

(1) Any two spheres that intersect do so at an angle of $\pi/2$.

(2) Whenever the intersection is nonempty, spheres A_i, A_i' intersect planes X_1, X_1', Y_1, Y_1' at angle $\pi/4$. Any other pair of spheres or planes with nonempty intersection intersects at angle $\pi/2$.

(3) R is completely covered by the closed balls bounded by the spheres. This means that P has finite volume and has only finitely many points in its boundary at infinity.

(4) P has 36 vertices at infinity, which correspond to points not covered by the open balls. Their position is illustrated in Figure 9.

It is not obvious right away that P also has finite vertices. This is because many sets of four hyperplanes bounding the polyhedron P meet at one point. For example, sides A_1, B_1, C_1 and A_2' meet at the point $(0, 1, -3\sqrt{2}/2, \sqrt{2}/2) \in \mathbb{H}^4$, and sides A_1, C_1, D_1 and Y_1' meet at the point $(-1, 2, -3\sqrt{2}/2, \sqrt{2}/2)$. Figure 1 depicts the section of P where $t = \sqrt{2}/2$ and $z = j\sqrt{2}/2$. Here j is any of -3, -1, 1, 3 as the section for every j is the same. We can see where four sides of P

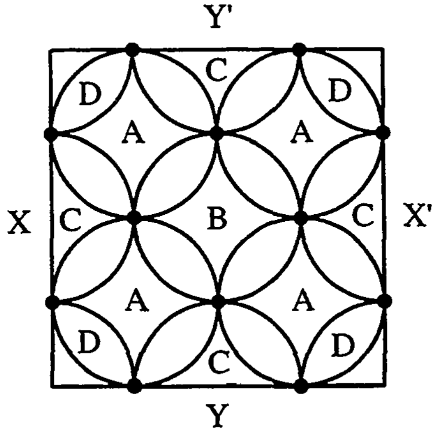

FIGURE 1. Section of P for $t = \sqrt{2}/2$, $z = \sqrt{2}/2$ showing the real vertices

intersect in a vertex and what letters those sides are labeled by. Figure 1 shows the location of all the vertices in one section — there being $4 \cdot 12 = 48$ in all.

Now we are ready to define two ways to pair sides of P. First, define the following isometries of $\mathbb{R}^3$:

$$q_0 = \text{reflection in plane } \{z = 0\}$$

$$q_1 = \text{reflection in plane } \{x - y = 0\}$$

$$q_2 = \text{reflection in plane } \{x + y = 0\}$$

$$s_1 = \text{rotation by } \pi \text{ about line } \{x + y = 0, z = 0\}$$

$$s_2 = \text{rotation by } \pi \text{ about line } \{x - y = 0, z = 0\}$$

$$t_0 = \text{translation by } 2\sqrt{2} \text{ in the } z \text{ direction.}$$

Use the same letters to denote the extensions of these maps to $\mathbb{H}^4$. (A Euclidean isometry $f : \mathbb{R}^3 \to \mathbb{R}^3$ extends to a hyperbolic isometry given by $(x, y, z, t) \mapsto (f(x, y, z), t)$.)

Let i_S denote the reflection in the hyperplane S. By s we denote the hyperbolic isometry that pairs the sides S and S' (it sends S to S'). We define Φ_1 to be the side-pairing given by

$$x_1 = \text{translation by 4 in the } x \text{ direction}$$

$$y_1 = \text{translation by 4 in the } y \text{ direction}$$

$$z_1 = t_0^2 = \text{translation by } 4\sqrt{2} \text{ in the } z \text{ direction}$$

$$b_1 = q_1 \circ t_0 \circ i_{B_1}$$

$$a_j = q_l \circ i_{A_j}, \text{ so that } l \equiv j \ (mod\ 2)$$

$$c_k = q_0 \circ i_{C_k}$$

$$d_k = q_1 \circ t_0 \circ i_{D_k},$$

where $j = 1, \ldots, 6,\ k = 1, \ldots, 4$ and $l = 1, 2$. The upper half of Figure 2 shows which sides are paired.

To get another side-pairing, Φ_2, we alter Φ_1 in the way the sides labeled by B's, C's and D's are paired. Refer to the lower half of Figure 2 to see which sides are paired. We define the new pairings b_1, c_k and d_k by

$$b_1 = q_0 \circ i_{B_1}$$

$$c_k = q_1 \circ t_0 \circ i_{C_k}$$

$$d_k = s_l \circ q_l \circ i_{D_k}, \text{ so that } l \equiv k \ (mod\ 2),$$

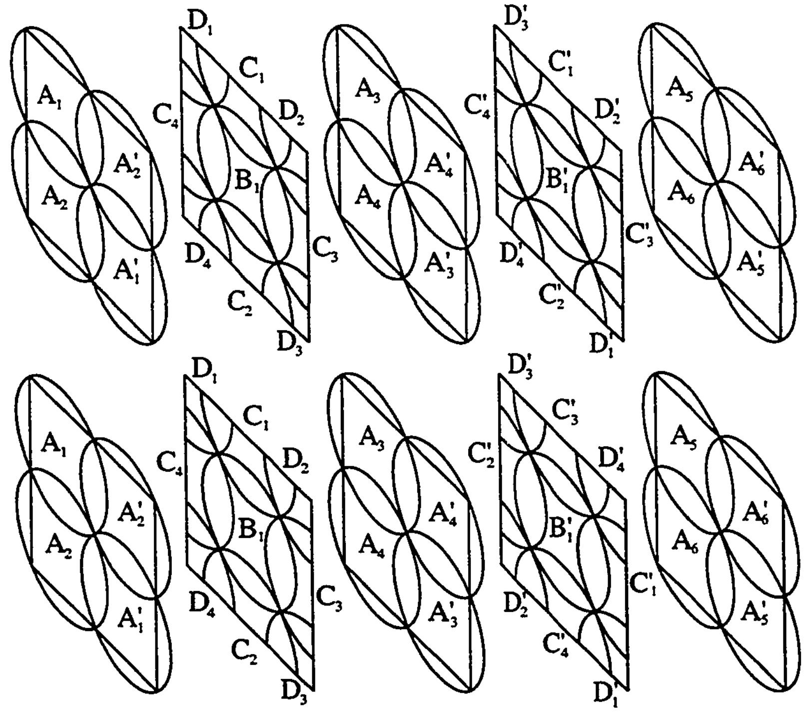

FIGURE 2. Side-pairings Φ_1 (top) and Φ_2 (bottom)

where $k = 1, \ldots, 4$ and $l = 1, 2$.

Next, we prove

Theorem 5.1.

(i) *The side-pairings Φ_1 and Φ_2 generate discrete torsion-free subgroups G_1 and G_2 of* $\operatorname{Isom} \mathbb{H}^4$ *whose fundamental polyhedron is P. Therefore, the quotient of $\mathbb{H}^4$ by the action of either of the groups is a complete hyper-*

bolic 4-manifold.

(ii) $\mathbb{H}^4/G_1$ *has seven ends while* $\mathbb{H}^4/G_2$ *has eight. In particular, the two manifolds are not homeomorphic.*

Proof. We use Theorems 3.1 and 3.2. First of all, the maps defined above really do map a side of P isometrically onto a side of P. To verify this for Φ_1, notice that each of its side-pairings $a_1, \ldots, a_6, c_1, \ldots, c_4$, is of the form $f \circ i_S$ where S is a side and f, *which preserves* P, is an extension to $\mathbb{H}^4$ of a Euclidean transformation on $\mathbb{R}^3$. The i_S keeps S fixed so $f \circ i_S(S) = f(S)$, and f, being an isometry of P, sends its sides to some other sides. We proceed similarly for the side-pairings $b_1, d_1, \ldots, d_4$: each of them is of the form $f \circ i_S$, but f is now an isometry that takes P_- to P_+, so it sends sides of P_- to sides of P_+. But the sides $B_1, D_1, \ldots, D_4$ are sides of both P_- and P, and, likewise, the sides $B_1', D_1', \ldots, D_4'$ are sides of both P_+ and P. For the side-pairing x_1 (y_1 and z_1 are done similarly), compose it with a reflection in the side X_1' to get an isometry of P whose image of X_1 is the same as by x_1. Therefore x_1 carries X_1 to another side of P, namely X_1'. The claim is proved in the same way for the side-pairing Φ_2.

Remark 5.2. Consider a set T of disjoint horospheres, each centered at a vertex at infinity of P. For the vertex ∞ choose, say, the horizontal plane $\{t = 3\}$. For the other vertices choose horospheres of the same radius that is small enough so that the horospheres intersect only those sides of P which contain the center of the horosphere. If the center of a horosphere is on the boundary of a hyperplane

S, then i_S preserves the horosphere. This combined with an argument like in the preceding paragraph can be used to show that the side pairings Φ_1 and Φ_2 satisfy the consistent horosphere condition.

Now we check the edge cycle condition for edges of P and the two side-pairings Φ_1 and Φ_2. With notation as above, we will always have $k = 1$ and $m = 1$. Hence, the second cycle length will always be the same as the first cycle length and they will be 4 and 8 respectively for edges with dihedral angles $\pi/2$ and $\pi/4$. Therefore, the sum of dihedral angles will be exactly 2π.

Firstly, we make sure that all edges are in cycles of said length. For all edges that are intersections of sides based on planes (i.e. "vertical" sides), this check is easy and boils down to checking the conditions of Poincaré's polyhedron theorem for a rectangular Euclidean parallelepiped with parallel sides paired by Euclidean translations. The check for any of the edges of type $A_i \cap Z_1$ is also straightforward. For all the other edges, we use the diagrams in Figures 4 and 5 to simplify and visualize the task of verifying.

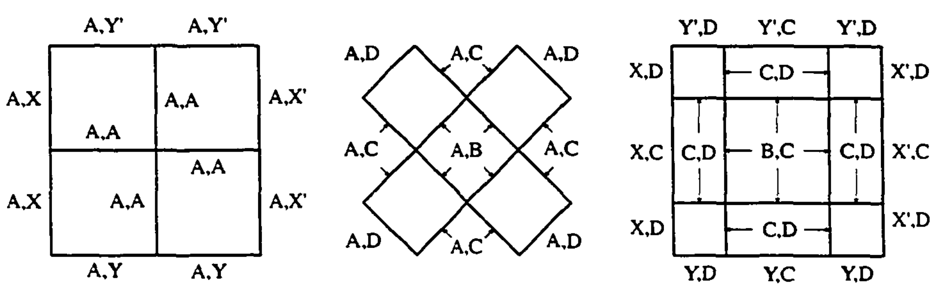

FIGURE 3. Representing edges in P by diagrams

Notice that the intersection of two hyperbolic hyperplanes is a codimension-2 hyperbolic subspace whose boundary at infinity is the intersection of the boundaries at infinity of the hyperplanes. Therefore, every edge E of P lies on a codimension-2 subspace determined either by the intersection of a sphere and a plane or by the intersection of two spheres in $\mathbb{R}^3$. (The spheres and planes are the boundaries at infinity of the sides that determine E.) These intersections are circles and they are represented by segments in Figure 3. The letters next to each segment indicate which side-types have generated the edge represented by it. To get the left and right diagrams we take intersections of planes and spheres labeled by the letters in the diagrams and then project them to the plane $\{z = 0\}$. For the middle diagram, we first project to the plane $\{z = 0\}$ the centers of those pairs of spheres whose labels are listed in it. Then we take the perpendicular bisector of the line joining those centers. To account for all the edges, we need several of these diagrams (Figure 4). Edge chasing for Φ_1 and Φ_2 is now performed on Figures 4 and 5 respectively. All edges in one cycle in each horizontal component of the pictures are labeled with the same letter. Some are not labeled because their cycles are similar to other labeled cycles. Also, Figure 5 omits some of the edges because their cycles are the same as for Φ_1.

For example, choose the edge $A_1 \cap A_2$. The edge chase, yielding the cycle labeled o in upper part of Figure 4 is

$$A_1 \cap A_2 \xrightarrow{a_2} A_2' \cap A_1 \xrightarrow{a_1} A_1' \cap A_2' \xrightarrow{a_2^{-1}} A_2 \cap A_1' \xrightarrow{a_1^{-1}} A_1 \cap A_2.$$

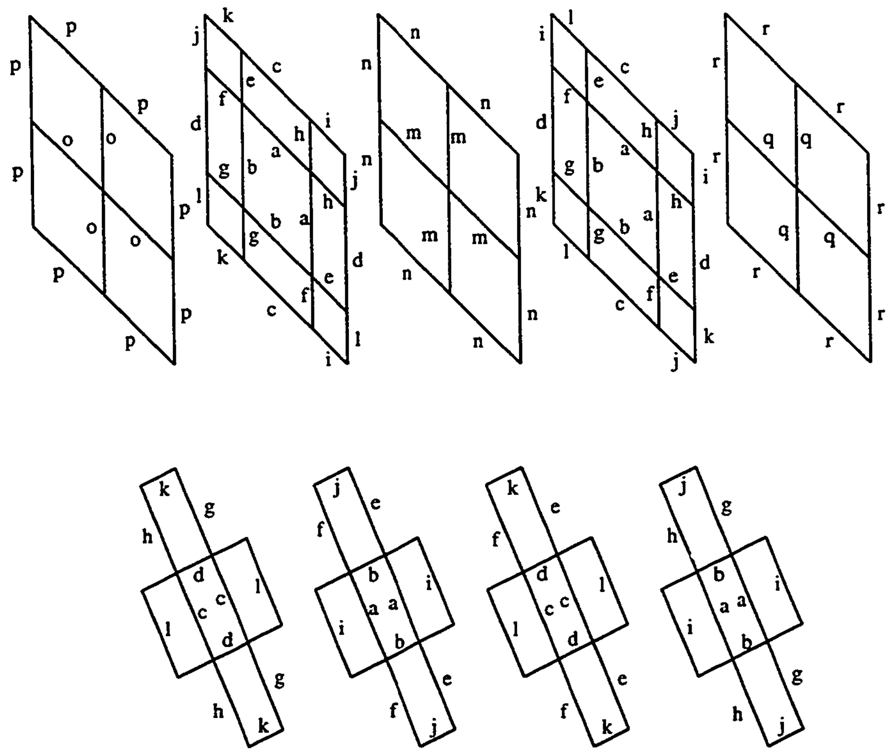

FIGURE 4. Edge chase for Φ_1

As another example, choose the edge $A_1 \cap D_1$. The edge chase, yielding the cycle labeled k in lower part of Figure 4 is

$$A_1 \cap D_1 \xrightarrow{d_1} D_1' \cap A_3' \xrightarrow{a_3^{-1}} A_3 \cap D_3' \xrightarrow{d_3^{-1}} D_3 \cap A_1' \xrightarrow{a_1^{-1}} A_1 \cap D_1.$$

Next, we check that for transformations $g_1, \ldots, g_q$ obtained by edge-chasing we have $g_q \circ \cdots \circ g_1|_{E_1} = 1$. As before, $g_i = f_i \circ r_i$, where f_i is the extension to $\mathbb{H}^4$ of a Euclidean transformation on $\mathbb{R}^3$ and r_i is either a reflection in a hyperplane containing E_i or the identity. Let $f = f_q \circ \cdots \circ f_1$. It is not difficult to see that f is always orientation preserving. Clearly $g_q \circ \cdots \circ g_1|_{E_1} = f|_{E_1}$, and it will be

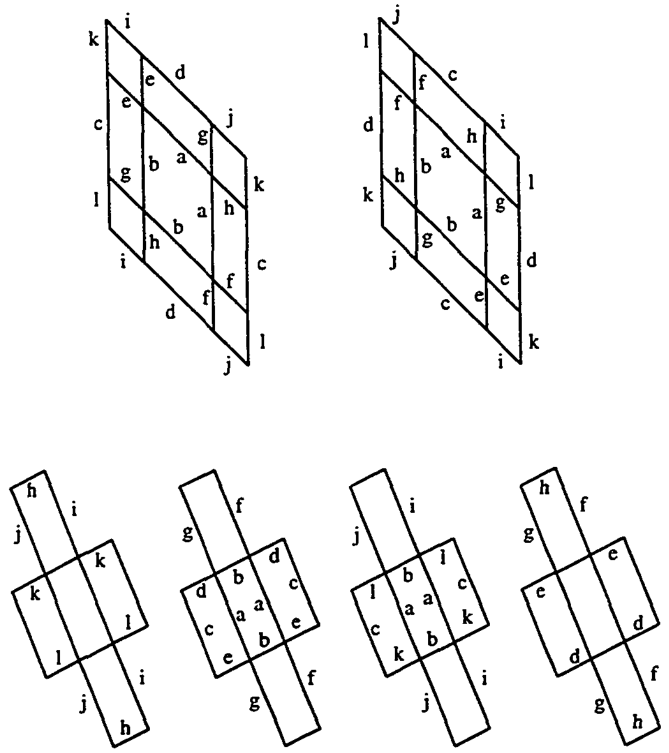

FIGURE 5. Edge chase for Φ_2

enough to show that $f = 1$. We will need the following easy lemma.

