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REGULARITY FOR ELLIPTIC SYSTEMS AND THE ANGLE CONDITION

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REGULARITY FOR ELLIPTIC SYSTEMS AND THE ANGLE CONDITION

A Dissertation APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

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Abstract of dissertation

In this dissertation we consider weak solutions of the System $A(u) + B(u) = 0$ on a bounded domain $\Omega \subset \mathbf{R}^n$ where $B(u)$ is a perturbation of critical growth, i.e. its growth exponent p equals the integrability exponent of the Sobolev space for which $A(u)$ is a coercive elliptic operator. Under certain structure conditions we get a higher integrability result for $1 < p < n$ and a regularity result for $p > 2$.

Introduction

If we consider multiple integrals of the form

$$J(u) = \int_{\Omega} F(x,u,Du)\,dx$$

where

$$F\,(x,u,p):\ \Omega \times \mathbf{R}^n \times \mathbf{R}^{nN} \rightarrow \mathbf{R},$$

the study of the stationary point of this integral is equivalent to the study of weak solution of the system

$$(1) \qquad -D_\alpha A_i^\alpha(x,u,Du) + B_i(x,u,Du) = 0$$

These integral arise in certain areas of Mathematical Physics.

The Euler equation for $J(u)$ is of the type of system (1), although systems of the type (1) are in general not Euler equations of a variational integral. In this thesis we study the case of natural growth conditions. (See Preliminaries for a complete discussion). In the case of an equation weak solutions in $H^{1,2} \cap L^\infty$ of nonlinear equations are smooth under natural growth conditions. Therefore we are led to consider $H^{1,2} \cap L^\infty$ as the natural class to consider. But for systems we cannot expect everywhere regularity.

E. De Giorgi's famous example of 1968 shows that, contrary to the scalar case, weak solutions of elliptic systems are not smooth in general even for linear problems. In the following decades it had been shown, however, that the set of non-regularity of the weak solution is rather small and for important systems with structure different from De Giorgi's example that the solution is smooth on the whole domain, as for instance for systems in diagonal form.

Perturbations often require additional conditions, too, such as the famous smallness condition for linear elliptic operators with perturbations of quadratic growth, i.e. critical growth since for linear operators the natural Sobolev space is $H^{1,2}(\Omega)$.

In this dissertation we deal with a situation where the solution is bounded but the L^∞-norm is not restricted, instead we impose a strict positivity condition on the perturbation assuing that the solution and the perturbation as vectors in the target space $\mathbf{R}^N$ have the same direction. For $p = 2$, results by Wiegner and Frehse secure everywhere Hölder continuity, using estimates of the Green function of the Laplace operator. Our main result for this chapter is a Caccioppoli type inequality for elliptic operators satisfying a certain structure condition which includes operators in diagonal form such as the p- Laplace operator.

For $p = 2$ this allows us to regain the results from Wiegner and Frehse using the 'direct method' developed in particular by Giaquinta and Giusti. In addition, our results extend to $p = 2$. In this case we obtain a 'reverse Hölder inequality' and then use the 'direct method' again to obtain almost everywhere regularity, i.e. the weak solution is Hölder continuous in an open subset Ω_0 such that the Hausdorff measure $H^{n-q}(\Omega \backslash \Omega_0) = 0$ for some $q > p$.

Preliminaries

We define

$$H^{1,p}(\Omega) : =$$

$\{u \in L^p(\Omega) \; | u$ has first derivatives in the sense of distributions in $L^p(\Omega)\}$,

and

$$\|u\|_{H^{1,p}(\Omega)} \; := \; (\int_\Omega |u|^p d\mathbf{x} + \int_\Omega |Du|^p dx)^{1/p} .$$

We also define

$$H_0^{1,p}\,(\Omega) \; := \; \overline{C_0^\infty(\Omega)}$$

where the completion is understood with respect to the norm $\| \cdot \| H^{1,p}(\Omega)$.

We note the Ω is a bounded and connected domain in $\mathbf{R}^n$. We consider bounded weak solutions of the system

$$-D_\alpha A_i^\alpha(x, u, Du) + B_i(x, u, Du) \; = \; 0,$$

i.e. vector valued functions $u \in L^\infty \cap H_0^{1,p}(\Omega)$ satisfying

$$\int_\Omega \sum_{i=1}^N \sum_{\alpha=1}^n A_i^\alpha(x, u, Du) D_\alpha \phi^i dx + \int_\Omega \sum_{i=1}^N B_i(x, u, Du) \phi^i \; dx = 0$$

for all

$$\phi \in H_0^{1,p}(\Omega\,; \; \mathbf{R}^N) \cap L^\infty(\Omega\,; \; \mathbf{R}^N).$$

The coefficent functions

$$A_i^\alpha : \Omega \mathrm{x} \mathbf{R}^N \mathrm{x} \mathbf{R}^{nN} \to \mathbf{R}$$

are subject to the following hypotheses:

$A1$) Growth Conditions

$$|A_i^\alpha(x,u,Du)| \leq \mu_1|Du|^{p-1} + l(x) \text{ for some}(\mu > 0)$$

with

$$l(\mathrm{x}) \in L^{p'}(\Omega);$$

where p' is the Sololev conjugate to p :

$$(p')^{-1} + p^{-1} = 1;\ 1 < p < \infty$$

A2) Ellipticity conditions

$$\sum_{i=1}^{N}\sum_{\alpha=1}^{n} A_i^\alpha(x,\eta,\xi)\xi_i^\alpha \geq \lambda|\xi|^p,\ \lambda > 0$$

for

$$\eta \in \mathbf{R}^N, \xi \in \mathbf{R}^{nN}.$$

A3) Structure Condition

$$\sum_{i=1}^{N}\sum_{\alpha=1}^{n} A_i^\alpha(x,\eta,\xi)[\xi_i^\alpha|\gamma|^2 - \gamma^i\sum_{l=1}^{N}\xi_\alpha^l\gamma^l] \geq 0$$

for all

$$\gamma, \mu \in \mathbf{R}^M, \eta \in \mathbf{R}^{nN}.$$

Conditions A1) and A2) are the usual condition providing regularity for weak solutions for problems with natural growth *c.f.* [2]. The structure condition A3) is rather restrictive. It is satisfied though, by systems in diagonal form, such as the Laplacian and the p-Laplacian. For further discussions and examples of this condition *c.f.* [7].

For the coefficient function $B_i(x, u, Du)$ we have the following assumptions:

B1) Natural (critical) growth condition:

$$|B_i(x, \eta, \xi)| \leq \mu_2(|\xi|^p + 1)$$

B2) Zero angle condition

$$\sum_{i=1}^{N} B_i(x, \mu, \xi)\ (\mu_i - \gamma_i) \geq 0$$

for all

$$\eta \in \mathbf{R}^{nN}, \mu, \gamma \in \mathbf{R}^N$$

with

$$|\gamma| \leq |\mu|$$

for all balls B_R with $B_R \subset\subset \Omega$.

The growth condition is called natural since it is satisfied by the operators of the Euler-Lagrange equation for a functional of the type

$$J(u) = \int_\Omega h(u)|Du|^p dx.$$

The name critical indicates that for a function u in $H^{1,p}(\Omega)$ the perturbation is only in $L^1(\Omega)$ and not in the reflexive spaces $L^q(\Omega), q > 1$.

Examples for which our conditions are satisfied are given by

$$-\Delta u + h(u)|Du|^2 u = 0$$

and

$$-\sum_{\alpha=1}^{N} D_\alpha(|Du|^{p-2} D\alpha\, u) + h(u)|Du|^p u = 0$$

for some non negative continuous function

$$h : \mathbf{R}^N \to \mathbf{R}^+.$$

We now verify $B1)$ for the example

$$-\Delta u^i + h(u)|Du|^2 u^i = 0, \quad i = 1, \ldots, N$$

We shall show

$$\sum_{i=1}^{N} h(u)|Du|^2 u^i(u^i - \gamma^i) \geq 0, \quad \text{for } |\gamma| \leq |u|$$

Note

$$\sum_{i=1}^{N} h(u)|Du|^2 u^i(u^i - \gamma^i) =$$

$$= h(u)|Du|^2 \sum_{i=1}^{N} u^i(u^i - \gamma^i) = h(u)|Du|^2[(u,u) - (u,\gamma)]$$

Now by the Cauchy inequality $(u,\gamma) \leq ||u||\ ||\gamma||$, hence by assumption $(u,\gamma) \leq ||u||\ ||u|| = ||u||^2 = (u,u)$, hence $(u,u) - (u,\gamma) \geq 0$ and so $h(u)|Du|^2(u, u-\gamma) \geq 0$.

