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A CRITIQUE OF THE SMSG AND UICSM SECONDARY SCHOOL MATHEMATICS PROGRAMS

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A CRITIQUE OF THE SMSG AND UICSM SECONDARY SCHOOL MATHEMATICS PROGRAMS

CHAPTER I

INTRODUCTION

Background of Problem

At the present time, the students of curriculum are witnessing what has already been described as a "revolution" in school mathematics.[1] The changes currently taking place in the mathematics curriculum are so extensive, so far-reaching in their implications, and so profound that the twentieth century will long be recorded as the period in which many of the very challenging problems of mathematics education will have been recognized and attacked by many study groups and authors in an effort to better the mathematics programs of American schools.

A first cause for such activity has been the tremendous advances made by mathematical research. Dr. Price suggests that "the twentieth century has been the golden age of

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[1] G. Bailey Price, "Progress in Mathematics and Its Implication for the Schools", The Revolution in School Mathematics (Washington, D. C.: National Council of Teachers of Mathematics, 1961), p. 1.

mathematics, since more mathematics, and more profound mathematics, has been created during this period than during all the rest of history."[1] The present century has seen the extensive development of subjects in pure mathematics such as topology, abstract algebra, measure theory, and functional analysis. Coupled with the extension of the body of pure mathematics has been rapid development in many fields of applied mathematics, e.g., probability and statistics, game theory, quality control, and linear programming.

> Chemists have found new uses and interpretations for mathematics; biologists are applying mathematics to the study of genetics; businessmen are using mathematics in scheduling production and distribution; sociologists are using complicated statistical ideas; game theory has been found to have important applications to human behavior; mathematical models give promise as a basis for the interpretation of phenomena in many disciplines.[2]

It is certain that modern research in mathematics has produced changes equally as profound as those made in chemistry, physics, and biology.

Secondary causes for the revolution in mathematics have been the automation revolution and introduction of the large-scale, high-speed, automatic digital computing machines. Automation has created the necessity for solving complicated design and development problems and the computer has

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[1] Ibid., p. 1.

[2] American Association of School Administrators, Administrative Responsibility for Improving Mathematics Programs (Washington, D. C.: American Association of School Administrators, 1965), p. 6.

contributed a tool for their solution. The automation revolution and the computer have produced a totally unprecedented demand for trained mathematicians and it is practically impossible to foresee an adequate supply of highly trained personnel to meet this demand.[1]

The demands created by the revolution in mathematics are not reserved solely for future professional mathematicians and scientists. The basic concern of general education is to help each student to realize and accept the challenge of his individual potential and to lay the foundation for successful pursuits of special aptitudes and interests so that he may function in maximized capacity as a responsible citizen. Mathematics educators have advocated that there does exist a body of mathematical subject matter that is of significance to every educable individual at any and all levels of instruction and that mathematics must be accepted as fundamental in general education. These educators have advocated that this body of knowledge can make significant contributions toward the attainment of the basic objectives of general education and that it can provide for the following:

> 1. Competence in the basic skills and understandings for dealing with number and form;
> 2. Habits of effective thinking--a broad term involving analytical, critical, and postulational thinking, as well as reasoning by analogies and the development of intelluctual curiosity;
> 3. Communication of thought through symbolic expression and graphs;

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[1] Price, pp. 3-5.

> 4. Development of the ability to make relevant judgments through the discrimination of values;
> 5. Development of the ability to distinguish between relevant and irrelevant data;
> 6. Development of intellectual independence;
> 7. Development of aesthetic appreciation and expression;
> 8. Development of cultural advancement through a realization of the significance of mathematics in its own right and in its relation to the total physical and social structure.[1]

Due to the rapid technological strides which are being made, it is now generally recognized that our society is so dynamic that the problems of technology to be solved fifty years from now will be vastly different from those of the present. This eminent change carries significance for the mathematics curriculum.

> Technology is subject to rapid change. Training in specifics can, and may, soon become obsolete. On the other hand, a person with fundamental training in mathematics will have the background for making adaptations to applications, even to those not now foreseen.[2]

The nature of the flexible society and the need for a well-informed mathematically literate citizenry would suggest that perhaps the emphasis in mathematics education should be directed toward the search for underlying principles and basic structures as guides to fundamental generalizations and abstractions which can be extrapolated beyond the needs of present society rather than upon rote mechanical arithmetic manipulation.

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[1] Charles H. Butler and F. Lynwood Wren, The Teaching of Secondary Mathematics (New York: McGraw-Hill Book Co., Inc., 1960), pp. 44-5.

[2] Ibid., p. 92.

The rapid development of mathematics, the ensuing technological revolution, and the need for a mathematically literate population have thus produced a direct challenge to our school systems.

> The technological revolution now in progress requires that new mathematics be taught in our schools, that the emphasis be shifted in the teaching of many subjects already included in our mathematics courses, and that we increase the production of mathematicians and mathematics teachers.[1]

The changing conditions of mathematics and industry demand that "new" mathematics be required in our schools and that there be changed emphasis in the "old" mathematics available for our society. Various curriculum study groups and authors have faced this challenge demanding that no efforts should be spared to insure that the mathematics education provided by our schools shall be adequate for the needs of our time.

Support for a changing emphasis in mathematics instruction has come from these and other additional sources.

1. In recent years serious questions have been posed regarding the advisability of a major emphasis upon making mathematics practical and relating it entirely to problems of living.

2. According to the statement of the Educational Policies Commission, the purpose which runs through and strengthens all other educational purposes is the development of the ability <u>to think</u>, and this purpose is realized only through the development of the rational powers of the individual.

3. The changing attitudes of the society due to the "space-race" with the other nations of the world

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[1]Price, p. 5.

have allowed our society to become more tolerant and cognizant of the need for change and, in short, have produced a more aware society.

4. Many foundations and government organizations have contributed heavily to curriculum study programs.[1]

During the past decade and in response to the need for a revised curriculum in elementary and secondary mathematics, several experimental groups, after conducting long and exhaustive surveys of the present curricula, have developed, tested, and made their contributions available for publication. The following is a partial list of those groups whose efforts seem to receive major predominance in the current literature.

1. The largest united comprehensive study in mathematics has been conducted by the School Mathematics Study Group. This group has sought to improve the mathematics programs in schools by preparing text and teacher materials. Its chief purpose has been to develop textbooks for the college-capable student, grades 7 through 12, although the group is now directing its attention to the lower grades and now has textbooks available for grades K-12 as well as various study guides and monographs for teacher enrichment and teacher-training programs.

2. The next largest united study has been that conducted by the University of Illinois Committee on School Mathematics which has prepared text and teacher materials for a new college-preparatory curriculum for grades 9-12.

3. The University of Maryland Project has focused on the development of experimental units for grades 7 and 8.

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[1]Butler and Wren, pp. 3-85.

4. The Ball State Teachers College Program has dealt with the preparations of learning materials for grades 7-12 with texts for grades 8, 9, and 10 having been published. Some work has been done in grades 1-3.

5. The Boston College Mathematics Institute is sponsoring a program for grades 7-12 although text materials are available only for grades 7-9.

6. The members of the University of Illinois Arithmetic Project have prepared materials for grades 1-6.

7. The Greater Cleveland Mathematics Program has as its purpose the development of a sequential program for the elementary and secondary schools although materials are now available only for grades K-6.

8. Other groups included are the (a) Geometry for Primary Grades Project, the Sets and Numbers Project, and Mathematical Logic for the School Project (including grades 1-5) located at Stanford, (b) the Syracuse-Webster Elementary Project with experimentation and preparation of materials in grades 3-10, and (c) the Ontario Mathematics Commission with materials for grades 9 and 10.[1]

Although all share common elements and all purport to be aimed at the improvement of mathematics instruction, each of the programs listed is somewhat unique in its approach and philosophy. They all stress unifying themes or ideas such as the structure of mathematics, operations and their inverses, sets, deductive reasoning, logic, valid generalizations, etc. Yet the prepared materials may differ radically in their presentation, placement, emphasis upon social applications,

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[1]National Council of Teachers of Mathematics, An Analysis of New Mathematics Programs (Washington, D. C.: National Council of Teachers of Mathematics, 1963).

structure, vocabulary, methods of presentation, and methods of proof.

The proliferation of modern experimental programs as indicated by the above listing as well as the explicit differences in those programs insofar as methods of presentation, placement of materials, attention to mathematical structures, vocabulary, etc., indicates that there is at present no existent generally-accepted philosophical model describing the nature of the ideal mathematics education program for the schools of the United States. In like vein, the National Council of Teachers of Mathematics, the professional organization of mathematics educators, has been prompt to insist that not all the areas of psychological significance insofar as mathematics curriculum is concerned have been identified or considered and that there does not exist a mutually-acceptable model adequately describing the learning process, the role of internal motivation of student, societal needs and social structure.

> How does the human brain and nervous system acquire its store of mathematical knowledge? How does the human organism use this store of knowledge once it has acquired it? These are fundamental questions to which the answers can be of great aid in the improvement of the instruction of mathematics. Although comparatively little is known about the answers, the little that is known should be studied by every teacher of mathematics on every level of instruction.[1]

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[1]The National Council of Teachers of Mathematics, The Learning of Mathematics, Its Theory and Practice (Washington, D. C.: National Council of Teachers of Mathematics, 1953), p. vii.

The authors further state:

> Until we know more, we must conceive of the aspects of learning--motivation, analysis, transfer of training, and practice--as fluid elements, as tentative workable explanations of a theory of learning. When we gain more knowledge of the operation of the brain, and of the manner in which human behavior is changed, these various aspects may shift position, change in their importance, and even new elements may enter the picture.[1]

In absence of such models, the decision to select or not select any of the several available modern curricula (and associated materials) is still a problem for the individual teacher whose decision must be made in terms of the logical order of mathematics and in the light of his own philosophy as to the desirable nature of the mathematics curriculum.

In 1959, the National Council of Teachers of Mathematics, recognizing the need for some criteria of analysis and comparison of modern programs, formed the Committee on the Analysis of Experimental Programs. This committee was delegated the responsibility of determining ways in which the National Council of Teachers of Mathematics could assist school staffs in the restructuring of mathematics curricula. Its report recognized and discussed several areas of disagreement which currently exist in the minds of mathematics curricula experts and stated these disagreements in several questions which a teacher must analyze, among others, in order to clarify his own view of mathematics education.

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[1] Ibid., p. 349.

Questions raised by this group were:

1. At a particular grade level, which topics can be most effectively developed and which are most appropriate; how does the sequence of presentation of particular topics at particular grade levels affect their value in the mathematical maturity of the students?

2. What emphasis should be placed on the study of mathematical structures in order to bring about better understanding and use of mathematics; how should such studies be timed?

3. What is the best method for presentation of mathematical ideas, i.e., what is the relative merit of a sequence of activities from which a student may independently recognize the desired knowledge as opposed to presenting the knowledge and then helping the student to rationalize it?

4. How early in his mathematical development can a student progress from the use of a generally unsophisticated language of mathematics to a highly precise and sophisticated use of such language?

5. At what level and with what degree of rigor and sophistication should a student be introduced to mathematical proofs and at what level should he be expected to comprehend the nature of a "good" proof and to independently construct such proofs?

6. What should be the ideal relationship existing in mathematics between the development of skill in the manipulation of symbols and skill in developing mathematical concepts?

7. What (and how much) emphasis should be placed on the social applications of mathematics and what should be the purpose and nature of these applications?[1]

The committee further suggested that the basic guiding philosophies of any writing group should clarify and reflect its position as to the role which mathematics is destined to play

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[1]National Council of Teachers of Mathematics, An Analysis of New Mathematics Programs, pp. 2-6.

as an integral part of the development and the role of mathematics in man's environment. In essence, the committee recommended that a person analyzing any of the modern programs in mathematics should pay special attention to eight variables which might be designated as guiding philosophies of the authors, placement of materials, attention to mathematical structures, methods of presentation, vocabulary, proofs, development of concepts and skills, and attention to social applications.

Although the authors of this report do not explicitly define these variables, their study implicitly evidences their interpretations and usage of terms involved. Placement of materials involves more than just where particular topics are introduced but also involves the sequence in which particular topics are studied as well as the rigor with which they are employed. The consideration of mathematical structures involves the study of the basic principles or properties of any system (not necessarily a number system) as well as the analysis of the basic properties common to different mathematical systems. Methods of presentation involves the particular pattern of approach to be used in communicating mathematical ideas to the student; in the main, these authors consider the general approach of demonstration, illustration, and description as opposed to the alternate pattern of experimentation, observation, and generalization. Vocabulary refers to the words and language of the mathematics profession as

well as the precise, rigorous, and non-ambiguous usage of such language in the communication of mathematical ideas. A person possesses a mathematical concept when he understands and appreciates a mathematical idea; he possesses skill only when he approaches the level of automatic response in the use of that concept. The term proof, recognizably having many meanings, is used in the mathematical sense and generally refers to an argument (according to prescribed logical processes) serving to establish the truth of a statement in view of certain definitions, assumptions, and intermediate statements. The term social application is used in a broad sense and describes the use of mathematics as a tool in solving the problems of society, as an instrument for describing the environment in which we live, and as an aid in the understanding and solution of problems which may arise in the possible technological life of an individual; this usage is opposed to the commonly conceived definition of social application in "everyday living experiences" of the general populace.

An analysis of the various modern experimental programs which have been listed in this chapter as well as other similar reviews of curriculum efforts reveals that the UICSM and the SMSG are the only experimental groups which have materials available for uses in grades 9 through 12. As a result, any instructor or school curriculum director wishing to establish a four-year secondary sequence of such modern

materials would have to choose either the UICSM or the SMSG materials.

Examination of the mathematics curriculum changes which have occurred during the past few years reveals that these two programs have (and probably will continue to do so in the future) exerted a tremendous influence on the curriculum efforts of other groups as well as commercial publishers. Since these programs have served such an important definitive role, it behooves any mathematics educator to realize fully the scope and sequence, structure, potential, and limitations of such programs.

Statement of Problem

The problem for this study is stated in the form of the following questions:

1. What are the major features of the SMSG and the UICSM programs insofar as guiding philosophies, placement of materials, attention to mathematical structures, methods of presentation, vocabulary, proofs, development of concepts and skills, and attention to social applications are concerned?

2. What are the major similarities and differences between the UICSM and SMSG programs insofar as guiding philosophies, placement of materials, attention to mathematical structures, methods of presentation, vocabulary, proofs, development of concepts and skills, and attention to social applications are concerned?

Definitions

In this study, secondary school designates the grade block 9-12. SMSG designates School Mathematics Study Group;

SMSG secondary program designates the sequence composed of First Course in Algebra, Geometry, Intermediate Mathematics, Elementary Functions, and Introduction to Matrix Algebra. UICSM designates University of Illinois Committee on School Mathematics; UICSM secondary program designates the sequence composed of Unit 1, The Arithmetic of Real Numbers; Unit 2, Generalizations and Algebraic Manipulation; Unit 3, Equations and Inequations; Unit 4, Ordered Pairs and Graphs; Unit 5, Functions and Relations; Unit 6, Geometry; Unit 7, Mathematical Induction; Unit 8, Sequences; Unit 9, Elementary Functions: Powers, Exponentials, and Logarithms; Unit 10, Circular Functions and Trigonometry; and Unit 11, Complex Numbers.

Purpose

The purpose of this study is to present an analysis of the SMSG and UICSM secondary programs so that a teacher or curriculum director contemplating the inclusion of one of these programs in their curriculum might make an intelligent decision as to which of the programs is better suited to his particular situation in terms of the criteria discussed.

Scope and Limitations

A universally-accepted model describing desirable outcomes for any mathematics education program is not presently available and the areas of internal motivation of the student, learning theory, societal needs, and social structure

have not been adequately explored to the mutual satisfaction of all educators. Any statement given concerning appropriateness (or lack of it) of grade placement or method of presentation can not be made until such a model is developed. This implies that any discussion of any of the "new" curricula (mathematics or otherwise) must be restricted with respect to the variable of rationality or logic of the discipline concerned. This study readily admits this restriction of discussion.

A further limitation has been placed upon this study in terms of the level of sophistication of mathematical discussion. The presentation of the materials (with remarks) of the SMSG and UICSM programs has been done within the limits of understanding of the teachers usually found in the secondary classroom and who are, in general, not familiar with both sets of modern textual materials. Such an extensive and comprehensive presentation has not been attempted previously.

This paper recognizes that the teacher plays a vital and dynamic role in the classroom and that his agreement or disagreement with the guiding philosophies of any textbook author might detract from the effectiveness or ineffectiveness of that author's presentation. Insofar as this study is concerned, it must be assumed that qualities of teaching of the two programs will be compatible.

This study is expository in nature and, in no sense, attempts to determine which is the better of the two programs

for a particular classroom situation. Rather, efforts are directed singularly toward an exposition of the materials revealing basic differences and similarities.

Rationale Behind Approach to Problem

The report of the Committee of the Analysis of Experimental Mathematics Programs has suggested some eight major areas of disagreement existing in the minds of mathematics curriculum experts: (1) guiding philosophies of the authors, (2) placement of materials, (3) attention to mathematical structures, (4) methods of presentation, (5) vocabulary, (6) proofs, (7) development of concepts and skills, and (8) attention to social applications.[1] Inasmuch as this influential committee has recommended that any effective analysis of any of the modern programs should be based upon these variables, these eight areas of comparison were chosen for this study of the SMSG and UICSM programs.

Survey of Pertinent Literature

In the spring of 1958, the president of the American Mathematical Association, after consulting at length with the presidents of the National Council of Teachers of Mathematics and the Mathematical Association of America, selected a committee (with Dr. Edward G. Begle as chairman) of educators and professional mathematicians (primarily university

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[1] Ibid., pp. 2-6.

mathematicians) to organize the School Mathematics Study Group. The primary objective of this group was the improvement of the mathematics instruction in the schools of the United States through an improved curriculum. It postulated essentially that greater substance should be introduced earlier in the mathematics sequence. The president of the American Mathematical Association also appointed an advisory committee which consisted of university and college mathematicians, high school mathematics teachers, educational theorists, and scientific and technological personnel. The group was charged with the responsibility of constructing a curriculum which was to take proper account of the increasing use of mathematics and science in technology and other areas of knowledge and at the same time to reflect advances in mathematics itself.

The textual materials prepared by the School Mathematics Group were outlined and written by teams of authors representing all facets of mathematics instruction. The original writings of the group were tested through widely-distributed actual classroom use under the direct supervision of local people who were interested in such efforts. Writing groups received constructive criticism from these teachers and, in turn, conducted appropriate rewriting of these materials. It was soon recognized that teachers needed additional training for effective teaching of the materials; consequently, teacher materials and supplementary aids were provided.

Although the first project was to prepare what optimistically were to be model textbooks for grades 7-12 to be used as patterns for commercial authors, the efforts of the group have been extended to the production of textual materials and teacher's commentaries for grades K-12 with two levels of materials in grades 7-9 as well as the units Introduction to Secondary School Mathematics, Programmed First Course in Algebra, Geometry with Coordinates, Analytic Geometry, and SMSG: The Making of a Curriculum. These books are available upon order for general classroom use by any teacher desiring to use them. Various teacher-enrichment productions, e.g., Concepts of Informal Geometry, Number Systems, Intuitive Geometry, Concepts of Algebra, and Geometry, have been prepared and distributed.[1] The basic financial support for the efforts of this group has been provided through grants from the National Science Foundation.

The University of Illinois Committee on School Mathematics was organized in December, 1951, through the joint efforts of the Colleges of Engineering, Education, and Liberal Arts and Sciences at the University of Illinois. The prime purpose motivating the group was to investigate the content and teaching of high school mathematics in grades 9-12. Dr. Max Beberman was selected as director of the project.

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[1]School Mathematics Study Group, New SMSG-Yale Publications (New Haven: Yale University Press, 1965-66), pp. 3-20.

Since that time the major efforts (representing efforts of both mathematicians and teachers) of this group have been directed toward the development of instructional materials and their experimental trials and rewritings in schools throughout the country. Summer institutes, including both content and pedagogy, have been conducted for the purpose of preparing teachers for the use of the UICSM materials. Although these textbooks have been available for unrestricted classroom use since 1958, UICSM prior to that time distributed these textbooks in classroom quantities only to teachers who had special instruction in their use and who had agreed to evaluate and criticize the texts. Dr. Beberman warns, however;

> We have introduced some new content, rearranged some of the traditional content and have developed many promising pedagogical techniques and approaches. . . . Although we do not consider the present editions as experimental, we still recommend that they be used with caution and preferably only by teachers who have had an opportunity to study them under the supervision of a person who has had classroom experiences in their use or who has made an intensive study of their contents and the implicit pedagogy.[1]

Since early in 1961, the committee has been preparing and evaluating self-instruction materials as well as designing and producing appropriate filmstrips, etc. Financial support has come initially through grants from the Carnegie Foundation with supplemental grants from the National Science Foundation and the United States Office of Education.

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[1] Ibid., p. 58.

Some initial differences in the SMSG and the UICSM secondary materials are illustrated by the content and sequence (and tentative grade placement) of these materials as indicated by the unit titles and/or short descriptions of the units.

UICSM:

Grade 9. The arithmetic of real numbers; generalizations and algebraic manipulations; equations and inequations, applications; ordered pairs and graphs.
Grade 10. Relations and functions; geometry.
Grade 11. Mathematical induction; sequences.
Grade 12. Exponential and logarithmic functions; circular functions and trigonometry; polynomial functions and complex numbers.[1]

SMSG:

Grade 9. First course in algebra, based on properties of the real number system; equations through quadratic equations in one variable and linear equations in two variables; graphs of linear and quadratic functions.
Grade 10. Geometry, basically Euclidean geometry, but including considerable solid geometry and an introduction to analytic plane geometry.
Grade 11. Intermediate mathematics, including trigonometry and college algebra with stress on number systems, discovery exercises, and proof; the function concept given spiral development and coordinate geometry introduced early for use as a tool in trigonometry; vectors developed as a mathematical system.
Grade 12. Elementary functions, developed in a method to prepare for the calculus and introduction to matrix algebra.[2]

In 1961, the National Council of Teachers of Mathematics published what has become, in a sense, a definitive

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[1]Dorothy M. Fraser, Current Curriculum Studies in Academic Subjects (Washington, D. C.: National Education Association, 1962), p. 32.

[2]Ibid., p. 34.

manual for modern mathematics programs. This publication, the reporting of eight Regional Orientation Conferences in Mathematics held in various parts of the United States, was published to outline the movements in secondary mathematics curricula and to aid school administrators in making decisions with respect to these questions: (1) What caused the revolution in school mathematics? (2) What has been done to implement this revolution? (3) What adminstrative decisions are involved for local school systems? Dr. Kenneth E. Brown, in his contribution to the report in which he discussed briefly the efforts of eight different experimental groups, suggested:

> All the programs we have discussed attempt to avoid the presentation of new material as a string of unrelated topics. Indeed, they stress unifying themes or ideas in mathematics such as the following: structure, operations and their inverses, measurement, extensive use of graphical representation, systems of numeration, properties of numbers, development of the real number system, set-language and elementary theory, logical deductions, valid generalizations.[1]

Dr. Brown directed little attention to the basic differences among the individual programs and devoted only approximately two pages to the efforts of SMSG and UICSM.

In a report prepared for the Project on Education, Dorothy M. Fraser has provided information regarding the rather astonishing number of projects and studies dealing

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[1]Dr. Kenneth E. Brown, "The Drive to Improve School Mathematics," The Revolution in School Mathematics (Washington, D. C.: National Council of Teachers of Mathematics, 1961), p. 22.

with the academic subjects in the school curriculum. Included in the report are descriptive statements based on materials provided by the sponsors of each project and reflect the goals and judgments of those sponsors. In the section dealing with modern elementary and secondary mathematics programs, the author states:

> They are designed to bring mathematics programs up to date in content and methods of presentation. The secondary-school projects described in this chapter have worked toward this goal through different approaches. In each of these projects, considerable changes in the mathematics curriculum are recommended, and some of the recommendations call for much more drastic changes than others.[1]

Her report terminates with very brief (approximately one page each) surveys of some ten such modern programs.

The 1961 publication, An Analysis of New Mathematics Programs, prepared by the National Council of Teachers of Mathematics to assist teachers and school administrators in their consideration of program changes, evaluated briefly eight curriculum-revision projects (among which were the UICSM and SMSG programs). Although each of these eight programs was evaluated, little attention was directed to specific details of the sequence of materials and little discussion was dedicated to the similarities and/or dissimilarities of the UICSM and SMSG sequences.[2]

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[1]Fraser, pp. 27-28.

[2]National Council of Teachers of Mathematics, An Analysis of New Mathematics Programs, pp. 6-68.

A recent publication sponsored jointly by the American Association of School Administrators, Association for Supervision and Curriculum Development, National Association of Secondary-School Principals, and the National Council of Teachers of Mathematics, has stated:

> The emphasis in mathematics programs today is upon mathematical structures learned in an atmosphere of active inquiry. The student is encouraged to think for himself and to realize that there are often many ways to reach a solution. He meets many basic mathematical ideas very early, and he broadens and deepens these concepts as long as he continues in the mathematics sequence.[1]

Little attention is directed to the details of the curriculum efforts of individual study groups although their influence upon the modern mathematics curriculum is recognized.

While three of the above-mentioned publications, The Revolution in School Mathematics, An Analysis of New Mathematics Programs, and Studies in Academic Subjects have been prepared specifically to orient educators to modern mathematics programs and various other pertinent publications and articles in periodicals have appeared, no materials are available which present specific and exhaustive surveys of the UICSM and SMSG programs. No materials are available (other than the materials themselves) which will allow a potential teacher or curriculum director to compare intelligently the UICSM and SMSG programs.

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[1] American Association of School Administrators, Administrative Responsibility for Improving Mathematics Programs, p. 7.

Organization of Study

As an introduction to the reporting of the study, Chapter I contains a discussion of the background of the problem, the statement of the problem and associated definitions and delimitations, a statement of the rationale behind the approach to the problem, and a review of literature pertinent to the study. Chapter II contains a detailed critique of the SMSG materials based on mathematical content and sequence and directed in particular toward the philosophy of the authors, placement of materials, attention to mathematical structures, vocabulary, social application, concepts and skills, methods of presentation, and proofs. Chapter III presentsa similar study of the UICSM program. Chapter IV contains a detailed over-all comparison of the two programs and Chapter V concludes the study with the summary of the results of the study and the list of derived recommendations.

CHAPTER II

THE SMSG SECONDARY MATHEMATICS PROGRAM

General Characteristics of the SMSG Materials

The set of SMSG textbooks selected for this study is composed of student Units 9 and 10, First Course in Algebra; Units 13 and 14, Geometry; Units 17 and 18, Intermediate Mathematics; Unit 21, Elementary Functions; and Unit 23, Introduction to Matrix Algebra. Teachers' commentaries for these units are Units 11, 12, 15, 16, 19, 20, 22, and 24, respectively. The textbooks were presumably prepared with Units 9 and 10 planned for the ninth-grade student, Units 13 and 14 for tenth-grade consumption, Units 17 and 18 for eleventh-grade use, and Units 21 and 23 for seniors.

The following statements by the SMSG are somewhat indicative of the philosophies guiding the efforts of that group.

> The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. . . . One of the prerequisites for the improvement of the teaching of mathematics in the schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time reflects recent advances in mathematics itself. . . . This healthy fusion of the old age and the new should lead students to a better understanding of the basic concepts and

> structure of mathematics and provide a firmer foundation for understanding the use of mathematics in a scientific society.[1]

Thus, SMSG dedicates its efforts toward both understanding and application in mathematics.

The commentaries which have been prepared for the teachers are correlated page-by-page with the student texts. They contain fairly complete solution sets, and constantly suggest pedagogical techniques as well as enrichment topics. Questions to anticipate from students as well as ones which might be posed are common. Suggested test items for most chapters are included. Although little formal mathematics is presented, references are made quite often to appropriate mathematical treatises.

First Course in Algebra, Units 9 and 10

Introduction

A more meaningful analysis of the SMSG's First Course in Algebra may be made in view of the objectives of the course as stated by the authors:

> The principal objective of the FIRST COURSE IN ALGEBRA is to help the student develop an understanding and appreciation of the algebraic structure exhibited by the real number system, and use of this structure as a basis for the techniques of algebra. More specifically, we are interested in an exploration of the

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[1] School Mathematics Study Group, First Course in Algebra, Unit 9 (New Haven: Yale University Press, 1960), Foreword.

> properties of addition and multiplication of real numbers and their order properties.[1]

The authors recognize that very few students will possess sufficient mathematical maturity, experience, and insight and knowledge of the real numbers to utilize a purely axiomatic, deductive approach to the algebraic study of the reals. Therefore, the authors undertook this study of algebra with the assumption that the student was fairly familiar with the real numbers but that he possessed only very vague notions regarding their algebraic structure although he is fairly proficient with the "arithmetic" operations. Their realization of the non-maturity of the students causes the authors to "intend to be quite informal and intuitive, but not incorrect."[2]

In view of the objectives of the course and the level of mathematical maturity of the students involved, these two units have been written in "spiral" form. These texts are so written that the student must read regularly and carefully the several pages of textual material--including discussions and informal questions--presented prior to each set of exercises in order to understand fully the concepts being developed. The many exercises are correlated and interrelated with the text to promote understandings of the concepts already

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[1] Ibid., p. ix.

[2] Ibid., p. x.

presented and also to open avenues for consideration of future topics. Student inquiry is emphasized.

Sets and the Number Line

The student is introduced to the concept of set by the definition: "A set is merely a collection of objects."[1] Little attention is directed toward the necessary "well-defined" property of sets. Convenient symbols are presented to indicate the description and formation of sets, e.g., braces to enclose the names of the elements or members of a set provided that it is convenient to list the members of the set completely or to indicate enough elements to establish an "and so forth" pattern, capital letters to name sets, and $\emptyset$ to abbreviate the empty, or null, set.

In view of the modern elementary programs in which these symbols are commonly used, it seems strange that the authors do not introduce the union and intersection of sets except in certain exercises and even then the terms per se are not used. Also the conventional set notation is not introduced, e.g., $\cup$ to denote union of two sets, $\cap$ to denote intersection of two sets, $\subset$ to denote is a subset of, $\in$ to denote is an element of, and $\notin$ to denote is not an element of. In like vein, the absence of set-builder notation or the use of set-generators is noticeable, e.g., $\{y = 3x : x$ is an even integer$\}$, is not utilized.

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[1] Ibid., p. 1.

The student is introduced to the number line (or, at least, the positive half of the number line) used in traditional texts with the exception that emphasis is placed on the idea that the number associated with a point on the line is called the coordinate of the point and that, consequently, a coordinate is not a point. At this early point, the authors consider the graph of a set of numbers to be the corresponding points on the number line whose coordinates are the numbers of the set. Thus the student becomes acquainted with the terms coordinate, associated with, and corresponding to. The student is led to develop addition and multiplication procedures involving non-negative reals by using the number line. In view of the fact that a line extends indefinitely far, the authors would be a bit more consistent if they would plant a "barb" on the ends of the segment used in each "picture" of a line to indicate that the line extended past the edge of the sheet.

The student is also introduced to the previously troublesome distinction between finite and infinite by the statement that a set will be considered finite if the members of that set can be counted with the counting coming to an end or if the set is empty. Any non-finite set is to be regarded as infinite. An exercise develops the abstract parallel that every finite set can be placed into a one-to-one correspondence with a finite set of natural numbers--a property which does not hold for an infinite set. The fact that every

infinite set can be placed into a one-to-one correspondence with a proper subset of itself later permits the intuitive formulation by the student of the more sophisticated definition: "A set is said to be infinite if and only if it can be placed into a one-one correspondence with a proper subset of itself."[1]

The counting numbers (or natural numbers) have the property that they may be arranged in such an order that each number is followed by its successor. Hence, there exists no largest counting number. This, in turn, implies that the set of counting numbers is infinite.

A fraction is defined as being a symbol which indicates the quotient of two numbers, and a number which can be represented by a fraction indicating the quotient of two whole numbers (excluding division by zero) is a rational number. The notion of the denseness of the rationals is used to intuitively guide the student to an appreciation of the fact that there are infinitely many points on the number line whose coordinates are whole numbers and/or rationals. Any discussion of irrationals and negatives is postponed until later although it is mentioned that every point has a number coordinate which may or may not be rational.

In one of the sets of exercises at the end of the chapter, considerable attention is placed on the property of closure of a set under certain given binary operations.[2]

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[1] Ibid., p. 14. [2] Ibid., p. 17.

Numerals and Variables

This chapter points out, and emphasizes quite strongly, the difference between number and numeral, between symbol and referrent, etc., and makes quite adequately the point that a numeral is a nonunique name for the abstract number. Since a numeral is a name of a number, a given number may have many names. However, no undue pressure is placed on the student to force the distinction between number and numeral and it is conveniently accepted that a symbol may represent either the number or the numeral representing it in the event that no confusion may arise. It is indicated, however, that indicated sums such as "4 + 2," indicated products such as "3 x 2," indicated quotients such as "6 ÷ 2," or combinations of these, are actually themselves names of numbers, with the understanding that an "equals" sign between two numerals indicates that these two numerals name the same number. It is furthermore postulated that an expression may not bear the distinction of being a numeral unless it represents a definite number.

The term numerical phrase is used to denote any numeral given by an expression which involves other numerals along with the signs for operation. It is illustrated that punctuation in the form of parentheses, braces, etc., often must be introduced to eliminate ambiguity in numerical expressions and to achieve numeral status.

A particularly useful innovation--that of numerical sentences--is introduced. A numerical sentence is simply the result of combining numerical phrases to make complete statements about numbers. The important fact about such a sentence involving numerals is that it is either true or false, but not both. Much discussion is devoted to the analogy between an English sentence and a numerical sentence; one wonders why the authors do not punctuate their numerical sentences with the period at the end and make the analogy more complete!

By relying on the accepted rules of arithmetic, the student is led to confirm that for various choices of arithmetic numbers a, b, and c, $(a + b) + c = a + (b + c)$. It is emphasized that all such sentences of this form have a common pattern and the student is directed to conclude that every sentence having this pattern is true. This property of addition is named the association property of addition.

Similar consideration of patterns leads to the "discovery" of the commutative property of addition, and the association property of multiplication over addition. Since the discussion has been restricted to the arithmetic numbers, the student does not at this point quantify these properties, i.e., he is not compelled to state the domain of the property. Neither does the student at this point use letters to represent "general" numbers but rather verbalizes the properties. Many examples and exercises are presented to help the student make a habit of using these properties to facilitate

computation and to help him become aware that the algorithms of arithmetic are possible because of these properties.

In spite of the informality of these discussions the authors illustrate very lucidly that the distributive property in the pattern already studied, i.e., $a(b + c) = ab + ac$, does not allow manipulation of $(a + b)c$ without closer examination. It is illustrated that an alternate pattern may be obtained, i.e., $(a + b)c = ac + bc$. In a sense, the authors have developed and illustrated a need for both a right and a left distributive property.

The concept of variable is introduced by this definition:

> A letter used to denote one of a given set of numbers is called a variable. In a given computation involving a variable, the variable is a numeral which represents a definite though unspecified number from a given set of admissible numbers.[1]

The set of values admissible for the variable is called the domain of the variable.

Sentences and Properties of Operations

The numerical sentence (restricted to specified, definite numerals and being either true or false) introduced in the previous chapter is extended in this chapter to the open sentence. An open sentence is essentially a sentence involving one or more variables with the property that the truth or falsity of the statement is undetermined unless

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[1] Ibid., p. 37.

additional information is given, e.g., "x + 3 = 7" is neither true nor false until additional information regarding the domain and value of x is supplied. Great emphasis is placed on the necessity for the statement of the domain of the variable when open sentences are being utilized.

The truth set of an open sentence in one variable is defined as the set of all those numbers from the domain of the variable which make the sentence true, e.g., the truth set of the sentence "x + 7 = 9" is {2}. This notion is extended to the graph of the truth set of an open sentence containing one variable (later referred to as the graph of a sentence) as being the set of all points on the number line whose coordinates are the values of the variable which make the open sentence true. The early study of sentences involving inequalities--an innovation fairly unique to modern texts--is introduced quite naturally without unnecessary fanfare. This introduction is facilitated by the use of the truth set rather than the traditional "the solution of an equation."

Much attention is directed toward preserving the analogy between open sentences and/or numerical sentences and the sentences of the English language. The variable involved in an open sentence is suggestive of a pronoun (since a variable is a name of a number) and symbols such as =, ≠, >, ≯, <, ≮, etc., as being verb forms.

Discussion is directed toward the study of compound sentences with the connectives and and or. The truth set of "Sentence A and Sentence B" is established to be the intersection of the truth set of Statement A and the truth set of Statement B, whereas the truth set of "Sentence A or Sentence B" is established to be the union of their respective truth sets. Similar extensions are made to include the graphs of these compound sentences. (Although the authors, indicating that such notation is not necessary, have not introduced conventional set notation such as symbols for union, intersection, etc., this discussion could have been made much more precise and more easily manipulated had such symbolism been introduced.)

The remainder of this chapter is devoted toward a more general statement of the properties of the arithmetic numbers as derived and informally stated in an earlier chapter. The student is now forced to quantify his statements by describing the domain of the sentence and to use variable symbols in their statement. The student intuitively recognizes and formally states the closure property of addition and multiplication, the addition and multiplication properties of 0, and the multiplication property of 1.

It is of interest to note that, while in this chapter the student determines the truth sets of many open sentences, no "rules" are used for such evaluations. The student is encouraged to "sift" the basic addition and multiplication

facts, check their effect on the truth or falseness of the sentences involved, and thereby determine the truth sets. It is also refreshing, from the mathematical point of view, to note exercises involving binary operations "defined" on rather unusual domains in which the student must examine the properties of these operations.

This chapter serves the purpose of causing the student to begin thinking of the system of arithmetic numbers more often in terms of its basic properties so that eventually everything that he does with such numbers will be done with these properties in mind.

Open Sentences and English Sentences

This particular topic deals very effectively with the generally frustrating problem of "thought" or "word" questions which inevitably cause great concern to the average algebra student. Due to the fact that the open sentence was used earlier, this problem is simplified by reliance upon the existing analogy between a mathematical sentence and an English sentence. The symbol "+" may be thought of as being synonymous with the English expression of "sum of," "more than," "increased by," "older than," etc. Similar equivalents for "x," "-," and "•" exist.

Much practice material is provided the student in translating such phrases as "x + 5" into English equivalents (<u>open phrases</u> into <u>word phrases</u>), e.g., "5 more than the

number of boys in a certain class," and "5 miles farther in one direction than another." Conversely, English phrases such as "6 feet longer than wide" may be written as "y + 6" provided that y represents the number which is the measure of the width. In the ensuing discussion and exercises the student is encouraged to extend this process to sentences and to learn to translate a word problem into a mathematical description (or open sentence) with the truth set of the open sentence consisting of the solution to the problem. As in previous exercises dealing with truth sets of open sentences, the truth sets are determined by reasoned guessing and graphing on the number line. Open sentences having as verb forms other order relations are treated simultaneously with those having the verb "equals."

The authors implicitly emphasize that a great distinction exists (and needs to be recognized) between a geometrical entity (line segment, rectangle, etc.,) and its various measures. It is also simultaneously apparent that the conversion of an English sentence into an open sentence removes from the sentence its physical connotations, i.e., the mathematical sentence has no connection with physical "reality" but is rather an abstraction symbolized by appropriate notation. In several of the available exercises in this chapter, the authors introduce irrelevant information to promote an awareness of that which is necessary and pertinent in the structuring of a proper decision.

The Real Numbers

In previous chapters the student was led to recognize that all counting numbers, zero, and rational numbers may be represented as coordinates of points of a half-line to the right of (and including) a given point labeled "0" on a number line. It was suggested earlier that there are points on this half-line whose coordinates cannot be represented by such numbers and that their coordinates are called irrational. The union of all such numbers which may be used to represent coordinates of points on this half-line are given the title of arithmetic numbers. The first four chapters are devoted to an examination of the properties of this system of numbers under the binary operations of addition and multiplication.

SMSG suggests that the points to the left of 0 may be named in a similar fashion and labeled with symbols similar to those used for points to the right except that an upper dash "-" will indicate that the point is to the left of 0, e.g., $^{-}7$ (read "negative 7") is the coordinate of a point 7 units to the left of the point labeled 0, thereby obtaining a new set which is effectively a "mirror image" of the numbers of arithmetic with respect to the point named 0. The numbers which are coordinates of points to the right of 0 are renamed the positive real numbers, the numbers which are coordinates of points to the left are named the negative real numbers, and the set of all numbers associated with points on the number line is called the set of real numbers. Thus the

numbers previously referred to as arithmetic numbers are now referred to as the non-negative real numbers. (The authors permit some notation that is a bit confusing, e.g., does $^{-}3/2$ mean $^{-}(3/2)$ or $^{-}3 + 2$?[1] The second interpretation would be premature in that no operations as yet have been defined for the negatives. They also use point, coordinate, and number interchangeably.)

This particular development of the reals is somewhat unorthodox (with respect to standard algebra texts) in that no particular physical interpretation is demanded, i.e., no attempt is made to consider negatives as numbers "smaller than zero." The notation "^{-}a" is particularly powerful in that no conflict can arise from inadvertent usage of the negative "sign" to indicate the operation of subtraction. One also notes that there are no "plus numbers," i.e., there exists no need for such notation as "+6" since the non-negative reals are the arithmetic numbers. Those interested in the rigor involved will note that it will not be necessary to prove that the properties of the arithmetic numbers hold for the non-negative reals since they are the same sets. Neither will it be necessary to prove the isomorphism of these two sets.

In view of the use of the number line to define negatives it is not surprising that the number line is used to

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[1] Ibid., p. 99.

define the "less than" order relation, i.e., "a < b" is defined to mean that a is to the left of b on the number line. This definition is extended to permit the formulation of the transitive property of the reals and the comparison property (or the trichotomy property) of the reals.

The opposite of a non-zero real number is defined to be the other real number which is at an equal distance from 0 on the real number line. The lower dash "-" is introduced to mean "the opposite of." Hence, if a is positive, $\bar{a}$ and -a ("negative a" and "opposite of a") are different names for the same number. But $\bar{a}$ is undefined if a is a negative real although -a is not undefined, e.g., $-(\bar{4}) = 4$. (It follows that "-a" may be positive or negative according as to whether a is negative or positive, respectively). SMSG suggests that it seems "natural" to retain the "opposite of" symbol to mean either "negative" or "opposite of" when the number in question is positive. (The authors insist that the student read "-a" as "opposite of a" rather than "minus a", and further cautions him that "taking the opposite" of a number is not "changing its sign.") The opposite of 0 is defined to be 0.[1]

The absolute-value of a non-zero real number is defined to be the greater of that number and its opposite. The absolute value of 0 is defined to be 0. The student formulates the equivalent definition: "If $x \geq 0$, $|x| = x$;

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[1] Ibid., pp. 109-110.

if $x < 0$, $|x| = -x$."[1] Also $|x|$ is illustrated to be the non-negative number which is the measure of the distance between 0 and x on the number line. (It is noted that the absolute value of any real number is a non-negative real or an arithmetic number.)

Properties of Addition

Since SMSG has earlier extended the system of arithmetic numbers to the system of real numbers, SMSG attempts in this chapter to develop an extension of the operation of addition from the arithmetic numbers to the real numbers in such a way that the basic properties of addition are preserved. Consequently, all definitions for addition of real numbers are developed using the absolute values of the reals. It is emphasized that a well-planned and appropriate definition for addition of the reals is formulated in order that the familiar properties of the operation of arithmetic may be proved as theorems for the reals in view of this definition.

SMSG attempts to lead up to a general definition of addition of the reals in a plausible way by making full use of the number line and the interpretation of the absolute value of a real number as the distance between the zero point and the number on the number line. Many examples employing gains and losses suggest how addition involving negatives

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[1] Ibid., p. 115.

must be defined in order not to conflict with physical interpretation.[1]

The general results of these many examples are stated as definitions both in English and in the language of algebra:

(a) If a and b are both negative numbers, then:
$a + b = -(|a| + |b|)$.
(b) If $a \geq 0$ and $b < 0$, then:
$a + b = |a| - |b|$, if $|a| > |b|$,
and $a + b = -(|b| - |a|)$ if $|b| > |a|$.
(c) If $b \geq 0$ and $a < 0$, then:
$a + b = |b| - |a|$, if $|b| > |a|$
and $a + b = -(|a| - |b|)$, if $|a| > |b|$.[2]

These definitions, complicated though they appear, contain only operations which are known by the student from previous experiences, i.e., operations defined on the arithmetic numbers. SMSG is careful to use quite often the term "definition of addition."

The student is led to examine several examples and to accept without proof the commutative property of addition, the associative property of addition, the addition property of opposites, i.e., for every real number a, $a + (-a) = 0$, the addition property of 0, and the addition property of equality, i.e., for any real numbers a, b, and c, if $a = b$, then $a + c = b + c$. The proofs of these properties are not given although the student is informed (and challenged to do so) that such properties could be proved by reliance on the definition of addition and the axioms of the arithmetic numbers.

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[1] Ibid., pp. 124-29. [2] Ibid., p. 127.

The addition property of equality is used to determine the truth sets of various open sentences (or equations). Since the authors have not yet introduced equivalent equations, SMSG insists that the expression, "If the equation is true for some number x, then . . ." be written each time the addition property of equality is used. This form emphasizes that the addition property of equality claims only that if a number makes an equation true, then it will make a new equation formed by the addition property of equality true, but not necessarily conversely. Therefore, any member of the apparent truth set must be checked to determine whether or not it is a solution. The "process" of transposition, a technique often vexing to a conscientious algebra student, is never mentioned and the text is worded in such a fashion that the student will invariably use the techniques of "adding opposites" rather than "carrying-across-and-changing-the-sign."

The additive inverse of a real number x is defined to be the number y which when added to x yields 0. The first formal proof of the text establishes the uniqueness of the additive inverse of any real number and that, furthermore, the additive inverse of a real is its opposite. (The existence of the additive inverse is assumed; the uniqueness of the additive inverse is proved.) A further proof establishes that the additive inverse of a sum is the sum of the additive inverses of the addends.[1]

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[1] Ibid., pp. 135-38.

Properties of Multiplication

By use of the absolute values of the real numbers and the operations of the arithmetic numbers, SMSG defines multiplication of real numbers in such a way that the properties of multiplication of arithmetic numbers still hold for real numbers. No effort is made to make the definition "plausible" except to emphasize that an extension of the arithmetic number system to the real system demands a particular definition if the structure of the real number system is to be the same as it was for the arithmetic number system. Using this definition and the axiomated properties of the arithmetic numbers, SMSG proves the multiplication property of 1, the multiplication property of 0, and the associative and commutative properties of multiplication for the reals.

It is stated that the distributive property holds for all real numbers. (However, this could not be proved at this point since if $a \geq 0$, $b \geq 0$, $c < 0$, $a(b + c) = |a|\,(|b| - |c|) = |a|\,(b - |c|)$. But the distributive property for arithmetic numbers does not permit the statement "$|a|\,(b - |c|) = |a|\,b - |a|\,|c| = ab - ac$" since multiplication has not been shown to be distributive over subtraction!)

The multiplication properties and the theorem establishing that $(-1)\,a = (-a)$ are used extensively to justify and perform a variety of algebraic simplifications (including multiplication of polynomials by monomials, binomials by

binomials, etc.) as well as to determine the truth sets of various open sentences.

The existence and the uniqueness of the multiplication inverse of all non-zero reals is stated without proof. The multiplication property of equality, e.g., for any real numbers a, b, and c, if a = b, then ac = bc, is presented without proof.

Examination of various examples leads the student to formulate the notion that if to both members of an equation one adds a real number or multiplies by a non-zero number (in a sense, "reversible" processes), the new sentence obtained is equivalent to the original sentence, i.e., their truth sets are the same. Many exercises using the notion of equivalent sentences are presented to provide practice in finding the truth sets of open sentences. Various of these exercises are structured to emphasize that operations other than the ones listed above often lead to erroneous "truth" sets or to extraneous solutions.

The symbol "1/a" is used to abbreviate the reciprocal of a or the "multiplication inverse of a," i.e., for $a \neq 0$, $a(1/a) = 1$. In view of this agreement, it is shown that $1/1/a = a$, $(1/a)(1/b) = 1/ab$, $\left(\frac{1}{-a}\right) = -\left(\frac{1}{a}\right)$, etc. The law of cancellation (though not referred to as such) is proved and used to determine the truth sets of various sentences. This chapter also introduces the indirect and reductio ad

absurdum proof as well as a preliminary discussion of inductive reasoning.

Properties of Order

This chapter concludes the study of the structure of the real number system with an examination of the less than order relation. The less than relation has been defined earlier for number pairs in view of their relative positions on the number line, i.e., $a < b$ if and only if a lies to the left of b on the number line. This notion is used to study the addition property of order, i.e., if a, b, c are real numbers and if $a < b$, then $a + c < b + c$; and the multiplication property of order, i.e., if a and b are real numbers, and if $a < b$, then $ac < bc$ if c is positive and $bc < ac$ if c is a negative number. Many exercises regarding the truth sets of inequalities are presented.

(During the past three chapters in particular of this unit one notes a transition from an inductive to that of a deductive approach to the study of the real number system. It would seem that a student might be somewhat confused at this point insofar as trying to decide what has been proved and what has not been proved--some statements regarding the system have been referred to as properties and others which appear just as fundamental have been called theorems while some properties have been "proved" and others accepted without proof.)

SMSG now presents a notion amazingly sophistocated for freshmen algebra students which should serve to eliminate much of this confusion. It is suggested that the real number system might be considered abstractly as a set of elements for which binary operation of addition and multiplication along with an order relation less than are given with the properties that (a) addition has closure, commutativity, associativity, an identity element 0, and there exists a unique additive inverse for each element; (b) multiplication has closure, commutativity, associativity, an identity element 1, and there exists a unique inverse for every non-zero number; (c) multiplication is distributive over addition; that (d) the trichotomy property (or comparison property) and transitive property hold for the less than order relation; and that (e) the addition and multiplication properties hold for the less than relation. Thus the system of real numbers might be defined independently of the arithmetic numbers and studied as a system in its own right. Under such a consideration, real number, addition, multiplication and order relation become undefined terms, the fundamental properties become axioms, and all the other properties derived as logical consequences of these axioms become theorems. (It is noted that the level of maturity of the student prohibits the consideration of the other postulate necessary for the structure of a well-ordered field, i.e., the completeness property, i.e., if S is any set of real numbers for which there is an upper

bound, then there exists a least upper bound for S.) It is emphasized that mathematics is concerned with the study of the properties of number systems and that it is by means of proofs that one "bridges the gap" between the basic properties of the defined systems and the theorems which grow out of them.[1]

Subtraction and Division for Whole Numbers

The real number system has been described earlier as a system with the two operations of addition and multiplication. For convenience, subtraction and division are introduced and defined directly in terms of the more basic operation of addition and multiplication.

Subtraction of the real number b from the real number a is defined as adding the opposite of b to a, i.e., $a - b = a + (-b)$. This is proved to be equivalent to the theorem often given as the alternate definition (emphasizing that subtraction is the inverse of addition) for subtraction: $a - b = c$ if and only if $a = b + c$. It is shown that subtraction is non-commutative and non-associative.[2]

Division is defined in respect to multiplication in much the same way as subtraction to addition, i.e., "For any real numbers a and b ($b \neq 0$), "a divided by b" means "a multiplied by the reciprocal of b. . . . $a/b = a(1/b)$, $b \neq 0$."[3]

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[1] Ibid., pp. 202-204. [2] Ibid., pp. 209-18.

[3] Ibid., p. 223.

These two definitions and some related properties are used to show that if $b \neq 0$, $d \neq 0$, then $(a/b)\ (c/d) = \frac{ac}{bd}$. Many exercises are presented to develop skill in simplifying numerical expressions to ones in which there are no indicated operations remaining which can be performed, and in which there is at most one indicated division (if any) with the numbers being involved in the indicated division having no common factors, i.e., writing the simplest numeral for the expressions. These exercises include what have been traditionally referred to as complex fractions.

Factors and Exponents

This chapter presents a discussion which is concerned primarily with determining the prime factorization of integers. This discussion is essential because SMSG uses the ideas of prime factorization for "reducing" fractions, finding the lowest common denominator of fractions, and simplifying radicals. The discussion is fairly orthodox while introducing the terms factor, proper factor, common factor, and prime and utilizing the sieve of Eratosthenes to locate the primes. The Fundamental Theorem of Arithmetic is used to assure one that the prime factorization of an integer is unique. One of the most important aspects of this discussion lies in the statement "usually factoring over the positive integers gives us the most interesting results, and so when we speak of "factoring" a positive integer, we shall always mean over

the positive integers."[1] Many problems involving fractions in which the student determines the least common denominator by use of the least common multiples are presented.

A fairly traditional introduction to exponents is found in this chapter. The student is introduced to positive integral exponents and led to formulate "rules" for manipulation of powers. The system of exponents is extended to non-positive integral exponents by defining $a^0 = 1$ and $a^{-n} = 1/a^n$.

The many exercises involving simplification of algebraic expressions presented demand that the student state explicitly the domain of the variables involved so that the expressions will be numerals for all values of the restricted domain. Many of the exercises involved demand a sophisticated appreciation of the material of the chapter and are used as theorems in later proofs in other chapters.

Radicals

The square root operation is introduced as the inverse of the squaring operation studied earlier and is motivated by consideration of the truth set of $X^2 = b$, $b > 0$. The existence of two square roots is discussed with the positive square root of b being denoted as the square root of b

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[1]School Mathematics Study Group, First Course in Algebra, Unit 10 (New Haven: Yale University Press, 1960), p. 252.

and designated $\sqrt{x}$. This yields the observation that $\sqrt{x^2} = |x|$, $-\sqrt{x^2} = -|x|$.

The square root of 2 is shown to be irrational, i.e., it is proved by contradiction that any number whose square is 2 is not rational. (At the time of the axiomization of the real number system in Chapter 8, SMSG authors stated: "The completeness axiom is needed, for example, to prove the existence of $\sqrt{2}$. In other words, we cannot prove, using only the fifteen properties stated above, that there is a real number a for which $a^2 = 2$."[1] Therefore, this proof of the irrationality of $\sqrt{2}$ does not establish the existence of $\sqrt{2}$ but rather asserts only that any number whose square is 2 is irrational.)

Proof of theorems establishing that for non-negative numbers a and b, $\sqrt{a}\sqrt{b} = \sqrt{ab}$ and that for $a \geq 0$, $b > 0$, $\sqrt{a/b} = \sqrt{a}/\sqrt{b}$ authorizes the simplification (including rationalization of the denominator of fractions) of expressions involving radicals. Great care is exercised in the statement of the domain for the variables involved in order that the phrases involved be meaningful.

A very excellent (and easily employed) technique for the approximation of $\sqrt{x}$ is presented. In summary, this method contains these steps:

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[1] School Mathematics Study Group, First Course in Algebra, Unit 11 (New Haven: Yale University Press, 1960), p. 226.