Lemma 5.3. *Let f be a nontrivial orientation-preserving Euclidean isometry of $\mathbb{R}^3$ that preserves a circle. Then it is a rotation about a line called the* axis *of f. Moreover, we have:*

(i) *If we write f as $Ux + u$, where U is an orthogonal transformation and $u \in \mathbb{R}^3$ then the axis of f is parallel to the axis of U.*

(ii) *The axis of f passes through the center of the circle and is either in the plane of the circle, or perpendicular to it. In the first case, the rotation is by angle π.*

(iii) *If f preserves a line l, then its axis either orthogonally intersects l or it is that line.* $\square$

Let $f = f_q \circ \cdots \circ f_1$ as above and suppose it is nontrivial. In what follows we interchangeably view f as a Euclidean isometry on $\mathbb{R}^3$ or as a hyperbolic isometry of $\mathbb{H}^4$.

We know that f preserves the circle that is the base of the edge $E = E_1$. Also, f preserves the family of planes $V = \{(x, y, z) \in \mathbb{R}^3 \mid x = 4k + 2 \text{ or } y = 4k+2,\ k \in \mathbb{Z}\}$ because it is a composite of maps from $\{x_1, y_1, z_1, q_0, q_1, q_2, s_1, s_2\}$. The rotational part U of f is a composite of maps from $\{q_0, q_1, q_2, s_1, s_2\}$, each of which preserves the axes l_1 and l_2 of s_1 and s_2, so U preserves them too. Now, looking at possible positions of the circle we get one of the following cases:

Case 1. When E is one of the sides represented in the lower half of Figure 4 then, by (i) and (ii) of Lemma 5.3, the axis l of U must either have direction vector $(\pm 1, \pm 1, \sqrt{2})$ or is in the plane perpendicular to that vector. Since U preserves l_1 and l_2, by part (iii) of the lemma, both of l_1 and l_2 must be either perpendicular or identical to l. If $l \perp l_1$ and $l \perp l_2$, then l is the z-axis, which contradicts the possible positions of l. If l is equal to either l_1 or l_2, then one can see that the axis of f is going to pass exactly through the segment that represents E in the middle diagram of Figure 3. However, it is clear that no rotation about these segments can preserve the family of planes V so we must have $f = 1$.

Case 2. When E is one of the sides represented in the upper half of Figure 4 the axis l of U lies, by parts (i) and (ii) of Lemma 5.3, in one of the the planes $\{x = 0\}$ or $\{y = 0\}$ or it is the x- or y-axis. Clearly $l \neq l_1$ and $l \neq l_2$, so applying part (iii) of Lemma 5.3 again we get $l \perp l_1$ and $l \perp l_2$, which means l is the z-axis. As long as E is not of the forms $C_i \cap X_1$, $C_i \cap Y_1$, $D_i \cap X_1$ or $D_i \cap Y_1$ the center of the circle on which E is based has odd x and y coordinates. However, a rotation about an axis parallel to z through such points cannot preserve the family of planes V and we again get $f = 1$.

Case 3. In the remaining cases, if E is of form $D_i \cap X_1$ or $D_i \cap Y_1$, regard f as an isometry of $\mathbb{H}^4$ and examine its action on vertices of P that are on E. Looking at Figure 1, it is clear that f, being by the above a rotation about an axis parallel to z, must send vertices that are on E to points whose either x or y coordinate falls out of $[-2, 2]$, a contradiction with the fact that f preserves E. Finally, if E is of form $C_i \cap X_1$ or $C_i \cap Y_1$ we just compute f: it is always $q_0^2 = 1$ for the side-pairing Φ_1 and it is always $q_1^2 = 1$ for the side-pairing Φ_2.

Thus, we have shown that $f = 1$ in all possible cases and the edge cycle condition has been verified.

Since the consistent horosphere condition is fulfilled by Remark 5.2, we may apply theorem 3.1 to get that the groups G_1 and G_2 generated by the side-pairings Φ_1 and Φ_2 are discrete, and that P is the fundamental polyhedron for both of them. What we do not yet know is whether G_1 and G_2 are torsion-free, that is, whether $\mathbb{H}^4/G_i$, $i = 1, 2$ are hyperbolic manifolds and not just orbifolds.

The check of torsion-freeness utilizes Theorem 3.2 and Remark 3.3. For an $x \in P$ that is in the interior of 3 or 4-sides of P, it is clear that $\omega[x] = 1$. For an x in the interior of 2-sides, this follows from the edge-cycle condition. This leaves 0- and 1-sides to be checked.

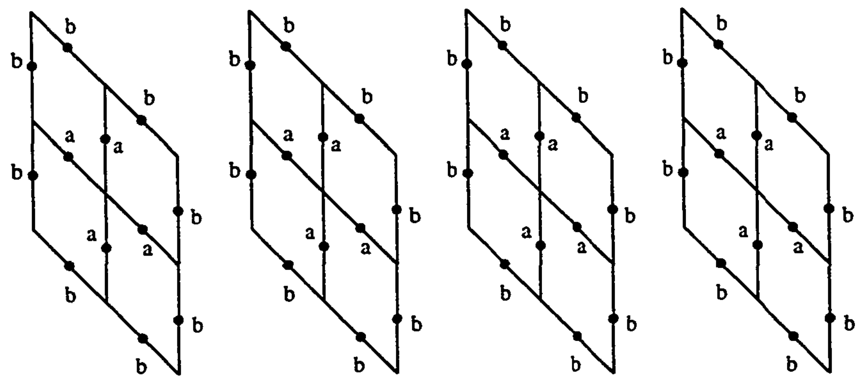

FIGURE 6. Cycles of real vertices

Figure 6 shows the sections of P for $t = \sqrt{2}/2$ and $z = -3\sqrt{2}/2$, $-\sqrt{2}/2$, $\sqrt{2}/2$, $3\sqrt{2}/2$. These sections contain all the vertices (0-sides) of P. All vertices in the same cycle (there are only two cycles) are labeled by the same letter — the cycles are the same for both Φ_1 and Φ_2. Vertices in the cycle labeled a occur as the intersection of four hyperplanes, each pair of which meets at angle $\pi/2$. Normalize P so that a vertex x from cycle a is the origin in the ball model $\mathbb{B}^4$ of hyperbolic space. We see that the normalized solid angle at x is the same as the normalized solid angle at $0 \in \mathbb{B}^4$ subtended by the four coordinate planes, and that is $1/16$. Vertices in the cycle labeled b are always intersections of four hyperplanes where one pair of them intersects at angle $\pi/4$ and all other pairs intersect at angle $\pi/2$. Normalizing as before, we see that the normalized solid angle at x is the same as the one at 0 subtended by the hyperplanes $\{x_2 = 0\}$,

$\{x_3 = 0\}$, $\{x_4 = 0\}$, $\{x_1 - x_2 = 0\}$, and that is $1/32$. Since cycles a and b contain 16 and 32 points respectively, we are done.

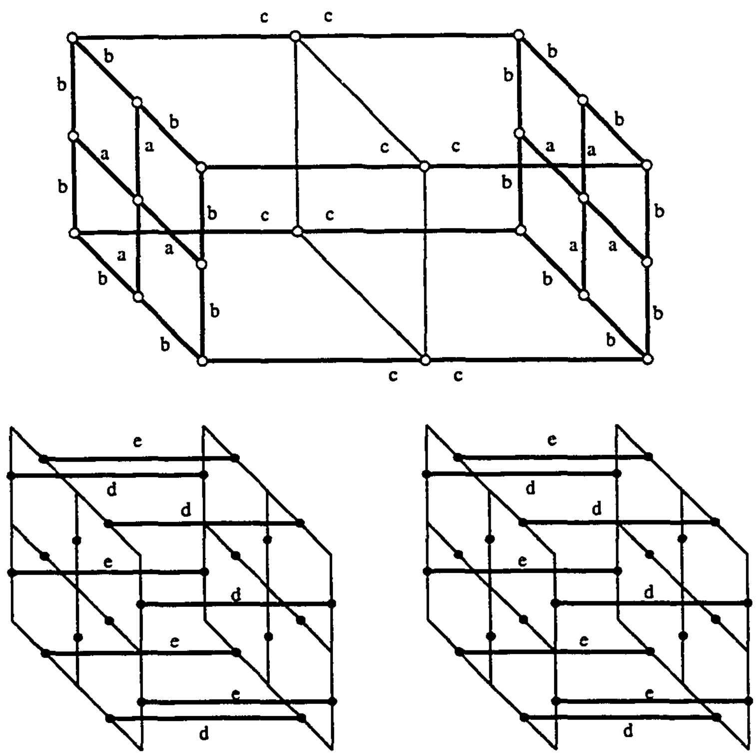

FIGURE 7. Cycles of some 1-sides for Φ_1

Now consider 1-sides. There are three cases depending on whether a 1-side F connects an ideal and a real vertex, two ideal vertices, or two real vertices.

For the first case, let F be a 1-side of P that is a geodesic half-line between one real and one ideal vertex of P. If for some $x \in \operatorname{int} F$ we have $\omega[x] < 1$, then x is a fixed point of some nontrivial $g \in G$. The isometry g must preserve F — otherwise, we would have $g(F) \cap F = \{x\}$ and this contradicts the fact that translates of P meet only along i-sides. However, this implies that the real vertex on F is fixed under g, a possibility we just proved cannot happen. So, we are left with checking 1-sides that have as endpoints either both real or both ideal vertices of P.

Every 1-side is the intersection of three different sides of P. Hence, to find all 1-sides with both endpoints real or ideal, we have to find pairs of real or ideal vertices lying on the same three sides. Figure 7 schematically depicts those 1-sides of P. A boldface line segment joining the real or ideal vertices indicates the existence of a 1-side joining them. The three sides on which the 1-sides lie are easily deduced from their position in the picture. (For example, the 1-sides labeled c are intersections of sides labeled by D's, X's and Y's.) As before, the letters on the 1-sides indicate to which cycle of 1-sides they belong.

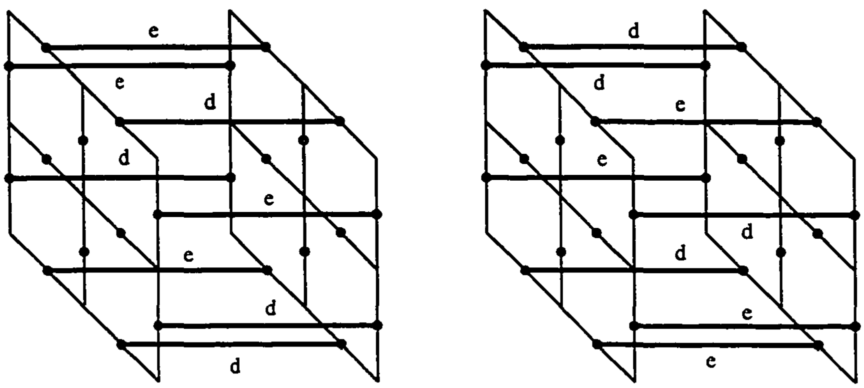

FIGURE 8. Cycles of some 1-sides for Φ_2

It takes a bit of checking to see that we have found all the needed 1-sides. For example, to see that no 1-side joins an ideal vertex in the plane $\{z = 0\}$ to an ideal vertex in the plane $\{z = \sqrt{2}\}$ we note that every vertex in the plane

$\{z = 0\}$ lies on only one of the B, C or D-sides, and some X, Y, Z or A-sides, while every vertex in the plane $\{z = \sqrt{2}\}$ lies on only one of the A-sides and some number of B, C or D-sides. Therefore, any pair of vertices from those two planes cannot belong to the same three sides.

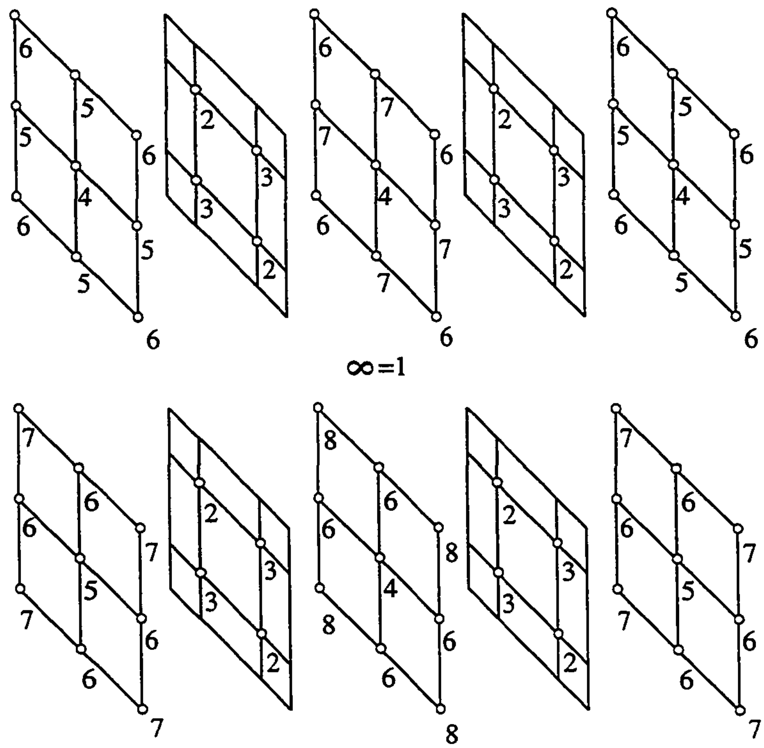

FIGURE 9. Cycles of ideal vertices for Φ_1 (top) and Φ_2 (bottom). In both cases, the vertex ∞ is labeled by 1

The 1-sides in the cycle labeled a are intersections of three sides meeting pairwise at angles $\pi/2$ (two sides labeled by A and one by Z). Taking an x from a side in the cycle, and normalizing in $\mathbb{B}^4$ so that $x = 0$ and the three sides are the first three coordinate hyperplanes, we see that $\omega(x) = 1/8$. Since there are 8 1-sides in the cycle, we get $\omega[x] \geq 1$. Other cycles are checked in the same way. For an x on a 1-side in the cycles b, c, d, e we get normalized solid angles of respectively 1/16, 1/8, 1/8, 1/8 with 16, 8, 8, 8 1-sides in the cycle, so $\omega[x] \geq 1$. Thus, we have proved that side-pairings Φ_1 and Φ_2 give rise to hyperbolic 4-manifolds.

Assertion (ii) of Theorem 5.1 is verified by counting cycles of ideal vertices for Φ_1 and Φ_2. The cycles are shown in Figure 9. $\square$

Thus, we have produced two finite-volume hyperbolic 4-manifolds. Before these constructions, the examples of hyperbolic manifolds with dimension higher than three were restricted to constructions via arithmetic groups (see, for example, [A2], [Mi]), or via "interbreeding" arithmetic groups to get nonarithmetic ones ([G-P]) and there was only one (compact) example using side-pairings, that of Davis in [D]. In the meantime, other constructions in dimension 4 using side-pairings have been done (see [N] and [R-T]) and they have been used to prove interesting theorems.