We also verify A3) for the example

$$-\sum_{\alpha=1}^{n} D_\alpha(|Du|^{p-2} D_\alpha u^i) + h(u)|Du|^p u^i = 0 \; i = 1, \ldots, N$$

Note that

$$\sum_{\alpha=1}^{n}|\xi|^{p-2}\left(\sum_{i\leq 1}^{n}\xi_i^\alpha\xi_i^\alpha|\gamma|^2(\sum_{\alpha=1}^{n}\xi_i^\alpha\gamma i)(\sum_{\alpha=1}^{n}(\xi_i^\alpha\gamma i)\right)$$

$$=|\xi|^{p-2}\sum_{\alpha=1}^{n}(|\xi|^2|\gamma|^2-(\xi^\alpha,\gamma))$$

$$=\sum_{\alpha=1}^{n}|\xi|^{p-2}\sum_{i<l<N}(\xi_i^\alpha\gamma^l-\xi_l^\alpha\tau^i)^2\geq 0$$

by the Lagrange identity.

The main result of chapter 1 states that the estimates on the gradient provided for $p=2$ in [2] are also valid for operators with $1<p<n$.

Theorem 1: Suppose that u is a weak solution of system [1] with

$$A(u)=-(\sum_{\alpha=1}^{n}D_\alpha A_i^\alpha(x,u,Du))_{i=1}^{N}$$

and

$$B(u)=(B_i(x,u,Du))_{i=1}^{N}$$

satisfying A1)-A3) and B1),B2) respectively. Then there is a constant c and an exponent $q>p$ such that

$$(\fint_{B_{R/2}}|Du|^q dx)^{1/q}\leq c(\fint_{B_R}|Du|^p dx)^{1/p}$$

Remark: Note that for $p\geq 2$, L^∞-estimates are available $c.f.$[8]. Using the arguments of [2] chapters 4 and 5 for $p=2$, we can now regain the results of Wiegner and Frehse as we point out in Chapter 3. For $p\neq 2$ we are not able to establish the everywhere regularity, however we will show the almost everywhere regularity for certain systems. To this end we consider systems with a perturbation B subject to the conditions B1), B2) and the p-Laplacian as elliptic operator, i.e.,

$$A(u) = -\sum_{\alpha=1}^{n} D_\alpha(|Du|^{p-2}D_\alpha u).$$

We have

<u>Theorem 2</u> Suppose that $A(u)$ is the p-Laplacian and that $B(u)$ satisfies B1) and B2). Then for a weak solution of (1) there is an open set $\Omega_0 \subset \Omega$ such that

$$u \in C^{0,\alpha}(\Omega_0, \mathbf{R}^N)$$

for all $\alpha < 1$ and the $n-q$ Hausdorff measure of $\Omega \backslash \Omega_0$ is zero for some $q > p$.

<u>Remark:</u> In our proof we need the regularity result for the unperturbed problem of [10], hence somewhat more general operators $A(u)$ can be dealth with, $c.f.$ [11] and [10].

In order to prove theorem 2 we use the following theorems. These can be found, for instance, in [3] and [10] respectively.

<u>Theorem A:</u> If $f \in L^1_{loc}$, then

$$H^{n-q}\{x \in \Omega \ : \ \liminf_{R\to 0} R^{q-n} \int_{B_R} |f|dx > 0\} = 0$$

where $q > p$.

<u>Theorem B:</u> Let B_R be a ball with radius $R \in (0,1]$ such that $B_{3R} \subset \Omega$. Let u be a weak solution of system (1). Then there is a positive constant c such that

$$M^p = \operatorname*{ess\,sup}_{B_{2R}} |Du|^p \leq C \cdot R^{-n} \int_{B_{3R}} (1 + |Du|)^p dx.$$

See [10] for a complete statement of this theorem. We also present:

Theorem C: Let B_R be a ball in $\mathbf{R}^n$ with radius R.

(1) If

$$u \in H_o^{1,p}(B_R)\ ,\ 1 < p < +\infty,$$

then

$$\int_{B_R} |u|^p dx \leq C(n,p) R^p \int_{B_R} |Du|^p dx$$

(2) If

$$u \in H^{1,p}(B_R)\ ,\ 1 < p < +\infty,$$

then

$$\int_{B_R} |u - u_R|^p dx \leq C(n,p) R^p \int_{B_R} |Du|^p dx \qquad \text{(Poincare inequalities)}$$

where

$$u_R = \frac{1}{|B_R|} \int_{B_R} |u| dx.$$

We also introduce the notion of Hausdorff measure.

Definition Let $E \subset \mathbf{R}^n, 0 \leq k < +\infty, 0 < \xi \leq +\infty,$ and let $\{F_h\}$ denote countable families of open sets in $\mathbf{R}^n$. Then the k-dimensional Hausdorff measure $H^k(E)$ in $\mathbf{R}^n$, is defined by:

$$H^k(E) = 2^{-k} w_k \sup_{\epsilon > o}\ \inf\{\sum_{h=o}^{\infty} (\text{diam } F_h)^k ; \bigcup_{h=o}^{\infty} F_h \supset E, \text{diam } F_h < \epsilon\},$$

Here w_k is the measure of the unit ball of $\mathbf{R}^k$.

Theorem D: If for $\alpha \in (0,1]$ there is a number R_o) such that

$$\int_{B\rho(x)} |u - u_{x,\rho}|^p dx \leq c\rho^{n+p\alpha}$$

for all x in an open set Ω and all $\rho < \min\{R_o, \mathrm{dist}(x, \partial\Omega)\}$, then u is locally Hölder continuous with exponent α in Ω.

Theorem E: Let the operator L satisfy

$$\alpha^{ij}(x)\xi_j\xi_j \geq \lambda|\xi|^2 \ \forall x \epsilon \Omega, \xi \epsilon \mathbf{R}^n, \sum |\alpha^{ij}(x)| \leq \Lambda^2$$

where

$$Lu = D_i(\alpha^{ij}(x)D_j u + b^i(x)u) + c^i(x)D_i u + d(x)u$$

and suppose that $f^i \epsilon L^{q/2}(\Omega), g \epsilon L^{q/2}(\Omega)$ for some $q > n$. If u is a $W^{1,2}(\Omega)$ supersolution of the equation $Lu = g + D_i f^i$ in Ω, non negative in a ball $B_{4R}(y) \subset \Omega$ and $1 \leq p < \frac{n}{2-n}$, then we have

$$R^{-n/p}\|u\|_{L^p(B_{2R(y)})} \leq C(\inf_{B_R(y)} u + K(R))$$

where $C = C(n, \Lambda/\lambda, \nu R, q, p)$.

Chapter 1

Higher integrability

The goal of this chapter is to prove the following theorem.

Theorem: Suppose that u is a weak solution of the system

$$-D_\alpha A_i^\alpha(x,u,Du) + B_i(x,u,Du) = 0 \qquad i = 1, \ldots, N$$

with conditions $A1 - A3$ and $(B1), (B2)$ satisfied. Then there is a constant c and an exponent $q > p$ such that

$$u \in H^{1,q}_{loc}(\Omega, \mathbf{R}^N)$$

and

$$\Big(\fint_{B_{R/2}} |Du|^q dx \Big)^{1/q} \le c \Big(\fint_{B_R} |Du|^p dx \Big)^{1/p}$$

for all balls B_R with $B_R \subset\subset \Omega$.

In the proof of this theorem we will use test functions defined by a generalized cut off procedure. For a function $u : \Omega \to \mathbf{R}^N$ we introduce $[u]_l$ by ($l \in \mathbf{R}^+$)

$$[u]_l = \begin{cases} u & \text{if} |u| \le l \\ \frac{l}{|u|} u & \text{if} |u| > l \end{cases}$$

Important for estimates to be derived is the following lemma.

Lemma A: The ellipticity of $A(u) = -D_\alpha A_i^\alpha(x,u,Du)$ and condition A3) provide the following pointwise estimate

$$\sum_{i=1}^N \sum_{\alpha=1}^n A_i^\alpha(x,u,Du) D_\alpha([u]_\theta)^i \ge 0 \text{ for } u : \Omega \to \mathbf{R}^N, \theta \in \mathbf{R}^+ \text{ and } |u| \ge \theta.$$

Proof of Lemma: By [7] we have for $\theta \in \mathbf{R}^+$

$$D_\alpha([u]_\theta)^i = \begin{cases} D_\alpha u^i & \text{if } |u| \le \theta \\ \frac{\theta}{|u|}(D_\alpha(u^i) - u^i \sum_{l=1}^{N} \frac{u^l}{|u|^2} D_\alpha(u^l)) & \text{if } |u| > \theta \end{cases}$$

Therefore

$$\sum_{i=1}^{N}\sum_{\alpha=1}^{n} A_i^\alpha(x,u,Du) D\alpha([u]_\theta)^i =$$

$$\frac{\theta}{|u|}\sum_{i=1}^{N}\sum_{\alpha=1}^{n} A_i^\alpha(x,u,Du)(D_\alpha u^i - u^i \sum_{l=1}^{N} \frac{u^l}{|u|^2} D_\alpha u^l)$$

However we notice that by the (A3) condition

$$\sum_{i=1}^{N}\sum_{\alpha=1}^{n} A_i^\alpha(x,u,Du)(D_\alpha u^i - u^i \sum_{l=1}^{N} \frac{u^l}{|u|^2} D_\alpha u^l) \ge 0$$

so therefore

$$\sum_{i=1}^{N}\sum_{\alpha=1}^{n} A_i^\alpha(x,u,Du) D_\alpha([u]_\theta)^i \ge 0 \text{ if } |u| > \theta$$

This completes the proof of the lemma.