1. To approximate $\sqrt{x}$, write the number x as a number between 1 and 100 times an even power of 10; $\sqrt{x} = \sqrt{a} \times \sqrt{10^{2n}}$.

2. Approximate $\sqrt{a}$ by some integer p between 1 and 10. Define $q = a/p$, and determine the average $(p + q)/2$ as a second approximation to $\sqrt{a}$; $\sqrt{x} \approx [(p + q)/2] \times 10^n$.

3. If a higher degree of accuracy is desired, let the second approximation assume the role of the first approximation p in step (a) and repeat the process until desired accuracy is achieved.

The Teacher's Commentary actually establishes that the error involved in these repeated approximations tends to 0, i.e., the sequence formed by these approximations converges to $\sqrt{x}$. (A very minor error is found in the text: "Thus if x is positive or zero, then $\sqrt{x^2} = x$, a positive number; . . . ."[1] If $x = 0$, $\sqrt{x} = 0$ which is neither positive nor negative.)

Polynomial and Rational Expressions

This chapter develops factoring of expressions as being analogous to the prime factorization of positive integers. The underlying definition of this chapter is that which defines a polynomial over a particular domain as being a phrase formed from variables and members of that domain with no indicated operations other than addition, subtraction, multiplication, or taking opposites. Therefore, the expression "polynomial over the integers" is a different set of expressions than "polynomials over the rationals" although the first set is a proper subset of the second. To factor a

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[1] SMSG, Unit 10, p. 292.

polynomial over a certain domain is to write a given polynomial as an indicated product of polynomials over the same domain.

The distributive property is used to structure the many techniques of factorization of polynomials including the "standard" forms found in most algebra texts for factoring quadratic polynomials, e.g., difference of squares, perfect squares, and completing the square. These techniques of factorization and the cancellation law are employed to determine the truth sets of open sentences. (SMSG includes the very welcome innovation of forcing the student to realize that the sentence "$x^2 - 6x + 8 = 0$" is equivalent to the sentence "$x - 4 = 0$ or $x - 2 = 0$"; the truth set of the sentence is therefore $\{2, 4\}$.)

A rational expression is considered as being a phrase involving real numbers and variables with at most the operations of addition, subtraction, multiplication, division, and taking opposites. Thus it is that the traditional problem of adding, subtracting, multiplying, and dividing "fractions" from conventional texts essentially becomes that of simplifying rational expressions. The analogy between rational expressions and rational numbers and between polynomials and integers is constantly emphasized.

The "long" division process for polynomials is structured on the successive subtraction of polynomial multiples of the divisor, obtaining at each step a polynomial of lower

degree. The Division Algorithm for polynomials is considered as the basic premise for the operation.

Truth Sets of Open Sentences

SMSG recalls that the procedure for solving a sentence consists of performing permissible operations (adding a real number to both members and/or multiplying both members by a non-zero real number) on the sentence to yield an equivalent sentence whose truth set is obvious. Many examples and exercises are given demanding careful attention to the concept of equivalent sentences and forcing the student to be careful to keep a record of the domain of the variable in order to maintain equivalence. The addition property and multiplication property of inequalities are used to yield equivalent inequalities.

Many exercises are used to emphasize that solution to sentences might inadvertently involve the solution sets of compound sentences, e.g., "$a/x = b$" is equivalent to the sentence "$a/x = b$ and $x \neq 0$" since a/x is not a numeral if $x = 0$; "$x^2 = 16$" is equivalent to "$x - 4 = 0$ or $x + 4 = 0$"; "$ac = bc$ is not equivalent to "$a = b$" but rather to "$a - b = 0$ or $c = 0$." Many problems using simplifying processes which do not maintain equivalence, e.g., squaring both members, are presented.

Graphs of Open Sentences in Two Variables

This chapter extends the graph of sentences from the line to the plane by introducing the Cartesian coordinate axes and associating points of the plane with ordered pairs of real numbers (and conversely). The truth set of a sentence in two variables is defined to be the set of all ordered pairs which make the sentence true and the graph of the sentence is the set of all points whose coordinates are members of their truth set.

Some work is presented with graphs of open sentences involving integers only along with graphs of sentences involving absolute values and inequalities. Certain compound sentences are also graphed by taking the union of the graphs of the component simple sentences if the connective is or and by determining the intersection of their graphs if the connective is and.

Systems of Equations and Inequalities

This chapter contains a graph-oriented study of systems of simultaneous linear equations. Probably the most important agreement is that the system

$$\left.\begin{array}{l} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{array}\right\}$$

may be represented as the conjunction "$Ax + By + C = 0$ and $Dx + Ey + F = 0$." Since the truth set of the conjunction may be determined by finding the intersection of their graphs, a

graphical process for solving systems of sentences is available.

It is shown that if $Ax + By + C = 0$ and $Dx + Ey + F = 0$ are the sentences of lines intersecting at exactly one point and if h and k are any two real numbers, then $h(Ax + By + C) + k(Dx + Ey + F) = 0$ is the sentence of a line passing through the point of intersection of the two lines. Therefore, one may choose multipliers h and k to obtain a line which is vertical (or horizontal) and which passes through the intersection point. Consequently, one may "solve" a system by determining two lines one of which is vertical and the other horizontal and both of which pass through the intersection of the lines of the system, e.g., "$2x + y - 4 = 0$ and $x - y + 1 = 0$" may be solved by writing the line $h(2x + y - 4) + k(x - y + 1) = 0$, choosing $h = 1$, $k = 1$ to obtain the sentence "$x = 1$" and then choosing $h = 1$, $k = -2$ to obtain the sentence "$y = 2$." This method, of course, is the graphical analogy of the method of addition for solving systems. The substitution method is also developed.

Quadratic Polynomials

In this chapter the parabola is examined by observing the changes which occur when in the simple sentence $y = x^2$, the polynomial is multiplied by a, as some number h is added to x, and as some number k is added to x^2. This leads to the changing by the completion of the square of any quadratic

sentence $y = Ax^2 + Bx + C$ into the standard form $y = A(x - h)^2 + k$--a form which facilitates graphing of the sentence. The relationship between the truth set of $Ax^2 + Bx + C = 0$ and the points of intersection with the x-axis of the graph of $y = Ax^2 + Bx + C = a(x-h)^2 + k$ is noted.

The standard form of the quadratic polynomial is used to solve quadratics. "$x^2 - 2x - 2 = 0$" may be written as "$(x - 1)^2 - 3 = 0$." But this sentence is equivalent to "$(x - 1 - \sqrt{3})(x - 1 + \sqrt{3}) = 0$", which in turn is equivalent to "$x - 1 - \sqrt{3} = 0$ or $x - 1 + \sqrt{3} = 0$." Therefore, the truth set is $\{1 + \sqrt{3}, 1 - \sqrt{3}\}$. (It is interesting to note that $x = \frac{-b \pm b^2 - 4ac}{2a}$ is never used to determine the truth set of a quadratic sentence except in an exercise in Chapter 17--a chapter which will probably not be studied by a majority of students.)

Functions

This chapter developing the concept of function is included here only for the consideration of selected classes of better students as well as exceptional individual students.[1] A function is defined as follows:

> Given a set of numbers and a rule which assigns to each number of this set exactly one number, the resulting association of numbers is called a function. The given set is called the domain of definition of the function, and the set of assigned numbers is called the range of the function.[2]

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[1] Ibid., p. 573. [2] Ibid., p. 576.

The functions utilized for consideration in this chapter are real functions since the range and domain are restricted to the reals. The essential idea of the function as presented by SMSG is found in the actual association from numbers in the domain to numbers in the range and not in the particular way in which the association happens to be described and it is emphasized that not all functions can be represented by algebraic expressions. The terminology of independent and dependent variable so often used where the variables x and y are related by some function f, i.e., $y = f(x)$, is avoided. The student considers linear and quadratic functions as well as "bracket" functions involving absolute values.

Evaluation of Units 9 and 10

The keynote of these units is the development of the materials by the use of a discovery approach. Many of the exercises are independent of the preceding problems and in some of the exercises there is very little continuity but, in general, the review exercises at the end of each chapter and the very excellent textual discussions (abounding in "How?," "Why?," "How do you know?," "Do you notice?," etc.) compensate for the occasionally weak exercises. The spiral approach and the "from inductive to deductive" approach to proofs with their associated "rules" should serve to implement mastery of concepts and skills. By the presentation of leading questions and through the verbalization presented in the texts, the

student is given ample opportunity to discover the basic structure of algebra.

The SMSG First Course in Algebra maintains an excellent balance between concepts and skills. Many of the "drill" exercises common to traditional texts are now needless (or at least to the degree exercised in the past) in view of the developed understanding of the algorithms on the part of the student. In most instances, however, new ideas and their significance are emphasized by multitudes of well-placed, well-structured exercises, which develop and demand understanding of the topics under consideration. In addition, the teacher's commentary provides suggested test items and review problems for each chapter.

First Course in Algebra attempts to develop in the student the ability to think in terms of general patterns and structures so that he may make pertinent social applications. The materials presented stress the ability to construct and manipulate in an original fashion the tools of structural mathematics rather than to manipulate in a rote fashion the symbols involved. Insofar as social applications are introduced with each section of exercises rather than in sets all of which follow a certain algorithm for structure and solution, very few sets of exercises involving just "applied" problems are found. Some critics have shown alarm over the lack of such problems but the writer of this paper has

counted at least 267 "story" problems in addition to the multitude of other types of problems.

The mathematics materials found in these texts are, with very few exceptions such as prime numbers, functions, and deductive proofs, the same as those found in traditional texts except that of course a vastly different and modernistic approach is utilized. Very few topics covered in traditional texts are omitted. However, the traditional texts would not be concerned with the general, abstract proofs nor would they exercise as much precision in the definition and language used in the discussion of variables, order, relations, absolute values, sentences, etc., and factoring although the topics would be covered.

The vocabulary for this course is fairly similar to that found in traditional texts although a considerably more precise language is used. New terms introduced include sets, sentences, phrases, open sentences, opposites, inverses, equivalent, truth sets, etc. The student should become very aware of the role of the definition in mathematical thought. Extensive use is not made of symbols such as $\cup$, $\cap$, $\forall$, $\exists$, $\ni$, $\Rightarrow$, etc., although their antecedents are verbalized quite regularly.

The "proofs" involved progress from inductive to deductive in nature. It is not until Chapter 6 that the student sees a deductive proof but the textual material and exercises from that point on provide ample opportunity for

development of direct deductive proofs, indirect proofs, proofs by contradiction, etc., and the use of the tools of precise definitions, axioms, and theorems to sharpen these proofs. It appears, however, that the student may have considerable difficulty in ascertaining just what has been proved and what has not been proved since many "fundamental" theorems are presented without proof whereas many which appear equally "fundamental" are presented along with fairly rigorous proofs. Of course, students at this level are incapable of exploring and proving everything about the real number system.

Geometry, Units 13 and 14

Introduction

Any fruitful consideration of the scope and sequence of the SMSG Geometry requires an understanding of the goals and objectives of the authors. The preface to Geometry contains these revealing statements by the authors:

> We began and ended our work with the conviction that the traditional content of Euclidean geometry richly deserves the prominent place which it now holds in high-school study, and have made changes only when the need for them appeared to be compelling. . . . The basic scheme in the postulates is that of G. D. Birkhoff. In this scheme, it is assumed that the real numbers are known, and they are used freely for measuring the distance and angles. . . . If we assume the real numbers, as in the Birkhoff treatment, then the handling of our postulates becomes a much easier task, and we need not face a cruel choice between mathematical accuracy and intelligibility. . . . It seems a good idea in itself to connect up geometry with algebra at every reasonable opportunity so that knowledge in one of the fields will make its natural contribution to the understanding of both. . . . We

> have, therefore, based the design of both the text and the problems on our belief that intuition and logic should move forward hand in hand.[1]

In view of these philosophies, the authors assume some knowledge on the part of the student of the real number system. This very modern approach to Euclidean geometry places great emphasis on the fundamentals of geometry and the texts attempt (though not demanding a logically complete system) to give a complete foundation of postulates and definitions as well as constant reinforcement of developed basic concepts through a multitude of well-constructed exercises emphasizing discovery and understanding. The texts are quite verbose as compared with traditional texts.

The Teacher's Commentary (Units 16 and 17) has several unusual characteristics. In addition to a running commentary containing pertinent questions to consider, solutions to all but the simplest of the exercises, discussion of questions which are likely to arise, and the mention of points which should be emphasized, these commentaries contain some rigid proofs of some of the theorems whose rigor forces the discussions to be logically incomplete for student consumption. These commentaries also classify every exercise in the text as being members of one of three classes: (1) problems that relate directly to the text; (2) problems that are similar to, and yet a bit more difficult than the first group

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[1]School Mathematics Study Group, Geometry, Unit 13 (New Haven: Yale University Press, 1960), Preface.

and which may be used for additional drill problems and as challenge problems for better students; and (3) enrichment problems that develop ideas as extensions of the information provided in the text. These commentaries also provide "Talks to the Teachers," a series of short, teacher-enrichment essays dealing with some topics, e.g., "Facts and Theory," "Equality, Congruence, and Equivalence," "Introduction to Non-Euclidean Geometry," and "Miniature Geometries," that cannot be dealt with conveniently and directly with particular topic discussions in the text. Also provided are lists of review exercises and possible text items.

Common Sense and Organized Knowledge

This chapter is evidently constructed with the purpose in mind of presenting the difference in magnitude of difficulty existing between various mathematical problems. It is impressed upon the student that once the basic information regarding a problem has been analyzed and organized, it is necessary to remain within the framework of this information to "solve" the problem.

The structure of a definition is explained from the viewpoint of "substitution," i.e., a definition is merely a mutually-agreed-upon substitution of a single word (or words) to serve as a synonym for a more complicated (and perhaps lengthier) expression. As a result, it is never necessary to "prove" a definition--a definition is merely an inter-subjective agreement to be used in communication. In view of

the fact that all definitions utilize other words and terms in their statements, it is apparent that it is impossible to "define" all ideas, concepts, and other entities. Consequently, some terms must be utilized without definition; these, of course, are to be regarded as undefined terms and their meanings must be common to all undertaking a discussion. The undefined terms of the SMSG treatment of geometry are introduced as the point, line, and plane. The authors should also admit that they are regarding straight as undefined when line is restricted to mean straight line--an agreement necessary to the later "definition" of between.

The general nature of the mathematical proof suggests that these definitions, undefined terms, and resulting theorems will be used to construct and prove other theorems. It becomes apparent that there can be no "first" theorems proved since no theorems would be available for use as foundations. Consequently, certain basic geometrical "statements" will be used and accepted as true without further substantiation as to the "truth" of these statements. The oft-quoted definition of these postulates as being "self-evident truths" is not suggested. SMSG Geometry suggests that any geometry is an invention to describe a particular system, e.g., the Euclidean geometry is an excellent approximation to 2-dimensional and 3-dimensional physical environment and physical space but other systems are more functional under different conditions. It is suggested that the "purpose of stating

postulates is to make it clear just where we are starting, and just what sort of mathematical objects we are studying."[1] Thus, one can then build up a solid, organized body of "facts" about those mathematical objects.

Sets, Real Numbers, and Lines

Chapter 2 immediately presents a radical departure from the traditional text in plane geometry. A very excellent, though brief, discussion of the fundamentals and language of set theory is introduced. No attempt is made to present this topic as one of great sophistication but rather an effort is made to cause the student to recognize that this is merely a convenient tool--a mathematical shorthand--to be utilized as need arises. Considerable rigor is displayed in examples and problems but rigorous notation and symbolism is not emphasized, e.g., the symbols $\cup$, $\cap$, and $\subset$, (union, intersection, and contained in or is a subset of) are not used. Plane and line (and indeed all geometric figures) are considered as sets of points. Sufficient terminology is introduced to provide the capable student with a working group of point-set theory although the study is not exhaustive.

Another unorthodox feature (as compared to traditional texts) of this chapter is the presentation of an excellently-written topic dealing with the real number system, discussing quite adequately the positive, negative, and zero integers,

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[1] Ibid., p. 9.

rational numbers, irrational numbers, and the notion that the set of real numbers is merely the union of these sets. The concept of number itself is left dormant and the assumption is made that either the student already appreciates the abstraction of number or that such rigor is not needed at this point.

The result of this discussion is presented in the statement of the "ruler postulate" which is usually reserved for more sophisticated discussions and which permits the coordinatization of a line: "The points of a line can be placed in a correspondence with the real numbers in such a way that to every point of the line, there corresponds exactly one real number and to every real number there corresponds exactly one point of the line."[1] This postulate, along with the developed properties of the real number system of uniqueness of order, transitivity of order, inequality relations, existence of square roots, absolute value, etc., enables the authors to lend an algebraical flavor to many geometrical concepts.

The Ruler Postulate and the two subsequent postulates, the Ruler Placement Postulate ("Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive."[2]) and the Distance Postulate ("To every pair of different points there corresponds a unique positive number."[3])

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[1] Ibid., p. 36. [2] Ibid., p. 40. [3] Ibid., p. 34.

anticipate the consideration of the distance between two points A and B (or the length AB) as being the absolute value of the difference of the corresponding numbers on the real number scale. The introduction of the real numbers in these postulates serve as a pedagogical device at this level to avoid the necessarily sophisticated study of measure theory.

The evasive term of "betweenness on a line" is also dealt with very adequately by reliance upon the introduction of correspondence of points with the set of real numbers and the undefined terms straight line, or line. B is to be considered as being between A and C if and only if A, B, C are distinct points on the same line and the measure of the distance between A and B added to the measure of the distance between B and C is equal to the measure of the distance between A and C. The segment $\overline{AB}$ is considered as the set of points having as elements the end points A and B together with all points that are between A and B. Considerable emphasis is placed on the difference between the segment (a geometrical figure, or set of points) and the length (a real number) of the segment. Similar statements suffice to define a ray $\overrightarrow{AB}$ as the set which is the union of segment $\overline{AB}$ and the set of all points C such that B is between A and C. This definition allows the definition of opposite rays, i.e., if A

is between B and C, then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called opposite rays.[1] Sufficient notation is introduced to serve as sign vehicles for these ideas, e.g., the line through A and B is represented by $\overleftrightarrow{AB}$, the line segment connecting A and B is represented by $\overline{AB}$, the ray emanating from A and containing B by $\overrightarrow{AB}$, and the measure of $\overline{AB}$ by AB.

The student as of yet has not been required to prepare formal proofs but rather merely to answer searching questions which have been prepared in such a fashion as to demand a working knowledge of the materials presented.

Lines, Planes, and Separation

As any person who has ever taught plane and solid geometry realizes, much effort must be spent in plane geometry in "forcing" the student to restrict his thinking to a plane while an equal amount of time must be spent in solid geometry trying to stimulate the student to enlarge his field of thought to 3-space. SMSG, feeling there should be no separation of solid geometry from plane geometry, faces the issue squarely and discusses some of the ideas of solid geometry early in the course. It is with this view in mind that, remembering the undefined terms of point, line, and plane, this definition is presented: "The set of all points is called space."[2]

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[1] Ibid., p. 46.

[2] Ibid., p. 53.

In this chapter is found a very "modern" concept, namely that of convex set. A set of points A is referred to as convex if for any two points P and Q of A, the entire segment $\overline{PQ}$ lies in A, i.e., $\overline{PQ}$ is a subset of A. This contrasts vividly with the standard usage that a polygon is convex if all its angles are less in degree-measure than 180. The modernistic idea of convex sets as presented by SMSG does not restrict itself to plane polygons and lends itself quite readily to discussion of regions, interiors of circles and spheres, etc.

Finally, the separation of a plane into two half-planes (which are themselves convex sets) by one of its lines and the separation of a space into half-spaces (also convex sets) by one of its planes is postulated. This again is a concept usually mentioned briefly (or not at all) in standard texts on the pretext that either it is too sophisticated for secondary students or that it is too obvious for discussion. Its importance in the SMSG text lies in the discussion of the idea of "oppositeness," i.e., opposite sides of a line or plane, a notion often assumed as undefined but which is troublesome to rigorous geometers.

Angles and Triangles

In traditional high school texts, interior of an angle is taken for granted since a point "obviously" lies inside an angle or it lies outside the angle. The earlier discussion

of betweenness and same side of by SMSG allows the authors to adequately define the interior of an angle.

This chapter also presents some very careful definitions for angles, triangles, etc. An angle is defined to be the union of two rays having the same endpoints but not being subsets of the same line; hence no possibility for a "straight" angle or a "zero" angle. A triangle is defined as the union of the segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, where A, B, and C are any three non-collinear points.[1] (These definitions demand the rather unusual result that, although a triangle determines three angles, it cannot be said that, since the sides of an angle are rays of infinite length, the triangle contains the three angles.) A very careful discussion is presented as to the conditions under which a point lies inside a triangle, a discussion leading to the consideration of the elements of the interior of a triangle as the intersection of the interiors of the three angles determined by the triangle.

The Angle Measure Postulate, the Angle Construction Postulate, and the Angle Addition Postulate permit the coordinatization of an angle in a fashion analogous to that used to coordinatize a line. The real number x ($0 < x < 180$) corresponding to a particular angle $\angle ABC$ is defined as the measure (actually the degree-measure) of the angle (and written $m(\angle ABC) = x$). It again will be noted that under

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[1] Ibid., pp. 71-72.

the definition of the measure of an angle, there is no angle whose measure is 0 or 180.

If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays and $\overrightarrow{AD}$ is another ray then $\angle BAD$ and $\angle DAC$ form a linear pair, and it is postulated that if two angles form a linear pair, they are supplementary and hence the sum of their measures is 180. A right angle is defined as an angle which is one of a linear pair of two angles having the same measure; hence the degree-measure of a right angle is 90.

Another novel definition is presented in this chapter in the definition of two angles as being congruent if their measures are the same real number. This treatment is triggered by the gross misuse of the word equal. The statement that $\angle ABC \cong \angle CDE$ indicates simply that these angles (or both sets of points) have the same measure, i.e., m ($\angle ABC$) = m ($\angle CDE$), and may or may not be equal sets.

Congruences

The conventional plane geometry text describes two congruent figures as being figures such that one may be made to coincide with the other without "changing their shapes." Such a demand implies assumption of various properties of geometrical figures which are invariant under such transformations.

The SMSG materials present an approach to congruency somewhat in contrast with this approach. Two triangles,

$\triangle ABC$ and $\triangle DEF$, for example, are considered congruent if at least one congruence exists between them. A congruence exists between these two triangles if (but not only if) there is a one-one correspondence ABC $\leftrightarrow$ DEF between the vertices such that the corresponding angles are congruent and such that $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\overline{BC} \cong \overline{FE}$. It is further emphasized that $\triangle ABC = \triangle DEF$ implies that there exists at least one such correspondence between these two triangles but there might exist more than one congruence, e.g., the triangles may be equilateral in which case A $\leftrightarrow$ D, B $\leftrightarrow$ E, C $\leftrightarrow$ F (or ABC $\leftrightarrow$ DEF) and A $\leftrightarrow$ E, B $\leftrightarrow$ D, C $\leftrightarrow$ F, (or ABC $\leftrightarrow$ EDF) would both be suitable one-one correspondences between the two triangles.) In view of this approach the statement $\triangle ABC \cong \triangle DEF$ yields this wealth of information: $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, $BC \cong EF$, $\angle A \cong \angle D$, $\angle B \cong \angle E$, $m(\angle A) = m(\angle D)$, $m(\angle B) = m(\angle E)$, $m(\angle C) = m(\angle F)$, $AB = DE$, $AC = DF$, and $BC = EF$. The notation ABC $\leftrightarrow$ DEF is particularly applicable in that it not only identifies correspondence (or congruence) but also within itself labels the "corresponding sides" and "corresponding angles," properties which often cause the student considerable difficulty in identification.[1] The Side Angle Side Postulate is stated in practically the same language utilized by Hilbert and is introduced as the preliminary step for the attack of the problems of determining the congruence of triangles.

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[1] Ibid., p. 115.

It is in this chapter (and not before) that the student is required to construct proofs. Although some "paragraph" proofs are used by the authors, the form utilized for proof by the student is essentially that of the traditional synthetic proof with the exception that the rigorous language already introduced is used.

A Closer Look at Proof

Prior to this point, the student has had only casual contact with complicated and rigorous proofs although a very inclusive vocabulary and a broad set of postulates (and several theorems) have been introduced. It is in this chapter, however, that the student is given an introduction into the rigor of a "good" deductive proof and the place and role of the arbitrarily-stated postulate and mutually-agreed-upon definition are made apparent. It is also suggested that in the solution of a geometry problem, the problem is translated into a special language to be utilized for solution but that the solution of the problem is independent of the environment leading to the problem. This then places the logic, symbol manipulation, etc., into a separate realm than the geometry problems themselves.

Logic and its structure is discussed adequately through informal processes. Consequently, the only apparatus of logic that is introduced is that which is necessary to the use of geometrical proofs. Very little of the symbolism of

logic is introduced and the treatment is somewhat that of an intuitive approach. The indirect proof is introduced as a powerful tool in "contradiction-of-assumption" proofs.

An unorthodox innovation of this text lies in its expenditure of considerable time and discussion in the clarification and recognition of the necessity of consideration of the properties of existence and uniqueness and its emphasis that one does not necessarily imply the other. The careful concern over justifying existence and uniqueness becomes particularly important when points, lines, segments, and other auxiliary sets not accounted for by the conditions of a problem are introduced into a proof. Emphasized also are the necessity for proof of "obvious" statements and the inherent dangers which lie in proofs which are based upon the examination of a geometric figure.

Geometric Inequalities

This particular chapter deals with the standard theorems developed in plane geometry relating to the inequalities inherent in geometrical considerations. The material covered in this chapter is quite similar to that found in corresponding chapters of traditional texts with the major innovation being that line segments and/or angles are compared through the property of their measure--a result of the discussion of the real number system and the consideration of the Ruler Postulate for the coordinatization of the line.

Therefore, although the inequalities being considered describe and quantify geometric relations, they involve only real numbers.

Perpendicular Lines and Planes in Space

The material presented in this brief chapter deals primarily with lines and their relationships with planes--perpendicular, parallel, etc.,--in 2-dimensional and 3-dimensional geometry. The treatment is largely intuitive although some effort is directed toward examination of the existence and uniqueness of lines perpendicular to planes, etc. Of course, this general topic traditionally has belonged to solid geometry but the postulates and theorems of earlier chapters make its consideration very feasible and advantageous at this point.

Parallel Lines in a Plane

This chapter presents the standard definitions and theorems (except that they are couched in set language) of "parallel" properties of lines in a plane along with the discussion of transversals, alternate interior angles, etc., as developed in all geometry texts. Accompanying this discussion is the very carefully-stated and concise definition of a quadrilateral as a union of line segments which leads to the proofs of the various theorems involving the many types of quadrilaterals, e.g., the square, the rhombus, and the rectangle.

Parallels in Space

This chapter develops, through a fairly conventional treatment, a study of the properties of parallelism and perpendicularity of lines and planes in space and extends these properties to the consideration of the projections of figures on a plane. The major departure from traditional texts is that set-language is utilized for the definition of dihedral angles (as the union of a line and two non-coplanar half-lines having this line as their common edge) and the measure of a dihedral angle as being the measure of any of its plane angles. This, of course, is equivalent to the coordinatization of a dihedral angle. It is noted that this entire chapter dealing essentially with 3-dimensional geometry would be found traditionally in a solid geometry text rather than a plane geometry text.

Areas of Polygonal Regions

Due emphasis is placed in this chapter upon the difference between a polygon and the region that it bounds, or a polygon and its interior. Such emphasis is necessary in that the polygon has been defined as a set of points and as such does not "contain" anything except the points in its sides. Consequently, SMSG uses as the basic tool the triangular region defined as the union of a triangle and its interior points since one can fairly easily define interior insofar as a triangle is concerned. A polygonal region can be considered

always as the union of a finite number of coplanar triangular regions such that if any two of these regions intersect, the intersection is either a segment or a point. It is further postulated that to every polygonal region there corresponds a unique positive number with the area of the polygon being defined as that number, and it is duly emphasized that the measure of the region is the property being discussed when one mentions the area of a polygon. It follows that the area of a polygonal region may be determined by considering the sum of the areas of the component triangular regions. One notes that the SMSG approach to the study of area is by postulation rather than by attempting to derive areas from a definition based on a measurement process. It is further postulated that the area of a rectangle (a real number) is the product of the length of its base (or at least its measure) and the length (or measure) of its altitude.[1] Consideration of the rectangle and various transformations allows the structuring of "formulas" (stated in theorem form) for determining the measures of the areas of squares, parallelograms, triangles, trapezoids, etc., along with a proof and discussion of the Pythagorean Theorem. A multitude of mensuration problems using these developed theorems is provided.

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[1]School Mathematics Study Group, Geometry, Unit 14 (New Haven: Yale University Press, 1961), p. 322.

Similarity

The most drastic difference between this topic and its counterpart in traditional texts is the very definition of similar triangles: "Given a correspondence between the vertices of two triangles. If corresponding angles are congruent and the corresponding sides are proportional, then the correspondence is a similarity, and the triangles are said to be similar."[1] It must be noted that the definition is a particularly apt one since the duality of conditions leads to a quite natural extension to similarity of general polygons. It is later shown that the existence of either of the properties of similarity for triangles implies the other but that such is not the case for the general polygon in which both conditions must be satisfied.

Circles and Spheres

Chapter 13 develops the standard terminology and theorems for circles as developed in conventional texts except that distinction is made between a circle and its interior. The interior is defined very adequately as being the union of the circle and the set of all points in the plane of the circle whose distances from the center are less than the radius. The common properties and theorems regarding a circle are extended to its 3-dimensional counterpart, the sphere, and it is in this extension that departure from tradition is noted.

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[1] Ibid., p. 365.

Characterization of Sets Construction

This chapter deals very efficiently with the traditional material of loci and simple straight-edge-and-compass constructions with a largely conventional treatment. The really significant innovation is the use of characterization of a set rather than locus--as a matter of fact, the term locus is not used. Therefore, the authors concern themselves with defining, or characterizing, a given set of points by means of the property which each and every element of the set must satisfy, e.g., a circle is characterized as the set of points in a given plane at a given distance from a given point.

The basic theorems on concurrence of angle bisectors, side bisectors, medians, etc., of triangles are considered. The techniques of Euclidean construction (using compass and straight edge) are introduced and the impossible construction problems of antiquity--trisection of the angles, duplication of the cube, and the squaring of the circle--are discussed.

Areas of Circles and Spheres

The area and the circumference of a circle are defined as being the limiting value of the area and perimeter, respectively, of an inscribed regular polygon as the number of sides increases without limit. No rigid discussion of the meaning of (or the conditions for the existence of) a limit is present but an excellent intuitive approach is utilized.

A similar idea involving the sum of equal chords of a circular arc is used to intuitively derive the expression for the measure of an arc and the area of a circular sector.

Volumes and Solids

This chapter, a chapter never found in traditional plane geometry texts, develops quite rigorous definitions and proofs concerning various 3-dimensional solids. Cavalieri's Theorem is postulated and serves as the key to the derivation of relations for determining the volume measures of prisms, pyramids, cylinders, and cones. Relations for the area and volume measures of spheres are developed by reliance upon the intuitive limit notion discussed earlier.

Plane and Coordinate Geometry

The SMSG text presents a well-written discussion of 2-dimensional, coordinate geometry which is "just about enough to give you an idea of what it is like and how it works."[1] Sufficient study is devoted to the system to develop plotting of points on a coordinate plane, the distance formula, the midpoint formula, the slopes of a line and its use in determining perpendicularity and parallelism, equations of straight lines, the canonical equation of the circle, and analytic proofs of some of the theorems proved earlier by synthetic techniques. This chapter is not essential to the

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[1] Ibid., p. 567.

continuity of the course but serves rather as an enrichment topic.

Evaluation of Units 13 and 14

SMSG's Geometry is devoted mainly to Euclidean plane geometry with some chapters on solid geometry and a short introduction to analytic geometry. The chapter on analytics is not necessary to the development and continuity of the course and might be considered as enrichment. All of the theorems and corollaries found in traditional texts are present in these texts. A very precise point-set language is maintained throughout along with a careful statement of postulates based on the Birkhoff postulates. Through the use of the real numbers for coordinatization of lines and angles, a pedagogical instrument is available to discuss distances, lengths, inequalities, etc., and, consequently, a flavor of algebra is noted in this treatment. There is an adequate display of material for a full year's work.

These units emphasize mathematical structure and the student should achieve a better understanding of the very nature and philosophy of mathematics. The student intuitively examines a concept, observes its verbalization into a precise language, and then studies the application thereof.

Most of the definitions presented are first discussed from an intuitive viewpoint and then stated in precise terminology. This precision in statement is maintained without

excessive length and the materials are consistent in the use of these precise definitions once they have been formulated. Many terms new to conventional treatments are found, e.g., set, member, union, intersection, half-line, half-plane, separation, half-space, one-to-one correspondence, congruence, ordered-pair, linear pair, measure, region, convex set, region, oppositeness, uniqueness, betweenness, etc. The authors never use the term plane geometry but rather the simple, more-inclusive term geometry. Intensive use of symbolism is not employed, e.g., $\cup$, $\cap$, $\in$, $\subset$, $\Rightarrow$, $\Leftrightarrow$, $\exists$, and $\forall$ are never introduced and set-builder notation is not available.

Again the keynote of these texts is student discovery and appreciation with the student participating in the intuitive processes that establish conjectures, and then helping to formulate formal proofs. The student is not given a system to use and apply but rather it is hoped that the student will become a contributing part of the development of the system and thereby appreciate more fully the materials. Although these texts employ a metric approach and contain considerably more algebra than traditional texts, the traditional structure of geometry is preserved.

There are very few problems (other than several mensuration problems) that are of a social nature. These problems that are "applied" in nature generally occur in the

intuitive discussions and occasionally a physical situation is described to illustrate a theorem being formulated.

The proofs, stated in essentially the same form as the traditional synthetic proofs, are as a whole, fairly complete and rigorous for the students at this level. The authors do not attempt to camouflage proofs which are too involved and rigorous for student appreciation but rather emphasize these "gaps." The textual material engages, as a rule, the "paragraph" proofs accompanied with adequate discussions. Formal logic is not included.

Intermediate Analysis, Units 17 and 18

Introduction

Units 17 and 18, Intermediate Analysis are the first units of the SMSG materials in which the SMSG curriculum deviates in a drastic fashion from its traditional counterparts. Several very important characteristics of the text are immediately ascertained by a glance through the chapters. Since only one year is devoted at the tenth grade level to both plane and solid geometry, additional time is present for eleventh grade consideration of additional topics. The authors have elected to devote this extra time to trigonometry, vectors, and a more extensive treatment of complex numbers than ordinarily attempted at this grade level. The controlling philosophy of the writers seems to be that one of the fundamental objectives of these two units is to advance

the student's understanding of number systems and their structures. These units demand that a student possess good manipulative understanding of the real number system as well as a fairly adequate background in geometry. The scope and sequence of the materials reflect a highly optimistic attitude on the part of the writers in that there is an abundance of material to be digested in the course. As a matter of fact, any student who thoroughly understands and possesses hoped-for manipulative skills with the materials could, with very little enrichment, pursue a beginning course in the calculus.

Number Systems

This chapter is dedicated to the careful study of the natural numbers, the integers, the rational numbers, and the real numbers. The treatment of this chapter of number systems never utilizes the number line. This treatment does not begin with the natural numbers being "given" and then extended, successively, to each of the other mentioned systems but rather studies the systems through postulation and definition and then examines the properties which these systems share. However, the student should easily notice that the new systems being constructed would have all the algebraic properties of the old system, would include all the numbers of the old system in such a way that the new and old algebraic operations when applied to numbers of the old system, would be the same, and contain new numbers of the kind

needed for some purposes for which the old system was inadequate.

The system N of natural numbers is defined as a set having as members the numbers 1, 2, 3, 4, 5, 6, . . . and having the two operations of addition and multiplication, an equals relation, and the order relation less than defining a to be less than b ($a < b$) if and only if there exists a natural number c such that $a + c = b$. The equals relation possesses the dichotomy property (E_1), the reflexivity property (E_2), the symmetry property (E_3), the transitivity property (E_4), the addition property (E_5) which states that if $a = b$, then $a + c = b + c$, and the multiplication property (E_6) stating that if $a = b$, then $ac = bc$. The operation of addition possesses the closure property (A_1), the commutativity property (A_2), and the associative property (A_3). The multiplication is defined to have the closure property (M_1), the commutativity property (M_2), the associativity property (M_3), and a multiplicative identity (M_4) with the additional property that multiplication is distributive over addition (D). The less than relation has the properties of trichotomy (O_1), transitivity (O_2), addition (O_3) stating that if $a < b$, then $a + c < b + c$, and multiplication (O_4 (N)) stating that if $a < b$, then $ac < bc$, c in N. Additionally, the system possesses the Archimedean property (O_5) that if a and b are any given natural numbers such that $a < b$, there is a natural number n such that $na > b$, and the well-order property (O_6).

Accompanying these postulates are several theorems which may be (and are) proved within the framework of these definitions and properties. In addition the cancellation properties (C_1, C_2, C_3, C_4) for equality and order--actually the converses of E_5, E_6, O_3, and O_4, respectively--are considered and C_2, C_3, C_4 are illustrated as being proved from the other properties.

The E, A, M, D, C, and O properties listed above are regarded as forming a logical basis of the natural number system. The authors state:

> In organizing the natural number system deductively, these basic properties may be assigned the role played by the axioms and postulates in the deductive organization of geometry. From them we may derive as theorems the other algebraic properties of the natural number system.[1]

In view of this statement and the statement by SMSG in an earlier unit suggesting that the desirable characteristics of a postulate system are simplicity, paucity, consistency, independence, and completeness,[2] it seems strange that the authors list three order properties (O_2, O_3, O_4) which may be proved directly from the "addition" definition of less than and the cancellation properties for equality and order (C_1, C_2, C_3, and C_4) as basic properties of the natural number system. In other words, the basic properties of the natural

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[1]School Mathematics Study Group, Intermediate Mathematics, Unit 17 (New Haven: Yale University Press, 1960), p. 11.

[2]School Mathematics Study Group, Geometry, Unit 16 (New Haven: Yale University Press, 1960), pp. 583-86.

number system (as identified by SMSG) when considered as axioms or postulates do not yield themselves to the SMSG's notion of a desirable postulate system since some of them may be proved within the framework of the others, i.e., they are not independent.

The system I of integers has defined as its members the numbers . . . , -3, -2, -1, 0, 1, 2, 3, . . . with the same operations and relations as the naturals. The system I possesses all the E, A, M, D, properties of the naturals and, additionally, the additive identity property (A_4) that a + 0 = a for any member of I and the subtraction property (A_5)--used to "define" subtraction--that for each pair of integers a and b, there is exactly one integer c such that a + c = b.[1] The additive inverse of an integer a is defined by the equation a + x = 0 (and assumes the role played by the opposite in earlier units). Since the well-order property is not valid for the integers, it is replaced by the discrete property stating that if a and b are integers and $a < b$, then $1 \leq b - a$.

The system Q of rational numbers is defined to have as elements all the integers and "numbers" of the form a/b ($b \neq 0$), a and b integers. Equality of rationals is carefully defined: $a/b = c/d$, $b \neq 0$, $d \neq 0$, if and only if $ad = bc$. The sum and product of the elements of Q are defined:

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[1]SMSG, Unit 17, pp. 24-42.

$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ and $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Less than is similarly defined: $a/b < c/d$ if and only if $ad < bc$.[1] From these definitions, it follows that Q possesses all the E, A, M, D, C, and O properties of Q and also the division property (M_5) which postulates that for each pair of rational numbers, a, b, $b \neq 0$, there is exactly one rational number c such that $bc = a$, i.e., division as the inverse of multiplication is closed in Q. (Of course, the discrete property of N and I no longer hold; this system Q is dense, i.e., between any pair of distinct rational numbers, there are infinitely many rational numbers.)

The construction of the real number system R is approached by the consideration of decimal expressions and the set of real numbers is considered as the set of all decimal expressions. The relations equals and less than are defined in view of nth place truncation to rationals, i.e., these definitions are based on those relations which we already possess for the rationals. In view of these definitions, the authors merely suggest that, formidable though it is to prove such, this set of numbers does possess all of the E, A, M, D, O properties of Q and also possess one new order property O_7 (R):

> If $\{a_0, a_1, a_2, \ldots, a_n, \ldots\}$ and $\{b_0, b_1, b_2, \ldots b_n, \ldots\}$ are two sequences of real numbers with the properties

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[1] Ibid., pp. 43-64.

(i) $a_0 \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$
(ii) $b_0 \geq b_1 \geq b_2 \geq b_3 \geq \ldots \geq b_n \geq \ldots$
(iii) $a_n < b_n$ for every natural number n
(iv) $b_n - a_n < 1/10^n$ for every natural number n
then there is one and only one real number c such that $a_n \leq c \leq b_n$, for every natural number n.[1]

The chapter also presents a thorough review of polynomials and their factors and rational expressions as well as the determination of truth sets of sentences (some including absolute values) by consideration of the definitions of additive and multiplication of the reals and the application of the E, A, M, D, O properties.

(It should be noted that nothing really new is found in this "review" chapter except the degree of sophistication is considerably higher than in earlier treatments of the reals by SMSG. Little attention is given to intuition and the purely deductive approach to the real numbers is employed. This chapter is, therefore, a highly abstract review of that which the student previously recognized and learned somewhat intuitively in Units 9 and 10.)

An Introduction to Coordinate Geometry in the Plane

This introduction to the study of analytics is approached by the utilization of the standard terminology, e.g., quadrant, x-axis, y-axis, horizontal and vertical axes, and the origin, with the major refinement being that the one-to-one correspondence between the set of all points in the plane

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[1] Ibid., p. 76.

and the set of all ordered pairs of real numbers is established (and emphasized). Thus it is that the ordered pair of real numbers corresponding to a point serves as the coordinates of the point. Actually, the entire chapter is predicated on the concept of ordered number pairs.

A very careful study is made of the "distance" formula which determines the distance (denoted $d(P_1, P_2)$) between two points P_1 and P_2. Accompanying this discussion is the derivation of the "midpoint" formula.

The graph of an equation or inequality is regarded as being the subset of the coordinate plane whose coordinates are truth values for the given equation or inequality (or any other restricting relation). It is at this point that "set-builder" notation is introduced for the first time, e.g., the equation $3x + 2y = 7$ may be denoted as the set $\{(x, y) : 3x + 2y = 7\}$. The standard tests for symmetry of graphs with respect to the origin and the axes are examined and many of the simpler theorems of plane geometry are established analytically.

A minor inconsistency is noted at this point. Since the coordinate axes are lines, they extend infinitely far. The SMSG Geometry emphasized this idea by "barbing" the axes to indicate that the line actually extended past the margin of the pages of the text but this unit does not consistently follow this convention. The curves which extend indefinitely

might also be "barbed" to indicate the extensions past the margin.

The Function Concept and the Linear Function

The student perhaps has studied already in Chapter 17, Unit 10, the concept of function as applied to numbers. In this chapter, however, the student is given the more formal, more abstract definition:

> Let A and B be sets and let there be given a rule which assigns exactly one member of B to each member of A. Then the rule, together with the set A, is said to be a function and the set A is said to be its domain. The set of all members of B actually assigned to members of A by the rule is said to be the range of the function.[1]

Under this definition the existence of a function requires a domain set and a rule for pairing a member of the range with each member of the domain. The word rule is understood to cover many different kinds of procedures for making assignments and no specific instructions are given about which sets are to be used in the construction of functions. Consideration is directed toward the constant function, the identity function, and multiplication as a function when domain is the set of ordered pairs of real numbers and whose range is the set of reals. Although the traditional study of function has tended to concentrate exclusively on functions defined by equations, it is made quite clear that not all equations define functions nor are all functions defined by equations.

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[1] Ibid., p. 166.

In view of the fact that the previous chapter has emphasized the distinction between a point and its coordinate, one is surprised that the graph of a function is defined as a set of ordered pairs rather than a set of points.[1] It follows then that for any function having as domain and range the reals, the term "graph of the function" may have either of two meanings: the geometric figure whose coordinates are the ordered pairs of the function or the set of ordered pairs $(x, f(x))$. It follows, of course, that if one should consider a function which does not pair real number with real numbers, the ordered pairs could not be assigned a geometric coordinate interpretation. This ambiguity is further propagated by the notion of defining functions geometrically in which it is accepted that a set of points (on the Cartesian plane) defines a function if and only if no two points of the set have the same x-coordinates. It would appear that this ambiguity could be erased by a more careful use of terminology.

A thorough study of linear functions defined by the equation $y = ax + b$, $a \neq 0$, is presented along with a consideration of functions defined by physical processes and functions defined by composition. Considerable emphasis is found throughout the chapter on the concept that the function defined by an equation and an equation defining a function are different notions.

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[1] Ibid., p. 173.

Quadratic Functions and Equations

The study of the quadratic functions as defined by the equation $y = ax^2 + bx + c$ ($a \neq 0$, a, b, c, real numbers) is based on the primary consideration of the quadratic function as a special kind of pairing of the real numbers with the real numbers. The general quadratic function is examined by successive consideration of a series of special cases, namely $y = x^2$, $y = ax^2$, $y = ax^2 + c$, $y = a(x - h)^2$, and finally, $y = a(x - h)^2 + p$, and the effects produced on the graph of the function as the various constants, a, c, h, and p are "inserted" and varied, e.g., the location of the vertex, relative position with respect to the axes, and concavity. The properties of these functions are analyzed by the use of properties of the real numbers and the coordinate graph is used as a display tool for these properties. It is assumed that these curves are "smooth" curves.

From the consideration of properties of the quadratic function, the quadratic function, the quadratic equation $ax^2 + bx + c = 0$ is shown by the completion of the square to have the truth set $\left\{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right\}$ and the significance of the discriminant is evaluated.

Quadratic equations are also solved by factoring. Equations, e.g., ones containing "fractions" and radicals, which are transformable into quadratic equations by transformations which may enlarge the solution set, diminish it, or

leave it invariant, are treated simultaneously along with quadratic inequalities.

The Complex Number System

The first section of this chapter reviews the inadequacy of the real number systems when general quadratics are concerned, e.g., $x^2 + 1 = 0$ has no solution in the system of reals. Consequently, the system of reals is extended to include those numbers of such a nature that every quadratic equation will have a solution. A system C is constructed so that C will include as a proper subset the set of reals, will allow the reals to still possess all the algebraic properties which they possessed as members of the real number system, and additionally to contain a solution to the equation $x^2 + 1 = 0$, i.e., an element i such that $i^2 = -1$. It is postulated that each element of C can be written in the form $Z = a + bi$, where a and b are real numbers. In this standard form of a complex number $Z = a + bi$, a is called the real part of Z and b is called the imaginary part. The extension to C is postulated with the objective that the new system should have as operations and relations ones which are defined in terms of the operations and relations of the reals. This consideration leads to theorems which give formulas for the sum, product, difference, and quotient of the complex numbers. The text then shows that under this extension of the system

from the reals to the complex, any quadratic equation will now have "meaningful" solutions.

The Argand diagram representation of complex numbers is introduced to yield geometrical interpretation of statements regarding complex numbers and to express geometric statements in terms of complex numbers. Geometric consideration for the sum and difference of two complex numbers are structured accordingly with the absolute value $|Z|$ of the complex number $Z = a + bi$ being defined as the distance from the origin to the point (a, b).[1] (SMSG, being a trifle careless in terminology at this point, might better have stated that the amplitude is the distance between the origin and the point (a, b) and thus eliminate the directed-distance connotation associated with the term "from the origin to the point.")

The Fundamental Theorem of Algebra is stated without proof and is used to illustrate the ultimate significance of the system of complex numbers for algebra. The chapter concludes with the optional discussion of the outline of Gauss's construction of the complex number system by utilization of number pairs with addition and multiplication being defined by $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$ respectively.

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[1] Ibid., pp. 275-78.

Equations of the First and Second Degree in Two Variables

This chapter is devoted to a systematic study of equations of the first and second degree and their graphs (considered as geometric figures). Although a considerable amount of work was done in Chapter 4 with quadratic functions and their graphs, greater emphasis was placed then on the numerical properties with the graph being used only as a display board for these numerical properties. This chapter is fairly orthodox in its study of these equations although, of course, the language is concise and crisp.

A thorough examination (utilizing the earlier defined concept of slope) of the general linear equation $Ax + By + C = 0$, $A^2 + B^2 \neq 0$, is presented along with the verification that the graph of every linear equation is a straight line and that every straight line is the graph of a linear equation. (The condition "$A^2 + B^2 \neq 0$" essentially states that either A or B is non-zero.)

The general conic is defined by the "focus-directrix-eccentricity" approach. Choose a point F (a, b) as a <u>focus</u>, a straight line $Ax + By + C = 0$ as directrix, a constant <u>e</u> to be the <u>eccentricity</u> and define a <u>conic</u> to be the set of points P(X, Y) such that $d(P, F) = e \cdot d(P, Q)$ where D(P, Q) represents the distance between the point P(X, Y) and the directrix. If $e = 1$, the conic is a parabola; if $0 < e < 1$, the conic

is an ellipse; if $e > 1$, the conic is a hyperbola.[1] Standard forms (or cononical forms) of the equation of the conics are introduced and it is illustrated that the equation of all conics having a vertical or horizontal directrix are of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ but that a non-vertical or non-horizontal directrix will yield a "cross-product" term of the form Bxy.

This chapter completes a description of all graphs of first and second degree equations as being straight lines and the conics (or their degenerates) although translation and rotation are not introduced to serve to examine conics of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, $B \neq 0$. Considerable practice is demanded in the "sketching" of conics utilizing the standard forms obtained by "completing the square," the concepts of symmetry, and the asymptotes for hyperbolas.

Systems of Equations in Two Variables

The basic working premise of this chapter is the definition of a solution set of an equation (or inequality) in two variables as being the set of ordered pairs of real numbers which satisfy the equation (or inequality)--thereby determining a one-to-one correspondence between the members of the solution set of the equation (or inequality) and the points of its graph. The geometric properties of the graphs

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[1] Ibid., p. 330.

of the equations developed in the previous chapters are used freely.

The solution set of a system of two-variable equations (or inequations) is regarded as the intersection of the truth sets of the component equations (or inequalities). If S_1 and S_2 are the truth sets of two sentences of a system and if $S_1 = S_2$, then the system of linear equations is dependent (and the graphs of the equations are coincident); if $S_1 \cap S_2 = \emptyset$, the systems are inconsistent (and the graphs of the component equations do not intersect), and the system is consistent if $S_1 \cap S_2 \neq \emptyset$. (It is at this point that SMSG uses the intersection symbol "$\cap$" for the first time.)[1]

Systems in two variables containing component equations one of which is linear and the other of which is quadratic as well as systems both components of which are quadratic are "solved" by both geometrical and algebraical methods. The graphs of the equations are used extensively to predict the number of elements in the solution set of a system by "counting" the intersection points of the graphs.

Systems of First-Degree Equations in Three Variables

The SMSG discussion of the solution of systems of first degree equations in three variables forces a quite heavy reliance upon geometry in such a way that statements

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[1] Ibid., p. 363.

about solutions of the systems are, for all practical purposes, actually synonymous with statements about the various configurations of planes and their intersections. The types of intersections of planes, i.e., the intersection is empty, a point, a line, or a plane, and parallelism are used to anticipate the types of solution sets that are to be obtained in algebraical solutions.

A one-to-one correspondence between ordered triples of real numbers and the points in space is "established" by the construction of a right-handed coordinate system. The formula for distance between these points in space is developed in the same fashion as its 2-dimensional counterpart and the first degree equation $Ax + By + Cz + D = 0$ is derived by reliance upon the geometrical property of a plane as being the set of points equidistant from two distinct points of space. The standard tests for parallelness, coincidence, and non-parallelness of planes are constructed.

The method of triangulation (attributed to Gauss) is utilized to determine general solutions to systems of three first-degree equations with three variables rather than Cramer's rule commonly found in many texts.[1]

Logarithms and Exponents

The treatment of logarithms and exponents in this text is radically different than in most traditional texts

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[1] Ibid., p. 432.

primarily in that this treatment begins with the study of the theory of logarithms and derives from it the theory of exponents and exponential functions. This approach is more modernistic and rigorous in that the traditional treatment of the laws for rational and irrational exponents are at the best unsatisfactory and, consequently, the theory of logarithms based on these foundations are equally unsatisfactory. This treatment permits the consideration of all logarithm functions simultaneously. The theory of logarithmic and exponential functions are emphasized in this chapter with the use of logarithms for computation being minimized.

The logarithm function is defined for positive x as follows:

(a) For each $x > 1$, the corresponding value of y is the area of the region bounded by the x-axis, the hyperbola $y = k/x$ and the vertical lines at 1 and x.
(b) For $x = 1$, the value of y is 0.
(c) For each x such that $0 < x < 1$, the value of y is the negative of the area bounded by the x-axis, the hyperbola $y = k/x$, and the vertical lines at 1 and at x.[1]

Although this function is defined in terms of the rather sophisticated mathematical concept of area, the texts take full advantage of the student's intuitive understanding of area, i.e., the text avoids any use of the terms "calculus," "integral," or "limit." The approximate area under the

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[1]School Mathematics Study Group, Intermediate Mathematics, Unit 18 (New Haven: Yale University Press, 1960), p. 455.

curves considered is found by "counting" squares under the curve $Y = k/x$ between the indicated ordinates.

Two particularly important logarithmic functions are considered: if $k = 1$, the corresponding function (designated $\ln x$) is called the natural logarithm function, and if $k = 1/\ln 10$, the corresponding logarithm function (designated $\log_{10} x$) is called the common logarithm function. The development of these two logarithm functions is completely independent of any notion of base and depends only on the value of the k-parameter being used.

The authors substantiate by use of the definition and many examples (and only indicate that a general proof could be made) that $\log(ab) = \log a + \log b$ for any arbitrary k. Many exercises involving computation with common logarithms and the sketching of graphs and the examination of the properties of the functions with arbitrary values of k are presented for student consumption.

It is of interest to note that this entire chapter is predicated on these fundamental properties: $y = \log x$ is defined and continuous for $x > 0$; $y = \log x$ is an increasing function; $\log 1 = 0$, $\log k = 1$; and for every two positive numbers x and y, $\log(xy) = \log x + \log y$ for any arbitrary k. The authors use only an intuitive notion of continuous--no "breaks" or "jumps" in a continuous curve--and do not attempt any rigorous definition of its meaning.

Since it may be shown that log x = klnx for any arbitrary k > 0, then (log x)/(log a) = (klnx)/(klna) = (lnx)/(lna) is independent of the k used to define the function. Therefore, $f_a(x) = \log x/\log a = \log_a x$, a > 0, a ≠ 1, x > 0, is defined to be the logarithm function with base a. (The motivation for defining $\log_a x$ as the ratio log x/log a is that this ratio depends only on x and a and not on the arbitrary k used to define log x.) It follows that under this definition the natural logarithm function with base e is the same as the natural logarithm function and that the logarithm function with base 10 is the common logarithm function. A compelling result of this definition--proved geometrically by an intuitive argument--is that for each real number S, the equation $\log_a x = S$ has a unique solution. It may be shown that logarithms with base a satisfy the general properties listed above.