6. Hyperbolic 4-manifolds sharing a fundamental polyhedron. The volume function

In this section we use the construction of the previous section to build more complicated noncompact hyperbolic 4-manifolds which will allow us to prove two theorems.

Choose a positive integer n, and let $Q_1, \ldots, Q_n$ be n copies of the polyhedron P. Call each of them a *block*. To each block, assign either of the side-pairings Φ_1 or Φ_2 and call it a block *of type* Φ_1 or Φ_2. Now form a new polyhedron Q by attaching side Z_1 of Q_i to side Z_1' of Q_{i-1}, $i = 2, \ldots, n$, i.e. by stringing the blocks together in a linear fashion in the direction of the z-axis. If the rectangle R and the spheres defined in §5 are thought to represent the polyhedron P, then all the translates of R and the spheres by $4\sqrt{2}j$ in the z-direction, $j = 1, \ldots, n-1$ will represent the polyhedron Q.

Q has on it a side-pairing induced by the side-pairings on each block. Clearly, the sides that were attached have vanished, so they don't fall under this rule: the remaining sides Z_1 of Q_1 and Z_1' of Q_n are paired by the translation t^{2n}. A moment's reflection convinces us that the side-pairing on Q generates a torsion-free group G — the proof is basically an n-fold repetition of the proof of the same result for the side-pairings Φ_1 and Φ_2. Except for i-sides, $i = 0, 1, 2$, that are contained in the sides Z_1' of Q_1 and Z_1 of Q_n, all the cycles are just inherited cycles from pairings on each block. The special cases are easily dealt with — they follow patterns established for Φ_1 and Φ_2.

How many cycles of vertices at infinity does Q with the side-pairing just defined have? Let there be k blocks of type Φ_1 and $n-k$ blocks of type Φ_2 among $Q_1, \dots, Q_n$. Starting with $Q_1, \dots, Q_l$, adding a block of type Φ_1 onto Q_l will add three new cycles of vertices at infinity: there are seven cycles on Φ_1 but four fall into cycles already existing on $Q_1, \dots, Q_l$. Similarly, adding a block of type Φ_2 adds four new cycles of vertices at infinity. It is now easy to see that Q will have $4 + 3k + 4(n-k) = 4 + 4n - k$ cycles of vertices at infinity. We have complete freedom of choice for k, so by varying k from 0 to n we can get manifolds with anywhere from $4 + 3n$ to $4 + 4n$ ends. Therefore, we have obtained at least $n+1$ nonhomeomorphic manifolds with the same fundamental polyhedron Q. (We likely get many more, since we completely ignored the various orderings of blocks of the two types that are possible when constructing Q.) This yields

Theorem 6.1. *Given any number N, there exist more than N nonhomeomorphic, noncompact, complete hyperbolic 4-manifolds of finite volume that share the same fundamental polyhedron in $\mathbb{H}^4$. In particular, they have the same volume.*

The last claim of the theorem is especially interesting in light of Wang's theorem (Theorem 2.2). Its corollary is that only finitely many different manifolds may have the same volume — our theorem shows that there is no bound on the number of manifolds having the same volume. In dimension 3, the same theorem has been proven by Wielenberg (see [Wi2]) for the noncompact case, and by Apanasov and Gutsul ([A-G]) for the compact one. In both papers, for N's that can be made arbitrarily large, polyhedra are constructed in $\mathbb{H}^3$ and different side-pairings are given on them whose quotient spaces are N nonhomeomorphic hyperbolic manifolds.

To get volume of the manifolds obtained from P using either of the side-pairings Φ_1 or Φ_2, we just need to, by the Gauss-Bonnet formula (Theorem 2.3), compute their Euler characteristic . (Since they share a fundamental polyhedron, they have the same volume, so it is enough to compute the Euler characteristic for one of them.)

First, we prove a lemma to assist us in finding the Euler characteristic.

Recall that for a finite CW-complex X, $\chi(X)$ may be computed either as an alternating sum of the numbers of i-cells in X, or as the alternating sum of the ranks of the i-th homology groups of X.

Let P be a finite-sided n-dimensional hyperbolic polyhedron with a side-pairing defined on it. Now let X be the CW-complex obtained from P in the obvious way, with 0-cells the real and ideal vertices of P, 1-cells the 1-sides of P together with their points at infinity, and so on. Then X inherits identifications by side-pairings of P, which give rise to a quotient space Y, also a CW-complex (even if P yields a manifold by identification, Y will not be one). We then have

Lemma 6.2. *If M is obtained from a side-pairing of P and Y is as above, then*

$$\chi(M) = \chi(Y) - \textit{number of ends of } M.$$

In other words, we may compute $\chi(M)$ directly from the polyhedron by taking the alternating sum of numbers of cycles of i-sides and ignoring the cycles of ideal vertices.

Proof. We see that $Y = \overline{M} \cup V$, where V is a disjoint union of cones over Euclidean manifolds that are the boundary of $\overline{M}$, so that $\overline{M} \cap V$ is a disjoint union of Euclidean manifolds. Consider the absolute Mayer–Vietoris sequence for M and V (see [Do, Proposition 8.15]). Enumerate the terms so that the k-th homologies of $M \cup V$, the sum of M and V, and $M \cap V$ correspond to indices $3k$, $3k+1$ and $3k+2$ respectively ($k \geq 0$). Let c_j and z_j denote respectively the rank of the j-th term and the rank of the kernel of the homomorphism joining the j-th and the $(j-1)$-st terms of the sequence. By exactness of that sequence we have $c_j = z_{j-1} + z_j$. Use this equality to see that $\sum(-1)^k c_{3k} - \sum(-1)^k c_{3k+1} + \sum(-1)^k c_{3k+2} = 0$, which is exactly $\chi(\overline{M} \cup V) - (\chi(\overline{M}) + \chi(V)) + \chi(\overline{M} \cap V) = 0$. Since V is contractible, $\chi(V)$ =number of components of V=number of of boundary components of $\overline{M}$. The fact that $\overline{M} \cap V$ is a disjoint union of Euclidean manifolds implies $\chi(\overline{M} \cap V) = 0$ which yields the desired formula. $\square$

Now we just have to count cycles of i-sides for Φ_1. The polyhedron P from §5 has one 4-side, 36 3-sides, 168 2-sides, 216 1-sides and 48 real 0-sides. Each 3-side is paired to exactly one other one, which yields 18 cycles of 3-sides. Among 2-sides, there are 24 with dihedral angle $\pi/4$, giving 3 cycles, and 144 with dihedral angle $\pi/2$, yielding 36 cycles. Among 1-sides, there are 80 with normalized solid angle $1/16$ giving 5 cycles and 136 with normalized solid angle $1/8$, yielding 17 cycles. From Figure 6 we know there are 2 cycles of 0-vertices. Thus, $\chi(P/\Phi_1) = 2 - (17+5) + (36+3) - 18 + 1 = 2$, so $\mathrm{Vol}(P/\Phi_1) = 2 \cdot 4\pi^2/3$.

Remark 6.3. Note that the same reasoning as in the above paragraph may be used to see that, when *the side-pairing of an n-polyhedron P yields a manifold M*, $\chi(M)$ depends only on the alternating sum of normalized solid angles of P over all the i-sides of P. In particular, it doesn't depend on the side-pairing of P. Indeed, the sum of normalized solid angles for each cycle of an i-side is exactly 1, which is contributed to the count of cycles of i-sides.

Now we can easily compute the volume the manifold obtained from any of the side-pairings of Q described above. The volume of such a manifold is the same as $\mathrm{Vol}\, Q$, and $\mathrm{Vol}\, Q = n \cdot \mathrm{Vol}\, P = 2n \cdot 4\pi^2/3$. This gives us

Theorem 6.4. *The set of all volumes of hyperbolic 4-manifolds contains the even multiples of* $4\pi^2/3$.

As remarked in the introduction, Ratcliffe and Tschantz have shown that the set of all volumes of hyperbolic 4-manifolds is the set of all multiples of $4\pi^2/3$. Using side-pairings on a certain polyhedron, they first construct a hyperbolic manifold M with Euler characteristic 1 and show that its first homology group has nonzero first Betti number. This implies that it has coverings of arbitrary index i, whose volume is, of course, $i \cdot \mathrm{Vol}(M)$.

It is interesting to consider the function $V(n)$ = the number of hyperbolic

4-manifolds with volume $n \cdot 4\pi^2/3$. The construction at the beginning of this section gives $V(2n) \geq n+1$. The count that we used was pretty rough — there are probably many more manifolds obtained from Q by the various side-pairings than $n+1$. We get 2^n examples, but cyclic permutations of an assignment of pairings Φ_1 and Φ_2 to the n blocks will clearly yield the same manifold. It may yet turn out that $V(2n) \geq 2^n/n$.

7. A GEOMETRIC INTERPRETATION OF THE CONSTRUCTION

In this section we analyze the construction of the manifolds in the previous section from a gluing-of-manifolds perspective. In the process, we prove a convenient theorem along the lines of Poincaré's polyhedron theorem, telling us when a hyperplane that intersects a fundamental polyhedron projects down to an embedded totally geodesic surface.

Let $M = \mathbb{H}^n/G$, where G is a discrete torsion-free subgroup of $\operatorname{Isom}\mathbb{H}^n$. A *totally geodesic hypersurface* is a subset $N \subset M$ so that for every $x, y \in N$ every geodesic connecting x and y is also contained in N. We are interested in *embedded* totally geodesic hypersurfaces which are the ones for which $p^{-1}(N)$ is a disjoint union of hyperplanes in $\mathbb{H}^n$, where $p : \mathbb{H}^n \to M$ is the standard projection. Let H be one of those hyperplanes, and $J \subset G$ its stabilizer in G, that is the subgroup $J = G_H = \{j \in G \mid j(H) = H\}$. Then H is *precisely invariant under* J, i.e. $g(H) = H$ when $g \in J$ and $g(H) \cap H = \emptyset$ when $g \in G \setminus J$. In particular, we want to look at some hypersurfaces with $\operatorname{Vol} N < \infty$, so they will correspond to subgroups $G_H \subset G$ that act on a hyperplane H as a hyperbolic lattice.

Conversely, we may start with a subgroup $J \subset G$ and a hyperplane H precisely invariant under J. Then H/J is an embedded totally geodesic hypersurface in M.

Let $M_i = \mathbb{H}^4/G_i$, where G_i is generated by the side-pairing Φ_i, $i = 1, 2$, and let H be the supporting hyperplane of side Z_1 of the polyhedron P. Suppose we know that H is precisely invariant under the subgroup $J \subset G_1$, $J = \langle a_1, a_2, x_1, x_2 \rangle$. Then H/J is a two-sided totally geodesic hypersurface in M_1 and we may cut M_1 along this hypersurface to get a connected manifold M_1' that has two boundary components which are isometric 3-dimensional hyperbolic manifolds given by H/J. We may do the same with M_2 to get M_2' which is also connected (the subgroup J in question doesn't change).

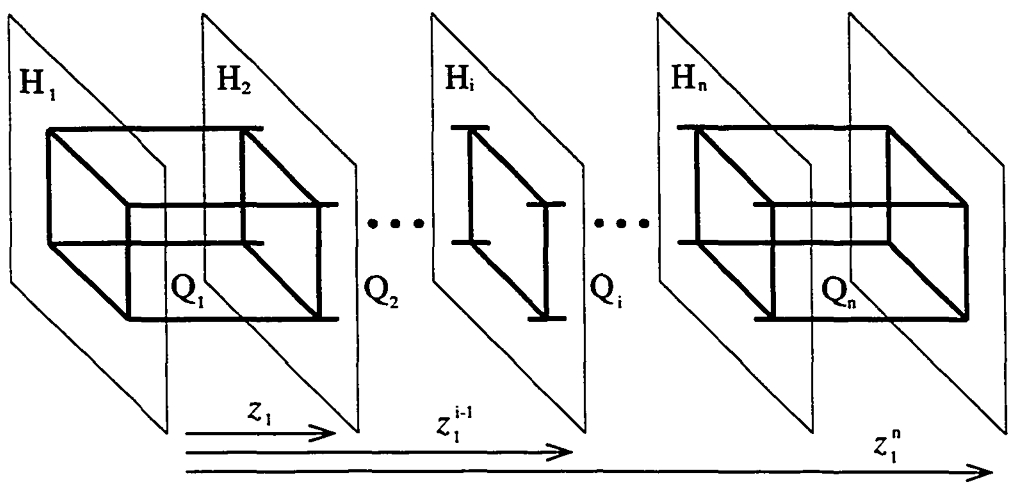

FIGURE 10. The planes H_i in the polyhedron Q

Now let $M = \mathbb{H}^4/G$, where G is a subgroup generated by any of the side-pairings of the polyhedron Q defined in the previous section. We identify the block Q_1 with P and its side-pairing with Φ_1 or Φ_2. Let H_i be the hyperplane supporting the side Z_1 of the block Q_i, $i = 1, \ldots, n$. So, $H_i = z_1^{i-1}(H)$ and it is kept invariant by the subgroup $J_i \subset G$, $J_i = z_1^{i-1} J z_1^{-(i-1)}$ (see Figure 10). Suppose we know that H_i is precisely invariant under the subgroup $J_i \subset G$. Then we may cut M along the hypersurfaces H_i/J_i. The hyperplanes H_i are exactly the ones that separate the polyhedron Q into blocks $Q_1, \ldots, Q_n$ (see Figure 10). Under identification of paired sides, we see that each block yields a submanifold of type M_1' or M_2'. Therefore, after cutting along H_i/J_i, $i = 1, \ldots, n$, we will get n pieces, each isometric to either M_1' or M_2'. Thus, we will have shown

Proposition 7.1. *Any of the manifolds constructed in §6 are obtained by gluing n copies of either M_1' or M_2' so that each copy of M_1' or M_2' is glued to another copy along one of their two totally geodesic boundaries i.e. they are strung together in a circular fashion. Here M_1' and M_2' are manifolds that we obtain by cutting hyperbolic 4-manifolds M_1 and M_2 along a totally geodesic hypersurface.* □

Remark 7.2. Let G_{0i}, $i = 1, 2$ be the subgroup generated by the same generators as G_i, but with z_1 omitted. Then, by using Maskit's combination theorems (see theorems 6.19 and 6.24 in [A2]) and fairly standard arguments (see, for example, [A3]) it is possible to show that

$$G = \left(G_{0i_1} \underset{J_2}{*} (z_1 G_{0i_2} z_1^{-1}) \underset{J_3}{*} (z_1^2 G_{0i_3} z_1^{-2}) \underset{J_4}{*} \cdots \underset{J_n}{*} (z_1^{n-1} G_{0i_n} z_1^{-(n-1)}) \right) \underset{z_1^n}{*} .$$

Here $\underset{J_i}{*}$ denotes the free product with amalgamation, while the last $\underset{z_1^n}{*}$ is the HNN-extension of the free amalgamated product in the parentheses by z_1^n, where

z_1^n conjugates subgroups J_1 and $z_1^n J_1 z_1^{-n}$. The index i_k is 1 or 2 depending on whether the block Q_k is of type Φ_1 or Φ_2.

Remark 7.3. The hyperbolic 3-manifold H/J has been described by Wielenberg, see Example 3 in [Wil]. It is the complement of a certain four-component link in S^3. Thus, we are gluing 4-manifolds along boundaries that are link complements in S^3.

The only thing left to do is to verify that the hyperplanes $H, H_1, \ldots, H_n$ are precisely invariant, respectively, under the subgroups $J, J_1, \ldots, J_n \subset G$. To this end we use the technical theorem stated below.