Proof of the Theorem: In order to invoke the reverse Hölder inequality we have to first prove the following Caccioppoli type inequality.

$$\int_{B_{R/2}} |Du|^p dx \le \epsilon \int_{B_R} |Du|^p dx + \frac{K(\epsilon)}{R^p} \int_{B_R} |u - \overline{u}|^p dx$$

We verify the inequality using test functions $\phi = ([u]_l - d_l)\eta^p$ and obtain the result by a finite induction over the increasing sequence of cut off levels

$$l = k\delta, \;\; k = 1, 2, 3, \ldots, M.$$

$$M = \min\{m \in \mathrm{N} \,|m\delta \geq ||u||\infty\}.$$

Here d_l vectors in $\mathbf{R}^N$ defined by

$$[d]_l = \begin{cases} \overline{u} & \text{if } |\overline{u}| \leq l \\ \frac{l}{|\overline{u}|}\overline{u} & \text{otherwise} \end{cases}$$

with

$$\overline{u} = \frac{1}{|B_R|}\int_{B_R} u\, dx,$$

the average of u over the ball B_R. The function η is the usual localization function, i.e., $\eta \in C_0^\infty(B_R)$ with $0 \leq \eta \leq 1, \eta \equiv 1$ in $B_{R/2}$and $|D\eta| \leq \frac{C}{R}$ for some fixed C greater than 2.

For an approppropriate $\delta > 0$ we get the following estimate for all $\in > 0$

$$\int_{B_R \cap \{|u|<k\delta\}} |Du|^p \eta^p dx \leq \epsilon \int_{B_R} |Du|^p \eta^p dx + \frac{K(\epsilon,p)}{R^p} \int_{B_R} |u - \overline{u}|^p dx$$

where the constant $K(\epsilon, p)$ does not depend on $\mathbf{R}$. The Caccioppoli inequality then follows. We first verify the above inequality for $k = 1$. We invoke Lemma A and ellipticity to obtain:

$$\int_{B_R \cap \{|u|\leq\delta\}} |Du|^p \eta^p dx \leq \int_{B_R \cap \{|u|\leq\delta\}} \sum_{i=1}^{N}\sum_{\alpha=1}^{n} A_i^\alpha(x,u,Du) D_\alpha u^i \eta^p dx$$

$$+ \int_{B_R \cap \{|u|>\delta\}} \sum_{i=1}^{N}\sum_{\alpha=1}^{n} A_i^\alpha(x,u,Du) D_\alpha([u]_\ell)^i \eta^p dx$$

$$= \int_{B_R} \sum_{i=1}^{N}\sum_{\alpha=1}^{n} A_i^\alpha(x,u,Du) D_\alpha([u]_\ell - d_\ell)^i \eta^p dx$$

$$= \int_{B_R} \sum_{i=1}^{N} \sum_{\alpha=1}^{n} A_i^\alpha(x,u,Du) D_\alpha \left(\eta^p([u]_\ell - d_\ell)^i\right) dx$$

$$- \int_{B_R} \sum_{i=1}^{N} \sum_{\alpha=1}^{n} A_i^\alpha(x,u,Du) p D_\alpha \eta \eta^{p-1}([u]_\ell - d_l)^i dx$$

$$= - \int_{B_R} \sum_{i=1}^{N} B_i(x,u,Du)([u]_\ell - d_\ell)^i \eta^p dx$$

$$- \int_{B_R} \sum_{i=1}^{N} \sum_{\alpha=1}^{n} A_i^\alpha(x,u,Du)\; p\; D_\alpha \eta\; \eta^{p-1}([u]_\ell - d_\ell)^i dx$$

$$= - \int_{B_R \cap \{|u| \leq \delta\}} \sum_{i=1}^{N} B_i(x,u,Du)([u]_\ell - d_\ell)^i \eta^p dx$$

$$- \int_{B_R \cap \{|u| > \delta\}} B_i(x,u,Du) \cdot ([u]_\ell - d_\ell)^i \eta^p dx$$

$$- \int_{B_R} \sum_{i=1}^{N} \sum_{\alpha=1}^{n} A_i^\alpha(x,u,Du)\; p\; D_\alpha \eta\; \eta^{p-1}([u]_\ell - d_\ell)^i dx \qquad = (*)$$

We note that by the Zero angle condition on the set $B_R \cap \{|u| > \delta\}$ we have

$$\sum_{i=1}^{N} B_i(x,u,Du)([u]_\ell - d_\ell)^i \geq 0.$$

This is because $|d_\ell| \leq l = \delta = |u_\ell|$ so the zero angle condition applies. Hence we get

$$- \int_{B_R \cap \{|u| \leq \delta\}} \sum_{i=1}^{N} B_i(x,u,Du)(u_\ell - d_\ell)^i \eta^p dx \leq 0$$

therefore

$$(*) \qquad \leq - \int_{B_R \cap \{|u| \leq \delta\}} \sum_{i=1}^{N} B_i(x,u,Du)([u]_\ell - d_\ell)^i \eta^p dx$$

$$-\int_{B_R}\sum_{i=1}^{N}\sum_{\alpha=1}^{n}A_i^\alpha(x,u,Du)\,p\,\eta^{p-1}D_\alpha\eta([u]_\ell-d_\ell)^i dx$$

$$\le \int_{B_R\cap\{|u|\le\delta\}}\sum_{i=1}^{N}|B_i(x,u,Du)|\;|[u]_\ell-d_\ell|\eta^p dx$$

$$+\int_{B_R}\sum_{i=1}^{N}\sum_{\alpha=1}^{n}|A_i^\alpha(x,u,Du)|p\;\cdot|D\eta|\;\eta^{p-1}|[u]_\ell-d_\ell|dx$$

This implies that

$$\lambda\int_{B_R\cap\{|u|\le\delta\}}|Du|^p\eta^p dx\le 2\delta\mu_2\int_{B_R\cap\{|u|\le\delta\}}|Du|^p\eta^p dx$$

$$+\int_{B_R}\mu_1|Du|^{p-1}\frac{2p}{R}\eta^{p-1}|[u]_\ell-d_\ell|dx+\int_{B_R}l(x)\frac{2p}{R}\eta^{p-1}|[u]_\ell-d_\ell|dx$$

Applying Young's inequality to the last two expressions we get:

$$\lambda\int_{B_R\cap\{|u|\le\delta\}}|Du|^p\eta^p dx\le 2\delta\mu_2\int_{B_R\cap\{|u|\le\delta\}}|Du|^p\eta^p dx$$

$$+\epsilon^{\frac{1}{1-p}}k(p)\int_{B_R}|Du|^p\eta^p dx+k(\epsilon,p)\int_{B_R}R^{-p}|[u]_\ell-d_\ell|^p dx$$

or

$$\lambda\int_{B_R\cap\{|u|\le\delta\}}|Du|^p\eta^p dx\le\epsilon^{\frac{1}{1-p}}k(p)\int_{B_R}|Du|^p\eta^p dx\le$$

$$k(\epsilon,p)\int_{B_R}R^{-p}|[u]_\ell-d_\ell|^p dx+2\delta u_2\int_{B_R\cap\{|u|\le\delta\}}|Du|^p\eta^p dx$$

This implies that

$$(\lambda-2\delta\mu_2)\int_{B_R\cap\{|u|\le\delta\}}|Du|^p\eta^p dx\le\epsilon^{\frac{1}{1-p}}k(p)\int_{B_R}|Du|^p\eta^p dx$$

$$+k(\epsilon,p)\int_{B_R} R^{-p}|[u]_\ell - d_\ell|^p dx$$

Now choose $\delta = \frac{\lambda}{4\eta_2}$ denote $\epsilon^{\frac{1}{1-p}} = \bar{\epsilon}$ and divide $\lambda - 2\delta\mu_2$ to get:

$$\int_{B_R\cap\{|u|\leq\delta\}} |Du|^p\eta^p dx \leq \bar{\epsilon}k(p)\int_{B_R} |Du|^p\eta^p dx$$

$$+k(\epsilon,p)\int_{B_R} R^{-p}|[u]_\ell - d_\ell|^p dx$$

Note that the above k may change from estimate to estimate but is not depending on R. We can write the last inequality as

$$\int_{B_R\cap\{|u|\leq\delta\}} |Du|^p\eta^p dx \leq \bar{\epsilon}k(p)\int_{B_R} |Du|^p\eta^p dx + k(\epsilon,p)\int_{B_R} R^{-p}|u-\overline{u}|^p dx$$

