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LATTICES IN BAUMSLAG-SOLITAR COMPLEXES

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Abstract

This dissertation classifies the pairs of nonzero integers (m, n) for which the locally compact group of combinatorial automorphisms, $\mathrm{Aut}(X_{m,n})$, contains incommensurable torsion-free lattices, where $X_{m,n}$ is the combinatorial model for Baumslag-Solitar group $BS(m, n)$. In particular, we show that $\mathrm{Aut}(X_{m,n})$ contains abstractly incommensurable torsion-free lattices if and only if there exists a prime $p \leq \gcd(m, n)$ such that either $\frac{m}{\gcd(m,n)}$ or $\frac{n}{\gcd(m,n)}$ is divisible by p. In all these cases, we construct infinitely many commensurability classes. Additionally, we show that when $\mathrm{Aut}(X_{m,n})$ does not contain incommensurable lattices, the cell complex $X_{m,n}$ satisfies Leighton's property.

We further describe the structure of uniform lattices in the combinatorial automorphism group of a generalized Baumslag-Solitar complex and exhibit an example of a uniform lattice in $\mathrm{Aut}(X_{8,16})$ that is not virtually torsion-free.

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Chapter 1

Introduction

In the 1990s, Bass and Kulkarni initiated the systematic study of lattices in the automorphism group of a locally finite tree and found that any two uniform lattices in the automorphism group of locally finite trees are commensurable up to conjugacy [BK90]. Ten years later, Burger and Mozes [BM00a], and Wise [Wis96, Wis07] made strong advances in the study of lattices for the product of two trees. They discovered many exotic examples, showing that this situation is vastly more complicated than the case of a single tree. This thesis builds on these foundational results by examining the lattice theory in the combinatorial automorphism group of a generalized Baumslag-Solitar complex, denoted $\mathrm{Aut}(X_{(A,\lambda)})$ for a labeled graph (A, λ).

The primary goal of this article is to address the following questions posed in Section (9) of [For24]:

Question 1.0.1. *For which pairs* (m, n) *does* $\mathrm{Aut}(X_{m,n})$ *contain incommensurable lattices?*

Question 1.0.2. *What are the uniform lattices in* $\mathrm{Aut}(X_{m,n})$ *with torsion? Are there uniform lattices that are not virtually torsion-free?*

1.1 Torsion-free uniform lattices in $\mathrm{Aut}(X_{m,n})$

Chapter 1 focuses on torsion-free uniform lattices in the combinatorial automorphism group $\mathrm{Aut}(X_{m,n})$, where $X_{m,n}$ is a combinatorial model for the Baumslag-Solitar group $\mathrm{BS}(m,n)$. In this chapter, we answer the question 1.0.1. The group $\mathrm{BS}(m,n)$ acts on $X_{m,n}$ freely and cocompactly. Furthermore, since $\mathrm{BS}(m,n)$ is finitely presented, $X_{m,n}$ is a locally finite two-dimensional cell complex. Figure 1.1 shows a local picture of $X_{2,4}$.

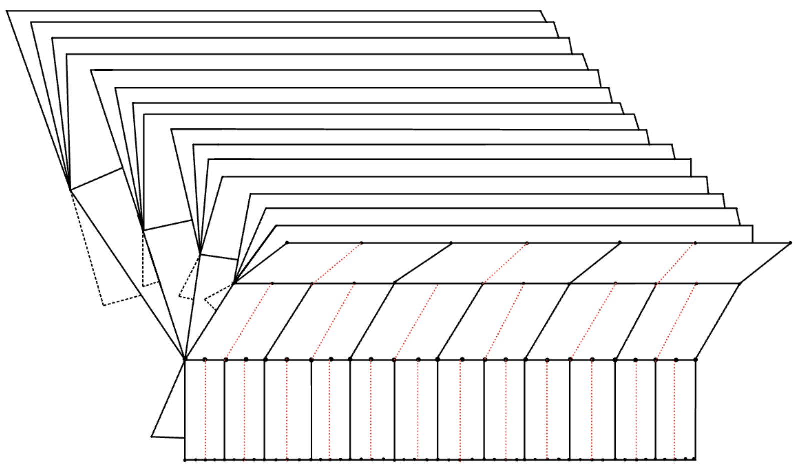


Figure 1.1: $X_{2,4}$

In [For24], Forester established the existence of incommensurable torsion-free uniform lattices in $\mathrm{Aut}(X_{m,n})$ for certain pairs of integers (m,n) by proving the following result:

Theorem 1.1.1. (Forester) *The group* $(\mathrm{Aut}(X_{d,dn})$ *contains uniform lattices that are not abstractly commensurable if one of the following holds:*

1. $\gcd(d,n) \neq 1$,

2. *n has a non-trivial divisor $p \neq n$ such that $p < d$, or*

3. $n < d$ *and* $d \equiv 1 \pmod{n}$*.*

We will completely classify for which pairs of integers (m, n) the locally compact group $\mathrm{Aut}(X_{m,n})$ of combinatorial automorphisms contains incommensurable lattices. We will write any pair of positive integers in the form (dm, dn) where $d = \gcd(dm, dn)$ (equivalently, $\gcd(m, n) = 1$). The main result of Chapter 1 is the following theorem:

Theorem 1.1.2. *The locally compact group* $\mathrm{Aut}(X_{dm,dn})$*, for* m *and* n *coprime, contains abstractly incommensurable torsion-free uniform lattices if and only if there exists a prime* $p \leq d$ *such that either* $p \mid m$ *or* $p \mid n$*. Furthermore,* $\mathrm{Aut}(X_{dm,dn})$ *contains infinitely many commensurability classes of lattices when* $p \mid m$ *or* $p \mid n$*.*

Theorem 1.1.2 shows that many complexes $X_{m,n}$ serve as combinatorial models for incommensurable groups and this behavior is only known in a few other cases: products of locally finite trees [BM00b], [Wis07], [Wis96], and also some examples of Dergacheva and Klyachko [DK23]. In fact, we will show that when m and n have no divisors less than or equal to d, then any two lattices in $\mathrm{Aut}(X_{dm,dn})$ are commensurable up to conjugacy.

We will say that a space satisfies the *Leighton property* if any two compact spaces covered by X admit a common finite sheeted covering up to isomorphism. This definition is motivated by Leighton's theorem, which is equivalent to the fact that trees satisfy the Leighton property.

Theorem 1.1.3. *[Lei82]*(Leighton's theorem) *Let* G_1 *and* G_2 *be finite connected graphs with a common cover. Then they have a common finite cover.*

Woodhouse established the Leighton property for a family of CAT(0) cube complexes exhibiting symmetry and homogeneity similar to regular graphs [Woo23], and for trees with fins [Woo21]. Shepherd, Gardam, and Woodhouse proved the Leighton property for "trees of objects" X when $\mathrm{Aut}(X)$ has finite edge isometry groups [She22]. Additional variations on Leighton's theorem can be found in [BS22], [She22], [Woo23], [Woo21], [Neu10], [SW24], [DK23], [BM00b], [Wis96], [Wis07]. In the complementary case to Theorem 1.1.2, when incommensurable lattices do not exist, we show that Leighton's property holds:

Theorem 1.1.4. *The Baumslag-Solitar complex $X_{dm,dn}$ with m and n coprime has the Leighton property if and only if m and n have no divisors less than or equal to d.*

Lastly, in addition to the above result, we have some new results on the invariant defined by Casals-Ruiz, Kazachkov, and Zakharov [CRKZ21]. In their work, the authors solved the isomorphism problem for the following class of GBS groups by giving an isomorphism invariant vector (we will call it the CRKZ invariant) well-defined up to cyclic permutation;

$$BS(1, n^l) \vee BS(n^{a_1}, n^{a_1}) \vee BS(n^{a_2}, n^{a_2}) \vee \cdots \vee BS(n^{a_{k-1}}, n^{a_{k-1}})$$

for every $l \geq 1$, $n \geq 2$, $k \geq 2$, $0 \leq a_1, a_2, \cdots, a_{k-1} \leq l-1$. This class of groups is denoted by $\mathcal{C}_{n,l}$. We show that this vector is commensurability invariant up to scalar multiplication.

Theorem 1.1.5. *Suppose G_1 and G_2 are two groups in $\mathcal{C}_{n,l}$. Then G_1 is commensurable to G_2 if and only if $c_1\vec{X}^l(G_1) = c_2\vec{X}^l(G_2)$ for some $c_1, c_2 \in \mathbb{N}$.*

The complete statement of this result is given in Theorem 3.5.8.

1.1.1 Methods

X satisfies Leighton's property if and only if any two torsion-free lattices in $\mathrm{Aut}(X)$ are commensurable up to conjugacy. Also, if there exists a prime $p \leq d$ such that either $p \mid m$ or $p \mid n$, then we will provide infinitely many examples of abstractly incommensurable lattices in $\mathrm{Aut}(X_{dm,dn})$. Therefore, to prove theorems 1.1.2, and 1.1.4, it suffices to prove the "if" direction in both theorems.

Section 3.2 is devoted to a general construction, which produces a prime power index subgroup of a p-unimodular GBS group.

The converse in Theorem 1.1.4 is proved in Section 3.3. For $(dm, dn) \neq (1, 1)$, if X is a compact cell complex covered by $X_{dm,dn}$, then the fundamental group of X is virtually a p-unimodular GBS group for all primes $1 \leq p \leq d$. We will use the result of Section 3.2 and Leighton's graph theorem to construct isomorphic finite index subgroups of the fundamental groups of any two compact cell complexes covered by $X_{dm,dn}$. The proof for $X_{1,1}$ follows from the fact that $\mathrm{Aut}(X_{1,1})$ is a discrete group and any two lattices in a discrete group are commensurable.

While proving the converse in Theorem 1.1.2, without loss of generality, we can assume that $p \mid m$. We split the proof into four cases and provide examples of incommensurable lattices in $\mathrm{Aut}(X_{dm,dn})$ for each case using different methods.

Case I $p \mid d$;

Case II $p \nmid d$, $n = 1$, and $m \neq p$; .

Case III $p \nmid d$, $n = 1$, and $m = p$;

Case IV $p \nmid d$ and $n > 1$.

Cases I and II are proved in Section 3.4. In these cases, we employ depth profiles to construct incommensurable lattices in $\mathrm{Aut}(X_{dm,dn})$. The depth profile is a commensurability invariant, developed by Forester in [For24], taking the form of a subset of the natural numbers, depending on a choice of elliptic subgroup.

In Section 3.5 we generalize the CRKZ invariant vector for a larger class of GBS groups denoted by $\mathcal{C}_{n,l}$, whose image under the modular homomorphism is generated by $1/n^l$ for some $l \geq 1$, and the index of an edge group in a vertex group is n^i. We will prove that the CRKZ invariant provides a commensurability invariant for certain GBS groups in $\mathcal{C}_{n,l}$. For case III we will construct incommensurable lattices belonging in $\mathcal{C}_{n,l}$.

Case IV is proved in Section 3.6. The argument in this case depends on the notion of a p–plateau and slide equivalence of certain GBS groups.

1.2 Uniform Lattices in $\mathrm{Aut}(X_{m,n})$

In chapter 4, we establish that a uniform lattice in $\mathrm{Aut}(X_{(A,\lambda)})$ admits a graph of groups decomposition over 2-ended groups. Here, $X_{(A,\lambda)}$ is the combinatorial model for the generalized Baumslag-Solitar group associated to the labeled graph (A, λ). In particular, this implies that any uniform lattice in $\mathrm{Aut}(X_{m,n})$ is the fundamental group of a graph of 2-ended groups.

We further construct an explicit example of a graph of 2-ended groups that acts faithfully, cocompactly, and discretely on $X_{8,16}$, thereby yielding a uniform lattice in $\mathrm{Aut}(X_{8,16})$ that is not virtually torsion-free. Hence answering question 1.0.2 completely.