With this background regarding logarithm functions and the consideration of the existence of their inverses, the exponential functions are defined as the inverses of the logarithmic function. Utilizing this definition and the properties of the logarithm functions, SMSG indicates the rather sophisticated derivation of the laws of exponents.

In retrospect, it is seen that the authors rely quite heavily on the student's intuition as well as using several geometrical proofs which are not rigorous as foundations for a quite rigorous approach to logarithmic functions.

Introduction to Trigonometry

This chapter is an introduction to trigonometry and does not profess to do more than present a brief survey. With the exception of the inverse functions (or arc-functions), however, most of the topics of the traditional trigonometry course are treated to some degree. As in other topics, the major innovation here has been the introduction of concise language which permits the separation of the concept of angle and its measure.

In order to adequately introduce the trigonometric functions of angles, SMSG introduces a path (P, d) as the "motion" described by letting a point R start at an initial point P of a circle and move a distance d (the measure of the path) counterclockwise or clockwise along the circle (according as d is positive or negative) to some terminal position Q. Two paths (P_1, d_1) and (P_2, d_2) are equivalent if and only if $d_1 = d_2$ but are equal if they are equivalent and $P_1 = P_2$. The sum of two paths is defined to be another path: $(P_1, d_1) + (P_2, d_2) = (P_2, d_2 + d_2)$. This innovation allows in a sense, a non-unique coordinatization of the arc of a circle.

If θ is the measure of a path whose initial point is P and whose terminal point is Q along a unit circle having center at A, the angle $\overrightarrow{AP} \cup \overrightarrow{AQ}$ is designated (A, P, θ). Since θ is a measure of the angle, the angle designated (A, P, θ) is referred to as a signed angle. Two signed angles (A_1, P_1, θ_1) and (A_2, P_2, θ_2) are equivalent if and only if

they are determined by equivalent paths, i.e., $\theta_1 = \theta_2$. If (A_1, P_1, θ_1) and (A_2, P_2, θ_2) are equivalent, then the geometric angles $P_1 A_1 Q_1$ and $P_2 A_2 Q_2$ are congruent. (It is interesting to note that the " $\angle$ " symbol is not used to denote angles.) The real number θ is the radian measure of (A, P, θ) if a unit circle is used to assign the measure. A similar device is used to assign degree-measures to signed angles. Under this assignment, "straight" angles and "zero" angles as well as "negative" angles are assigned measures. This particular approach essentially establishes a function whose domain is the set of angles and whose range is the set of all real numbers.

An angle is considered to be in <u>standard position</u> in a plane coordinate system if and only if its vertex is at the origin and its initial side is a subset of the x-axis. Such an angle is <u>primary</u> if and only if $0 \leq \theta < 360$ degrees.

The six trigonometric functions of an angle are defined in the traditional method by considering the unique angle in standard position to which it is equivalent. The <u>function</u> aspect of these definitions is emphasized, e.g., <u>sin θ</u> is a function whose domain is the set of all angles (P, x, θ) and whose range is the set $\{x: -1 \leq x \leq 1\}$.

The remainder of the chapter is dedicated to the consideration of specific angles, use of trigonometric tables, solution of triangles, the law of sines, the law of cosines, and the addition and "half-angle" formulas. Cursory attention

is directed toward the solution of trigonometric equations and the definition and proofs of identities.

The System of Vectors

The most unorthodox feature of this brief treatment of vectors is its inclusion in a secondary school mathematics textbook.

As the introductory tool for consideration, a line segment $\overline{AB}$ is said to gain a sense of "direction" if one end point is designated as an initial point and the other is the terminal point. This directed line segment (or vector) is designated $\overrightarrow{AB}$. (Note that this vector is designated $\overrightarrow{AB}$ but that $\overrightarrow{AB}$ symbolizes a ray.) Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equivalent if their measures (designated by $|AB|$ and $|CD|$, respectively) are equal and the rays $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel, i.e., the lines containing $\overline{AB}$ and $\overline{CD}$ are parallel or coincident. $\overrightarrow{AB} \doteq \overrightarrow{CD}$ denotes that $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$. "By the sum of two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ we mean the directed line segment $\overrightarrow{AX}$ where X is the unique point such that $\overrightarrow{BX} \doteq \overrightarrow{CD}$."[1] Geometric figures are used to lend credence to the associative and commutative properties of the addition of vectors and the distributive and associative properties of the defined multiplication of vectors by scalars. Vectors are then used as a device for solving certain geometric problems and proving certain geometric theorems.

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[1] Ibid., p. 631.

The component notation for vectors, e.g., $[X_1, Y_1]$ is the vector whose initial point is (0, 0) and whose terminal point is (X_1, Y_1) or, more generally, any vector whose horizontal component is X_1 and whose vertical component is Y_1, is introduced and the sum of two vectors $[X_1, Y_1]$ and $[X_2, Y_2]$ is redefined to be $[X_1 + X_2, Y_1 + Y_2]$. This, in actuality, introduces the notion of free vectors although SMSG does not use the term. Introduced also are the vectors $\bar{i} = [1, 0]$ and $\bar{j} = [0, 1]$ to serve as a base for all vectors in the plane. This new consideration of vectors is utilized to prove algebraically the theorems illustrated earlier by geometrical processes. The inner product (commonly known as the "dot" product of two vectors) is defined in terms of the magnitude of the vectors and the angles determined by them. The vector product (or cross-product) is not defined.

The chapter treats quite extensively several types of problems which are of a physical nature involving resultants, forces, velocity, acceleration, etc., by means of an assumed isomorphism between the set of forces and the set of vectors treated as a formal mathematical system. (One notes in the chapter a slight discrepancy with respect to earlier usage. In this unit, $|AB|$ is used to indicate the distance from A to B (and from B to A) whereas Unit 13, page 34, indicates that the distance between P and Q shall be denoted by PQ.)

Polar Form of Complex Numbers

An earlier chapter (Chapter 5) has introduced complex numbers, discussed equality, addition, subtraction, multiplication, and division of such numbers, and the Argand diagram representation of complex numbers was utilized to portray addition and subtraction of these elements. This particular chapter introduces and studies the polar representation of complex numbers in a traditional manner with the unorthodox feature again being its inclusion in a secondary text.

The central tool for the chapter is, of course, the representation of a complex number $Z = X + Yi$ in its <u>polar form</u> $Z = |Z| (\cos \theta + i \sin \theta)$, where $\cos \theta = x/\sqrt{X^2 + Y^2}$ and $\sin \theta = Y/\sqrt{X^2 + Y^2}$. The central theorem utilized is, as expected, <u>de Moivre's</u> theorem stating that for any natural number <u>n</u>, $(\cos \theta + i \sin \theta)^n = \cos (n\theta) + i \sin(n\theta)$. SMSG "illustrates" this theorem by an inductive process without ever attempting a proof in even a semi-rigorous proof except in the teacher's commentary. Extension of this definition and this theorem lead directly to the finding of n<u>th</u> powers (<u>n</u> a natural number) and roots of complex numbers and the solution of quadratic equations with complex coefficients.

Sequences and Series

This chapter treats in an informal manner different aspects of the topic of sequences and series and introduces several of the related terms by appeal to the student's intuition.

The key definition of the chapter is this:

> A finite sequence of n terms is a function a whose domain is the set of numbers $\{1, 2, 3, \ldots, n\}$. The range is then the set $\{a(1), a(2), \ldots, a(n)\}$, usually written $\{a_1, a_2, \ldots, a_n\}$. The elements of the range are called the terms of the sequence.[1]

It follows then that an infinite sequence is one whose domain is the set of positive integers. A finite sequence is abbreviated $\{a_k\}_{k=1}^{n}$ and an infinite sequence $\{a_k\}_{k=1}^{\infty}$. The indicated sum (abbreviated $\sum_{k=1}^{n} a_k$) of a finite sequence is referred to as a finite series. The text develops the necessary definitions and formulas for the general terms and the sum of n terms of arithmetic and geometric sequences (or progressions).

The authors introduce an intuitive notion of the concept of limit by considering a correspondence between the terms of a sequence of numbers and a set of points in the number line and thereby lending a geometrical connotation to the concept. They consider a sequence to have a limit L (written $\lim_{n\to\infty} a_n = L$) if and only if the nth term of the sequence becomes and remains arbitrarily close to L as n increased. The non-introduction of the (ϵ, δ) notation for definition of a limit prohibits a "nice" proof of such conventional theorems as the one stating that if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then $\lim_{n\to\infty} (a_n + b_n) = A + B$, and, consequently,

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[1] Ibid., p. 731.

several such theorems involving the sequences found by taking the sum, product, and quotient of convergent series "term-by-term" are stated without proof.

A series is defined to be convergent if and only if it is "summable," i.e., if the sequence of partial sums of its sequence is convergent. (If the sequence $\{a_k\}_{k=1}^{\infty}$ is considered, the sequence of partial sums is $\left\{\sum_{k=1}^{n} a_k\right\}_{n=1}^{\infty}$.) The sums of several series, e.g., certain infinite geometric series, are investigated by utilization of inductively determined formulas.

Permutations, Combinations, and the Binomial Theorem

In order to approach the study of permutations and combinations, the authors introduce ordered m-tuples by first considering the $N_1 \times N_2$ elements (ordered couples or 2-tuples) of the Cartesian cross-product (although the term is not used) of two finite sets having cardinal numbers N_1 and N_2 respectively. Ordered triples (a, b, c) are considered as ordered couples of the form ((a,b), c)--although the Cartesian product is not associative--and an ordered quadruple (a, b, c, d) is considered as the ordered couple ((a, b, c), d). Inductive extensions allow the formation of ordered m-tuples with the result that if $B_1, B_2, \ldots, B_m$ are finite sets having $N_1, N_2, N_3, \ldots, N_m$ as their respective cardinal numbers, one might form $N_1 \times N_2 \times \ldots \times N_m$ ordered m-tuples

such that the first component of each comes from B_1, the second from B_2, etc.

The traditional "combination of m things taken n at a time (denoted C(m, n)) is now considered in set-terminology as the number of n-element subsets of a set having m elements. Similarly, the "permutations of m things taken n at a time" is now considered as the number of those ordered m-tuples of elements of a set having n elements which have no duplication. Derivation of the formulas involved rely heavily upon those definitions and inductive arguments pedagogically assisted by rectangular arrays.

The binomial theorem is derived at this point for the first time in the SMSG texts and is a direct result of the theories of combinations. As a matter of fact, the theorem is stated in "combination" form, i.e., if N is a natural number then $(X + Y)^n = \sum_{m=0}^{n} C(n, m) X^{n-m} Y^m$.[1] (It should be noted that while this chapter--and, in fact, the last several chapters--relies quite heavily upon induction, the "proofs" are not "proofs by induction" but rather are only expository arguments that point out apparent patterns which are then accepted as "truths.")

Algebraic Structures

This chapter, definitely intended only as a supplementary chapter to be used if time permits, reviews the

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[1] Ibid., p. 823.

nature of the fundamental algebraic operations already used by seeking to abstract that which is essential and common to several number systems. This exposition serves as the last effort to clarify the role of the definition and axiom in a mathematical structure.

Some attention is directed toward the properties of the one-operation structure called a group. Examples of finite groups are demonstrated as well as those of both Abelian and non-Abelian groups. Equal attention is focused on the definitions and properties of the two-operation structure called a field.

Evaluation of Units 17 and 18

Units 17 and 18 essentially contain discussions of all the topics ordinarily encompassed by a study of second-year algebra and trigonometry although the treatment of trigonometry is a very brief and compact one. In addition to the traditional topics, however, these materials abound in discussions which ordinarily would not be encountered at this level. An abundance of effort is expended to develop and display the structure of the systems of natural numbers, the integers, the rationals, and the reals with an adequate and sophisticated extension to the complex numbers. Additionally, the student is led through an extensive consideration of selected topics from analytic geometry and vector algebra.

One of the key aspects of these units is their constant and unremitting emphasis on the study and importance of mathematical structures. Implicitly emphasized in the chapters which deal with number systems--naturals, integers, rationals, reals, and complex--are the conditions required for an extension of a number of system. Vector algebra is approached as a mathematical system with a brief analysis of the assumed isomorphism between the system of vectors and physical forces as well as the discussion of the relationship between the system of vectors and the system of complex numbers. Some time is devoted to the development of commutative and non-commutative groups and fields. In all instances, simplification of algebraic expressions and the solution of systems of equations are performed on and justified by the number system (and its properties) which is the domain for the discussion.

It appears that the authors have attempted to develop both concepts and skills with the primary emphasis upon the understanding of concepts and their development. Each topic presented is discussed extensively with exercises following serving to reinforce the concepts involved. In most instances the concepts are first presented in a fairly rigorous fashion with the related skills being developed later. It appears that fewer exercises of an exploratory nature are found in these units than in earlier units.

The language used in these units is fairly precise and sophisticated although one gains the impression that this aspect was not of prime importance to the authors. The role of the definition is implicitly emphasized although the student has very little opportunity to formulate definitions on his own. Symbols to denote mathematical notions are again used only rarely, i.e., one never meets the symbols $\forall$, $\exists$, $\Rightarrow$, $\Leftrightarrow$, etc., although these notions are used extensively. Even the symbols $\cup$ and $\cap$ are used sparingly.

Earlier units of SMSG materials have encouraged and demanded student involvement in the development of the materials in that the student is led to discover and formulate concepts for himself. In these units, however, the authors appear to be content to present material for student consumption rather than student participation. Although the expositions are excellent, the student is "lectured to" by the texts although the occasional "challenge exercises" call for some exploration on the part of the student.

Fairly rigorous proofs are quite abundant throughout the texts with some few being left as exercises. Since it is assumed that the student is already acquainted with good proofs, little discussion is devoted to the nature of a good proof. Those exercises demanding proofs call for little ingenuity on the part of the student since most of them follow the same pattern as those example proofs found in the textual exposition. It is worthy of mention that the authors do not

attempt to "gloss over" proofs which are beyond the level of the materials but rather call attention to these inadequacies.

These units do not in any sense emphasize social application. As a matter of fact, little attention is directed toward social application although one finds a few problems related to the physical sciences, e.g., problems involving falling bodies, mixtures, and time rates. The one notable exception is the chapter involving vectors where one finds a multitude of problems involving forces, velocities, work, etc.

Elementary Functions, Unit 21

Introduction

Elementary Functions follows generally the outline recommended by the Commission on Mathematics for the first semester of the twelfth grade although some disagreements with respect to point of view are noted. The integration of the topics of trigonometry and solid geometry into earlier units leaves the twelfth grade open for the fairly sophisticated study of "elementary" functions--in particular, polynomial functions, exponential and logarithmic functions, and certain periodic functions--which again plateau the spiral approach utilized by the SMSG.

Unit 21 actually assumes only a minimum amount of mathematical background in SMSG units although some knowledge of trigonometry is assumed, and of course, the basic techniques of algebra and geometry. As a matter of fact, several

topics studied earlier by students of SMSG materials, e.g., logarithmic and exponential functions, are approached in this unit from a diametrically opposite view than that utilized in earlier units.

Functions

This chapter presents a thorough discussion of functions concluding the spiral approach to functions in general and the linear, constant, and absolute value functions in particular. A function from a set A to a set B is represented as being a mapping from a domain set A onto a range set B and suitable notation is introduced to serve as the communication while for such representation, e.g., $f : x \rightarrow 2x + 7, x \in R$, describes the mapping of a given set (in this instance, the real numbers) to another set $\{f(x) : f(x) \in R\}$. For the purpose of this study, functions are restricted to the reals or subsets of the reals although more general mappings are discussed.

The graph of a function is reintroduced to serve as an aid for understanding functions. A graph is the graph of a function if and only if no line parallel to the Y-axis meets it in more than one point.[1] This "test" implies that the domain of a function is forced by definition to "lie" along the horizontal axis--a convention evidently subscribed

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[1]School Mathematics Study Group, Elementary Functions, Unit 21 (New Haven: Yale University Press, 1960), p. 34.

to by the authors in their textual discussions and examples. Yet the authors introduce exercises which force the student to consider graphs of functions which have the domain represented along the Y-axis!) The graph of a constant function is seen to be a straight line parallel to the X-axis and that of a linear function as being a straight line which may be parallel to neither axis. Attention is directed toward composition functions and the concept of such functions is used to define the inverse of a function which is shown to be consistent with the general tenor of the topic in that an inverse function is viewed as a "reverse" mapping.

Polynomial Functions

This chapter presents an intensive and sophisticated study--at least as far as conventional secondary texts are concerned--of what traditionally has been considered as the theory of polynomial functions and concerns itself primarily with developing methods for ascertaining the existence of and the location of (or approximation of) zeros of polynomial functions.

The process of synthetic division is structured by repeated use of the distributive law with actual proof of the validity of the process being presented only for third-degree polynomials with an intuitive induction utilized to generalize the process. This process is used to develop the Remainder Theorem (at least for third-degree polynomials), the Factor

Theorem, and these results to "prove" the Fundamental Theorem of Algebra.[1]

The Location Theorem advocating that "if f is a polynomial function and if a and b are real numbers such that f(a) and f(b) have opposite signs, then there is at least one zero of f between a and b" is made plausible on the basis of graphical representation of polynomial functions. Only an intuitive appeal is made for the continuity of the functions involved in this instance and the authors do not attempt a rigid definition of continuity. The Location Theorem, coupled with various search techniques including those which locate the potential integral and/or rational zeros, is used to determine or approximate the zeros for polynomial functions.

Tangents to Graphs of Polynomial Functions

This chapter serves to complete the study of the theory of polynomial equations as presented by SMSG and deals primarily with the problem of determining the equation of the tangent to the graph of a polynomial function. To find the slope of such tangents, the authors, in view of the fact that no calculus is available, resort to a simple (yet sophisticated) and somewhat intuitive process. By considering the function

$$f : x \rightarrow a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

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[1] Ibid., p. 74.

and writing, by repeated use of synthetic division, this function in powers of x - a, i.e.,

$f : x \rightarrow b_0 + b_1 (x - a) + b_2 (x - a)^2 + \ldots + b_n (x - a)^n$,

one may find the equation of the tangent at (a, f(a)) by observing that as x "approaches" a, powers of (x - a) may be made arbitrarily small and that the first power of (x - a) dominates the higher powers. Consequently, the equation of the tangent at (a, f(c)) is $f(x) = b_0 + b_1 (x - a)$.

By again utilizing the process of synthetic division, the pertinent coefficient b_1 (actually the slope of the tangent to the curve at the point (a, f(a)) may be determined and the resulting associated function (actually the first derivative of the function $f : x \rightarrow f(x)$)

$$f : x \rightarrow a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^n$$

is designated as the slope function. Actually, this result is proved only for second, third, and fourth degree polynomials and intuitively inducted to higher degree polynomials.

The slope function is used to determine relative maximum and minimum points of curves and to solve various related "practical" problems. The slope function is used also to introduce Newton's iterative method for determining the real irrational zeros of polynomial functions. As in the preceding chapter, the authors make only an intuitive appeal for continuity of functions and utilize to the maximum degree the consideration of a function as being a mapping.

Exponential and Logarithmic Functions

This development discusses what the authors identify as "a totally new class of functions"[1]--the exponential and logarithmic functions--although approximately 104 pages of Unit 18 have already been devoted to this same subject. This earlier discussion has defined a logarithmic function by reliance upon the measure of the area under a branch of an equilateral hyperbola and the exponential function as the inverse of that logarithmic function.

This later discussion utilizes exactly the opposite approach--a novelty which seems very inconsistent. The exponential function is defined by $f : x \rightarrow ka^x$, $a > 0$. The conventional interpretation is defined for rational exponents, i.e., $a^{p/q}$ (p, q integers, $q \neq 0$) is interpreted as the q-th root of the p-th power of a. a^x, x irrational, is considered as the limiting value of $a^{p/q}$, p, q integers, as p/q approaches x, e.g., $a^{\sqrt{2}}$ is defined to be the point of convergence of the sequence a^1, $a^{1.4}$, $a^{1.41}$, $a^{1.414}$, $a^{1.4142}$, . . . (Again, the authors appeal only intuitively to the notions of limit and continuity.)

The logarithmic function $f^{-1} : x \rightarrow \log_a x$ is defined as the inverse of the exponential function $f : x \rightarrow a^x$. The advantages and disadvantages of different bases, particularly the base e, are discussed. Very little attention is paid to logarithmic computations. Accompanying these mathematical

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[1] Ibid., p. 145.

discussions are dozens of problems treating important physical phenomena such as growth, radioactive decay, cooling, and compound interest utilizing exponential and logarithmic formulas.

Circular Functions

This chapter is devoted to a completely new approach (as compared to the earlier study in Units 17 and 18) of plane trigonometry. The circular functions $\sin : x \rightarrow \sin x$ and $\cos : x \rightarrow \cos x$ are defined by choosing the points A (1, 0) and P (u, v) on a unit circle with measure of the counterclockwise arc being $\underline{x}$. Then $u = \cos x$ and $v = \sin x$. These two functions are established to be periodic, i.e., functions $\underline{f}$ such that $f(x + a) = f(x)$, with fundamental periods of each being 2π. In like vein $\tan : x \rightarrow \frac{\sin x}{\cos x}$ is discovered to have the fundamental period π. The graphs of these functions are thoroughly examined and their properties explored.

A unique feature of this chapter is the accompanying summarization of the properties of the class of plane vectors which are of unit magnitude and which emanate from the origin. A class of rotation functions--actually determined by linear combinations of these unit vectors--is studied and a simple algebra of these classes structured. These rotation functions are then used to derive the formulas for $\sin(X \pm Y)$ and $\cos(X \pm Y)$ in a manner usually found in vector analysis texts.

A fairly intensive study of identities is then structured on these and other fundamental identities.

The reader should note several inconsistencies in this chapter which could have been eliminated by even fairly careful editing. Among these are the following:

1. "An angle is defined in geometry as a pair of rays or half-lines with a common end point."[1] This definition is in conflict with an earlier SMSG definition: "An angle is the union of two rays which have the same endpoint but do not lie in the same line."[2]

2. Earlier units have decreed that two different angles whose measures are equal will be considered as being congruent (not equal) and yet one finds this statement: "Clearly $\measuredangle P_0 0P_1 = \measuredangle P_1 0P_2 = \measuredangle P_2 0P_3 \ldots$."[3] Consistency demands that this statement be read "Clearly $\angle P_0 0P_1 \cong \angle P_1 0P_2 \cong \angle P_2 0P_3 \ldots$"

3. Earlier units have indicated that the symbol $0P$ will indicate the measure of the line segment $\overline{0P}$ and not the segment itself. Yet one finds $0P$ being used to represent the vector $\overrightarrow{0P}$ without suitable clarification.[4]

Evaluation of Unit 21

The materials presented in Elementary Functions are within the realm of reasonableness in that previous units of the SMSG series have laid an adequate groundwork for their presentation. The structure of mathematics is recognized consistently and easily throughout this unit. The language used

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[1] Ibid., p. 243.

[2] School Mathematics Study Group, Unit 13, p. 71.

[3] School Mathematics Study Group, Unit 21, p. 246.

[4] Ibid., p. 254.

in this unit is considerably more sophisticated and precise than that in earlier units and the proofs are generally more rigorous although some of the language utilized is inconsistent with that of earlier units. It appears that the authors assume that adequate (and considerable) attention has been given to proofs in the earlier years and, consequently, more time may be alloted to rigorous proofs at this stage. This emphasis is directed toward the mathematical maturity of the student and should provide an adequate foundation for proofs later in his mathematical career.

Throughout this unit, SMSG authors stress the basic concepts with their devoting less space than usual to mere acquisition of skills, i.e., "drill" is used less often than in the past with a higher percentage of theoretical observations and proofs demanded. The student is forced to recognize basic principles and to develop necessary proofs which are superior to that of simply stating the principles and applying them to routine-type exercises. In some instances, e.g., logarithmic and exponential functions, diametrically-opposed approaches are taken as compared to earlier developments of the same topics.

Although such is not the primary purpose of the writers, examples from the sciences and other areas, e.g., maxima and minima, laws of cooling, radioactive decay, and harmonic motion, are used quite effectively (and extensively) to remind the student of the usefulness of various topics under study.

It appears that these examples of application cause the mathematical theory to unfold in a logical sequence as need arises with a resulting balance between theory and its application.

Introduction to Matrix Algebra, Unit 23

Introduction

This unit represents the introduction into the twelfth-grade mathematics curriculum of a topic not heretofore taught in high schools except in isolated cases. This unit is structured in such a way that a student will gain facility in matrix manipulation and also simultaneously an appreciation of the several systems of mathematics already encountered. The properties of the algebra of matrices are studied such that the consideration of the properties of other systems seems to arise quite naturally. This study of matrices "ties together much that the student has learned already concerning algebra, geometry, trigonometry, and functions, and thus it furnishes a fitting capstone to his secondary-school study of mathematics."[1] Although the unit is designed for only a half-year course for college-capable students, an extremely large amount of sophisticated mathematics is met in a short period of time in this unit.

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[1]School Mathematics Study Group, Introduction to Matrix Algebra, Unit 24 (New Haven: Yale University Press, 1960), p. x.

In addition to the regular textual material, this unit contains in an appendix a list of very challenging "research" exercises dealing with such topics as quaternions, non-associative algebras, and set theory. These problems are designed to be student group projects and are very sophisticated in nature.

Matrix Operations

In this chapter the system of matrices is developed as an "invented" number system which has arisen (as has all other number systems studied earlier) from certain needs. One notices an implicit emphasis on the notion that the system of matrices was invented with a purpose.

A matrix is defined in the conventional sense except no mention is made of the field from which elements of the matrix are to be selected, i.e., a m x n matrix is defined as a rectangular array of numbers (in this case, real numbers) having m rows and n columns. Ensuing discussions define and discuss the order of a matrix, the transpose of a matrix, and the equality relation for matrices, along with defined rules for determining the sum of matrices. Attention is directed toward the product of numbers and matrices (and the resulting properties) although such multiplication of a matrix by a number is defined only in a left-hand sense, e.g., one studies (and establishes) the rule that $X(A + B) = XA + XB$, X a real

number and A and B matrices, but does not consider the product (A + B) X.[1]

Multiplication of matrices is defined in the traditional manner with adequate discussion being directed toward the study of the properties of this class of elements having this defined operation. Probably the outstanding feature of this chapter is the demonstration (and proof) that the commutative law for multiplication and the cancellation law do not hold.

In all cases, the authors present a multitude of manipulative examples to suggest a conjecture which is then climaxed by a fairly rigorous proof utilizing the general $[a_{ij}]$, i = 1, 2, 3, 4, . . . , m; j = 1, 2, 3, . . . , n, notation for matrices. All definitions are approached similarly, i.e., certain notions are illustrated by examples and formal statements of such are then formulated from the derived conjectures.

The Algebra of 2 x 2 Matrices

SMSG in this chapter uses the study of 2 x 2 matrices as the vehicle for studying several very important notions common in modern mathematics but which are foreign to the traditional secondary curriculum. Imbedded in this chapter is a discussion of the closure of sets under given operations--a

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[1] School Mathematics Study Group, Introduction to Matrix Algebra, Unit 23 (New Haven: Yale University Press, 1960), pp. 19-24.

notion certainly not new to a SMSG student. However, one finds also a listing of the field postulates, the ring postulates (and the proof that 2 x 2 matrices form a ring under addition and multiplication) and the group postulates (and the proof that the invertible 2 x 2 matrices form a group under multiplication).[1] The determinant function for 2 x 2 matrices is described as the mapping $\delta : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow ad - bc$ of the set of matrices having real elements into the set of real numbers. The introduction of the matrices $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ empower the student, through consideration of the correspondence $x(i) + y(j) \leftrightarrow xI + yJ$, to prove that the set of complex numbers is isomorphic to the set of 2 x 2 matrices and that, consequently, the algebra of complex numbers is imbedded in the algebra of matrices.

Matrices and Linear Systems

This chapter reviews the role which matrices may play in determining the truth sets of systems of linear equations. The authors approach this presentation by first examining the familiar algebraic methods of determining solution of such systems and then illustrating that this procedure may be synthesized by using matrices and row operations performed upon them.

It is illustrated that any elementary operation, i.e., interchanging of any two rows, multiplication of all elements

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[1] Ibid., pp. 53-99.

of any row by a non-zero constant, or addition of an arbitrary multiple of any row to any other row, may be considered as the result of pre-multiplication (or left-multiplication) by an elementary matrix, i.e., a square matrix obtained by performing a single elementary row operation on the identity matrix.

It is demonstrated that one may represent a system of linear equation by matrix techniques, e.g., the system

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\}$$

may be represented by the matrix equation

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} .$$

One may then by utilizing row operations on matrices (or by multiplication by elementary matrices) "diagnolize" the above system to the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} ,$$

from which one derives the equivalent system

$$\left.\begin{aligned} X &= s_1 \\ Y &= s_2 \\ Z &= s_3 \end{aligned}\right\} .$$

Similarly, one might "triangularize" the system into

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}$$

from which the solution set can be read fairly quickly.

Two significant factors are introduced in this section. The first is the representation of a system of linear equations in matrix form. Secondly, the consideration of a system of linear system in matrix form leads to the consideration of a matrix function which has both domain and range as sets of matrices, i.e., the matrix equation AX = B actually defines a function f : X $\longrightarrow$ B where the domain consists of column matrices X and the range of column vectors Y.

Representations of Column Matrices as Geometric Vectors

This sufficiently sophisticated chapter forms a very adequate basis for an introductory study of vector analysis. The representation of vectors in two-space by 2 x 1 column vectors actually yields a geometric interpretation, i.e., as directed line segments, for 2 x 1 matrices.

In general, the vectors in this chapter are treated as <u>free vectors</u>, i.e., most of the considerations involved allow the study of <u>located vectors</u> $\overrightarrow{AB}$: (a, b) (c, d)--vectors having their "tail" at (a, b) and "tip" at (c, d)--by considering the vectors $\overrightarrow{OP}$: (0, 0) (h, k) where h = c - a and k = d - b. Such machinery permits the consideration of the geometric interpretation of the multiplication of a vector by a

number and of the addition of two vectors, direction numbers and direction cosines of lines, the definition of the inner product $\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = ac - bd$, and an examination of the properties of the system of 2 x 1 matrices with respect to this inner product.

The striking feature of this chapter is the discussion of the postulates for a general vector space and a subspace as well as discussions and exercises involving linear combinations of vectors, bases for vector spaces, linear independence of vectors, and the spanning of vector spaces by basis vectors.

One notices that the notation utilized here to symbolize the vector quantities contradicts that used in Unit 18, e.g., $\overrightarrow{AB}$ in this unit abbreviates a vector having tail at A and tip at B whereas Unit 18 would have interpreted this symbol as the ray emanating from A and containing the point B.

Transformations of the Plane

This chapter presents another mathematically sophisticated chapter dealing with properties of the system of two-dimensional vectors utilizing a matrix representation and dealing particularly with transformation functions whose domain and range are both subspaces of the vector space H. The notion that such functions, called transformations of the plane into itself, associated with each point P of a plane the point

P' (or maps P onto P') yields a geometric interpretation to the study of such functions. This chapter deals entirely with such functions and does not consider more general mappings such as those carrying a point into a line, a circle, or other geometric configurations. This chapter again utilizes the free vector matrix representation, i.e., the located vector $\overrightarrow{PQ}$: (a, b) (c, d) having "tail" point at (a, b) and "tip" point at (c, d) is represented by the locater vector $\overrightarrow{OP}$: (0, 0) (e, f) where e = c - a and f = b - d.

A linear transformation on H is defined to be a function f from H into H such that for every pair of vectors U and V in H, f(U) + f(V) and for every real number r, f(rV) = rf(V).[1] Utilization of this definition and the consideration of the function V $\rightarrow$ AV, A a 2 x 2 matrix and V a 2 x 1 vector matrix, yields the theorem that every matrix transformation A is linear. This theorem establishes machinery to allow matrix representation for vector function by utilizing matrix multiplication, e.g., the function V $\rightarrow$ aV, $a \in R$, $V \in H$, may be represented by matrices as

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

and the "rotation" function which rotates a vector through an angle θ about the origin as

$$\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

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[1] Ibid., p. 190.

Such representation authorizes the use of matrix properties to study extensively the properties of various types of linear transformations, particularly those which leave unchanged the lengths of the vectors transformed, e.g., the reflection of the plane in the x-axis, a rotation of the plane through any given angle about the origin, and a reflection in the x-axis followed by a rotation about the origin.

Evaluation of Unit 23

Considering the detail and carefulness of approach utilized by the authors producing this unit, one probably would conclude that the topics of this unit are neither too difficult nor too abstract for a high school senior although traditionally such topics have not been included in the mathematics curriculum. The topics are developed by such an approach that a person who had never before studied thoroughly the real number system could do so at this time. The knowledge of the real number system is strengthened through consideration of (and comparison with) the characteristics of matrix multiplication and addition. Structural properties of systems are illustrated by the introduction of various abstract sets along with their operations and the consideration of the properties of closure, commutativity, associativity, etc., and the determining of their ring, field, and group categories. The many ties between structural systems with which the student is already acquainted and those being

introduced are emphasized, e.g., isomorphism is discussed in terms of the correspondence between complex numbers and matrices. An implicitly powerful notion is that 2 x 2 matrix theory can lead to (or be extended to) a system as complex as a vector space. The vocabulary, though not unsophisticated, would seem to be fairly intelligible to properly-placed students and such students should be able to follow explanations, proofs, and problems without too much difficulty although some of the notation is contradictory to that used in earlier units.

In the most part, the basic theorems are proved for the student although the student does prove various auxiliary theorems. Many of the exercises seem to have imbedded in them the opportunities to discover the necessary or convenient ideas for subsequent exercises, i.e., the discovery approach is used consistently. Many of the concepts to be developed fairly rigorously in subsequent discussions are developed in an intuitive fashion prior to their rigorous formalization. The distinction between the proof of a theorem and that of its converse is emphasized by the rending of such proofs into two separate theorems, one being the converse of the other. The indirect method of proof is used only rarely. The text seems to have struck a balance between development of concepts and skills in that adequate opportunity is provided for original development on the part of the student as well as a plentitude of routine reinforcement problems.

Some sound applications are indicated. The use of matrices and column vectors in industrial and business problems are pointed out at the beginning of the unit and then not mentioned further after such applications have been demonstrated.

Evaluation of the SMSG Secondary Program

The SMSG units which have been discussed in some detail in this chapter were "written explicitly for college-capable students"[1] though they are not necessarily college-preparatory in nature, i.e., they also may be considered as sound terminal courses for students of mathematics. The subject matter and the method of presentation are based on this early statement of philosophy by SMSG authors:

> First, we need an improved curriculum which will offer students not only the basic mathematical skills but also a deeper understanding of the basic concepts and structure of mathematics. Second, mathematics programs must attract and train more of those students who are capable of studying mathematics with profit. Finally, all help possible must be provided for teachers who are preparing themselves to teach these challenging and interesting courses.[2]

In summary of the critique of the SMSG secondary program, one notes several characteristics of these materials insofar as content and placement are concerned.

1. First Course in Algebra (which does not presuppose a study of the SMSG Junior High School program) emphasizes the structure of algebra. This study

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[1] School Mathematics Study Group, Newsletter No. 4, 1960, p. 8.

[2] Ibid., p. 4.

is based on an intensive exploration of the behavior of numbers and pays careful attention to the language of the subject. This material, though comprised of materials making up a traditional first algebra course, is organized radically different than its traditional counterparts due to its fundamental-structures motivation.

2. Geometry is devoted mainly to plane geometry (based on the postulates of George D. Birkhoff) and includes some chapters on solid geometry and and introduction to analytic geometry. Selected topics from solid geometry are introduced early in the unit with such topics integrated on occasion with the plane geometry and, in other cases, in special chapters. This development of geometry, evidently written in the belief that Euclidean geometry still deserves a prominent place in the secondary curriculum, assumes student familiarity with the number line since geometry is connected with algebra at every reasonable opportunity through coordinatization of both segments and angles for purposes of assigning measures. The definitions and theorems are exact although exactitude never appears to be an obstacle to the student.

3. In the eleventh-grade Intermediate Mathematics, chapters are devoted to trigonometry, vectors, logarithms, mathematical induction, and complex numbers. These chapters are written with a vastly higher degree of sophistication than those texts usual for this level. The units, written in the same vein, though on a higher plateau, as the ninth-grade materials, emphasize structure at all times. A preliminary chapter reviewing elementary algebra from this point of view serves to introduce students who are new to mathematical structures to the realization that the earlier studied reals form an abstract number system. Much attention has been directed toward giving the student insight into the nature of mathematical thought as well as preparing him to perform certain manipulations with ease and understanding.

4. Elementary Functions, though overlapping with earlier units, and, unfortunately, in some cases conflicting with them is designed for use for the first semester of the twelfth grade. The central theme is that of studying polynomial, exponential,

logarithmic, and trigonometric functions. The course, with its many simple but geometrically meaningful innovations for studying areas, tangents, maximum and minimum points, etc., should furnish a student with an adequate intuitive background for an introductory course in calculus.

5. Introduction to Matrix Algebra, though devoted to a second-semester, twelfth-grade study of matrices including applications to the solutions of systems of equations and to geometry, also pays careful attention to algebraic structures. The majority of the mathematics presented is new to the student and should put him close to the frontiers of mathematics as well as to provide him with many examples of patterns that arise in varied circumstances. In retrospect, one notes that, with the exception of the unit dealing with matrices, SMSG has presented very little mathematics which is actually new to the secondary curriculum although the presentation, rigor, arrangement, language, symbolism, emphasis, etc., are vastly different.

Although the mathematical content of the SMSG materials is fairly conventional, the textual material and exercises abound with terms which probably would not have appeared or, at least, would not have been used so frequently in texts of twenty years ago, e.g., set, subset, pattern, opposite, reflexive, symmetric, transitive, axiom, postulate, open sentence, truth set, binary, convex, half-line, half-space, union, intersection, region, measure, continuous, unique, limit, Abelian, basis, spanning, matrix, vector, conformable, embedded, equivalent, field, ring, isomorphism, kernel, circular function, domain, range, image, inverse, least upper bound, greatest lower bound, mapping, mathematical model, one-to-one correspondence, and integral polynomial zeros. SMSG seldom deigns to "invent" new terms as vehicles for

communicating ideas and concepts but rather uses the terms which are commonly accepted by mathematicians. (It does seem rather strange, however, that SMSG seldom utilizes mathematical symbols for communication of mathematical tools. Only occasionally does one find in these texts the symbols $\cup$ and $\cap$ for union and intersection, respectively, and even the highly utilitarian "set-builder" notation for describing sets is seldom utilized. Although all statements and theorems are quantified and the terms "imply" and "if and only if" are used frequently, one never finds the symbols $\forall$, $\exists$, $\Rightarrow$, and $\Leftrightarrow$ abbreviating for any, there exists, implies that, and if and only if, respectively.)

The proofs utilized by the SMSG authors provide another departure from traditional texts. In the units for the ninth grade, the proofs seem to be largely intuitive and lack rigor since the student is not yet fully acquainted with the characteristics of a good mathematical proof. However, the ninth-grade units, in spite of their intuitive appeal insofar as proofs are concerned, use progressively more sophisticated arguments until, by the end of the year's program, one notes an explicit concern for rigorous proofs. The tenth-grade study of geometry concerns itself still more with good, concise proofs as well as pointing out the "loopholes" in various proofs. The coordinatization of lines and angles as well as the introduction of the coordinatized plane permits proofs which are more numerically-oriented than those found

in traditional geometry. The units prepared for later consumption are characterized by clean, crisp proofs for which the foundations have been laid during the first two years. A student having studied this program will have become acquainted with the various types of proof although he will not have seriously studied the logic involved in such proofs. The same student should also be aware of the role of the definition, the axiom and postulate, and the undefined term in any mathematical proof and also realize that the truth of a mathematical "fact" is relative to the abstract system under consideration.

When compared with traditional treatments of secondary mathematics, the really striking difference as evidenced by the SMSG program is the method of presentation of material in both textual discussions and exercises. In most instances, the comparatively lengthy textual discussions are written in a manner which seems to suggest a conversation with the student. The student, in reading these textbooks, must continually contribute answers to "Why?" and "How?" if he is to follow the development of the topics. The authors apparently wrote these discussions with the ultimate goal in mind of helping to train the student to find out things for himself--a goal motivated by the heuristic philosophy that a student retains and understands more completely those things which he discovers for himself. The exercises, written in similar vein, often embody extensions of the material already studied

and serve to assist the intuitive formulation of concepts to be precisely verbalized later as well as to reinforce the concepts already verbalized. (One does note, in this respect, the results of several authors contributing separate sections which were "tied together" with a minimum of editing and thereby damaging, to some extent, the continuity of the presentation.)

One must predicate a discussion of "concepts versus skills" in the SMSG program upon the notion that, in the eyes of SMSG, the development and verbalization of concepts is a desired goal, i.e., in a sense, the grasping and formulation of concepts <u>is</u> a skill. However, even the "drill" fancier would find very little to criticize in these textbooks since an abundance of such exercises is presented although not to the degree as programs of the past years--a manifestation of the belief that drill based upon understanding, i.e., meaningful drill, should provide desired reinforcement of skills with fewer exercises than non-meaningful drill. It appears that, in most instances, a pleasing balance has been achieved between drill exercises and ones structured to emphasize the underlying concepts.

SMSG makes very little direct effort to "apply" to the physical environment the mathematical concepts derived. Conversely, no particular effort is expended to emphasize the abstract nature of mathematics and the recognized characteristic of modern mathematics as having no necessary connection

with the physical world. (It should be noted, however, that these materials contain many exercises which are of an applied nature and have a ring of "practicality" in that they are stated in an environmental language.) SMSG's guiding philosophy with respect to this problem has been reflected in the statement: "Since no one can predict with certainty his future profession, much less foretell which mathematical skills will be required in the future by a given profession, it is important that mathematics be so taught that students will be able in later life to learn the new mathematical skills which the future will surely demand of many of them."[1]

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[1] Ibid., p. 3.

CHAPTER III

THE UICSM SECONDARY MATHEMATICS PROGRAM

General Characteristics of the UICSM Program

The UICSM secondary mathematics program is one which is based on several explicitly-stated operating principles:

> We believe that students should be given an opportunity to discover a great deal of the mathematics which they are expected to learn. . . . We believe firmly that the learning of mathematics should be a delightful experience for youngsters, and that this delight is the reward for hard work. . . . Since we believe that interest is a necessary condition for learning, we have tried to set the development of mathematical ideas in situations which are inherently interesting to young people. . . . There is a time in the student's mathematical career when he needs to bring rigor into the pursuit of mathematics and the mathematics he learns prior to this time must be so organized that when rigor is finally introduced, it will not be necessary for him to throw out anything he has learned at an earlier level.[1]

These books, written for college-bound students, reflect these operating principles in their textual structure.

The UICSM set of mathematics texts is divided into eleven units: Unit 1, The Arithmetic of the Real Numbers; Unit 2, Generalizations and Algebraic Manipulations; Unit 3, Equations and Inequations; Unit 4, Ordered Pairs and Graphs;

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[1]University of Illinois Committee on School Mathematics, High School Mathematics, Unit 1, Teacher's Edition (Urbana: University of Illinois Press, 1960), pp. 1-4.

Unit 5, Relations and Functions; Unit 6, Geometry; Unit 7, Mathematical Induction; Unit 8, Sequences; Unit 9, Elementary Functions, Powers, Exponentials, and Logarithms; Unit 10, Circular Functions; and Unit 11, Complex Numbers. The fact that the units are bound separately should not cause the reader to regard the units as separate texts but rather as separate sequential and integrated chapters of High School Mathematics. These units are not written for particular grade levels but rather for experience levels although Units 1-4 comprise roughly the introductory high school algebra course and Unit 6 that of a high school plane geometry course. (In view of this parallel, Units 1-4 will be examined as a continuous block of study.)

The teacher's commentaries are major textual productions within themselves. Rather than being separate units, the commentaries have been designed so that they can be integrated page-by-page with the pages of the student edition with each of the commentary pages containing a discussion relevant to one or more pages of the text. These commentary sheets describe the experience-based notions as to what to expect from the student and contain mathematical background and appropriate references to mathematical treatises for the teachers as well as pedagogical suggestions for the presentation of mathematical concepts.

The Arithmetic of Real Numbers, Unit 1

Unit 1 is devoted to an investigation of the arithmetic of the real numbers and, consequently, that which is usually called algebra is not even mentioned until Unit 2. It evidently has been the experience of the writers that in the earlier grades the students have learned very few things about numbers and the general properties of numbers and possess little understanding of the processes involved although they have learned various algorithms and computational procedures. However, an "algebraic" treatment of real numbers demands knowledge of some of the general properties of numbers such as associativity, commutativity, distributivity, equality and inequality relations, as well as an appreciation of deductive proof. It is toward this particular set of needs that Unit 1 is directed.

Introduction

In order that the student may gain an intuitive idea regarding the difference between number and number-symbol, the writers present a very provocative account of an interchange of letters between two persons--one of whom is trying to teach mathematics to the other by mail. Various misunderstandings develop due to the inadvertent interchanging of symbols and referrents leading to a maximum degree of confusion. The student is led to recognize that the source of

this confusion lies in the symbolism and language rather than the abstraction of the numbers themselves.[1]

Names of numbers are called numerals or numerical expressions. It follows directly that any given number has many numerals, e.g., "4," "8 ÷ 2," "8 x ½," the "product of 2 by 2" and "3 + 1" are all names for the number 4. This discussion of the set of numerals naming the number 4 is the beginning of an effort on the part of UICSM to illustrate that the symbol "3 + 1" does not command the reader to add 3 and 1 but rather is a name for 4. "2 + $\sqrt{2}$" is a numerical expression representing the sum of 2 and $\sqrt{2}$ and a shorter correct name for the sum of these two real numbers is not available. Similarly, "1/3 + 1/5" is a name for the number which names the sum of 1/3 and 1/5, a shorter name for this sum being "8/15."

The exercises at the end of this topic are typical of the kind of exercises which occur throughout the text. The exercises are designed to reinforce the already presented concepts and yet at the same time to force the student to extend these ideas still further. The student must be alert to the discussion at hand in order to complete successfully the exercise.

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[1]University of Illinois Committee on School Mathematics, The Arithmetic of Real Numbers, Unit 1 (Urbana: University of Illinois Press, 1962), p. A-N.

Distance and Direction

The writers do not make any attempt to formally define a number but only present the idea of the numbers of arithmetic through consideration of measurements of magnitudes such as lengths, areas, volumes, speeds, etc., and, in actuality, accept the arithmetic numbers as being undefined. These arithmetic numbers, which have already been used by the student since the first grade, are illustrated as being capable of measuring distances for trips along a "road" (a straight line), measuring from some arbitrary point as departure point provided that alterations can be made to account for the direction of travel. The symbols $\rightarrow$ and $\leftarrow$ are "invented" to solve this need and a system is devised whereby $\overrightarrow{a}$ represents a number signifying a trip of a units in a given direction (perhaps west) and $\overleftarrow{a}$ represents another number of a units along the opposite (east) direction. These numbers representing both direction and magnitude are defined as real numbers. One notes that reliance is being placed intuitively upon the number line whose coordinates are real numbers and also that this extension to the real numbers treats the set of non-negative real numbers as different from, although isomorphic to, the set of numbers of arithmetic.

The authors are very careful at this point to prevent the formation of the notion that a "real number is an arithmetic number with a sign in front" since a number is an

abstraction and has no "front."[1] It is evidently for this reason that the term signed-number is never used by UICSM when referring to the real numbers.

The "directed trips" and the associated right-real and left-real numbers are renamed once the student has become familiar with the underlying associated ideas. The right-real numbers are named positive numbers (indicated by ^{+}a instead of $\overrightarrow{a}$) and the left-real numbers are named negative numbers (indicated by ^{-}b instead of $\overleftarrow{b}$). It is emphasized that positive and negative numbers come in pairs, each pair containing the positive number ^{+}a and the negative number ^{-}a, each corresponding to the arithmetic number a.

Addition of Real Numbers

The addition of real numbers is structured on an extension of the number scale as is ordinarily presented (although the number scale is not used), except that the symbolical representation of the real numbers is different. The addition of two real numbers ^{+}a and ^{-}b (represented by $^{+}a + {}^{-}b$) is presented by the use of two successive trips with the sum being the number representing the total displacement, i.e., a single trip that would cause the same displacement as the combination of the two trips. As emphasized by the writers, the operation of addition of real numbers is different from the operation of addition of arithmetic numbers and, since the

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[1] Ibid., p. 3.

operations are different, it is not surprising that the new operation does not have the same properties as the old, e.g., addition does not necessarily involve increasing, and that the addition of real numbers is a combination of the operations of addition and subtraction of arithmetic numbers. The care exercised by the UICSM writers is illustrated by their insistence that when one adds $^{+}5$ to $^{+}3$, one must write "$^{+}3 + {}^{+}5$" rather than "$^{+}5 + {}^{+}3$, i.e., <u>adding</u> $^{+}5$ is written in such a way that the student associates this operation in the same way that they associate <u>subtracting</u> $^{+}5$.

It is evident that the superscript sign convention prevents ambiguity between <u>operation-symbol</u> and <u>number-symbol</u>. This notation separates the naming of the indicated operation from the "sense" of the real number to which the operation is applied--a very radical and useful departure from traditional texts.

The real number <u>0</u> is introduced as the sum of any two opposite real numbers ^{+}a and ^{-}a. It is pointed out that the real number <u>0</u> and the arithmetic number <u>0</u> are not the same since the real number <u>0</u> is the sum of two real numbers whereas the arithmetic number <u>0</u> is the difference of two arithmetic numbers. The authors prudently suggest, however, that no attempt will be made to differentiate symbolically between the two meanings.[1]

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[1] <u>Ibid</u>., p. 14.

The student is compelled to formulate the "rules" for the addition of real numbers, i.e., the text does not present at this time the formal rules for this operation. The writers do not even mention the process of subtraction at this point but rather wait until the concept of inverse operations has been developed.

Multiplication of Real Numbers

Certain examples using a water pump (pumping in or out), a water tank (filling or emptying), a movie projector (running backwards or forwards), and consideration of hypothetical interactions of such permit the student to formulate rules for determining the products of real numbers. It is emphasized that the physical interpretation which leads to the rules for multiplying real numbers does not prove that the rules are correct but that the rules themselves are consequences of the definition of real numbers and the operations, and these definitions are implicit in their physical interpretation.

Numbers of Arithmetic and Real Numbers

This particular topic deals with perhaps one of the most sophisticated mathematical ideas in the unit. The purpose of the Exploration Exercises of the section and the textual discussion is to present the point of view that the system of numbers of arithmetic and the system of non-negative real numbers are different systems which have the same

structure. This, of course, is an illustration of the concept of isomorphism. It is demonstrated that if the system of the numbers of arithmetic is isomorphic to the system of non-negative real numbers, the two systems have the same structure, and are "abstractly" identical. There is no reason, however, for saying that the numbers in one system are the same as the numbers in the other. The term isomorphism is not used in the text at this point.

This presentation allows considerable simplification in many expressions in that the non-negative real numbers act as do the numbers of arithmetic with respect to addition and multiplication and their names may even be interchanged under these circumstances, e.g., $^{+}9 + {}^{-}3 = {}^{+}6$ might be written as $9 + {}^{-}3 = 6$. It also suggests that such statements as $7 \times 5 = 35$ are ambiguous since, without further information, it is impossible to determine whether these are arithmetic numbers or real numbers. However, if no confusion can exist as a result, the reader is urged to interpret the expression in either fashion.

Punctuating Numerical Expressions

This particular topic is rather conventional in its presentation although the language conforms with the more modernistic definitions. It is illustrated that in mathematics the ambiguity of expressions such as $8 \times 3 + 2 \times 5$ may be removed by punctuation of the expression by use of parentheses,

brackets, braces, etc. It is mentioned that $\{ [(7 \times 2) + 6] \times 5\} + 7$ is a <u>name</u> for a number and that its equivalent 107 (also a name for the same number) is considerably simpler.

The conventional rules for precedence of arithmetic operations, grouping, and simplification are then postulated but are restricted by the examples to the arithmetic numbers--perhaps somewhat confusing to a student who has so recently absorbed assorted pieces of information regarding the real numbers. The novelty of the presentation lies in its use of the terms <u>punctuation</u> (using symbols to remove ambiguities), <u>equivalent</u> (another expression naming the same number), and <u>unabbreviating</u> (tantamount to a procedure for deciding upon order of operations).

Principles for the Numbers of Arithmetic

This topic, also restricted to arithmetic numbers, develops, through an abundance of exercises, an awareness of the <u>associative principles of addition and multiplication</u> and the <u>commutative principles for addition and multiplication</u>. In essence, these principles are stated as postulates in the undefined system of arithmetic numbers.

In its development of the distributive principle for multiplication over addition, the UICSM material departs from the conventional in one respect. Most texts suggest that $ax(b + c)$ is the "same" as $(b + c)xa$ since the results of

these operations are the same, but since the UICSM always writes the name of the multiplier to the right of the multiplication sign, the distributive principle for multiplication is stated as (a + b)xc = axc + bxc, whereas the left-distributive principle for multiplication over addition is stated as cx(a + b) = cxa + cxb.[1] Other principles are postulated and illustrated: (a) the principle for adding 0 (a + 0 = a), (b) the principle for multiplying by 1 (axl = 1), and (c) the principle for multiplying by 0 (ax0 = 0).

An interesting notation innovation is the introduction of the abbreviations cpa (commutative principle for addition), apa (associative principle for addition), cpm (commutative principle for multiplication), ldpma (left-distributive principle for multiplication over addition), etc. One also notes with interest the pattern which is used, i.e., a sentence written with only operations indicated and grouping symbols indicated with empty spaces to be filled, one at a time, as justification for each step is made.

Principles for the Real Numbers

This topic extends the preceding discussion to the field of real numbers and allows and encourages the student to verify (though not prove) that the principles developed for the arithmetic numbers hold also for the system of real numbers. The better student is encouraged to understand that

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[1] Ibid., p. 52.

perhaps the operations on the reals were formulated as they were in order that just such an extension of principles would hold, i.e., it was desirable that the principles of arithmetic numbers (since they were isomorphic to a subset of the reals) would be "included-in" the principles of the reals.

Inverse Operations

In this topic, another very sophisticated mathematical concept is utilized in the introduction of ordered pairs and the subsequent definition of various new operations. As an illustrative example, consider the operation adding 3. This operation actually entails taking all pairs of numbers such that the second member of the pair is the sum of the first member and the number 3, e.g., (3, 6), (2, 5), (6, 9). Consider now the operation of subtracting 3 defined to be the taking of the set of all pairs of numbers such that the first member of the pair is the sum of the second member and the number 3, e.g., (3, 0), (6, 3), (8, 5). Certainly these operations are not the same although there does seem to be similarity (in some unknown way) between the two. Certainly if the operation adding 3 were followed by the operation subtracting 3, or conversely, the results would merely nullify each other, i.e., the subtracting 3 operation "undoes" the adding 3 operation. Subtracting 3 is then defined as the inverse of adding 3. Extensions and further discussions of inverses are made but the major point is that an inverse of

an operation is an operation which "undoes" the effect of that operation.