Theorem 7.4. *Let P be a fundamental polyhedron for a discrete group $G \subset \mathrm{Isom}(\mathbb{H}^n)$ that is generated by some side-pairing of P. Let H be a hyperplane in $\mathbb{H}^n$ so that $\mathrm{int}_H(H \cap P) \neq \emptyset$ and let J be a subgroup of G that keeps H invariant. Assume the following three conditions hold:*

(1) *$H \cap P$ is a fundamental polyhedron for the action of J in H.*

(2) *If H contains a side S of P and s is the side-pairing corresponding to S then $s(H) \neq H$.*

(3) *Suppose H contains an edge E of P. Let $\{\sigma_i = (E_i, S_i, R_i, g_i)\}_{i=1,2\ldots}$ be the sequence obtained by the edge-chase corresponding to E as in the edge cycle condition. (That is, $E_1 = E$, S_i and R_i are the sides that determine E_i, $E_i = S_i \cap R_i$, $g_i(E_i) = E_{i+1}$ and g_i is the side-pairing that pairs R_i and S_{i+1}.) Let α be the angle between H and S_1 and let θ_i be the dihedral angle of P at the edge E_i. If another edge E_{l+1} in the*

cycle of E is contained in H and β is the angle between H and S_l, then $\theta_1 + \cdots + \theta_l + \beta - \alpha = k\pi$ must be satisfied for some integer k.

Then H is precisely invariant under J.

Proof. Let $f \in G$ and suppose that $K = f(H) \cap H \neq \emptyset$. We want to show that $f \in J$. There are three cases depending on how K intersects the elements of the tiling $\{g(P),\ g \in G\}$. We will repeatedly use the fact that $H \subset \cup_{j\in J}\, j(P)$ and $f(H) \subset \cup_{j\in J}\, fj(P)$ which follows from the assumption that $H \cap P$ is a fundamental polyhedron for J in H.

Case 1. There exists an $x \in K$ so that $x \in \operatorname{int} g(P)$ for some $g \in G$. Since $x \in H$, there must be a translate of P under J that contains x. The only possible candidate is g, so we conclude $g \in J$. Likewise, since $x \in f(H)$, there is a $j \in J$ so that $x \in fj(P)$. But this can only happen if $fj = g$, so $f = gj^{-1} \in J$.

Case 2. There exists an $x \in K$ that is contained in the interior of a side R of some translate of P. Then R is common to exactly two translates of P. If R is not contained in H then H cuts into the interior of both of those translates. Again, the parts of H that are in the interiors of these translates must be covered by translates of P under J so the translates abutting R are of form $j(P)$ and $j'(P)$ for some $j, j' \in J$. Furthermore, there must be a $j'' \in J$ so that $x \in fj''(P)$. Since x is in only two translates of P, this means that either $fj'' = j$ or $fj'' = j'$. In both cases we get $f \in J$.

If, on the other hand, H does contain R, then at least one of the two translates of P abutting R is of form $j(P)$, $j \in J$. Assuming $f(H) \neq H$ gives us that $f(H)$

intersects the interior of $j(P)$. The portion of $f(H)$ in $\operatorname{int} j(P)$ must be in some $fj'(P)$ for some $j' \in J$ so we get $fj' = j$, which forces $f \in J$, contradicting $f(H) \neq H$. Therefore $f(H) = H$. Now the two translates of P that abut R are of form $j(P)$ and $js^{-1}(P)$, where s is the side-pairing of the side S for which $j(S) = R$. One of those translates is also of form $fj'(P)$ for some $j' \in J$. If $fj' = j$ then $f \in J$. The other case, $fj' = js^{-1}$ implies $s = j'^{-1}f^{-1}j$, so s preserves H. This, however, contradicts assumption (2), because H contains S, since $R \subset H$, $S = j^{-1}(R)$ and $j^{-1}(H) = H$.

Case 3. If neither case 1 nor 2 occurs, we get that K is contained in translates of edges of P, which are $(n-2)$-dimensional. Since $\dim K \geq n-2$ we get that K must be $(n-2)$-dimensional, which implies $f(H) \neq H$. Furthermore, there exists an $x \in K$ and an edge E' of some $g(P)$ so that x is in the interior of E'. As before, one of the translates of P that contains x must be of the form $j(P)$. Move everything by j^{-1} so that x is now on an edge E of P and $E \subset j^{-1}f(H) \cap H$. The translates of P that abut E are P, $g_1^{-1}(P)$, $g_1^{-1}g_2^{-1}(P), \ldots$, so as before, there must be a $j' \in J$ and an integer l so that $j^{-1}fj' = g_1^{-1} \circ \cdots \circ g_l^{-1}$. But then $j^{-1}f(H) = j^{-1}fj'(H) = g_1^{-1} \circ \cdots \circ g_l^{-1}(H)$, so $E \subset g_1^{-1} \circ \cdots \circ g_l^{-1}(H) \cap H$. From $E \subset g_1^{-1} \circ \cdots \circ g_l^{-1}(H)$ we get that $E_{l+1} = g_l \circ \cdots \circ g_1(E) \subset H$, so E_{l+1} is in the cycle of E and is contained in H. Let $K^{\perp}$ be the 2-dimensional orthogonal complement of K through x. The intersections of translates of P that abut E with $K^{\perp}$ are angles with rays emanating from a single vertex x. Intersections of H and $g_1^{-1} \circ \cdots \circ g_l^{-1}(H)$ with $K^{\perp}$ are two lines and the angle between them is $\theta_1 + \cdots + \theta_l + \beta - \alpha$. Condition (3) now says that this angle is $k\pi$, so the lines are identical and so are the hyperplanes that they represent. From here it follows that $f(H) = H$, a contradiction with $f(H) \neq H$. Therefore, case 3 never occurs and $f \in J$ by cases 1 and 2. $\square$

Actually, we proved

Corollary 7.5. *Theorem 7.4 is valid for $g \subset \operatorname{Isom} X$, where $X = \mathbb{R}^n$ or S^n.*

Proof. The proof of Theorem 7.4 did not use any hyperbolic space-specific properties, only the fact that P was a fundamental polyhedron. Therefore it also applies in the other two constant curvature settings, that is, for fundamental polyhedra of discrete isometry groups of the n-sphere and Euclidean n-space. $\square$

Remark 7.6. Notice that the group G in the theorem did not have to be torsion-free. However, if H contains an edge of P, condition (3) allows the number k that was defined in the edge cycle condition to only be 1 or 2.

Examples. We give several applications of the theorem that include the claims of precise invariantness needed for Proposition 7.1. All except example 7.9 have as P the polyhedron defined in §5.

Example 7.7. Let $G = G_1$ or G_2, H=the hyperplane based on the plane $\{z = 0\}$, $J = \langle x_1, y_1, a_3, a_4 \rangle$. Clearly H is invariant under J. By applying (now in dimension 3) Poincaré's polyhedron theorem to $H \cap P$ and restrictions of x_1, y_1, a_3, a_4 to H we may easily see that $H \cap P$ is a fundamental polyhedron for J in H. (Here conditions (2) and (3) from the theorem do not apply.) Therefore, H/J is a totally geodesic hypersurface embedded in M_1 or M_2.

Example 7.8. Let $G = G_1$ or G_2, $H = Z_1$ and $J = \langle x_1, y_1, a_1, a_2 \rangle$. As in Example 7.7 we check that $H \cap P$ is a fundamental polyhedron for J in H. Here we also need to verify condition (3) of Theorem 7.4. (Condition (2) clearly holds.) Taking, for example, $E = Z_1 \cap A_1$ whose cycle is $\{Z_1 \cap A_1,\ A_1' \cap Z_1,\ Z_1' \cap A_5,\ A_5' \cap Z_1',\}$ we see that $\alpha = 0,\ l = 1$ and $\beta = \pi/2$, so condition (3) is satisfied. Using Theorem 7.4 gives us that H/J is a totally geodesic hypersurface embedded in M_1 or M_2.

Example 7.9. It is now easy to see that the hyperplanes $H_1, \dots, H_n$ (in above notation) are precisely invariant under the subgroups $J_1, \dots, J_n \subset G$. The proof for H_1 and J_1 corresponds to the one in Example 7.8, while the other cases correspond to Example 7.7. This completes the proof of Proposition 7.1.

Example 7.10. Let $G = G_1$ or G_2 and let H_1, H_2 be the hyperplanes based respectively on the planes $\{x - y = 0\}$ and $\{x + y = 0\}$. We may use Theorem 7.4 to verify that H_1 is precisely invariant under $\langle a_2, a_4, a_6, b_1, d_2, d_4, y_1 x_1 \rangle$ and that H_2 is precisely invariant under $\langle a_1, a_3, a_5, b_1, d_1, d_3, y_1^{-1} x_1 \rangle$. Again, condition (1) of Theorem 7.4 is verified by the Poincaré polyhedron theorem in dimension 3. Note that condition (3) of that same theorem applies.

8. Hyperbolic manifolds as codimension-k complements

The material that follows had as its inspiration two sources. One is the fact that many noncompact hyperbolic 3-manifolds are, topologically, complements of links inside S^3. Thus, one could hope noncompact hyperbolic 4-manifolds to be complements of surfaces inside some closed 4-manifold (say S^4).

The other source was an early attempt at a construction of a hyperbolic 4-manifold, similar to the ones in §5. That "example" had as one of its boundaries the Hantzsche-Wendt manifold $\mathbb{R}^3/\mathcal{G}_6$ described in §4. For the manifold $\mathbb{R}^3/\mathcal{G}_6$ it could easily be seen that it was not an S^1-bundle, so it could not be the boundary of a tubular neighborhood of any surface, which was necessary in order to be able to think of the hyperbolic manifold as the complement of that surface.

This led to the investigation of the general question: when may we think of a noncompact hyperbolic $(n+1)$-manifold M as a complement of a closed codimension-k submanifold A inside a closed $(n+1)$-manifold N? (Note that $\dim M = n+1$ now, since most of what follows will concern the flat manifolds that bound M and so those will have dimension n.) We assume that A has as many components as there are ends of M. This is to rule out the following type of situation that we do not want to consider: let $k = 1$ and let $\overline{M}$ be bounded by two homeomorphic flat manifolds $E \cong E_1 \cong E_2$ — then we may glue the boundary components together to get a closed N in which M will be a complement of E. If $M = N - A$ we will say that M is a *codimension-k complement.*

First, it is clear that whether M is a codimension-k complement will depend only on the ends of M, i.e. on the flat manifolds bounding it. Secondly, if every manifold bounding M is an S^{k-1}-bundle then M is easily seen to be a codimension-k complement. Consider one of the ends $E \times [0,\infty) \subset M$, and let it be an S^{k-1}-bundle over a manifold B with $p : E \to B$ the bundle projection. Then construct the disc bundle $P = E \times [0,1] \cup A/((x,1) \sim p(x))$: this will be a compact $(n+1)$-manifold with $\partial P = E$ and we will have $E \times [0,\infty) = P - B$. By gluing P, in any chosen way, to $M - (E \times (0,\infty)$ along their boundaries (E is the boundary of $M - (E \times (0,\infty))$ we will have 'filled in' one end of M. We can do the same for every end of M to get a closed manifold N inside which M will be a complement of the union of B's.

More importantly, the converse to this observation holds. Before we state it, we recall a fact that will be used several times (see [S, Theorem 7.2.10]).

Fibration homotopy sequence. Let E be a fiber bundle (or, more generally a fibration) over B with fiber F. Then there exists a map ∂ so that the sequence

$$\ldots \to \pi_i F \xrightarrow{i_*} \pi_i E \xrightarrow{p_*} \pi_i B \xrightarrow{\partial} \pi_{i-1} F \to \ldots,$$

is exact, where $F \xrightarrow{i} E$ is the inclusion and $E \xrightarrow{p} B$ is the projection.

Proposition 8.1. *Assume that $M = N - A$. Let B be any component of A and*

$E \times [0,\infty)$ *the end of* M *corresponding to* B. *If* $n \neq 4$ *then* E *is an* S^{k-1}*-bundle over* B.

Proof. First consider the case $n \neq 3, 4$. Set $E_0 = E \times [0,\infty)$, where $E \times [0,\infty)$ is the end of M corresponding to B. Now $B \cup (E \times (0,\infty))$ is an open neighborhood of B, so there is a tubular neighborhood V of B that is contained in it. We can write $V - B = F \times [0,1)$ where F is the boundary of V. Let $F_0 = F \times [0,1)$. Since F is compact, there is an $s \in [0,\infty)$ so that $F \subset E \times [0,s)$. Similarly, by compactness of E, there is a $t \in (0,1)$ so that $E \times \{s\} \subset F \times [0,t)$. Let $E_s = E \times [s,\infty)$ and $F_t = F \times [t,1)$. We clearly have $F_t \subset E_s \subset F_0 \subset E_0$. Let i denote the inclusion of any of those sets to any of its supersets. Notice that there exist retractions $E_0 \xrightarrow{p} E_s$ and $F_0 \xrightarrow{r} F_t$. Define $g : E_s \to F_t$ by setting $g = E_s \xrightarrow{i} F_0 \xrightarrow{r} F_t$. We show that g and $f = F_t \xrightarrow{i} E_s$ induce isomorphisms on all homotopy groups. Really, $F_t \xrightarrow{f} E_s \xrightarrow{g} F_t = F_t \xrightarrow{i} E_s \xrightarrow{i} F_0 \xrightarrow{r} F_t = 1_{F_t}$. Hence, $g_* f_* = 1$ so g_* is surjective and f_* is injective. On the other hand, $g_* = \pi_q E_s \xrightarrow{i_*} \pi_q F_0 \xrightarrow{r_*} \pi_q F_t$. Clearly r_* is an isomorphism and i_* is injective being the first map in the composite $\pi_q E_s \xrightarrow{i_*} \pi_q F_0 \xrightarrow{i_*} \pi_q E_0$ which is an isomorphism. Therefore, g_* is bijective from which it follows that g_* and f_* are inverses of each other.

To finish the proof, we apply Farrell and Hsiang's theorem (Theorem 4.2): since E is flat, it is aspherical, and we have $\pi_q F \cong \pi_q F_t \cong \pi_q E_s \cong \pi_q E$. Therefore, F is an aspherical n-manifold with the same fundamental group as E, so it must be homeomorphic to E.

If $n = 3$, we just need to show that F is irreducible. But F is the boundary of a regular neighborhood of the codim k manifold B, so it is an S^{k-1}-bundle over B, $k = 1, 2, 3, 4$. Hence, F is either S^3, an S^2-bundle over S^1, an S^1-bundle over a surface B or an S^0-bundle over a 3-manifold B. The first two cases and the third for $B = S^2$ or P^2 are ruled out by using the fibration homotopy sequence since $\pi_1 F$ contains a free abelian group of rank three. Continuing with the third case, all surfaces B except S^2 or P^2 have $\mathbb{R}^2$ as their universal cover. Let $\mathbb{R}^2 \xrightarrow{p} B$ be the covering map. We form the pullback bundle p^*F and notice that it also a covering of F. However, $p^*F = \mathbb{R}^2 \times S^1$ due to contractibility of $\mathbb{R}^2$. The universal cover of $\mathbb{R}^2 \times S^1$ is $\mathbb{R}^3$ and it is easy to see that the composite $\mathbb{R}^3 \to p^*F \to F$ is a covering map. Therefore, the universal cover of F is $\mathbb{R}^3$, so F must be irreducible. The fourth case is dealt with as in Proposition 12.1.

We have shown that in all cases $E \cong F$, and F is the boundary of a regular neighborhood of a codim k manifold B, hence an S^{k-1}-bundle. □

The next proposition shows that M can only be a codimension-1 and -2 complement, since flat manifolds can only be S^0 and S^1-bundles.