Note $|[u]_\ell - d_\ell| \leq |u-\overline{u}|$. For the case $|[u]_\ell| \leq |u|$, $|d_\ell| \leq |\overline{u}|$ cf.: figure 1: By the law of cosines $c \leq b = |u-\overline{u}|$. Therefore $l < r_2$ implies that

$$a = |[u]_\ell - d_\ell| \leq c \leq |u-\overline{u}|$$

So the assumption of induction now reads

$$\int_{B_R\cap\{|u|\leq(k-1)\delta\}} |Du|^p\eta^p dx \leq \bar{\epsilon}k(p)\int_{B_R} |Du|^p\eta^p dx + k(\epsilon,p)R^{-p}\int_{B_R} |u-\overline{u}|^p dx$$

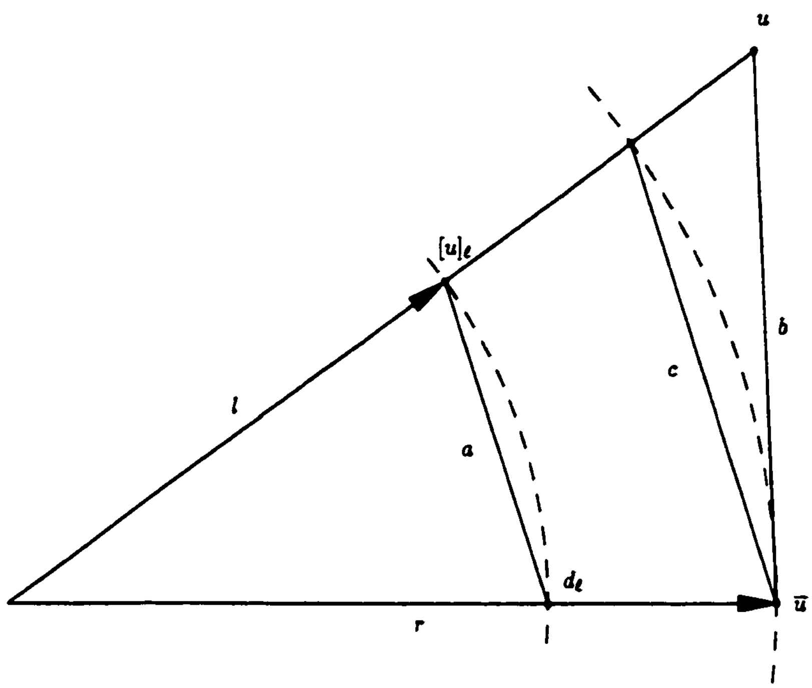

Fig. 1

We estimate using ellipticity and Lemma A:

$$\lambda \int\limits_{B_R \cap \{(k-1)\delta < |u| \leq k\delta\}} |Du|^p \eta^p dx$$

$$\leq \int\limits_{B_R \cap \{(k-1)\delta < |u| \leq k\delta\}} \sum_{i=1}^{N} \sum_{\alpha=1}^{n} A_i^\alpha(x, u, Du) D_\alpha u^i \eta^p dx$$

$$+\int\limits_{B_R\cap\{|u|>k\delta\}}\sum_{i=1}^{N}\sum_{\alpha=1}^{n}A_i^\alpha(x,u,Du)D_\alpha([u]_\ell)^i\eta^p dx$$

$$+\int\limits_{B_R\cap\{|u|\le(k-1)\delta\}}\sum_{i=1}^{N}\sum_{\alpha=1}^{n}A_i^\alpha(x,u,Du)D_\alpha([u]_\ell)^i\eta^p dx$$

$$\le\int\limits_{B_R}\sum_{i=1}^{N}\sum_{\alpha=1}^{n}A_i^\alpha(x,u,Du)D_\alpha([u]_\ell-d_\ell)^i\eta^p dx$$

$$\le-\int\limits_{B_R\cap\{|u|>k\delta\}}\sum_{\alpha=1}^{n}B_i(x,u,Du)D_\alpha([u]_\ell-d_\ell)^i\eta^p dx$$

$$-\int_{B_R\cap\{(k-1)\delta<|u|\le k\delta\}}\sum_{i=1}^{N}B_i(x,u,Du)D_\alpha([u]_\ell-d_\ell)^i\eta^p dx$$

$$-\int\limits_{B_R}\sum_{i=1}^{N}\sum_{\alpha=1}^{n}A_i^\alpha(x,u,Du)p\,\eta^{p-1}D_\alpha\eta([u]_\ell-d_\ell)^i dx$$

$$\le-\int\limits_{B_R\cap\{|u|\le(k-1)\delta\}}\sum_{i=1}^{N}B_i(x,u,Du)\ ([u]_\ell-d_\ell)^i\eta^p dx$$

$$-\int\limits_{B_R\cap\{(k-1)\delta<|u|\le k\delta\}}\sum_{i=1}^{N}B_i(x,u,Du)\left(\frac{k\delta}{|u|}u-d_\ell\right)^i\eta^p dx$$

$$-\int\limits_{B_R\cap\{(k-1)\delta<|u|\le k\delta\}}\sum_{i=1}^{N}B_i(x,u,Du)\ (u-\frac{k\delta}{|u|}u)^i\eta^p dx$$

$$+\int\limits_{B_R}\sum_{i=1}^{N}\sum_{\alpha=1}^{n}|A_i^\alpha(x,u,Du)|\ p\ |D\eta|\ \eta^{p-1}|[u]_\ell-d_\ell|\eta^p dx$$

Since

$$(B,u_l-d_l)=(B,\frac{k\delta}{|u|}u-d_l)+(B,u-\frac{k\delta}{|u|}u)$$

where (.,.) denotes the scalar product of vectors.

We note that on the set $\{(k-1)\delta < |u| \leq k\delta\}$ we have $|u - \frac{k\delta}{|u|}u| < \delta$ also $(B, \frac{k\delta}{|u|}u - d_l) \geq 0$ since $|\frac{k\delta}{|u|}u| = k\delta = l \geq |d_l|$.

So

$$\int_{B_R \cap \{(k-1)\delta<|u|\leq k\delta\}} \sum_{i=1}^{N} B_i(x,u,Du) \cdot (\frac{k\delta}{|u|}u - d_\ell)^i \eta^p dx \leq 0$$

So

$$(**) \leq \int_{B_R \cap \{|u|\leq (k-1)\delta\}} \mu_2 |Du|^p 2\delta \eta^p dx + \delta\mu_2 \int_{B_R \cap \{(k-1)\delta<|u|\leq k\delta\}} |Du|^p \eta^p dx$$

$$+\bar{\epsilon}k(p) \int_{B_R} |Du|^p \eta^p dx + k(\epsilon,p) \int_{B_R} R^{-p} |[u]_\ell - d_\ell|^p dx$$

And using the induction assumption

$$\leq \mu_2 2\delta\bar{\epsilon}k(p) \int_{B_R} |Du|^p \eta^p dx + k(\epsilon,p) R^{-p} \int_{B_R} |u - \overline{u}|^p dx$$

$$+\delta\mu_2 \int_{B_R \cap \{(k-1)\delta<|u|\leq k\delta\}} |Du|^p \eta^p dx + \bar{\epsilon}k(p) \int_{B_R} |Du|^p \eta^p dx$$

$$+k(\epsilon,p) \int_{B_R} R^{-p} |[u]_\ell - d_\ell|^p dx$$

$\Rightarrow$

$$(\lambda - \delta\mu_2) \int_{B_R \cap \{(k-1)\delta<|u|\leq k\delta\}} |Du|^p \eta^p dx \leq \mu_2 2\delta\bar{\epsilon}k(p) \int_{B_R} |Du|^p \eta^p dx$$

$$+\mu_2 2\delta k(\bar{\epsilon},p) R^{-p} \int_{B_R} |u - \overline{u}|^p dx + \bar{\epsilon}k(p) \int_{B_R} |Du|^p \eta^p dx$$

$$+k(\epsilon,p) \int_{B_R} R^{-p} |[u]_\ell - d_\ell|^p dx$$

This implies that

$$\int\limits_{B_R\cap\{(k-1)\delta<|u|\leq k\delta\}} |Du|^p\eta^p dx \leq \bar{\epsilon}k(p)\int\limits_{B_R}|Du|^p\eta^p dx + k(\bar{\epsilon},p)R^{-p}\int\limits_{B_R}|u-\overline{u}|^p dx$$