Pairs of real numbers whose sum is 0 are defined as being opposites. Therefore, every real number has an opposite.

Subtraction of Real Numbers

By using the results of the preceding topic, the process of subtraction of a real number is defined as an operation which is the inverse of adding that given number, i.e., the principle of subtraction states that subtracting a real number is precisely the same as adding its opposite. This treatment is very modernistic in that the student should recognize that this presentation "constructs" subtraction by use of only the principles of addition, i.e., the operation of addition has been extended to serve as a definitive tool for inverse operations.

Opposites

An important fact about the real numbers is that, for each real number, there is a real number (the opposite of the first) which when added to the first gives 0. Finding the opposite of a real number is defined to be an operation just as adding 6 or multiplying by 6 are operations. The symbol "—" is introduced as the symbol naming the oppositing operation. Some potentially troublesome notation is introduced here; the sign "—" for opposite is approximately twice as long as the

sign "-" for negative, e.g., —($\bar{}$9) is the opposite of negative 9.

Since $^{+}$a has already been shown replaceable by a when no ambiguity is present, the above principles of opposites allows one to rename the negative number $\bar{}$b as -b (since the opposite of a positive number is a negative number).[1] This then allows the authors to rename the negative real numbers as being the opposites of positive reals unless ambiguity is produced. This means that the minus sign is to be used in three different ways: (1) when naming a negative number, (2) when naming the opposite of a real number, and (3) when indicating the subtraction operation. The result of the introduction of these many operations has been to define carefully the various uses of the minus sign which are usually thrown into the traditional algebra texts without differentiating explanations. The authors also discuss briefly the sameing operation which is analogous to the oppositing operation, i.e., just as oppositing takes a real number to its opposite, the sameing operation may be thought of as the taking of a number to itself.

Division of Real Numbers

Just as subtracting a is defined to be the inverse of adding a, dividing by b, $b \neq 0$, is defined to be the inverse of multiplying by b. A fraction is defined to be a type of

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[1] Ibid., p. 86.

numeral having three parts: numerator, fraction-bar, and denominator. This form indicates division of the numerator by the denominator. This definition requires that a fraction be a numeral rather than a number and is important in that 1/4 is a different fraction from 2/8; however, 1/4 is the same number as 2/8. With this concept in mind, a quotient of the number a by the number b, $b \neq 0$, may be named as the fraction a/b.

Comparing Numbers

UICSM presents a scheme somewhat different from SMSG to consider order relations. Consider a pair of any two real numbers a and b, $a \neq b$. The number of the pair to which one must add a positive number to get the other is defined to be the smaller whereas the other of the pair is the larger. One notes that this ordering principle for the reals does not rely upon the number line for meaning--a unique feature in that SMSG relies heavily upon representation of the set of reals as being coordinates of a number line.

The Number Line

It is at this point that the authors formally introduce a device commonly introduced early in the semester in most texts--the number line--in which the ordered set of real numbers is thought of as being superimposed upon a straight line. This device is of great value in the clarification of the greater than and less than relations.

It has already been seen that to each arithmetic number there corresponds a unique real number and to that real number corresponds a unique opposite. With this in mind, the authors develop a very unusual (insofar as traditional treatments are concerned) definition: The number of arithmetic which corresponds with a real number is called the absolute-value of the real number.[1] This definition demands that the absolute value of a real number is not a non-negative real number but rather a number of arithmetic--a quite unorthodox result. It is also noted that absolute-valuing is an operation "matching" each real number to a single corresponding number of arithmetic. Rigorous discussion of this operation and the possibility of its inverse operation is presented.

Generalizations and Algebraic Manipulations, Unit 2

This particular unit treats one of the most important and valuable ideas in mathematics--the concept of the variable. The concept is treated with especial care since this is the first time that most students will have done formal study in the use of variables and one notes several unorthodox features of the UICSM position insofar as the nature and role of the variable is concerned. A major purpose of the unit, in addition to displaying the properties of the variable, is to help students develop the manipulative skills which are traditionally taught in beginning courses in

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[1] Ibid., p. 104.

algebra. It is noted, however, that UICSM does not use the term variable but rather a new term which is coined to serve as vehicle for this concept.

Sentences

In the earlier topics, it has been mentioned that if two numerals are the names of the same number, the conventional equality sign would be used to indicate this fact. A sentence which consists of an equality sign between a pair of numerals (or numeral expressions) is a true sentence if the numerals are names of the same number; it is a false sentence if the numerals are the names of different numbers. Under this agreement some statements are neither true or false, e.g., $9 + \triangle = 15$ is neither true nor false. This, of course, is recognized as being the standard introduction to the concept of variable--yet the authors do not introduce the term at this point. The student is encouraged to recognize that an open statement containing a "hole", although neither true nor false as it presently stands, may be converted into a sentence which is either true or false (but not both) by putting a numeral into (or in the place of) the "hole." The many figures such as $\square$, $\triangle$, $\star$, $\bigcirc$, etc., which might be used to indicate "holes" in a statement are given the name frames. One notes that, under the UICSM treatment, the frame does not "stand for a number" but rather "holds a place for" a numeral, i.e., the frame is a placeholder for a numeral.

Subsequent developments will characterize the variable (a special type of which will be called pronumeral) as a place-holder for the name of a number rather than the traditional consideration of a variable as "the name of a number."

The authors give a very precise description of what is meant by substitution: "To substitute a numeral for a frame in an expression or sentence is to replace each occurrence of the frame by copies of the numeral."[1] A multitude of exercises is provided which illustrate the use of frames and pattern sentences for the many principles of the reals.

Pronouns

A sentence which is either true or false is called a statement. Sentences containing holes or frames which can be converted into statements by filling the holes or frames with names (or by the process of substitution) are called open sentences. It follows then that an open sentence is neither true nor false. Actually, then, since the holes in an open sentence hold places for nouns, they are called pronouns, and in the event that mathematical sentences are used, the frames are the holding places for nouns which are the names of numbers, i.e., they hold places for numerals and as such are called pronumerals. It is very conveniently suggested that,

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[1]University of Illinois Committee on School Mathematics, Generalization and Algebraic Manipulations, Unit 2, Teacher's Edition (Urbana: University of Illinois Press, 1962), p. 13.

although the work of mathematics could be done adequately by use of frames such as □, △, and ○, it might be more convenient to use the lower-case letters of the alphabet to perform the chore of placeholding in open sentences. The expressions and sentences which can be generated from pronumerals are given the name pronumeral expressions. The authors stress these properties of pronumerals: (a) a pronumeral is a mark; (b) a pronumeral is not a numeral, and (c) a pronumeral is a mark which holds a place for numerals.[1]

One notices that this treatment of variable is rather unorthodox in that the variable is no more than a mark on the paper and not something which is denoted by a mark as a number is denoted by a numeral. This conflicts with the usual consideration of the variable as being the name of a number. Also a variable per se does not have a referrent since it is not a name and is, under this consideration, no more than a blank in an expression which serves to hold the place for some object.

Many exploratory exercises are provided at this point which lead the student to formulate generalizations (using pronumerals and open sentences) of those ideas which he has already developed regarding the principles and rules of manipulation of the reals and to state these generalizations

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[1]University of Illinois Committee on School Mathematics, Generalization and Algebraic Manipulations, Unit 2 (Urbana: University of Illinois Press, 1962), pp. 14-18.

in exact, precise, non-ambiguous language, e.g., the student is led to formulate such precise rules as: "For each x, for each y, if x is negative and y is negative, then $x + y = -(|x| + |y|)$."[1] Considerable emphasis is placed upon the quantifiers for each, and for every.

Generalizations

The student is motivated to study in this topic the process of generalization. A generalization is obtained by writing a qualifying phrase as a "hypothesis" phrase for an open sentence, i.e., a generalization is obtained by writing a quantifying phrase such as "for each x" in front of a sentence. "For every y, $y + y = 2y$," is an open generalization sentence. The student is given practice in the confirmation (by use of test-patterns) of generalizations and the very idea of generality is presented intuitively, yet effectively, and shows that every instance of generalization regarding the reals must be a consequence of the developed principles of the real numbers. The reader cannot help but be impressed by the very logical, axiomatic approach to the process of generalization by utilization of what the authors refer to as test-patterns, which are actually "proofs" (using frames) structured on the axioms (earlier referred to as basic principles) and their logical applications. These proofs of generalities are structured on the basic axioms and are built logically

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[1] Ibid., p. 29.

from them as foundation blocks. It is also emphasized that these proofs are only as strong as the axioms--a point which often seems to be under-emphasized (and even neglected) in many of the treatments of the structure of the number system.

Simplification of Expressions

It is suggested that one of the major uses of pronumerals and pronumeral expressions is to write formulas which can be used in solving problems. Quite often, however, such expressions may be so long and cumbersome as to be unwieldy and the principles for real numbers are convenient to justify simplification of the expressions, i.e., generalization procedures may be utilized. These simplifications must be results of the principles of the real number system.

Equivalent numerical expressions are numerals for the same numbers and equivalent pronumeral expressions are ones such that for each substitution of a numeral for a pronumeral, both expressions have the same value. The rationale behind this definition is apparent since to simplify any expression is merely to transform (by use of the basic principles) it into an equivalent one which is simpler.

Theorems and Basic Principles

In this topic--a remarkably sophisticated one--are collected the basic principles for real numbers from which all other principles concerning the operations of addition, multiplication, opposition, subtraction, and division can be

obtained. The student is forced to derive many such principles, use them to justify computational short-cuts, and to gain skill in applying these shortcuts.

The procedure of taking a known subject matter (in this instance, the real numbers) and organizing it deductively by choosing some "true" statements from it as basic principles (or postulates) and deriving others (theorems) from them is illustrated and discussed here. It is also pointed out that in the application and construction of mathematics, one forgets the "known subject matter," and considering the postulates and theorems merely as sentences in an uninterpreted language, concentrates on the logical connections among these sentences--a process which pays strict attention to the structure exhibited by the original subject matter. The authors also introduce the symbol " $\forall$ " to represent the quantifier <u>for each</u>, and states the basic principles in terms of this usage.

Oppositing and Subtracting

By utilization of the basic principles already introduced the student is encouraged to prove such theorems as the following: (a) $\forall_x \forall_y \forall_z$ if $x = y$, then $x + z = y + z$ (the <u>uniqueness principle for addition</u>); (b) $\forall_x \forall_y \forall_z$ if $x + z = y + z$, then $x = y$ (<u>cancellation principle for addition</u>); (c) $\forall_x \forall_y$ if $x = +y$, then $-x = -y$ (<u>uniqueness principle for opposition</u>); (d) $\forall_x \forall_y$ if $x + y = 0$, then $-x = y$ (<u>0-sum</u>

theorem). These theorems are then utilized to provide tools for writing equivalent simpler expressions.

Division

In an earlier topic the process of division has been defined as the inverse of multiplying--a concept discussed by the utilization of ordered pairs. It has been shown that it is always possible to subtract a second real number from a first real number but that one can divide a first real number by a second real number if and only if the second is non-zero. It is concluded, from an analysis of the non-existence of the inverse of multiplying-by-0 that dividing-by-zero is not an operation.

It is shown that the process of multiplying by each non-zero real number has an inverse such that "$\forall_x \forall_{y\neq 0}$ there is just one z such that zy = x," and this theorem is presented along with a "proof" of the division theorem "$\forall_x \forall_{y\neq 0} \forall_z$, if zy = x, then z = x ÷ y." These and other results are utilized to prove the cancellation principle "$\forall_x \forall_y \forall_{z\neq 0}$, if xz = yz, then x = y."

Principles for multiplying fractions, determining the least common denominator, dividing fractions (justification of the "invert-and-multiply" rule), division and opposition (opposite of a quotient is the quotient of opposites), etc., are developed as generalizations of the theorems and basic principles. These studies allow the student to procure a very deep

understanding of the processes utilized for working with fractions and lead him naturally to the study of "algebraic" fractions.

Comparing Real Numbers

In Unit 1 the concept of greater than was considered by the construction of ordered pairs. This concept is now formalized into a subtraction procedure: "$\forall_x \forall_y$, (a) if $x - y$ is a positive number, then $x > y$, and (b) if $x - y$ is not a positive number then $x \not> y$."[1]

Although several exercises demand it, no basic principles yet have been given from which students could derive theorems regarding absolute values. However, it follows that "$|\ .\ .\ .\ |$" (an arithmetic number) may be used as an abbreviation for "$^{+}|\ .\ .\ .\ |$" and the principle "$\forall_x$, if $x > 0$, then $|x| = x$ and if $x < 0$, then $|x| = -x$," follows directly.

Equations and Inequations, Unit 3

Graphs and Coordinates

In previous topics the student has already thought of the real numbers as being ordered on a line. In this topic the subject is elaborated in that a one-to-one correspondence is indicated between the real numbers and the points on a line. Each point on the line (represented by a dot)

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[1] Ibid., p. 109.

corresponds with a unique real number and conversely. The real number associated with a particular point on the line is considered as the coordinate of that point.

Solution Set of a Sentence

At this particular point, a concept is introduced which is very basic to the modern development of mathematics and one which has contributed a tremendous amount to the revolution of the structure of mathematics--the concept of the set. If an open sentence is considered, each real value of x can be used to convert the open sentence into a true statement or a false statement. A value of x which will convert the sentence into a true statement is said to satisfy the open sentence or to be a solution of the open sentence. The set of all real numbers satisfying the open sentence is called the solution set of the sentence. The student is introduced to the notion that some solution sets contain many elements, that some solution sets have only one element (and are called singletons), and that some solution sets are empty (or null). Suitable notation for the elements of a set is developed, i.e., $\{a, b\}$ is a sufficient way to describe the set whose elements are a and b, whereas the set-builder $\{x : x + 5 < 9\}$ describes the set of all numbers x such that the sum of x and 5 is less than 9.

Since the UICSM authors have not yet introduced exponential notation, one finds expressions such as 2xx and

3xxx, rather than $2x^2$ and $3x^3$, respectively. From this point forward, the authors make extensive use of set notation and language and are consistent in their usage.

Graph of a Sentence

The term graph of a sentence is an unusual one in secondary texts although the associated idea is rather orthodox. The picture (or representation by dots on the number line) made up of the graphs of the numbers in the solution set of a sentence is called the graph of the sentence. The geometric term locus is introduced as a possible alternative for the term solution set of a sentence. Convenient notation for discussion of these sets, or loci, is presented, e.g., $\{x : 1 < x < 3\}$ names the set of numbers each of which is greater than 1 and also simultaneously less than 3. This same set might also be labeled by $\overline{1, 3}$ and, in the event that one wished the set to include the numbers 1 and 3, one might use $\bullet\overline{1, 3}\bullet$ whereas $\bullet\overline{1, 3}$ would describe the set $\overline{1, 3}$ with the set $\{1\}$ unioned to the set.

Equations

Probably no definition of mathematics is more confusing than the one ordinarily given for equation. (What does it mean to say that an equation is a statement indicating that two numbers or quantities are equal? Two numbers cannot be equal since the only thing to which a number can be equal is itself!) This difficulty has been corrected by the UICSM

definition of an equation: "An equation is a sentence obtained by connecting expressions by an equality sign."[1] Under this definition, the truth or falsity of a statement is entirely irrelevant as to whether or not the sentence is an equation. The term equation refers to the form of the sentence and not to its content. The numbers which satisfy (or comprise the solution set of) an equation are given the title of roots of the equation (and are numbers, not numerals).

It is at this point that the carefulness with which the concept of equivalent pronumeral expressions was developed begins to bear fruit. The student sees and realizes the power of the tool of replacing an expression by an equivalent expression. Its application toward finding the solution set of a sentence is evident.

Although at this point the student has had considerable practice with the solutions of equations, he has not yet seen a formal statement of the ordinary axioms usually seen, e.g., "When equals are added to equals, the results are equal." He has, however, formulated his own procedures and techniques for rewriting an expression into an equivalent expression. He has at the same time explored the significance of the very term equivalent by considering such examples as these: If equation (a) is transformed into an equivalent equation (b), the roots

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[1]University of Illinois Committee on School Mathematics, Equations and Inequations, Unit 3 (Urbana: University of Chicago Press, 1962), p. 19.

of (a) are roots of (b), and the roots of (b) are roots of (a). He has confirmed that certain transformation principles available to him hold the solution set invariant while altering the form of the expression. These transformation principles give him authority to simplify a "complicated" expression into a simpler, equivalent one.

UICSM introduces at this point the is a subset of relation. This relation is abbreviated by the regular notation, i.e., "A $\subseteq$ B" indicates that set A is a subset of set B.

Equivalent Equations

This topic develops the further application of the student's reasonably good acquaintance with the equation transformation principles and guides him in the application of this acquaintance in deriving equations which are equivalent to a given one. An equation whose roots are the same as the roots of a given equation is said to be equivalent to it (and conversely). The process of transformation by use of the basic principles is utilized until an equivalent one whose root is "obvious" is obtained. The problem of solution of equations containing fractional expressions is also solved by transforming them into equations which do not contain fractions.

It is pointed out very dramatically that when one transforms an equation by multiplication, one may not get an equivalent equation, e.g., $x^2 = 2x$ is not equivalent to $x = 2$

since the solution set for the first equation is $\{0, 2\}$ but the solution set of the second is $\{2\}$. The student is encouraged to always be on the alert for transformations which do not yield equivalent expressions.

Transforming a Formula

This topic treats the subject often referred to as the solving of literal equations. Again the concept of equivalence is utilized in that formulas such as $C = 5/9 (F - 32)$ and $F = 1.8C + 32$ are given as examples of equivalent formulas since either of them can be transformed into the other by using transformation principles developed earlier.

Solving Problems

The authors of the UICSM materials admit readily that they do not know any universal formula for solving "word" or "verbal" problems but subscribe to the philosophy that the most effective teaching technique is to provide the student plenty of material with which to practice. This treatment does, however, steer away from processes which succeed in getting students to solve certain kinds of problems in a mechanical way. It is pleasing to find several "word" problems in which insufficient data is present, problems in which the data is purposely inconsistent, and problems in which more information is provided than is necessary for the solution of the problem. Schematic diagrams are utilized to a high

degree in the construction of equations. The process of "translation" from English statements to open sentences is emphasized.

By use of the familiar principles for real numbers, pronumeral expressions may be expanded, e.g., $(x + 10)(x + 2) = x(x + 2) + 10(x + 2) = xx + 2x + 10x + 20 = xx + 12x + 20$. It is here that the exponent symbol, or exponent, is formally introduced for the first time in the UICSM texts. It must be noted that rules such as "The square of a binomial is the square of the first plus twice the product plus the second squared," "Sum of two terms times difference of same two terms is difference of squares," are not introduced and the student resorts to the basic principles, e.g., distributive, associative, and commutative principles of the reals for derivation although the student will probably formulate these rules on his own initiative. It is important to note also that various exercises are given which use frames as variables, e.g., $(\square + \triangle)^2$, $(\bigstar - \bigcirc)^2$, $(\bigstar + \square)(\bigcirc + \hexagon)$.

The topic of factoring of quadratic pronumeral expressions is developed as essentially a trial-and-error process relying primarily upon the distributive principles for multiplication. A more thorough treatment is reserved for a later unit.

Quadratic Equations

An implicit application of the factoring transformation principle "$\forall_x \forall_y$, $x = 0$ or $y = 0$ if and only if $xy = 0$," plus the 0-product theorem is made to solve a select group of quadratic equations of the form $ax^2 + bx + c = 0$ such that $ax^2 + bx + c = 0$ may be transformed into an equivalent sentence $(dx + e)(fx + g) = 0$. Consequently, "$ax^2 + bx + c = 0$" is equivalent to the sentence "$dx + c = 0$ or $fx + g = 0$."

Solving Inequations

The solution of inequations is a process that is fairly unique to modern treatments of secondary algebra. Certainly the concept of solution sets of sentences (which may even be intervals) gives a tremendous boost to the technique and also the carefulness with which the basic principles of the reals have been developed again lends support.

It is shown that the addition transformation principle and factoring transformation principle for inequations are entirely analogous to those for equations but that the multiplication principle is more complicated, i.e., the generalizations "$\forall_x \forall_y \forall_{z<0}$ $xz < yz$ if and only if $x > y$," and "$\forall_x \forall_y \forall_{z>0}$ $xz > yz$ if and only if $x > y$" must be introduced.[1] These theorems are obtained informally in the text discussion.

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[1] Ibid., p. 102.

Square Roots

This very sophisticated treatment of a very familiar topic should provide opportunities for some inspired teaching since the student is led to think about some very important notions, see some fascinating theorems proved, and to learn an important iterative process for the computation of square roots. Although the authors do not actually prove (due to the need for principles of completeness of the reals) the fact, the idea that each positive number is the square of at least one positive number is utilized.

The text contains an explicit presentation of a study of degrees of approximation. A typical statement is this: "For each positive number x, there is just one integer y such that $y \leq x < y + 1$. This number y is the approximation to x correct to the units place." This is further refined in such statements as "The approximation to x correct to the nearest unit is the integer y such that $y - 0.5 \leq x < y + 0.5$."[1] Suitable extensions are made for the hundredth place, etc.

A very excellent, though brief, topic is presented on estimate of errors. Typical exercises and solutions require discussions of this type:

> Assume that you know that a number y is the approximation to a number x correct to 2 decimal places. You know that $y \leq x < y + 0.01$, i.e., $0 \leq (x - y) < 0.01$.

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[1] Ibid., p. 112.

> Therefore you know that the size of the error $x - y$ is between 0 and 0.01, i.e., $|x - y| < 0.01$.[1]

Various iterative processes are developed for finding approximations to square roots although the traditional process is not introduced. One of these methods is essentially the same as the "dividing-and-averaging" method discussed by the SMSG writers.

The non-negative square root of a non-negative real number is named the principal square root, or, simply, the square root. The following generalization is developed: "For each x, $\sqrt{x^2} = x$ if $x \geq 0$ and $\sqrt{x^2} = -x$ if $x < 0$, i.e., for each x, $\sqrt{x^2} = |x|$."[2] No introduction is made of the negative and fractional exponents often found in traditional treatments although the student is adequately introduced to the simplification and manipulation of radicals.

Ordered Pairs and Graphs, Unit 4

This unit serves as a good introduction to the number plane and presents an excellent overview of the graphing of sentences. This unit refines the concept of ordered pairs and graphs and presents a quite new approach to graphing. Several important ideas are presented in the Introduction.

The authors consider a pair of numbers a and b and write them in the style (a, b), i.e., order the number a to occupy the first position and b to occupy the second position.

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[1] Ibid., p. 118. [2] Ibid., p. 132.

One is then forming an ordered pair of numbers such that of the ordered pair the first component is a and the second component is b. The ordered pair (a, b) is not the same as the set $\{a, b\}$ since set $\{a, b\}$ is the same set as $\{b, a\}$ but (a, b) is not the same ordered pair as (b, a). In general, (a, b) = (c, d) if and only if a = c and b = d. The set of all ordered pairs with first components from some set A and second components from some set B is called the Cartesian product of set A by set B and is named A $\times$ B.

Lattices

If one considers a Cartesian product of two sets A and B and represents the ordered pairs as corresponding to points on a rectangular array of "dots," one forms what is ordinarily considered as a lattice. The number of columns in the rectangular array is usually the same as the number of elements in the first factor of the product A $\times$ B and the number of rows is the same as the number of elements in the second factor of the Cartesian product. The first component of each of the set of ordered pairs is named the first coordinate and the second component the second coordinate.

Consider the set of all ordered pairs whose components are integral real numbers, i.e., the Cartesian product of the set of all real integers by itself. This lattice, infinite in dimension, is named the number plane lattice. Since each point of the lattice corresponds to an ordered

pair whose coordinates are integers, and conversely, the process of graphing a set of ordered pairs is the process of locating the points of the lattice which corresponds with (or has as coordinates) the given ordered pairs, a one-to-one correspondence between select points of a plane and the set of ordered pairs of integers is established. The set of points in the lattice having second component 0 is called the first component axis and that set having first component 0 is named the second component axis (as opposed to the traditional naming of these axes as the x-axis and y-axis) while the point in common between the two axes is named the origin.

The intersection of two sets is defined as the set of elements which belong to both of the given sets with the union of two sets being the set of elements belonging to either (one or other or both) of the given sets. Proper notation, i.e., "$\cup$" and "$\cap$" to designate union and intersection respectively, is introduced to express these notions. Various exercises are utilized to demonstrate these concepts with lattices being utilized to give "graphical" significance to them with various motivating lattice games being suggested. The authors also design an opportunity to discuss the number (more properly, the cardinal number) of a set and use the notation n(A) to represent the cardinal number of the set A.

The Number Plane

In this topic the idea of the lattice formed by the cross-product of two sets composed of integral numbers is extended to the number plane--the Cartesian square of the set of all real numbers. (It must be noted that UICSM at this point implicitly appears to regard the number plane as the Cartesian square of the set of all real numbers and does not concern itself with the problem of distinguishing between a point and its coordinates as evidenced by these statements:

> This Cartesian square is called the number plane.[1]
>
> An ordered pair of real numbers is a point of the number plane.[2]

Yet the authors refer to the "components of the points to be plotted"[3] and later establish the traditional definition for coordinate--a somewhat inconsistent usage. A vertical grid line pictures a set of ordered pairs of real numbers which have the same first component and a horizontal grid line pictures a set of ordered pairs which have the same second component. Any pair composed of elements from a vertical grid line and a horizontal grid line may serve as components for the number plane. Any ordered pair of real numbers is a point of the number plane and corresponds with a "dot" which can be positioned on the plane. The "dot" is the graph of

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[1] University of Illinois Committee on School Mathematics, Ordered Pairs and Graphs, Unit 4 (Urbana: University of Illinois Press, 1962), p. 20.

[2] Ibid., p. 21. [3] Ibid.

the ordered pair, and the components of the ordered pair are the coordinates of the dot. The first coordinate of each point is named the abscissa and the second coordinate the ordinate. This development of the number plane is very lucid and functional in that the locus of a bi-pronumeral sentence (conventionally referred to as an equation with two variables) is simply the graph of all ordered pairs which satisfy the sentence.

One may see immediately that the solution of a pair of simultaneous equations (or sentences) is that set of ordered pairs which satisfy both sentences, e.g., the solution of the simultaneous system of equations

$$\left.\begin{array}{r} x + y = 6 \\ 3x + y = 2 \end{array}\right\}$$

is the intersection of the two sets of ordered pairs one of which satisfies the first equation and the other the second of the equations, i.e., $\{(x, y) : x + y = 6\} \cap \{(x, y) : 3x + y = 2\} = \{-2, 8\}$. Graphically, this intersection represents the intersection of the graphs of the sentences. Similar discussions are presented to graph inequations in which cases the solution sets may represent areas of the plane rather than particular points of the plane as well as graphs of equations and inequations involving absolute values, etc.

Graphs of Formulas

This topic virtually duplicates the preceding one except that generalizations are graphed and the statement of the domain of the pronumeral is emphasized. It is important to note that step-graphs and other piecewise continuous graphs are studied.

Factors

In order to lend rigor to the topic of factorization, the number systems already studied by the students are evaluated in view of their being subsets of more inclusive systems, i.e., $\{ x : x \text{ is a positive integer} \} \subset \{ y : y \text{ is an integer} \} \subset \{ z : z \text{ is a rational number} \} \subset \{ w : w \text{ is a real number} \}$. Very close attention is paid to their properties and the following sophisticated definition is developed: "For each set S of numbers, $\underline{x}$ is a factor of $\underline{y}$ with respect to S if and only if $\underline{x}$ and $\underline{y}$ are in S, and if there is a $\underline{z}$ in S such that $y = xz$."[1] One notes as the major implication of this definition the fact that the domain of factorization must be defined for factorization to be non-ambiguous.

A very comprehensive study of even and odd integers is provided along with the convention that a positive integer which has exactly two factors with respect to the set of positive integers is a <u>prime number</u> and any number which is neither a prime number or 1 is a <u>composite number</u>. A study

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[1] <u>Ibid</u>., p. 49.

is made of the prime factorization of composite numbers and the Fundamental Theorem of Arithmetic is introduced and utilized in the accompanying exercises. Scientific notation (often called powers-of-ten) is presented as a computational tool and to provide practice with exponent manipulation. (At this point the student is introduced briefly to zero and negative exponents.)

Factoring

If the factor-sets (set of all factors) of two non-zero integers A and B are considered, the highest common factor of A and B is defined to be a positive integer x such that x is a multiple of each common factor of the numbers of the given sets. Similar discussions are made to define the least common multiple of two integers.

The concept of the highest common factor is applied to the factorization of pronumeral expressions such as

$$15x^2y^2z^2 + 20x^3y^2z^5 - 35x^2y^3z = 5x^2y^2z\,(3z + 4z^4x - 7y).$$

It is at this point that the authors first consider briefly the factoring of pronumeral expressions. None of the traditional rules for factoring is stated and one might suggest that not enough of the exercises are provided for reinforcement of the developed concepts although the Supplementary Exercises furnish an added number of such exercises. The concept of least common multiple is used to simplify sums and

differences of fractional expressions into equivalent expressions containing only one fraction.

Evaluation of UICSM Units 1-4

These four units, originally designed for ninth-grade consumption, are concerned with a detailed study of the real number system and its properties, an introduction to precise statement of principles and theorems with the accompanying proofs of several of them, sets and set notation, and solution sets and graphs of equations and inequations. As an end result, these units differ little, with respect to subject-matter content, from the conventional algebra texts except that the language approaches and techniques are substantially different.

These units exhibit clearly their fundamental aim as being the consideration of the structure of mathematics. The aim is evidenced by the careful attention paid to the development of the real number system, the careful foundation of vocabulary and fundamental concepts, and the emphasis on proofs based on the principles of the real number system.

Accompanying this aim is the evidenced genuine concern with the development of a precise vocabulary in the language of mathematics and the implicit belief that a student can be led early from a generally unsophisticated use of language to a precise and sophisticated use of it. Great care is taken to distinguish between an object and its name and a

symbol and its referrent--the strict distinction between numeral and number is consistent. The universal quantifier as well as quantifier phrases occur early in the materials. The theorems and principles are stated in precise and sophisticated language in order to adequately state generalizations. Many terms new to traditional texts are "coined" or "borrowed" to present new notions, e.g., pronumerals, pronumeral expressions, oppositing, sameing, pattern sentences, generalizations, transformation principles, number plane lattices, and grid lines.

The numbers of arithmetic are postulated and the principles of the system are verified in Unit 1 with deductive proofs being introduced in Unit 2. There is considerable rigor in the proofs especially in regard to the precision of language and the demand for the student's authentication of each statement in a proof by reliance upon the basic principles and proven theorems, although no formal logic is included. The use of the quantifier and other special symbols hold the notation at a fairly high level. The students are led to make independent proofs by the end of Unit 4 with the much earlier formulation of simple proofs.

Student discovery is the keynote of these UICSM units. Exploration Exercises, a novel feature of this production, appear frequently and encourage and guide the student in the discovery of generalizations. The "rules for signed numbers" are not stated as such but are to be discovered by the

student through several interpretations of the symbols. Students are led to discover their own solutions to many problems, e.g., solution of equations and inequations, thereby exhibiting UICSM's belief that the learning process is deepened by presenting a sequence of activities from which students may independently recognize some desired knowledge or concept. In many instances, special activities involving human characters are hypothesized which, through their interest appeal, serve to lead the student to recognize ambiguities.

Great emphasis is placed upon the development of concepts and this development occupies a more prominent role than the development of manipulation skills although the development of skills is not neglected. Lengthy lists of supplementary exercises accompany each unit, providing adequate drill for reinforcement of skills.

Numerous exercises (at least 429 such exercises in Unit 3) illustrate the social application of mathematics. These exercises, covering many fields and requiring critical thinking and careful analysis, serve to illustrate the mathematical principles. It appears, however, that the variety of application is of secondary importance compared to the principles.

Relations and Functions, Unit 5

As indicated by the title, this unit is devoted to a discussion of relations (as sets of ordered pairs) and the study of a special kind of relation, the function. Included are sections dealing with a simple algebra of sets, illustrations of relations arising from geometric problems, and ones introducing the notions of domain, range, and converses of relations as well as a close scrutiny of the properties of reflexivity and symmetry. In addition to these topics, this unit deals briefly with variable quantities, functional dependence, linear functions, quadratic equations, and systems of equations.

Relations

Although the first few pages of this topic essentially recall and couch in a more precise language some of the notions of earlier units, UICSM immediately proceeds in this unit to introduce a sophisticated and non-traditional treatment of relations and functions. It is recalled that the notion of ordered pairs of elements has been introduced and utilized earlier, e.g., and operation was considered as "a set of ordered pairs no two of which have the same first component."[1] The then explored implications of this definition are several. First, a relation is a set of ordered

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[1]University of Illinois Committee on School Mathematics, Relations and Functions, Unit 5 (Urbana: University of Illinois Press, 1962), p. 1.

pairs. Second, a relation which has been named, e.g., the greater-than relation, has to be described in some fashion and its domain specified before one can know what the name means. Third, a relation defined on a particular set is essentially a subset of the Cartesian product of the set with itself. Accompanying the discussion of these implications of the relation definitions are many fairly abstract exercises which reinforce this modernistic presentation and which emphasize the number-pair approach to the study of functions.

It is at this point that UICSM uses for the first time the term variable--defined simply to be a pronoun (and not necessarily a pronumeral). In the eyes of UICSM, the variable is only a mark (or placeholder) to hold a place in a sentence or expression for names of things (not necessarily numbers). The set of all things whose places are held by the variable is considered to be the domain of the variable.

Principles for Sets

Through comparison with the principles of the real numbers as studied in Unit 2, the student is led to realize that the replacement of + by $\cup$, $\cdot$ by $\cap$, 0 by $\emptyset$, and 1 by S in certain of these statements produces several of the principles for operating with subsets of a set S as discovered in the unit, e.g., "$\forall_x \forall_y \forall_z \ (x + y)\cdot z = x\cdot z + y\cdot z$," as established for the reals has as its analogous counterpart "$\forall_X \forall_Y \forall_Z \ (X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$ for subsets of S.

Test patterns are used to establish still other principles, e.g., the principles of complements of sets, which may not have analogous counterparts in the system of reals.

An optional section (which incidentally includes the structuring of test-patterns for proving De Morgan's Laws) shows that one can use a minimum of seven basic principles (or axioms and definitions--though not called such) to "prove" all the basic principles needed for operating with sets and subsets. An unorthodox feature of this treatment is that the sets being considered in this unit have as elements numbers, lines, rays, segments, curves, regions, populations or any defined membership. Also, UICSM has utilized symbolic abbreviations to the utmost in the presentation of these principles.

Relations and Geometric Figures

This brief section provides a few opportunities for the student to apply through geometric considerations that which he has learned about relations. Although the topic is brief, several characteristics of the UICSM materials are displayed. First, in the examples involving the various inequalities associated with side-measures of the triangles, it is emphasized implicitly that a side of a triangle is a segment and that one of the properties of a segment is its length and that, therefore, measures of geometric entities are numbers (in particular, numbers of arithmetic). Second, the

student is forced to realize the existence of the structure of systems by being reminded of the isomorphism between the system of numbers of arithmetic and the system of non-negative reals. Third, the UICSM attention to careful detail is illustrated by its insistence upon clockwise orientation when giving the side-measures of a triangle, e.g., the triangle (4, 7, 2) is a "different" triangle than the triangle (4, 2, 7) due to the clockwise orientation.

It does seem somewhat inconsistent, however, that the authors will in the same statement discuss the degree-measure of an angle in such a way as to indicate that the measure is a real number and then in the same sentence say that "an angle is 50°," e.g., "What is the relation of a degree-measure of one angle to the degree-measure of another angle of a triangle whose third angle is an angle of 50°?"[1]

Properties of Relations

The authors, having defined a relation R among the elements of a set S as being the subset of the Cartesian square S × S (or as a set of ordered pairs) utilize this definition to present a sharp definition of the domain and range of a relation. The domain (designated as $\mathcal{D}_R$) of a function R on a set S is essentially defined to be the set of members of S which are first components of members of R. Similarly, the set of members of S which are second components of R is

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[1] Ibid., p. 30.

the range (designated as $\mathcal{R}_R$) of R. The existential quantifier $\exists$ abbreviating "there is at least one" is introduced and used immediately. A rather unusual set, that of the field of R $\mathcal{J}_R = \mathcal{D}_R \cup \mathcal{R}_R$ is introduced to consist of the total set of elements of S involved in the given relation R and is actually the smallest subset of S whose Cartesian square contains all members of R. The examples and exercises presented to reinforce understanding of these definitions have a variety of subject matters, e.g., real numbers, people, sets, geometric figures, and measures.

An exercise develops the notion of the converse of a relation.[1] The converse of a relation R is the relation whose members are obtained by reversing the order of the components of the members of R, i.e., the converse of R is $\{ (x,y) \in \mathcal{R}_R \times \mathcal{D}_R : x R y \}$.

The remainder of the section deals with the reflexive and symmetric properties which a function may have, i.e., a relation R among the members of a set S is reflexive if and only if for each $x \in \mathcal{J}_R$, the pair (x, x) is in R and the relation R among the set of members of a set S is symmetric if and only if for every (x, y) in S $\times$ S, (x, y) in R implies that (y, x) is in R. The graphical connotations of these definitions are examined thoroughly along with the

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[1] Ibid., p. 39.

consideration of a multitude of exercises leading to student appreciation of these definitions.

Functions

As a result of having been persistent and precise in their treatment of relations as being sets of ordered pairs, UICSM now has the vocabulary available to formulate the modernistic definition of function. "A function is simply a set of ordered pairs no two of which have the same first component."[1] (This definition is illustrated to have the graphical significance that a function is a relation such that each vertical line crosses its graph in at most one point.) Many exercises serve to illustrate that functions can be sets of ordered pairs of any kind of elements and not necessarily pairs of numbers. UICSM assigns the term argument of a function to members of the domain of a function and the values of the function to the range. A multitude of functions are illustrated many of which are merely finite sets of pairs and which are defined by listing the pairs while many of them are infinite sets of pairs and therefore demanding set-builder notation for definition of function.

Immediately following this general discussion of the function is the consideration of a function having a domain A and a range B as determining a mapping of A on B with the value of the function for a given argument being the unique

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[1] Ibid., p. 50.

image of this value. This alternate definition allows various pictorial representations of the effects of function and allows a meaningful exploration of the composing of functions and terminating in the definition of the operation of composition. Having considered earlier the concept of the converse of a function, UICSM introduces the more stringent concept of the inverse of a function which, generally speaking, involves the understanding that if the converse of a function is also a function, the converse is called the inverse of f and designated f^{-1}.

The student at this point has concluded through a series of amazingly abstract, yet meaningful, exercises that for functions f and g, there may be obtained the composition function f·g. Exploration Exercises indicate that given functions g and h, it is sometimes possible to find a function f such that h = f·g. Under such a condition, h is defined to be functionally related to g, or, more briefly, h is a function of g and h depends only on g. Consideration of the necessary conditions for the existence of such functions culminates in a theorem which is indicative of the tenor of the entire topic:

> For each function h, for each function g, there is a function f such that h = f·g if and only if $D_k \subseteq D_g$, and, for all X_1 and X_2 in D_g such that $g(X_1) = g(X_2)$, if either X_1 or X_2 belongs to D_k then both belong to D_k and $h(X_1) = h(X_2)$.[1]

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[1] Ibid., p. 91.

Variable Quantities

As indicated by the title, this topic serves as an introduction to variable quantities and to a sophisticated function-based analysis of uses of and operations on the formula functions used to define area-measures and volume-measures. The student has already recognized that the range of a function may be any set but that some of the more useful functions are numerical-valued, i.e., their range consists of numbers. Such numerical-valued functions are referred to by UICSM as variable quantities. Many variable quantities are considered as functions whose ranges are sets of numbers of arithmetic and others whose ranges are sets of real numbers, e.g., the area-measure of a square.

Since the singulary operations, e.g., oppositing, squaring, absolute-valuing, and reciprocating, and the binary operations, e.g., addition and multiplication, can be applied to a number or a pair of numbers obtaining unique results, such operations are, in view of the development of functions, simply functions. The essence of the discussion lies in the notion that each operation on numbers can be used to induce an operation on functions, e.g., the operation oppositing introduced earlier has been a singulary operation defined on numbers but the analogous oppositing operation for functions is defined on functions. The term constant variable quantity, or constant, is introduced to describe variable quantities named by numerals. (The reader notes some apparent conflict

between the terms constant variable quantity and constant function. In view of the definitions presented, although a variable quantity is a function, a constant variable quantity may not be a constant function since a constant function has been defined earlier to have the set of real numbers as a domain.[1]) After this subtle introduction to variable quantities, UICSM indicates that variable quantities can be multiplied and added and that operations on real-valued variable quantities satisfy principles analogous to the basic principles for real numbers, and that formulas can be manipulated by rules similar to those used for manipulating numerical equations.

Some apparent ambiguity is noted. "So, whenever we have a formula in which both numerals and names of variable quantities occur, we shall interpret each numeral as standing for a variable quantity whose domain is that of the variable quantifier and whose range is the set consisting of the number named by the numeral."[2] It follows then that each numeral is not only a name for a number but also a name for any function whose range consists of this number. The authors attempt to justify the need for such distinction by demanding that if one considers the formula $C = \pi d$, $\underline{C}$ and $\underline{d}$ variable quantities, then $c/d = \pi$ makes sense only if π is a name for a variable quantity. (Actually the function approach utilized

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[1] Ibid., p. 83. [2] Ibid., p. 104.

here demands that $c = \pi d$ is a product-function such that for each circle X, $C(X) = \pi d(X) = \pi(X) \cdot d(X)$. But $\pi(X) = \pi$ since $\pi(X)$ is a constant function.) This connotation forces the reinterpretation of numerals in formulas as being names of variable quantities.

Linear Functions

This comparatively simple unit gives a somewhat standard treatment of linear functions with the major differences being the emphasis placed on the precision of the language involved and reliance upon the ordered-pair approach to functions. The brace-language for describing functions is utilized consistently as indicated by this definition: "f is a linear function of a real variable if and only if there are real numbers $a \neq 0$ and b such that $f = \{(x, y) : y = ax + b\}$."[1] The coefficient a is defined to be the slope of the function (free of geometrical significance) and b is demonstrated to be the intercept of f. Exploration exercises and textual discussion yield the result that if (x_1, y_1) and (x_2, y_2) are any two distinct pairs of a linear function, then $a = \frac{y_2 - y_1}{x_2 - x_1}$, a property sometimes used to define the slope of a function by reliance upon geometrical analogy. (One notes that under such a definition, the functions $g = \{(x, y) : 3x + 2 = 0\}$ and $h = \{(x, y) : y = 5\}$ are not linear functions although

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[1] Ibid., p. 120.

their graphs are straight lines, i.e., the UICSM definition of linear function demands that the graph of such a function be oblique to the axes.)

Applications of Linear Equations

By the applications of the machinery developed in the preceding topic, UICSM develops a fairly orthodox treatment of proportionality with one major difference again being the language involved. Two variable quantities M and N are <u>proportional</u> if and only if they have the same domain and there exists a non-zero constant <u>h</u> such that $M = hN$ and M is <u>inversely proportional</u> to N if and only if their domains are the same and there is a number $h \neq 0$ such that for each $e \in \mathcal{D}_N$, $M(e) \cdot N(e) = h$. Suitably precise definitions yield an approach to joint variation.

Quadratic Functions

This topic concerns itself primarily with the graphs of quadratic functions. The graph of a quadratic function is analyzed through the examination of many progressively complicated examples. The suggestive analysis utilized shows intuitively that one might determine the axis of symmetry, the vertex (called the <u>extreme point</u>), and the concavity of the graph by transforming the set selector $y = ax^2 + 6x + c$ by completion of the square to an equivalent $y = a(x - p)^2 + q$ with the extreme point being (p, q), the axis of symmetry being $\{(x, y) : x = p\}$, and having upward or downward

concavity according as a is positive or negative, respectively.

Quadratic Equations

The solution of quadratic equations written in the standard form $ax^2 + bx + c = 0$, $a \neq 0$, and, consequently, all quadratics which may be transformed to this as an equivalent form, is developed by the non-novel technique known traditionally as "completion of the square" and the resulting quadratic formula is formulated as follows: "$\forall_{a \neq 0} \forall_b \forall_c$ such that $b^2 - 4ac \geq 0$, $\{x : ax^2 + bx + c = 0\} = \left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right\}$.[1]

Systems of Equations

The determining of solution sets of systems of linear equations leans heavily on the graphical interpretation of those linear equations--an innovation particularly applicable in the analysis of consistency or non-consistency and dependence and independence of equations. To solve a system of two equations is to find the set of ordered pairs (x, y) which satisfy both equations of the system, i.e., the solutions of the system

$$\left.\begin{array}{l} ax + by + c = 0 \\ dx + ey + f = 0 \end{array}\right\}$$

involves finding the members of the set $\{(x, y) : ax + by +$

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[1] Ibid., p. 192.

$c = 0\} \cap \{(x, y) : ey + f = 0\}$ or, equivalently, to determine the set $\{(x, y) : ax + by + c = 0 \text{ and } dx + ey + f = 0\}$. The scheme introduced for obtaining such solutions is actually the traditional method of "addition and subtraction" although veiled in a more sophisticated garb. To determine the component $\underline{x}$, for example, choose real constants $\underline{m}$ and $\underline{n}$ such that the equation

$$m(ax + by + c) + n(dx + ey + f) = 0$$

is equivalent to the equation $a'x = b'$ which, in turn, is equivalent to $x = b'/a'$. Many applied problems whose solutions are defined by quadratic equations terminate the unit.

An Evaluation of Unit 5

This particular unit contains an extremely smooth extension of the development of the concepts of relations and functions through repeated use of the underlying basic concepts of sets and set operations. The materials involved lean quite heavily on the materials of the earlier units and assume at least a very adequate familiarity with sets and set operations as well as the basic principles of the real number system. Although linear equations are introduced and the development of complicated quadratic functions are pursued in detail, systems of quadratics are not treated in this unit. This unit follows quite naturally and with amazing continuity the materials of Units 1-4. Consequently, both teachers and

students would probably be seriously handicapped without previous experience with the prior units.

Set theory and the associated language of sets are used continually throughout this unit. Although some such preliminary work was done in earlier units, this unit sees a more formal presentation of set theory and even proofs of many of the theorems of union, intersection, and complementation. The basic structural concepts of this unit is the set and, particularly, sets of ordered pairs which serve as the vehicle for the definition and discussion of relations and functions thereby allowing modernistic discussions of many of the topics of traditional second-year algebra. Unit 5 seems dedicated to the stressing of a few major underlying ideas, e.g., set, relation, function, and inverse, and their applications to special situations. The unit consequently places a strong emphasis on the mathematical structure of the system involved.

The precise language introduced in earlier units is continued in this unit. It does appear, however, that some attempt has been made to use more of the nomenclature of traditional treatments of secondary mathematics with fewer unorthodox words being coined to communicate particular ideas. Such a trend is evidenced, for example, for the use of the numerical variable to replace the earlier-introduced pronumeral.

The UICSM belief that good mathematics requires precision of language for exactness and clarity is evidenced and demonstrated by the careful naming of sets, uses of well-stated definitions, and insistence upon the statement of the domain of definitions and theorems. Again, since the language is not all standard insofar as conventional treatments are concerned, a fairly thorough acquaintance with earlier units is demanded.

The early set-algebra proofs of the unit, based on the assumption of the principles of real numbers, are well-done and mathematically-sound for this level although they would require more mathematical sophistication and awareness of proof techniques than one would ordinarily expect of a student at this level. The text evidently is written on the implicit assumption that the student already knows how to construct fairly rigorous proofs. After the first few topics of the unit have been completed, the proofs are relatively unstructured, i.e., proofs of the formal sort are relatively rare although the student is expected to back up the answers to questions with reasons.

Unit 5, as were earlier units, is written in accordance with the belief that students may work certain well-structured and well-chosen problems and thereby discover important principles. However, in many instances in the unit, it appears that the student is forced to make generalizations from a minimum of problems and in many instances to confirm a

generalization already stated by the authors without prior intuitive examples. In several instances, however, the Exploration Exercises provide a challenging avenue of meaningful entry into the next topic to be studied.

This unit deals almost exclusively with the development of concepts rather than skills although there is an adequate supply of practice materials--57 of the 278 pages of the text form an appendix of supplementary exercises--and problems where skills need to be developed. Most of the exercises in the text proper are for the purpose of developing an understanding of the concepts.

This unit does not place a strong emphasis on social applications although many of the sections involve some discussion of non-mathematical areas for applications of the topics being studied. The supplementary exercises provide enough "thought" problems to provide necessary exercises for this type of problem.

Geometry, Unit 6

This unit in the study of geometry attempts to lead students to see geometry as a mathematical theory abstracted from physical experience and deductively organized and to help students gain the insight which will enable them to guess probable consequences of assumptions as well as to embrace an understanding of logic which will aid them in establishing that their guesses are properly consequences of their

assumptions. It is hoped by the authors that the traditional chasm between algebra and geometry will be recognized as non-realistic and thereby eliminated.[1]

Introduction

The introduction, which within itself could rightly be called a chapter, begins with an account of a well-planned journey to a planet Glox and fictionalizes certain communication problems which might arise. The example is carefully constructed to illustrate the deductive process of drawing conclusions in an orderly fashion from a given set of systematically-arranged observations. The identical problem is then completely restated with radically new environmental conditions, i.e., the situation containing the problem is completely revamped without changing the problem itself. This construction serves to intuitively establish that statement, analysis, and solution processes of a problem may be deducted from well-arranged data and that the process is isolated from the environment of the problem. In essence, these examples serve to establish that it does not matter what axioms talk about as long as the objects being discussed have the properties expressed by these axioms and theorems. It then follows directly that the discussion of any model requires the assumption of certain basic statements from which

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[1]University of Illinois Committee on School Mathematics, Geometry, Unit 6 (Urbana: University of Illinois Press, 1960), p. i.

conclusions may be drawn in accordance with certain orderly processes of implication.

The abstractions of point, line, plane, and straight are introduced with no effort whatsoever to formally define these concepts. A line is considered as a set of points and a plane a flat surface extending "forever" in all directions. No apologies or explanations are offered for acceptance of these terms yet the impression is created in the student's mind that these are the fundamental abstractions which he must universally accept in forthcoming discussions.

One notes also the early introduction and practically total reliance on the set notation--a vehicle for expression which until the past few years has been reserved for students on the upper plateaus of mathematical thought. This reliance on point-set notation for geometrical communication is an outstanding characteristic of this unit.

The concept of <u>betweenness</u> is intuitively developed by a series of illuminating examples and searching questions forcing the student to realize that betweenness is a notion pertaining to <u>collinear</u> points only. The development of betweenness allows the authors to develop notation clarifying ideas usually found troublesome.[1]

1. $\overleftrightarrow{PQ}$ represents the set of all points of the straight line containing the distinct points P and Q.

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[1] <u>Ibid</u>., pp. 9-17.

2. $\overrightarrow{PQ}$ represents the set of all points on the straight line PQ which are on the Q-side of P, i.e., all points z such that Q lies between P and z. This set is given the name of half-line.

3. $\overrightarrow{PQ}$ represents $\overrightarrow{PQ} \cup \{P\}$, or the half-line $\overrightarrow{PQ}$ unioned with its end-point. The set $\overrightarrow{PQ}$ is named ray and the point P the vertex of the ray.

4. $\overline{PQ}$ represents the set of all points between P and Q and is defined as an open interval, i.e., an open interval determined by points P and Q is the set of all points z such that z lies between the points P and Q.

5. $\overline{PQ}$ represents $\overline{PQ} \cup \{P, Q\}$ and is called a closed interval or segment, i.e., the segment $\overline{PQ}$ is a geometric figure containing the points P and Q and all points between P and Q.

These definitions and symbols allow traditionally troublesome definitions to be stated more precisely, e.g., two lines l_1 and l_2 may be defined to be parallel if and only if $l_1 \cap l_2 = \emptyset$.

Another bothersome idea is discussed fairly rigidly, although intuitively. The discussion of the separation of a plane into two half-planes by a straight line permits the student to verbalize explicitly such concepts as that indicating that two points C and D are on opposite sides of the line L if and only if $\overline{CD} \cap L = \emptyset$. This again illustrates UICSM's attention to details which are often neglected in traditional treatments of geometry.

The chapter contains fifteen Introduction Axioms stating some of the facts about points and lines and characterizing some of the concepts which students have discovered while studying the Introduction. These axioms (and the twenty accompanying Introduction Theorems which may be derived directly

from the axioms) are, for the most part, concerned with the properties of geometric sets which may be seen from geometric figures. The purposes of the Introduction evidently are to build up good intuition about problems concerning collinearity, order of points on a line, separation, betweenness, etc., to give practice in using set notation, and to enlarge the student's understanding of the nature of the proof of a theorem as being an argument starting from axioms, or previously proved theorems, and shows how the theorem in question follows, by accepted logical principles, from just these axioms and/or theorems.

Measures of Segments

In view of the facts that line segments are sets of points and that "=" always means <u>is</u> in the sense of <u>is the same as</u>, it is impossible to say that two line segments are equal unless they are exactly the same sets in which case the segments are coincidental. (This would also hold true for angles, rays, and, in fact, any geometric sets.) Therefore, the UICSM authors introduce early some of the aspects of measure theory in order to adequately cope with metric properties of geometric concepts.

Such statements as "$\overline{AB}$ = 6 inches"--an erroneous usage of terms which would have expressed traditionally that the line segment $\overline{AB}$ has length 6 inches--are more rigorously and precisely stated by the introduction of the <u>measure</u> of a

line segment. Appropriate notation to express the assignment of a real number to a segment is introduced, e.g., inch-measure $(\overbullet{AB}) = 6$, foot-measure $(\overbullet{AB}) = 1/2$, yard-measure $(\overbullet{AB}) = 1/6$, etc. In essence, the UICSM method utilizes the mapping process to assign measures by considering the set of all line segments as a domain, the set of all non-negative reals (which in an earlier unit has been shown to be isomorphic to the arithmetic numbers) as the range, and a function (or mapping) which associates with each element of the domain a unique element of the range. Call this function unit-measure. Under this terminology, the measurement of a line segment $\overbullet{AB}$ is simply the assigning of a non-negative real number (abbreviated by $m(\overbullet{AB})$ or simply by AB) to the segment by the unit-measure function. Measurement in different units is accomplished by the introduction of different mapping functions. It follows also that the distance between points A and B may be defined as $m(\overbullet{AB})$ or the measure of the segment $\overbullet{AB}$.