Proposition 8.2. *Let E be a flat manifold that is a fiber bundle over some manifold with fiber S^l, the l-dimensional sphere. Then*

(i) *The number l must be 0 or 1.*

(ii) *When $l = 1$, the map induced by inclusion of a selected fiber $i_* : \pi_1 S^1 \to \pi_1 E$ is injective and the image $i_*(\pi_1 S^1)$ is a normal subgroup of $\pi_1 E$.*

Proof. (i). The universal cover of E is $\mathbb{R}^n$. If E were fibered by spheres S^l, $l \geq 2$, each fiber would by virtue of $\pi_1 S^k = 1$ lift to an embedding $S^l \hookrightarrow \mathbb{R}^n$ and we would get a fibering of $\mathbb{R}^n$ by l-spheres. This contradicts a result of Borel and Serre (see [B-S]) which says that $\mathbb{R}^n$ cannot be a fiber bundle with any compact fiber other than a point.

(ii). Assuming that fibers of E are circles, suppose that $i_* : \pi_1 S^1 \to \pi_1 E$ is not injective. Since $\pi_1 E$ is torsion-free, the image of a generator of $\pi_1 S^1$ must be $1 \in \pi_1 E$. Then we can lift $S^1 \hookrightarrow E$ to a map $S^1 \hookrightarrow \mathbb{R}^n$. Doing this for every fiber would give us a fibration of $\mathbb{R}^n$ by circles, again contradicting [B-S]. Hence, i_* is injective. The last claim follows from the fibration homotopy sequence, since $i_*(\pi_1 S^1)$ is the kernel of $\pi_1 E \xrightarrow{p_*} \pi_1 B$. $\square$

9. Hyperbolic manifolds as codimension-2 complements. A criterion for flat manifolds to be S^1-bundles

We first treat the case $k = 2$, as this is the analogue of the much-investigated case of hyperbolic structure on link complements. Thus, we must consider when a flat n-manifold E is an S^1-bundle. The following theorem gives a useful criterion.

Theorem 9.1. *Let $E = \mathbb{R}^n/G$ be a compact flat n-manifold, where G is a discrete subgroup of* $\operatorname{Isom}\mathbb{R}^n$. *Then the following are equivalent:*

(i) *E is an S^1-bundle over some base manifold B.*

(ii) *There is an element $f \in G$ so that $\langle f \rangle$ is a normal subgroup of G and for every $g \in G$, $g^k \in \langle f \rangle$ implies $g \in \langle f \rangle$.*

Furthermore, if $n \neq 4, 5$ and (i) *holds, the manifold B is homeomorphic to a flat manifold.*

Proof. (i)$\Rightarrow$(ii). Assume E is an S^1-bundle over B. Let f be a generator for $\pi_1 S^1 \subset \pi_1 E$. Proposition 8.2 tells us that $\langle f \rangle$ is normal in $G = \pi_1 E$. The second part of condition (ii) is equivalent to $G/\langle f \rangle$ being torsion-free. Since $G/\langle f \rangle \cong \pi_1 B$, we need to check that $\pi_1 B$ is torsion-free.

The fibration homotopy sequence for $S^1 \hookrightarrow E \to B$ together with Proposition

8.2(ii) gives immediately that $\pi_k B = 0$ for $k \geq 2$. It follows that the universal cover $\tilde{B}$ of B is also aspherical and $\pi_1 \tilde{B} = 1$. Whitehead's theorem then tells us that $\tilde{B}$ is contractible. If $\pi_1 B$ had a finite order element, it would have an element of prime order. This would give a deck transformation b on $\tilde{B}$ of prime order. However, a theorem of P.A. Smith asserts that a prime order homeomorphism of a contractible space must have a fixed point, which contradicts b being a deck transformation. Hence, $\pi_1 B$ cannot have any torsion and we are done.

(ii)$\Rightarrow$(i). Assume condition (ii) holds. We will construct a flat manifold B over which E fibers with fibers S^1. First of all, f must be a translation. This is true because the normality of $\langle f \rangle$ implies that f has at most two conjugates $f^{\pm 1}$ and the subset of translations of G is characterized by the property that they have finitely many conjugates (see §4).

Let $f(x) = x + v$. If $g \in G$ and we write $g(x) = Ax + a$, then v is an eigenvector of A by normality of $\langle f \rangle$ in G. Really, $gfg^{-1} = f^{\pm 1}$ reads as $A(A^{-1}x - A^{-1}a + v) + a = x \pm v$ in coordinates, so $Av = \pm v$.

Since every A is orthogonal, the action of G on $\mathbb{R}^n$ splits into a product action: choose an orthonormal basis $\{e_1, \ldots, e_n\}$ so that $e_1 \parallel v$. Then $Ax + a$ may be written as

$$\begin{bmatrix} \pm 1 & 0 \\ 0 & A' \end{bmatrix} \begin{bmatrix} x_1 \\ x' \end{bmatrix} + \begin{bmatrix} a_1 \\ a' \end{bmatrix}$$

where $x_1, a_1 \in \mathbb{R}$, $x', a' \in \mathbb{R}^{n-1}$ and A' is an $(n-1) \times (n-1)$ orthogonal matrix.

Let $H \subset \operatorname{Isom} \mathbb{R}^{n-1}$ be the subgroup of elements of form $A'x' + a'$ where each such was obtained from an element $g \in G$ as above. Then H is discrete since G is discrete and we have a homomorphism $\pi : G \to H$. Furthermore, H acts freely on $\mathbb{R}^{n-1}$. Indeed, suppose there is an element $h \in H$ with a fixed point. By normalizing we may assume that the fixed point is 0 and $h(x') = A'x'$. An element $g \in G$ so that $\pi(g) = h$ must have form

$$\begin{bmatrix} 1 & 0 \\ 0 & A' \end{bmatrix} \begin{bmatrix} x_1 \\ x' \end{bmatrix} + \begin{bmatrix} a_1 \\ 0 \end{bmatrix}.$$

(A -1 in the upper left corner is ruled out by the fact that $a' = 0$ and g has no fixed points.) Since A' has finite order, there is a k so that $g^k(x) = x + \lambda e_1$. Now by discontinuity of action of G the translations g^k and f must be commensurate, that is, there are numbers l and m so that $g^{kl} = f^m$. The second part of (ii) now implies $g \in \langle f \rangle$, which gives $A' = I$, that is, $h = 1$. Let us note here that the preceding argument also shows that if we pick $x \in \mathbb{R}^n$, $x = (x_1, x')$ and a small enough ball $U \subset \mathbb{R}^{n-1}$ centered at x' on which H acts freely, then $U \times \mathbb{R} \subset \mathbb{R}^n$ is precisely invariant under $\langle f \rangle$ in G.

Let $B = \mathbb{R}^{n-1}/H$. Since H is discrete and acts freely $\mathbb{R}^{n-1}/H$ is a flat manifold. Furthermore, it is compact, because the subgroup of translations in H has rank $n-1$.

Now let $q : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the standard projection. We define $p : \mathbb{R}^n/G \to \mathbb{R}^{n-1}/H$ by $p[x] = [q(x)]$. This is clearly well defined and E is an S^1-bundle over B. Really, an open ball V in B is covered by a union of disjoint open balls in $\mathbb{R}^{n-1}$. Pick one of them and name it U. We showed above that $U \times \mathbb{R}$ is precisely invariant under $\langle f \rangle$ in G. Therefore, $p^{-1}(V) = U \times \mathbb{R}/G = U \times \mathbb{R}/\langle f \rangle = V \times S^1$.

To prove the last statement: assume (i), then (ii) is satisfied. The proof of (ii)$\Rightarrow$(i) produces a flat manifold B' over which E is an S^1-bundle. This may not be the same as the manifold B, since the fibering of E could be different (the preimages of points in the fibering $\mathbb{R}^n \to B'$, described above, are straight lines, while there is no assumption on the preimages of points in the fibering $\mathbb{R}^n \to B$). However, from the proof it is easy to see that $\pi_1 B \cong \pi_1 E / \langle f \rangle \cong \pi_1 B'$. We now combine this with Theorem 4.2 to get $B \cong B'$. Notice that to get the result for $n = 4$ (i.e. $\dim B = 3$) we should also show that B is irreducible, which we have not yet been able to do. $\square$

Remark 9.2. After proving the theorem, the author found out that direction (ii)$\Rightarrow$(i) has been proven in greater generality by Vasquez ([V]): if $L \subset K$ is a subgroup of the maximal abelian subgroup K of G so that G/L is torsion-free, then E is a rank L-torus bundle over the flat manifold $\mathbb{R}^{n-\operatorname{rank} L}/(G/L)$.

The following is easily extracted from the proof of Theorem 9.1:

Corollary 9.3. *Let $E = \mathbb{R}^n/G$ be a compact flat n-manifold that is an S^1-bundle over a manifold B. If $f(x) = x + v$ is the element that corresponds to the loop given by a fiber, and $g(x) = Ax + a$ is any element of G, where $A \neq I$ and $a \in \ker(A - I)$, then $Av = \pm v$ and $a \nparallel v$.*

Proof. We refer to Theorem 9.1 and its proof: we saw that $Av = \pm v$. If $Av = -v$ then a, being an eigenvector of A, cannot be parallel to v. If we had $Av = v$ and $a \parallel v$ then some nontrivial power of g would be a translation in the direction of v implying $g^k \in \langle f \rangle$ and $g \notin \langle f \rangle$, which contradicts (ii). $\square$

10. EXAMPLES

In this section we give some examples of flat manifolds that are and some that are not S^1-bundles.

Example 10.1. Up to affine equivalence, there are 10 compact 3-dimensional flat manifolds. Each is of the form $\mathbb{R}^3/G$, where G is a discrete, torsion-free subgroup of $\operatorname{Isom}\mathbb{R}^3$. The list of groups G that generate the manifolds may be found in [Wo] or [H-W] and we use the notation of those two sources.

Using Corollary 9.3, the manifolds $\mathbb{R}^3/\mathcal{G}_3$, $\mathbb{R}^3/\mathcal{G}_4$ and $\mathbb{R}^3/\mathcal{G}_5$ are clearly not S^1-bundles since there are nontranslational elements in each group whose rotational part has only a 1-dimensional 1-eigenspace and no (-1)-eigenspaces. Then the translational part of g must point in the direction of the 1-eigenspace, violating Corollary 9.3.

For $\mathcal{G}_6$, the vectors that are ± 1-eigenvectors of the rotational part of every $g \in \mathcal{G}_6$ point in the directions of the three elements of the orthogonal basis $\{(1,1,0),(1,-1,0),(0,0,1)\}$. However, each of the basis elements is also a 1-eigenvector of a rotational part of some element that has 1-eigenspace of dimension only 1, so as above, $\mathbb{R}^3/\mathcal{G}_6$ is not an S^1-bundle.

All the other flat manifolds are S^1-bundles over either a torus or a Klein bottle, and sometimes both. Clearly $\mathbb{R}^3/\mathcal{G}_1 = T^3$ is an S^1-bundle. For the other flat manifolds, a geometric argument that they are S^1-bundles will be more instructive than check the criterion from Theorem 9.1. We give arguments for $\mathbb{R}^3/\mathcal{B}_1$ and $\mathbb{R}^3/\mathcal{B}_4$, while the others are done similarly.

For both of these groups we may take the cube C from §4 as the fundamental polyhedron. In what follows, let P and Q be, respectively, reflections in the planes $\{x_2 = 0\}$ and $\{x_3 = 0\}$. The side-pairings that generate $\mathcal{B}_1$ are:

$$s : \{x_1 = -1\} \to \{x_1 = 1\},\ s(x) = Qx + (2, 0, 0)$$

$$t_2 : \{x_2 = -1\} \to \{x_2 = 1\},\ t_2(x) = x + (0, 2, 0)$$

$$t_3 : \{x_3 = -1\} \to \{x_3 = 1\},\ t_3(x) = x + (0, 0, 2).$$

One can see that the restriction of the projection $p_2 : \mathbb{R}^3 \to \{x_2 = 0\}$ to C induces a map $p : \mathbb{R}^3/\mathcal{B}_1 \to$ a Klein bottle, which is a fiber bundle. Similarly, restricting the projection $p_3 : \mathbb{R}^3 \to \{x_3 = 0\}$ induces a map $p : \mathbb{R}^3/\mathcal{B}_1 \to$ a torus, also a fiber bundle. Thus, $\mathbb{R}^3/\mathcal{B}_1$ fibers into an S^1-bundle over both a Klein bottle and a torus. Notice that both projection maps have sections: a Klein bottle and a torus are contained in $\mathbb{R}^3/\mathcal{B}_1$ as images of the squares $\{x_2 = 0\}$ and $\{x_3 = 0\}$ in $C/\mathcal{B}_1$ respectively.

The side-pairings for $\mathcal{B}_4$ are:

$$s_1 : \{x_1 = -1,\ x_3 \in [0, 1]\} \to \{x_1 = 1,\ x_3 \in [-1, 0]\},\ s_1(x) = Px + (2, 0, -1)$$

$$s_2 : \{x_1 = -1,\ x_3 \in [-1, 0]\} \to \{x_1 = 1,\ x_3 \in [0, 1]\},\ s_2(x) = Px + (2, 0, 1)$$

$$s_3 : \{x_2 = -1,\ x_3 \in [0, 1]\} \to \{x_2 = 1,\ x_3 \in [0, 1]\}, s_3(x) = Qx + (0, 2, 1)$$

$$s_4 : \{x_2 = -1,\ x_3 \in [-1, 0]\} \to \{x_2 = 1,\ x_3 \in [-1, 0]\},\ s_4(x) = Qx + (0, 2, -1)$$

$$t_3 : \{x_3 = -1\} \to \{x_3 = 1\},\ t_3(x) = x + (0, 0, 2).$$

The translation that represents a fiber loop in $\mathcal{B}_4$ must be in direction of the x_3-axis. Really, using Corollary 9.3 and its notation, the vector v can be either a linear combination of e_1 and e_3 (i.e. in the 1-eigenspace of P), or in the direction of e_2 ((-1)-eigenvector of P). Since e_1 and e_3 are eigenvectors of different eigenvalues of Q, this leaves as the only possibility a vector in direction of one of e_1, e_2 and e_3. The first one is ruled out because it spans the 1-eigenspace of the rotational part of s_1s_3. The second fails because $s_3^2(x) = x + (0,2,0)$, so if we write s_3 in the normalized way this would contradict the criterion from Theorem 9.1. Now, looking at the fundamental cube C, we see that the projection $p_2 : \mathbb{R}^3 \to \{x_2 = 0\}$ induces a map $\mathbb{R}^3/\mathcal{B}_4 \to$ Klein bottle. The preimage of each fiber in C is a vertical segment. This time, however, there is no section for this fibering, as the following group-theoretic argument shows.

Edge-chasing gives us the presentation of $\mathcal{B}_4$:

$$\left\langle s_1, s_2, s_3, s_4, t_3 \mid s_2^{-1}s_1t_3,\ s_2^{-1}t_3s_1,\ s_1^{-1}s_4s_1s_3,\ s_2^{-1}s_3s_2s_4,\ s_4^{-1}s_3t_3,\ s_3^{-1}t_3s_4 \right\rangle.$$

We get $\pi_1 B = \pi_1(\mathbb{R}^3/\mathcal{B}_4)/\langle t_3 \rangle = \left\langle s_1, s_3 \mid s_1s_3s_1^{-1} = s_3^{-1} \right\rangle$ from the fibration homotopy sequence. Were there a section of $\mathbb{R}^3/\mathcal{B}_4 \to B$, there would be a map $i : B \to \mathbb{R}^3/\mathcal{B}_4$ which would induce a section of the map $\pi_1(\mathbb{R}^3/\mathcal{B}_4) \xrightarrow{p_*} \pi_1 B$. The images of $s_1, s_3 \in \pi_1 B$ under i_* would have to be of the form $s_1t_3^k$ and $s_3t_3^l$ in $\pi_1(\mathbb{R}^3/\mathcal{B}_4)$ for some $k, l \in \mathbb{Z}$. Since i_* is a homomorphism,

$$(s_1t_3^k)(s_3t_3^l)(s_1t_3^k)^{-1} = (s_3t_3^l)^{-1}$$

must be valid in $\pi_1(\mathbb{R}^3/\mathcal{B}_4)$. The first two relators in the above presentation of $\mathcal{B}_4$ give $s_1t_3s_1^{-1} = t_3$, the last two give $s_4t_3s_4^{-1} = t_3^{-1}$. The third is $s_1s_3s_1^{-1} = s_4^{-1}$

and the last one says $s_3 s_4^{-1} = t_3$. Using these, we may simplify the displayed equation to get $t_3^{1+2(l-k)} = 1$, which is impossible, since t_3 has infinite order.