Since

$$\int\limits_{B_R\cap\{|u|\leq k\delta\}} |Du|^p\eta^p dx = \int\limits_{B_R\cap\{|u|\leq k-1)\delta\}} |Du|^p\eta^p dx$$

$$+ \int\limits_{B_R\cap\{(k-1)\delta<|u|\leq k\delta\}} |Du|^p\eta^p dx$$

We get (conclusion of induction)

$$\int\limits_{B_R\cap\{|u|\leq k\delta\}} |Du|^p\eta^p dx \leq \bar{\epsilon}k(p)\int\limits_{B_R}|Du|^p\eta^p dx + k(\bar{\epsilon},p)R^{-p}\int\limits_{B_R}|u-\overline{u}|^p dx$$

Since $\|u\|_\infty \leq M\delta$ we get the Cacciopoli inequality

$$\int\limits_{B_R}|Du|^p\eta^p dx \leq \bar{\epsilon}k(p)\int\limits_{B_R}|Du|^p\eta^p dx + k(\bar{\epsilon},p)R^{-p}\int\limits_{B_R}|u-\overline{u}|^p dx$$

So

$$(1-\bar{\epsilon}k(p))\int\limits_{B_R}|Du|^p\eta^p dx + k(\epsilon,p)R^{-p}\int\limits_{B_R}|u-\overline{u}|^p dx$$

Choosing $\bar{\epsilon}$ such that $(1-\bar{\epsilon}k(p)) > 0$ for all $k(p)$ we get:

$$\int\limits_{B_R}|Du|^p\eta^p dx \leq k(\epsilon,p)R^{-p}\int\limits_{B_R}|u-\overline{u}|^p dx$$

which can be rewritten as:

$$\fint_{B_{R/2}} |Du|^p dx \leq k \fint_{B_{R/2}} |u - \overline{u}|^p dx,$$

using properties of η. Applying the Sobolev-Poincare inequality we obtain

$$\fint_{B_{R/2}} |Du|^p dx \leq c(\fint_{B_R} |Du|^{\frac{pn}{n+p}} dx)^{\frac{n+p}{n}}$$

We are now is a position to apply the following theorem taken from [2].

<u>Theorem:</u> Let Q be an n-cube. Suppose that

$$\fint_{Q_R} g^p dx \leq b\{(\fint_{Q_{2R}} g dx)^p + \fint_{Q_{2R}} f^p dx\} + \theta \fint_{Q_{2R}} g^p dx$$

for each $x_o \in Q$ where

$$Q_R = \{x \in \mathbf{R}^n | |x_i| < R, \ i = 1, \ \dots \ , n\}$$

and each

$$R < \max \{\frac{1}{2} \text{ dist } (x_o, \partial Q), R_o\}$$

where R_o, b, θ are constants with

$$b > 1, \ R_o > 0 \quad 0 \leq \theta < 1,$$

with

$$g \in L^p(Q), \ \ p > 1, \ \ f \in L^r(Q) \ \ r > p.$$

then $g \in L^q_{\text{loc}}(Q)$ for $q \in [p, p+\epsilon)$ and

$$(\fint_{Q_R} g^q dx)^{\frac{1}{q}} \leq b_1\{(\fint_{Q_{2R}} g^p dx)^{\frac{1}{p}} + (\fint_{Q_{2R}} f^q dx)^{\frac{1}{q}}\}$$

$Q_{2R} \subset Q$, $R < R_o$, where b_1 and ϵ are positive constants depending only on b, q, n and r.

We set

$$q = \frac{pnr}{n+p}\,,\ g = |Du|^{\frac{pn}{n+p}},\ f \equiv 0\,,\ \theta \equiv 0$$

This implies by the above theorem that $|Du|^{\frac{pn}{n+p}} \in L^r_{loc}(B)$ for every

$$r \in [\frac{p+n}{n}, \frac{p+n}{n} + \epsilon]$$

ϵ a small chosen constant and

$$(\fint_{B_{R/2}} |Du|^{\frac{pnr}{n+p}} dx)^{\frac{1}{r}} \leq C(\fint_{B_R} |Du|^p dx)^{\frac{n}{n+p}}$$

writing $\frac{1}{r}$ as $\frac{n+p}{pnr} \cdot \frac{pn}{n+p}$ we obtain by simple algebra

$$(\fint_{B_{R/2}} |Du|^q dx)^{1/q} \leq C(\fint_{B_R} |Du|^p dx)^{1/p}$$

where $q > p$; this completes the proof of the theorem.

Chapter 2

Regularity of weak solutions

The goal of this chapter is to prove regularity of weak solutions of System (1) with coefficiant functions

$$A_i^\alpha(x,u,Du) = |Du|^{p-2} D_\alpha u^i(x),$$

i.e. $A(u)$ is the p-Laplacian plus a perturbation. We present a variant of the argument presented in [3] based on the convexity of the potential of the p-Laplacian to deal with all p, $1 < p < n$, simultaneously.

In order to prove theorem 2, we first prove a proposition similar to a lemma presented in [2].

<u>Proposition:</u> For every $x_o \in \Omega$ and for every ρ and R with

$$\rho < R < \max\{\mathrm{dist}(x_o, \partial\Omega, 1\}$$

we have the inequality

$$\int_{B\rho(x_o)} (1 + |Du|^p)dx \leq c\left[\left(\frac{\rho}{R}\right)^n + \chi(x_o, R)\right] \int_{B_{R(x_o)}} (1 + |Du|^p)dx$$

where

$$\chi(x_o, R) = g(R^{p-n} \int_{B_{R(x_o)}} |Du|^p dx)$$

for an increasing continuous function $g : \mathbf{R}^+ \to \mathbf{R}^+$ and $g(t)$ going to zero as t goes to zero.

Proof of Proposition:

First we note that $|\xi|^p$ defines a convex mapping from $\mathbf{R}^{nN}$ into $\mathbf{R}$. Therefore we have for any two $\xi, \hat{\xi} \in \mathbf{R}^{nN}$ and $p > 1$:

$$|\xi|^p \leq |\hat{\xi}|^p + p\sum_{i=1}^{N}\sum_{\alpha=1}^{n} A_i^\alpha(\xi)(\xi - \hat{\xi})_i^\alpha$$

and

$$0 \leq \sum_{i=1}^{N}\sum_{\alpha=1}^{n}(A_i^\alpha(\xi) - A_i^\alpha(\hat{\xi})(\xi - \hat{\xi})_i^\alpha$$

Now locally we compare the weak solution u of system (1) with a weak solution v of the unperturbed system

$$\begin{cases} -\sum\limits_{\alpha=1}^{n} D_\alpha A_i^\alpha(Dv) = 0; & \text{in } B_{R/2} \text{ for } i = 1, \ldots, N \\ v = u & \text{in } \partial B_R/2 \end{cases}$$

For $\rho < \frac{R}{3}$ we have by theorem B $c.f.$: [10] that

$$\int_{B_\rho} |Dv|^p dx \leq \rho^n \text{ess} \sup_{B_{R/3}} |Dv|^p \leq c(\frac{\rho}{R})^n \int_{B_{R/2}} (1 + |Dv|^p) dx$$

We also note that $||v||_{L^\infty(B_{R/2})} \leq ||v||_{L^\infty(\Omega)}$; see [8].

Using the convexity of $|\xi|^p$ we get:

$$\int\limits_{B_{R/2}} |Dv|^p dx \leq p \int\limits_{B_{R/2}} \sum_{i=1}^{N}\sum_{\alpha=1}^{n} A_i^\alpha(Dv) D_\alpha(v-u)^i dx$$

$$+ \int\limits_{B_{R/2}} |Dv|^p dx = \int\limits_{B_{R/2}} |Du|^p dx,$$

since v is a weak solution of the unperturbed system.

For $w = u - v$ in $H_o^{1,p}(B_{R/2})$ we have

$$\int\limits_{B_{R/2}} |Du|^p dx \le c \int\limits_{B_{R/2}} |Du|^p dx$$

and we note that the trivial extension of w is in $H_o^{1,p}(\Omega)$. Next we claim that

$$* \int\limits_{B_R/2} (1 + |Du|^p)|w| dx \le c \int\limits_{B_R} (1 + |Du|^p) dx\, \chi(x_o, R).$$

Indeed using first Hölder's inequality and then the reverse Hölder inequality verified for u in chapter 1 we get

$$\int\limits_{B_{R/2}} |w|\,(| + |Du|^p) dx \le c[(\int\limits_{B_{R/2}} dx)^{p/q} + (\int\limits_{B_{R/2}} |Du|^q dx)^{\frac{p}{q}}](\int\limits_{B_{R/2}} |w|^{\frac{q}{q-p}} dx)^{\frac{q-p}{q}}$$

$$\le c(1 + \int\limits_{B_R} |Du|^p dx)(\int\limits_{B_{R/2}} |w|^p dx)^{\frac{q-p}{pq}}$$

In the last estimate we used the fact that $|w|^{\frac{q}{q-p}} \le c|w|$ since $q > p$ and applied Hölder's inequality to $\int\limits_{B_{R/2}} |w| dx$.