The authors also emphasize that for the majority of the work utilized in plane geometry, the unit of measurement itself is fairly unimportant but that the basic properties common to all measures are the ones of paramount importance, e.g., $m(\overbullet{AB}) + m(\overbullet{BC}) = m(\overbullet{AC})$ and $m(\overbullet{AB}) + m(\overbullet{BC}) > m(\overline{BC})$ if A, B, and C are collinear. The measures of segments are then utilized to prove rigorously (several by algebraic processes)

many of the various inequality theorems ordinarily introduced in conventional plane geometry textbooks.

One notices immediately in the development of the theorems the elimination (or a severe modification) of the general pattern of synthetic proof traditionally utilized. Instead of the standard arrangement of "figure, statement of hypothesis, statement of desired conclusion, argument, validating statements, and, finally, the conclusion," the UICSM program introduces the column proof (a modified form of the standard block proof) and also a paragraph proof which essentially follows the pattern of proof in logic. The point-set notation lends itself quite naturally to such an arrangement. These methods of arranging proofs are used interchangeably whenever the situation deems one of the methods more feasible than the other.

. Angles and Their Measures

Since the UICSM authors have developed rigorous language to define segments, rays, etc., their definition of an angle is straightforward: "An <u>angle</u> is the union of two noncollinear rays which have the same vertex. The rays are the <u>sides</u> of the angle and their common vertex is the <u>vertex</u> of the angle."[1] In other words, $\angle ABC$ represents the union of the sets of points $\overrightarrow{BA}$ and $\overrightarrow{BC}$, or $\overrightarrow{BA} \cup \overrightarrow{BC}$. (One notices that the use of the word <u>collinear</u> excludes the existence of a

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[1] <u>Ibid</u>., p. 51.

straight angle since, under this definition, a straight angle would be merely a straight line having no unique vertex, no unique interior, nor a unique bisector.)

The measure of an angle is introduced by considering the mapping of a domain of all angles into the range of the real numbers with the degree-measure function being the measure-function of primary concern. The degree-measure of $\angle ABC$ is denoted by $^{o}m(\angle ABC)$ or, simply, $m\angle ABC$. It is axiomated that the range of the degree-measure function will be the set of all real numbers x such that $0 < x < 180$. Supplementary angles are defined as pairs of angles whose measures total 180; complementary angles are those pairs whose measures total 90. An angle whose measure is equal to that of its supplement is defined to be a right angle. The interior of an angle $\angle ABC$ is defined to be the intersection of the set of all points on the C-side of $\overleftrightarrow{BA}$ with the set of all points on the A-side of $\overleftrightarrow{BC}$.

Traditional plane geometry texts have, without due regard to the variant or invariant properties of geometrical figures under translation and rotation, etc., considered two geometric figures to be congruent if they may be made to coincide. The UICSM unit states simply:

> Angles are said to be congruent if and only if they have the same measures. Segments, also, are said to be congruent if they have the same measure.[1]

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[1] Ibid., p. 58.

Another "unorthodox" definition is that of perpendicularity: $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ if and only if their union contains a right angle. It is seen that these and other precisely stated definitions are in keeping with the tenor of the unit.

Triangles

This particular chapter, as indicated by the title, deals with triangles, classification of triangles, and their various properties. In this chapter are developed the theorems dealing with the congruence of triangles--a concept which is a very basic foundation-stone in the traditional study of geometry.

A triangle is considered as the union of three line segments whose endpoints are three noncollinear points, i.e., $\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$. Each of the angles which contains two sides of the triangle is an angle of the triangle. (A consequence of this definition is that a triangle does not contain an angle since the sides of a triangle are merely proper subsets of the sides of the angle which are themselves rays.) Further examination leads to the consideration of the interior of a triangle as being the intersection of the interiors of the three angles of the triangle--a consideration which forces a distinction between a triangle and its interior.

UICSM demands that two triangles shall be considered congruent if and only if the vertices of one can be matched

with the vertices of the other in such a way that corresponding sides are congruent and corresponding angles are congruent. If one considers the triangles $\triangle ABC$ and $\triangle DEF$ and then examines the mapping $A \rightarrow D$, $B \rightarrow E$, $G \rightarrow F$ and determines that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$, $\angle CAB \cong \angle FDE$, $\angle CBA \cong \angle FED$, and $\angle ACB \cong \angle DFE$, then one says that a <u>congruence</u> exists between the two triangles. It follows that six such mappings of the vertices of one triangle to the vertices of another may exist whereas several, only one, or, possibly, none may establish a congruence. The corresponding-parts dilemma which accompanies the need for determining corresponding parts of two congruent triangles is partially solved by the use of the notation $ABC \leftrightarrow DEF$ to indicate the correspondence such that A matches D, B matches E, and C matches F. In the event that the correspondence indicates congruence, this notation directs that $\overline{AB}$ and $\overline{DE}$ are corresponding sides and that $\angle ABC \cong \angle DEF$, etc.

With this particular idea of congruence in mind, the authors pursue the question of the establishment of congruence and the conditions necessary to prove congruence. The familiar side-side-side, side-angle-side, and angle-side-angle theorems for congruence are proved in view of the definition of congruence.

Geometric Inequations

This particular chapter deals primarily with the various inequalities between measures of angles of triangles, measures of sides of triangles, interior and exterior angles, etc., present in geometric considerations. Axiom B stating that " $\forall x \forall y \forall z$ if $y \notin \overleftrightarrow{xz}$, then $xy + yz > xz$,"[1] is the basic tool for proofs of the theorem involving these inequalities. In most aspects, this UICSM chapter is fairly similar to those presenting the traditional treatment. Of course, the language is converted to that which is compatible with the definitions introduced and the set notation utilized throughout the book.

It is in this chapter that the authors deal with the medians, altitudes, and angle-bisectors of triangles and discuss the "distance of a point to a line." Further theorems of congruence are also developed.

Parallel Lines

In the conventional textbook of geometry, it is postulated that through any point exterior to a given line, there exists one and only one line parallel to the given line. This basic postulate of Euclidean geometry had survived without serious question or doubt until the advent of geometries such as those submitted by Reimann, Lobachevsky, and Bolyai who doubted the sacredness of this (and other) postulates of

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[1] Ibid., p. 32.

Euclid. It is of interest that the axioms present in the UICSM material are sufficient to prove, in the light of these axioms, that there exists such a line and that there exists at most one such line.

This chapter, other than its total reliance on set theory as the vehicle for proof with the utilization of the modern language introduced earlier, is essentially standard in its results. The standard theorems regarding parallel lines and transversals are developed along with several miscellaneous related topics. In the last few pages of the chapter students are given the opportunity through exercises to search out, and prŏve, theorems of their own devising.

Quadrilaterals

After the consideration of a polygon as being the union of segments such that each endpoint is an endpoint of just two segments with no two segments intersecting except at an endpoint, and such that no two segments with a common endpoint are collinear, this chapter begins a detailed study of the classes and properties of quadrilaterals. Various definitions are presented for the parallelogram, rhombus, rectangle, and square from enumeration of various classes of quadrilaterals. The UICSM materials make a definite effort to make the definition include as few of the properties of the class as is needed to make the definition clear and concise and develop

the remainder of the properties of each particular class as theorems.

One definition of this chapter seems out-of-character with the remainder of the chapter.

> . . . the polygon has been drawn in a flat wooded surface and that nails have been driven part way into the wood at the vertices. Think of a noose placed loosely around the nails. The polygon is convex if and only if the noose touches all the nails when it is tightened.[1]

This definition hardly seems to carry the precise notion that a polygon is convex if and only if, for each of its sides, all of its points not on this side belong to the same one of the two half-planes whose common edge contains the side in question.

Similar Polygons

After spending some several pages on the notion of sufficient and necessary conditions for statements or sentences, the UICSM authors pursue the topic which is ordinarily entitled similar polygons. One notes in this chapter that, while doing the work preparatory to the study of similar geometric figures, the UICSM textbook emphasizes that the real numbers u, v, x, and y ($v \neq 0$, $y \neq 0$) are "in proportion" if and only if $u/v = x/y$. This concept of proportionality extends, quite naturally, to the ratio of geometric line segments by the convention that two line segments are in a given

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[1] Ibid., p. 162.

ratio if the measures associated with these segments form that given ratio. Since a measure is a real number, the ideas of ratio and proportion for segments then becomes just application of the techniques of ratio and proportion for real numbers.

Two polygons are similar if there exists a matching of the vertices of the first polygon to the vertices of the second for which the corresponding angles are congruent and the corresponding sides are proportional, i.e., there may be demonstrated a similarity. From this definition, the standard theorems regarding similarity and properties of similar triangles are developed as well as theorems dealing with transversals, mean proportionals, and the Pythagorean Theorem.

Trigonometric Ratios

This chapter presents another radical departure from traditional geometry texts. In this chapter, three of the trigonometric ratios (sine, cosine, and tangent) usually developed near the end of a geometry text (or even reserved for a later course) are discussed in a coordinate-free manner by use of the ratios of measures of the sides of a right triangle. However, application of these ratios is restricted to the analysis and solutions of problems directly or indirectly involving right triangles. The functions secant, cosecant, and cotangent are never mentioned. Emphasis is placed on the difference between the trigonmetric function and the

trigonometric ratio in that sine of A indicates a particular ratio associated with angle A whereas sine is the mapping function relating the angle and its (opposite leg) / (hypotenuse) ratio. The domain of these three functions is limited to the non-zero angles which are acute thereby implying that trigonometric ratios for zero, obtuse, and negative angles are not to be considered at this point.

Rectangular Coordinate Systems

One of the more recurrent criticisms of plane geometry during the last century has been directed toward its total reliance upon the synthetic method of proof. It has been argued that, in many instances, reliance upon analytic procedures would be more desirable and yet just as meaningful and rigorous as reliance upon the synthetic proof. This challenge is met by the UICSM materials by the introduction of an excellently written chapter dealing with the fundamentals of rectangular coordinates, coordinate geometry, and analytic techniques made possible by this introduction. Reliance on set theory is again noted in that each point on a plane is considered as being matched with an ordered pair of real numbers called the coordinates of that point. The rather unique notation introduced involves letting $x(u)$ and $y(u)$ represent the abscissa and ordinate, respectively, of the point u and $d(\overline{AB})$ naming the distance between points A and B.

The distance formula (for determining the measure of the distance between two points) is developed by the use of the metric concept of assigning a measure to the line segment connecting any two arbitrary points on the plane. The midpoint formula is introduced and then generalized to the general division formula. Various theorems of plane geometry are then proved by the use of analytic tools and some introductory work is done with the concept of slope and the writing of various equations whose graphs are straight lines. Many of the exercises develop still a greater number of the common theorems of analytic geometry.

Circles

This particular chapter, with a few notable exceptions, develops the standard theorems dealing with circles and their properties. A circle, like any other geometric figure, is defined as a set of points. The circle with center C and radius r is considered as the set $\{P : d(\overline{CP}) = r\}$. The minor arc $\overset{\frown}{AB}$ of a circle with center C is defined to be the set having as elements the points A and B of the circle and all points of the circle in the interior of the angle $\angle ACB$. To name a major arc, UICSM authors innovate and write the names of the end points and, between them, write a name of another point of the arc, e.g., $\overset{\frown}{AB}$ is the name of a minor arc whereas $\overset{\frown}{AKB}$ is the name of a major arc. Two circles are

congruent if they have the same radius--the radius being the measure of a radial segment.

Another distinction is noted in that this chapter utilizes both synthetic and analytic proofs for development of theorems and relies quite heavily on the equation of the circle for determining points of intersection, the radical axis, etc. In summary, the chapter encompasses every theorem dealing with circles, tangents, inscribed angles, etc., and related properties found in traditional texts.

The standard method for defining the measure of the circumference of a circle has been to consider an inscribed regular polygon and allow the number of sides to increase indefinitely. As the number of sides increases, it is hopefully mentioned that the sum of the measures of the sides of the polygon will approach as a limit the circumference of the circle. The term limit is, of course, one of complexity and only an intuitive development is ever presented. UICSM appeals intuitively to the theory of sequences for a discussion in this sense: The perimeter of the regular polygons inscribed in the circle are represented as a monotonically-increasing sequence $P_3, P_4, P_5, P_6, \ldots, P_n, \ldots$, where n represents the number of sides, and P_n represents the perimeter of an inscribed polygon of n sides. There will exist upper bounds to the sequence. The least upper bound of the sequence is defined as the circumference of the circle. Also the chapter presents a few problems involving the techniques

of ruler-and-compass construction although a rigorous discussion is not presented.

Measures of Regions

The term triangular region is introduced by UICSM to name the union of the triangle and its interior. Extensions of this idea are made to care for the general polygon and the circle. This carefulness in naming a triangular region (and other regions) is introduced to insure that the student's attention is directed to the fact that the domain of the area-measure function consists of the regions rather than the triangles (or other simple closed curves) themselves.

The concept of least upper bound of a sequence is utilized to define the area-measure function for a circular region. The sequence $A_3, A_4, A_5, \ldots, A_n, \ldots,$ is considered with A_n being the area-measure of a regular inscribed polygon having n sides and the least upper bound of the sequence is defined as the area-measure of the circular region. The standard formulas for the measures of sectors, annular disks, etc., are then developed.

Appendix

Also included in Unit 6 as an Appendix is a very lucid presentation of the fundamentals of decision-making and the rules of reasoning. Various inference schemes are considered among which are substitution rules for equations, conditionalizing and discharging assumptions, hypothetical

syllogisms, contraposition, double denial, biconditionals, conjunctives, alternation, denying an alternative, test-pattern principles, principle of identity, and the law of excluded middle. Many excellent exercises which are designed for practice in logic and use of set theory notation are provided for the students' and instructor's use. The Appendix is provided for the purpose of review but if a student has not studied earlier UICSM units (or similar textbooks), it would be desirable that the student study the Appendix before beginning the textbook proper.

An Evaluation of Unit 6

Unit 6 of the UICSM program presents in what is optimistically designated a one-semester course a study of the geometrical topics common to a traditional high school course in plane geometry. This unit can be taught either before or after Unit 5 but would be practically incomprehensible to a student not having studied Units 1-4. The point of view of the authors apparently is that this unit should not be taught unless a proper background in proof has been laid and that geometry should be taught at a particular experience level rather than at a particular grade level. Furthermore, the primary emphasis of the unit is on proof rather than geometrical content and, in spite of the fact that traditional content is used, one gains the impression that the content is

more a vehicle for proof than an essential element of content within itself.

This unit strives to lead students to see geometry as a deductively-organized mathematical theory abstracted from physical experience as well as to aid them in the development of an insight which will enable them to logically ascertain the ramifications of assumptions as well as to attain a deeper understanding of that very logic which aids them in ascertaining these consequences. The unit never presents geometry in such a way that even hints at a distinction between the branches of mathematics. Implicit in the unit is the notion that the way to a better understanding and use of mathematics is through the conscious and continual study of the structure of the system. This approach to plane geometry is a point-set approach based largely on Hilbert's axioms.

The unit demonstrates continually a basic and fundamental belief in the paramount importance of precise and sophisticated language. All theorems are accompanied by quantifier statements indicating the domain of the theorem. Set theory and its language along with the associated symbols are commonly used. The authors do not hesitate to coin new words and symbols to communicate particularly troublesome ideas.

The unit assumes considerable experience in proof prior to the consideration of the unit. Some understanding of the methods of mathematical proof, e.g., the basic logical

principles governing its use, the use of variables and quantifiers, and the role of test-patterns as proofs of universal generalization as well as some formal consideration of conditional sentences, must have been arrived at earlier by the student. The unit, in its effort to be rigorous at all times in its proofs, admits that many of the proofs contain logic gaps which might not be apparent to the student. Consequently, the authors point out loopholes or continuity breaks in the structure or statements which are true but nonetheless cannot be proved at that point of presentation, i.e., the unit seems to advocate that unconscious usage of poor reasoning is much more abhorrable than admittedly poor reasoning. The burden of proof is placed on the student at an early time in the unit. The Introduction Axioms are formulated from desired figure-oriented properties which are then stated in a precise form and accepted as axioms for the structuring of the unit. The text uses the conventional synthetic geometric proof, the column and paragraph proof, and analytic proofs interchangeably.

The unit strongly emphasizes a sequence of activities from which the student independently recognizes, verbalizes, and communicates a desired knowledge. The exercises, particularly the Exploration Exercises, foster student discovery of concepts to be formally studied in later chapters and multitudes of examples are exhibited. The textual material is essentially a written-lecture type of presentation.

The unit places primary emphasis on the development of the geometrical concepts with great demand placed upon the development of manipulative skills using these concepts in writing mathematical proofs. There is absolutely no demand for development of skills in applying these concepts to any other situation.

Unit 6 presents no evidence of the belief of the authors that mathematics at this level need have social application. One notices no application to motivate the study of any of the topics and the few exercises of this type are found in the unit after the applicable theory has already been developed.

Mathematical Induction, Unit 7

Since really effective proofs of many of the generalizations regarding the integers (both negative and non-negative) require mathematical induction and recursive definitions, UICSM has developed Unit 7 to cater mainly to this need. Coupled with this achievement is that of providing for the student maximum opportunity to practice the proving of generalizations regarding the integers and inequality relations. This unit continues the spiral deductive organization of the student's inductive knowledge of the real number system which began in Unit 2. One gains the impression, by a preliminary inspection, that, in contrast with earlier units, the central exercise of Unit 7 is the proving of theorems.

The Real Numbers

This segment of UICSM Unit 7 begins what is a new look (a higher level in the spiral study) of the real number system by a reconsideration of the intuitively-true basic principles and non-rigorously proved theorems of earlier units. Units 1-4 (and particularly Unit 2) have stated the ten basic principles (the two commutative principles, the associative principles, the right and left distributive principles, the principles for $\underline{0}$ and $\underline{1}$, the principles for subtraction, and the principle for quotients), and have used these principles to prove in a non-rigorous fashion some 78 theorems regarding the system of reals.

In this considerably more rigorous step of the spiral treatment of the reals, the student is forced to realize that, since the above basic principles involve explicitly only the number $\underline{0}$ and $\underline{1}$, all the other numerals for whole numbers must be defined with an appropriate and advantageous scheme being to define $\underline{2}$ as an abbreviation for $\underline{1 + 1}$, 3 as an abbreviation for $\underline{2 + 1}$, . . . , 10 as an abbreviation for $\underline{9 + 1}$, etc.

By the utilization of this numeral abbreviation definition and by repeated use of the ten basic principles of the reals and selected ones of the 78 theorems of Unit 2, the student is led to justify, in a step-by-step manner which intuitively suggests the desirability of the inductive method, the traditional algorithms for determining the computing facts for addition and multiplication as well as the

associated and analogous algorithms for simplifying certain types of algebraic expressions. The analysis of the Division-with-Remainder Algorithm and the techniques for determining division computing facts as presented in earlier units reveals that these techniques have implicitly utilized the theorem "$\forall_x$ $x/1 = x$ and $\forall_x$ $x/-1 = -x$," and that implicit in this theorem is the assumption that $1 \neq 0$ and $-1 \neq 0$--two inequations which cannot be derived from the ten basic principles! Accordingly, the basic principle "$1 \neq 0$ and $-1 \neq 0$" is added to the list of basic principles in order to axiomate the desired property for the system and to prove the associated theorem stating that "$\forall_x$ if $x \neq 0$ then $-x \neq 0$."[1]

The Positive Numbers

The earlier section of this unit has concluded that the eleven now-present basic principles are not sufficient to prove statements such as "$2 \neq 0$." In order to approach more adequately the axiomatic study of the reals, UICSM authors introduce four basic principles which express the relationship between the oppositing operation and the property of being positive and which adequately distinguish between the set P of positive reals and the set N of negative reals.

(P_1) $\forall_x$ $[x \neq 0 \Rightarrow$ either $x \in P$ or $-x \in P]$.

(P_2) $\forall_x$ not both $x \in P$ and $-x \in P$.

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[1]University of Illinois Committee on School Mathematics, Mathematical Induction, Unit 7 (Urbana: University of Illinois Press, 1961), pp. 16-20.

(P_3) $\forall_x \forall_y$ [(x $\in$ P and y $\in$ P) $\Rightarrow$ x + y $\in$ P].

(P_4) $\forall_x \forall_y$ [(x $\in$ P and y $\in$ P) $\Rightarrow$ xy $\in$ P].[1]

These four principles combined with earlier principles, supply sufficient axioms for the establishment as a theorem of what is traditionally axiomized as the trichotomy law by most modern algebraists. The exercises for this section seem to be particularly strong in their demand for good deductive reasoning and provide an opportunity for various proof techniques.

Inequations

The greater than relation " > " as studied in Unit 2 was defined such that, for two real numbers a and b, a > b if and only if a - b is positive. This definition is now introduced as a basic principle:

(G) $\forall_x \forall_y$ [y > x $\Leftrightarrow$ y - x $\in$ P].[2]

This principle, along with the principles (P_1), (P_2), (P_3), and (P_4), are used to establish various theorems regarding inequalities and, particularly, the addition, multiplication, and factoring transformation principles for inequalities.

The Positive Integers

UICSM Unit IV contained many proofs regarding properties of the set of positive integers $I^+ = \{1, 2, 3, 4, \ldots\}$. These proofs implicitly accepted the following assumptions

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[1] Ibid., p. 74. [2] Ibid., p. 30.

although the students probably were not disturbed by accepting such:

(1) The positive integers belong to P, that is $I^+ \subseteq P$.
(2) I^+ is closed with respect to addition and multiplication.
(3) For all positive integers m and n, if $n > m$ then $n - m$ is a positive integer.
(4) 1 is the only positive integer between 0 and 2.
(5) Each non-empty set of positive integers has a least member.
(6) For each real number x, there is an integer k such that $k \leq x < k + 1$.
(7) Each positive integer other than 1 has just one prime factorization.[1]

Since these have not been included as basic principles (nor can they be deducted from the sixteen basic principles of this unit) the UICSM authors lead an exploratory search for certain basic principles which are simpler in nature and more homogeneous in content which would be adequate for proving any desired theorem regarding the positive integers. After several pages of examination, the following basic principles are formulated:

(I_1^+) $1 \in I^+$;

(I_2^+) $\forall_n\ n + 1 \in I^+$;

(I_3^+) $\forall_S [(1 \in S \text{ and } \forall_n [n \in S \Rightarrow n + 1 \in S]) \Rightarrow \forall_n\ n \in S]$.[2]

(Close examination of these basic principles in view of the context of the materials reveals that these basic principles constitute essentially a rewording of Peano's Axioms for the

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[1] Ibid., p. 47. [2] Ibid., p. 49.

natural numbers (or positive integers). Using only these axioms, it is possible to define addition and multiplication in the set I^+ and to prove that I^+ has all the properties of an integral domain except that it does not have a zero and its elements do not have additive inverses. It is noted also that I_3^+ says precisely that $I^+ \subseteq K$ where $K = \{1, 1 + 1, 1 + 1 + 1, 1 + 1 + 1 + 1, \ldots\}$, i.e., K is the set of all real numbers which can be proved to exist by using I_1^+ and I_2^+. Examination shows that $I^+ = K$ and, therefore, every positive integer can be postulated from (I_1^+) and (I_2^+).)

After having studied these principles for the set I^+, the unit introduces the standard three-part proof by <u>mathematical induction</u> (involving the <u>initial step</u> establishing F(1), the <u>induction step</u> establishing that $F(n) \Rightarrow F(n + 1)$, and the <u>final step</u> establishing $\forall_n F(n)$ of a function F and uses this technique to prove closure for addition and multiplication for I^+ as well as to examine numerous recurrence relations.

The Relation Greater Than for the Positive Integers

In the closer examination of the <u>greater than</u> relation for the positive integers, detailed attention is paid to the existence (or non-existence) of lower bounds and greatest lower bounds for various sets of numbers with the major product of the study being that a non-empty set of positive reals may have lower bounds but no greatest lower bound whereas

every non-empty set of positive integers has a least member. Motivation for such a study is furnished by the proof indicating that there does not exist a rational number whose square is 8 and the accompanying realization that nothing can yet be proved regarding the existence of a positive real number whose square is 8.

The reader may note that the list of basic principles heretofore introduced is complete enough to describe adequately the set of positive integers but they do not, in any manner, describe the distribution of those integers among the reals. Consequently, the authors introduce the next-to-last basic principle of the unit, the Cofinality Principle: "(C) $\forall_x \exists_n n > x$, n a positive integer, x a positive real number."[1] (This principle is easily seen to be a rather special case of the Archimedean Property: "If A and B are positive real numbers, there exists a positive integer n such that $nA > B$."[2] Had UICSM chosen to have introduced the completeness principle which states that each non-empty set of reals having an upper bound has a least upper bound, the cofinality principle could have been proved as a theorem.)

The Integers

The student's earlier knowledge of the integers as a particular subset of the reals is summarized in the last

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[1] Ibid., p. 89.

[2] Neal H. McCoy, Introduction to Modern Algebra (Boston: Allyn and Bacon, Inc., 1960), p. 102.

basic principle of the unit: "(I) $\forall_x [x \in I \Leftrightarrow (x \in I^+$ or $x = 0$ or $-x \in I^+)$,"[1] i.e., an integer is a real number which is a positive integer or zero or whose opposite is a positive integer. Various theorems are presented describing the properties of the set as well as a study of the greatest integer function and the fractional-part function.

The transitive, reflexive, and non-symmetric properties of the divisibility relation are recalled and the analogies between the divisibility relation and the less-than-or-equal-to relation are examined and arrayed. Supplemental exercises develop the notion of the highest common factor. Accompanying the development of the Euclidean Algorithm for the computations of the highest common factor are associated methods of solving simple linear Diophantine equations.

An Evaluation of Unit 7

This unit is a considerably more sohpisticated step than Units 1-4 in the spiral approach to the study of the real number system and devotes itself to a deductive organization of the student's somewhat intuitive existing knowledge of the reals with the formulation of a set of axioms (called basic principles) which serve to characterize the structure of that system. The major goal of this unit is the promotion of the student's practicing in the proving of generalizations concerning the positive reals, the positive integers, and the

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[1] UICSM, Unit 7, p. 94.

integers along with the inequality relations. These proofs rely heavily upon the process of induction and, consequently, the inductive process and recursive relations are prevalent. The UICSM believes, evidently, that the mathematical content of this unit is not academically beyond the secondary student with an adequate background in Units 1-6 although the textbook reads as though it were a traditional treatment of introductory modern algebra. Certainly it would be futile for a student to attempt this unit without having studied earlier units. Since the entire unit is devoted to the axiomatic development of the real number system, it is evident that UICSM is dedicated to the study of mathematical structures as a basis for a better understanding and appreciation of mathematics.

Both the student's textbook and the teacher's guides are consistent in their use of precise language and the symbols $\forall$, $\exists$, $\Rightarrow$, $\Leftrightarrow$, etc., are used whenever applicable. Apparently UICSM advocates that precise language and mathematical symbolism are desirable at this level as the vehicle for communicating mathematical ideas.

The proofs involved in this unit are sufficiently rigorous and varied in nature with comparable proofs never seen in traditional curriculum sequences short of a modern algebra course. The unit utilizes the methods of proof-by-contradiction, paragraph proofs, column proofs, and inductive proofs and demands that the student construct a multitude of

such proofs to establish some 129 theorems of the real numbers by the use of twenty-one basic principles. UICSM evidently assumes that earlier units have been successfully comprehended by the student since no effort is made at this point to elaborate (other than by examples) upon the nature of a good proof.

This unit, as does all earlier units, includes an abundance of Exploration Exercises designed to motivate the student to formulate independently the ideas of the following topic. The textual material frames many questions which demand the student's consideration for his full understanding of the materials. One gains the impression that the reading of the textual material demands as much concentration on the part of the student as does the actual solution of the exercises.

Since the primary and principal activity of this unit is the proving of theorems, the exercises are largely theoretical in nature. The exercises are well-structured to develop both the concepts and their dependent manipulative skills. Quite often, the problems involve extension of the theory and demand appreciation prior to the next topic.

UICSM makes no effort to motivate the study of the concepts and basic principles of this unit by the use of social applications. Evidently UICSM considers the understanding of the concepts as a more desirable objective than their potential social applications.

Sequences, Unit 8

This unit includes a study of the usual material on arithmetic progressions (although as a special case of more general sequences), integral exponents, geometric progressions (though structured as depending on exponentiation sequences), and the process of assigning sums to infinite geometric progressions, as well as concentrated study of combinatorial processes. This material, though certainly not new to a secondary program, is far from traditional in its approach, depth, and mathematical rigor.

Continued Sums

This unit takes as its fundamental working definition that of a sequence a as "a function whose domain is I^+, the set of positive integers."[1] The sigma-notation $\sum_{i=1}^{n} a_i$ is introduced immediately and used to indicate the sum of the first n terms of a sequence. An auxiliary sequence called the continued sums sequence for a given sequence a is defined recursively by

$$\sum_{p=1}^{1} a_p = a_1 \,, \quad \forall_n \sum_{p=1}^{n+1} a_p = \sum_{p=1}^{n} a_p + a_{n+1} \,.$$[2]

This continued-sums sequence whose nth term is the sum of the first n terms of the sequence a, is used to compute by

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[1] University of Illinois Committee on School Mathematics, Sequences, Unit 8 (Urbana: University of Illinois Press, 1961), p. 4.

[2] Ibid., p. 9.

induction the sums of the first n terms of various sequences.

After the student has become familiar with the concept of sequence, the domain of the function is extended to $I^+ \cup \{0\}$, the set of non-negative integers, and even, on occasion, the set I of integers, and a more general definition is extended for the sigma-notation.

> For each $j \in I$ and each function a whose domain includes $\{k : k \geq j\}$
>
> $$\sum_{i=j}^{j-1} a_i = 0,$$
>
> $$\forall_{k \geq j-1} \sum_{i=j}^{k+1} a_i = \sum_{i=j}^{k} a_i + a_{k+1}.$$ [1]

With this basic definition, the left distributive principle for continued sums, the sum-rearrangement theorem, the associative transformation principle for continued sums, and the translation transformation principle for continued sums are formulated in order to authorize computational techniques.

UICSM introduces a rather novel approach to the study of arithmetic progressions through its definition of a difference sequence Δa having terms $(\Delta a)_i$ which is formed from a sequence a having terms a_i by the formula $(\Delta a)_p = a_{p+1} - a_p$ to determine the pth term of the difference sequence. It then follows that a sequence whose first difference sequence is a constant sequence is an arithmetic progression with Δa being the common difference. The treatment culminates in the

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[1] Ibid., p. 36.

proof of the regular formulas for the nth term and the sum of the first n terms of an arithmetic progression.

An overview of even the first topic of this unit suggests that a student must rely heavily upon the inductive process developed in Unit 7. This cursory examination also yields the suggestion that the exercises, in most instances, are considerably less sophisticated than the textual discussions.

Continued Products

The continued products sequence of a given sequence a having terms a_i is defined in a way analogous to the continued sums sequence of the earlier topic.

> For each $j \in I$ and each function a whose domain includes $\{k : k \geq j\}$,
>
> $$\prod_{i=j}^{j-1} a_i = 1,$$
>
> $$\forall_{k \geq j-1} \prod_{i=j}^{k+1} a_i = \prod_{i=j}^{k} a_i \cdot a_{k+1}.$$ [1]

(The Greek letter Π is used to denote continued products just as Σ was used to indicate continued sums.) Although this definition of the continued products sequence seems unwieldy, cumbersome and somewhat sterile, its immediate applications are many, e.g., k! may be defined by $\forall_{k \geq 0} k! = \prod_{p=1}^{k} p$ which in turn implies that 0! = 1 and the exponential sequence may be defined by $\forall_x \forall_{k \geq 0} x^k = \prod_{p=1}^{k} x$ which adequately defines

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[1] Ibid., p. 94.

a non-negative integral power of a base x and in turn implies that $\forall_x x^0 = 1$. UICSM leads an exploratory search for theorems for continued products sequences analogous to those proved earlier for continued sums sequence.

UICSM supplements the above mentioned definition for the exponential sequence by defining $\forall x \neq 0 \; \forall k < 0 \; x^k = 1/x^{-k}$ in order that exponential expressions involving negative integral exponents may be manipulated by the inductively-proved theorems $\forall x \neq 0 \; \forall_j \; \forall_k \; x^j \; x^k = x^{j+k}$, $(xy)^k = x^k \; y^k$, $(x/y)^k = x^k/y^k$ $(x/y)^{-k} = (y/x)^k$.[1] UICSM again provides a vast proliferation of exercises to develop manipulative facility of these definitions and theorems.

The language of sequences enables UICSM to define recursively a geometric progression as a sequence a such that, for some x, $a_1 \neq 0$, $\forall_n \; a_{n+1} = a_n r$. (The student is expected to prove inductively, in original exercises, the standard formulas for the nth term and the sum of the first n terms of a geometric progression.) The sum of an infinite geometric progression whose common ratio r such that $|r| < 1$ is determined by the traditional type of argument except that a fairly rigorous definition of the concept of a limit is implicit in this definition: "In general, for any sequence a, and any number s, $\sum_{p=1}^{\infty} a_p = s$ if and only if $\forall_{y>0} \exists_m \forall_{n \geq m} |s - \sum_{p=1}^{n} a_p| < y$."[2]

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[1] Ibid., p. 114. [2] Ibid., p. 143.

The function C(j, k), $j \geq 0$, $k \geq 0$, used to determine the number of k-membered subsets of a j-membered set, may be defined recursively by

$$\forall_{j \geq 0} \; C(j,0) = 1,$$

$$\forall_{j \geq 0} \forall_{K \geq 0} C(j,K+1) = C(j,K) \frac{j-K}{K+1}.$$[1]

This basic definition authorizes the solution of problems traditional to the study of combinations. Similarly, the derived theorem (as well as supporting theorems) stating that "$\forall_{j \geq 0} \quad \forall_{k \geq 0} \quad P(j, k) = \prod_{i=0}^{k-1} (j - i)$," and used to number the permutations of j things taken k at a time is used to solve traditional permutation problems.

The text proper of this unit culminates with an inductive proof of the binomial theorem: $\forall_x \forall_y \forall_{j \geq 0}$ $(x + y)^j = \sum_{k=0}^{j} C(j, k) x^{j-k} y^k$. This topic, although somewhat unique in its presentation, is fairly traditional in approach and certainly provides sufficient practice for students pursuing the unit.

Throughout this unit are certain enrichment supplements which, though not vital to the continuity of the text, certainly contribute to its effectiveness. Among these are the consideration of the Fibonacci sequence, sum of deviations of the terms of a sequence from a given number, base-m representations of positive integers, recursive relations for finding sums of powers, and prime and composite integers.

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[1] Ibid., p. 168.

An Evaluation of Unit 8

As in earlier units, UICSM does not specify that Unit 8 be taught at any particular grade level but rather at a particular mathematical experience level. This unit, devoted largely to sequences and their uses in examining many topics such as exponential functions, arithmetic and geometric progressions, factorial functions, combinations and permutations, and the binomial theorem, definitely presupposes an adequate understanding of earlier units, particularly Units 6 and 7. UICSM's emphasis on the continued development of the real number system as a mathematical structure forces one to conclude that the understanding of the mathematical structure of the reals is one (if not the) prime objective.

UICSM Unit 8 formulates several new terms, e.g., continued sum sequence, difference sequence, and continued products sequence, to communicate certain applicable notions. The unit makes extensive use of the sigma-notation $\sum_{i=1}^{n}$, the product-notation $\prod_{i=1}^{n}$, and the now familiar $\Rightarrow$, $\Leftrightarrow$, $\exists$, and $\forall$. The unit also relies quite heavily upon recursive definitions to define particular concepts involving an iterating process.

The rigorous proofs, predominantly inductive in nature, are the prime characteristics of this unit. The extension of the ideas of the reals to develop sequences indicates UICSM's concern with the importance of critical examination of good proofs. UICSM definitely assumes that the appreciation of a

good proof is already a working part of the student's mathematical background.

The method of presentation of this unit is not appreciably different from that of Unit 7. Although the student is guided along possible avenues of proof, the burden of the proof must be assumed by the student. Again the primary purpose of the unit seems to be to train the student in the process of good mathematical proof and to provide adequate practice in this endeavor. This unit, as does earlier units, provides a wealth of Exploration Exercises to lead the student to discover ideas independently.

Since one of the prime objectives of this unit seems to be the acquisition of skills in writing proofs, this unit, through its multitude of exercises, certainly provides adequate opportunity for the achievement of these desired skills. As in Units 6 and 7, UICSM apparently does not subscribe to the notion that social application is to be a major objective at this experience level. This unit does, however, state many of the exercises in a physical-environment language.

Elementary Functions, Powers, Exponentials, and Logarithms, Unit 9

This unit, initially designed to be the preliminary unit of study for the twelfth-grade, concerns itself with a study of principal roots, rational numbers and rational exponents, exponential functions and their inverses, the logarithm functions, and completes the list of postulates (or

principles) necessary for the defining of the real number system. The unit contains approximately 190 pages of textual material and exercises and nearly that many additional pages devoted to Appendices which contain, among other topics, UICSM's only consideration of solid geometry. The unit is in harmony with earlier units and is written in accordance with the philosophy of the UICSM in that it stresses discovery of generalizations by the students rather than presenting generalizations and then explaining them.

Definite Description

The basic definition for this topic is that of the square root: "$\forall_{x \geq 0}\ \sqrt{x}$ = the real number z such that ($z \geq 0$ and $z^2 = x$.)."[1] This definition demands the consideration of both an existence condition ($\forall_{x \geq 0}\ \forall_z$ ($z \geq 0$ and $z^2 = x$) and a uniqueness condition ($\forall_{x \geq 0}\ \forall_x\ \forall_y$ (($y \geq 0$ and $y^2 = x$) and ($z \geq 0$ and $z^2 = x$) $\Rightarrow y = z$). UICSM immediately suggests (and proves later) that the existence condition cannot be considered a consequence of the already adopted basic principles although the uniqueness condition may be thus established. In order to establish the existence condition one needs the completeness principle of the reals and UICSM forces the students to see the need for such a basic principle.

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[1] University of Illinois Committee on School Mathematics, High School Mathematics, Unit 9 (Urbana: University of Illinois Press, 1962), p. 3.

The Need for a New Basic Principle

UICSM proves that the presently-existing basic principles for the reals are insufficient to establish an existence condition for the reals by showing that all of these basic principles still hold if they are restricted to describing the rationals but that the statement "$\forall_{x \geq 0} \exists_z (z \geq 0$ and $z^2 = x)$," becomes false when restricted to the rationals. Therefore the system of basic principles as already formulated is not sufficient to establish existence of the square roots of all non-negative reals.

In preparation for the Completeness Principle, the student is led, through Exploration Exercises, to discover that the set $S = \{ x \geq 0 : x^2 < 2 \}$ has no greatest member but that $\underline{b}$ being a non-negative number whose square is 2 assures one that $\underline{b}$ is an upper bound of S, that $\forall_{x \geq 0} [x < b \Rightarrow x \in S]$, that no number less than $\underline{b}$ is an upper bound of S and that, additionally, $\underline{b}$ is the least upper bound of S. These are used as tools for proving that the least upper bound $\underline{b}$ of S is such that $b^2 = 2$, i.e., $b = \sqrt{2}$.

The Least Upper Bound Principle

UICSM attempts to establish that, generally, "$\forall_{y > 0} \exists_{z \geq 0} z^2 = y$" and "$\forall_{y > 0} \{x \geq 0 : x^2 < y\}$ has a least upper bound" are equivalent statements.[1] Therefore, the assumption of the completeness principle (called by UICSM the

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[1] Ibid., p. 31.

least upper bound principle), "Every nonempty set which has an upper bound has a least upper bound," justifies the work with square roots.

Exploration Exercises concerned with the domains and ranges of various functions as well as with monotonic decreasing and monotonic increasing functions and their inverses motivate the following:

> A function f is continuous at x if and only if $x \in \mathcal{D}_f$ and f(x) differs arbitrarily little from $f(x_0)$ for each $x \in \mathcal{D}_f$ which is sufficiently close to x_0. A function is continuous if and only if it is continuous at each of its arguments.[1]

One notes that arbitrarily little and sufficiently close are undefined although the standard and rigorous definition for continuity is considered in the accompanying appendix.

It is interesting to note that UICSM introduces and uses at this point some theorems, e.g., "Each monotonic function has a monotonic inverse of the same type," and "Each positive integral power function is continuous," which are accepted without proof (except in the Appendix)--a highly unusual occurrence for this group.

Principal Roots

UICSM's study of roots other than square roots is predicated upon the so-called (PR) relation defining the "$\sqrt[n]{\ }$" operator: $\forall_n \forall_{x \geq 0}$ $(\sqrt{x} \geq 0$ and $(\sqrt[n]{x})^n = x)$."[2] This definition of this operator, along

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[1] Ibid., p. 42. [2] Ibid., p. 49.

with earlier-considered theorems, yields a uniqueness-theorem: $\forall_n \forall_{x \geq 0} \forall_y [(y \geq 0$ and $y^n = x) \Rightarrow y = \sqrt[n]{x}]$. This (PR) relation and the uniqueness theorem provide the necessary tools for manipulation of radical expressions.

The Rational Numbers

The rationality or irrationality of a real number is determined by tests structured from a rather unique definition (or basic principle) of a rational number: $\forall_x [x \in R \Leftrightarrow \exists_n xn \in I]$.[1] (R is the set of rational numbers; $n \in I$). The denseness of the rationals in the reals as well as the denseness of the reals is considered.

Rational Exponents

Earlier discussions of this unit have defined for each $\underline{x}$ the exponential sequence with base $\underline{x}$ as a sequence whose domain is the set of nonnegative integers. A later definition extended the domain of sequence functions to ones whose domain is the set of all integers. This topic deals with the extension of these exponential sequences to ones having rational numbers. The method of attack is simple and nonnovel--the student is forced to assign a meaning to such sequences in a manner so that the laws which held for integral exponents will still hold. The result of such examination is stated in a rather nontraditional, i.e., insofar as

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[1] Ibid., p. 62.

wording is concerned, defining principle: $\forall_{x>0} \forall_r \forall_{m,\ rm \in I} x^r = (\sqrt[m]{x})^{rm}$.[1] The student is then led to prove those theorems for rational exponents which are analogous to those for integral exponents.

The Exponential Functions

At this point, the student will have become familiar with many facts regarding sequences with rational arguments. These facts encompass the following central ideas. First, for $0 < a \neq 1$, the exponential function having base a and rational arguments is a continuous monotonic function which is decreasing if $0 < a < 1$ and increasing if $a > 1$. Second, for $0 < a < 1$, given $M > 0$, there is an N such that, for each $r > N$, $a^{-r} > M$ and $0 < a^r < \frac{1}{M}$. Third, for $a > 1$, given $M > 0$, there is an N such that, for each $r > N$, $a^r > M$ and $0 < a^{-r} < \frac{1}{M}$. These facts are used to structure the following adoption:

(a) $\forall_{x>1} \forall_u x^u$ = the least upper bound of $\{y : \exists_{r<u}\ y = x^r\}$,
(b) $\forall_{0<x<1} \forall_u x^u = (1/x)^{-u}$,
(c) $\forall_u 1^u = 1$ and $\forall_{u>0} 0^u = 0$.[2]

This adoption (or definition) essentially removes the "$u \in R$" restriction on the definition of an exponential function with base x and rational argument u, i.e., $\{(u, y), u \in R : y = x^u$. UICSM is now confronted, by this definition with these three

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[1] Ibid., p. 92. [2] Ibid.

problems if rigorous consistency is to be maintained. First, the existence of the least upper bound of the set in (a) must be shown. Second, these new definitions must be shown to be consistent with the old. Third, the theorems proved by utilizing earlier definitions must hold in view of these definitions. The last two of these three tasks is reserved for the Appendix although the textbook adequately develops the first.

The Logarithm Functions

This topic presents a rather traditional, except for language, introduction to the theory of logarithms. It is shown, by consideration of many examples, how one might use the inverse of an exponential function to perform multiplication, division, and exponentiation computations by addition, subtraction, and multiplication, respectively. The values of the "powers of a" are approximated by reference to a well-constructed graph of $\{(x, y) : a^y = x\}$. Linear interpolation techniques are also introduced to further the process of approximation of powers--authorized by the fact that the inverse function, as well as the exponentiation function, is continuous.

The UICSM approach to the formal study of logarithms is based on the notion of an inverse exponential function, i.e., the exponential function with base a is $\{(x, y) : y = a^x\}$ and the logarithm function with base a is $\{(x, y) : x = a^y\}$. As a tool for proving the theorems regarding logarithms, UICSM

proposes a defining principle (L): $\forall_{0 < a < 1} \forall_{x > 0}$ $a^{\log_a x} = x$. It is concluded, after some examination, that the domain of each logarithmic function is the set of positive numbers and its range is the set of reals and that each such function is continuous and monotonic increasing if the arbitrarily-chosen base is greater than 1 and monotonic decreasing if the arbitrarily-chosen base is positive and less than 1.

By using the above principle (L), the student is forced to construct proofs for the computational theorems of logarithms. Most of the computation is done by use of the common-logarithm-function (to the base 10). The exercises of the topic involve a variety of general proofs as well as an ample supply of exercises involving logarithmic computation and the solution of exponential equations, etc.

Some Laws of Nature

This presentation, somewhat out of character with the earlier UICSM units, illustrates ways by which mathematics might help discover physical laws by the abstraction of empirical data into mathematical functions. Boyle's and Gay-Lussac's laws are shown to describe adequately the functional relation between the empirically-obtained readings for temperature, pressure, and volume of various types of gasses.

In a similar vein, the consideration of the decay of radioactive substances demands that one understand the

behavior of the expression $(1 + x)^{\frac{1}{x}}$ as x tends to 0. This limit, actually the least upper bound of the set $\{y: \exists_{x > 0} y = (1 + x)^{1/x}\}$ which, of course, is 2.718281829459..., is defined to be the ubiquitous number e. The examination of Newton's law of cooling, the study of transient currents in simple circuits, the examination of the adiabetic compression of gasses and even compound interest all illustrate the importance of this number e. The vast realm of physical application also indicate the relative convenience of e as a base for logarithms rather than the common base 10.

Appendices

The Appendices (comprising a total of 146 pages of material as compared to 189 pages of textual material in the unit) is actually a unit of study within itself. With one exception, the five Appendices concern themselves with rigorous proofs of theorems used earlier without proof in the unit. Included also are more sophisticated statements of the definitions involved, e.g., continuity.

Appendix A deals more completely with principal roots and, after a rather sophisticated discussion of the concepts of monotonicity, increasing and decreasing functions, continuity, etc., establishes that each monotonic function has a monotonic inverse of the same type and that each continuous monotonic function f whose domain is a segment $\overline{a, b}$ has a continuous monotonic inverse of the same type whose domain is

the segment $\overline{f(a), f(b)}$. These two results have been used in textual discussions without proof.

Appendix B places its main emphasis on the study of the irrational numbers. The one-to-one correspondence between two sets traditionally used to define the relation "having the same number of elements as" and "having more elements than" is utilized to approach the properties of infinite and countably infinite. UICSM demonstrates that, subject to these definitions, the set of rationals is countably infinite but that the set of irrationals (as well as the set of reals) is not countably infinite.

Appendix D contains a very brief (actually only 42 pages) synopsis of what usually is referred to as solid geometry and devotes itself primarily to an abbreviated development of the surface area and volume formulas for simple solids. The approach to solid geometry is a point-set one and utilizes a language compatible with Unit 6. Cavalieri's Principle is axiomized and used quite extensively to develop the basic formulas regarding volume-measure of solids. This very brief development presents, in capsule form, the standard mensuration formulas for triangular regions, quadrilateral regions, regular polygonal regions, circular regions, prismatic solids, pyramidal solids, prismatoidal solids, and spherical solids. The outstanding characteristic of the presentation is its total devotion to precise point-set vocabulary. The exercises

presented are largely computational in nature and present a call for very few formal proofs.

An Evaluation of Unit 9

This unit has led the student through the completion of the set of basic principles for the real numbers by the statement of the least upper bound principle, has brought into prominence more of the principles of logic, has discussed and applied the properties of monotonicity and continuity of functions, has defined and explored power functions, exponential functions, and logarithmic functions, and has brought to the student's attention the realization that functions of these kinds occur, in a sense, in nature. The Appendix, if pursued, has exhibited more of the properties of rational and irrational numbers, has talked about mensuration formulas, and the use of logarithms in computational work. The study of this unit, if to be in the least comprehensible to the students, must have been preceded by the past several units of the sequence in that the type of presentation, the language, the structure of the material, the technique of proofs, etc., are assumed from previous units.

UICSM has, in context with the earlier units, continued to insist upon the exhibition of the structure of the real number system. The inclusion in this unit of the least upper bound principle (or completeness principle) completes the axiomization of the real number system and emphasizes the

reliance of the computation and simplification algorithms upon this list of basic principles and the derivable theorems.

The vocabulary, as in all the previous units, is one of precision and rigor. No attempt is made to avoid sophisticated terminology and the unit fairly bristles with terms such as least upper bound, monotonic, decreasing function, continuous, infinite, uncountability, and operators. The unit also employs extensive symbolism appropriate to the discussions. All statements and theorems are accompanied by the appropriate quantifiers stated in symbolic forms.

This unit assumes much experience in proofs of all types, e.g., inductive and reductio ad absurdum, prior to the level at which the material is used. The text makes a valiant effort to be rigorous in its various proofs yet must rely upon select theorems which are beyond the experience level of the student and whose proofs must be relegated to the Appendix. It must be noted that, in spite of these loopholes in the development of the unit, the text points to these loopholes and demands that the student recognize the inadequacies.

This particular unit seems to present to the student a series of written lectures which develop the desired concepts and then reinforce this concept with a variety of drill exercises. It appears that fewer exercises demand student proofs, but rather, more of the exercises which, though simple in nature, are designed for reinforcement. As in all earlier

units, the Exploration Exercises play a vital role in permitting the student to intuitively build an anticipatory insight into the material to be more rigorously attacked in the next section.

This unit appears, in addition to developing a tremendous number of mathematical concepts, to provide maximum opportunity for development of associated skills. Literally hundreds of exercises provide opportunity to apply the techniques of logarithmic computation, reduction and simplification of radical expressions, etc.

The unit concludes with a consideration of several of the laws of nature and illustrates how one may quantify nature in many instances and describe (and anticipate) physical phenomena by a mathematical model abstracted from empirical data. Accompanying these discussions are many exercises which deal with the physical environment and are, in a sense, applied in nature.

Circular Functions and Trigonometry, Unit 10

This unit, destined to be a part of the twelfth-grade mathematics program, develops what has been called for the past many years plane trigonometry. The over-all results of the presentation are roughly those of such a traditional course but this unit pays more careful attention to detail and structure and the use of the winding-functions (or wrapping-functions) to serve as an angle-free vehicle for the

definition of the trigonometric functions is an innovation worthy of special attention. The actual unit itself is written in accordance with the UICSM discovery approach although more careful editing would have corrected several contradictory usages of terms which have been defined earlier and are now used in a conflicting manner.

Functions

This section, essentially a review for students having studied Unit 5, re-introduces a function as being a set of ordered pairs no two of which have the same first component. Although not stated explicitly by UICSM, the statement of the rule defining the value of the function corresponding to each of its arguments is essential to the description of the function--an essential sometimes accomplished by the introduction of an algebraic expression and sometimes by listing the ordered pairs. Many examples of different functions, among which is the greatest integer function (to be used extensively later in the unit) defining $[x]$ to be the real integer n such that $n \leq x < n + 1$, are presented along with the consideration of many functions whose domains and ranges are not both necessarily numbers.

Circular Functions

As a basic tool for the definition of sine and cosine functions, UICSM introduces the rather novel winding function W which determines W(x) by laying off on a unit circle

(centered at the origin) an arc whose measure is x (starting at (1,0)) in a counterclockwise direction if $x > 0$ and a clockwise direction if $x < 0$ with W(x) being the endpoint of the arc. The added condition that W(0) = (1, 0) completes the description of W(x). The first coordinate of W(x) is defined to be cos(x) and the second component sin(x); hence W(x) = (cos(x), sin(x)), with the domain of x being the set of real numbers. The reader notes that the symbol $\overset{\frown}{ABD}$ is used to designate the arc from A to D in the counterclockwise direction--an innovation that seems somewhat out of character since "$\frown$" would perhaps intuitively suggest clockwise!) As in earlier units, the name m is used to designate the function whose domain is the set of all segments and arcs and whose value for each of its arguments is the measure of that argument.

One notes that this definition of the sine and cosine functions is independent of the consideration of an angle. Due to the close connection between these functions and the unit circle, UICSM uses the term circular function more often than trigonometric function. The term also serves to de-emphasize their computational use in solving triangles and hopefully permits the later value of the functions in describing periodic phenomena.

Some Properties of Cos and Sin

By total reliance upon the angle-free winding function $W(x)$ earlier introduced, UICSM leads in the discovery of many of the properties of the sine and cosine functions. Included in these are: (a) sin and cos are periodic functions since $W(x + 2\pi) = W(x)$ for every x and, consequently, $\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$; (b) the sine is an odd function and the cosine is an even function since $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ for every x; (c) for every x and y such that $x + y = \pi/2$, $\sin x = \cos y$; (d) the sine function is monotonically increasing in the interval $\overline{0, \pi/2}$ and the cosine is monotonically decreasing in that interval and (e) for every x, $\sin^2 x + \cos^2 x = 1$. The unique part of this presentation lies not in the final results but rather in the use of the winding function approach which allows the definition of the circular functions without ever considering, in any sense, angles.

The remaining four circular functions of traditional trigonometry are introduced by definitions based on the sine and cosine functions: $\tan x = \sin x/\cos x$ and $\sec x = 1/\cos x$ with the domain of both functions being the set of all real x such that $x \neq (2n + 1)(\pi/2)$ and $\cot x = (\cos x)/(\sin x)$ and $\csc x = 1/(\sin x)$ are introduced as functions having as domains the set of real x such that $x \neq (2n)(\pi/2)$. These definitions provide ample opportunity for transformation of expressions containing names of circular functions to

simpler equivalent forms--actually the traditional identities. UICSM does not use the term identity since "we don't know of any simple definition of 'identity' which agrees sufficiently well with the intuitive notion that an identity is an equation which is generally true, we shall not use the word 'identity' in this unit."[1]

Geometric Applications of Circular Functions

After spending considerable time discussing the radian-measure of an angle (actually a very natural consequence of the winding function used to define the function) and the difference between a line segment and its measure as well as the difference between a region and its measure, UICSM proceeds to develop the cosine law and sine law of traditional texts. These laws are then used as tools for solving triangles as in those traditional texts except that great care is exercised to distinguish between a side of a triangle and its measure and every statement (and exercise) is worded in terms of measures.

The degree-measure function is introduced by the conversion formula postulating that an angle whose radian-measure is π has a degree-measure of 180, i.e., if A is an angle such that $m(A) = 1$, then $^{\circ}m(A) = 180$. Since the circular functions have been stated essentially in radian-measure

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[1] University of Illinois Committee on School Mathematics, Circular Functions and Trigonometry (Urbana: University of Illinois Press, 1963), p. 38.

language, UICSM finds it necessary to introduce new functions, called degree-sine and degree-cosine, etc., to make it possible to work directly with angles whose measures are radian-measures without having to convert their measures to radian-measures.