For the rest of the 3-dimensional examples, we get that $\mathbb{R}^3/\mathcal{G}_2$ is an S^1-bundle over the Klein bottle, with a section, that $\mathbb{R}^3/\mathcal{B}_2$ is an S^1-bundle over both the torus and the Klein bottle, with sections, and that $\mathbb{R}^3/\mathcal{G}_3$ is an S^1-bundle over the Klein bottle, with no section.

It is easy to find flat manifolds that are S^1-bundles in any dimension n: just take $E^{n-1} \times S^1$, where E^{n-1} is a flat $(n-1)$-manifold. We now verify the expected, but not readily transparent fact that there are orientable examples in every dimension that are *not* S^1-bundles.

Example 10.2. We start with a 4-dimensional example. Let $C = [-1,1]^4$. We define the following side-pairings on C:

$$t_1 : \{x_1 = -1\} \to \{x_1 = 1\},\ t_1(x) = x + (2,0,0,0)$$

$$t_2 : \{x_2 = -1\} \to \{x_2 = 1\},\ t_2(x) = x + (0,2,0,0)$$

$$r : \{x_3 = -1\} \to \{x_3 = 1\},\ r(x) = Rx + (0,0,2,0)$$

$$s : \{x_4 = -1\} \to \{x_4 = 1\},\ s(x) = Sx + (0,0,0,2),$$

where $R(x_1,x_2,x_3,x_4) = (-x_2,x_1,x_3,x_4)$ and $S(x_1,x_2,x_3,x_4) = (x_2,x_1,-x_3,x_4)$. We can use Poincaré's polyhedron theorem (Theorem 3.1) to verify that $G = \langle t_1,t_2,r,s\rangle$ is a discrete subgroup of $\operatorname{Isom}\mathbb{R}^4$. We must be careful to check for that the group is torsion-free: this is accomplished by verifying that the normalized solid angle sum is 1 for dimension-0 and -1 faces

(Theorem 3.2). However, $\mathbb{R}^4/G$ is not an S^1-bundle due to Corollary 9.3: the only possible candidates for v are vectors in subspaces generated by e_3 and e_4, but both fail by the last part of the corollary.

Example 10.3. Next, we give an odd-dimensional example. Define side-pairings of $C = [-1,1]^{2j+1}$ as follows:

$$t_i : \{x_i = -1\} \to \{x_i = 1\},\ t_i(x) = x + 2e_i,\ i = 1, \ldots, 2j,$$

$$r : \{x_{2j+1} = -1\} \to \{x_{2j+1} = 1\},\ r(x) = Rx + 2e_{2j+1}$$

where $R(x_1, x_2, \ldots, x_{2j-1}, x_{2j}, x_{2j+1}) = (x_2, -x_1, \ldots, x_{2j}, -x_{2j-1}, x_{2j+1})$, i.e., R is a rotation by $\pi/2$ in each 2-dimensional plane spanned by e_{2i-1} and e_{2i}, $i = 1, \ldots, j$. Using Theorem 3.1 again and Corollary 9.3 we see that $\mathbb{R}^{2j+1}/\langle t_1, \ldots, t_{2j}, r\rangle$ is a flat $(2j+1)$-manifold that is not an S^1-bundle. (Torsion-freeness comes from the fact that every element of the group can be written as $r^k t$, where t is a translation in $\mathbb{R}^{2j} \times \{0\}$.) Notice that for $j = 1$ this is the manifold $\mathbb{R}^3/\mathcal{G}_4$.

We need the following lemma:

Lemma 10.4. *Let $G \subset \operatorname{Isom}\mathbb{R}^m$ and $H \subset \operatorname{Isom}\mathbb{R}^n$ be discrete, torsion-free groups yielding compact flat manifolds. If neither $\mathbb{R}^m/G$ nor $\mathbb{R}^n/H$ are S^1-bundles, then $\mathbb{R}^{m+n}/(G \times H)$ is not an S^1-bundle provided that whenever $e \in G$ and $f \in H$ are translations so that $\langle e\rangle$ and $\langle f\rangle$ are normal subgroups in G and $g \in G$, $h \in H$ are nontranslational elements with $g^k \in \langle e\rangle$, $h^l \in \langle f\rangle$, then $\gcd(k, l) \neq 1$.*

In particular, if the holonomy groups of G and H are p-groups then $\mathbb{R}^{m+n}/(G\times H)$ is not an S^1-bundle .

Proof. Apply Theorem 9.1 to $G\times H$. Assuming that $\langle(e,f)\rangle$ is normal in $G\times H$ we have $\langle e\rangle \lhd G$ and $\langle f\rangle \lhd H$. Since $\mathbb{R}^m/G$ and $\mathbb{R}^n/H$ are not S^1-bundles, there exist nontranslational elements $g\in G$ and $h\in H$ so that $g\notin\langle e\rangle$ and $h\notin H$, but there exist k and l (and we take the smallest positive ones) such that $g^k\in\langle e\rangle$ and $h^l\in\langle f\rangle$. The assumption of the lemma then gives a $d=\gcd(k,l)\neq 1$. Let $k=dq$ and $l=dr$. Then $(g^q,h^r)^d\in\langle(e,f)\rangle$ but $(g^q,h^r)\notin\langle(e,f)\rangle$, which violates the second part of (ii) in Theorem 9.1. □

Example 10.5. We can now construct an example of a non-S^1-bundle in any dimension $n\geq 3$. Let $n=3q+r$, where $r=3,4,5$, $q\geq 0$. Denote by E_4, G_4 the manifold and group, respectively, obtained in Example 10.2. Denote by E_i, G_i the i-dimensional manifold and its group from Example 10.3 (i odd). Using Lemma 10.4, we can inductively see that $E=E_3\times\cdots\times E_3$ (q factors) is not an S^1-bundle, since the holonomy groups of G_3, G_4 and G_5 are $\mathbb{Z}_4$, $\mathbb{Z}_8$ and $\mathbb{Z}_4$, respectively. In the same way, we see that $E\times E_r$ is not an S^1-bundle.

11. Hyperbolic structure on torus and Klein bottle complements inside S^4

In this section we show that there exist at most finitely many complements of unions of tori and Klein bottles in S^4 that have hyperbolic structure on them.

The fact that there exist flat 3-manifolds that are not S^1-bundles (§10) together with the theorem that every flat 3-manifold bounds a finite volume noncompact hyperbolic 4-manifold (see [N]), shows that, unlike hyperbolic 3-manifolds, some hyperbolic 4-manifolds may not be embedded as a codimension-2 complement.

It would be interesting to see some examples of 4-manifolds M that are codimension-2 complements inside a "familiar" closed 4-manifold N. In particular, we would like N to be orientable.

We have already seen (Theorem 2.3) that $\mathrm{Vol}(M) = 4\pi^2/3 \cdot \chi(M)$, so it is reasonable to first try with manifolds of smallest volume, i.e. those with Euler characteristic 1. Many (1171) such examples were found by Ratcliffe and Tschantz in [R-T], where they also listed the types of ends of each one. Unfortunately, while there are plenty of nonorientable ones that are codimension-2 complements, all the orientable ones have as at least one component of boundary the manifold $\mathbb{R}^3/\mathcal{G}_6$, which is not an S^1-bundle.

There are, in fact, plenty of examples of orientable manifolds that are codimension-2 complements. Every example constructed in §6 is such, and so are the orientable examples from [N] with boundaries that are S^1-bundles. However, if we want M to be a codimension-2 complement inside S^4, then the number of possible candidates is limited, which is the consequence of the following proposition.

Proposition 11.1. *Let M be a finite-volume noncompact hyperbolic manifold. If $M = N - A$, where A is a codimension-2 submanifold of N, then $\chi(M) = \chi(N)$.*

Proof. By Proposition 8.1 every boundary component E_i of M is an S^1-bundle over a component B_i of A, $i = 1, \dots, m$. Let P_i denote a 2-disk-bundle neighborhood of B_i inside N. Then $N = \overline{M} \cup P_1 \cup \cdots \cup P_m$. The Mayer-Vietoris sequence for $\overline{M}$ and $P_1 \cup \cdots \cup P_m$ (see Lemma 6.2) gives $\chi(\overline{M} \cup (\cup_{i=1}^m P_i)) = \chi(\overline{M}) + \chi(\cup_{i=1}^m P_i) - \chi(\overline{M} \cap (\cup_{i=1}^m P_i)$, which gives $\chi(N) = \chi(M) + \sum_{i=1}^m \chi(P_i) - \sum_{i=1}^m \chi(E_i)$. However, since every E_i is a flat manifold and every P_i a disk bundle over a flat manifold (so P_i is homotopy equivalent to B_i), we get $\chi(P_i) = 0$ and $\chi(E_i) = 0$ for $i = 1, \dots, m$. This implies $\chi(M) = \chi(N)$. $\square$

We now have

Theorem 11.2. *Let M be a finite-volume noncompact 4-hyperbolic manifold. If $M = N - A$, where A is a codimension-2 submanifold of S^4, then A is a disjoint union of tori and Klein bottles and $\chi(M) = 2$.*

Furthermore, there are only finitely many manifolds M with those properties.

Proof. In the notation of proof of Proposition 11.1, every B_i is a flat 2-manifold due to the last statement of Theorem 9.1, thus, every B_i is a torus or a Klein bottle. Moreover, $\chi(M) = \chi(S^4) = 2$.

The last statement follows immediately from Wang's theorem (Theorem 2.2): since $\mathrm{Vol}(M) = 4\pi^2/3 \cdot \chi(M)$, there are only finitely many hyperbolic manifolds M with $\chi(M) = 2$. $\square$

The only explicitly given examples of manifolds M with $\chi(M) = 2$ that this author is aware of are those of his own, presented in §5 (in §6 it is proved that their Euler characteristic is 2). Further work may yet identify these as codimension-2 complements inside the 4-sphere.

12. Hyperbolic manifolds as codimension-1 complements

Now we turn our attention to the other case, $k = 1$. Assume $M = N - A$ is a codimension-1 complement. As we have seen in §8, every flat n-manifold E bounding M must then be an S^0-bundle over some n-manifold B. That is, E is a twofold cover of B. We have

Proposition 12.1. *Let B be an n-manifold doubly covered by a flat manifold E. If $n \neq 4$ then B is a flat manifold.*

Proof. Since $\mathbb{R}^n$ is the universal cover of E, it is also the universal cover of B. The group $\pi_1 B$ cannot, therefore, have torsion as this would contradict Smith's theorem. Now $\pi_1 B$ has $\pi_1 E$ as an index 2 subgroup, so the maximal free abelian subgroup of $\pi_1 E$ has finite index in $\pi_1 B$. But (see [C]) a torsion-free group with a finite-index free abelian subgroup of rank n is necessarily a Bieberbach group, which, by a theorem of Auslander and Kuranishi (see [C]) is the fundamental group of an n-dimensional flat manifold. Now Theorem 4.2 implies that B itself is flat. Notice that, when $n = 3$, B is irreducible because its universal cover is $\mathbb{R}^3$. $\square$

We may assume that N was formed like in the introduction. More precisely, let $\partial\overline{M} = E_1 \cup \cdots \cup E_m$ and suppose $E_i \xrightarrow{p_i} B_i$ is a twofold covering for $i =$

$1, \dots, m$, where $B_1, \dots, B_m$ are components of A. Define P_i to be $(E_i \times [0,1] \cup B_i)/(x,1) \sim p_i(x)$. Then $E_i = \partial P_i$ and $N = \overline{M} \cup_{E_1} P_1 \cup \dots \cup P_m$, where each P_i is attached to $\overline{M}$ along its boundary. Let $G_i = \pi_1 P_i \cong \pi_1 B_i$ and $H_i = \pi_1 E_i$. We have that every H_i is a subgroup of both G_i and $\pi_1 M$.

By van Kampen's theorem, the fundamental group of N is going to be the amalgamated product of $\pi_1 N$ with $G_1, \dots, G_m$ with amalgamating subgroups $H_1, \dots, H_m$, respectively. It will therefore contain free abelian groups with rank greater than 1 (those that are contained in the H_i's), so the manifold N cannot be hyperbolic, because the fundamental group of a closed hyperbolic manifold never contains such subgroups.

From the above decomposition of N we may deduce

Proposition 12.2. *The universal cover of N is $\mathbb{R}^{n+1}$.*

Proof. Recall that a horoball in $\mathbb{H}^{n+1}$ is a Euclidean ball in $\mathbb{H}^{n+1}$ touching $\partial\mathbb{H}^{n+1}$. The point of touching is not in $\mathbb{H}^{n+1}$, so the boundary of a horoball (called a horosphere) is homeomorphic to $\mathbb{R}^n$. Let $\mathbb{H}^{n+1} \xrightarrow{p} M$ be the universal covering. Then (see [A2]) $p^{-1}(E_i \times [0,\infty)) = \mathcal{A}_i$, where $\mathcal{A}_i$ is a collection of disjoint horoballs. Let $\mathcal{A} = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_m$ — note that the horoballs from the collections $\mathcal{A}_i, i = 1, \dots, m$ are all disjoint. We have $p^{-1}(\overline{M}) = \mathbb{H}^{n+1} - (\cup_{A \in \mathcal{A}} \operatorname{int} A)$.

If we pick one of the horoballs A in $\mathbb{H}^{n+1}$ and perform an inversion in it the outside of the horoball will be mapped to the inside and $\partial\mathbb{H}^{n+1}$ is mapped to a horosphere inside A that is the boundary of some horoball A', centered at the same point as A. The collection $\mathcal{A}$ maps under this inversion to a collection of balls $\mathcal{C}$ contained in $\operatorname{int} A$ that is outside A' but each ball in the collection touches the boundary of A'. Let $U = A - (A' \cup (\cup_{C \in \mathcal{C}} C))$ — it is clear that $U \cong p^{-1}(\overline{M})$.

Now note that the universal cover of each I-bundle P_i is $\mathbb{R}^n \times [-1, 1]$, which is homeomorphic to a region between two horoballs. Call such a region V, for example $V \cong A - \operatorname{int} A'$ from above. We construct the universal cover of N as follows. Into every missing horoball of $p^{-1}(\overline{M})$ insert a region of type V. Now smaller horoballs are missing from $\mathbb{H}^{n+1}$. Into each insert a region of type U: now even smaller horoballs are missing, along with some balls that touch their boundary (those corresponding to the collection $\mathcal{C}$ in U). Let W_1 be the resulting subset of $\mathbb{H}^{n+1}$. Now repeat the procedure: insert regions of type V into the smaller balls and then regions of type U into the even smaller resulting balls to get W_2. Continuing inductively, we get a sequence of $W_1 \subset W_2 \subset \ldots$. It is clear that $\cup_i W_i$ is the universal cover of N and it is not difficult to see that this space is homeomorphic to $\mathbb{R}^{n+1}$. $\square$

The principal goal of the rest of this section is to show that, under appropriate conditions, there are only finitely many N's inside which M is a codimension-1 complement. We concentrate on one end $E_i \times [0, \infty)$: let $E = E_i$, $P = P_i$, $B = B_i$, $G = G_i$ and $H = H_i$. Notice that $\pi_1 P = \pi_1 B$. We think of E as the boundary of P, $E = \partial P$. Let $f, g : \partial P \to \partial \overline{M}$ be two homeomorphisms taking $E \subset P$ to $E \subset \overline{M}$. We can form two manifolds, $\overline{M} \cup_f P$ and $\overline{M} \cup_g P$, by identifying ∂P with component E of $\partial \overline{M}$ by the two homeomorphisms. If

$\partial P \xrightarrow{g^{-1}f} \partial P$ extends to a homeomorphism $P \to P$, then $\overline{M} \cup_f P$ and $\overline{M} \cup_g P$ will be homeomorphic by a homeomorphism that is the identity on $\overline{M}$. Let $\operatorname{Homeo} E$ denote the homeomorphisms of E and $\operatorname{Homeo}_{eP} E = \{f \in \operatorname{Homeo} E \mid f \text{ extends to P}\}$. There is a surjection from $\Pi_{i=1}^{m}(\operatorname{Homeo} E_i / \operatorname{Homeo}_{eP_i} E_i)$ to the set of all possible N's obtained in the described way. Thus, it will be enough to show that $\operatorname{Homeo}_{eP_i} E_i$ has finite index in $\operatorname{Homeo} E_i$ for $i = 1, \ldots, m$.