Conversely, we ask whether every fundamental group of a graph of 2-ended groups can be realized as a uniform lattice in the automorphism group of some GBS complex. Our result shows that such groups can always be realized as uniform lattices in the automorphism group of an *almost* GBS complex. The main result of Chapter 4 is stated below.

Theorem 1.2.1. *If G is the fundamental group of a graph of* 2*-ended groups, then a quotient of G by a finite normal subgroup is a uniform lattice in the automorphism group of an almost GBS complex. Conversely, every uniform lattice in the automorphism group of a GBS complex is the fundamental group of a graph of* 2*-ended groups. Furthermore, every uniform lattice in the orientation-preserving automorphisms of a GBS complex is the fundamental group of a graph of orientable* 2*-ended groups.*

For the forward direction, we start with a group G, which is the fundamental group of a graph of 2-ended groups, and we equip it with a graph of space structure. Let X denote the total space of this graph of spaces, and consider its universal cover $\tilde{X}$. Since each vertex and edge group is 2-ended, the corresponding vertex and edge spaces in the graph of spaces are covered by spaces that are quasi-isometric to $\mathbb{R}$. Hence, $\tilde{X}$ inherits a tree of spaces structure, in which each vertex and edge space is quasi-isometric to $\mathbb{R}$. We assign a copy of $\mathbb{R}$ to each vertex and edge space and connect them via vertical edges (coming from lifts of paths), resulting in an almost GBS complex.

For the converse direction, suppose G is a uniform lattice $\mathrm{Aut}(X_{(A,\lambda)})$, where (A, λ) is a GBS complex. Since $X_{(A,\lambda)}$ is homeomorphic to $T \times \mathbb{R}$, the action on G on $X_{(A,\lambda)}$ descends to an action on T. The quotient of this action yields a finite graph, and the vertex and edge stabilizers are 2-ended subgroups of G, producing a graph of groups decomposition for G with 2-ended vertex and edge groups.

Chapter 2

Preliminaries

In this chapter, we provide the basic prerequisites and results needed for the subsequent chapters.

2.1 Graphs

A *graph* A consists of two sets $V(A)$ and $E(A)$, called the *vertices* and *edges* of A, respectively. It also includes an involution on $E(A)$, which sends $e \in E(A)$ to $\overline{e} \in E(A)$, where $e \neq \overline{e}$, and maps $\partial_0, \partial_1 : E(A) \to V(A)$, satisfying $\partial_1(e) = \partial_0(\overline{e})$. For an edge e, the vertices $\partial_0(e)$ and $\partial_1(e)$ are called *initial* and *terminal* vertices of e, respectively. We say e joins the initial vertex $\partial_0(e)$ to the terminal vertex $\partial_1(e)$. An edge e is called a *loop* if $\partial_0(e) = \partial_0(\overline{e})$. For each vertex $v \in V(A)$, define $E_0(v) = \{e \in E(A) : \partial_0(e) = v\}$.

A *directed graph* is a graph A together with a partition $E(A) = E^+(A) \sqcup E^-(A)$ that separates every pair $\{e, \overline{e}\}$. The edges in $E^+(A)$ are called *directed edges*. For each $v \in V(A)$, we define $E_0^+(v) = \{e \in E^+(A) : \partial_0(e) = v\}$ and $E_0^-(v) = \{e \in E^-(A) : \partial_0(e) = v\}$.

A *labeled graph* (A, λ) is a finite graph with a label function $\lambda : E(A) \to (\mathbb{Z} - \{0\})$, hence each $e \in E(A)$ has a label $\lambda(e)$, which is a nonzero integer.

2.2 Bass-Serre theory

A *graph of groups* consists of a graph A together with a family of group $\mathcal{G}$, assigning a group G_v to each vertex $v \in V(A)$ and a group G_e to each edge $e \in E(A)$ such that $G_e = G_{\overline{e}}$. For each $e \in E(A)$, there is an injective homomorphism $f_e : G_e \longrightarrow G_{\partial_0(e)}$.

A *graph of spaces* consists of a graph A together with a family of topological spaces $\mathcal{X}$, assigning a space X_v to each vertex $v \in V(A)$ and a space X_e to each edge $e \in E(A)$ such that $X_e = X_{\overline{e}}$. For each $e \in E(A)$ there is a π_1-injective continuous map $\phi_e : X_e \longrightarrow X_{\partial_0(e)}$. The *total space* corresponding to this graph of space is the quotient topological space

$$X_A = \left(\bigsqcup_{v \in V(A)} X_v \ \sqcup \ \bigsqcup_{e \in E(A)} (X_e \times [0,1]) \right) \Big/ \sim$$

where the equivalence relation $\sim$ identifies $(x, 0) \in X_e \times [0,1]$ with $\phi_e(x) \in X_{\partial_0(e)}$, and $X_e \times \{1\}$ is identified with $X_{\overline{e}} \times \{1\}$ via identity map. Note that X_v and X_e are subspaces of X_A, where X_e is identified with $X_e \times \{1\}$.

Let $\tilde{X}_A$ be the universal cover of X_A with the covering map $p : \tilde{X}_A \longrightarrow X_A$. Then $p^{-1}(X_v)$ consists copies of universal cover of X_v and $p^{-1}(X_e) = p^{-1}(X_e \times \{1\})$ consists of copies of universal cover of X_e. Therefore, $\tilde{X}_A$ is a total space of a tree of spaces where all vertex spaces are universal covers of vertex spaces in $\mathcal{X}$ and edge spaces are universal covers of edge spaces in $\mathcal{X}$.

A graph of spaces $(A, \mathcal{X})$ corresponds to a graph of groups $(A, \mathcal{G})$ if for each vertexx $v \in V(A)$, we have $\pi_1(X_v) = G_v$, and for each edge $e \in E(A)$, $\pi_1(X_e) = G_e$, such that the induced map on fundamental groups satisfies $(\phi_e)_* = f_e$. In this case, the fundamental group of the total space X_A is called the *fundamental group of the graph of groups* $(A, \mathcal{G})$.

2.3 GBS groups

A *generalized Baumslag–Solitar group* or a *GBS group* is the fundamental group of a graph of groups where all edge and vertex groups are $\mathbb{Z}$. Any GBS group can be represented by a labeled graph (A, λ), where the inclusion from the edge group G_e to the vertex group $G_{\partial_0(e)}$ is given by multiplication by the nonzero integer $\lambda(e)$.

The Baumslag-Solitar group $BS(m, n)$ is the GBS group represented by the labeled graph with one vertex and one edge with labels m and n.

For a labeled graph (A, λ), a *fibered* 2-*complex*, denoted by $Z_{(A,\lambda)}$, is the total space of a graph of spaces in which the vertex and edge spaces are circles. If C_v is the oriented circle for $v \in V(A)$ and C_e is the oriented circle for $e \in E(A)$, and M_e is the mapping cylinder for the covering map $C_e \to C_{\partial_0(e)}$ of degree $\lambda(e)$, then

$$Z_{(A,\lambda)} = \sqcup_{e \in E(A)} M_e \Big/ \sim$$

where M_e and $M_{\overline{e}}$ are identified along C_e and $C_{\overline{e}}$ for each $e \in E(A)$, and all copies of C_v are identified by identity map for each $v \in V(A)$. The fundamental group of $Z_{(A,\lambda)}$ is the GBS group represented by (A, λ).

Notation: We denote the GBS group defined by the labeled graph having one vertex and k loops each with labels m_i and n_i by $\bigvee_{i=1}^{k} BS(m_i, n_i)$.

A labeled graph is called *reduced* if every edge e with $\lambda(e) = \pm 1$ is a loop.

A *G–tree* is a simplicial tree X on which G acts without inversions, i.e., if $g \in G$ fixes an edge, then it fixes every point on this edge. For a given G-tree X, an element $g \in G$ is called *elliptic* if it fixes a vertex, and *hyperbolic* otherwise. Every hyperbolic element has a g-invariant line on which it acts via a non-trivial translation. A subgroup $H < G$ is called *elliptic* if there exists a vertex $v \in V(X)$ such that $hv = v$ for all $h \in H$.

A GBS group is called an *elementary* GBS group if it is isomorphic to $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, the Klein bottle group or the union of infinitely ascending chain of infinite cyclic groups; otherwise, it is called a *non-elementary* GBS group. If G is a non-elementary GBS group, then any two G-trees produce the same set of elliptic (and hyperbolic) elements in G. Therefore, for a non-elementary GBS group, we can define the notion of elliptic (and hyperbolic) elements independently of its G-trees.

Let G be a GBS group with a G-tree X and a quotient labeled graph (A, λ). Fix an elliptic element $a \in G$. Then, for any $g \in G$, we can find nonzero integers m and n such that $g^{-1}a^m g = a^n$. The *modular homomorphism* is a map $q : G \to \mathbb{Q}^*$ defined by $q(g) = \frac{m}{n}$. This map is independent of the choice of elliptic element a. The restriction of the modular homomorphism to elliptic elements is a trivial map and Q^* is abelian; therefore, it factors through $H_1(A)$. If $g \in G$ maps to $\alpha \in H_1(A)$, which is represented by a $1-$cycle $(e_1, e_2, \cdots, e_n)$, then

$$q(g) = \prod_{i=1}^{n} \frac{\lambda(e_i)}{\lambda(\overline{e}_i)} \tag{2.1}$$

If V is any non-trivial elliptic subgroup of G, then we have a formula:

$$|q(g)| = \frac{[V : V \cap V^g]}{[V^g : V \cap V^g]} \tag{2.2}$$

2.4 Lattices in combinatorial automorphism groups

For a CW complex X, the *topological automorphism group* (denoted as $\mathrm{Aut}_{top}(X)$) consists of homeomorphisms of X that preserve the cell complex structure of X.

The *combinatorial automorphism group*, denoted $\mathrm{Aut}(X)$, is obtained by quotienting $\mathrm{Aut}_{top}(X)$, where two automorphisms are considered the same if they induce the same permutation on the set of cells of X. For a connected and locally finite CW complex X, the combinatorial automorphism group $\mathrm{Aut}(X)$ is locally compact.

In a locally compact group G, a discrete subgroup $H < G$ is called a *lattice* if G/H carries a finite positive G-invariant measure, and a *uniform lattice* if G/H is compact. A subgroup H in $G = \mathrm{Aut}(X)$ is discrete if and only if every cell stabilizer $H_\sigma = \{h \in H : h\sigma = \sigma\}$ is finite. In this case, define the covolume of H to be:

$$\mathrm{Vol}(X/H) = \sum_{[\sigma] \in cell(X/H)} 1/|H_\sigma|$$

where the sum is taken over a set of representatives of the H–orbits of cells of X. The next proposition follows from [BL01, 1.5-1.6].

Proposition 2.4.1. *Let X be a connected locally finite CW complex. Suppose that $G = \mathrm{Aut}(X)$ acts cocompactly on X and let $H < G$ be a discrete subgroup. Then*

1. *H is a lattice if and only if Vol$(X/H) < \infty$*
2. *H is a uniform lattice if and only if X/H is compact.*

Note that, if H is a torsion-free lattice in $\mathrm{Aut}(X)$, then every cell stabilizer H_σ is the trivial subgroup. Therefore $\mathrm{Vol}(X/H) < \infty$ if and only if X/H is compact. Hence, the above Proposition implies that a torsion-free lattice is uniform.