Since $w|_{\partial B_{R/2}} = 0$, we use Poincare's inequality to estimate

$$\int\limits_{B_{R/2}} |w|^p dx \le cR^p \int\limits_{B_{R/2}} |Dw|^p dx \le cR^p \int\limits_{B_{R/2}} |Du|^p dx$$

Multiplying this inequality with R^{-n} estimate $(*)$ follows with

$$\chi(x_o, R) = (R^{p-n} \int\limits_{B_R(x_o)} |Du|^p dx)^{\frac{q-p}{q}}$$

Also, because of our growth condition we estimate with Young's inequality and Hölder's inequality

$$\int_{B_\rho} \sum_{i=1}^{N} \sum_{\alpha=1}^{n} A_i^\alpha(Dv) D\alpha(u-v)^i dx$$

$$\leq c \left(\int_{B_\rho} (|Dv)|^{p-1})^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{B_\rho} |Du|^p dx \right)^{1/p}$$

$$\leq c\left(\int_{B_\rho} |Dv|^p dx \right) + \frac{1}{2p} \left(\int_{B_\rho} |Du|^p dx \right)$$

Now putting these results together we get

$$\int_{B_\rho} |Dv|^p dx \leq \int_{B_\rho} |Dv|^p dx + p \int_{B_\rho} \sum_{i=1}^{N} \sum_{\alpha=1}^{n} A_i^\alpha(Du) D_\alpha(u-v)^i dx$$

$$\leq \frac{1}{2p} \int_{B_\rho} |Du|^p dx + C \int_{B_p} |Dv|^p dx + p$$

$$\int_{B_p} \sum_{i=1}^{N} \sum_{\alpha=1}^{n} [A_i^\alpha(Du) - A_i^\alpha(Dv)] D_\alpha(u-v)^i dx$$

$$\leq \frac{1}{2} \int_{B_p} |Du|^p dx + C \int_{B_p} |Dv|^p dx$$

(since v is a weak solution of the unperturbed system.)

$$+p \int_{B_{R/2}} \sum_{i=1}^{N} \sum_{\alpha=1}^{n} A_i^\alpha(Du) D_\alpha(u-v)^i dx$$

$$= \frac{1}{2} \int_{B_\rho} |Du|^p dx + C \int_{B_\rho} |Dv|^p dx$$

$$+p \int_{B_{R/2}} \sum_{i=1}^{N} Bi(x,u,Du)w^i dx \le \frac{1}{2} \int_{B_\rho} |Du|^p dx + C$$

$$\int_{B_\rho} |Dv|^p + p \int_{B_{R/2}} (1+|Du|^p)|w| dx$$

using the definition of weak solution and the growth condition on the perturbation.

With (*) we finally establish the desired inequality of the proposition:

$$\int_{B_p} |Du|^p dx \le C(\frac{\rho}{R})^n (\int_{B_{R/2}} |Dv|^p dx)$$

$$+C \int_{B_R} (1+|Du|^p) dx\, \chi(x_o,R) \le C[(\frac{\rho}{R})^n + \chi(x_o,R)] \int_{B_R} (1+|Du|^p) dx$$

We shall indicate here how this inequality implies regularity. With $o < \alpha < 1$ choose a fixed τ such that $2C\tau^{p-p\alpha} = 1$, with C being the same constant as in the proposition just proved. We define Ω_o to be the set

$$\{x_o \in \Omega |\text{ there is } R_o < \min(1, \text{dist}\,(x_o, \partial\Omega)) \text{ such that } \sup_{R<R_O} (x_o,R) < \tau^n\}$$

We shall prove that the weak solution is Hölder continuous on Ω_o. We introduce for x_o and $R < R_o$ the quantity

$$\phi(R) = R^{p-n} \int_{B_R(x_o)} (1+|Du|^p) dx$$

For $o < \gamma < 1$ we have:

$$\phi(\gamma R) \le C\tau^p \phi(R)(1+\chi(x_o,R)\tau^{-n})$$

To see this note that our proposition implies that

$$\phi(\rho) \leq C\phi(R)\{(\frac{\rho}{R})^n + chi(x_o, R)\}$$

So

$$\phi(\tau R) \leq C\phi(R)\{(\frac{p}{R})^n + \chi(x_o, R)\} = C\phi(R)(\tau^n + \chi(x_o, R))$$

multiplying by γ^{-n} and dividing by γ^{-n} we obtain

$$C\phi(R)(\tau^n + \chi(x_o, R)) = C\phi(R)(\tau^n + \chi(x_o, R))\tau^{-n}$$

$$= \tau^n C\phi(R)(1 + \tau^n\chi(x_o, R)) \leq \tau^p C\phi(R)(1 + \tau^n\chi(x_o, R),)$$

since $n > p \;\; 0 < \tau < 1$.

Since $\sup\limits_{R<R_o} \chi(x_o, R) < \tau^n$ we have that $\phi(\tau R) \leq \gamma^p\phi(R)$ and since $2C\tau^{p-pk} = 1$ implies $C\tau^p = \tau^{p\alpha}$ we deduce $\phi(\tau R) \leq \tau^{p\alpha}\phi(R)$. By iteration we get $\phi(\tau^k R) \leq \tau^{kp\alpha}\phi(R)$ for all $k \in \mathrm{N}$. This implies the estimate

$$\phi(\rho) \leq C(\frac{\rho}{R})^{p\alpha}\phi(R)$$

for all ρ with $0 < \rho < R$. Indeed for $\rho < R$ there exists a k such that $\tau^{k+1}R \leq \rho < \tau^k R$.

Further

$$\phi(\rho) = \rho^{p-n}\int_{B_\rho}(1 + |Du|^p)dx \leq \rho^{p-n}\int_{B_{\tau^k R}}(1 + |Du|^p)dx$$

$$\leq (\tau^{k+1}R)^{p-n}\int_{B_{\tau^k R}}(1 + |Du|^p)dx = \tau^{p-n}\phi(\tau^k R).$$

Now

$$\phi(\tau^k R) \leq \tau^{kp\alpha} Q\phi(R) = (\frac{1}{\tau})^{p\alpha}\tau^{(k+1)p\alpha}\phi(R)$$

$$= (\frac{1}{\tau})^{p\alpha}\frac{\tau^{(k+1)p\alpha}}{(\frac{\rho}{R})^{p\alpha}}(\frac{\rho}{R})^{p\alpha}\phi(R) = \tau^{-p\alpha}(\frac{\rho}{R})^{p\alpha}\phi(R)$$

Hence the estimate holds with $C = \tau^{-n-p(\alpha-1)}$.

It follows that for $x \in B(x_o, R)$ we have

$$\int_{B_\rho(x)} |Du|^p dy \leq C\int_\Omega (1 + |Du|^p)dy\ \rho^{n-p+p\alpha}$$

To see this observe that by definition of $\phi(\rho)$ we have

$$\int_{B\rho(x)} |Du|^p dy \leq \phi(\rho)\rho^{n-p} \leq C\phi(R)(\frac{\rho}{R})^{p\alpha}\frac{\rho^{n-p}}{R^{n-p}}\int_\Omega (1 + |Du|^p)dy$$

$$\leq \rho^{n-p+p\alpha}C_1\int_\Omega (1 + |Du|^p)dy$$

and so, for every $x \in B(x_o, R)$

$$\int_{B_\rho(x)} |u(y) - u_{x,\rho}|^p dy \leq C_2\rho^{n+p\alpha},$$

this is because by theorem C (Poincare's inequality)

$$\int_{B_\rho(x)} |u(y) - u_{x,\rho}|^p dy \leq C\rho^p \int_{B_\rho(x)} |Du|^p dy$$

$$\leq C\rho^p\rho^{n-p+p\alpha}C_1 \int_{B_\rho(\hat{x})} |u(y) - u_{x\rho}|^p dy$$

$$\leq C_3\rho^{n+p\alpha}$$

Hölder continuity follows from Theorem D.

Finally we shall point out why $H^{n-q}(\Omega \backslash \Omega_o) = 0$. First we show that

$$(R^{p-n} \int\limits_{B_R} |Du|^p dx)^{1/p} \le (R^{q-n} \int\limits_{B_R} |Du|^q dx)^{1/q}$$

By Hölder's inequality we have:

$$\int\limits_{B_R} |Du|^p dx \le (\int\limits_{B_R} |Du|^q dx)^{p/q} |B_R|^{1-p/q}$$

so

$$\frac{1}{R^{n-p}} \int\limits_{B_R} |Du|^p dx \le \frac{C}{|B_R|^{\frac{n-p}{n}}} (\int\limits_{B_R} |Du|^q dx)^{p/q} |B_R|^{1-p/q}$$

$$= (\frac{1}{|B_R|^{\frac{n-p}{n}-1-q/p}} \int\limits_{B_R} |Du|^q dx)^{\frac{p}{q}}$$

$$\le (\frac{1}{R^{n-q}} \int\limits_{B_R} |Du|^q dx)^{\frac{1}{q}}$$

Note that by the result of chapter 1, $|Du|^q \in L^1_{loc}$ with $q < p$.