Choose a basepoint $e \in E \subset P$ and consider an $f \in \operatorname{Homeo} E$ such that $f(e) = e$. If $f_* : \pi_1(E, e) \to \pi_1(E, e)$ does not extend to a homomorphism $\pi_1(P, e) \to \pi_1(P, e)$, then f cannot extend to P. Hence, our first concern will be to see which automorphisms of $H = \pi_1(E, e)$ extend to $G = \pi_1(P, e) \cong \pi_1 B$. We define $\operatorname{Aut}_{eG} H = \{h \in \operatorname{Aut} H \mid h \text{ extends to G}\}$.

Example 12.3. Let $E \to B$ be the standard double covering of the Klein bottle by the torus. Then we may write $G = \langle a, b \mid aba^{-1} = b^{-1} \rangle$ and $H = \langle a^2, b \rangle$. Since H is the maximal abelian subgroup in G, for every $f \in \operatorname{Aut} G$ we have $f(H) = H$. Using this fact, it is easy to see that $f \in \operatorname{Aut} G$ if and only if $f(a) = a^{\pm 1} b^l$ and $f(b) = b^{\pm 1}$ for some $l \in \mathbb{Z}$. The restriction to H then always has the form $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$. This means that such elements of $\operatorname{Aut} H$ are the only ones that extend to G. Now, letting $f_k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ for $k \in \mathbb{Z}$ we see that $f_k^{-1} f_l = f_{l-k}$. Then all the automorphisms f_k are in different classes of $\operatorname{Aut} H / \operatorname{Aut}_{eG} H$ giving $[\operatorname{Aut} H : \operatorname{Aut}_{eG} H] = \infty$. Since all elements of $\operatorname{Aut} H$ are induced by homeomorphisms of E and all elements of $\operatorname{Aut}_{eG} H$ by homeomorphisms of P, we get $[\operatorname{Homeo} E : \operatorname{Homeo}_{eP} E] = \infty$, leaving no hope for finiteness of N's.

In order to get $[\operatorname{Aut} H : \operatorname{Aut}_{eG} H]$ finite, we need to put an additional requirement on B for a given E — a good one turns out to be that G and H have the same holonomy group. Let $H \subset G$ be a finite-index subgroup of G and let G have holonomy group Φ. It follows from the algebraic characterization of crystallographic groups that H is crystallographic if and only if G is. We will say that G and H have the *same holonomy group*, or that G is an extension (H is a subgroup) with the same holonomy group, if the restriction of $G \xrightarrow{p} \Phi$ to H is surjective.

We start with the simplest situation:

Proposition 12.4. *Let G be a free abelian group and let $H \subset G$ be a subgroup of finite index. Then $\operatorname{Aut}_{eG} H$ has finite index in $\operatorname{Aut} H$.*

Proof. Let $\operatorname{rank} G = \operatorname{rank} H = n$. We may choose bases $u_1, \ldots, u_n$ of H and $v_1, \ldots, v_n$ of G so that $u_1 = \alpha_1 v_1, \ldots, u_n = \alpha_n v_n$ where every integer α_{i+1} divides α_i for $i = 1, \ldots, n-1$ (see [K]). Let $\alpha = \alpha_1$ and embed G into the group $H' = \frac{1}{\alpha} H$ with the basis $u'_1, \ldots, u'_n$ where $v_i = \frac{\alpha}{\alpha_i} u'_i$, so that $u_i = \alpha u'_i$ for every $i = 1, \ldots, n$. We have $H \subset G \subset H'$. It is clear that every homomorphism $f : H \to H$ extends to a homomorphism $f' : H' \to H'$ and that f will extend to G if and only if $f'(G) = G$. In fact, there exists a bijection $\operatorname{Aut} H / \operatorname{Aut}_{eG} H \to \{f'(G) \mid f \in \operatorname{Aut} H\}$ given by $[f] \mapsto f'(G)$. Furthermore, subgroups $K \subset H'$ containing H with index $[G : H]$ are in one-to-one correspondence with subgroups of $H'/H = \mathbb{Z}_\alpha^n$ of order $[G : H]$. Since every $f \in \operatorname{Aut} H$ induces a homomorphism $f^* : H'/H \to H'/H$ we have a one-to-one correspondence between images $\{f'(G) \mid f \in \operatorname{Aut} H\}$ and images $\{f^*(G/H) \mid f^* \text{ is induced by } f \in \operatorname{Aut} H\}$. The latter set is the subset of all order $[G : H]$ subgroups of H'/H so it is finite. This proves the proposition. $\square$

The general case now follows from Proposition 12.4.

Theorem 12.5. *Let G be a crystallographic group, and H its subgroup of finite index. If G and H have the same holonomy group, then $\operatorname{Aut}_{eG} H$ is a finite index subgroup of $\operatorname{Aut} H$.*

Proof. Let K and L be respectively the maximal free abelian subgroups of G and H. Since G and H have the same holonomy group, we have $K \cap H = L$ and $G = \langle H, K \rangle$. As in the proof of Proposition 12.4, form a free abelian group L' so that $L \subset K \subset L'$. If, as in §4, H is represented as $\Phi \times L$ with the multiplication rule $(\sigma, l)(\sigma', l') = (\sigma\sigma', \sigma \cdot l' + l + c(\sigma, \sigma'))$, then $H' = \Phi \times L'$ can be given a group structure using the same rule. The same cocycle c can be used since $L \subset L'$. Since $L \subset K \subset L'$, we get $H \subset G \subset H'$. Given $f \in \operatorname{Aut} H$, its restriction to L satisfies $f(c(\sigma, \sigma')) = c(\sigma, \sigma')$. The restriction to L extends to $f' : L' \to L'$, so $(\sigma, l') \mapsto (f \cdot \sigma, f'(l'))$ defines an extension of f to H', which we also call f'. As above, there is the bijective correspondence $\operatorname{Aut} H / \operatorname{Aut}_{eG} H \to \{f'(G) \mid f \in \operatorname{Aut} H\}$. However, $f'(G) = \langle H, f'(K) \rangle$ so the latter set has only finitely many elements, since, as we saw in Proposition 12.4, the set $\{f'(K) \mid f \in \operatorname{Aut} L\}$ is finite.

Note that the group H' does not have to be torsion-free, even if H was such, nor must H be a normal subgroup of H'. $\square$

Corollary 12.6. *With assumptions as in Theorem 12.5, if* $\operatorname{rank} G = \operatorname{rank} H = n$ *and* $[G : H] = 2$*, then* $[\operatorname{Aut} H : \operatorname{Aut}_{eG} H] \leq 2^n - 1$.

Proof. Let us first assume that G and H are both free abelian. As in the proof of Proposition 12.4, $H \subset G \subset H' = \frac{1}{2}H$, and there is a bijection $[\operatorname{Aut} H : \operatorname{Aut}_{eG} H] \to \{f^*(G/H) \mid f \in \operatorname{Aut} H\}$. If we choose bases for H and G as in the proof of 12.4, then G/H is the first factor $\mathbb{Z}_2 \times \{0\}$ in $H'/H = \mathbb{Z}_2^n$, and $f^*(G/H)$ is an order two group inside $\mathbb{Z}_2^n$ for every $f \in \operatorname{Aut} H$; therefore, it is determined by an order two element of $\mathbb{Z}_2^n$, of which there are $2^n - 1$ (all but zero). For every element $v \in \mathbb{Z}_2^n$ there is an automorphism sending the first basis element e_1 of $\mathbb{Z}_2^n$ to v that is induced by an automorphism of H'. (If j is the first nonzero coordinate of v, the automorphism sending the j-th basis element e_j to v and leaving all other basis elements fixed will do if we precede it by the automorphism swapping e_j and e_1.) Thus, we get that $\{f^*(G/H) \mid f \in \operatorname{Aut} H\}$ has $2^n - 1$ elements.

For crystallographic G and H with K and L their maximal free abelian subgroups we have, as in Theorem 12.5,

$$[\operatorname{Aut} H : \operatorname{Aut}_{eG} H] = |\{f'(G) \mid f \in \operatorname{Aut} H\}| =$$

$$|\{f'(K) \mid f \in \operatorname{Aut} H\}| \leq |\{f'(K) \mid f \in \operatorname{Aut} L\}| = 2^n - 1.$$

The inequality in the above computation comes from the fact that not every automorphism of L is a restriction of an automorphism of H. $\square$

Now we are ready to return to the topological side of our considerations.

Proposition 12.7. *Let $n = 2$ or 3 and let E be a flat n-manifold that is the double cover via map $E \xrightarrow{p} B$ of a (necessarily flat) manifold B that has the same holonomy group as E. If $P = E \times [0,1] \cup B/(x,1) \sim p(x)$ is the I-bundle induced by p, then $[\operatorname{Homeo} E : \operatorname{Homeo}_{eP} E] \leq 2^n - 1$.*

Proof. Recall that for a space X, two maps $f, g : X \to X$ are pseudoisotopic ($f \sim g$) if there exists a homeomorphism $F : X \times I \to X \times I$ so that $F|_{X\times\{0\}} : X \times \{0\} \to X \times \{0\}$ is equal to f and $F|_{X\times\{1\}} : X \times \{1\} \to X \times \{1\}$ is equal to g.

Let J denote the subgroup $\{f : E \to E \mid f \text{ is pseudoisotopic to the identity}\}$ and set $\operatorname{Pseudo} E = \operatorname{Homeo} E/J$, $\operatorname{Pseudo}_{eP} E = \operatorname{Homeo}_{eP} E/J$. With $\operatorname{Inn} A$ denoting the inner automorphism group of a group H, let $\operatorname{Out} H = \operatorname{Aut} H/\operatorname{Inn} H$ and $\operatorname{Out}_{eG} H = \operatorname{Aut}_{eG} H/\operatorname{Inn} H$ (note that $\operatorname{Inn} H \subset \operatorname{Aut}_{eG} H$). We will also need the group $J_* = \{f : E \to E \mid f \text{ is homotopic to the identity}\}$. There is a map $q : \operatorname{Pseudo} E \to \operatorname{Out} H$ that takes $\operatorname{Pseudo}_{eP} E$ to $\operatorname{Out}_{eG} H$. We have a sequence of maps

$$\operatorname{Homeo} E/\operatorname{Homeo}_{eP} E \xrightarrow{\cong} \operatorname{Pseudo} E/\operatorname{Pseudo}_{eP} E \xrightarrow{q} \operatorname{Out} H/\operatorname{Out}_{eG} H.$$

The last quotient is finite by Theorem 12.5 and the fact that $\operatorname{Out} H/\operatorname{Out}_{eG} H \cong \operatorname{Aut} H/\operatorname{Aut}_{eG} H$. Now $[\operatorname{Pseudo} E : \operatorname{Pseudo}_{eP} E] = [\operatorname{Pseudo} E : \ker q] \cdot [\ker q : \operatorname{Pseudo}_{eP} E]$ and clearly $[\operatorname{Pseudo} E : \ker q] = [\operatorname{Out} H : Out_{eG}H] \leq 2^n - 1$. Note that $\ker q = \langle \operatorname{Pseudo}_{eP} E, J_*/J \rangle$, so $[\ker q : \operatorname{Pseudo}_{eP} E]$ will be finite if and only J_*/J if is finite. For $n = 2$ it is a theorem of Baer that $J_* = J$ (see [E]). When $n = 3$, this is a result of Waldhausen ([W, Theorem 7.1]). This gives

$J_* \subset \mathrm{Pseudo}_{eP}\, E$, so $[\ker q : \mathrm{Pseudo}_{eP}\, E] = 1$ and the proposition has been proved. $\square$

Remark 12.8. It is unlikely that $[J_* : J]$ is finite for higher dimensions. There are examples (see [B]) of simply connected manifolds (for example, $S^2 \times S^3$) for which this index is infinite.

Given a flat manifold E, it is not clear right away whether there exist only finitely many nonhomeomorphic I-bundles P over B such that $\partial P = E$ and B has the same holonomy group as E. The following proposition answers this question.

Proposition 12.9. *Let P be an I-bundle over a flat n-manifold B with $\partial P = E$. Assume that $E = \mathbb{R}^n/H$, $H \subset \mathrm{Isom}\,\mathbb{R}^n$. Then P is homeomorphic to $E \times [0,1]/(x,1) \sim r(x)$, where $r : E \to B$ is a covering map given by $\mathbb{R}^n/H \to \mathbb{R}^n/G$, where $G \subset \mathrm{Isom}\,\mathbb{R}^n$. Furthermore, if we fix an E and additionally require that B and E have the same holonomy groups (i.e. that their fundamental groups have the same holonomy groups) , then there exist at most $2^n - 1$ nonhomeomorphic manifolds P.*

Proof. It is not difficult to see that P must have form $E \times [0,1]/(x,1) \sim p(x)$, where $E \xrightarrow{p} B$ is a double covering. Clearly $B = \mathbb{R}^n/G'$, where $H \subset G''$, but we need not have $G'' \subset \mathrm{Isom}\,\mathbb{R}^n$. By Theorem 4.2, there is a homeomorphism $\bar{f} : B \to \mathbb{R}^n/G'$, where $G' \subset \mathrm{Isom}\,\mathbb{R}^n$. Taking a lift $f : \mathbb{R}^n \to \mathbb{R}^n$ of $\bar{f}$ we get that $fG''f^{-1} = G'$, $fHf^{-1} = H'$ for some $H' \subset G' \subset \mathrm{Isom}\,\mathbb{R}^n$. Bieberbach's third theorem (4.1) allows us to find an affine transformation $a : \mathbb{R}^n \to \mathbb{R}^n$ so that

$aH'a^{-1} = H$ and $aG'a^{-1} = G$, where $G \subset \operatorname{Isom}\mathbb{R}^n$. This gives an equivalence of coverings $E \xrightarrow{p} B$ and $E \xrightarrow{r} \mathbb{R}^n/G$ which immediately implies the existence of a homeomorphism $P \to E \times [0,1]/(x,1) \sim r(x)$.

The second part of the theorem is answered if we find out how many groups $G \subset \operatorname{Isom}\mathbb{R}^n$ contain H as an index-2 subgroup. If, as in proof of Theorem 12.5, L and K denote the maximal free abelian subgroups of H and G, then $G = \langle H, K\rangle$. It follows from the proof of Corollary 12.6 that for a lattice L in $\mathbb{R}^n$ there are exactly $2^n - 1$ lattices in $\mathbb{R}^n$ that contain L as an index-2 subgroup so we are done. □

Putting Propositions 12.7 and 12.9 together we get

Theorem 12.10. *Let $n = 2$ or 3 and let M^{n+1} be a noncompact finite-volume hyperbolic manifold with m ends. Assume that each component E_i, $i = 1, \dots, m$ of the boundary of M doubly covers some flat manifold with the same holonomy group. Then there are at most $(2^n - 1)^{2m}$ nonhomeomorphic manifolds N so that $M = N - A$, where $A = B_1 \cup \cdots \cup B_m$ is a disjoint union of m flat manifolds so that every B_i has the same holonomy group as E_i.*

Proof. As before, if $N = M - A$, then $N = \overline{M} \cup P_1 \cdots \cup P_m$, where each P_i is an I-bundle over B_i with $\partial P_i = E_i$. By Proposition 12.9, for a given E_i, there are at most $2^n - 1$ nonhomeomorphic P_i's possible, each of which can be glued to $\overline{M}$ to yield at most $2^n - 1$ different manifolds by Proposition 12.7. □

13. Criteria for groups of flat manifolds to have an index-2 torsion-free extension. Examples

In this section we investigate when the fundamental group H of a flat manifold E has an index-2 extension G (that is, $H \subset G$ and $[G : H] = 2$). In particular, in connection with §12, we give a few sufficient conditions for G to be torsion-free and have the same holonomy group.