2.5 The 2−complex $\mathbf{X_{m,n}}$

For $m, n > 0$, we define $Z_{m,n}$ as the presentation 2−complex corresponding to the group presentation $\langle a, t : t^{-1}a^m t = a^n\rangle$ for the Baumslag-Solitar group $BS(m, n)$. The complex $Z_{m,n}$ consists of a single vertex, two edges labeled as a and t, and a 2−cell attached along the boundary word $t^{-1}a^m t a^{-n}$. We structure $Z_{m,n}$ as a cell complex by subdividing the 2-cell into $\gcd(m,n)$ 2-cells (see Figure 2.1 for an example). We will denote the universal cover of $Z_{m,n}$ by $X_{m,n}$, and the cell complex structure is inherited from $Z_{m,n}$. For any nonzero integers m and n define $X_{m,n}$ to be $X_{|m|,|n|}$.

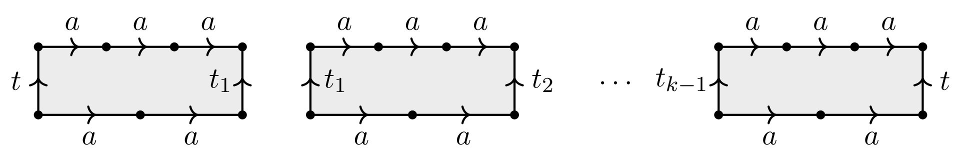


Figure 2.1: The cell structure for $Z_{m,n}$ when $m = 3k$, $n = 2k$.

The following theorem gives a sufficient condition for a GBS group to be a lattice in $\mathrm{Aut}(X_{dm,dn})$. We will use this theorem extensively throughout all sections to construct uniform lattices in $\mathrm{Aut}(X_{dm,dn})$.

Theorem 2.5.1. *[For24] Let G be the GBS group defined by labeled graph (A, λ), and suppose there is a directed graph structure $E(A) = E^+(A) \sqcup E^-(A)$ on A such that*

1. *for every $v \in V(A)$,*

$$\sum_{e \in E_0^+(v)} |\lambda(e)| = dm \ \text{ and } \sum_{e \in E_0^-(v)} |\lambda(e)| = dn$$

2. *for every $e \in E^+(A)$, let $n_e = |\lambda(e)|$, $m_e = |\lambda(\bar{e})|$, and $k_e = \gcd(m_e, n_e)$; then*

$$n_e/k_e = n \text{ and } m_e/k_e = m.$$

Then G is a lattice in $\mathrm{Aut}(X_{dm,dn})$.

The following theorem from [For24] provides the general description of a torsion-free uniform lattice within the combinatorial automorphism group $\mathrm{Aut}(X_{dm,dn})$ as a GBS group, for $(m,n) \neq (1,1)$.

Theorem 2.5.2. *[For24] Suppose $m \neq n$ and let G be a torsion-free group. Then G is isomorphic to a uniform lattice in $Aut(X_{dm,dn})$ if and only if there exists a compact GBS structure (A, λ) for G, a directed graph structure $E(A) = E^+(A) \sqcup E^-(A)$, and a length function $l : V(A) \sqcup E(A) \to \mathbb{N}$ satisfying $l(e) = l(\overline{e})$ for all $e \in E(A)$ such that the following holds.*

(1) For every $v \in V(A)$:

$$\sum_{e \in E_0^+(v)} |\lambda(e)| = dm \text{ and } \sum_{e \in E_0^-(v)} |\lambda(e)| = dn \tag{2.3}$$

(2) For every $e \in E^+(A)$,

$$l(\partial_0(e)|\lambda(e)| = ml(e)$$
$$l(\partial_1(e)|\lambda(\overline{e})| = nl(e)$$

(3) For every $v \in V(A)$, let $k_0(v) = \gcd(l(v), m)$ and $k_1(v) = \gcd(l(v), n)$; then there exist partitions

$$E_0^+(v) = E_1^+ \sqcup \cdots \sqcup E_{k_0(v)}^+ \text{ and } E_0^-(v) = E_1^- \sqcup \cdots \sqcup E_{k_0(v)}^-$$

such that the sums $\sum_{e \in E_i^+} l(e)$ are all equal for all i, and the sums $\sum_{e \in E_j^-} l(e)$ are all equal for all j.

Remark 2.5.3. By the proof of Proposition 4.6 in [For24], the conditions (1) and (2) in Theorem 2.5.2 together imply that the directed graph is strongly connected. In particular, every directed edge is contained in a directed circuit.

2.6 The GBS complex $X_{(A,\lambda)}$

Let (A, λ) be a labeled graph. Choose a directed graph structure $E(A) = E^+(A) \sqcup E^-(A)$ on A. We define a 2-dimensional cell complex associated to (A, λ) as follows. For each vertex $v \in V(A)$, introduce a vertex w_v and a loop a_v based at w_v. For each edge $e \in E^+(A)$ with $\partial_0(e) = v_1$, $\partial_1(e) = v_2$, attach $\gcd(\lambda(e), \lambda(\overline{e}))$ 2-cells as illustrated in figure 2.2. It is straightforward to verify that, after disregarding the orientation of the edges, the resulting complex is independent of the chosen directed graph structure on A, and is homeomorphic to the fibered 2-complex $Z_{(A,\lambda)}$. We define the generalized Baulslag-Solitar complex $X_{(A,\lambda)}$ to be the universal cover of this cell complex, with the induced cell structure. The orientation of the edges in the universal cover is not considered.

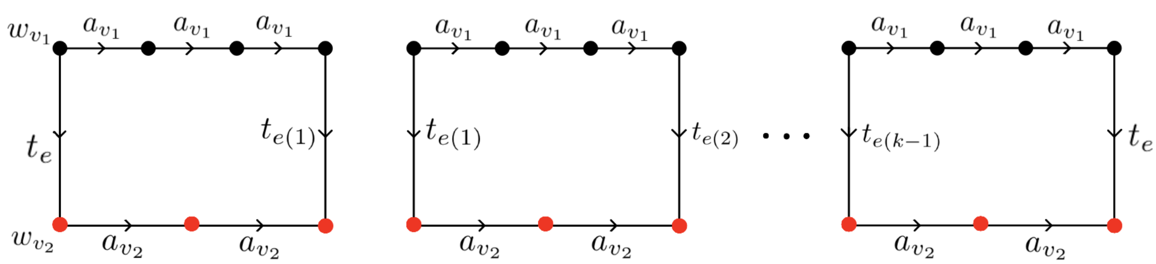


Figure 2.2: Part of the cell structure for $Z_{(A,\lambda)}$ when $(\lambda(e), \lambda(\overline{e})) = (3k, 2k)$.

We now describe the combinatorial description of $X_{(A,\lambda)}$, generalizing the description of $X_{m,n}$ from Section 4 of [For24]). Let $T_{(A,\lambda)}$ denote the Bass-Serre tree for G with the labeled graph (A, λ). Then $X_{(A,\lambda)}$ is homeomorphic to $T_{(A,\lambda)} \times \mathbb{R}$. Let $\pi : X_{(A,\lambda)} \longrightarrow T_{(A,\lambda)}$ denote the projection map. The refer to the edges in $X_{(A,\lambda)}$ that project to vertices in $T_{(A,\lambda)}$ as horizontal edges, and those projecting to edges in $T_{(A,\lambda)}$ as vertical edges. For a vertex $v \in V(T_{(A,\lambda)})$, the preimage $\pi^{-1}(v)$ is called a branching line; for an edge $e \in E(T_{(A,\lambda)})$, the preimage $\pi^{-1}(e)$ is called a strip. Each branching line is tiled by horizontal edges. Let $q : T_{(A,\lambda)} \longrightarrow A$ be the quotient map.

For an edge $e \in E(T_{(A,\lambda)})$ with the image $q(e) = f \in E(A)$, the strip $\pi^{-1}(e)$ in $X_{(A,\lambda)}$ is an (i,j)-strip where

$$(i,j) = \Big(\frac{\lambda(f)}{\gcd(\lambda(f), \lambda(\overline{f}))}, \frac{\lambda(\overline{f})}{\gcd(\lambda(f), \lambda(\overline{f}))} \Big)$$

(see Figure 2.3 for an example of (i,j)-strip).

Now fix a vertex $v \in V(T_{(A,\lambda)})$, with image $q(v) \in V(A)$, and fix $f \in E_0(q(v))$. There are $\lambda(f)$ edges $f_1, f_2, \cdots f_{\lambda(f)} \in E(T_{(A,\lambda)})$ with initial vertex v and mapping to $f \in E(A)$ under q. The strips $\pi^{-1}(f_1), \pi^{-1}(f_2), \cdots \pi^{-1}(f_{\lambda(f)})$ are attached to the branching line $\pi^{-1}(v)$. Let $k = \frac{\lambda(f)}{\gcd(\lambda(f),\lambda(\overline{f}))}$. Then each strip $\pi^{-1}(f_i)$ is attached to branching line $\pi^{-1}(v)$ along the coset $i + k\mathbb{Z}$, where we identify the vertices $\pi^{-1}(v)$ with $\mathbb{Z}$. Thus, the $\lambda(f)$ strips are joined to $\pi^{-1}(v)$ along cosets $i + k\mathbb{Z}$ for $i = 1, \cdots, \lambda(f)$. The description of the neighborhood of each branching line, together with the structure of the Bass-Serre tree $T_{(A,\lambda)}$, completely determines the cell complex $X_{(A,\lambda)}$.

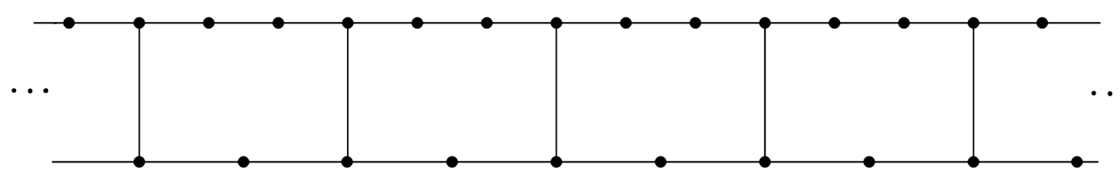
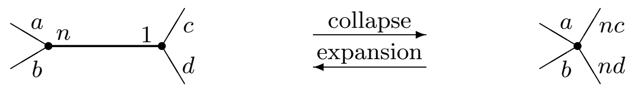


Figure 2.3: A (2,3) strip.

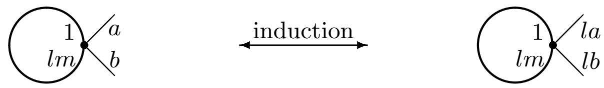
2.7 Deformation moves

Any two GBS trees for a non-elementary GBS group G are related by an elementary deformation. That is, they are related by a finite sequence of elementary moves, called elementary collapses and expansions. There are also slide and induction moves, which can be expressed as an expansion followed by a collapse.

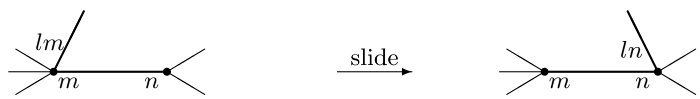
Collapse and expansion moves are as follows:

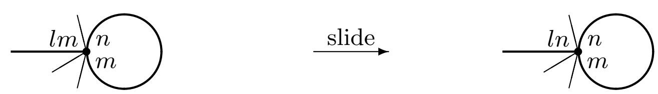


The induction move is as follows:



There are two slide moves:





We will use the following theorem by Forester [For06] in section 3.6, which provides a sufficient condition for two reduced labeled graphs to be related only through slide moves.

Proposition 2.7.1. *Suppose* (A_i, λ_i) *are compact reduced labeled graphs representing the same GBS group* G*, for* $i = 1, 2$*. If* $q(G) \cap \mathbb{Z} = 1$*, then* A_1 *and* A_2 *are related by slide moves.*

2.8 Index of a segment

Let X be a locally finite G–tree. A *segment* is an edge path $\sigma = (e_1, \cdots, e_k)$ with no backtracking. Its initial and terminal vertices are $\partial_0(\sigma) = \partial_0(e_1)$ and $\partial_1(\sigma) = \partial_1(e_k)$, respectively. The pointwise stabilizer of σ is $G_\sigma = G_{\partial_0(\sigma)} \cap G_{\partial_1(\sigma)}$. The index of σ is the number $i(\sigma) = [G_{\partial_0 \sigma} : G_\sigma]$. One can compute the index of any segment by applying the remark below iteratively, which can be found in [For24].