The reader notes several inconsistencies in this particular unit.

1. The discussion of the degree-measure of angles in terms of radian-measures is inappropriate since the student has already studied the degree-measure of angles in Unit 6 and used the term measure to indicate degree-measure--the term measure is used here to indicate radian-measure!

2. Mention is made (and use presumed) of empty angles and straight angles although the definitions of Unit 6 have decreed that such angles are not to be defined! In this sense, the statement "π radians = 180°," although a helpful mnemonic, is somewhat meaningless.

3. UICSM uses the sum and difference of angles--usage which definitely contradicts the definition of angles as sets as presented in Unit 6 and which removes the possibility of addition of angles since the only admissible operations for sets are union and intersection.

4. Also, the word prove in the Exploration Exercises of this section seems to indicate that an intuitively-convincing argument is sufficient--certainly contradictory to the rigor of earlier units.

Consequently, this particular section seems to be out-of-character with earlier units and, in some instances, even contradictory.

A Basic Theorem About Circular Functions

By reliance upon the winding function $W(x)$, UICSM is able to establish that for every $\underline{x}$ and $\underline{y}$ such that $W(x) \neq W(y)$, one of the two arcs $\overset{\frown}{W(y), W(x)}$ is congruent to one of the two arcs $\overset{\frown}{W(0), W(x - y)}$. An immediate consequence of this proof is that for every $\underline{x}$ and $\underline{y}$, $m(\overset{\frown}{W(y), W(x)}) = m(\overset{\frown}{W(0), W(x - y)})$ from whence the student may show algebraically the authenticity of the subtraction principle for the cosine which states that for every $\underline{x}$ and $\underline{y}$, $\cos (x - y) = \cos x \cos y + \sin x \sin y$. This subtraction principle for the cosine, along with the basic properties of the function as earlier developed, provides the foundation for the proofs of the addition principle of the cosine, and the addition and subtraction laws for the sine and the tangent--actually the basic fundamental identities of traditional trigonometry.

Inverse Circular Functions

UICSM presents in this section a semi-traditional approach to the inverse trigonometric functions with the major innovation being the language used. It is discovered that the sine, cosine, tangent, and cotangent functions do not themselves have inverses (as seen from the ordered-pairs approach) but that each of the functions have subsets which do have inverses.

An Evaluation of Unit 10

This unit on trigonometry does not necessarily presuppose the knowledge on the part of the student of Unit 7, 8, and 9, and it is conceivable that a student might adequately comprehend the unit without having studied even Unit 5 in that the concepts required from that unit are reviewed thoroughly in this particular unit. The unit ties itself nicely to Unit 6 although several notions of Unit 10 are contradictory to those of Unit 6, e.g., the use of the empty angle (an angle whose degree-measure is 0), the straight angle whose degree-measure is 180, and the sums and differences of angles. UICSM's emphasis upon structure is evidenced by the careful attention paid to the statements of domain and range for the functions as well as the emphasis, by use of the winding function, of the notion that the argument (or domain) of the sine and cosine functions is the set of real numbers rather than a set of angles.

The language of this unit is far less sophisticated than earlier units although careful attention is paid to the statement of quantifiers for the theorems; the measures of arcs, segments, and angles; the ordered number-pair definition for functions, etc. However, little use is made of the earlier oft-used set-builder notation and the symbols used to indicate for every, there exists, such that, implies that, if and only if are not used here as in earlier units. Earlier units have demanded that the letters m, n, p, and q represent

elements in I^+ (the set of positive integers) and that $\underline{i}$, $\underline{j}$, and $\underline{k}$ are elements in I (the set of integers), a convention not followed by this unit.

The proofs in this unit, although fairly rigorous in nature, are certainly not of the sophistication of the earlier units. Most of the proofs, mainly analytic in nature, seem merely to give convincing arguments in many instances as opposed to the meticulous attention to detail in earlier units.

As does all previous UICSM units, this unit puts to good use the Exploration Exercises to allow the student to discover (and intuitively comprehend) the concepts of the section to be studied. The textual material itself is continually posing questions which force a student, if he actually reads the material, to formulate (and authorize) many of the concepts for himself. Exercises are many and varied in nature and seem designed to reinforce the concepts already studied rather than to extend the student's body of knowledge.

The unit is fairly well-balanced in its discussion of concepts and its demand for skill-building. One does notice that little direct attention is paid to the process of determining the truth sets of trigonometric equations, the graphs of trigonometric functions, the study of trigonometric reductions by means of the related angle, the logarithms of trigonometric functions--topics usually covered extensively in such a course. It would appear that these omissions might be

potential causes for concern for those planning to study more advanced mathematics.

UICSM makes little, if any effort to apply this unit to a physical environment. In fact, the long, tedious triangle-solving problems and logarithmic computations of the traditional trigonometry course are minimized practically to their non-inclusion.

Complex Numbers, Unit 11

This comparatively brief unit culminates the UICSM sequence with a fairly thorough study of complex numbers by a number-pair approach. The general mode of presentation is in accordance with the previous units.

Linear and Quadratic Equations

As an introduction to the study of complex numbers, UICSM reviews briefly the techniques for determining the truth sets of linear and quadratic equations and illustrates quite vividly that the meaningfulness of a solution depends upon the number system in which the solution is being sought. The student is informed (and constantly reminded) from the beginning of the unit that the objective of the unit is to construct a number system (called the complex number system) which will be constructed with certain defined operations such that a proper subset of that system will be isomorphic to the system of reals with respect to addition and multiplication and in which every member will have a square root.

The Complex Numbers

UICSM essentially introduces the complex numbers as being simply a system of ordered pairs of real numbers--actually just points of the number plane. If $z_1 = (x, y)$ and $z_2 = (u, v)$ are two complex numbers, then the sum $z_1 + z_2$ is defined to be $(x + u, y + v)$, the opposite of z_1 is defined to be $-(x, y) = (-x, -y)$, and the difference $z_1 - z_2$ is defined to be the complex number w such that $z_1 = w + z_2$. By using these definitions, the student is forced to determine the authenticity of the associative and commutative properties of addition.

Complex Numbers and Geometry

UICSM suggests that this section, dealing with the definition and implications of the product of a complex number (x, y) by a real number t might be omitted in the interest of economy of time. This definition for the multiplication of a complex by a real is motivated by a set of Exploration Exercises which shows, among other things, that if $(x_0, y_0) \neq (0, 0)$, then the line $\overleftrightarrow{(0, 0)\ (x_0, y_0)}$ is $\{(x, y) : \exists_t\ (tx_0, ty_0) = (x, y)\}$. It follows that one may define a line containing points z_0 and z_1 as $\{z : \exists_t\ z = z_0 + t(z_1 - z_0)\}$ and that the points z_0, z_1, and z_2 are collinear if there is a real t such that $z_2 = z_0 + t(z_1 - z_2)$. The notion of linear dependence of complex numbers is imbedded in this statement:

For all complex numbers z_1 and z_2, z_1 and z_2 are <u>linearly</u> dependent if and only if $\exists t_1, \exists t_2 \left[(t_1, t_2) \neq (0, 0) \text{ and } t_1 z_1 + t_2 z_2 = (0, 0) \right]$.[1]

The notion of the distance between complex numbers and the closely related notion of a complex number are shown to be useful in proving many theorems about triangles and parallelograms, e.g., the medians of a triangle are concurrent.

One notes here another word usage which is contradictory to that of Unit 6. Unit 11 considers two lines L and L' to be parallel if and only if $L \cap L' = \emptyset$ <u>or</u> $L = L'$.[2] Unit 6 has considered two lines L and L' as being parallel if and only if $L \cap L' = \emptyset$--a troublesome and unnecessary error for the careful student.

Definition of Multiplication of Complex Numbers

In its effort to extend the real number system to the system of complex numbers so that these complex numbers will have some of the properties of addition and multiplication of the reals, UICSM causes examination of the need for a defined operation $\times$ to be called multiplication so that this operation is distributive with respect to addition, is associative and commutative, that there is an identity element <u>e</u> such that for every $z \neq (0, 0)$, $z \times e = e \times z = z$, and that it is possible to define a reciprocation operation so that for every

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[1] University of Illinois Committee on School Mathematics, <u>Complex Numbers, Unit 11</u> (Urbana: University of Illinois Press, 1963), p. 32.

[2] <u>Ibid</u>., p. 40.

$z \neq (0, 0)$, there will be a number z^* such that $(z) \times (z^*) = e$. This multiplication must be defined in such a way that there will exist a proper subset of this set which will be isomorphic with respect to the reals under the operations of multiplication and addition.

Examination shows that, for all complex numbers (x, y) and (u, v), the definition $(x, y) \times (u, v) = (xu - vy, xv + yu)$ will suffice as a suitable definition for the desired operation. (The letter i is introduced to abbreviate $(0, 1)$; $i^2 = ii = (0, 1) \times (0, 1) = (-1, 0)$. In view of the defined operation, for any complex number (x, y), $(x, y) \times (1, 0) = (x, y)$; hence, there exists the multiplicative identity $(1, 0)$.) Since any complex number (x, y) might be considered as $(x, 0) + y(0, 1)$, the conventional symbol $x + yi$ is introduced to abbreviate (x, y). The reciprocation operator "/" is defined in such a way that $z \times (/z) = (1, 0)$, $/(x + yi) = \left(\frac{x}{x^2 + y^2}\right) - \left(\frac{y}{x^2 + y^2}\right) i$. This authorizes the definition of division such that for z_1 and $z_2 \neq (0, 0)$, $z_1 \div z_2 = z_1 \times (/z_2)$.

UICSM shows intuitively that, in view of these definitions, the set of real complex numbers (complex numbers having 0 as their second component) is isomorphic to the set of reals. It follows easily that any real number, negative or nonnegative, has two square roots in this system and that, consequently, any quadratic equation with real coefficients has meaningful solutions in the system of complex numbers.

Quadratic Equations with Real Complex Coefficients

This section, containing only a few minor innovations, is probably the most traditional treatment of any section in the entire UICSM sequence in its discussion of quadratic equations having real complex coefficients. The name real complex coefficient is shortened to real coefficient since the real complex system and the real number system are isomorphic. The term imaginary is sacrificed in favor of unreal--an innovation which should prove helpful to both student and instructor. The text leads a rather typical discussion of the uses of the discriminant of a quadratic equation, the sums and products of roots of a quadratic equation, equations involving radicals and possible extraneous solutions, and written problems whose solution sets are the truth sets of quadratics.

Quadratic Equations in Two Variables

In this section, the graphs of the parabola, the circle, the ellipse, the hyperbola, and the general conic are studied. Particular attention is paid to symmetry of the graphs, the extreme point of the parabola, the center and radius of the circle, the foci and axes of the ellipse, and the center and asymptotes of the hyperbola. The solution sets of systems of these equations are determined both graphically and algebraically with heavy emphasis upon geometric interpretation of the results.

Systems of Equations

This section briefly discusses the several techniques for solving systems of quadratics. Included among these are the traditional method of substitution, the method of addition, and the transforming of systems into equivalent ones which contain equations reduced to a pair of linear equations, e.g., the elimination of constants technique.

An Evaluation of Unit 11

This unit, although the last in a series of eleven units, definitely does not presuppose an intensive knowledge of the first ten units and, as a matter of fact, probably could be adequately comprehended by a good student who has completed (and understood) Unit 5. The unit provides a comprehensive, though not exhaustive, study of the complex numbers through the number-pair approach rather than the vector (or Argand's) approach.

UICSM, on several occasions in the unit, emphasizes that the complex number system is to be regarded as an extension of the real number system and is to be designed such that the addition and multiplication operations, notions of equality, etc., are to be formed such that the operations will be commutative, associative, that there will be additive and multiplicative identities, etc., and that this invented system will contain a subsystem isomorphic to the set of reals. This structuring of the complex number system is motivated by

the need for a system which will authorize a meaningful solution set for any quadratic equation. The observant student should note that the extension from the real numbers to the complex numbers is analogous to earlier extensions structured in response to demonstrated needs.

This unit is typified by UICSM's constant use of crisp, nonambiguous language with each theorem and definition accompanied by the now-usual quantifier. The unit relies more heavily on set-language than did Unit 10 but not as heavily as did Units 6-9.

The proofs are very adequate for the material presented and are fairly rigorous in nature with the majority being direct proofs with some few relying upon the inductive process. Some few of the theorems, particularly those dealing with the exhibition of isomorphisms between various systems, are somewhat intuitive in nature and do not bother with rigorous presentations.

The unit makes maximum use of the Exploration Exercises in helping the student to discover the desirable properties and definitions which this new number system should have if its subsystem, the system of reals, is to retain the properties of the reals. The textual material presented reads easily, is sequentially organized, and conversational in nature.

This unit, although doing a more-than-adequate job of developing concepts, has an unusually large number of what

one might term drill exercises. Certainly any person opposing modern programs on the basis that little drill is provided for reinforcement of concepts would be at a loss for substantiating evidence in this unit.

In this unit, UICSM does not use the social applications of mathematics to motivate the study of ideas, to develop basic principles, and, therefore, does not consider social application of mathematics as a major objective at this level. The unit does provide, through two sets of applied exercises, some opportunities for applying some of the principles after they have been developed.

An Evaluation of the UICSM Secondary Program

The UICSM units which have been examined in some detail in this chapter are the results of a group formed to "prepare text materials for a new college preparatory mathematics curriculum, grades 9 through 12."[1] The director of the program, Max Beberman, has suggested, however, that the role of UICSM is not purely that of a college preparatory one since the UICSM production is "an attempt to bring to the mind of the adolescent some of the ideas and modes of

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[1]Dorothy M. Fraser, Current Curriculum Studies in Academic Subjects (Washington, D. C.: National Education Association, 1962), p. 32.

thinking which are basic in the work of the contemporary mathematician."[1]

Insofar as placement of the materials is concerned, the UICSM tentatively recommends the units concerned with the arithmetic of the real numbers, generalizations and algebraic manipulations, equations and inequations, and ordered pairs and graphs (Units 1-4) for grade 9; the units dealing with relations and functions, and geometry (Units 5 and 6) for grade 10; the units on mathematical induction and sequences (Units 7 and 8) for grade 11; and the units on exponential and logarithmic functions, circular functions and trigonometry, and polynomial functions and complex numbers (Units 9-11) for grade 12. These units, though bound separately, were written to form a thoroughly sequential and integrated set of mathematical experiences. UICSM constantly stresses the sequential nature of its spiral approach to the development of a four-year curriculum and warns against attempting to use parts of it to supplement another program already in existence. The sequential nature of the units will allow different grade placement of the units and the appropriateness of any particular unit at any particular grade level must be judged in terms of the mathematical experiences which the students have had prior to that point and not in terms of the grade level itself.

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[1]Max Beberman, An Emerging Program of Secondary School Mathematics (Cambridge: Harvard University Press, 1962), p. 44.

It is this continuity of these materials which seems to be one of the strongest points of these units.

Insofar as subject matter is concerned, the UICSM materials align themselves roughly into these divisions. Units 1-4 present a detailed study of the real number system, precise statement of principles (or axioms and definitions) and theorems and the proofs of certain theorems, sets and their notation, solutions of linear and simple quadratic equations, and graphing of equations and inequations. Unit 5, a unit providing an extremely smooth flow of the development of the concepts of relation and function through the use of sets and operations on them, introduces linear functions and proceeds to the more complicated quadratics which are studied in detail. Unit 6 presents a metric-oriented approach to many of the concepts of Euclidean geometry with the emphasis being on proof rather than content. By considering the machinery authorized by a study of sequences and induction, Units 7 and 8 complete the development of the real number system (begun in Unit 2) with the exception of the principle of completeness. Unit 9 provides a rigorous discussion of elementary functions (power, exponential, and logarithmic) and their applications and presents the completeness principle. Unit 10 contains a treatment of the circular functions based on a winding function with the emphasis being on such properties as periodicity, evenness and oddness, monotonicity, and on analytic trigonometry rather than triangle-solving.

Unit 11 devotes itself to an axiomatic, number-pair approach to the system of complex numbers. In retrospect, one notes that Units 1-5 differ little from conventional textbooks in content but the approach and techniques are substantially different, that Unit 6 presents a considerably more metric approach to plane geometry than traditional texts, that solid geometry is relegated to merely an appendix topic in Unit 9, that the units on mathematical induction and sequences are essentially additions to the curriculum, and that the unit on complex numbers is more inclusive than those of traditional programs. These materials, rigorous in nature and precise in language, are certainly, insofar as content is concerned, adequate to prepare a college-bound student for the elementary calculus.

The UICSM units were developed with the demonstration of the structure of mathematics as a prime target. In Unit 1, the careful attention given to the development of the rational numbers and the subsequent emphasis laying a careful foundation of vocabulary and fundamental concepts to permit rigorous proofs in Unit 2 early demonstrate the logical structure of mathematics. In Unit 5, the ideas of functions and relations are developed in terms of sets and the entire unit is heavy in its emphasis on structure. Unit 6, in its approach to geometry through postulates based on those of Hilbert, indicates that a way to a better understanding of the use of mathematics is through continual study of structure

and, through this consideration, removes the traditional algebra-geometry division. The permeation of geometry with algebraic techniques illustrates the basic unity of mathematics. Subsequent units complete the development as a structure of the real number system which was begun in Unit 2.

Probably the most significant innovation of the UICSM materials lies not in the subject matter but in the manner and method of presentation. UICSM seems to believe that a student will come to understand mathematics better when his textbook and his instructor use precise, nonambiguous language and when he is directed to discover generalizations for himself.

> These two disiderata--discovery, and precision in language--are closely connected, for new discoveries are easier to make once previous discoveries are crystallized in precise descriptions (it is easier to discover how to solve equations when you know what an equation and a variable are!) and skill in the precise use of language enables a student to give clear expression to his discoveries.[1]

In answer to this belief, UICSM pays careful attention to the problem of distinguishing between the use and the mention of symbols and attempts to be rigorous at all times. Also in answer to this need, UICSM has attempted to allow the student to play an active part in developing and inventing mathematical ideas and procedures. Unorthodox as it may seem, UICSM apparently subscribes to the notion that it is unnecessary to require a student to verbalize his discovery to determine

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[1] Ibid., p. 4.

whether or not he is aware of a rule--hence fewer stated rules than in traditional texts, e.g., schemes for solving equations and step-by-step descriptions of algorithms for manipulations and simplifications. Also one notes the UICSM insistence that a student become aware of a concept before a name has been assigned to the concept--hence much discussion leading to the formalization and final verbalization of a concept rather than discussion preceded by the statement of a concept. In short, UICSM predicates its entire program upon student participation, development, understanding, verbalization, and generalization.

The UICSM materials are characterized by terms which are not too common in the traditional secondary texts. A scanning of the units reveals such terms as opposites, sameing, punctuating, pronumerals, test-patterns, open sentences, truth sets, universal quantifiers, equivalent, generalization, operator, set, subset, union, intersection, transformation, ordered pairs, Cartesian product, finite, infinite, countably infinite, lattices, symmetric, transitive, reflexive, existential quantifier, field and converse of a relation, composition, mapping, measure, binary, biconditional, modus ponens, monotonicity, recursion, sequence, closure, continuous, and dense. UICSM, on occasion, invents new names for several notions and concepts which reflect the use to which such notions and concepts are to be put, e.g., the name of a numeral is referred to as a pronumeral and the general form for proving

or disproving certain basic theorems is referred to as a test-pattern. From the very beginning, UICSM introduces and uses symbolism to its fullest, e.g., $\cup$, $\cap$, $\in$, $\subseteq$, $\exists$, $\forall$, $\Rightarrow$, and $\Leftrightarrow$ are common. In the same vein, the units introduce adequate notation to distinguish between a line, a ray, a vector, and a segment although the units are not consistent in their usage. It is apparent that UICSM is dedicated to a need for an adequate and appropriate nonambiguous mathematical vocabulary for all students at the secondary level and to the belief that students at such a level may appreciate a precise vocabulary.

In the structuring of proofs, maximum use is made of the test-pattern and the elements of simple logic. The role of the axiom (or postulate), the definition, and the theorem in any proof is evidenced by the rigor of the proofs. As an example, Unit 6 is essentially an outline of a purely deductive treatment of Euclidean plane geometry in which the student learns that "a deductive theory can be obtained by abstraction of postulates from a model and deduction of theorems from these postulates without reference to the model, and that such a deductive theory can then be reinterpreted to yield information about other models."[1] Several types of proofs can be found, e.g., paragraph proofs, proofs by contradiction, indirect proofs, column proofs, and proofs by

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[1] Ibid., p. 43.

induction, and all display maximum possible rigor. An outstanding feature of these proofs is the insistence of the UICSM upon the necessity of pointing out the inadequacies of many of the proofs of the units in the apparent belief that the student's recognition of nonrigor in a proof will compound that student's appreciation of a good proof. Since the four-year UICSM approach to the structure of the real number system is a spiral one, these proofs grow increasingly sophisticated as the pages are turned. Another outstanding feature is the insertion of metric methods in the unit on geometry--an innovation which, as mentioned earlier, emphasizes the basic unity of mathematics and lends an algebraic flavor to geometry.

The concepts-versus-skills aspect of these materials is difficult to analyze in that UICSM feels that the acquisitions, manipulation, and application of concepts are within themselves skills and thus it is that UICSM makes no effort to establish a dichotomy between the two types of exercises but rather intermixes them to a pleasant balance. An examination of a set (chosen at random from Unit 1) of exercises reveals a total of 97 drill problems dealing directly with operations on real numbers followed by a set of Exploration Exercises--largely discovery type in nature--which causes the student to anticipate and appreciate the concepts to be verbalized in the next chapter. It appears that even a drill fancier would have little to complain about regarding these

presentations although the sets of exercises are not drill-oriented. One notices, however, very few sets of exercises in which a stereotyped example is given which may be followed as a pattern by practically every exercise in the set. Rather the student usually discovers and formulates his own rules as he studies exploratory exercises. It is evident, of course, that certain units such as the one treating mathematical induction do not yield themselves to mechanical, rote-drill exercises.

In that a "child delights in the what-would-happen-if type of question, and, if he can given consistent answers to such questions, he regards this work as being eminently practical,"[1] UICSM introduces many mathematical concepts by imbedding them in student-centered situations of "fantasy" which are sensible for the grade level involved. It is hoped that this approach, in accordance with the discovery approach, will develop interest in mathematics and power in mathematical thinking. Because of the student's independence of rote rules and routines, it also develops versatility in applying mathematics. Very few exercises are couched in a concrete, environment-oriented language in what has been traditionally referred to as sensible socially-applied problems since a sensible problem for an adult is not necessarily a sensible problem for an adolescent.

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[1] Ibid., p. 37.

CHAPTER IV

A COMPARISON OF THE SMSG AND UICSM SECONDARY MATHEMATICS PROGRAMS

A comparison of the SMSG and the UICSM programs is difficult to construct due to several reasons. One of the foremost difficulties is created by the vast mass of textual material comprising these programs. The SMSG student textbooks contain a total of 2756 pages with the teacher's commentaries containing a total of 2640 pages whereas the UICSM student editions contain 2404 pages and the teacher's commentaries at least 2300 for a grand total of at least 10,100 pages of written materials in the two programs. The different scope and sequence arrangements of the two programs reflect the realization that there is no universal agreement as to "what should be taught where and when." Specific SMSG units are constructed for specific grade-level consumption, while usage of particular UICSM units is predicated upon readiness alone.

However, the obstacle of paramount importance lies in the realization that a comparison of the two programs should be made in view of a completely unbiased, objective frame of reference if the programs and their positions are to be

adequately assessed. The state of flux currently existing in mathematics curriculum precludes the existence of any uniform agreement as to universally acceptable criteria for judgment of factors in a "proper" secondary mathematics program. Further, evaluation of the programs must be made in reference to the conditions in the environment where the programs are used. Hence, this chapter attempts not to evaluate but only to analyze and compare these programs with respect to the variables designated as philosophies of authors, placement of materials, attention to mathematical structures, vocabulary, proof, methods, concepts and skills development, and attention to social applications. These variables, admittedly not independent, and the associated discussions, necessarily somewhat redundant, hopefully serve to orient a reader to the general characteristics, similarities, and differences between the programs.

Philosophies of the Authors

An analysis of the SMSG and UICSM programs will be accelerated by an appreciation of the guiding philosophies of the two groups. Both groups have at different times stated their philosophies in various ways.

The SMSG authors, early in their program development, stated:

> The world of today demands more mathematical knowledge on the part of more people than the world of yesterday and the world of tomorrow will make still greater demands. Our society leans more and more heavily on

> science and technology. The number of our citizens skilled in mathematics must be greatly increased; an understanding of the role of mathematics in our society is now a prerequisite for intelligent citizenship.[1]

Thus, from its beginning, SMSG has been dedicated to the notion that the study of mathematics is to be predicated upon the needs of society and has implicitly directed that such awareness be communicated to the student. This awareness is coupled with a profound implication stated by the SMSG authors:

> Since no one can predict with certainty his future profession, much less foretell which mathematical skills will be required in the future by a given profession, it is important that mathematics is to be so taught that students will be able in later life to learn the new mathematical skills which the future will surely demand of many of them. . . . First, we need an improved curriculum which will offer students not only the basic mathematical skills but also a deeper understanding of the basic concepts and structures of mathematics.[2]

In like vein, SMSG recently stated:

> Since 1958, the School Mathematics Study Group has concerned itself with the improvement of teaching of mathematics in the schools of this country. . . . One of the prerequisites for the improvement of the teaching of mathematics is an improved curriculum--one which takes account of the increasing use of mathematics and science in technology and in other areas of knowledge, and at the same time, reflects advances in mathematics itself. . . . These textbooks were designed to improve substantially the curriculum of school mathematics by offering the student not only the basic mathematical skills but also a deeper

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[1] School Mathematics Study Group, Newsletter No. 1 (New Haven: Yale University Press, 1959), p. 4.

[2] Ibid.

> understanding of the basic concepts and structures of mathematics.[1]

As the director of UICSM, Dr. Max Beberman, in his now-famous Inglis Lecture of 1958, stated:

> In 1952, a few of us at the University of Illinois asked ourselves: Can able mathematicians together with skillful teachers develop materials of instruction and train highschool teachers in their use so that the products of the program are enthusiastic students who understand mathematics?[2]

In this lecture, Dr. Beberman emphasized that the construction of such a curriculum effort demands obeisance to several guiding principles which must be reflected in the products. The curriculum architect must have in mind an image of the student and a catalogue of his knowledge and misknowledges at each particular grade level. This architect must have in mind the expectation of what a college-bound high school student should know of mathematics at graduation. The architect must be cognizant of the application (perhaps relatively far in the future) of the mathematical knowledge to a vast host of physical and environmental problems some of which are yet unanticipated. The architect must be cognizant of and conversant with the drastic changes occurring within the field of mathematics. The prime and foremost result of that

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[1] School Mathematics Study Group, New SMSG-Yale Publications (New Haven: Yale University Press, 1965-66), p. 3.

[2] Max Beberman, An Emerging Program of Secondary School Mathematics (Cambridge: Harvard University Press, 1962), p. 1.

architect's effort must be that the student understand his mathematics. His lecture concluded with these statements:

> The UICSM program is a product of the combined efforts of mathematicians and teachers. It is an attempt to determine what the teacher must do to bring to mind some of the ideas and modes of thinking which are basic to the work of the contemporary mathematicians.[1]

Analysis of the philosophies of the two groups as referred to the desirable nature of the high school curriculum reveals equivalent beliefs. Both groups subscribe to the idea that mathematics be oriented in some degree to the needs of the society yet the current needs of the society is not the only objective. Both groups take the position that, due to the impending technological and scientific strides in our society, the student currently studying mathematics may not now be able to identify adequately and to appreciate his needs. Certainly, neither group advocates the "tearing-away" of mathematics from its social implications but rather suggests that the changing society and the uncertainty of the mathematical needs of the future force one to study a mathematics so structured that it will answer present needs and also can be extrapolated to a newer and radically-different environment. Then mathematics, if not to allow the present environment to "cast a millstone about its neck," must, though being completely cognizant of the present needs of the society, transcend that society. Both groups indicate their

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[1] Ibid., pp. 43-44.

belief that this need may be met only through student understanding of that which he studies. This understanding, however, is to be accompanied by the acquisition of those skills requisite to the application of that understanding. Thus it is that the guiding philosophies of the two groups are so nearly equivalent as to be identical. It follows that these two different programs, being based on common philosophies, must structure their differences on bases other than the desired end-products of the four-year study by the student of such programs.

Placement of Materials

It will be recalled that placement of mathematical topics is defined in a broad way so as to consider more than just when or where a particular content topic is introduced but also to involve the sequence in which particular topics are studied as well as the rigor with which they are employed. In a sense, the depth and degree of sophistication to which an idea, concept, or tool is explored may be as important (or perhaps more so) as the point of entry of that idea, concept, or tool.

In Chapter I of this paper are found very brief course outlines and/or unit titles of the SMSG and UICSM programs. Due to the compactness and brevity of these outlines, they seem at first glance to describe drastically different programs of study insofar as mathematical content is

concerned. Yet the extensive, practically voluminous critiques of the SMSG and UICSM programs presented in Chapters II and III, respectively, of this paper, have provided some fairly interesting results insofar as the mathematical content of these sequences are concerned.

The SMSG ninth-grade course, although provocative, meaningful, and mathematically sound as well as being characterized by new techniques of presentation, covers essentially the same basic material as does a conventional first-year algebra textbook. Although differing in comparison with conventional textbooks with respect to content, postulational scheme, and manner of treatment, SMSG Geometry is still basically a treatment of synthetic Euclidean geometry. The SMSG eleventh-grade units, though structured differently than the conventional texts, display a basic mathematical content essentially that of trigonometry and second-year or college algebra. Similarly, although the treatment is novel and the approach more sophisticated, the basic subject matter of the first-semester, twelfth-grade unit is somewhat in context with the conventional program. At this point, the SMSG program has adequately prepared for, though not trespassed upon, the elementary calculus. Of course, as pointed out earlier, SMSG's Introduction to Matrix Algebra, being composed of mathematics which is new to the student and new to the secondary curriculum, is intended to put the student close to the frontiers of mathematics during his senior year and to

provide examples of mathematical patterns that arise in varied circumstances.

Similarly, UICSM's unit-by-unit study of the arithmetic of the real numbers, generalizations and algebraic manipulations, equations and inequations, ordered pairs and graphs, relations and functions, circular functions and trigonometry, and polynomial functions and complex numbers introduces only a few really new mathematical topics to the secondary curriculum. Therefore, these new materials, though presenting some new topics, utilizing new mathematical approaches and pedagogical techniques, and directing new emphases, are basically sufficient to prepare a student for the elementary calculus though not encroaching upon that area.

Since both SMSG and UICSM utilize the spiral approach to the development of secondary mathematics, several areas which hopefully will illustrate some of the major similarities and dissimilarities between the two programs may be chosen for comparison of the two. The study of the several following areas, chosen to represent the development of mathematical content, the structuring of mathematical tools, and the adoption of algorithms, serves to indicate some of the differences insofar as content placement is concerned. Since any two areas chosen are likely to overlap, the discussions will, in some instances, be somewhat redundant.

One of the most universally accepted innovations in modern mathematics programs has been the introduction of

"set-language" as a tool to build a more precise, rigorous communication vehicle. Both SMSG and UICSM are characterized by their early introduction of such tools but examination of their usages of such reveals some major differences.

SMSG introduces on the very first page of their materials the set as being a collection of elements with some common characteristic. Within the next fifty pages, SMSG uses empty (or null) sets, finite and infinite sets, and defines the truth set (later referred to as the solution set) of an open sentence to be the set of all members chosen from the domain of the variable and which make the sentence true. However, the only method ever utilized to identify the elements of a set is the enumeration of the names of the elements of the set in braces or by verbal description of the elements of the set. The set-builder notation is not used in First Course in Algebra. SMSG's Geometry begins immediately (and continues steadfastly) the usage of set language in its study of geometry and although the is a subset of relation and the union and intersection operations are defined on sets, these entities are verbalized but never symbolized. All geometric figures, e.g., straight lines, rays, segments, angles, and circles, are characterized as sets of points and the theorems are verbalized in set language. Early in Intermediate Mathematics, sets are employed (among a multitude of other usages) to describe the traditional locus and to define the solution set of a system of simultaneous equations (or sentences) to

be the intersection of the solution sets of the component sentences. (It is at this point that $\cap$ is used for the first time to symbolize intersection.) It is not until Appendix 1, Elementary Functions, that SMSG formally introduces the symbols $\cup$ and $\in$ and provides the set-builder notation to describe sets. The Cartesian cross-product is not regarded as important enough to be granted index-status in any of the units although lattices are used quite frequently in several of the units. Thus, an examination of the SMSG materials reveals that although SMSG relies upon set language from the very first page, the usage is somewhat informal and little attention is directed toward incorporating the associated symbolism into the textual materials and exercises.

On the other hand, Units 1 and 2 of the UICSM program do not employ set language for the first two units of study. Unit 3 immediately introduces the set as an undefined term and uses the concept to define intervals, open intervals, half-open intervals, half-lines, rays, etc., as sets of real numbers. The set-builder notation is introduced early in the unit with the is a subset of relation being defined (and symbolized by $\subseteq$) on sets. The Cartesian cross-product of two sets A and B is brought into focus early in Unit 4 and used to discuss lattices whose coordinates are integers, lattices whose coordinates are rationals, etc., with the discussion culminating in the construction of the number plane as the Cartesian square of the reals. The union and intersection

operations are defined and symbolized early in Unit 4. From Unit 4 through the remainder of the sequence, practically every page of the UICSM textbook resounds with set language and notation with the set-builder notation being a universal tool. Every geometric figure is defined as a set of points and all theorems (both geometric and non-geometric) are phrased in set language whenever feasible. Thus it is that UICSM makes a very formal tool of set language and continually makes precise usage of the instrument.

It follows, therefore, that both programs make fairly extensive usage of set language. Although SMSG introduces the language earlier in the first year of study, UICSM makes a vastly more formal usage of such in later units and employs both set language and the associated symbolism to a much higher degree than SMSG.

Since practically every page of any mathematics text uses directly or indirectly the concept of variable, it is interesting to note the different approaches taken by the two groups. This difference, though a subtle one, is probably the greatest single difference between the texts prepared by each group for ninth-grade consumption.

According to the approach used by SMSG, a variable is a letter used to denote one of a given set of numbers and, in a given computation involving a variable, the variable is a numeral which represents a definite, though unspecified, number from a given set of admissible numbers. (Generally

speaking, SMSG restricts the domain of the variable to sets of numbers or, as in the last unit of study, to sets of matrices.) SMSG's approach to variable suggests that the solution set of the sentence $x + 7 = 9$ is equal to the solution set of the sentence $x = 2$--a set whose elements are "obvious."

In its approach to variable, UICSM, having used frames (or "holes") to hold places for numerals in open sentences, points out that since the "holes" in an open English sentence hold places for nouns, they are called pronouns. Similarly, the pronoun x in the open sentence $x + 7 = 9$ holds a place for those nouns which are names of numbers, i.e., x holds a place for numerals, and, consequently, is given the name pronumeral. Pronumerals (a term used by UICSM is the sense in which the variable has been used in conventional texts) in an open sentence hold places for numerals. Therefore, in the eyes of UICSM, a pronumeral is merely a symbol which can be replaced by a numeral. UICSM's Unit 5 sharpens this notion even more by defining a variable as simply a pronoun. A variable is little more than a "mark" which holds a place in a sentence or in an expression for names of things. It follows that a pronumeral is, in actuality, a numerical variable but that the term variable is a more inclusive one than pronumeral. (This approach to variable and pronumeral suggests that if one says that $\{2\}$ is the solution set of the sentence $x + 7 = 9$, then one means

that if x were replaced in the sentence by the numeral 2, the sentence would be converted from open to true.)

In comparative summary, the SMSG text uses the approach that the variable is the name of a definite though often unspecified number, while the UICSM textbook states that the pronumeral (actually a numerical pronoun or numerical variable) holds the place for the name of a number. Minor point though it is, the variable as introduced by UICSM is a more inclusive one than that used by SMSG since, to the UICSM, a variable is simply a pronoun which may have as elements in its domain any type of object whatsoever.

Since both programs make extensive use of the number line, number plane, and graphs from the very beginning of their programs, an examination of their views as to their notions of such is appropriate. These differences, though again subtle, create associated language structure differences throughout the two sequences.

In the early chapters SMSG assumes a one-to-one correspondence between the set of real numbers and the points on a straight line. The real numbers associated with the points label the points and coordinatize the line. The line on which points are labelled is referred to as the real number line and the number associated with the point is called the coordinate of the point. The graph of a set of numbers is the corresponding set of points on the number line whose coordinates are the numbers of the set and the graph of the

truth set of an open sentence is the set of all points whose coordinates are the values of the variable which make the open sentence true. In like vein, the real number plane is constructed by coordinatizing a plane through an axiomated one-to-one correspondence between the points of a plane and the set of ordered pairs of real numbers. The plane being coordinatized is the real number plane and the pairs of real numbers are the coordinates of the points. Similarly the graph of a sentence in two variables is the set of all points (usually marked with a heavy dot) whose coordinates satisfy the sentence. In essence, the SMSG program refers to the line and plane being coordinatized by the reals as the real number line and the real number plane, respectively, and the points of the number line (or number plane) are "geometric" points having the real numbers (or ordered pairs of reals) as coordinates.

UICSM displays a somewhat different approach to the real number line. After UICSM has used real numbers to measure directed trips, the real number line is defined as the ordered set of real numbers. Therefore, the points of the number line are real numbers. Pictures of segments of straight lines are used to represent the number line with the real numbers being designated as coordinates of the points on the pictorial representation. In order to distinguish between number line and number line picture (in a sense, to distinguish between abstraction and "picture"), an element of

the real number line is referred to as a point on the real number line and an element of the geometrical figure whose coordinate is the point (or real number) as a dot. Each dot on a line is referred to as the graph of the real number line which is its coordinate.

In like fashion, UICSM defines the real number plane as the Cartesian square of the reals, i.e., the points of the plane are ordered pairs of real numbers corresponding with dots which can be marked on pictures of the plane. The dot is the graph of the ordered pair and the components of the ordered pair are the coordinates of the dot. The authors are consistently careful to refer to a page or sheet of paper used to represent a section of a number plane as a picture of the real number plane.

In summary, SMSG uses a more geometrical notion for the real number line than UICSM since SMSG refers to the number line as a "geometrical" straight line coordinatized by the reals whereas UICSM considers the number line to be the set of reals itself. Similarly, SMSG refers to a plane coordinatized by ordered pairs of reals as the real number plane (whose elements are geometric points) whereas UICSM considers the real number plane to be the Cartesian square (whose elements are ordered pairs of real numbers) of the reals.

In several instances throughout the SMSG units, one finds the term relation, e.g., Unit 9 speaks of the relation

is less than for real numbers as an order relation. However, the term order relation is used with little, if any, explicit attention to the word relation although order is adequately defined. A glance through even the first few units reveals the use of many relations, e.g., equals, is congruent to, is similar to, greater than, not greater than, less than or equal to, is the square root of, divides, and is a factor of, although the term relation is never considered important enough per se to warrant explicit discussion or even the award of index status.

The UICSM emphasis upon the concept of relation is a completely different one. After having discussed (among others) the is equal to, is not equal to, is greater than, and is less than relations without having explicitly defined the term, UICSM devotes an entire unit (Unit 5) of study to relations and functions. In view of the fact that UICSM has already made early use of the Cartesian cross-product of sets, it is feasible to define a relation as a set of ordered pairs, i.e., a UICSM relation is a subset of the Cartesian product of two sets. UICSM describes a relation by either listing the total set of ordered pairs comprising the relation, by graphing the ordered pairs of the relation, or by employing the brace or set-builder notation. All three forms emphasize the necessity of the description of the domain (the set of elements used for the first components) and the range (the set of elements used for the second components) of the

relation. Suitable attention is directed toward the existence or nonexistence of the reflexive, symmetric, and transitive properties of certain relations along with exhibition of several equivalence relations as well as the examination of the union and intersection of relations. (As will be seen later, UICSM uses this fairly intensive study of relations as a springboard for its definition of function in the same unit.) The use of relation as an ordered pair of elements pervades all the remaining units of the UICSM units and allows a sophistication of language which is somewhat unexpected in a secondary project.

Thus it is demonstrated that SMSG essentially uses relation as an undefined term with little attention to explanation. On the other hand, UICSM, in its devotion to detail, uses the ordered-pair notion of relation in an explicit and utilitarian fashion to build other concepts and to provide a sophistication in its language.

In modern programs, the proper understanding and usage of the function has become of major importance. UICSM and SMSG take full advantage of the concept but, again, their approaches and emphases differ.

The SMSG sequence, in its unfolding, reveals an increasingly sophisticated spiral approach to the concept of the function. The term function is not used until the last chapter of the materials prepared for the ninth-grade. At that time a function is constructed by using some "rule" to

assign to each element of a set A of numbers exactly one number. The association of numbers along with the rule, the given set of numbers (the domain) and the set of assigned numbers (the range) is defined as the function (actually a real-valued function). Using the notion of a function as an association, the student is led to study the linear function ($y = ax + b$, $a \neq 0$), the quadratic function ($y = ax^2 + bx + c$, $a \neq 0$), etc. Thus, during the ninth grade the SMSG student is acquainted only with real-valued functions.

SMSG Geometry, though not making an explicit issue of the subject, employs several measure-functions. This usage is very informal and is only considered implicitly as that of a function. The Distance Postulate actually postulates the existence of a measure-function whose domain is the set of line segments and whose range is the set of positive reals. The Angle-Measure Postulate axiomates the existence of a measure-function whose domain is the set of angles and whose range is $\{x: 0 < x < 180\}$; the area-measure formula $A = bh$ actually defines a function whose domain is the set of rectangular regions and whose range is that of the positive reals. No special notation is employed in this usage with the major reflection of function theory being the notion of correspondence.

Early in the units prepared for eleventh-grade consumption, SMSG formally broadens the concept of function to include nonreal-valued functions by allowing the domain and

range to be any sets whatsoever. It is at this point that f is first used to name a function and f(x) to symbolize the element assigned to x by the function f. Although the function definition is not restricted to reals, the majority of the text concerns itself with real-valued functions, e.g., linear functions, quadratic functions, and composition and inverse functions. Exceptions involve the assignment of measures and paths to signed angles and the six trigonometric functions whose domains are signed angles and whose ranges are appropriate subsets of the set of reals.

In the twelfth-grade units, SMSG considers a function as a mapping and symbolizes the function f by the notation $f : x \rightarrow f(x)$. Elementary Functions applies the concept of mapping to polynomial, exponential, logarithmic, and trigonometric functions. Introduction to Matrix Algebra carries this concept still further to define the determinant function (having 2 x 2 matrices and the set of reals as the range)

$$\delta : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow ad - bc$$

and the inner-product function (having the set of ordered pairs, or two-dimensional vectors, as domain and the set of reals as range).

$$\text{Dot} : \left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right) \rightarrow ac + bd.$$

Therefore, it is evident that the SMSG spiral approach to functions initiates in the real-valued functions

described in Unit 10 and terminates in the mapping interpretation of Unit 23. Regardless of the language used, SMSG considers a function from set A to set B to be a "correspondence" of the elements of A and B such that to each element of A there corresponds exactly one element of B.

As did the SMSG authors, the UICSM writers chose not to introduce function until after the student had considerable depth in the program. (UICSM's first use of function comes in Unit 5--a unit probably studied at the first of the tenth grade.) The UICSM student's first acquaintance with function lies in the definition that a function is a set of ordered pairs, i.e., a relation, no two of which have the same first component. The student sees the function as a special kind of subset of the Cartesian product of two sets and studies a multitude of examples where neither the domain or range of the function is the set (or a subset) of the reals. The set-builder notation is immediately applied and continually employed to describe functions, and membership of an ordered pair in a function is described in several ways, e.g., if $F = \{ (x, y) : y = 2x - 1 \}$, then $(3, 5) \in F$, 3F5, or $F(3) = 5$. As an example of the typical uses to which UICSM subjects the ordered pair function, the area-function for rectangles is described as $A = \{ (x, y) \in R \times N$: y is the area-measure of x, R is the set of rectangles, N is the set of numbers of arithmetic $\}$.

Within just a few pages after the introduction of the function as a set of ordered pairs no two of which have the same first component, UICSM declares that every function determines a mapping of its domain onto its range. UICSM, using the ordered pair notation, is able to very gracefully state that a function has an inverse if and only if its converse is a function or if the mapping from domain to range is a one-to-one mapping.

As did the SMSG authors, the UICSM writers implicitly employ functions to discuss length-measures, area-measures, angle-measures, etc., in their unit on geometry. Even the sine function in trigonometry is defined in this unit as $\sin = \{ (x, y) \in X \times Y : x = \text{set of acute angles}, Y = \overline{0, 1} \}$.

Unit 7 sees the introduction of the greatest integer function $[\![x]\!]$ which maps each real number onto the greatest integer not greater than the real number and the fractional part function $\{ x \}$ which maps a real number x onto $x - [\![x]\!]$. Recursive definitions for certain types of functions are also employed. Later units direct attention to increasing functions, nth power functions, monotonic functions, continuous functions, principal nth root functions, principal square root functions, logarithmic and exponential functions, winding functions, and trigonometric functions. In practically all instances the ordered pair or mapping aspects of the function are of primary application with extremely careful attention being paid to the domain and range of such

functions. (In that the ordered pair and mapping concepts were introduced practically simultaneously, it appears that the spiral approach to functions is not a characteristic of the UICSM materials although the functions studied later in the sequence are more sophisticated than those near the point of entry of the function.)

This view of the approaches to function as evidenced by the two groups reveals that SMSG, utilizing a highly spiral approach, initially considers a function as a special sort of correspondence between sets of numbers, later generalizes the domain and range of function to general sets, and plateaus in the twelfth-grade with the function considered as a mapping. The developmental work with function is therefore spread across four years of work. The notation used is somewhat informal. UICSM, on the other hand, insists upon the function as a set of ordered pairs of elements no two of which have the same first component (early in Unit 5) and immediately characterizes a function as a mapping. The notation, primarily the set-builder notation, to describe sets is used in a very formal and precise manner throughout the remainder of the units.

In that both programs devote a large percentage of their writing to exhibiting the structure of number systems as being composed of a <u>set</u> of numbers, <u>operations</u> defined on the numbers in the set, and <u>rules</u> governing these operations, it is appropriate to examine their views on the meaning and

usage of the term operation. Their relative appreciation of the term again reflects different points of view and emphases.

Throughout the SMSG texts, statements such as "Addition on the number line is such an operation," "It contains only operations which we know how to do from previous experience," "We now want to define a new and very useful operation on a single real number," etc., reveal an awareness of the term. However, the term operation is used freely without an iota of explicit concern for the definition thereof. In essence, SMSG essentially regards operation as undefined and uses it in the dictionary sense as being a process which involves a change or transformation in a quantity or process or action that is a part of a series in some work. Consequently, SMSG pays little attention to the existence and role of operators.

UICSM, however, directs explicit attention to the definition of operation (defined on numbers) and attempts to illustrate the significance and results of the concept. In Unit 1, the authors view (though do not formally state) an operation as a set of ordered pairs of numbers no two of which have the first component. Unit 1 contains in its textual discussion references to many operations, e.g., adding, multiplying, oppositing, sameing, positiving, negativing, and squaring. Unit 5, in a more rigorous discussion pursuing a somewhat different avenue, defines a singulary operation (actually applied to a single element) on a set S to be a

mapping (when applicable) of a subset of the set S onto a subset of S, e.g., the square-root operation applies only to the subset of nonnegatives of the set of reals, and a binary operation (applied to ordered pairs of numbers) as a mapping of a subset of the Cartesian square of S onto a subset of S. Since whenever an operation can be applied to a number or an ordered pair of numbers, the result is unique, an operation defined on a set of numbers or ordered pairs is simply a function whose domain and range are subsets of the set. From this point, a very simple extension authorizes the operations (themselves functions) to be defined in functions as well as numbers. Throughout the later units, UICSM treats +, −, $\sqrt{\ }$, $\sqrt[m]{\ }$, $|\ldots|$, etc., as operators defining the mapping involved in the operation function.

Thus it is noticed that whereas SMSG assumes student appreciation of the concept of operation, UICSM devotes considerable discussion to the notion. UICSM's discussion, by nature an abstract one, portrays a careful attention for sophisticated detail on the part of the authors.

Pervading the entire SMSG sequence is the construction and analysis of number systems--a development based on a spiral approach. Unit 9 essentially axiomates the system of arithmetic numbers though it spends several dozens of pages reviewing its operation, relations, and properties. The arithmetic-number operations of grade-school experience are interpreted in view of number line experiences. The negatives

are introduced by the use of labels for the left half of the number line with the union of the set of negatives and the set of arithmetic numbers being named the set of reals. (Hence, it is noted that the set of arithmetic numbers is the set of nonnegative reals.) The operations of addition and multiplication are defined and verbalized although such definitions are structured so that desirable number-line experiences may be formalized. Although many of the properties of the reals, e.g., the associative and commutative properties of addition and multiplication, the addition and multiplication properties of equality, the addition properties of zero and opposites, and the distributivity of multiplication over addition, are stated in a generalized form, these properties are exhibited and verified rather than proved. Therefore, it is apparent that the early SMSG approach to the structure of number systems is an informal one.

Unit 17 presents a formal, abstract development of the number systems which were informally constructed and analyzed in the earlier units. The system of natural numbers is characterized by a list of nine definitions and twenty "basic properties" which, though not necessarily independent, serve to define the abstract system. The system of integers (whose consideration is motivated by the nonclosure of the naturals under subtraction) is likewise defined by supplementing the list of basic properties for the naturals and definition of the opposites of the naturals. The system of

rationals (whose study is motivated by the nonclosure of the integers under division) is defined by extending the properties of the integers and introducing the definition of the multiplicative inverses of the nonzero integers. The development of the reals is climaxed by the consideration of the reals as being the set of all decimal expressions, i.e., the union of the rationals and the irrationals. The system of complex numbers is constructed as an answer to the need for a still more inclusive system which contains the real system with all its properties and also containing a number satisfying the sentence $x^2 + 1 = 0$. Since every real number is defined to be a member of the complex number system, the system of reals is a subsystem of the system of complex numbers.

Therefore SMSG uses the term extension of systems in just that sense. The extension of each system, motivated by a need for a new type of number and of a more inclusive system, involves constructing a new system which would have all the algebraic properties of the old system; includes all the numbers of the old system in such a way that the new and old algebraic operations, when applied to the numbers of the old system, would be the same; and contains new numbers of the type needed.

As did SMSG, UICSM utilizes a spiral approach to the study of number systems though differing in several respects. These differences, as do many others, cause a considerable variance in language structure of the two programs.

In Unit 1, the arithmetic numbers, accepted as undefined, are pictured as measuring distances along the number line. The real numbers are defined informally as numbers which may represent directed distances along the number line. (Therefore, the numbers of arithmetic, though isomorphic to the nonnegative reals, are not the nonnegative reals and the set of arithmetic numbers is not a subset of the reals.) Units 2-4 concern themselves with the examination of the properties of the reals, forming and justifying generalizations describing the reals, and practicing manipulations pertinent to a study of the system.

Units 6, 7, and 8 reflect a continued concern for the derivation of a list of basic principles and theorems which will completely characterize the abstract system referred to as the reals--a task that is essentially completed in the appendix of Unit 9. However, the approach to the formal study of the subsystems of the reals seems a unique one. These studies are motivated by these questions: Which of the reals are positive numbers? Which of the positive numbers are positive integers (or naturals)? Which of the reals are integers? Which of the reals are rational? UICSM proceeds to establish basic principles which will "sift" the positives from the reals, the positive integers from the positives, the integers from the reals, etc., and to study theorems describing each subsystem. In essence, this approach examines the

special properties which some kinds of reals possess and which other kinds of reals do not possess.

UICSM defines the complex number system as being simply the set of all ordered pairs of real numbers, i.e., the Cartesian square of the reals, subject to the equals relation (a, b) = (c, d) if and only if a = c and b = c; the product definition (a, b) · (c, d) = (ac - bd, bc + ad), etc. Addition and multiplication are defined in such a way that there is a proper subset of C such that the number system C with respect to its multiplication and addition is isomorphic with the real numbers and its defined operations of addition and multiplication. Just as the set of arithmetic numbers is not a subset of (but is isomorphic to a subset of) the set of reals, the set of reals is not a subset of (though isomorphic to a subset of) the set of complex numbers.

It is therefore evident that UICSM is of the opinion that systems are not "extended" in the sense employed by SMSG. Whereas SMSG considers each system to be a subsystem of progressively more inclusive systems, UICSM forces each system to be isomorphic to subsystems of progressively more inclusive systems.

In association with the introduction of new systems, both programs employ a somewhat unusual and meaningful tool to introduce the negatives and positives. These innovations produce a distinct language difference in the two sequences during the first several chapters.

SMSG's introduction of the negatives as being numbers which might be labels for the points on the left half of the number line (just as the arithmetic numbers were the labels for the points on the right ray) causes the arithmetic numbers to play the role of what has been traditionally referred to as the positive reals. SMSG, therefore, while labeling the negatives with an upper prefixed dash, e.g., $\bar{}2$, need not label the positive numbers with any symbol since they are actually the arithmetic numbers. The opposite of any real number is defined by saying that two numbers are opposites if they are on opposite sides of zero in the number line and equidistant from that point. The opposite of a number is labeled by prefixing the lower dash, e.g.; the opposite of $\bar{}2$ is $-\bar{}2$). After discussing the possible ambiguities (and inaccuracies) promoted by interchanging the symbols, SMSG agrees that since $\bar{}a$ and $-a$ are the same if a is positive and since the "opposite" dash is applicable to any real number whereas the "negative" dash applies only to positive numbers to name negatives, the lower dash may be used in most instances to name either "the opposite of" or "the negative of" unless ambiguity or inadvertent error demands more careful use. In this context, the text never uses such expressions as "add the negative of" but rather "add the opposite of."

UICSM's insistence upon the numbers of arithmetic as not being real numbers, produces a somewhat more cumbersome, though not unwieldy, situation. The right-real numbers are

designated as positive by a prefixed upper dash, e.g., $^{-}2 = {}^{+}2$, and the left-real numbers are designated as negative and designated by an upper prefix dash, e.g., $^{-}2 = {}^{-}2$. Pairs of numbers whose number line sum is zero are designated as opposites and the lower prefixed dash is used to name the opositing operation, e.g., the opposite of $^{-}2$ is $—{}^{-}2$). After suitable discussion to the need for such distinctions, UICSM agrees that since $^{-}9$ and $—9$, etc., are both names for the same number, they could be used interchangeably. Similarly, since +9 and 9, etc., are members of isomorphic sets, they also can be used interchangeably unless relatively rare ambiguities occur in which case the more precise notation can be used. As does SMSG, UICSM uses such expressions as "add the opposite of" to the total exclusion of "add the negative of," etc.