We conclude that

$$\{x_o \in \Omega | \liminf_{R \to 0} \frac{1}{R^{n-p}} \int\limits_{B_R(x_o)} |Du|^p dx > 0\}$$

$$\subset \{x_o \in \Omega | \liminf_{R \to 0} \frac{1}{R^{n-p}} \int\limits_{B_R(x_o)} |Du|^p dx > 0\} =: E$$

so by theorem A of the introduction $H^{n-q}(E) = 0$ and hence $H^{n-q}(\Omega \backslash \Omega_o) = 0$.

This completes the proof of the theorem.

Chapter 3

Everywhere regularity

In certain cases everywhere regularity holds; i.e. for systems of quadratic growth ($p = 2$) with special structure and assumptions. We will consider systems of partial differential equations of the form

$$\sum_{\alpha,\beta=1}^{n} D_\alpha(A^{\alpha\beta} D_\beta u^i) = f_i(x, u, Du) \quad i = 1, \dots, N. \tag{3.1}$$

These type of systems are called 'diagonal' since $A_{ij}^{\alpha\beta} = A^{\alpha\beta}\delta_{ij}$ where δ_{ij} is the Kronecker symbol. We assume the following conditions:

(E_1)

$$A^{\alpha\beta}(x) \in L^\infty(\Omega)$$

(E_2)

$$\sum_{\alpha,\beta=1}^{n} A^{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq \lambda|\xi|^2, \ \forall \xi \in \mathbf{R}^n; \lambda > 0, \text{ (ellipticity)}$$

(E_3)

$$|f_i(x, u, Du)| \leq a|Du|^2 + b; \quad a, b \text{ constants } i = 1, \dots, N \text{ (quadratic growth)}$$

We also assume that the L^p-estimate for the gradient of a weak solution of (1) is true. By definition we have

$$\Omega \setminus \Omega_o = \{x_o \in \Omega : \liminf_{R\to 0} R^{2-n} \int_{B_R(x_o)} |Du|^2 dx > \epsilon_0\}$$

where ϵ_0 is a positive constant. To show $\Omega = \Omega_o$ we show that for every x_o there exists ρ such that

$$\rho^{2-n} \int\limits_{B_\rho(x_o)} |Du|^2 dx \leq \epsilon_o.$$

We choose $\phi = u\eta$ with $\eta \in C_o^\infty(B_{2R}), \eta \geq 0$, in the weak formulation

$$\int\limits_\Omega A^{\alpha\beta} D_\alpha u^i D_\beta \phi^i dx = \int\limits_\Omega f_i(x, u, Du)\phi^i dx \quad i = 1, \ldots, N$$

to get

$$\int\limits_\Omega A^{\alpha\beta} D_\alpha(u^i) D_\beta(u^i\eta) dx = \int\limits_\Omega f_i(x, u, Du) u^i \eta dx$$

or

$$\int\limits_\Omega A^{\alpha\beta} D_\alpha(u^i) D_\beta(u^i)\eta dx + \int\limits_\Omega A^{\alpha\beta} D_\alpha(u^i) D_\beta \eta u^i dx = \int\limits_\Omega f_i(x, u, Du) u^i \eta dx$$

where $\sup\limits_\Omega |u(x)| \leq M$. We take note that the function

$$z = M(2R) - |u|^2, M(t) := \sup_{B_{t,(xo)}} |u|^2$$

is a nonnegative super solution of an elliptic operator with right-hand side

$$\int\limits_{B_{2R}} A^{\alpha\beta} D_\alpha z D_\beta \eta dx \geq 2\bar{b} \int\limits_{B_{2R}} \eta dx$$

for every $\eta \in C_o^\infty(B_{2R}), \eta \geq 0$

From the weak Harnack inequality (see theorem 8.18 in [5]) we have

$$R^{-n} \int\limits_{B_{2R}} z dx \leq C[\inf_{B_R} z \ + R^2]$$

Now let $w \in H_o^1(B_{2R})$ be the solution of the equation

$$(*) \qquad \int\limits_{B_{2R}} A^{\alpha\beta} D_\alpha w D_\beta \phi dx = \frac{1}{R^2} \int\limits_{B_{2R}} \phi dx \ \ \forall \phi \in H_o^1(B_{2R})$$

Take $\phi = wz$, we get:

$$\int\limits_{B_{2R}} A^{\alpha\beta} D_\alpha w D_\beta (wz) dx = \frac{1}{R^2} \int\limits_{B_{2R}} wz dx$$

or

$$(3.2) \qquad \frac{1}{2} \int\limits_{B_{2R}} A^{\alpha\beta} D_\alpha w^2 D_\beta z dx + \int\limits_{B_{2R}} z A^{\alpha\beta} D_\alpha w D_\beta w dx = \frac{1}{R^2} \int\limits_{B_{2R}} wz dx.$$

We have $w \leq \alpha_1$ in B_{2R} by the Maximum principle and $w \geq \alpha_2 >$ in B_R by the weak Harnack inequality. Moreover, α_1 and α_2 do not depend on R.

<u>Lemma</u>: For a weak solution of $(*)$ there is an $\alpha_2 > 0$ such that $w \geq \alpha_2$ in B_1 with α_2 depending only on λ and M with $|A^{\alpha\beta}(x)| \leq M$.

<u>Proof:</u> To see this; we perform the change of independant variables $x = Ry$. The function $v(y) = w(Ry)$ is a solution of the Dirichlet problem

$$\begin{cases} -D_\alpha(a^{\alpha\beta} D_\beta v) = 1 & \text{in } B_2 \\ v = 0 & \text{on } \partial B_2 \end{cases}$$

where $a^{\alpha\beta}(y) = A^{\alpha\beta}(Ry, u(Ry))$.

As a consequence of the weak Harnack inequality we have the following Lemma, implying the independence of α_1 and α_2 from R.

We first show that

$$(*)' \int\limits_{B_2 R} A^{\alpha\beta} D_\beta w D_\alpha \phi dx = \frac{1}{R^2} \int\limits_{B_2 R} \phi dx$$

if and only if

$$\int\limits_{B_2} \alpha^{\alpha\beta}(y) D_\beta v(y) D_\alpha \phi(y) dy = \int\limits_{B_2} \phi(y) dy$$

Observe the following

$$\int\limits_{B_2} \alpha^{\alpha\beta}(y) D_\beta v(y) D_\alpha \phi(y) dy = \int\limits_{B_2} \phi(y) dy$$

if and only if

$$\int\limits_{B_2} \alpha^{\alpha\beta}(y) D_\beta w(Ry) D_\alpha \phi(y) dx = \int\limits_{B_2} \phi(y) dy \quad (\text{since } v(y) = w(Ry))$$

if and only if

$$\int\limits_{B_2} \alpha^{\alpha\beta}(y) R D_\beta w(Ry) D_\alpha \tilde{\phi}(y) dy = \int\limits_{B_2} \tilde{\phi}(y) dy \quad \text{where } \tilde{\phi}(y) = \phi(Ry)$$

if and only if

$$R \int\limits_{B_2} A^{\alpha\beta}(Ry, u(Ry)) D_\beta w(Ry) D_\alpha \tilde{\phi}(y) dy = \int\limits_{B_2} \tilde{\phi}(y) dy$$

if and only if

$$R \int\limits_{B_2} A^{\alpha\beta}(Ry, u(Ry)) D_\beta w(Ry) D_\alpha (\phi(Ry) dy = \int\limits_{B_2} \tilde{\phi}(y) dy$$

if and only if

$$R^2 \int\limits_{B_2} A^{\alpha\beta}(Ry, u(Ry)) D_\beta w(Ry) D_\alpha (\phi(Ry) dy = \int\limits_{B_2} \phi(Ry) dy$$

Now let

$$x = Ry = F(y)\ ,\ J = \det DF(y) = \det \begin{pmatrix} R & & \\ & \ddots & \\ & & R \end{pmatrix} = R^n$$

then the last equation holds if and only if

$$\frac{R^2}{R^n} \int\limits_{B_{2R}} A^{\alpha\beta}(x, u, (x)) D_\beta w(x) D_\alpha \phi(x) dx = \frac{1}{R^n} \int\limits_{B_{2R}} \phi(x) dx$$

if and only if

$$\int\limits_{B_{2R}} A^{\alpha\beta}(x, u, (x)) D_\beta w(x) D_\alpha \phi(x) dx = \int\limits_{B_{2R}} \phi(x) dx$$

This proves $(*)'$.