Remark 2.8.1. When $\sigma = (e_1, e_2)$ with $n_j = \lambda(e_j)$ and $m_j = \lambda(\bar{e}_j)$, for $j = 1, 2$; then $i(\sigma) = n_1 n_2 / \gcd(m_1, n_2)$.

2.9 Subgroups of GBS groups

If G is a GBS group represented by a labeled graph (A, λ), then there is a one-to-one correspondence between conjugacy classes of GBS subgroups of G (excluding hyperbolic cyclic subgroups) and admissible branched coverings $(B, \mu) \to (A, \lambda)$. An *admissible branched covering* from labeled graph (B, μ) to (A, λ) consists of a surjective graph morphism $\pi : B \to A$ and a degree map $d : V(B) \sqcup E(B) \to \mathbb{N}$ satisfying $d(e) = d(\bar{e})$ for $e \in E(B)$, and if $e \in E(A)$ with $v = \partial_0(e)$, $u \in \pi^{-1}(v)$, $k_{u,e} = \gcd(d(u), \lambda(e))$ then

1. $|\pi^{-1}(e) \cap E_0(u)| = k_{u,e}$
2. If $e' \in \pi^{-1}(e) \cap E_0(u)$ then $\mu(e') = \lambda(e)/k_{u,e}$, and $d(e') = d(u)/k_{u,e}$.

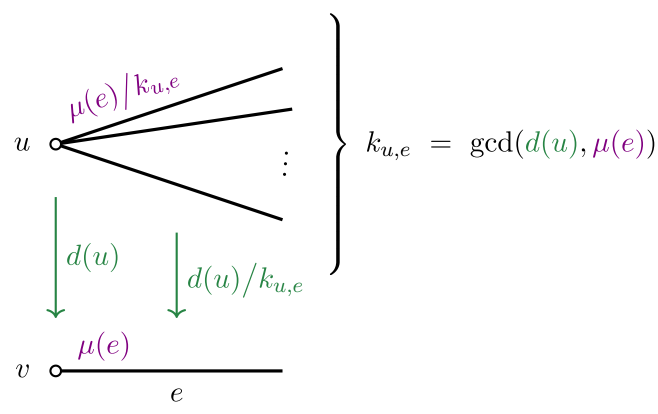


Figure 2.4: The admissibility condition. Each edge of $p^{-1}(e) \cap E_0(u)$ has label $\mu(e)/k_{u,e}$ and degree $d(u)/k_{u,e}$. There are $k_{u,e}$ such edges.

Let (A, λ) be a labeled graph, and $\pi : B \to A$ be a covering map in the topological sense. For any $e \in E(B)$, we can define the label $\mu(e) = \lambda(\pi(e))$. Then (B, μ) is labeled graph, and any constant degree map c, coprime to all edge labels of A, makes $\pi : (B, \mu) \to (A, \lambda)$ into an admissible cover.

Remark 2.9.1. An admissible branched cover from labeled graph (B, μ) to labeled graph (A, λ) describes a topological covering of fibered 2-complexes $Z_{(B,\mu)} \longrightarrow Z_{(A,\lambda)}$.

Finite index subgroups of GBS groups without a proper p–plateau have a nice description. The notion of a *plateau* was introduced in [Lev15]. For a labeled graph (A, λ) and a prime number p, a non-empty connected subgraph $P \subseteq A$ is a *p–plateau* if for every edge $e \in E(A)$ with $v = \partial_0(e)$ belonging to the vertex set of P: $p \mid \lambda(e)$ if and only if $e \notin E(P)$. The subgraph $P \subset A$ is called a *plateau* if it is a p–plateau for some prime p. A plateau P is considered *proper* if $P \neq A$.

Proposition 2.9.2. *[Lev15] Given a connected labeled graph (A, λ), the following conditions are equivalent:*

- *every admissible covering $\pi : \overline{A} \to A$ is a topological covering;*
- *A contains no proper plateau.*

2.10 Depth Profile

The notion of depth profile is introduced in [For24] as a commensurability invariant of GBS groups. One defines an equivalence relation on the set of subsets of $\mathbb{N}$ by declaring that $S \subset \mathbb{N}$ is equivalent to the set $\{n/\gcd(r,n) : n \in S\}$ for each $r \in \mathbb{N}$ and taking the symmetric and transitive closure. Let us denote the set $\{n/\gcd(r,n) : n \in S\}$ by S/r.

Proposition 2.10.1. *[For24] Two subsets S and S' are equivalent if and only if there exist $r, r' \in \mathbb{N}$ such that $S/r = S'/r'$.*

Given $S \subset \mathbb{N}$ and $k \in \mathbb{N}$ such that $\gcd(x, k) = 1$ for all $x \in S$, define

$$S[k] = \{xk^i : x \in S, \text{ and } i \geq 0\}. \tag{2.4}$$

Lemma 2.10.2. *Suppose $S, S' \subset \mathbb{N}$, $k \in \mathbb{N}$, and $\gcd(s, k) = \gcd(s', k) = 1$ for all $s \in S$ and $s' \in S'$. Then S and S' are equivalent if and only if $S[k]$ and $S'[k]$ are equivalent.*

Proof. Proposition 2.10.1 implies that if S is equivalent to S', then $S/r = S'/r'$ for some $r, r' \in \mathbb{N}$. We can assume that $\gcd(r, k) = \gcd(r', k) = 1$ since replacing r by $r/\gcd(r, k)$ and r' by $r'/\gcd(r', k)$ does not change the sets S/r and S'/r', as k is coprime to all integers in S and S'. This implies that $\gcd(r, sk^i) = \gcd(r, s)$ and $\gcd(r', s'k^i) = \gcd(r', s')$. Thus, the following sequence of equalities, together with Proposition 2.10.1, implies that $S[k]$ is equivalent to $S'[k]$.

$$(S[k])/r = (S/r)[k] = (S'/r')[k] = (S'[k])/r'$$

For the converse, suppose $S[k]$ is equivalent to $S'[k]$. Then, by Proposition 2.10.1, we have $(S[k])/r = (S'[k])/r'$ for some $r, r' \in \mathbb{N}$. We claim that $S/r = S'/r'$. Assuming the claim, S is equivalent to S' by Proposition 2.10.1. Since $S/r \subset (S[k])/r = (S'[k])/r'$, for every $s \in S$, we have

$$\frac{s}{\gcd(r, s)} = \frac{s'k^j}{\gcd(r', s'k^j)} = \frac{s'}{\gcd(r', s')} \frac{k^j}{\gcd(r', k^j)}$$

for some $j \geq 0$ and $s' \in S'$. This is possible only if $k^j/\gcd(r', k^j) = 1$ since $\gcd(s, k) = 1$. Therefore, $s/\gcd(s, r) \in S'/r'$. Thus, we have $S/r \subseteq S'/r'$, and similarly $S'/r' \subseteq S/r$. $\square$

Let G be a non-elementary GBS group, and $V < G$ is a non-trivial elliptic subgroup. For an element $g \in G$, define its *$V-$depth* as $D_g(V) = [V : V \cap V^g]$, where V^g denotes the subgroup $\{gxg^{-1} : x \in V\}$. Define the *depth profile*

$$\mathscr{D}(G, V) = \{D_g(V) : g \in G \text{ is hyperbolic and } q(g) = \pm 1\} \subset \mathbb{N}.$$

The depth profile is commensurability invariant, i.e., if two non-elementary GBS groups G_1 and G_2 are commensurable, then the sets $\mathscr{D}(G_1, V_1)$ and $\mathscr{D}(G_2, V_2)$ are equivalent in the above sense (proved in [For24]).

Chapter 3

Torsion-Free Lattices in Baumslag-Solitar Complexes

In this chapter, we investigate for which pairs of integers (m, n) all torsion-free uniform lattices in $\mathrm{Aut}(X_{m,n})$ are commensurable. When $\mathrm{Aut}(X_{m,n})$ contains incommensurable lattices, we show using three distinct techniques that it actually contains infinitely many commensurability classes of such lattices.

In the following section, we introduce the p-modular homomorphism and show that the p-modulus of a path depends only on its endpoints. This result will be used to construct an admissible branched cover for p-unimodular labeled graphs.

3.1 The p-modular homomorphism

For a prime number p, the *p–adic valuation* on the field of rational numbers is the map $\nu_p : \mathbb{Q}^* \to \mathbb{Z}$ defined by $\nu_p(\frac{a}{b}) = \nu_p(a) - \nu_p(b)$, where $\nu_p(m) = \max\{e \in \mathbb{N} : p^e \mid m\}$.

Definition 3.1.1 (p–modular homomorphism). For a GBS group G and a prime number p, the *p–modular homomorphism* is the map $q_p : G \to \mathbb{Z}$ defined as $q_p = \nu_p \circ q$, where q is the modular homomorphism on G, and ν_p is the restriction of the p-adic valuation on Q^*. The GBS group G is called *p–unimodular* if $q_p(G) = 0$.

Let (A, λ) be a labeled graph. For an edge path $(e_1, e_2, \cdots, e_k)$, define

$$q_p^A(e_1, e_2, \cdots, e_k) = \nu_p \circ q_A(e_1, e_2, \cdots, e_k),$$

where we define q_A by the right hand side in formula (2.1).

Lemma 3.1.1. *Let G be a p-unimodular GBS group represented by labeled graph (A, λ). For edge paths $(e_1, e_2, \cdots, e_k)$ and $(f_1, f_2, \cdots, f_l)$ with the same initial and terminal vertices, i.e., $\partial_0(e_1) = \partial_0(f_1)$ and $\partial_1(e_k) = \partial_1(f_l)$, we have*

$$q_p^A(e_1, e_2, \cdots, e_k) = q_p^A(f_1, f_2, \cdots, f_k).$$

Therefore, for a fixed vertex $w_0 \in V(A)$, we get a well-defined function $h_p : V(A) \to \mathbb{Z}$ (with respect to w_0) defined by

$$h_p(v) = q_p^A(e_1, e_2, \cdots, e_k) \tag{3.1}$$

for any path $(e_1, e_2, \cdots, e_k)$ in A from w_0 to v.

Proof. Since $(e_1, e_2, \cdots, e_k)$ and $(f_1, f_2, \cdots, f_l)$ are two edge paths in A from w_0 to v, the path $(e_1, \cdots, e_k, \overline{f}_l, \overline{f}_{l-1}, \cdots \overline{f}_1)$ represents a $1-$cycle in $H_1(A)$. Then,

$$\begin{aligned} 0 &= q_p(e_1, \cdots, e_k, \overline{f}_l, \overline{f}_{l-1}, \cdots \overline{f}_1) \\ &= q_p^A(e_1, \cdots, e_k) + q_p^A(\overline{f}_l, \overline{f}_{l-1}, \cdots \overline{f}_1) \\ &= q_p^A(e_1, \cdots, e_k) - q_p^A(f_1, f_2, \cdots, f_l) \end{aligned}$$

Thus, $q_p^A(e_1, \cdots, e_k) = q_p^A(f_1, f_2, \cdots, f_l)$. $\square$

Remark 3.1.2. We can define the map $q_n^A : G \to \mathbb{Z}$ for any positive integer $n \geq 1$, and Lemma 3.1.1 also holds for the map q_n^A.

3.2 p-unimodularity and coverings

In this section, we present a general result regarding an admissible branched cover of a p-unimodular labeled graph. This result will be used in the subsequent section to demonstrate that all uniform lattices in $\mathrm{Aut}(X_{dm,dn})$ are commensurable when m and n have no divisor less than or equal to d.