It is apparent therefore that both programs make heavy application of the concept of opposites although their views on the role of negatives and positives differ somewhat. SMSG's approach to the negatives removes the necessity for labeling the positives whereas UICSM finds it necessary to distinguish between the positives and the nonzero arithmetic numbers. This latter, of course, produces a somewhat more abstract language in the texts of the UICSM.

The notion of the absolute value of a real number is one of prime importance to both programs in that both programs employ the absolute value to state the "rules" for

manipulation of the reals. According to the SMSG approach, the absolute value of a nonzero real number is the greater of that number and its opposite with the absolute value of $\underline{0}$ defined to be $\underline{0}$. Hence the absolute value of any real number is $\underline{0}$ or a positive real number. The number line evaluation of the absolute value yields the interpretation that the absolute value of any real number is the measure of the nondirected distance between $\underline{0}$ and the point whose coordinate is the number. UICSM, by use of the real number line, indicates that a given real number corresponds to a given number of arithmetic since the set of nonnegative reals is isomorphic with the set of arithmetic numbers with the elements of the set of negative reals being the opposites of the set of positive reals. The absolute value of a given real number is defined by UICSM to be the number of arithmetic corresponding to the given real number. This implies that the absolute value of a real number is not a real number but rather a number of another system, the system of arithmetic numbers. Of course, these different interpretations of the absolute value is necessitated by the different approaches employed to "extend" the number systems involved.

Since the first year of study as advocated by both programs has as one of its major purposes the development of an informal appreciation of the real number system, the development of the arithmetic operations of the reals is of paramount importance in the comparison of the program.

These developments, though not contradictory, display somewhat different emphases.

SMSG, prior to any consideration of the operations to be defined on the reals, presents a comprehensive review and summary of the properties and operations of the arithmetic numbers and structures a number line interpretation of the addition and multiplication operations. Immediately upon introduction of the reals, SMSG, after exhibiting several instances in which the addition of reals might be environmentally applicable, describes geometrically the operation of number line addition of the reals. To add two real numbers a and b, start from the point whose coordinate is a and move b units to the right or left according as b is positive or negative, respectively, with the coordinate of the terminal point being defined as a + b. The student is led to translate immediately into the language of algebra the "rules" which he has discovered for the addition of the reals.

Insofar as multiplication is concerned, the student is led, through textual exposition, to phrase a meaning for the product of two real numbers in such a way that whatever meaning is given to that product, that meaning must agree with the products which are already present for the nonnegative reals and that the properties of multiplication discovered for the nonnegative reals must still hold for all real numbers. The product of real numbers a and b is defined, consequently, to be $|a|\,|b|$ if both a and b are both positive (or both

negative) and $-(|a|\,|b|)$ if one is nonnegative and the other negative. Little attempt, if any, is made to make the operation "intuitively" plausible by using environment-based applications of such an operation.

Subtraction of the reals is extended from the system of arithmetic numbers in such a way that it still has the properties known from arithmetic and using only ideas with which the student is already familiar. After examination of such desired properties, the student is led to recognize that subtracting the real number b from the real number a must be equivalent to adding the opposite (or additive inverse) of b to a. Insofar as the number line is concerned $a - b$ is the measure of the directed distance from b to a and $|a - b|$ is the nondirected distance between a and b. In a similar fashion, division, being the inverse of multiplication, of the real number a by the real number b ($b \neq 0$) is defined to be the product of a by the reciprocal (or multiplicative inverse) of b.

UICSM, on the first few pages of Unit 1, introduces the real numbers and immediately proceeds to the task of structuring definitions for the addition and multiplication operations in the reals. After such definitions are somewhat complete, UICSM then presents an examination of the principles for the members of arithmetic to allow a more precise examination of the operations defined in the reals which have been conceptualized (but not yet verbalized) by the student.

The initial appreciation of the desired properties of the operation of addition of reals is through number-line construction used along with business application such as profit and loss and other business applications, gain and loss in football yardage, above and below sea-level, etc. The student is led to discover the "rules" for such addition without the verbalization of such until Unit 2, i.e., the student uses a nonverbalized concept for an entire unit without explicit concern for the verbalization thereof. As in the case of the addition of real numbers, UICSM students come to a nonverbal awareness of the operation of multiplication through physical interpretation. This interpretation, based on a projector which can be run forward or backward containing a film depicting a pool being filled or emptied, is so designed as to lead the student to the formulation of computing rules which are in accord with accepted procedures for multiplication of real numbers. The verbalization of the "rules" for such, though used continually through Unit 1, is not demanded until Unit 2.

Subtraction initially names the number-pair inverse of the addition of a given real number, e.g., subtracting 2 is the inverse of adding 2. This notion culminates in the Unit 1 nonverbalized realization that subtracting a real number is equivalent to adding its opposite--a notion verbalized in Unit 2 by "$\forall_x \forall_y x - y = x + -y$." Similarly, division in Unit 1 is considered to be the number-pair inverse of

multiplication and Unit 2 defines division by the principle of quotients "$\forall_x \forall_{y \neq 0}$, $(x \div y) \cdot y = x$," and the associated theorem "$\forall_x \forall_{y \neq 0} \forall_z$ if $z \cdot y = x$, then $z = x \div y$."

An examination of these two approaches indicates that the UICSM authors introduce the number-line and construct environmental situations to create a nonverbal awareness of the "desirable" definitions for the arithmetic operations on the reals. This creation is followed by a fairly precise examination of the generalizations of the system of arithmetic numbers. The complete Unit 1 utilizes these nonverbalized definitions with no formal statements of such until Unit 2. On the other hand, SMSG precedes its discussion of the reals with the analysis of the properties of the arithmetic numbers and then promotes the simultaneous discovery and verbalization of the rules for performing the operations on the reals.

An examination of the use of relations as reflected by the two programs has revealed already that UICSM uses a highly number-pair oriented approach to relations (and pays high tribute to the concept) while SMSG makes little explicit issue of the use of the term although definitely using the concept in its textual materials. The less than relation should serve to reflect the difference in the actual definitions of relations employed by the two groups.

SMSG's Unit 9 defines the less than relation by issuing the directive that a is less than b if and only if a is

to the left of b on the number line although a theorem (often used by other programs as a definition) proved later in the unit states that if x is less than z, then there is a positive real number y such that $z = x + y$. In Unit 17, SMSG, in its more abstract and rigorous development of number systems, decrees that the natural number a is less than the natural number b if and only if there exists a natural number c such that $a + c = b$. Similar extensions are made for the integers, the rationals, and the reals. UICSM, in its brief mention of the counting numbers, uses less than to declare that a is less than b if and only if a comes before b in the counting process but in its Unit 1 study of the reals immediately defines the lesser of two reals to be the one to which one must add a positive real to obtain the other. Unit 5 introduces the number-pair definition of the less than relation to be $\{(x, y) \in R \times R : y - x \text{ is a positive number}\}$ and Unit 7 considers the basic defining principle for less than as $\forall_x \forall_y (x < y \Leftrightarrow y - x \in P\}$. In essence, therefore, SMSG introduces and, for at least two years of study, uses the comparatively informal number-line approach to less than before ascending to the "positive difference between larger and smaller" definition. On the other hand, UICSM early in its program introduces and uses exclusively this outlook.

Since the limit is a sophisticated tool used in practically all areas of mathematics, both SMSG and UICSM make heavy application of the term. The groups again place

vastly different emphasis upon the degree of sophistication with which the concept is to be initiated and exercised.

SMSG usage of the limit is a fairly informal, though adequate, one. In its Unit 10 structure of the square root of a algorithm as being one which "could be continued without end, each time finding two rational numbers, closer and closer together, with $\sqrt{a}$ lying between them."[1] The circumference C of a circle is the limit of the perimeters p of the regular inscribed polygons since "if we decide how close we want p to be to C, we ought to be able to get p to be this close to C merely by making n large enough."[2] A similar tool defines the area of a circular region to be πr^2 and the length of an arc $\overset{\frown}{AB}$ as the limit of the sum of chords whose endpoints are points of the arc, etc. The definition in Unit 17 of the equality relation for reals through nth place truncation of decimal representations, the limiting forms of the conics, and the derivation of the asymptotes of the hyperbola all imply the informal use of limit, e.g., the asymptotes of a hyperbola are straight lines such that "the curve gets closer and closer to these lines as x increases."[3] A somewhat more rigorous reliance on the limit is reflected in the definition that a sequence of terms, a_i, "has a limit a_n if

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[1] School Mathematics Study Group, Unit 10, p. 300.

[2] School Mathematics Study Group, Unit 17, p. 345.

[3] School Mathematics Study Group, Unit 14, p. 516.

a_n becomes and remains arbitrarily close to A as n gets larger and larger."[1] Although several pages of exercises contain many applications of limits, the notion of "arbitrarily close" is the predominant feature with arbitrarily remaining formally undefined; Unit 21 hopes that n can be chosen "large enough so that for a given x

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

will differ from e^x by an arbitrarily small amount."[2] There are many other discussions given similar to the example cited. These definitions, though informal, could be stated very easily in the conventional (ϵ, δ) form and thus it should follow that the student studying mathematics from the SMSG texts are gaining the correct calculus concepts although they are not stated in a rigorous language.

Insofar as area of applications are concerned, UICSM's limit is used in a fashion similar to that of SMSG except that its language of description is considerably more formal. After having used the limit very informally in Unit 3 to discuss the square root algorithm, UICSM, in its unit on geometry, defines the circumference of a circle as the least upper bound (assuming the completeness principle) of a sequence of perimeter-measures of regular polygons inscribed in the circle and the area-measure of a circular region as the least upper

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[1] School Mathematics Study Group, Unit 18, p. 576.

[2] School Mathematics Study Group, Unit 21, p. 213.

bound of a sequence of area-measures of inscribed regular polygons with successively more sides. Unit 8 sees a sequence a_i converging to a real number s if and only if

$$\exists_y > 0 \; \exists_m \; \forall_n \geq m \; \left| s - \sum_{p=1}^{n} a_p \right| < y,$$

and Unit 9, in its attempt to define "arbitrarily close" and "sufficiently close" in its definition of continuity, employs very sophisticated language essentially involving the conventional (ϵ , δ) notion of the limit. Therefore, the difference in the SMSG and the UICSM usage lies in the sophistication of the language and not in the basic areas of application.

Closely akin to the limit used by both programs is the property of continuity of functions and/or graphs. SMSG, in its study of the logarithmic and exponential functions, refers to the graph of the logarithmic function (and its inverse, the exponential function) as being continuous since it has no breaks, jumps, gaps, or holes in it. Several of the theorems regarding polynomial functions, e.g., the Location Theorem (actually a loosely stated form of Rolle's Theorem described by another name) assumes, though not concerning itself with a formal definition for such, continuity of the polynomial function. Hence, SMSG does not concern itself with answering a need for a formal definition of continuity and the discussions, when necessary, are graph oriented. However, UICSM's attack is one climaxing in emphatic rigor. The

first eight units use continuity in a way analogous to that of SMSG but early in Unit 9 UICSM states that "a function $\underline{f}$ is continuous at x_0 if and only if $x_0 \in \mathcal{D}_f$ and $f(x)$ differs arbitrarily little from $f(x_0)$ for each $x \in \mathcal{D}_f$ which is sufficiently close to x_0,"[1] and later sharpens the condition by stating that a function $\underline{f}$ "is continuous at a_0 if and only if $a_0 \in \mathcal{D}_f$ and

$$\forall_{c>0} \exists_{d>0} \forall_{x \in \mathcal{D}_f} [|x - a_0| < d] \Rightarrow |f(x) - f(a_0)| < c.$$[2]

Therefore, UICSM's use of continuity in the latter units is predicated upon a rigorous function-based definition that permits a high degree of mathematical sophistication.

Since mathematics and logic have been shown to be inseparable, the student of modern curricula probably would expect to find logic reflected in the modern programs. Although SMSG makes good use of the connectives and and or in the solution of equations and systems of equations and is always concerned with determining the factors pertinent to good deductive reasoning, the SMSG units reflect little explicit attention to the rules of formal logic. This is not to imply that logical reasoning is not a desirable objective of the SMSG program but rather that the rules of logical reasoning and the structure of logic are not studied per se. On the other hand, UICSM furnishes (and recommends that the topic be

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[1] University of Illinois Committee on School Mathematics, Unit 9, p. 42.

[2] Ibid., p. 211.

studied in conjunction with Unit 6 as a vital part of the unit's work) a thirty-eight page appendix (complete with exercises) dealing with some of the rules of reasoning and the principles of logic used in proving theorems. Included in this discussion are brief acquaintances with universal instantiation, the conditionalizing and discharging of assumptions, modus tollens, contraposition, rules of double denial, converses, biconditional sentences, conjunctive sentences, alternation sentences, etc. The study of this particular appendix is not necessary to the continuance of the squence but does illustrate UICSM's attention to the role of formal logic in mathematical reasoning.

In Unit 18, SMSG states, in its proof of De Moivre's Theorem:

> . . . continuing in this way, we may derive, one after the other, similar formulas for z^4, z^5, z^6, . . . , z^n, for each natural number n. The formula of De Moivre states the general result. $z^n = r^n (\cos n\theta + i \sin n\theta)$.[1]

This example is typical of the many references made by SMSG to notions which could be proved only by the process of inductive proof. Yet formal inductive proof is not introduced by SMSG until the Appendix of Unit 21--a discussion entailing a total of 16 pages with a grand total of 18 exercises involving the First and Second Principles of mathematical induction and an acquaintance with recursive definitions.

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[1]School Mathematics Study Group, Unit 18, p. 696.

UICSM's Unit 7, entitled Mathematical Induction, continues the deductive organization of the real number system. Students practice the proving of generalizations concerning the positive numbers, the inequality relations, the positive integers, and the integers. Although the unit is not devoted entirely to inductive proof, induction is central to much of the unit since the proofs of many of the generalizations about the positive integers require the use of mathematical induction and the notion of recursive definitions. Unit 8, relying heavily upon induction, concerns itself with sequences and with proving theorems about sequences, particularly continued sums sequences, e.g., the arithmetic progression, continued products sequences, e.g., factorial sequences, exponential sequences, geometric progressions, and infinite geometric progressions, and the binomial theorem along with combinations and permutations. Later units refer to the method of inductive proof without fanfare.

SMSG includes in its eleventh-grade materials a quite intensive and inclusive chapter dealing with vectors and vector techniques. Vectors, initially defined as directed line segments, are equivalent if they have the same lengths and the rays of which they are subsets are parallel and in the same direction. The properties of addition of two vectors (the sum of two vectors being defined in the geometric sense, i.e., $\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AX}$, where X is the unique point such that $\overrightarrow{BX}$ is equivalent to $\overrightarrow{CD}$) and the multiplication by

scalars are verified somewhat informally and the results used to establish vector-based proofs for many of the theorems of plane geometry. Following such discussion is the introduction of the more rigorous component notation (or coordinate notation) for vectors with the properties of addition, e.g., associativity and commutativity and the existence of a vector zero and additive inverses, algebraically demonstrated and proved in view of comparable properties of the system of real numbers. Through the standard definition of the inner product, the conventional tests for perpendicularity of free vectors, etc., are developed along with vector tools for finding the angle between two vectors, etc. After a demonstration of the role of vectors in physical applications, the system of vectors is abstracted as a formal mathematical system and the chapter plateaus its study of vectors with the introduction and analysis of the axioms for a two-dimensional vector space along with the theorem stating that any system S satisfying the axioms for a two-dimensional vector space is isomorphic to the system of vectors in a plane. It is noted that this development of vectors builds the "algebraic" or component representation of vectors upon the geometrical interpretation of such entities.

After using rotation vectors in Unit 21 to develop the formulas for sin (x + y) and cos (x + y), SMSG, in Unit 23, sees the system of plane vectors considered as the set of 2 x 1 column matrices with real numbers as elements with the

addition and multiplication by a scalar of such vectors defined by the matrix operations. The prime objective of the chapter appears to be the demonstration of the association between column vectors and directed line segments with the geometric interpretation of addition and multiplication by scalars being preceded by algebraic interpretation. The chapter, including also a matrix-based definition for the dot-product of two vectors as well as exploration of areas of applications for such, culminates in the examination of the axioms for vector spaces and subspaces.

UICSM apparently does not recognize a need for the tools provided by consideration of vectors and, consequently, uses the concept of vector very sparingly. As a matter of fact, UICSM does not consider the term vector important enough to award it a position in its "index--table of contents" listing of topics and terms introduced. Therefore, where SMSG makes quite an issue of vector techniques, UICSM does not employ the tool with its textual developments.

Inasmuch as both programs make early and extensive use of the Cartesian coordinate system and analytic techniques, it is of interest to note the consideration of conics as evidenced by each of the programs. Although, as in many earlier-discussed areas, their approaches are not contradictory, they do evidence a difference in structure and emphasis.

SMSG, while confirming itself to nonslant conics, nevertheless pays explicit attention to the study of the

conics as well as their limiting forms and degenerates through the avenue of the exploration of the graphs of two-variable quadratic expressions. Although in early units the parabola is discussed informally as the graph of a quadratic polynomial in one variable and the ellipse as being the locus of points the sum of whose distances from two fixed points is constant, the SMSG approach to the conics is the focus-eccentricity-directrix one with a conic being defined as the set of points P with the property that the distance from P to a fixed point (the focus) is equal to a constant e (the eccentricity) times the distance from P to a fixed line (the directrix). If $e = 1$, the conic is a parabola; if $e < 1$, the conic is an ellipse; if $e > 1$, the conic is a hyperbola. Canonical (or standard) forms of the equations of the loci are developed for each of the conics and a multitude of exercises hopefully develop facility in finding the vertex, focus, and directrix of any given parabola; the eccentricity, transverse and conjugate axes, and the asymptotes of any given hyperbola; and the eccentricity, vertices, co-vertices, center, foci, and lengths of major and minor axes of any given ellipse.

UICSM, in its Unit 11 analysis of quadratics, pays explicit attention to the general quadratic of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ and examines various loci whose equations might lead to this form. Among such loci are those involving the locus of a point moving such that (1) the sum of its distances from two fixed points is

constant, (2) the absolute difference of its distance from two fixed points is constant, and (3) the distance between the point and a fixed straight line is the same as the distance between the point and a given fixed point. These curves are named ellipse, hyperbola, and parabola, respectively. The general shape of the curves are examined through the properties of symmetry with canonical forms for the equations being developed for nonslant conics. The discriminant test to determine the nature of a slant conic is developed and used for such analysis with slant conics being sketched by addition-of-the-ordinates techniques. UICSM's study of conics seems to arise as a side-result of the study of systems of equations and not as a result of explicit desire to illustrate the conics though the treatment is fairly comprehensive.

Both SMSG and UICSM emphasize the study and analysis of functions. Central to (and typical of) such analyses are their study of the logarithmic and exponential functions.

Early in the third year of study, SMSG introduces the logarithm function on the domain of positive reals by constructing the correspondence that for each $x > 1$, the logarithm of x is the area (used as an undefined term) of the region bounded by the x-axis, the hyperbola $y = h/x$, and vertical lines at 1 and x; for $x = 1$, the logarithm of x is 0; and for x such that $0 < x < 1$, the logarithm of x is the negative of the area bounded by the x-axis, the hyperbola

$y = h/x$ and the vertical lines at 1 and x. Values of such logarithms (for various positive values of h) are approximated by "intelligently guessing" the area under carefully graphed hyperbolas. The natural logarithm function is defined by choosing $h = 1$; the common logarithm function is one whose value is caused to be 1 when $x = 10$ by choosing $h = 1/\ln 10$. The standard computational algorithms are justified in view of these definitions. Although such discussion is very informal and largely geometry-based, the logarithm function (with $h > 1$) is demonstrated to be monotonically increasing and hence has an inverse. The inverse is defined to be the exponential function.

Early units have successively defined a^x for x positive integral, for x integral, and for x rational. Unit 21 of SMSG defined the exponential function $f: x \rightarrow ha^x$, $h > 0$, $a > 0$, for an arbitrary real x by informally "pinching down" on a^x, $m < x$, m rational, as m approaches x, and the equivalent limit of the sequence a^n, $n > x$, n rational, as n approaches x. This function, also either monotonic increasing or monotonic decreasing (or, as SMSG says, strictly increasing or strictly decreasing) has an inverse which is named the logarithm function.

UICSM's earlier units have defined x^h, h rational, through the use of the continued products sequence for h nonnegative and integral, through the reciprocal definition for h negative and integral, and for h rational by the

principle $\forall_{x > 0} \forall_r \forall_m$, rm $\in$ I, $x^r = (\sqrt[m]{x})^{rm}$. Unit 9 completely redefines the exponential function as follows:[1]

(4_1) $\forall_{x > 1} \forall_u x^u$ = the least upper bound of $\{y: \exists_{r < u} y = x^r\}$,

(4_2) $\forall_{0 < x < 1} \forall_u x^u = (1/x)^{-u}$,

and

(4_3) $\forall_u 1^u = 1$ and $\forall_{u > 0} 0^u = 0$.

UICSM therefore is forced (if its usual rigor is to be maintained) to establish (1) the existence of the least upper bound postulated in the first phase of the definition, (2) that the new definition is consistent with those already adopted, and (3) that the "laws of exponents" still hold for real number exponents. (The formal proofs of (2) and (3) are very sophisticated ones and are reserved for the appendix.) An equally sophisticated proof (also reserved for the appendix) insures the existence of the inverse of the function. This inverse function, the <u>logarithm function</u>, authorizes the standard algorithms for computation.

These approaches to the logarithmic and exponential functions somewhat typify the two programs. Although the subject matter is essentially equivalent, the precision of language and degree of rigor of UICSM is more pronounced than that of SMSG. SMSG, in its approach to logarithmic and exponential function, displays no explicit concern for a

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[1]University of Illinois Committee on School Mathematics, <u>Unit 9</u>, p. 92.

rigorous definition of continuity but rather is ready to accept a geometry-based argument to conclude that the graph of the function has no holes or gaps in it; UICSM demands that a fairly sophisticated notion of continuity be verbalized and incorporated into the proof. SMSG, in its "pinching down" on x^h, h irrational, is willing to accept the existence of a limit for the associated sequences; UICSM insists on rigorously proving that their utilized least upper bound exists.

SMSG's eleventh-grade study of complex numbers, motivated by the empty solution set of $ax^2 + bx + c = 0$, $a \neq 0$, with respect to the system of reals if $b^2 - 4ac < 0$, constructs an extension of the real number system so that, with respect to that extended system, every quadratic equation with real coefficients has a nonempty solution set regardless of the value of the discriminant. The student is led to seek a new number system which contains the system of real numbers with its familiar properties and also an element satisfying the sentence $x^2 + 1 = 0$. Such an extension demands the introduction of elements of the form $a + bi$, a and b real and $i^2 = -1$. The standard rules are deduced for calculating with these complex numbers with the introduction of the additive and multiplicative identities and inverses. Following such deductions, maximum usage is made of Argand's representation of the complex $a + bi$ as the point (a, b) on the Cartesian plane to yield a geometric interpretation to the sum and difference of two complex numbers and to verify associativity,

commutativity, etc. (An optional topic at the end of the chapter contains a completely abstract number-pair construction of the complex number system along with number-pair definitions for equality, sum, product, etc.) The SMSG consideration of complex numbers is climaxed by the presentation (though not a proof of) the Fundamental Theorem of Algebra and the "proof" of De Moivre's Theorem based on the polar form of complex numbers. Thus, SMSG utilizes three successive forms to express a complex number x: (1) $z = a + bi$, where a and b are reals and $i^2 = -1$, (2) $z = (a, b)$, an ordered pair of reals, and (3) $z = r(\cos \theta + i \sin \theta)$.

UICSM's treatment of the complex numbers, reserved until the last unit of study in the sequence, is motivated by the study of quadratics. The elements of the complex number system are initially defined to be ordered pairs of reals subject to certain restrictions, e.g., $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$, $(a, b) + (c, d) = (a + c, b + d)$, $(a, b) \times (c, d) = (ac - bd, ad + bc)$, $-(a, b) = (-a, -b)$, $(a, b) - (c, d) = (a - c, b - d)$, etc. The study of the properties of this system subject to these definitions precedes the introduction of $x + yi$ to symbolize (a, b) with the division of complex Z_1 by Z_2 defined to be $Z_1 \div Z_2 = Z_1 \times /Z_2$, where $/Z = / x + yi = (x/x^2 + y^2) - (y/x^2 + y^2)i$, authorizes the analysis of the solutions of quadratic equations and properties of the conics. No mention is made of De Moivre's theorem or the polar form of complex numbers and Argand's

representation is used very sparingly. UICSM, though entitling an entire unit as Complex Numbers, certainly does not grant the entire unit to the topic and the resulting treatment is less inclusive than that of SMSG's eleventh-grade treatment although reserved for the last topic of study.

Table 1 displays a partial list of the symbols, their referrents, and the respective units of entry of such symbols used in textual discussions and exercises by SMSG and UICSM, respectively. (The unit of entry designates the unit in which the symbol is first introduced.) Although some few of the indicated symbols are used sparingly, most of them are found frequently after their introduction. Examination of this partial list of symbols yields some insight into the nature of the usage of mathematical symbolism by the two programs. (Such commonplace symbols as +, -, ·, =, $>$, $<$, $\sim$, Δ, and $\cong$ are not listed.)

The UICSM texts rely more heavily than do those of SMSG on the symbols ordinarily used in logic and usually reserved for higher courses in mathematics, e.g., the universal quantifier $\forall$, the existential quantifier $\exists$, and the implies that and if and only if symbols $\Rightarrow$ and $\Leftrightarrow$, respectively. UICSM's symbolism is used to convey very detailed distinctions in a manner more precise than that of SMSG, e.g., UICSM distinguishes between the interval $\overline{a, b}$ (not containing a and b) and the closed interval $\overline{a, b}$ which contains a and b, the half line $\overrightarrow{AB}$ (not containing A) and the ray $\overrightarrow{AB}$ (which contains

TABLE 1

MATHEMATICAL SYMBOLS AS UTILIZED BY SMSG AND UICSM

Referrent	SMSG		UICSM	
	Symbol	Unit of Entry	Symbol	Unit of Entry
Absolute value of $\underline{a}$	$\|a\|$	9	$\|a\|$	1
Absolute value of $z = a + bi$	$\|z\|$	17	$\|(a, b)\|$	11
Arc with endpoints $\underline{A}$ and $\underline{B}$	$\overset{\frown}{APB}$	14	$\overset{\frown}{APB}$	6
Cardinal number of $\underline{A}$	--	--	$n(A)$	4
Cartesian product of sets $\underline{A}$ and $\underline{B}$	--	--	$A \times B$	4
Closed interval of reals $\{x: a \leq x \leq b\}$	--	--	(a, b)	3
Complement of set $\underline{T}$	--	--	$\tilde{T}$	5
Determinant of matrix $\underline{X}$	$\delta(X)$	23	--	--
Distance between P_1 and P_2	$d(P_1, P_2)$	17	$m(\overline{P_1, P_2})$	6
Domain of relation $\underline{R}$	--	--	$\mathcal{D}_R$	5
Empty set	$\emptyset$	9	$\emptyset$	3
Existential quantifier	--	--	$\exists$	5
Field of relation $\underline{R}$	--	--	$\mathcal{F}_R$	5
Half-line of number line ($\underline{a}$ is not an element)	--	--	$(\overrightarrow{a, b})$	3
Half-line ($\underline{P}$ is not an element	--	--	$\overrightarrow{PQ}$	6

TABLE 1--Continued

Referrent	SMSG		UICSM	
	Symbol	Unit of Entry	Symbol	Unit of Entry
Half-open interval of reals, $\{x: a \leq x < b$	--	--	(a, b)	3
If and only if	--	--	$\Leftrightarrow$	7
Implies that	--	--	$\Rightarrow$	7
Intersection of sets	$\cap$	21	$\cap$	4
Inverse of function f	f^{-1}	21	f^{-1}	5
Inverse of matrix A	A^{-1}	23	--	--
Is an element of	$\in$	21	$\in$	3
Is a subset of	--	--	$\subseteq$	3
Is approximately equal to	$\approx$	10	$\doteq$	3
Left-real a	--	--	$\overleftarrow{a}$	1
Line with elements P and Q	$\overleftrightarrow{PQ}$	13	$\overleftrightarrow{PQ}$	6
Line-interval (points between P and Q)	--	--	$\overline{PQ}$	6
Line segment with P and Q as endpoints	$\overline{PQ}$	23	$\overline{PQ}$	6
Matrix (m x n)	$[a_{ij}]_{mxn}$	23	--	--
Measure of arc	$m(\overset{\frown}{APB})$	14	$m(\overset{\frown}{APB})$	6
Measure of line segment	$m(\overline{PQ})$	13	$m(\overline{PQ})$	6
Negative of a	^{-}a	9	^{-}a	1

TABLE 1--Continued

Referrent	SMSG		UICSM	
	Symbol	Unit of Entry	Symbol	Unit of Entry
One-to-one correspondence	$\leftrightarrow$	13	$\leftrightarrow$	6
Open interval of reals, $\{x: a < x < b\}$	--	--	$\overline{(a, b)}$	3
Opposite of $\underline{a}$	-a	9	$-a$	1
Positive $\underline{a}$	--	--	^{+}a	1
Range of relation $\underline{R}$	--	--	$\mathcal{R}_R$	5
Ray of line (endpoint A)	$\overrightarrow{AB}$	13	$\overrightarrow{AB}$	6
Ray of number line (endpoint $\underline{a}$)	--	--	$\overrightarrow{a, b}$	3
Right-real $\underline{a}$	--	--	$\vec{a}$	1
Set	{...}	9	{...}	3
Transpose of matrix $\underline{A}$	A^t	23	--	--
Triangular region	--	--	$\blacktriangle$	6
Union of sets	$\bigcup$	21	$\bigcup$	4
Universal quantifier	--	--	$\forall$	2
Length of vector $\overrightarrow{AB}$	$\lvert\overrightarrow{AB}\rvert$	18	--	--
Vector with tip at $\underline{B}$ and tail at $\underline{A}$	$\overrightarrow{AB}$	18	--	--
Vector with components $\underline{a}$ and $\underline{b}$	[a, b]	18	--	--

A), and the triangle Δ and the triangular region $\blacktriangle$. Both programs are very careful, however, to distinguish between sets and their measures. Not only does UICSM introduce more symbols and utilize them to draw finer distinctions, the UICSM textual materials display a more prevalent usage of such symbols and seem never to use words to communicate notions, concepts, or operations which have been previously abbreviated by a symbol.

Both SMSG and UICSM structure an algorithm for obtaining the approximate square root of positive reals. The examination of their respective approaches to such an algorithm reveals another difference in the emphases and detail as advocated by the groups.

Both programs initially advocate what might be termed as "educated guessing" to approximate solutions, i.e., the "pinching-down" on the square root of a real by successive approximations involving a search process. Both groups present, as a refined tool, an iterative process illustrating that, if y_1 is an approximation to $\sqrt{x}$, then $y_2 = (y_1 + x/y_1)/2$ is an even better approximation, and, in general, $y_{i+1} = (y_i + x/y_i)/2$ is a better approximation than y_i for $\sqrt{x}$. The differences in the two treatments lie mainly in UICSM's attention in the student text to the degrees of approximation and the estimate of errors involved in the process. Fairly precise proofs of the convergence of $y_{i+1} = (y_i + s/y_i)/2$ to $\sqrt{x}$ are contained in the commentaries of

both programs but the UICSM textual treatment is more detailed than that of SMSG. SMSG essentially uses the "it can be shown" approach in its textual discussion with the student.

Both modern programs have subscribed to the belief that the study of primes and composites should be integrated into the high school curriculum both for its expository role in number theory and its utilitarian role in the operations on both arithmetic and algebraic fractions. SMSG's Unit 10, while paying proper attention to the domain of factorization and the Fundamental Theorem of Arithmetic (with no attempt at proof) uses the study of primes and composites primarily for operations on fractions and simplification of radical expressions as well as factorization of polynomial expressions. (The SMSG study of primes precedes that of factorization of polynomial expressions.) UICSM's Unit 4 study of primes and composites (along with the highest common factor and greatest common divisor), preceded by "trial and error" factorization of polynomial expressions, achieves its initial value in simplifying and facilitating operations on fractions. Unit 8, however, presents a more sophisticated look at some of the number theory aspects of primes and composites involving the Sieve of Eratosthenes, Euclid's proof that the number of primes is infinite, a discussion of Goldbach's conjecture, and culminating in a fairly rigorous proof of the Fundamental Theorem of Arithmetic. Therefore, it is evident that UICSM's treatment is more thorough and rigorous and emphasizes the

number theory aspects of the study of primes and composites more than does that of SMSG.

In that both programs devote entire units to the study of geometry, it is feasible to construct a comparison of the two efforts. SMSG devotes a full year to the study of geometry whereas UICSM dedicates only one semester to such a study per se.

SMSG's units on geometry reflect the conviction of the authors that the traditional content of Euclidean geometry serves as a significant part of the high school curriculum and that Euclidean geometry (though Euclid's postulates are not logically sufficient for geometry) rightly deserves the rather prominent place which it has traditionally held. Therefore, Geometry reflects the inclination on the part of the SMSG authors to make changes only when absolutely necessary. Since SMSG authors were of the opinion that the foundations of geometry are not a part of elementary mathematics and hence do not belong in the secondary curriculum, Geometry utilizes as its basic postulate scheme that proposed by G. D. Birkhoff rather than the more formal one advocated by Hilbert. This treatment, assuming the real numbers, makes the handling of the postulates a much easier task.

Geometry is a one year set-oriented study of geometry which, at every opportunity, connects geometry with algebra and, whenever feasible, treats geometric topics algebraically. In the main, the text is devoted to plane geometry and

includes all the topics of the conventional plane geometry texts although including a few chapters on solid geometry (to be discussed on later pages of this paper) and terminating with a brief, though intensive, introduction to analytic geometry. This introduction, involving slope, parallelness and perpendicularity, the distance formula, the midpoint formula, equations and intersection of lines, equations of circles, analytic proofs of selected theorems, etc., is not necessary to the continuity of the unit and contributes primarily as an enrichment addition to the unit. The proofs of the unit are primarily two-column proofs although one paragraph proofs are suggested.

UICSM Geometry optimistically presents a one-semester course including most of the topics common to high school courses in plane geometry. Since, in the eyes of UICSM, knowledge of the principles of logic and an appreciation of the role of the rules of reasoning in drawing valid inferences are prerequisites for the understanding of the nature of a good proof, an appendix on logic is included to be studied concurrently with the unit. UICSM's insistence upon good proofs causes the authors to insist upon pointing out the nonrigorous points of their proofs and to explore ways in which such "gaps" could be filled by the introduction of more sophisticated techniques. The one-column proofs and paragraph proofs are used to the uniform exclusion of the two-column proofs of traditional texts. The basic postulate scheme is

that of David Hilbert and the use of the real numbers to coordinatize lines, segments, rays, angles, arcs, etc., produces a highly algebraic treatment of plane geometry. The language of sets is used at every opportunity. An early introduction of analytic geometry (involving the distance formula, midpoint formula, slope, perpendicularity and parallelness, etc.) yields a tool which is utilized heavily in the textual development of a multitude of the theorems of geometry remaining to be proved in the unit and thus analytic techniques become a vital, integrated part of the unit. (This use of analytic tools, among other innovations, allows UICSM to cover in one semester all the topics common to a traditional plane geometry course.)

Therefore, SMSG and UICSM both present set-oriented, algebraic-flavored treatments which contain treatment of at least all the topics of the conventional high school plane geometry. Differences are noted however. SMSG uses the postulate treatment of Birkhoff; UICSM uses the postulate scheme of Hilbert. The SMSG units involve a treatment of some of the topics of solid geometry; UICSM restricts its treatment at this point to plane geometry. SMSG introduces analytics as a terminal study; UICSM introduces the tools of analytics early in its unit and structures it to be a vital, integrated part of the unit. SMSG utilizes the traditional two-column proof almost exclusively although occasionally using the paragraph proof; UICSM employs both the one-column

and paragraph proofs to the practically total exclusion of the two-column proof. SMSG, though insistent upon logical reasoning, devotes little direct attention to the rules of formal logic; UICSM devotes a lengthy appendix (to be studied concurrently with its unit on geometry) to development of some of the rules of logic.

The last several years have seen a decline in the popularity of solid geometry. This change in status of that subject is reflected in the work of both SMSG and UICSM.

Geometry, as published by SMSG, contains a few chapters on solid geometry which are inserted into the unit but which are basically isolated from the other chapters. The chapters concern themselves with such topics as lines and planes in space, perpendicular lines and planes in space, parallels in space, circles and spheres, and volumes of solids. This treatment of solid geometry is set-oriented, theorem-directed, and the proofs involved are of the same nature as those of the chapters dealing with plane geometry. Therefore, SMSG evidently feels that solid geometry should be a part of the secondary curriculum but as an integrated part of geometry rather than as a separate subject.

On the other hand, Geometry, as published by UICSM, has the standard study (though abbreviated and laden with innovations) of plane geometry but pays no attention to solid geometry. It is not until Unit 11 that there is found (in the appendix which probably would not be consumed if time

were short) an abbreviated development of the surface area-measure and volume-measure formulas for simple solids. The appendix does little more than give some nonrigorous justification for the formulas for the volume-measure formulas of simple closed regions, e.g., prism, cylinders, pyramids, cones, spheres, etc., and the area-measure formulas of such regions. The approach is largely axiomatic with no theorems to prove and no theoretical observation demanded. It appears, therefore, that UICSM advocates that traditional solid geometry *per se* has little merit in the secondary curriculum.

During the past several years, severe criticism has been directed from some quarters toward the undue emphasis upon triangle-solving, tedious logarithmic computations, etc., in the conventional courses in trigonometry. Both the UICSM and SMSG trigonometries demonstrate a reaction which should be pleasing to such critics.

SMSG does not devote a single unit nor even the composite of a semester's work to the exclusive study of trigonometry. The student's first acquaintance with trigonometry comes in Unit 18 with a four-week, 80-page study based on the definition of the sine and cosine function of a real number x defined as the abscissa and ordinate, respectively, of a point P located on a unit circle centered at the origin and such that the counterclockwise arc (measured from the intersection of the circle with the positive x-axis) has a measure x. This particular definition, along with suitable

refinements, serves to authorize the study of the graphs of the trigonometric functions, the law of sines, the law of cosines, and the addition formulas as well as an impressive amount of work with equations and identities. Radian measure for angles is used as often as degree measures. The presentation also looks (though does not emphasize) at the right-triangle definitions for the trigonometric functions.

Unit 21 returns to the study of trigonometry with a five-week, 75-page circular-function-oriented treatment also based on the wrapping, or winding, function. Both radian and degree angle-measures are commonplace. In addition to studying identities and arc functions, SMSG develops slope functions for the sine and cosine functions. In summary, the merging of the two SMSG segments involving the study of trigonometry would produce a unit displaying very few really dramatic differences between this and the conventional trigonometry course if the usual chapters on solution of right triangles and logarithmic computation were neglected. These omissions, plus precise language, abundant usage of radian measure, and basic initial winding function definitions of the trigonometric functions would be the most noticeable differences.

UICSM, though introducing right-triangle definitions of the trigonometric functions in Geometry to solve right triangles mainly as an illustrative tool of ratio and proportion, generally reserves the study of the circular functions

for a complete unit of work to be consumed during the senior year. As did SMSG, UICSM utilizes the winding function to structure angle-free definitions for the trigonometric functions sine and cosine and then uses these two functions to define the other four functions. The Law of Sines and the Law of Cosines are proved analytically. Right triangles are solved through consideration of their being a subset of the oblique triangles. The solutions of triangles is minimized. The unit, complete with the addition and subtraction formulas, double- and half-angle formulas, inverse functions, and identities and equations, is, as was SMSG's treatment, very comparable to that of the conventional trigonometry course insofar as subject matter is concerned although the basic functions are defined differently, radian measure is used heavily, computations with trigonometric functions and solutions of triangles (particularly right triangles) are minimized, and the entire unit is function-oriented in its approach.

Throughout its entire sequence, SMSG has displayed a willingness to accept informal proofs of many of the theorems which UICSM has explained with rigor and a high degree of sophistication. Likewise, SMSG has been willing to sacrifice thorough and extended discussions of several topics, e.g., sequences, relations, induction proofs, limits, continuity, and existence theorems regarding bounds, whereas UICSM has insisted upon comparatively sophisticated (and necessarily

time-consuming) examinations of these areas. Therefore SMSG has available a block of time to devote to an introduction to matrix algebra. This is probably the major content innovation of SMSG (as opposed to UICSM) in that practically all of the material in this one-semester unit is unique to the SMSG program.

The subject matter content of the unit involves the study of matrix operations and the resulting properties, the algebra of 2 x 2 matrices, matrices and linear systems, representation of column matrices as geometric vectors yielding a matrix interpretation of the algebra of vectors as well as the study of vector spaces and subspaces, and transformations of the plane. The development of the unit creates opportunities for the consideration of the axioms for fields, groups, and rings. As operation after operation is defined, the structure of mathematics is repeatedly emphasized and terms such as group, ring, field, and isomorphism are introduced when feasible to illustrate unifying concepts.

Attention to Mathematical Structures

For many years high school mathematics has consisted largely of the study of models or applications built upon mathematical systems. The properties of such systems certainly apply to the models but the properties of the system may be fairly obvious in the abstract system and yet hidden by the nature of the physical objects named in the models,

by an overt concern with the achievement of manipulative skills authorized by the model, emphatic attention to social applications, etc. The properties of mathematical systems are fundamental and enduring whereas the models or applications thereupon built may change as social needs vary. Therefore, advocates of modern mathematics programs have shown a concern for the necessity of promoting the teaching of the study of mathematical structures. Such a study of structures involves the study of the basic principles or properties of any system as well as the analysis of the basic properties or principles common to different (and perhaps all) mathematical systems. These mathematical systems are not necessarily number systems.

SMSG's First Course in Algebra reflects a serious attempt to communicate some appreciation of the structure of mathematics. In its initial study of the arithmetic numbers, SMSG shows the "doing and undoing" relatedness of addition and subtraction and of multiplication and division; displays the interrelatedness of the "four fundamental operations" by using the number line and by reviewing the operations defined in earlier grades; and adequately displays the commutative, associative, and distributive laws as they apply to the fundamental operations. The introduction of the real number system (in the same year of study) is characterized by the same treatment. The extension of the system of arithmetic numbers to the system of reals very definitely conveys the notion

that the familiar system of arithmetic numbers is a subsystem of the reals and shares certain displayed properties with that more inclusive system. Therefore, SMSG's first two units of study, by both the presentation of leading questions and through verbalization, give the student the opportunity to gain some glimpse of the basic structure of the two systems.

SMSG's Geometry effectively destroys the somewhat prevalent, yet erroneous, impression that geometry and algebra are completely disjointed realms of study. The early introduction and use of the reals to coordinatize lines, rays, segments, angles, etc., with the Ruler Postulate and the Angle-measure Postulate, etc., authorizing the precise use of geometric figures (which are sets of points) and measures (real numbers assigned to those sets), make a gigantic stride toward demonstrating the basic unity of mathematics. Since SMSG connects geometry with algebra at every reasonable opportunity, knowledge in either one of these areas should make a natural contribution to the understanding of both. However, an earlier and more inclusive use of analytic tools might have helped to create more of a feeling of a displayed unity of mathematics and have heightened the desired impression that geometry and algebra are interrelated and inseparable. Similarly, this impression could have been heightened by the utilization of fewer two-column proofs (as used by most conventional geometry texts) in favor of more one-column and paragraph proofs as used in preceding and succeeding units.

Approximately 25 to 30 per cent of the textual development of SMSG's Intermediate Mathematics deals specifically with the structure of mathematical systems, and the explicit attention paid to the study of mathematical structures plays a key role in the two units. The natural numbers, the integers, and the rational numbers are reviewed from the point of view of their basic structural properties. The real number system is characterized by its basic properties with the complex number system being developed after a brief, but inclusive, section outlining the conditions required for the extension of a number system. (This "extension of system" approach involves displaying the natural numbers as being a subset of the integers, the integers as a subset of the rationals, the rationals as a subset of the reals, and the reals as a subset of the complex numbers, and should again serve to convey the impression that each system is a subsystem of a still higher system and shares certain exhibited properties with that higher system.) Simplification of algebraic expressions, derivations of truth sets of equations and systems of equations, etc., are focused on the properties of the number system under consideration. The chapter on vector algebra, including a discussion of the assumed isomorphism between the system of vectors and system of physical forces, and between the system of vectors and the system of complex numbers with respect to certain operations, approaches the topic from the structural point of view. The last chapter

includes discussions of noncommutative as well as commutative groups and fields. SMSG's Elementary Functions continues the study of structures through its function-oriented study of polynomials, exponents, logarithms, and trigonometry.

SMSG's crowning effort in exhibiting the structure of mathematics lies in its Introduction to Matrix Algebra and its study of many unifying concepts repeatedly emphasizing the structure of mathematics. Knowledge of the real number system is reinforced through a comparison with the characteristics and properties of matrix addition and multiplication. Some of the differences between the real number system and that of matrices, with their respective additions and multiplications, are illustrated by the differences between a field and a ring. The unit contains many examples of abstract sets along with their operations and the student is required to check those systems for the simple structural properties such as closure, commutativity, associativity, etc., and to determine whether or not they are fields or rings. The concept of group, reserved until after the study of rings and fields, is introduced through the study of 2 x 2 invertible matrices. The unit continually points out ties between newly-introduced structural systems and ones with which the student is already familiar. A correspondence between complex numbers and matrices is used to illustrate again the concept of isomorphism. After the development of such studies, the structure of general algebras is discussed.

In this particular unit, students can view the growth of comparatively simple 2 x 2 matrix theory into a system as complex as vector spaces.

As was the case for SMSG, the UICSM textbooks reflect a fervent belief that the study of mathematical structure is essential if a better understanding and use of mathematics is to be achieved. One of the underlying objectives of the UICSM program is the development of the real number system and an examination of its structure although many other systems are studied in the process. The fact that UICSM treats in a sense its entire sequence as a single textbook with the separate units playing the role of chapters does much to establish continuity of presentation and allows an increasingly sophisticated spiral treatment of the real number system.

Units 1-4 contain a rather informal, yet detailed, study of the real number system and the properties of that system along with precise statements of principles and theorems. The units develop the system of reals as a self-contained system complete with elements, operations, relations, and rules which are defined prior to formal review of the system of arithmetic numbers. The subsequent study of the properties and principles of the arithmetic numbers leads UICSM to demonstrate that the set of arithmetic numbers is isomorphic to the set of nonnegative reals with respect to their respective operations of addition and multiplication.

Thus, the nonnegative reals and the numbers of arithmetic, though members of different systems, share common properties such as associativity and commutativity of addition and multiplication, distributivity of multiplication over addition, etc., with respect to their operations. The study of inverse operations demonstrate the interrelatedness of addition and subtraction and of multiplication and division. Even Unit 1, early in the first year of study, evidences careful attention to the development of the properties of the system of rational reals. Unit 5 is characterized by set theory and set theoretical concepts. The ideas of relations and functions are developed in terms of sets and, as they apply, application of these concepts and further topics of traditional second-year algebra are interwoven. The approach of the unit stresses a few underlying ideas (set, relation, function) and a careful exploration of the properties of these general ideas.

UICSM's Geometry indicates a conscious attempt to construct an algebraic treatment of geometry. As did SMSG, UICSM introduces the reals to coordinatize lines, angles, etc., makes a valiant attempt to separate sets and measure of sets, treats geometry from a unifying point-set approach, and introduces algebraic processes whenever feasible. The unit is characterized by the early introduction and heavy reliance upon analytic tools to establish theorems traditionally established by two-column synthetic proofs. The one-column and

paragraph proofs lend even more informal support to the basic unity of algebra and geometry.

Units 7 and 8 complete the development of the real number system which was begun in Units 1 and 2, with the exception of the principle of completeness (which is reserved for Unit 9). Involved in this effort are formal rigorous discussions of the basic principles of the reals as well as the study of the subsystems of positive reals, positive integral reals, integral reals, and rationals. The basic scheme involved in the study of these systems involves the analysis and enumeration of those basic properties possessed by the positives and not possessed by the nonpositives, the properties possessed by the positive integral reals and not possessed by the nonpositive integral reals, the properties possessed by the integral reals and not possessed by the non-integral reals, etc. Imbedded in such examinations are the necessary examinations and enumerations of the properties shared by the various subsystems.

Complex Numbers constructs the system of complex numbers along with its defined operations such that the system of real numbers is isomorphic to a subset of the system of complex numbers with respect to their respective operations of addition and multiplication. The majority of the entire unit is dedicated to the study of the properties and principles of the complex number system.

UICSM's primary concern with the development of the real number system makes little use of vectors, matrices, fields, rings, groups, vector spaces and subspaces, etc., to demonstrate the properties common to different systems and to illustrate unifying concepts although many exercises and some textual attention is directed toward examination of abstract systems. Although SMSG makes heavy use of such systems, UICSM's comparative neglect of such is somewhat compensated by the study of the system of numbers of arithmetic being isomorphic to the system of nonnegative reals and the system of reals being isomorphic to a subset of the system of complex numbers along with consideration of the abstract and subsystem. UICSM's emphasis on proof makes it evident that the logical structure of mathematics is paramount.

Methods

Both SMSG and UICSM have insisted upon an important qualification to the study of secondary mathematics--the modern student must <u>understand</u> his mathematics and be enabled to discover generalizations for himself. Both groups further advocate that the student will understand mathematics better if he plays an active role in the development of mathematical ideas and procedures. This insistence has been reflected in practically every aspect of the two programs but probably more highly than elsewhere in the manner in which the texts have been written and arranged and the role assigned the

exercises. Although no text can within itself establish and dictate the method to be used in any particular situation, the orientation of the presentation can make some methods more feasible than others.

The critique of SMSG materials as presented in Chapter 2 of this paper has illustrated that First Course in Algebra develops most of its topics by using the discovery approach imbedded in the textual discussions. However, some of the exercises are independent of preceding problems and, in many instances, there is a minimum of evolution of ideas throughout the sequence of some of the problem sets. The review exercises at the end of the chapters and the "leading" discussions preceding the problem sets compensate, in some degree, for the difficulty. In Geometry the students continue to be participants in the development of the material and are led by the authors through the intuitive processes that establish conjectures, and then to construct proofs. The textbook exercises are well selected to cause student involvement in the system rather than to present a system to be used. In Intermediate Mathematics, the authors appear to be content to present material for student consumption rather than student participation in content development, i.e., the student is, in a sense, "lectured to" by the textbook. The style and the format of the text seem to suggest the presentation of the final product to the student with exercises built to fit precisely the concepts developed in textual materials

although exceptions are noted, e.g., the Challenge Exercises which, though not immediately related to textual materials, are not included for the purpose of developing basic material. (This is not to imply that the discovery approach is neglected and that understanding is minimized but rather that the student, while reading the textbook, is more likely to "follow" the formulation of a concept rather than to independently construct such.) Elementary Functions, written in the same general vein as Intermediate Mathematics, forces the student to recognize basic principles and to develop necessary proofs and the text goes far beyond simply stating the principles and how to apply them. In Introduction to Matrix Algebra, the basic theorems, for the most part, are proved for the student with the student expected to prove auxiliary theorems and to solve problems based on the already proved theorems. The attention to the discovery process is illustrated by the fact that many of the problems seem to have involved in them some of the necessary and basic concepts used in solving subsequent exercises. Many of the concepts to be developed rigorously in subsequent discussions are developed in an intuitive fashion prior to their rigorous formalization.

An overview of the entire SMSG sequence authorizes several conclusions regarding the methods utilized by the writing team. Although SMSG is thoroughly dedicated to the role of student discovery and, consequently, dedicates itself to the promulgation of such, the great bulk of the textual

materials, i.e., the mathematical content, is presented by means of textual exposition which leads the student through concept discoveries. These expositions are far superior to those normally found in conventional texts. They are written in suitable language for students at the particular grade level for which they are intended and the language, though not unwieldy and overly technical, is clear and precise from the mathematical point of view. Although the exercises present a multitude of opportunities for student exploration, the exploration exercises are, in the main, serving an enrichment function while their omission would detract little from the continuity of the materials.

For the most part, new concepts are introduced by first citing examples, then making conjectural generalizations, and then following with a deductive proof. In most instances, the generalizations are given to the student in the textual presentation but usually near the end of the discussion so that, as the presentation develops, the student still has a chance to come to the desired conclusion on his own. Often, several concepts are being discussed and generalized simultaneously. The desired generalizations are usually verbalized immediately after formulation.

Although both the SMSG exercises and textual materials often raise or border on questions that are not to be resolved until a later date, the SMSG materials generally have the exercises independent of the development of the textual

content sequence. Although these exercises concern themselves with the content material of the textbook, they seldom involve an extension of the mathematical theory being exposed but rather serve to clarify and intensify the student appreciation of such.

Very few of the chapters or topics begin with the introduction of definitions to be utilized in the study but, rather, are preceded by a study of the type of situation in which the intended definition will function. The definitions are not "thrown out to the students" but usually are preceded by the analysis of the reasons why a particular definition is made in a particular way, e.g., the definitions of the arithmetic operations as defined on the reals are preceded by the enumeration of the desirable characteristics to be imbedded in the definitions.

As was the case for SMSG, UICSM has as one of the fundamental concepts of its program the value attached to the principle of student discovery. UICSM displays, through its written materials, the belief that the learning process is deepened by providing a sequence of activities from which a student may independently recognize some desired segment of knowledge. UICSM's belief that the student will understand mathematics better if he plays an active role in developing the material causes UICSM to attempt to design both textual exposition and exercises in such a way that the student will discover principles and rules. The UICSM presentation,

though dedicated as was SMSG to the discovery method, reveals several different emphases in its approach.

Among the most noticeable features of the UICSM program are its exercise sets. UICSM introduces two general types of exercises as characterized by their usage.

UICSM's general exercises, usually immediately following textual expositions, serve the primary roles of reinforcing the concepts already gained (whether verbalized or not) and, in many instances, to serve as vehicles for the extension of the mathematical theory already studied. The results of such exercises often are used as theorems for later textual proofs and, consequently, as the basis for further theorems. (In these instances, these exercises must not be omitted in that the omission would detract from the continuity of the sequence.) UICSM's texts, supplying a tremendous volume of exercises and including long lists of supplemental problems at the end of each unit, are so constructed that the exercises are usually directed toward the mathematical material immediately at hand and often involve student generalizations and proofs of concepts not previously verbalized.

Insofar as method of presentation of mathematical concepts are concerned, the importance of the UICSM Exploration Exercises can hardly be overemphasized. These exercises, quite prevalent in all UICSM units, are constructed to produce student curiosity and awareness regarding the concepts to be studied later. The results of these exercises are

often incorporated into defining principles and theorems to be proved in later topics. These exercises serve also to develop an awareness of the inadequacies of past proofs (insofar as mathematical rigor is concerned) and/or mathematical systems. These Exploration Exercises, a vital part of the UICSM textbooks, definitely encourage discovery and insist on the student's playing an active part in developing and inventing mathematical ideas and procedures. (As a matter of fact, it is often quite difficult to locate the point at which the textual exposition begins and the Exploration Exercises cease.)