Note that if $|x| < 1$ then $x \in B_{\frac{1}{4}}(y)$ for some y with $|y| < 3/4$. We need a lower bound on $||v||_{L^1(B_{\frac{1}{2}})}(y)$. (See theorem 8.18 in [5])

Note that from

$$\begin{cases} -D_\alpha(a^{\alpha\beta} D_\beta v) = 1 & \text{on } B_2 \\ v = 0 \text{ on } \partial B_2 \end{cases}$$

We have the weak formulation

$$\int\limits_{B_2} A^{\alpha\beta} D_\alpha v D_\beta \phi dx = \int\limits_{B_2} \phi dx$$

Choose $\phi = \eta, \eta \in C_o^\infty(B_{\frac{1}{2}}(y),\ \eta \geq 0$ and fix $R = \frac{1}{4}$; this implies

$$\int\limits_{B_{\frac{1}{4}}(y)} a^{\alpha\beta} D_\alpha v D_\beta \eta dx = \int\limits_{B_{\frac{1}{4}}(y)} \eta dx$$

hence

$$C_1(N) \le \int\limits_{B_{\frac{1}{4}}(y)} \eta dx = \int\limits_{B_{\frac{1}{4}}(y)} a^{\alpha\beta} D_\alpha v D_\beta \eta dx$$

$$\le M\Big(\int\limits_{B_{\frac{1}{4}}(y)} |Dv|^2 dx)^{\frac{1}{2}}\Big)\Big(\int\limits_{B_{\frac{1}{4}}(y)} |D\eta| dx)^{\frac{1}{2}}$$

$$\le MC_2(N)\Big(\int\limits_{B_{\frac{1}{4}}(y)} |Dv|^2 dx)^{\frac{1}{2}}$$

so

$$\Big(\int\limits_{B_{\frac{1}{4}}(y)} |Dv|^2 dx^{\frac{1}{2}}\Big) \ge M((N).$$

Now take

$$\eta \in C_o^\infty(B_{\frac{1}{2}}(y)) \text{ with } 0 \le \eta \le 1, \;\; \eta \equiv 1, \text{ on } B_{\frac{1}{4}}(y), \text{ and} |D\eta| \le 4C,$$

then

$$\int\limits_{B_{\frac{1}{2}}(y)} \lambda |Dv|^2 \eta^2 dx \le \int\limits_{B_{\frac{1}{2}}(y)} A^{\alpha\beta} D_\alpha v D_\beta v \eta^2 dx$$

$$= \int\limits_{B_{\frac{1}{2}}(y)} A^{\alpha\beta} D_\alpha v D_\beta (v\eta^2) dx - \int\limits_{B_{\frac{1}{2}}(y)} 2\eta v A^{\alpha\beta} D_\alpha v D_\beta \eta dx$$

$$= \int\limits_{B_{\frac{1}{2}}(y)} v\eta^2 dx + \frac{\lambda}{2} \int\limits_{B_{\frac{1}{2}}(y)} |Dv|^2 \eta^2 dx + \frac{2}{\lambda} \int\limits_{B_{\frac{1}{2}}(y)} |v|^2 dx$$

so

$$\frac{\lambda}{2} \int\limits_{B_{\frac{1}{2}}(y)} |Dv|^2 \eta^2 dx \le \int\limits_{B_{\frac{1}{2}}(y)} v\eta^2 dx + \frac{2}{\lambda} \int\limits_{B_{\frac{1}{2}}(y)} |v|^2 dx$$

hence

$$\lambda \int_{B_{\frac{1}{2}}(y)} |Dv|^2\eta^2 dx \le 2 \int_{B_{\frac{1}{2}}(y)} v\eta^2 dx + \frac{4}{\lambda} \int_{B_{\frac{1}{2}}(y)} |v|^2 dx$$

$$\le 2 \int_{B_{\frac{1}{2}}(y)} v\eta^2 dx + C \int_{B_{\frac{1}{2}}(y)} |v|^2 dx$$

$$\le (2+\alpha C) \int_{B_{\frac{1}{2}}(y)} |v| dx$$

since one can write $|v|^2 \le \alpha |v|$, where $\alpha = \max_{B_2} |v|$.

Since we already have

$$\left(\int_{B_{\frac{1}{4}}(y)} |Dv|^2 dx \right) \ge (MC(N))^2$$

this implies that

$$(MC(N))^2 \le \int_{B_{\frac{1}{4}}(y)} |Dv|^2 dx) \le \int_{B_{\frac{1}{2}}(y)} |Dv|^2\eta^2 dx \le \frac{(2+\alpha C)}{\lambda} \left(\int_{B_{\frac{1}{2}}(y)} |v| dx \right)$$

So

$$0 < \frac{\lambda MC(N)^2}{2+\alpha C} \le \int_{B_{\frac{1}{2}}(y)} |v| dx \le C \inf_{B_{\frac{1}{2}}(y)} v.$$

From this it can be seen that α_2 does not depend on R.

In conclusion the relation (setting $\eta = w^2$ in (2))

$$\frac{1}{2} \int_{B_{2R}} A^{\alpha\beta} D_\beta \eta D_\alpha z dx + \int_{B_{2R}} z A^{\alpha\beta} D_\alpha w D_\beta w dx = \frac{1}{R^2} \int_{B_{2R}} wz dx$$

implies that

$$\int\limits_{B_{2R}} D_\beta \eta D_\alpha z dx \leq CR^{-2} \int\limits_{B_{2R}} z dx$$

Since w is bounded in B_{2R} and because of the ellipticity,

$$\int\limits_{B_{2R}} z A^{\alpha\beta} D_\alpha w D_\beta w dx \geq 0.$$

We also have

$$\begin{aligned}
\int\limits_{B_{2R}} |Du|^2 \eta dx \leq \int\limits_{B_{2R}} D_\beta \eta D_\alpha dx &\leq CR^{-2} \int\limits_{B_{2R}} z dx \\
&= CR^{-2}[R^n(\inf_{B_R} z + R^2)] \\
&= CR^{n-2}(\inf_{B_R} z + R^2) \\
&= CR^{n-2}[M(2R) - M(R) + R^2]
\end{aligned}$$

Observe that

$$\sum_{k=0}^{\infty}[M(2^{1-k}R) - M(2^{-k}R)] \leq M(2R) \leq \sup_{\Omega} |u|^2$$

so

$$\begin{aligned}
\sum_{k=0}^{\infty} \int\limits_{B_{2^{-k}R}} |Du|^2 dx &\leq CR^{n-2} \sum_{k=0}^{\infty}[M(2^{1-k}R) - M(2^{-k}R) + R^2] \\
&\leq CR^{n-2}(\sup_{\Omega} |u|^2 + R^2)
\end{aligned}$$

Therefore $\rho^{2-n} \int\limits_{B_\rho(x_o)} |Du|^2 dx \leq \epsilon_o$ with $\rho = 2^{-k}R$ for some k.

This implies $\Omega = \Omega_o$ and therefore the everywhere regularity of u.

It is natural to ask whether everywhere regularity holds for systems of the type $A(u) + B(u) = 0$ with $A(u)$ being the p-Laplace operator if $p \neq 2$. It does not seem that the methods's of that which has been discussed ($p = 2$) extend in a straight forward manner. More research is needed to answer this question.

Bibliography

[1] J. Freshse. On a class of nonlinear diagonal elliptic systems with critical growth and C^{α}-regularity. Essays in Honor of Ennio De Giorgi Volume II. Birkhäuser. 1998

[2] M. Giaquinta. Multiple integrals in the calculus of variations and nonlinear elliptic systems. Vorlesungsreihn SFB 72, No.6; University Bonn.

[3] M. Giaquinta., E. Giusti. Nonlinear elliptic systems with quadratic growth. Manuscripta-math.24 (1978), 323-349.

[4] M. Giaquinta. E Giusti. On the regularity of the minima of variational integrals, Acta Math. 148 (1982), 31-46.

[5] Gilbarg D., N.S. Trudinger. Elliptic partial differential equations of second order, Springer Verlag, Heidelberg, New York (1977).

[6] Ladyzenskaya, N.N. Ural'ceva. Linear and quasi-linear elliptic equations, New York and London 1968.

[7] R. Landes. On the existence of weak solutions of perturbated systems with critical growth. J. reine. angew. Math 393 (1989), 21-38.

[8] R. Landes. Some remarks on bounded and unbounded weak solutions of elliptic systems. Manuscripta math. 64, 227-234 (1989).

[9] Morrey, C.B. Jr. Multiple integral problems in the calculus of variations. Springer Verlag, Heidelberg, New York (1966).

[10] Tolksdorf, Peter. Everywhere-regularity for some quasilinear systems with

lack of ellipticity. Ann. Mat. Pura Appl. (4). 134 (1983), 241-266.

[11] Uhlenbeck, K. Regularity for a class of nonlinear elliptic systems. Acta Math. 138 (1977), 219-240.

[12] Wiegner, M, Das Existenz -and Regularitäts problem bei Systemen nichtlinearer elliptischer Differentialgleichungen. Habilitationsschrift, Bochum 1977.