Theorem 3.2.1. *Suppose (A, λ) is a finite labeled graph which is p-unimodular. Then there exists a finite admissible cover (A_p, λ_p) such that $p \nmid \lambda_p(e)$ for all $e \in E(A_p)$.*

Proof. First, perform expansion moves on the edges of (A, λ) to obtain a labeled graph (A_1, λ_1) such that if $p|\lambda_1(e)$ for some $e \in E(A_1)$, then $\lambda_1(e) = p$ and $\lambda_1(\bar{e}) = 1$.

If $e \in E(A)$ and $p \mid \lambda(e)$, then perform $\nu_p(\lambda(e))$ expansion moves as in Figure 3.1.

Figure 3.1: Example illustrating $\nu_p(\lambda(e)) = 3$ expansion moves for an edge $e \in E(A)$.

Consider the subgraph B_1 of A_1 with the following vertex and edge set

- $V(B_1) = V(A_1)$
- $E(B_1) = \{e \in E(A_1) : p \nmid \lambda_1(e), \lambda_1(\bar{e})\}$

Note that if $e \in E(A_1)$ but $e \notin E(B_1)$, then $\lambda_1(e) = p$ and $\lambda_1(\bar{e}) = 1$ or $\lambda_1(e) = 1$ and $\lambda_1(\bar{e}) = p$ Now we will define an admissible branched cover $(\tilde{A}_1, \tilde{\lambda}_1)$ of (A_1, λ_1)

with the property that $p \nmid \tilde{\lambda}_1(e)$ for all $e \in E(\tilde{A}_1)$. Figure 3.2 illustrates this construction with an example. If $B_1 = A_1$ (equivalently, $E(A_1) = E(B_1)$) then take $(\tilde{A}_1, \tilde{\lambda}_1) = (A_1, \lambda_1)$.

Claim: The function $h_p : V(A_1) \to \mathbb{Z}$ defined by equation (3.1) with respect to a fixed vertex $w_0 \in V(A_1)$ is constant on each component of B_1.

If $v_1, v_2 \in V(B_1)$ are in the same component of B_1 then there exists an edge path $(e_1, e_2, \cdots e_k)$ in B_1 with $\partial_0(e_1) = v_1$, $\partial_1(e_k) = v_2$. Since $e_i \in E(B_1)$, $p \nmid \lambda(e), \lambda(\overline{e})$. Therefore,

$$q_p^A(e_1, e_1, \cdots e_k) = \nu_p \left(\frac{\lambda(e_1), \lambda(e_2), \cdots, \lambda(e_k)}{\lambda(\overline{e}_1), \lambda(\overline{e}_2), \cdots, \lambda(\overline{e}_k)} \right) = 0 \tag{3.2}$$

Choose any edge path $(f_1, f_2, \cdots f_r)$ in A_1 from w_0 to v_1. Then the edge path given by $(f_1, f_2, \cdots f_r, e_1, e_1, \cdots e_k)$ has initial vertex w_0 and terminal vertex v_2. Then,

$$\begin{aligned} h_p(v_2) &= q_p^A(f_1, f_2, \cdots f_r, e_1, e_1, \cdots e_k) \\ &= q_p^A(f_1, f_2, \cdots f_r) + q_p^A(e_1, e_1, \cdots e_k) \\ &= q_p^A(f_1, f_2, \cdots f_r) \\ &= h_p(v_1). \end{aligned}$$

This completes the proof of the claim. For $k \in \mathbb{Z}$, define W_k to be the subgraph of B_1 spanned by vertices $v \in V(B_1)$ with $h_p(v) = k$. Since B_1 is a compact graph, and the p-modulus of each edge in B_1 is either 0, 1 or -1; $W_k \neq \varnothing$ only for $k \in \{\alpha, \alpha + 1 \cdots \alpha + \beta\}$ for some $\alpha, \beta \in \mathbb{Z}$.

Construct an admissible branched covering $(\tilde{A}_1, \tilde{\lambda}_1)$ of (A_1, λ_1) by taking a disjoint union of p^i copies of $W_{\alpha+i}$ for all $0 \leq i \leq \beta$ together with some new edges. These copies are denoted as $W^1_{\alpha+i}, W^2_{\alpha+i}, \cdots, W^{p^i}_{\alpha+i}$ with the same edge labels as $W_{\alpha+i}$. The surjective graph homomorphism $\pi_1 : \tilde{A}_1 \to A_1$ for this admissible branched covering

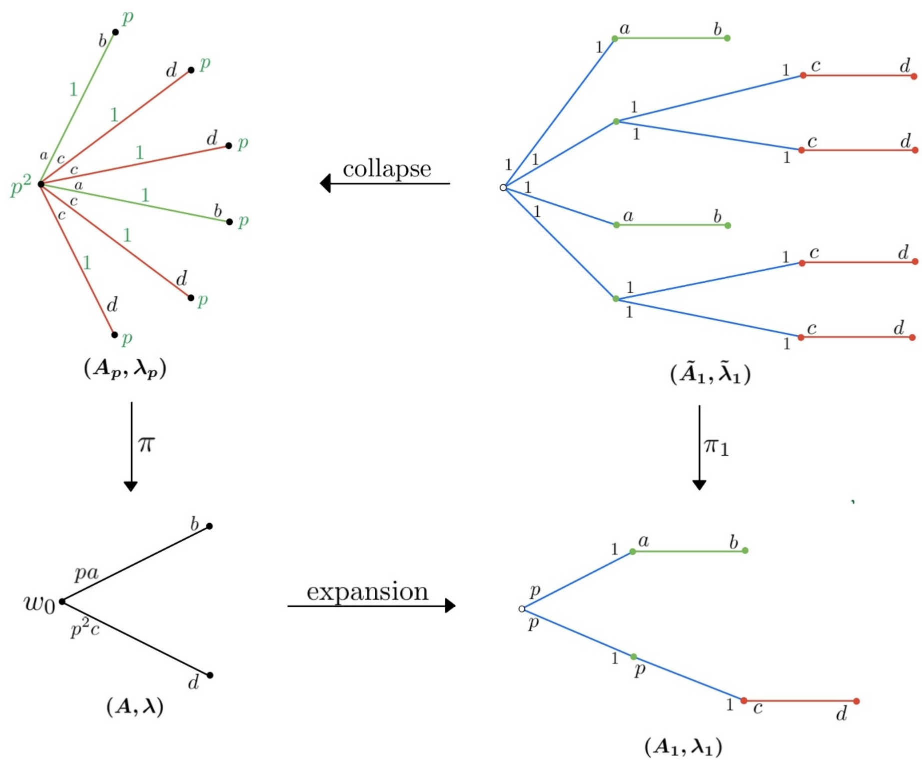


Figure 3.2: Construction of the admissible branched cover of the labeled graph (A, λ) when $p = 2$. Here W_0 is the white vertex, W_1 is the green subgraph and W_2 is the red subgraph. Hence $(\tilde{A}_1, \tilde{\lambda}_1)$ contains 2^0 copies of W_0, 2^1 copies of W_1, and 2^2 copies of W_2.

maps each copy $W^j_{\alpha+i}$ to $W_{\alpha+1}$ via the identity map. Define the degree d_1 of each vertex and edge in $W^j_{\alpha+i}$ to be $p^{\beta-i}$. Define

$$V(\tilde{A}_1) = \{v^j_{\alpha+i} : v \in V(W_{\alpha+i}),\ 0 \le i \le \beta,\ \text{and } 1 \le j \le p^i\}.$$

Consider the surjective homomorphism

$$\Phi : \frac{\mathbb{Z}}{p^{i+1}\mathbb{Z}} \to \frac{\mathbb{Z}}{p^i\mathbb{Z}}$$

defined by $\Phi([a]) = [a]$ for all $[a] \in \frac{\mathbb{Z}}{p^{i+1}\mathbb{Z}}$. The new edges in $\tilde{A}_1$ are given as follows: for each $e \in E(A_1) - E(B_1)$ with $\partial_0(e) \in V(W_{\alpha+i})$, $\partial_1(e) \in V(W_{\alpha+i+1})$ for some

$0 \leq i \leq \beta - 1$, we have $\lambda(e) = p$, and $\lambda(\overline{e}) = 1$. Then there are p^{i+1} new edges in $\tilde{A}_1$, $\{e^j : 1 \leq j \leq p^{i+1}\}$ with $\partial_1(e^j) = (\partial_1(e))^j_{\alpha+i+1}$ and $\partial_0(e^j) = (\partial_0(e))^{\phi(j)}_{\alpha+i}$. Define the label of these edges and their involution to be 1. π_1 maps the edge e^j to e with a degree $p^{\beta-i-1}$.

Let $v \in V(W_{\alpha+i}) \subset V(A_1)$. Then, for any $\tilde{v} \in \pi_1^{-1}(v)$, we have $d_1(\tilde{v}) = p^{\beta-i}$. For an edge $e \in E(A_1)$ with $\partial_0(e) = v$ and $\partial_1(e) = w$, one of the following conditions holds: either $w \in V(W_{\alpha+i})$, or $w \in V(W_{\alpha+i+1})$, or $w \in V(W_{\alpha+i-1})$.

If $w \in V(W_{\alpha+i})$, then $e \in E(W_{\alpha+i})$ and $p \nmid \lambda_1(e)$, hence $\gcd(\lambda_1(e), d_1(v)) = 1$, and any $\tilde{e} \in \pi_1^{-1}(e)$ is an edge in $W^j_{\alpha+i}$ for some $1 \leq j \leq p^i$. Therefore $|\pi_1^{-1}(e) \cap E_0(\tilde{v})| = 1$, $\tilde{\lambda}_1(\tilde{e}) = \lambda_1(e)$, and $d_1(\tilde{e}) = p^{\beta-i} = d(\tilde{v})$.

If $w \in V(W_{\alpha+i+1})$, then $\lambda_1(e) = p$, hence $\gcd(\lambda_1(e), d_1(v)) = p$, and any $\tilde{e} \in \pi_1^{-1}(e)$ is a new edge. Therefore $|\pi_1^{-1}(e) \cap E_0(\tilde{v})| = p$, $\tilde{\lambda}_1(\tilde{e}) = 1$, and $d_1(\tilde{e}) = p^{\beta-i-1}$.

If $w \in V(W_{\alpha+i-1})$, then $\lambda_1(e) = 1$, hence $\gcd(\lambda_1(e), d_1(v)) = 1$ and any $\tilde{e} \in \pi_1^{-1}(e)$ is again a new edge. Therefore $|\pi_1^{-1}(e) \cap E_0(\tilde{v})| = 1$, $\tilde{\lambda}_1(\tilde{e}) = 1 = \lambda_1(e)$, and $d_1(\tilde{e}) = p^{\beta-i}$. Hence $(\tilde{A}_1, \tilde{\lambda}_1)$ is an admissible branched cover of (A_1, λ_1) for the surjective map $\pi_1 : \tilde{A}_1 \to A_1$, with the degree function d_1 defined above.

Since all new edges have label 1, we can collapse all new edges to obtain a labeled graph (A_p, λ_p). Also, $p \nmid \lambda_1(e)$ for all $e \in E(\tilde{A}_1)$, and A_p is obtained by from $\tilde{A}_1$ via collapse moves therefore $p \nmid \lambda_p(e)$ for all $e \in E(A_p)$.