UICSM's textual expositions are, as are those of SMSG, vastly superior to those of conventional texts. The text, much more verbose than its conventional counterpart, and characterized by mathematically precise language, is laden with student-directed questions which are concerned with the topic immediately at hand. The treatment is insistent upon student discovery, understanding, generalization, and verbalization as evidenced by the demand that the student discover (and formulate for himself) the "rules" for the arithmetic operations of addition and multiplication on the reals, the "rules" for solving equations and inequations, and the "rules" for manipulating algebraic expressions, etc. These, and other, discoveries are guided by the text but the student is directed by well-constructed guideposts rather than led by the textual exposition.

Associated with UICSM's insistence upon the key role of discovery is UICSM's demonstrated belief that it is not necessary to require a student to verbalize his discovery to determine whether he is aware of a rule or concept and that sequences of properly constructed exercises can determine whether or not the desired awareness is present, e.g., UICSM students add real numbers on page 9 of Unit 1 and yet do not state (or even see stated) a "rule" for such until some 170 pages later in spite of the fact that the nonverbalized "rule" has been used extensively by the student in the meantime. (It is also difficult at times to distinguish between the "textual" material and the exercises in that the reading and understanding of the textual materials are as involved as the exercises.) Somewhat related to the process of discovery is the UICSM insistence that students become aware of concepts before assigning names to the concept. As mentioned earlier, UICSM predicates its entire program upon student participation, development, understanding, verbalization, and generalization and the discovery method as utilized by UICSM could be described as <u>experimentation</u>, <u>observation</u>, and <u>generalization</u> with the student being fully active in the three areas.

<u>Vocabulary</u>

Any student of mathematics realizes that the usage of precise nonambiguous language in the communication of

sophisticated concepts is of paramount importance since mathematics requires precision in language for exaction and clarity. Some disagreement is noted, however, in the level at which generally unsophisticated language can be replaced by sophisticated and precise use of the vehicle. Both UICSM and SMSG programs reflect their cognizance of the vocabulary and language problems which have developed through the years in mathematical curriculum. Their ensuing emphases and belief differ appreciably, however.

A retrospective examination of the language and vocabulary used by SMSG reveals a precise and utilitarian terminology in spite of the fact that there is no undue explicitly-directed emphasis upon precise and sophisticated language. The student is encouraged to learn to write mathematical statements that convey the information which they were designed to convey. In most instances, the new terms to be defined are discussed intuitively prior to their formal definition but are used precisely after their first formal introduction. The resulting statements of definitions (and theorems) are exact; the exactitude does not detract from the utilitarianism of the definitions (and theorems). The textual developments are such that definitions are precise without too much length, e.g., an angle is defined as the union of two non-colinear rays having a common endpoint--a definition which is authorized by a carefully constructed textual background.

Mathematically-sophisticated concepts whose abstraction is such that their formal statements are thought to be beyond the comprehension of the student of a particular unit are used, although only informally explained, in a manner serving to present misunderstandings although the concepts are never formally verbalized, e.g., the concepts of limit and continuous. All theorems are accompanied by the pertinent quantifiers although such quantifiers are usually imbedded in the textual discussion rather than in the body of the theorem, e.g., a chapter may indicate at the beginning that all the theorems of a particular chapter will involve a certain domain of application.

Although many terms are used which probably would not have been used so frequently in their conventional counterparts, SMSG seldom invents or "coins" new terms to identify new ideas and concepts but instead uses the terms which are commonly accepted by mathematicians. The general vocabulary in any particular unit seems to be at the level appropriate to the other academic subjects studied at the same period. Extensive use is not made of symbols and, although the language of sets is used extensively, very little of the associated symbolism is introduced and used generally.

As a whole, the SMSG vocabulary does not appear to be overly sophisticated and would appear to be fully intelligible to students properly placed in the course. The language and vocabulary are not particularly unique to the SMSG texts and

even a student entering the program in a unit late in the sequence would probably suffer very little in this respect.

UICSM's student-discovery-oriented textbooks implicitly postulate that student understanding will be achieved when he uses nonambiguous language and when the student is enabled to discover generalizations by himself. UICSM advocates that mathematical discoveries are facilitated by the formulation of precise descriptions and the accompanying skill in the precise uses of language helps the student to give clear expression to those discoveries. It is apparent that UICSM is dedicated to the belief that students at the secondary level may appreciate and incorporate into their communication process a precise mathematical vocabulary.

UICSM pays careful attention to the destruction of the ever-present ambiguities which are common to secondary materials. Discussions in great detail are directed toward the number versus numeral, the use versus mention of symbols, the geometric set versus its measure, and, in general, the name (or measure) of a described entity versus the entity itself. The usage of such distinctions is uniform throughout the textual expositions and all the exercises.

Not all the language, particularly in the earlier units, is standard. UICSM quite often "invents" terms to describe notions and concepts in such a way that the description somewhat conveys the use to which the notion and concepts are to be directed, e.g., the pronumeral, the operations

of sameing, oppositing, negativing, positiving, abbreviating, and unabbreviating. These terms are usually replaced by their more conventional counterpart in later units, e.g., the pronumeral becomes a numerical variable.

UICSM's materials show a concern for precise and subtle distinctions, e.g., the interval whose endpoints are A and B, respectively, designates the set of points between A and B in the line $\overleftrightarrow{AB}$ whereas the segment whose endpoints are A and B, respectively, designates the interval unioned with its endpoints, and carefully reflects these distinctions in their language. The materials also represent the belief that the meanings of very few terms should be "taken for granted" as being appreciated by the student, e.g., whereas SMSG essentially assumes the meaning of relation and operation, UICSM devotes several pages of material exclusively to these concepts.

The definitions, usually preceded by intuitive discussions of the concepts to be verbalized, are precise and sophisticated and often very verbose in their statement due to the inclusion of all the qualifying conditions. UICSM provides a vast number of opportunities for the student to construct rigorous definitions and to state precisely-worded theorems on his own initiative and as a vital part of his mathematical development.

The generalizations and principles (or theorems) are always accompanied (as an integral part of such) by the

quantifiers listing their domain of application. More involved topics such as composition of functions, intersection of relations and functions, etc., can be discussed due to UICSM's constantly applied vocabulary of basic set concepts and relations and functions.

In addition to the precise and sophisticated language used as a tool, a compact mathematical symbolism is introduced and used constantly to develop the textual exposition. From the very beginning, UICSM introduces and uses such symbolism to its fullest and seldom uses the "verbal form" to describe a notion previously named by a symbol.

Proof

Both SMSG and UICSM programs take proper consideration of the role played in modern mathematics by good crisp proofs. Both programs devote serious attention to the role of the definition, the undefined term, and the axiom (or postulate) in any mathematical proof and also are emphatic in insisting that the "truth" of a mathematical "fact" is relative to the system under consideration. In both programs, the proofs are much more rigorous than in their traditional counterparts.

The critique of the SMSG materials as presented in Chapter II of this paper has pointed out several characteristics of this program regarding its approach to proofs. SMSG's First Course in Algebra uses heavily the number line in the

development of the properties of the real number system and establishes a few simple generalizations based upon such although it is not until Chapter 6 that the student sees his first deductive proof. Chapter 7 introduces the student to inductive and deductive reasoning and challenges him to provide reasons for the steps displayed in a text-structured proof. Chapter 8 provides the student with proofs for the existence and uniqueness of identities and inverses. The student himself proves independently only very few theorems in the first year although SMSG is optimistically hopeful that he, in the meantime, is developing an appreciation for the process. The proofs in Geometry, highly algebraic in flavor in many instances due to the metric approach, are generally complete and rigorous. Little attempt is made to camouflage the "gaps" in the proofs but rather to direct attention to these "gaps" and promote student awareness of such. The student is forced into the construction of proofs as soon as possible. (This nonattempt to "gloss over" temporarily inadequate proofs beyond the level of the materials but rather to call attention to these inadequacies is a mathematically pleasant characteristic of these and other SMSG units.) The remaining units of the SMSG sequence present increasingly sophisticated proofs climaxing in the demand for very thorough, precise proofs in the last units. SMSG devotes little time to the explicit discussion of the nature of a

good proof but rather illustrates such through the textual examples and discussion.

A student having studied the entire SMSG program will have studied and become acquainted with the various types of direct and indirect proofs although he will not have studied explicitly the formal logic involved in good proofs. No attempt is made to stereotype the proof forms--particular methods of proof are used interchangeably as deemed feasible and economical by the circumstances involved. Some are precise, to the point, and arranged in an orderly format whereas others are presented in an informal, paragraph proof. In some instances, particularly in the later units, the concepts which will be handled fairly rigorously in a subsequent discussion are presented in an informal fashion in the exercises which should serve to help the students discover ideas to be used in the later proofs. The number line, the number plane, and graphs are used whenever feasible to structure informal proofs. Algebraic techniques are used fairly extensively in Geometry and analytic techniques are used frequently in Intermediate Mathematics and Elementary Functions. Indirect proof, though utilized in some cases, is used rarely. The "proofs" of several concepts which can be proved most economically by induction are saluted by indicating that the truth of such concepts could be shown--as a matter of fact, proofs are relatively rare in the SMSG treatments. Much attention is

directed towards the difference in the one-proof "If p, then q" theorems and the two-proof "p if and only if q" theorems.

A cursory examination of the proofs imbedded in the UICSM programs is likely to create the impression that the UICSM proofs are vastly more sophisticated and rigorous than those of SMSG. A more thorough examination reveals, however, that much of the apparent rigor is more a reflection of the sophistication of the subject matter content than of the proof form per se. As an example, UICSM insists upon detailed study (along with explicit verbalization) of such concepts as limit, continuity of functions, monotonicity, etc., whereas SMSG merely devotes relatively informal attention to such concepts. It follows, necessarily, that proofs utilizing the equivalent of the (ϵ, δ) definitions for limit and continuity will appear relatively abstract and overly unwieldy in secondary texts. The nature of the proofs in UICSM is influenced somewhat by the more inclusive use (as compared to SMSG) of the number-pair approach to functions and relations and the use of general sets rather than subsets of the reals for the domain and range of such relations and functions.

Some differences other than these are noted which are reflected particularly in the emphases placed in certain proof forms as well as the point of entry of rigorous proofs. The UICSM student is expected to make fairly rigorous proofs independently by the end of the first semester of study and to formulate simple proofs even earlier. The UICSM texts display

an early attempt to place the burden of the proof on the student. Geometry demonstrates a high regard for the rules of formal logic and studies such simultaneously with the subject of geometry. The proofs, most of which are paragraph and one-column proofs, for the remaining units are precise and rigorous and devote, as did SMSG, considerable attention to displaying the "gaps" in the proofs. Methods are suggested by which these "gaps," considerably fewer in number than in the SMSG units, might be filled in future studies.

The UICSM materials present more proofs regarding existence and uniqueness than do their SMSG counterparts. The proofs, being less graph-oriented, are more verbal and thorough than those of SMSG. UICSM, after having devoted an entire unit to the process of induction and topics which can economically utilize that process, makes frequent use of proof-by-induction in later units. As did SMSG, UICSM avoids any stereotyping of proof form and uses any particular type of proof which is feasible and economical, e.g., the tools of analytics are used extensively in Geometry to prove theorems traditionally established by the synthetic process. UICSM certainly provides opportunities for the student to practice the preparation and presentation of rigorous mathematical proofs. The concept of rigorous proof seems to be central to the development of the ideas presented in the UICSM units.

Concepts and Skills

The mathematics curriculum has long been a center of controversy in that proponents of various schools of methodology have disagreed as to the relationship which should exist in the mathematics programs between the function of concept development and skill development in the manipulation of symbols. Some have advocated that the student gains the full meaning of a concept only when he approaches the automatic response in the use of the concept while others have argued that such a response is not necessary for student understanding of mathematical concepts. Opponents of modern programs have insisted that such programs have neglected the development of skills while paying undue attention to mathematical structure, generalizations, and proofs. At the present time, no really exhaustive, statistically-valid study has been conducted (due to the vast number of variables involved) which will indicate the validity (or nonvalidity) of such claims although some studies have been conducted, e.g., the studies carried out by the Educational Testing Service and the Minnesota National Laboratory have shown that SMSG students, generally speaking, did at least as well as would be expected in achievement.[1] Whether these treatments are adequate is a question which can be answered only after further

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[1]School Mathematics Study Group, Newsletter No. 10 (Stanford: Leland Stanford Junior University, 1961).

testing and evaluation of the progress of the students involved.

An examination of the philosophy of the authors as well as the textbooks of the SMSG reveals that the development of concepts occupies a more prominent role in their program than their conventional counterparts. This does not preclude the development of manipulative skills in that SMSG insists that the acquiring of skills and the development of understanding of basic concepts can grow together.

Chapter II of this paper has suggested that the textual discussions, considerably more verbose than their traditional counterpart, seldom verbalize a "rule" until at least that usage has been recognized by the student. In only a relatively few instances does one find a displayed example of a detailed step-by-step problem-solving technique. Rather, the student is encouraged to analyze any given problem situation and to choose the most feasible method, either direct or indirect, for its solution and to be able to make mathematically feasible each step of the chosen technique. In short, the SMSG emphasis upon drill is upon meaningful drill and demands that students "earn the right" to manipulate symbols.

Although admittedly statistically insignificant, an examination of the SMSG exercises reveals several interesting facts. If one identifies a "drill" exercise as one which follows directly and without mathematically theoretical extension from the textual discussion, previous exercises or

examples, and a "discovery" exercise as one in which the student is compelled to extend the theoretical results of the textual discussion or to verbalize (or, at least, utilize) an independently-derived conclusion, corollary, or generalization, the following results appear.

First Course in Algebra contains approximately 73% drill and 27% discovery exercises; Intermediate Mathematics presents approximately 88% drill and 12% discovery exercises; Elementary Functions contains approximately 80% drill and 20% discovery; and Introduction to Matrix Algebra displays approximately 70% drill and 30% discovery. (Due to the arrangement of the exercises, each of the component parts of a numbered exercise was considered an exercise if the authors merited it important enough as an entity to assign an identifying letter to that part; the miscellaneous exercises and review exercises were not tabulated. This classification does not, in any sense, attempt to analyze the comparative difficulty and/or length of the individual exercises. The exercises in Geometry were not tabulated due to the fact that the stated main function of these units is the development of concepts.) This data would certainly indicate a preponderance of drill exercises but it must be noted that many of these exercises are actually instruments to provide drill for discovery exercises in which the student has independently recognized and formulated the working principles. SMSG, in its effort to develop both concepts and skills, seems to have

devoted less space than its traditional counterparts to the acquisition of skills and has placed vastly more stress on the development of basic concepts with the optimistic belief that the resulting meaningful drill will be successful in developing the necessary skills. In the same sense, the study of the exercises reveals that student-directed discovery exercises are followed by at least several exercises which should serve to reinforce the involved concepts, i.e., in all instances, problems provide for the development of skill of translation as well as for furthering the understanding of the concepts involved.

The critique presented in Chapter III of this paper as well as the statement of the guiding philosophy of the UICSM in this chapter have indicated that the entire UICSM approach has been predicated on the notion that the student is more likely to remember that which he discovers for himself and that, therefore, the discovery approach is the proper one for students. UICSM has further recommended that its materials are to be pursued at a rate compatible to the capacity of the students involved and these materials reflect a conscious effort to maintain continuity of mathematical experiences.

The textual materials, considerably lengthier in discussion than their traditional counterparts, continually lead a student from notion to notion by forcing that student to justify practically every sequential step from derived or accepted principles with the explicit emphasis being on

understanding of concepts. As was the case for SMSG, the UICSM materials seldom demand the association of a particular response with a particular stimulus and, in actuality, appear to avoid any stereotyping of any procedure. Certainly very few algorithms or "rules" are developed. The UICSM materials do exhibit many examples (structured in detail and more numerous than SMSG) for student consumption but the number and type is insufficient to lead the student to work the majority of the exercises. The materials are so written that, on occasion, it is impossible to separate the textual material from the exercises and hence the perusal of the textual materials is a requisite for the exercises, and conversely. As also evidenced by SMSG the UICSM authors demand a variety of proof forms with a total disregard for uniformity of method of proof but rather base the appropriateness of a given proof form upon the feasibility and economy of that usage.

Although a well-defined trichotomy does not exist, the UICSM exercises might be classified as either "drill" (following directly and in the same manner as an exhibited example or previously examined exercise), or "discovery" (forcing the student to extend the theoretical results of the textual discussions or to verbalize or utilize an independently-derived generalization regarding material already studied), or "exploration" (in which the student examines independently exercises which are to build an anticipatory awareness of concepts for future studies). An actual

count (performed in the same manner as earlier indicated for the SMSG materials) of the exercises in alternate units of the UICSM sequence presents an indication of the trend in exercises as evidenced by UICSM. Unit 1 reveals approximately 70% drill, 12% discovery, and 18% exploration exercises; Unit 3 contains approximately 74% drill, 14% discovery, and 12% exploration exercises; Unit 5 displays 63% drill, 27% discovery, and 10% exploration exercises; Unit 7 presents 69% drill, 14% discovery, and 17% exploration exercises; Unit 9 contains 68% drill, 21% discovery, and 11% exploration exercises; and Unit 11 concludes the sequence with 78% drill, 9% discovery, and 13% exploration exercises. The general pattern of presentation used by UICSM is the development of a mathematical idea followed by a large number of exercises to provide practice in applying the idea to theoretical mathematical problems. It is to be noted that the discovery exercises often contain as results theorems which, though to be independently proved by the student, are an integral part of the sequence. Similarly, the exploration exercises, being anticipatory in nature, foretell and create an awareness for following textual discussions.

A study of the UICSM materials indicate that the development of concepts occupies a more prominent and central role than the development of manipulative skills although UICSM professes to believe that a mathematics program should develop both the concepts and the necessary manipulative

skills. Certainly, UICSM provides a multitude of exercises which are "drill" in nature for those wishing to use them. It will be remembered that in the eyes of both UICSM and SMSG, the ability to make conjectural generalizations and then establish their validity (or nonvalidity) is within itself a desirable skill.

Social Applications

Mathematics educators as well as practicing professional mathematicians are often divided on the desirable nature of mathematics insofar as the emphasis to be placed on social applications and the purpose and nature of such applications are concerned. Some feel that the clear understanding of a subject such as mathematics can be gained only through social applications whereas others feel that the clarity of such understanding may be dimmed by overt attention to social application. This disagreement is further reflected in the fact that some people hold that the major purpose of mathematics is to serve as the "hand-maiden of the sciences" and to function as a tool in solving the problems of our society whereas others feel that the actual building of the mathematical tool is the more important. The structuring of a mathematics curriculum must, of necessity, reflect the author's philosophy with respect to such positions.

SMSG early admitted that mathematics is to be based on the needs of the society but that the rapidly-changing

needs of society create a practically insurmountable barrier to teaching society-oriented mathematics in that no person (or persons) could possibly anticipate the mathematical needs of a society of even fifty years from now. Therefore, SMSG's primary attention to the social role of mathematics has been through the avenue of correct decision making and structuring of valid generalizations.

This attention to good decision-making has produced as one of the major characteristics of the SMSG program an insistence on fairly rigorous generalizations and proofs. This insistence, though not overly-demanding, has promoted student awareness of the role of the undefined term, the definition, the postulate, and the theorem in mathematical decision-making and problem-solving. Included in this sequence has been a fairly simple presentation of some of the rules of informal logic which is so necessary in correct deductive reasoning. These mathematical problem-solving situations, necessarily abstract in nature, have implicitly demonstrated that decision-making, if to be properly performed, must be done in view of the factors germane to the situation at hand and that such a decision is achieved by manipulating abstractions within a model free from environmental stresses and stigmas. SMSG assigns an increasingly important role to mathematical philosophy as a factor in the evolution of our society yet does not give explicit attention to such a

role except indirectly through the desirable aspects of good deductive reasoning.

In terms of the total number of problems, SMSG textbooks contain relatively few social applications. Occasionally one finds a few problems related to the physical sciences, e.g., falling bodies, rates, mixtures, radioactive decay, maxima and minima problems, the laws of cooling, and applications of simple harmonic equations. Fleeting references are made to the quantification of physical phenomena, e.g., the laws of exponential growth and decay illustrating a use for logarithmic and exponential functions. Even the unit on matrices--a unit rich in potential exhibitions of such references--uses only few examples from the lives of the students to point out the existence of social applications and then omits further consideration of such. (The chapters devoted to vectors do present numerous problems dealing with forces, velocities, work, etc. However, one gains the impression that such a consideration has as its purpose the exposition of the assumed isomorphism between the mathematical system of vectors and the physical system of forces rather than the exhibition of the social applications of forces per se.) This is not to say that problems of the social nature and worded in environmental language are not present--actually at least 267 such are found in First Course in Algebra--but rather to imply that SMSG does not devote undue

attention to the role of mathematics in helping one understand his environment.

UICSM, in the adoption of a policy similar to that advocated by SMSG regarding the social application of mathematics, recognizes early in Unit 3 the application value of mathematics by including numerous exercises on the subject. Such an inclusion covers many areas and tends to require critical thinking as well as careful analysis. The emphasis is entirely upon mathematical principles; social application seems merely to illustrate them and it is clear that the variety of applications is of secondary importance. In the remainder of the units, UICSM, in general, does not use social applications of mathematics to motivate the study of ideas and to develop basic principles. Often, however, new sections of study are introduced by means of an allusion to a nonmathematical field to show where this particular topic might be applied with the sciences being used exclusively in this respect.

The exercises themselves which are of an "applied" nature quite often do more than simply state a situation in environmental terms but rather include scientific facts in their statement, i.e., physical principles are often stated as working assumptions. Although it is somewhat out of context in the UICSM materials, Unit 9 produces a quite dramatic presentation showing how mathematical models often can be abstracted from empirical data and then used to quantify and

predict physical phenomena, e.g., the study of Gay-Lussac's laws for gasses, absolute temperature, isopiestic and isothermal conditions, the general gas laws, radioactive decay, Newton's law of cooling, and transient currents in simple circuits. Implicit in this discussion is the role of mathematics in the development of the sciences although little direct attention is paid to the importance of mathematical thought in philosophy and as a factor in the evolution of our society. The emphasis, however, is still on the mathematical principles involved.

As did SMSG, UICSM pays careful attention to the process of logical reasoning in mathematics and hopefully expects the transfer to the social environment. Special attention is directed to the role of the definition, the postulate, and the theorem in decision-making. Considerable emphasis is placed upon the rules of formal logic and the associated symbolism. The many examples illustrating that intuitive shortcomings, environmental differences, and lack of precise communication vehicles may create problem-solving difficulties, seem to portray the role played by correct deductive reasoning in the physical world, although UICSM directs little explicit effort toward that particular usage of mathematics. UICSM does, however, adequately indicate that decision-making is essentially the manipulation of abstract entities which represent, in many instances, concrete objects and which, in other instances, represent abstractions.

CHAPTER V

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Statement of Problem

The problem for this study was stated in the form of the following questions:

1. What are the major features of the SMSG and the UICSM programs insofar as guiding philosophies, placement of materials, attention to mathematical structures, methods of presentation, vocabulary, proofs, development of concepts and skills, and attention to social application are concerned?

2. What are the major similarities and differences between the UICSM and SMSG programs insofar as guiding philosophies, placement of materials, attention to mathematical structures, methods of presentation, vocabulary, proofs, development of concepts and skills, and attention to social applications are concerned?

The primary purpose of the study was to prepare an analysis of the SMSG and UICSM secondary programs so that a teacher or curriculum director contemplating the usage of one of these two particular secondary mathematics programs might make a more intelligent decision in terms of the criteria discussed in the statement of the problem and determine which of the programs is better suited to their particular situation.

Procedure

The general procedure utilized in the completion of this study involved several steps. (1) In addition to the survey of available pertinent literature, a study was made of the history of the programs and particularly the guiding and motivating philosophies of the two groups. (2) The several newsletters of both groups were examined thoroughly in order that the writer might gain an "historical" perspective as to the developmental efforts and problems of the two groups. (3) Careful study was directed toward the textual discussions of the student textbooks; practically every exercise in both programs was solved and analyzed. (4) A thorough study of the commentaries was conducted with careful attention being pair to teacher-directed comments identifying the beliefs of the sponsoring groups as to the desirable role of mathematics in the secondary curriculum, the rationale motivating the approaches, and the suggested pedagogical devices. (5) The characteristics of the prepared materials of the two groups with respect to the variables identified in the statement of problem were evaluated for each unit within each program as well as the separate total programs, thus permitting identification of the major differences and similarities of the two programs with respect to each of the study variables.

Summary

Philosophies of the Authors

This study has shown that the guiding philosophies of the groups preparing the UICSM and SMSG programs with respect to the characteristics of a desirable secondary mathematics program were so nearly equivalent as to be identical although, in some instances, their techniques of implementation differed. In the eyes of both groups, the successful and desirable mathematics curriculum must create student appreciation of some of the ideas and methods of exploration and generalization basic to the work of the contemporary mathematician, must display and take account of the increasing demand upon mathematics by present and future scientific and technological strides as well as to reflect the tremendous advances and exponential growth of the discipline itself, and must develop an understanding of the role of mathematics in our society. Furthermore, this curriculum must be developed and so taught that the student, although presently unable to ascertain the totality of the uses to which his mathematical knowledge might be applied in the future, will be able in later life to extrapolate his present skills and to learn the new mathematical skills which shall be demanded of him in the increasingly complex society of the future. Both groups apparently were dedicated to the belief that the basic ingredient of such a curriculum must be the recognition of the logic of the

mathematics studied by the student and that the eventual success or failure of their programs depended upon the achievement or nonachievement, respectively, of such.

Both groups subscribed heavily to the tenet that the desirable outcomes of a successful mathematics curriculum should be interested, enthusiastic students who, whether college-bound or not, simultaneously possess a deeper understanding of the basic concepts and structures of mathematics and are well-grounded in the basic mathematical skills requisite to the utilization of such understandings. Both groups evidently subscribed to the premise that the construction of such an understanding-oriented curriculum should result in the mathematical literacy needed by both the professional mathematicians and the lay citizenry of the society. Both groups optimistically postulated that fewer students will be repelled by such a curriculum than would have been by the conventional manipulation-oriented curriculum.

Placement of Materials

This study has shown that the SMSG units, despite their many provocative and invigorating mathematical and pedagogical innovations, introduced, prior to the unit on matrix algebra, very little basic mathematical content not common to traditional secondary curricula. The ninth-grade First Course in Algebra contained essentially the same basic mathematical material as the traditional ninth-grade algebra

textbook; the tenth-grade Geometry was basically a study of Euclidean plane and solid geometry; and the eleventh-grade Intermediate Mathematics and the twelfth-grade Elementary Functions dedicated themselves to the comparatively more rigorous and inclusive development of concepts and techniques usually reserved for second-year algebra and trigonometry. The twelfth-grade Introduction to Matrix Algebra, however, was completely new to the student and the secondary curriculum. Interwoven into this sequence of units was a spiral study of the set of real numbers involving increasingly sophisticated and rigorous discussions, concepts, and proofs.

Similarly, the UICSM sequence introduced only a few really new mathematical topics to the secondary curriculum. However, UICSM's presentation placed emphasis upon various areas not heretofore emphasized in the traditional secondary programs, e.g., mathematical induction, formal logic, and sequences, and at the same time, minimized other areas, e.g., solid geometry; utilized new mathematical approaches and pedagogical techniques, e.g., a highly algebraic treatment of Euclidean geometry; and dramatically rearranged the order of introduction of various topics. UICSM also constructed an increasingly-sophisticated spiral development of the system of real numbers.

A major difference in the two programs was noted in the basic arrangement of the component units. The various SMSG units were constructed for specific grade-level

consumption, i.e., First Course in Algebra was intended for ninth-graders and Geometry was intended for tenth-graders, etc., and, as such, inadvertently conveyed the impression of being "separate-subjects" presentations. The units were so constructed that a student would experience only a minimum of difficulty in entering the SMSG sequence at any particular grade level although such is not recommended by the authors. The basic unity of mathematics was, in a sense, illustrated by the final unit Introduction to Matrix Algebra which served as a unifying "capstone" to the entire sequence.

The UICSM units, however, were written to form a thoroughly integrated and sequential course of study with the eleven component units being viewed as "chapters" in a single book entitled High School Mathematics. The appropriateness of the study of any unit was to be based upon mathematical-experience levels rather than grade levels and, therefore, grade placement of the units could be resolved in a variety of ways. Generally speaking, the UICSM sequence was constructed so as to create a smooth flow of mathematical concepts thus not conveying the idea of the "compartmentalization" of mathematics into relatively disjointed areas of concentration. The nature of the units would make it comparatively difficult for a student to enter the program at an intermediate position although better students might do so if provided some orientation as to the nature of previous units.

Since the basic mathematical contents of the two programs were essentially equivalent (with some exceptions to be mentioned later), the major similarities and dissimilarities in the subject matter content of the two programs lay in the order and time of presentation and the emphasis placed upon component concepts. For the purpose of this study, several areas of comparison were chosen which were somewhat illustrative of these variables.

Both SMSG and UICSM relied heavily upon the language and application of elementary set theory and used such at every opportunity at which such usage was feasible and economical. However, although the SMSG sequence introduced sets and terms such as is an element of, is a subset of, union, intersection, etc., frequently, SMSG seldom did more than verbalize these concepts, and until the last units, little attention was directed toward incorporating the associated symbolism into the exercises and textual discussions. Little attention was directed to such set notions as that of the Cartesian product. On the other hand, UICSM, although not using sets per se until Unit 3, made heavy and formal demands upon the language of sets and the associated symbolism. The UICSM units fairly bristled with set language and symbols using the set-builder notation as a basic vehicle of communication. Practially every generalization was stated in set-language with maximum utilization of symbols. The Cartesian cross-product was used frequently to authorize and implement

such tools as relations, functions, number planes, lattices, etc.

SMSG, in its approach to the study of algebraic expressions, considered the variable to be a numeral representing a definite, though unspecified, number from a given set of admissible numbers. In its analogous development, UICSM first introduced a pronumeral to be a symbol holding a place for (or which could be replaced by) a numeral. Later units in UICSM defined a variable as simply a pronoun having any set of elements as domain; hence a pronumeral is a numerical variable. In essence, where SMSG considered a variable to be a numeral, UICSM held that a variable was a placeholder for the name of any object and reserved the pronumeral as a placeholder for numerals.

Both UICSM and SMSG depended heavily upon the real number line, the real number plane, and graphs. Their different constructions of such entities forced associated language differences into the texts. SMSG, postulating a one-to-one correspondence between the set of real numbers and the points on a straight line, used the real numbers to coordinatize a straight line called the real number line; the graph of a set was the set of points on the real number line whose coordinates were the numbers of that set. Similarly, SMSG considered the real number plane to be the "geometric" plane coordinatized by ordered pairs of reals and the graph of a two-variable sentence to be the set of all points of the plane

whose coordinates satisfied the sentence. UICSM defined the real number line to be the set of all real numbers and the real number plane to be the Cartesian square of the reals, i.e., the set of all ordered pairs of reals. The graph of a sentence was the set of all "dots" on a "number line picture" (or "number plane picture") which were pictures of numbers (or pairs of numbers) forming the truth set of the sentence.

SMSG, although making frequent use of relations such as equals, greater than, less than, etc., used relation as an undefined term with little attention to explanation. UICSM directed a significant amount of explanation to the concept and defined a relation to be a set of ordered pairs (not necessarily real numbers) of elements, i.e., a subset of the Cartesian cross-product of two sets, and placed heavy dependence upon relations and their properties in its later definition of function. The extensive use of number-pairs was a significant characteristic of the UICSM materials.

The first two years of the SMSG sequence used the number line to define the less than relation for the reals by stating that $a < b$ if and only if a is to the left of b on the number line; the later units used the more sophisticated and abstract notion that $a < b$ if and only if there existed a real positive number c such that $a + c = b$. UICSM, early in its sequence, introduced and utilized the number-pair definitions of the less than relation to be $\{(x, y) \in R \times R : y - x \text{ is a positive number}\}$.

In its many uses of the function, SMSG considered a function from set A to set B to be a correspondence of the elements of set A and set B such that to each element of A there corresponded exactly one element of B. Most of the functions in the early units were real-valued with a function being considered as a mapping only in the last two units. UICSM initially considered a function from set A to set B to be the set of ordered pairs (with the first component of each being an element of set A and the second component of each being an element of set B) no two of which have the same first component. This view of a function was followed practically immediately (and used extensively thereafter) by the consideration of a function as a mapping from its domain to its range, with the domain and range being general sets and not necessarily sets of real numbers.

Both programs directed an appreciable amount of time to exhibiting the structure of number systems and the examination of the operation and rules defined on these systems. SMSG, in its interwoven discussion of the operations, essentially regarded the term operation as an undefined one and paid little or no attention to the existence and role of operators. UICSM, directing explicit and lengthy attention to a more abstract discussion of the many operations involved, used a variety of techniques to define operations, e.g., an operation (such as adding 3) may be considered a set of ordered pairs no two of which have the same first component, a

singulary operation on a set S is a mapping of a subset of S onto a subset of S, and a binary operation is a mapping of the Cartesian square of S onto a subset of S.

Both of the programs spent appreciable time in their spiral examination (spread across several units) of the extension of number systems although their approaches varied. Both groups essentially postulated the system of arithmetic numbers and then carefully examined the properties of that system. In the extension from the system of arithmetic to the system of reals and then from the system of reals to the system of complex numbers, SMSG, in each instance, constructed a new system such that the initial system was a proper subsystem of the new system, i.e., $\{\text{arithmetic numbers}\} \subseteq \{\text{real numbers}\} \subseteq \{\text{complex numbers}\}$. UICSM, however, utilized a more sophisticated extension method in that its study "extended" an intial system by constructing a new system such that the initial system was isomorphic to a proper subsystem of the new system, i.e., the system of arithmetic numbers was isomorphic to (but not equal to) the set of non-negative reals and the set of reals was isomorphic to (but not equal to) the system of complex numbers of the form $a + bi$, a a real number.

These different approaches to the extension of number systems forced different approaches to various concepts, e.g., the construction of the negatives. SMSG initially introduced the negatives as being labels for points on the left-half of

the number line since the positives played the role of being labels for those on the right-half. UICSM introduced the "right-reals" and the "left-reals" as being measures of the directed distance of points to "the right of" or "to the left of" the origin, respectively, on the number line. UICSM, due to its insistence that the set of arithmetic numbers was not the set of nonnegative real numbers but rather that the system of arithmetic numbers was isomorphic to the system of nonnegative reals, distinguished between right-reals and positives. Both groups granted a highly utilitarian role to the opposite.

Both groups utilized the absolute value quite heavily in the statement of generalizations. SMSG considered the absolute value of a nonzero real number to be the greater of that number and its opposite with the absolute value of 0 defined to be 0. In the eyes of UICSM, the absolute value of a real number was the number of arithmetic corresponding to the number (or its opposite if the number is negative) under the isomorphism between the system of arithmetic numbers and the system of nonnegative reals; hence the absolute value of a real number is not a real number.

Insofar as the arithmetic operations on the reals were concerned, SMSG used a number-line interpretation for addition with the subtraction of a number b being defined as the addition of the opposite of b. Multiplication was defined so as to phrase a meaning for the product in such a way that

its meaning would agree with the results of the addition operation; division by b ($b \neq 0$) was defined as the product by the reciprocal of b. UICSM used number-line interpretations and other environmental situations to create nonverbal awareness of the desirable properties of the operation and, as did SMSG, relied upon the notion of inverse to define subtraction and division.

Although SMSG and UICSM both made extensive use of the limit, their approaches and degree of sophistication differed. SMSG essentially assumed that the limit of a function of x as x approaches some a is L if and only if f(x) gets "arbitrarily close" to L as x approaches a. The discussion of limits, as presented by SMSG, could be easily amended to the conventional (ϵ, δ) form if desired but the authors evidently felt that the informal discussion was sufficient. UICSM, however, employed a very sophisticated verbalization (and provided practice in its usage) essentially employing the (ϵ, δ) notion of the limit.

SMSG did not concern itself with a formal definition of continuity and used graph-oriented approaches to illustrate the concept, e.g., a function is continuous in an interval if its graph has no "holes" or "gaps" within the interval. The UICSM attack on continuity of a function at a point a was a rigorous function-based definition that demanded the existence of the function at $x = a$ and that the limit of f(x) as x approaches a be f(a).

Although SMSG concerned itself implicitly with demonstrating good decisions and the rules of decision-making, no direct attention was given to the study of formal logic. UICSM furnished its student with a thirty-eight page appendix (to be studied in conjunction with the unit on geometry) dealing with the rules of reasoning and the simpler principles of logic.

Although the units involved many theorems which can be rigorously established only by induction, SMSG's approach to the mechanics of mathematical induction was informal with little special attention to the process until an appendix of Unit 21. UICSM Unit 7 dealt extensively with the process of mathematical induction and forced induction to play a vital role in the remainder of the units. A particularly heavy reliance upon mathematical induction was noted in the UICSM Unit 8 treatment of sequences.

In the topics dealing with conics, SMSG utilized the focus-eccentricity-directrix approach to the study of conics and restricted itself to the study of nonslant conics. UICSM defined the ellipse, hyperbola, and parabola as the locus of a point the sum of whose distance from two fixed points is constant, the difference of whose absolute distance from two fixed points is constant, and the distance from a fixed line and a fixed point is constant, respectively, but examined canonical forms for the equations of both slant and non-slant conics.

The SMSG study of the complex numbers, motivated by the empty solution set of $ax^2 + bx + c = 0$ if $b^2 - 4ac < 0$, involved an extension of the system of reals to the system of complex numbers which includes elements (designated alternately by SMSG as $a + bi$, (a, b), and $r(\cos \theta + i \sin \theta)$) such that the system of reals is a subsystem of the complex number system and the solution set of $ax^2 + bx + c = 0$ is always nonempty. UICSM's analogous study was based on the number-pair approach and involved constructing a system C such that $ax^2 + bx + c = 0$ will always have a nonempty solution set and such that the system of reals will be isomorphic to a proper subsystem of C.

Both SMSG and UICSM constructed an iterative algorithm to approximate the square root of a positive real, i.e., if y_1 is an approximation to $\sqrt{x}$, then $y_2 = (y_1 + x/y_1)/2$ is a better approximation. SMSG's treatment of the algorithm was informal; UICSM directed attention to degrees of approximation and the estimation of errors in the process.

SMSG and UICSM both sponsored studies of primes and composites as an aid in determining greatest common divisors and least common multiples, the factorizations of polynomial expressions, and simplification of fractions. UICSM's treatment was the more thorough and rigorous of the two and emphasized the number theory of the study of primes and composites.

SMSG's Geometry, a one-year set-oriented, algebraic-flavored study of geometry (both plane and solid) based on

the postulate scheme proposed by G. D. Birkhoff, reflected the SMSG belief that Euclidean geometry deserves its prominent place in the secondary curriculum and, consequently, presented a treatment which, though full of innovations, was fairly traditional. UICSM's Geometry, a one-semester, set-oriented, algebraic-flavored study based on the postulates of David Hilbert, covered most of the topics common to traditional plane geometry courses and was characterized and facilitated by the early introduction of coordinate geometry.

SMSG's Geometry, reflecting SMSG's belief that some consideration should be given to solid geometry, contained a few chapters dealing with a set-oriented approach to the study of solid geometry. UICSM, apparently dedicated to the belief that solid geometry deserves little prominence, devoted only an appendix discussion in Unit 9 to any of the topics of solid geometry and even then did little more than justify in a nonrigorous fashion some of the mensuration formulas associated with such a study. Both SMSG and UICSM presented circular-function-oriented studies of trigonometry which deemphasized the solution of right triangles and logarithmic computation. Both studies were characterized by precise language, abundant usage of radian measure, and analytic techniques. UICSM devoted an entire unit (Unit 10) to the study of trigonometry as a study of circular functions; SMSG devoted four weeks of study in Unit 18 and five weeks in Unit 21

to the study of topics traditional to the study of plane trigonometry.

Although both programs utilized symbolism frequently, the UICSM texts relied more heavily on the symbols ordinarily used in logic and traditionally reserved for higher courses in mathematics. The UICSM symbolism conveyed more precise and detailed distinctions than that of SMSG and the UICSM materials, seeming never to use a word to communicate a notion when a symbol was available, displayed a more prevalent usage of symbols than those of SMSG.

The major mathematical content unique to SMSG (as opposed to UICSM) was the inclusion of vectors, vector algebra, and vector techniques (culminating in the examination of the axioms for vector spaces) in the eleventh-grade units, and the unit on matrix algebra (Unit 23) dealing with matrix operations, the algebra of 2 x 2 matrices, linear systems, matrix interpretation of the algebra of vectors, etc. The unit on matrix algebra served as a vehicle for discussing the axioms for fields, groups, and rings, and to emphasize repeatedly the structure of mathematics. In order to gain the time to be used for these topics which UICSM did not pursue, SMSG was willing to accept informal proofs of many theorems proved with sophisticated rigor by UICSM and to sacrifice thorough time-consuming discussions of many topics which UICSM pursued with enthusiasm, e.g., sequences, relations,

induction proofs, limits, continuity, existence theorems, logic, and number theory.

Attention to Mathematical Structures

When compared with conventional treatments of secondary mathematics, both the SMSG and the UICSM texts were characterized by their attention to mathematical structures. In both treatments, it was evident that the display of the logical structure of mathematics was of paramount importance since each group directed an appreciable percentage of its efforts to the study of the structure of mathematical systems and the interrelations of the various mathematical systems considered.

SMSG's extension of number systems and the analysis of the shared properties of such systems were imbedded in a sequence of materials which saw each abstract system constructed, its algebraic properties examined and generalized, and that system then imbedded in a more inclusive system of which the particular system was a subsystem. The component systems were examined from the point of view of their basic structural properties with careful attention directed to the display and generalization of the properties shared by the various systems. At various points in its sequence, SMSG directed attention to the naturals, the wholes, the rationals, the reals, the complex numbers, and, finally, the system of matrices.

UICSM likewise spent considerable time in constructing and displaying extensions of mathematical systems although utilizing a slightly different approach to the study of the shared properties of these systems. The UICSM extension of systems involved the examination of a particular abstract system (along with the generalization of its properties) followed by the construction of a different, more-inclusive system such that the particular system was isomorphic to a proper subsystem of that more inclusive system. The UICSM array of extensions initiated in the extension from the arithmetic numbers to the reals and culminated in the development of the complex number system. The axiomization of the reals was followed by studies which "sifted" the positives from the reals, the integers from the reals, the rationals from the reals, etc., and which essentially examined the special properties which some kinds of reals possess and which other kinds of reals do not.

Methods

It has been shown that both groups were dedicated to the value of the discovery approach in the teaching of mathematics. Both groups insisted upon student understanding and have advocated that a student's understanding would be maximized if he has a part in the development of that which he pursues.

In the main, SMSG presented the great bulk of its mathematical content through a comparatively verbose textual presentation which carefully led the student through concept discoveries. However, the student was usually given an opportunity to verbalize the gained concept prior to its statement by the text. Generally, new concepts were introduced by citing examples, forming conjectural generalizations, and then deductively proving the generalizations. Often several concepts were generalized simultaneously. These generalizations were usually verbalized and displayed fairly quickly after formulation. Although such was not always the case, the exercises were generally independent of the textual content-development sequence since they seldom involved an extension of the basic mathematical theory being exposed. This does not imply that the exercises were never exploratory in nature but rather that they served, on occasion, in an enrichment role as well as a concept-reinforcing one.

The UICSM program particularly represented the belief that the learning process was aided if the student discovered the principles and rules for himself. The UICSM general exercises, usually directed to the content directly at hand, served the roles of reinforcing the concepts already gained and often to extend the mathematical theory already studied through the proof of theorems to be used in future proofs. The prevalent Exploration Exercises were constructed to point out inadequacies of existing systems, to formulate desirable

properties to be demanded of new systems and techniques, and to promote student curiosity and awareness regarding topics to be studied in the near future. Such exercises demanded that the student play an active part in the development of the concepts. In brief, the UICSM textual discussions, general exercises, and Exploration Exercises formed an integrated continuum no part of which could be omitted. The UICSM units also exhibited the belief that it was not necessary that a student verbalize a discovery immediately but rather that he might possess nonverbal awareness of a concept and manipulate it prior to its naming and verbalization. The UICSM method of presentation of mathematical concepts could be characterized by the three steps of experimentation, observation, and generalization.

Vocabulary

When compared with their conventional counterparts, both SMSG and UICSM units were characterized by a concern for a precise, mathematically-sophisticated, unambiguous language although UICSM directed more explicit consideration to the discussion of the need for such. Both programs used set language consistently, stated all generalizations with appropriate quantifiers, differentiated between <u>number</u> and <u>numeral</u>, <u>set</u> and <u>measure of set</u>, etc. SMSG used basically the language commonly used by contemporary mathematicians and often used terms whose formal definitions were evidently thought to

be beyond the comprehension of the student without formal, overly-sophisticated verbalization. No overt concern with symbolism was noted. UICSM, on the other hand, frequently "coined" terms to describe concepts in such a way that the name conveyed the potential use of the term and seldom assumed the meaning of any term without direct and precise attention. The UICSM language drew finer lines of distinction in many more instances than did SMSG, e.g., UICSM distinguished between the interval $\overline{AB}$, and the segment $\overline{AB}$ (containing A and B). From the very beginning, UICSM introduced and exploited symbolism to its fullest.

Proofs

The SMSG and the UICSM units were characterized from the very beginning by attention to "good" proofs which were significantly more precise and more rigorous than those of their traditional counterparts. Both appealed to the role of the undefined term, the definition, the postulate, and the theorem in good decision-making. Both groups utilized various forms of both direct and indirect proofs and avoided the stereotyping of proof forms since they used any particular type of proof which was economical and feasible at the point of discussion, e.g., both groups utilized a more highly algebraic approach to geometry than evidenced in the traditional programs. It was significant that, in addition to presenting a multitude of good proofs, both groups attempted to promote

an awareness of the "holes" and "gaps" (fewer in number in the UICSM texts than in the SMSG texts due to the intensity of the UICSM presentation and the concentration on fewer topics) in certain nonrigorous proofs necessitated by the sophistication of the concept involved. Both groups made frequent use of the number line and also the number plane to make intuitive generalizations which were then formalized into fairly crisp proofs. The concept of rigorous proofs seemed to be central to the development of both programs and, in both sequences, the student was led always to authorize his manipulation of symbols and to be able to support his generalizations.

However, several differences were noted in the proofs included in these two programs. Generally, UICSM expected precise and rigorous proofs earlier in its sequence than did SMSG, e.g., UICSM expected fairly rigorous proofs by the end of the first semester of study. More attention was paid by UICSM to the rules of formal logic. The language used by UICSM was more rigorous and discerning than that of SMSG, e.g., UICSM demanded that attention be directed to formal discussions of relations, operations, limits, and continuity whereas SMSG was willing to accept such terms as being somewhat undefined. UICSM's number-pair approach to relations and functions and its consideration of a multitude of non-real-valued functions and relations influenced the nature of its proofs. UICSM made a more prevalent usage of symbolism

in its proof forms than did SMSG. The precision and sophistication of the language made the UICSM proofs appear more involved and unwieldy than those of SMSG although the basic difference might lie in the subject matter rather than in the proof vehicle. The UICSM texts reflected a greater concern for uniqueness and existence proofs than did those of SMSG. The coordinate plane and the associated analytic tools were introduced earlier and used more extensively by UICSM than by SMSG. UICSM made an oft-employed tool of the theory of mathematical induction whereas SMSG hardly made use of the implement. UICSM early placed the burden of the proof on the student whereas SMSG usually led the student through the proof by posing well-framed leading questions and by suitable textual discussions.

Concepts and Skills

Although including a great number of drill exercises, SMSG devoted less space than its traditional counterparts to the acquisition of manipulative skills and placed high emphasis on the development of basic concepts. SMSG hopefully postulated that less drill is needed to develop basic skills if that provided drill is meaningful in nature and based on an understanding of the process. In all units, however, many exercises were provided which should reinforce and develop associated skills with respect to the involved concepts and techniques. Many of these exercises were, in reality,

exercises providing drill for concepts developed in discovery exercises in which the student had independently recognized and formulated the concept and/or working techniques. The sets of exercises were seldom stereotyped in form and approach.

Although UICSM postulated that a mathematics program should develop both the concepts and the manipulative skills, it was evident that the development of concepts played a more prominent and central role than the development of manipulative skills although a veritable multitude of drill exercises were presented (in greater abundance than in the SMSG units) for those desiring to use them. The texts contained as an integral part of the sequence exercises which forced the student to produce theorems which were a vital part of the content development. Many of the drill exercises, seldom stereotyped in nature, reinforced those concepts independently discovered by the student. UICSM demanded that all drill be based on understanding and that, hence, less formal drill was necessary.

Both groups, therefore, placed a premium on understanding of concepts which authorize meaningful drill. Further, in the eyes of both groups, the ability to make and validate conjectural generalizations was within itself a desirable skill and conscious effort was directed toward the development of that skill.

Social Applications

Both SMSG and UICSM, although admitting that mathematics is to be based on the present and future needs of society, utilized relatively few exercises exhibiting what might be referred to as social application and devoted only a minor part of the textual discussions to the role of mathematics in the exposition and understanding of the environment. This is not to imply that exercises of a social nature (and worded in an environmental language) did not appear but rather that the two groups did not devote a disproportionate amount of effort in helping one understand his environment and to solve problems of a social nature. Their major contributions in this respect were imbedded in the fact that both groups directed special attention to the role of the definition, the postulate, and the theorem in decision-making was well as the informal logic so necessary in correct deductive reasoning. In both programs, the authors seemed to optimistically hope for the transfer of such logical reasoning based on a sound mathematical background and the appropriate mathematical model whenever needed by the social environment.

Conclusions

Due to the multiplicity of intermediate variables involved, a precise statement of the major differences and similarities of the two programs with respect to the variables

identified for this study was difficult although some general conclusions were justified by the study.

1. The guiding philosophies of the SMSG and UICSM were so nearly equivalent as to be identical in that both groups predicated their efforts upon the need for a mathematically-informed population whose members not only possess the basic mathematical skills but also a deeper understanding of the basic mathematical concepts and structures of mathematics.

2. Insofar as basic mathematical content was concerned, the SMSG authors appeared willing to sacrifice some sophistication and depth of examination of certain concepts in order to include brief discussions of topics such as vectors, matrices, and the postulates for groups, rings, fields, and vector spaces, as well as all the topics common to traditional programs. The UICSM materials (built more as an integrated continuum of mathematical experiences than are those of the SMSG) were characterized by intensified attention to the basic concepts and, as a result, have a more narrow scope of basic content than those of SMSG.

3. The spiral approaches used by SMSG and UICSM to the study of the real number system and their constant analyses, comparisons, and extensions of mathematical systems made it evident that attention to mathematical structures and the exhibition of the logical structure of mathematics were of paramount importance in the eyes of both groups.

4. Both SMSG and UICSM were dedicated to the notion that a student understands better that which he helps discover and develop and, consequently, both groups employed the discovery method. SMSG, through the use of lengthy textual discussions, led the student to formulate and validate generalizations which usually were verbalized fairly quickly; UICSM, through leading questions (which the student must answer) and exercises often involving independent proofs of theorems, demanded individual student exploration and promoted student awareness and appreciation of concepts prior to their formal verbalization.

5. SMSG and UICSM made maximum usage of precise language and descriptive vocabularies; SMSG generally used the language employed by contemporary mathematicians whereas UICSM frequently invented new descriptive terms which were often more discerning than those of SMSG.

6. Both SMSG and UICSM reflected a proper consideration of the role played in mathematics by good proofs; both groups avoided stereotyping of proof forms and utilized various proof forms whose usages were dictated only by economy and feasibility. The UICSM proofs were, in general, more intensified and discerning than those of SMSG due, in part, to the fact that UICSM was often concerned with some of the comparatively sophisticated concepts whose validity SMSG was willing to assume in order that the necessary examination time be spent exploring new concepts.

7. Both SMSG and UICSM hopefully developed both basic concepts and the associated manipulative skills although they definitely devoted less space than their traditional counterparts to the acquisition of skills and placed vastly more stress on the development of basic concepts.

8. Although SMSG and UICSM demanded that mathematics play a heavy social role, their primary attention to the social role of mathematics has been through the avenue of correct decision-making and their textbooks contained relatively few problems directly reflecting social applications.

Recommendations

In view of the impressions gained in this study regarding the efforts of the SMSG and UICSM teams, the writer of this paper would offer the following recommendations:

1. No person contemplating the teaching of either of the programs should do so unless he is definitely in agreement with the philosophies guiding the two writing teams. The discovery approach utilized by UICSM and SMSG units demands the utmost cooperation between the textbook and the teacher-directed classroom experiences if the program is to be even moderately successful.

2. Since a potential teacher's reaction to these two programs is very likely to depend on his preparation to teach them, it would be well for a person contemplating the adoption of such a program to review briefly his own academic qualifications, willingness to learn, and inclination to modify his past approaches if changes in such are necessary.

3. The teacher contemplating the adoption of either of these programs should view and examine the sequences as a continuum of experiences rather than as a series of separate units since the nature of the units are such that, in a sense, the sum of the parts is less than the whole.

4. The person contemplating the adoption of either of these programs should do so only after he has considered the objectives of his curriculum.

5. The prospective teacher of either of these modern programs should understand and be able to clarify the role of mathematics in the development of the individual and his society.

6. The person contemplating the adoption of either of the two programs should be cognizant of the fact that neither of the programs was prepared to be the final answer to all the problems of mathematics curriculum but rather to reflect the thinking of these particular groups as to the nature of a desirable curriculum and to serve as definitive guideposts for other authors.

Suggested Areas for Further Research

This study recognizes the relevance of several theoretical questions which arise from the study of the background of the problem, the survey of pertinent literature, and the analyses of the SMSG and UICSM secondary programs. These areas, definitely nonindependent in nature, might well be considered as projects for further research.

1. Although mathematicians generally agree that mathematics and logic are inseparable, it would appear

feasible to devote further study to the question as to whether or not the secondary mathematics program should demand the development of mathematics as a logical system.

2. Attention should be directed primarily toward the construction of a theoretical model describing the desirable outcomes for a secondary mathematics program. Such a model could then detail the desired mathematical content of a secondary program, describe the placement of such content, and identify the roles played by the intermediate variables of internal motivation of the student, societal needs, and the interaction between mathematical reasoning and environmental needs, both present and future. Such a model would do much to insure maximum economy in the development of requisite mathematical understandings and skills.

3. Although the body of knowledge regarding the learning process in general is growing, further concentrated research should be directed to determine, if possible, how youngsters learn mathematics and the concepts involved. Since intuition plays such a vital role in mathematics, concentrated attention could well be directed toward the further exploration of the nature of intuition and how such intuition could be fostered in young people. A direct result of such a study would be the acquisition of practical and utilitarian methods of developing intuition and guiding such developments through curriculum efforts.

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