Let H be a crystallographic group — then H fits into an exact sequence $0 \to L \to H \to \Phi \to 1$ (see §4) where L is the maximal free abelian subgroup of H. As before, we assume $\operatorname{rank} L = n$. We think of Φ as the subgroup of $\operatorname{Aut} L$ (see §4). As in the proof of Proposition 12.4, let $L' = \frac{1}{2}L$. Notice that for every element $l \in L$, $\frac{1}{2}l$ is well-defined. An element σ of Φ extends to an automorphism σ' of L' which induces an automorphism $\sigma^* : L'/L \to L'/L$. Let $L^* = L'/L \cong \mathbb{Z}_2^n$ and $\Phi^* = \{\sigma^* \mid \sigma \in \Phi\} \subset \operatorname{Aut} L^*$. Index-2 extensions of L inside L' are in one-to-one correspondence with nonzero elements of L^* (see proof of Corollary 12.6).

We have

Theorem 13.1. *Let H be crystallographic group. The following are equivalent:*

(i) *H has an index-2 extension G with the same holonomy group.*

(ii) *There exists a vector in L^* that is a 1-eigenvector for all elements of Φ^*.*

Proof. The main ideas were already developed in the proofs of 12.4, 12.5 and 12.6 to which we refer.

(i)$\Rightarrow$(ii). Let $H \subset G$, $[G : H] = 2$ and let K be the maximal free abelian subgroup of G. We have $K \cap H = L$ and $[K : L] = 2$. Since $H \subset G$ and G has the same holonomy group, the action of Φ on L extends to K. Let $t^* \in L^*$ represent K. Every element σ in Φ extends to K which implies $\sigma^*(t^*) = t^*$ for all $\sigma^* \in \Phi^*$.

(ii)$\Rightarrow$(i). The fact that a common 1-eigenvector t^* exists for the action of Φ^* on L^* immediately implies that there exists a Φ-invariant index-2 subgroup $K \subset L'$. Let $H' = \Phi \times L'$ be the group defined in the proof of 12.5. Then $G = \langle K, H \rangle$ contains H as a subgroup and G and H clearly have the same holonomy group. Since H acts on L by conjugation and this action extends to K we have that for every $k \in K$ and $h \in H$ $hk = (hkh^{-1})h$, where $hkh^{-1} \in K$. Using this fact it is readily shown that $\langle K, H \rangle = KH$ and that every element of $G - H$ can be represented in form kh for some fixed $k \in K - H$ and some $h \in H$. It follows that H has index 2 in G. $\square$

Now let H be torsion-free. If condition (ii) from Theorem 13.1 is satisfied and we construct G, there is no guarantee, in general, that G is torsion-free. In order to achieve that, we will need additional conditions.

Assume (ii) from Theorem 13.1 holds. Let t^* be a 1-eigenvector for the action of Φ^* on L^* and let $t \in L'$ be any element such that $t + L = t^*$. Construct K and G as in 13.1. We will say that the extensions $K = \langle t, L \rangle \subset L'$ and $G = \langle t, H \rangle$ are *induced* by t. Every element of $G - H$ can uniquely be written as th, for some $h \in H$. Notice that H, being index-2 in G, is a normal subgroup of G so there is a homomorphism $\psi : G \to \mathbb{Z}_2$ with kernel H. The following theorem gives some sufficient conditions that ensure torsion-freeness of G.

Theorem 13.2. *Let H be a Bieberbach group and G an index-2 extension of H with the same holonomy group, induced by an element $t \in L' = \frac{1}{2}L$. Then G has torsion if and only if there exists an element $h \in H$ such that $(th)^2 = 1$, or, equivalently, $t(hth^{-1})h^2 = 1$.*

In particular, if any of the following holds, G is torsion-free.

(i) *The holonomy group of H has odd order.*

(ii) *The element t is inside a subgroup $L_1 \subset L'$ which is kept invariant by the action of H on L' and which contains no power of any nontranslational $h \in H$.*

(iii) *For every $h \in H$, $hth^{-1} = t^{\pm 1}$ and $h^k \in \langle t \rangle$ implies $h \in \langle t \rangle$.*

Proof. Since $(th)^2 = t(hth^{-1})h^2$ and th is nontrivial (because $t \notin H$) we have one direction of the equivalence. For the other one, let th be an element of G and let $(th)^k = 1$ for some positive integer k. We must have $k = 2l$, since otherwise $\psi((th)^k) = 1 \in \mathbb{Z}_2$. This gives $((th)^2)^l = 1$ which implies $(th)^2 = 1$ because $(th)^2 \in H$ and H is torsion-free.

Notice that to check for torsion-freeness of G we do not have to worry about translations h in H possibly satisfying $(th)^2 = 1$: if h is a translation, then th is a nonzero translation since $t \notin H$, so th has infinite order and cannot satisfy $(th)^2 = 1$. We now explain how each of the conditions (i)-(iii) imply the nonexistence of an $h \in H$ with $t(hth^{-1})h^2 = 1$.

(i). The equation $h^2 = (t(hth^{-1}))^{-1}$ implies that h^2 is a translation (hth^{-1} is in K since conjugation by elements in H extends to K). But h^2 cannot be a translation if h is a nontranslational element and Φ has odd order.

(ii). Assuming $h^2 = (t(hth^{-1}))^{-1}$ again gives that h^2 is a translation that should be in L_1, because both t and hth^{-1} are, contradicting the assumption of (ii).

(iii). We have $t(hth^{-1}) = 1$ or $t(hth^{-1}) = t^2$. If $t(hth^{-1})h^2 = 1$ were satisfied for an element $h \in H$ we would get either $h^2 = 1$ or $h^2 = t^{-2}$. The first is impossible by torsion-freeness of H, the second by the assumption of (iii). □

Corollary 13.3. *If a Bieberbach group H contains an element f so that $\langle f \rangle$ is a normal subgroup of G and for every $g \in G$, $g^k \in \langle f \rangle$ implies $g \in \langle f \rangle$, then it has a torsion-free index-2 extension G with the same holonomy group.*

In other words, if $n \neq 4$ and $E = \mathbb{R}^n/H$ is an S^1-bundle, then it is an S^0-bundle over a flat manifold with the same holonomy group as H.

Proof. Form $L' = \frac{1}{2}L$, where L is the translation subgroup of H and let $t \in L'$ be such that $t = \frac{1}{2}f$. The fact that $hfh^{-1} = f^{\pm 1}$ for every $h \in H$ gives $\sigma \cdot f = f^{\pm 1}$ for every $\sigma \in \Phi$, which in turn implies $\sigma' \cdot t = t^{\pm 1}$ for the extension σ' of σ to L'. This means that $\sigma^*(t^*) = t^*$ for every $\sigma^* \in \Phi^*$, so Theorem 13.1 gives an index-2 extension G of H induced by t. It is clear that $hth^{-1} = t^{\pm 1}$ and $h^k \in \langle t \rangle$ implies $h \in \langle t \rangle$ for every $h \in H$, hence G is torsion-free by Theorem 13.2(iii).

The topological wording of the theorem follows from Theorem 9.1. This theorem could also have been proved by elaborating the idea that we may define an involution on E by sending every point in a fiber to its antipodal point. $\square$

We use the above theorems to test which flat 3-manifolds are S^0-bundles.

Example 13.4. From §10 we know that $\mathbb{R}^3/\mathcal{G}_1$, $\mathbb{R}^3/\mathcal{G}_2$ and $\mathbb{R}^3/\mathcal{B}_i$, $i = 1, \ldots, 4$ are all S^1-bundles. Corollary 13.3 implies they are S^0-bundles over flat 3-manifolds with the same holonomy groups.

Example 13.5. Consider $\mathbb{R}^3/\mathcal{G}_3$. The fundamental polyhedron for $\mathcal{G}_3$ is an upright prism of height $1/3$ based on a regular hexagon. The group $\mathcal{G}_3$ is generated by transformations r, t_1 and t_2, where r is a translation in the z-direction by $1/3$ followed by a rotation by $2\pi/3$ about the z-axis and t_1 and t_2 are translations pairing two pairs of parallel vertical sides of the prism. Choose t_1 and t_2 so that the angle between the vectors of t_1 and t_2 is $\pi/3$. The translation subgroup L is generated by t_1, t_2 and $t_3 = r^3$; the holonomy group Φ of $\mathcal{G}_3$ is $\mathbb{Z}_3$. Let $\sigma \in \Phi$ be the image of r under $\mathcal{G}_3 \to \Phi$. Then the matrices of σ and σ^*, in basis vectors t_1, t_2, t_3 of L and basis vectors induced by t_1, t_2, t_3 on L^*, respectively, are

$$\sigma = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \sigma^* \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The only eigenvector of σ^* in L^* is $(0,0,1)$. According to Theorem 13.1, $G = \langle \frac{1}{2}t_3, \mathcal{G}_3 \rangle$ is an index-2 extension of $\mathcal{G}_3$ which is torsion-free by Theorem 13.2(i).

It is easily computed that G is generated by t_1, t_2 and r', where r' is a translation in the z-direction by $1/6$ followed by a rotation by $-2\pi/3$ about the z-axis. Notice that $G \cong \mathcal{G}_3$ since it has the same holonomy group as $\mathcal{G}_3$ and $\mathcal{G}_3$ is the only flat 3-manifold group with holonomy group $\mathbb{Z}_3$.

Example 13.6. The group $\mathcal{G}_5$ exhibits rather different behavior from $\mathcal{G}_3$. Let t_1 and t_2 have the same meaning as in Example 13.5, but let r now be a translation in the z-direction by $1/6$ followed by a rotation by $\pi/3$ about the z-axis. The translation subgroup L of $\mathcal{G}_5$ is spanned by t_1, t_2 and $t_3 = r^6$; the holonomy group Φ of $\mathcal{G}_5$ is $\mathbb{Z}_6$. Again, an upright hexagonal prism (this time of height $1/6$) is a fundamental polyhedron for $\mathcal{G}_5$. If σ is the the generator of Φ, then

$$\sigma^* = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which means that its only eigenvector in L^* is $(0,0,1)$. Then the only possible index-2 extension of H will be induced by an element $t = \frac{1}{2}t_3 \in L'$. However, this extension has torsion since $t^{-1}r^3$ is a rotation about the z-axis by angle π. This shows $\mathcal{G}_5$ has no index-2 extensions with the same holonomy group.

For illustrative purposes, we used a geometric argument to show that every index-2 extension of $\mathcal{G}_5$ must have torsion. Actually, we can consider $\mathcal{G}_5$ as an abstract group (not embedded in $\operatorname{Isom}\mathbb{R}^n$) and arrive at the same conclusion. The equation $rt_3r^{-1} = t_3$ holds in $\mathcal{G}_5$, which implies $rtr^{-1} = t$, since the action of Φ extends to $K = \langle t, L\rangle$. Now $(t^{-1}r^3)^2 = t^{-1}r^3t^{-1}r^3 = t^{-1}(r^3t^{-1}r^{-3})r^6 = t^{-2}r^6 = t_3^{-1}r^6 = 1$. Similar purely algebraic considerations can be carried out in Example 13.8 as well.

For any torsion-free extension G of $\mathcal{G}_5$ with a holonomy group different from that of $\mathcal{G}_5$, the order of the holonomy group of G would have to equal 12, but no such Bieberbach group exists in dimension 3.

Example 13.7. A fundamental polyhedron for the group $\mathcal{G}_4$ is the cube C

defined in §4. Its side-pairings (generating $\mathcal{G}_4$) are the two perpendicular translations t_1 and t_2 pairing the vertical sides of the cube and the transformation r that is a translation in the z-direction by 2 followed by a rotation by $\pi/2$ about the z-axis. The translation subgroup L is generated by t_1, t_2 and $t_3 = r^4$; the holonomy group Φ of $\mathcal{G}_4$ is $\mathbb{Z}_4$. If σ is the generator of Φ, then σ^* has as its matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

One of its eigenvectors is $(1,1,0)$. The index-2 G extension induced by the translation $t = \frac{1}{2}(t_1 + t_2) \in L'$, is torsion-free by virtue of Theorem 13.2(ii). (The subgroup L_1 from 13.2(ii) is contained in the plane spanned by t_1 and t_2 while every power of a nontranslational element that is a translation is a translation in direction of t_3.) Since the holonomy of G is $\mathbb{Z}_4$ and no flat 3-manifold group other than $\mathcal{G}_4$ has $\mathbb{Z}_4$ as its holonomy group, we conclude $G \cong \mathcal{G}_4$.

Example 13.8. Consider $\mathcal{G}_6$, whose fundamental polyhedron and generating transformations were described in §4. The subgroup L of translations is generated by $t_1 = f_1^2$, $t_2 = (f_1 f_3)^2$ and $t_3 = f_3^2$ and the holonomy group of $\mathcal{G}_6$ is $\Phi \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It is easy to see that the matrix of every element of Φ^* is I, so every nonzero vector t^* in L^* is an eigenvector for every element of Φ^*. However, no index-2 extension of $\mathcal{G}_6$ is torsion-free. Let $t^* = (1,0,0)$ and consider the group G induced by the element $t = \frac{1}{2}t_1 \in L'$. We have $(tf_1)^2 = 1$, since tf_1 is a rotation by π about an axis. If $t^* = (1,1,0)$, G is induced by the element $t = \frac{1}{2}(t_1 + t_2) \in L'$. But then $t(f_1^{-1} t f_1) = f_1^2$, yielding torsion by Theorem 13.2.

We reason similarly if t^* has zeroes in other coordinates. Finally, if $t^* = (1,1,1)$, G is induced by $t = \frac{1}{2}(t_1 + t_2 + t_3)$, and we get $t(f_1^{-1} t f_1) = f_1^2$ again.

Thus, $\mathcal{G}_6$ has no index-2 extension with the same holonomy group. Any index-2 extension with holonomy group different from that of $\mathcal{G}_6$ would have to have holonomy group of order 8. However, since no flat 3-manifold group has a holonomy group of order 8, this is impossible.

These examples yield

Corollary 13.9. *Among flat 3-manifolds, only* $\mathbb{R}^3/\mathcal{G}_5$ *and* $\mathbb{R}^3/\mathcal{G}_6$ *do not allow a free* $\mathbb{Z}_2$ *action on them, or, equivalently, they cannot doubly cover any manifold, i.e. they are not* S^0*-bundles. Every other flat 3-manifold* E *doubly covers a flat manifold with the same holonomy group.*

14. CONCLUDING REMARKS

We saw in §10 that there are plenty of flat n-manifolds which are not S^1-bundles and in §13 that there are two flat 3-manifolds which are not S^0-bundles. Most likely examples can also be constructed for every n of flat n-manifolds that are not S^0-bundles. Therefore, in general, there should be no shortage of hyperbolic manifolds M that cannot be thought of as a codimension-1 or -2 complement. How can this be overcome? One way is to consider finite covers of M. In this connection we have the following theorem from [A-F]:

Theorem 14.1. *(Aravinda-Farrell) For every finite-volume noncompact hyperbolic $(n+1)$-manifold M there exists a finite covering $\tilde{M}$ so that every manifold E bounding $\tilde{M}$ is an n-torus.* □

Thus, since every n-torus is an S^1-bundle over an $(n-1)$-torus and it doubly covers another n-torus (which has the same, trivial, holonomy group), we have

Corollary 14.2. *Every hyperbolic manifold has a finite cover that is a codimension-1 and -2 complement.* □

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