Now, we will demonstrate that (A_p, λ_p) defines an admissible branched cover of (A, λ). We have the natural map $\pi : A_p \to A$ obtained from the following sequence of maps:

$$A_p \xrightarrow{\textit{expansion}} \tilde{A}_1 \xrightarrow{\pi_1} A_1 \xrightarrow{\textit{collapse}} A$$

Define the degree of a vertex $u \in V(A_p)$ to be the maximum degree of all the vertices in $\tilde{A}_1$ which maps to u under the collapse map from $\tilde{A}_1$ to A_p. Every edge

$f \in E(A_p)$ corresponds to a unique edge $\tilde{f} \in E(\tilde{A}_1)$; define the degree of f to be the degree of $\tilde{f}$. Let $e \in A$, with $\partial_0(e) = v$, and $u \in \pi^{-1}(v)$. By the construction of $(\tilde{A}_1, \tilde{\lambda}_1)$, we have $\nu_p(d(u)) \geq \nu_p(\lambda(e))$, implying that $\gcd(\lambda(e), d(u)) = p^{\nu_p(\lambda(e))}$. Furthermore, it is evident that $|\pi^{-1}(e) \cap E_0(u)| = p^{\nu_p(\lambda(e))}$, and for $e_p \in \pi^{-1}(e) \cap E_0(u)$, we have $d(e_p) = d(u)/p^{\nu_p(\lambda(e))}$, and $\lambda_p(e_p) = \lambda(e)/p^{\nu_p(\lambda(e))}$. Therefore the labeled graph (A_p, λ_p) defines an admissible branched cover of (A, λ) whose edge labels are coprime to p. This finishes the proof of the Theorem.

□

3.3 The Leighton property of $X_{m,n}$

Assume m and n have no divisor less than or equal to d. We will show that for such a pair of numbers (dm, dn), all torsion-free uniform lattices in $\mathrm{Aut}(X_{dm,dn})$ are commensurable.

The main result of this section is Proposition 3.3.4, which implies that any torsion-free uniform lattice in $\mathrm{Aut}(X_{dm,dn})$ for $(m, n) \neq (1, 1)$ has a finite index subgroup represented by a directed labeled graph with edges labeled m at the initial vertex and n at the terminal vertex. It follows from Theorem 2.5.2(1) that the vertices of these graphs have d incoming and d outgoing edges incident to them. According to Leighton's theorem for graphs, any two labeled graphs with these properties share a common compact admissible cover.

Let's recall the statement of Leighton's theorem here,

Theorem 3.3.1. *[Lei82, Leighton's theorem] Let G_1 and G_2 be finite connected graphs with a common cover. Then they have a common finite cover.*

For the rest of the section, fix a general torsion-free uniform lattice G in $\mathrm{Aut}(X_{dm,dn})$, for $(m,n) \neq (1,1)$ unless otherwise stated. Let (A,λ) be a compact GBS structure for G given by Theorem 2.5.2 with directed graph structure $E^+(A)$ and length function $l : E(A) \sqcup V(A) \longrightarrow \mathbb{N}$; then A is strongly connected by Remark 2.5.3.

Lemma 3.3.2. *If $e \in E^+(A)$, then $|\lambda(e)| = \alpha m$ and $|\lambda(\overline{e})| = \beta n$ for some $1 \leq \alpha, \beta \leq d$.*

Proof. Let $\partial_0(e) = v_1$ and $\partial_1(e) = v_2$. Since A is strongly connected, there exists a directed cycle $(e_1, e_2, \cdots e_k)$ in (A,λ), with $e_i \in E^+(A)$ for each i and $e_1 = e$. For $1 \leq i \leq k$, assume $\partial_0(e_i) = v_i$, and $\partial_1(e_i) = v_{i+1}$.

Claim(1): For any $i \in \{1, 2, \cdots k\}$, $l(v_i) = \left(\frac{n}{m}\right)^{i-1} \left|\frac{\lambda(e_1)\lambda(e_2)\cdots\lambda(e_{i-1})}{\lambda(\overline{e}_1)\lambda(\overline{e}_2)\cdots\lambda(\overline{e}_{i-1})}\right| l(v_1)$.

We will prove the claim using induction on i. It is trivially true for $i = 1$. Now, assume that the claim holds for $i-1$. By Theorem 2.5.2(2),

$$l(\partial_0(e_{i-1}))|\lambda(e_{i-1})| = l(v_{i-1})|\lambda(e_{i-1})| = ml(e_{i-1}).$$

Therefore $l(e_{i-1}) = \frac{1}{m} l(v_{i-1})|\lambda(e_{i-1})|$. Furthermore,

$$l(\partial_1(e_{i-1})|\lambda(\overline{e}_{i-1}))| = l(v_i)|\lambda(\overline{e}_{i-1})| = nl(e_{i-1}).$$

Hence,

$$\begin{aligned}
l(v_i) &= n\frac{l(e_{i-1})}{|\lambda(\overline{e}_{i-1})|} \\
&= \frac{n}{m}\frac{|\lambda(e_{i-1})|}{|\lambda(\overline{e}_{i-1})|} l(v_{i-1}) \\
&= \frac{n}{m}\frac{|\lambda(e_{i-1})|}{|\lambda(\overline{e}_{i-1})|} \left(\frac{n}{m}\right)^{i-2} \left|\frac{\lambda(e_1)\lambda(e_2)\cdots\lambda(e_{i-2})}{\lambda(\overline{e}_)\lambda(\overline{e}_2)\cdots\lambda(\overline{e}_{i-2}}) \right| l(v_1) \\
&= \left(\frac{n}{m}\right)^{i-1} \left|\frac{\lambda(e_1)\lambda(e_2)\cdots\lambda(e_{i-1})}{\lambda(\overline{e}_1)\lambda(\overline{e}_2)\cdots\lambda(\overline{e}_{i-1})}\right| l(v_1).
\end{aligned}$$

This proves the claim. Since $(e_1, e_2, \cdots e_k)$ is a cycle, $v_1 = v_{k+1}$. Hence,

$$l(v_1) = l(v_{k+1})$$
$$l(v_1) = \left(\frac{n}{m}\right)^k \left|\frac{\lambda(e_1)\lambda(e_2)\cdots\lambda(e_k)}{\lambda(\overline{e}_1)\lambda(\overline{e}_2)\cdots\lambda(\overline{e}_k)}\right| l(v_1),$$

and so

$$\left|\frac{\lambda(e_1)\lambda(e_2)\cdots\lambda(e_k)}{\lambda(\overline{e}_1)\lambda(\overline{e}_2)\cdots\lambda(\overline{e}_k)}\right| = \left(\frac{m}{n}\right)^k. \tag{3.3}$$

Claim(2): $m \mid |\lambda(e_1)|$ and $n \mid |\lambda(\overline{e}_1)|$.

Assuming claim(2) is true we get $|\lambda(e)| = \alpha m$, and $|\lambda(\overline{e}_1)| = \beta n$, for some $\alpha, \beta \geq 1$. Also, by Proposition 2.5.2(1) $|\lambda(e)| \leq dm$, and $|\lambda(\overline{e}_1)| \leq dn$. Therefore, $\alpha m = |\lambda(e)| \leq dm$, and $\beta n = |\lambda(\overline{e})| \leq dn$, implying $\alpha, \beta \leq d$. This proves the lemma.

To prove claim(2) we will only prove $m \mid |\lambda(e_1)|$, and $n \mid |\lambda(\overline{e}_1)|$ can be proved in similar manner. Let $m = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ be the prime factorization of m. If $\nu_{p_i}(\lambda(e_1)) \geq r_i$ for all $1 \leq i \leq k$ then $m \mid |\lambda(e)|$. Assume there is some $1 \leq i \leq k$ for which $\nu_{p_i}(\lambda(e_1)) < r_i$.

Without loss of generality we can assume $\nu_{p_1}(\lambda(e_1)) < r_1$. Then by equation (3.3), $\nu_{p_1}(\lambda(e_i)) > r_1$ for some $2 \leq i \leq k$. Again without loss of generality, we can assume that $\nu_{p_1}(\lambda(e_2)) > r_1$. Since $|\lambda(e_2)| \leq dm < p_1^{r_1+1} p_2^{r_2} \cdots p_k^{r_k}$, there exists $2 \leq j \leq l$ such that $\nu_{p_j}(\lambda(e_2)) < r_j$. Let $s_j = \nu_{p_j}(\lambda(e_2))$, then the set $J = \{j : s_j < r_j\}$ is a nonempty set. Proposition 2.5.2(2) provides the following sequence of implications;

$$l(v_2)|\lambda(e_2)| = ml(e_2)$$

$$\nu_{p_j}(l(v_2)) + \nu_{p_j}(|\lambda(e_2)|) = \nu_{p_j}(m) + \nu_{p_j}(l(e_2))$$

$$\nu_{p_j}(l(v_2)) = r_j - s_j + \nu_{p_j}(l(e_2)) \geq r_j - s_j.$$

Therefore

$$k(v_2) = \gcd(l(v_2), m) \geq \prod_{j \in J} p_j^{r_j - s_j}. \tag{3.4}$$

By Proposition 2.5.2(3), $E_0^+(v_2) = E_1^+ \sqcup \cdots \sqcup E_{k(v_2)}^+$ such that $\sum_{e\in E_i^+} |\lambda(e)| = \sum_{e\in E_j^+} |\lambda(e)| = C$ for all $1 \le i, j \le k_{v_2}$.

$$\begin{aligned} dm &= \sum_{e\in E_0^+(v_2)} |\lambda(e)| \\ &= \sum_{e\in E_1^+} |\lambda(e)| + \cdots + \sum_{e\in E_{k(v_2)}^+} |\lambda(e)| \\ &= Ck(v_2) \end{aligned}$$

The fact that $e_2 \in E_0^+(v_2)$ and equation 3.4 imply that,

$$|\lambda(e_2)| \le C = \frac{dm}{k(v_2)} \le \frac{dm}{\prod_{j\in J} p_j^{r_j - s_j}} = d \prod_{j\notin J} p_j^{r_j} \prod_{j\in J} p_j^{s_j} \tag{3.5}$$

By our assumption, $|\lambda(e_2)| = cp_1^{r_1+\epsilon_1} p_2^{s_2} p_3^{s_3} \cdots p_k^{s_k}$ for some $c, \epsilon_1 \ge 1$. The fact that no prime less than or equal to d divides m gives the following inequality which contradicts inequality 3.5.

$$\begin{aligned} |\lambda(e_2)| &> dp_1^{r_1} p_2^{s_2} p_3^{s_3} \cdots p_k^{s_k} \\ &= dp_1^{r_1} \prod_{\substack{j\notin J \\ j\neq 1}} p_j^{s_j} \prod_{j\in J} p_j^{s_j} \\ &\ge d \prod_{j\notin J} p_j^{r_j} \prod_{j\in J} p_j^{s_j} \end{aligned}$$

where the last inequality follows from the fact that $s_j \ge r_j$ for $j \notin J$. □

Proposition 3.3.3. *The GBS group G represented by (A, λ) is p-unimodular for all primes $p \le d$.*

Proof. Suppose $(e_1, e_2, \cdots e_k)$ is a cycle in $H_1(A)$ (it need not to be a directed cycle). Let

$$\{e_1, e_2, \cdots e_k\} = \{e_{i_1}, e_{i_2}, \cdots, e_{i_r}\} \sqcup \{e_{j_1}, e_{j_2}, \cdots, e_{j_s}\}$$

where, $\{e_{i_1}, e_{i_2}, \cdots, e_{i_r}, \overline{e}_{j_1}, \overline{e}_{j_2}, \cdots, \overline{e}_{j_s}\} \subseteq E^+(A)$. Let $\partial_0(e_i) = v_i$ and $\partial_1(e_i) = v_{i+1}$. Then Theorem 2.5.2(2), and the fact that $l(e) = l(\overline{e})$ imply that

$$l(v_{i+1}) = \begin{cases} \frac{n}{m}\frac{|\lambda(e_i)|}{|\lambda(\overline{e}_i)|}l(v_i), & \text{if } e_i \in E^+(A) \\ \frac{m}{n}\frac{|\lambda(e_i)|}{|\lambda(\overline{e}_i)|}l(v_i), & \text{if } e_i \in E^-(A). \end{cases}$$

Since $(e_1, e_2, \cdots e_k)$ is a cycle, $v_1 = v_{k+1}$ and the same proof as in claim(1) of Lemma 3.3.2 implies

$$l(v_1) = l(v_{k+1}) = \left(\frac{n}{m}\right)^{r-s} \frac{|\lambda(e_{i_1})\lambda(e_{i_2})\cdots\lambda(e_{i_r})|}{|\lambda(\overline{e}_{i_1})\lambda(\overline{e}_{i_2})\cdots\lambda(\overline{e}_{i_r})|}\frac{|\lambda(e_{j_1})\lambda(e_{j_2})\cdots\lambda(e_{j_s})|}{|\lambda(\overline{e}_{j_1})\lambda(\overline{e}_{j_2})\cdots\lambda(\overline{e}_{j_s})|}l(v_1).$$

Therefore,

$$\begin{aligned} q(e_1, e_2, \cdots e_k) &= \frac{|\lambda(e_1)\lambda(e_2)\cdots\lambda(e_k)|}{|\lambda(\overline{e}_1)\lambda(\overline{e}_2)\cdots\lambda(\overline{e}_k)|} \\ &= \frac{|\lambda(e_{i_1})\lambda(e_{i_2})\cdots\lambda(e_{i_r})|}{|\lambda(\overline{e}_{i_1})\lambda(\overline{e}_{i_2})\cdots\lambda(\overline{e}_{i_r})|}\frac{|\lambda(e_{j_1})\lambda(e_{j_2})\cdots\lambda(e_{j_s})|}{|\lambda(\overline{e}_{j_1})\lambda(\overline{e}_{j_2})\cdots\lambda(\overline{e}_{j_s})|} \\ &= \left(\frac{m}{n}\right)^{r-s}. \end{aligned}$$

Since $p \leq d$, and $\gcd(m, i) = \gcd(n, i) = 1$ for all $1 \leq i \leq d$, we have $\gcd(m, p) = \gcd(n, p) = 1$. Therefore $\nu_p\left(\frac{m}{n}\right) = 0$, and

$$\begin{aligned} q_p(e_1, e_2, \cdots e_k) &= \nu_p \circ q(e_1, e_2, \cdots e_k) \\ &= \nu_p\left(\left(\frac{m}{n}\right)^{r-s}\right) \\ &= (r-s)\left(\nu_p\left(\frac{m}{n}\right)\right) \\ &= 0. \end{aligned}$$

□

Proposition 3.3.4. *For a fixed prime number $p \leq d$, there exists a finite-index GBS group $H_p \leq G$ with compact directed GBS structure (A_p, λ_p) and a directed graph* *structure $E(A_p) = E^+(A_p) \sqcup E^-(A_p)$ satisfying (1)-(3) of Theorem 2.5.2, such that for all $e \in E^+(A_p)$,*

$$|\lambda_p(e)| = \alpha_e m \text{ and } |\lambda_p(\overline{e})| = \beta_e n \tag{3.6}$$

for some $1 \leq \alpha_e, \beta_e \leq d$, coprime to p.

Proof. By Proposition 3.3.3, the graph (A, λ) is p-unimodular. Therefore, Theorem 3.2.1 guarantees the existence of $H_p \leq G$. As H_p itself is a uniform lattice in $\mathrm{Aut}(X_{dm,dn})$, (A_p, λ_p) admits a directed graph structure satisfying (1)-(3) of Theorem 2.5.2. Finally, Lemma 3.3.2 implies that the labels of edges in A_p satisfy equation 3.6. □

Corollary 3.3.4.1. *There exists a finite-index GBS group $H \leq G$ with a compact GBS structure $(\tilde{A}, \tilde{\lambda})$, and a directed graph structure $E(\tilde{A}) = E^+(\tilde{A}) \sqcup E^-(\tilde{A})$ satisfying (1)-(3) of Theorem 2.5.2, such that for all $e \in E^+(\tilde{A})$*

$$|\lambda(e)| = m, \text{ and } |\lambda(\overline{e})| = n.$$

Proof. The result follows from applying Proposition 3.3.3 iteratively for every prime $p \leq d$. □

Corollary 3.3.4.2. *Any two torsion-free uniform lattices in Aut$(X_{dm,dn})$ are commensurable up to conjugacy.*

Proof. Let G_1 and G_2 be torsion-free uniform lattices in $\mathrm{Aut}(X_{dm,dn})$. By Corollary 3.3.4.1, there exist finite index GBS groups $H_i \leq G_i$ with compact GBS structures $(\tilde{A}_i, \tilde{\lambda}_i)$ for $i \in \{1, 2\}$ satisfying

$$|\lambda(e)| = m \text{ and } |\lambda(\overline{e})| = n \tag{3.7}$$

for all $e \in E^+(\tilde{A}_i)$. By Theorem 2.5.2(2) we also have

$$\sum_{e \in E_0^+(v)} |\lambda(e)| = dm, \text{ and } \sum_{e \in E_0^-(v)} |\lambda(\bar{e})| = dn. \tag{3.8}$$

These equations imply that $\tilde{A}_1$ and $\tilde{A}_2$ are $2d$ regular graphs. Consequently, by Leighton's graph covering theorem, they also share a common finite-sheeted topological cover. Equation (3.8) guarantees that this common cover can be labeled to create a common admissible cover of $\tilde{A}_1$ and $\tilde{A}_2$. Thus, H_1 and H_2 are commensurable up to conjugacy in $\mathrm{Aut}(X_{dm,dn})$, implying the same for G_1 and G_2.

□

3.3.1 Torsion-free uniform lattices in Aut($X_{d,d}$)

This subsection addresses the last component of Theorem 1.1.2, namely that any two torsion-free uniform lattices in $\mathrm{Aut}(X_{d,d})$ are commensurable.

If $\mathrm{Aut}(X_{m,n})$ is a discrete group, it cannot contain incommensurable lattices [BL01, 1.7]. Also, $\mathrm{Aut}(X_{m,n})$ is discrete if and only if $\gcd(m, n) = 1$ [For24, Theorem 4.8]. Therefore, $\mathrm{Aut}(X_{1,1})$ is discrete, and any two lattices in $\mathrm{Aut}(X_{1,1})$ are commensurable.

Let Γ be a torsion-free uniform lattice in $\mathrm{Aut}(X_{d,d})$ for $d \geq 2$. Then, by Proposition 2.4.1, Γ acts on $X_{d,d}$ freely and cocompactly, providing a covering space action. Each branching line in $X_{d,d}$ covers a circle, and each strip covers either an annulus or a Möbius band, therefore the quotient space may not be orientable. However, by [[Bas93], Proposition 6.3] we can find an index 2 subgroup G of Γ that acts on $X_{d,d}$ without changing the sides of any strip. Therefore, the quotient obtained from the action of G on $X_{d,d}$ is a fibered 2-complex $Z_{(A.\lambda)}$ for some labeled graph (A, λ). For $e \in E(A)$, if $l(e)$ denotes the number of 2 cells tiling the annulus corresponding to e

and $l(v)$ denotes the combinatorial length of the circle corresponding to v, then we have the following result which is derived from the proof of Proposition 4.6 of [For24].

Proposition 3.3.5. *Suppose G is a torsion-free uniform lattice in $Aut(X_{d,d})$ for $d \geq 2$ such that the action of G on $X_{d,d}$ does not change the sides of any strip. Let (A, λ) be the labeled graph structure such that $X_{d,d}/G$ is homeomorphic to $Z_{(A,\lambda)}$ with the associated length function $l : V(A) \sqcup E(A) \mapsto \mathbb{N}$. Then, for any edge $e \in E(A)$ we have,*

1. $l(\partial_0(e))|\lambda(e)| = l(\partial_1(e))|\lambda(\overline{e})|$.

2. *G is unimodular.*

Proof. Condition (1) follows from the cell structure of the annulus corresponding to the edge e. This annulus is tiled by $(1, 1)$ cells whose boundary curves have length $l(e)$ which wrap $|\lambda(e)|$ and $|\lambda(\overline{e})|$ times onto the circles corresponding to $\partial_0(e)$ and $\partial_1(e)$, respectively.

To prove (2), let $(e_1. \cdots, e_k)$ be a cycle in A. Then, by (1)

$$\begin{aligned} q(e_1. \cdots, e_k) &= \frac{|\lambda(e_1) \cdots \lambda(e_k)|}{|\lambda(\overline{e}_1) \cdots \lambda(\overline{e}_k)|} \\ &= \frac{l(\partial_1(e_1)) \cdots l(\partial_1(e_1))}{l(\partial_0(\overline{e}_1)) \cdots l(\partial_0(\overline{e}_1))} \\ &= 1 \end{aligned}$$

where the last equality fallows form the fact that $\partial_0(e_{i+1}) = \partial_1(e_i)$ for $1 \leq i \leq k-1$ and $\partial_1(e_k) = \partial_0(e_1)$.

$\square$

Proposition 3.3.6. *[Lev07] If G is a non-elementary GBS group then G is unimodular if and only if it has a finite index subgroup isomorphic to $F_n \times \mathbb{Z}$ for some $n > 1$.*

Proposition 3.3.7. *Any two torsion-free uniform lattices in $Aut(X_{d,d})$, for $d \geq 2$ are commensurable.*

Proof. Let Γ_1 and Γ_2 are two torsion-free uniform lattices in $\mathrm{Aut}(X_{d,d})$. We can assume that Γ_i act on $T_{d,d}$ without inversion (possibly after passing to an index 2 subgroup of Γ_i) for $i = 1, 2$. By Proposition 3.3.5 Γ_i are unimodular, and hence contain a finite index subgroup isomorphic to $F_{n_i} \times \mathbb{Z}$ for some $n_i > 1$ by Proposition 3.3.6. Finally, since $F_{n_1} \times \mathbb{Z}$ and $F_{n_2} \times \mathbb{Z}$ are commensurable, it follows that Γ_1 and Γ_2 are commensurable.

□

3.3.2 Leighton's Property

Definition 3.3.1. We say that a cell complex X has the *Leighton property* if every pair of compact cell complexes, both having X as their common universal cover, admits a common finite-sheeted covering.

By Leighton's theorem, trees have the Leighton property.

Theorem 3.3.8. *When m and n have no prime divisor less than or equal to d, the cell complex $X_{dm,dn}$ satisfies the Leighton property.*

Proof. For cell complexes X_1 and X_2 with common universal cover $X_{dm,dn}$, the fundamental groups $\pi_1(X)$ and $\pi_2(X)$ are torsion-free since X_1 and X_2 are finite-dimensional aspherical cell complexes, hence defining torsion-free uniform lattices in $\mathrm{Aut}(X_{dm,dn})$. For $i = 1, 2$, we can choose labeled graphs (A_i, λ_i) with associated fibered 2-complexes X_i.

For $(m, n) \neq (1, 1)$, the results of Proposition 3.3.4 and its corollaries 3.3.4.1, 3.3.4.2 imply the labeled graphs (A_1, λ_1) and (A_2, λ_2) admit a common finite sheeted admissible branched covering. The fibered 2-complex associated with this common finite sheeted admissible branched covering provides a common finite sheeted cover of X_1 and X_2 by Remark 2.9.1.

For $(m, n) = (1, 1)$ and $d = 1$, all lattices in $\mathrm{Aut}(X_{1,1})$ are commensurable as it is a dicrete group. Therefore $X_{1,1}$ satisfies Leighton's property.

For $(m, n) = (1, 1)$ and $d \geq 2$, we can assume that X_i are orientable (possible after passing to degree 2 covering of X_i) for $i = 1, 2$. X_i is a fibered 2-complex associated to some labeled graph (A_i, λ_i) which is unimodular by Proposion 3.3.5, hence p-unimodular for all prime p. By Theorem 3.2.1, we can find a $2d$ regular admissible branched cover $(\tilde{A}_i, \tilde{\lambda}_i)$ of (A_i, λ_i) with edge labels 1. By Leighton's graph covering theorem, the graphs $\tilde{A}_1$ and $\tilde{A}_2$ admit a common finite topological cover, denoted as $\tilde{A}$. Assigning labels 1 to each edge of $\tilde{A}$ yields a labeled graph $(\tilde{A}, \tilde{\lambda})$ which defines an admissible branched cover of $(\tilde{A}_i, \tilde{\lambda}_i)$. Since the composition of admissible branched covers is again an admissible branched cover, $(\tilde{A}, \tilde{\lambda})$ defines a common admissible branched cover for (A_1, λ_1) and (A_2, λ_2). Thus, the fibered 2-complex associated with $(\tilde{A}, \tilde{\lambda})$ provides a common finite sheeted topological cover of X_i.

$\square$

3.4 Example of incommensurable lattices using the depth profile

We provide examples of incommensurable lattices in both Case (I) and Case (II), utilizing the commensurability invariant known as the depth profile.

3.4.1 Incommensurable lattices in Case (I)

In this subsection, we provide examples of incommensurable lattices in $\mathrm{Aut}(X_{dm,dn})$, when there is a prime number $p \leq d$ such that $p \mid d$, and $p \mid m$ or $p \mid n$. Without loss of generality, let p be a prime number that divides both m and d. Note that $p \nmid n$ since $\gcd(m, n) = 1$.

The lattice $\mathbf{\Gamma_1}$. Consider the lattice Γ_1 defined by the directed labeled graph (B_1, μ_1) in Figure (3.3). It is a bipartite graph with two vertices u_1 (white) and v_1 (black), and $2d$ directed edges. The edges $e_1, e_2, \cdots e_d$ are directed from u_1 to v_1 and the edges $f_1, f_2, \cdots, f_d$ are directed from v_1 to u_1. We have $\mu_1(e_i) = \mu_1(f_i) = m$ and $\mu_1(\overline{e_i}) = \mu_1(\overline{f_i}) = n$.

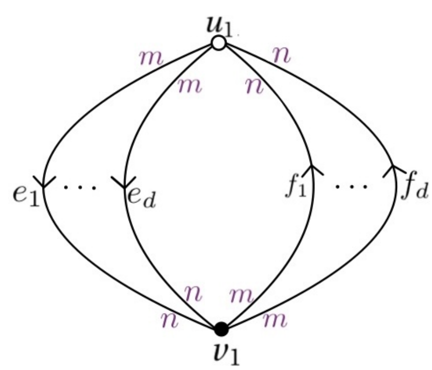


Figure 3.3: B_1

The lattice $\mathbf{\Gamma_k}$ for $\mathbf{k \geq 2}$. This group is defined by the directed labeled graph (B_k, μ_k) in Figure (3.4). It has k vertices $v_1, \cdots, v_k$, and $d(k-1) + \frac{d}{p}$ directed edge. There are d directed edges from v_i to v_{i+1} for $1 \leq i \leq k-1$ with initial label m and terminal label n and there are $\frac{d}{p}$ directed edges from v_k to v_1 with initial label pm and terminal label pn.

The groups Γ_k for $k \geq 1$ are all latties in $\mathrm{Aut}(X_{dm,dn})$ by theorem 2.5.1. We will compute the depth profiles of Γ_k, for $k \geq 1$, and will show that these depth profiles

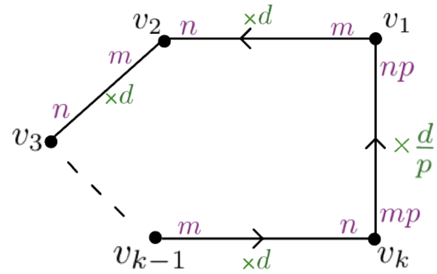


Figure 3.4: Graph B_l for $l \geq 2$, defining lattices in $\mathrm{Aut}(X_{dm,dn})$ for a prime number $p \mid d, m$. Here, $\times\alpha$ denotes the number of edges between vertices.

are not equivalent using Lemma 2.10.2. This will prove that these groups are pairwise incommensurable.

Definition 3.4.1. A segment σ in a $G-$tree is called *unimodular* if $i(\sigma) = i(\overline{\sigma})$.

Proposition 3.4.1. *[For24] Let G be a GBS group and X be its GBS tree. Suppose V is the stabilizer of a vertex $x \in V(X)$. Define the set*

$$\mathscr{I}(x) = \{i(\sigma) : \sigma \text{ is a non-trivial unimodular segment with endpoints in } Gx\}.$$

Then

$$\mathscr{D}(G,V) \subseteq \mathscr{I}(x) \subseteq \mathscr{D}(G,V) \cup \{1\}.$$

Proposition 3.4.2. *Let X_1 be the Bass-Serre trees for the given GBS structure on Γ_1. Suppose V_1 is the stabilizer of a vertex $x_1 \in V(X_1)$ which maps to the black vertex in the graph of groups B_1. Then $\mathscr{I}(x_1) = S_1[n] - \{1\}$ where*

$$S_1 = \left\{m^i : i \in \mathbb{N} \cup \{0\}\right\}.$$

Proof. Note that the unimodular segments in X_1 with both endpoints in $\Gamma_1 x_1$ have even lengths, and the vertices along these segments alternate between black and white. Additionally, for this entire proof, we only consider the unimodular segments from a black to another black vertex in X_1.

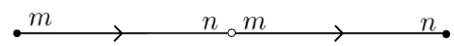


Figure 3.5: A segment τ in X_1.

Let's denote any length 2 segment in X_1 with the labels given in Figure 3.5 by τ. For such segments τ, the initial index $i(\tau)$ is m^2 and the terminal index $i(\overline{\tau})$ is n^2. Now every unimodular segment σ in X of length > 2 has one of the following forms:

(1) $\sigma = \sigma_1\sigma_2$ with σ_1, σ_2 unimodular segments in X

(2) $\sigma = \tau\sigma_1\overline{\tau}$ with σ_1 a unimodular segment in X

(3) $\sigma = \overline{\tau}\sigma_1\tau$ with σ_1 a unimodular segment in X.

Let D_1 denote the set $S_1[n] - \{1\}$ (see equation (2.4) for definition of $S[k]$). It is easy to verify that D_1 is closed under taking lcm. We will show that every unimodular segment σ has index in D_1 by induction on its length. Let's denote any edge in X_1 with an initial label m and a terminal label n by e. Then, the length 2 unimodular segments in X_1 whose end vertices are black are $e\overline{e}$ and $\overline{e}e$. By Remark 2.8.1 we can see that $i(e\overline{e}) = m \in D_1$, and $i(\overline{e}e) = n \in D_1$.

If σ is of type(1), then by Remark 2.8.1, we have $i(\sigma) = \operatorname{lcm}(i(\sigma_1), i(\sigma_2)) \in D_1$.

By Remark 2.8.1, and using the fact that $\gcd(\operatorname{lcm}(a, b), b) = b$, we see the following. If σ is of type(2), then its index is

$$i(\sigma) = n^{-2}m^2 \operatorname{lcm}(i(\sigma_1), n^2),$$

which lies in D_1. Similarly, if σ is of type(3), then

$$i(\sigma) = m^{-2}n^2 \operatorname{lcm}(i(\sigma), m^2),$$

which also lies in D_1. This shows that $\mathscr{I}(x_1) \subseteq D_1$.

Finally, consider the following unimodular segments in X_1:

(a) $e^i\overline{e}^i$

(b) $\overline{e}^j e^j$

These segments have indices m^i, and n^j, respectively. Therefore the index of a concatenation of segments of type(a) and type(b) is $\mathrm{lcm}(m^i, n^j) = m^i n^j$. Hence we also have $D_1 \subseteq \mathscr{I}(x_1)$. $\square$

Proposition 3.4.3. *Let X_k be the Bass-Serre tree for the given GBS structure for Γ_k, for $k \geq 2$. Suppose V_k is the stabilizer of a vertex x_k which maps v_1. Then $\mathscr{I}(x_k) = S_k[n] - \{p\}$ where*

$$S_k = \{pm^{ik}, m^{ik+1}, \cdots m^{ik+(k-1)} : i \geq 0\}$$

Proof. Note that, the unimodular segments in X_k with both endpoints in $\Gamma_k x_k$ have even lengths as they contain equal numbers of forward- and backward-oriented edges.

Let D denote the set $S_k[n] - \{p\}$. It is easy to check that D is closed under taking lcm. We will show that every unimodular segment in X_k with both endpoints in $\Gamma_k x_k$ has index in D by induction on its length. Let's denote edges in X_k with initial label m and terminal label n by e, and edges with initial label pm and terminal label pn by f.

For the base case, we will show that the index of unimodular segments of length $2l$, for $1 \leq l \leq k$, is contained in D. Observe that $i(\tau_1 e\overline{e}e\overline{e}\tau_2) = i(\tau_1 e\overline{e}\tau_2)$ for any segments τ_1 and τ_2. Therefore it is sufficient to compute the index of unimodular segments that are either $e^l\overline{e}^l$, $\overline{f}\overline{e}^{l-1}e^{l-1}f$ or concatenations of smaller unimodular segments. Since $i(e^l\overline{e}^l) = m^l$ and $i(\overline{f}\overline{e}^{l-1}e^{l-1}f) = pn^l$, and since D is closed under taking lcm, it follows from Remark 2.8.1 that the set of indices of unimodular segments of length $2l$ is contained in D.

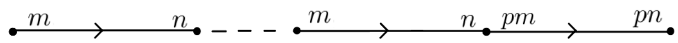


Figure 3.6: A segment τ in X_k

Consider the segment $\tau = e^{k-1}f$ as in Figure 3.6. Note that $i(\tau) = pm^k$ and $i(\overline{\tau}) = pn^k$. The index of every unimodular segment in X_k of length $> 2k$ is contained in the index of one of the following unimodular segments;

(1) $\sigma\sigma'$ with σ, σ' unimodular segments in X_k

(2) $\tau\sigma\overline{\tau}$ with σ a unimodular segment in X_k

(3) $\overline{\tau}\sigma\tau$ with σ a unimodular segment in X_k.

The index of an unimodular segment of type(1) is $\operatorname{lcm}(i(\sigma), i(\sigma'))$ which is contained in D as it is closed under taking lcm. Also, by Remark 2.8.1 and the fact that $gcd(lcm(a,b),b) = b$, we get $i(\tau\sigma\overline{\tau}) = (pn)^{-k}pm^k \operatorname{lcm}(pn^k, i(\sigma)) \in D$, and $i(\overline{\tau}\sigma\tau) = (pm)^{-k}pn^k \operatorname{lcm}(pm^k, i(\sigma)) \in D$. This shows that $\mathscr{I}(x_k) \subseteq D$.

Finally, consider the following unimodular segments in X_k for $i \geq 0$, and $0 \leq j \leq k-1$;

(a) $\tau^i\overline{\tau}^i$

(b) $\tau^i e^j \overline{e}^j \overline{\tau}^i$

(c) $\overline{\tau}^i e^j \overline{e}^j \tau^i$

(d) $\overline{\tau}^i \overline{f} \overline{e}^{j-1} e^{j-1} f \tau^i$

These segments have indices pm^{ik}, m^{ik+j}, pn^{ik}, and pn^{ik+j}, respectively. Therefore,

- Concatenation of segments of type(a) and type(c) has index $pm^{i_1k}n^{i_2k}$,

- Concatenation of segments of type(a) and type(d) has index $pm^{i_1k}n^{i_2k+j_2}$,

- Concatenation of segments of type(b) and type(c) has index $m^{i_1 k+j_1} n^{i_2 k}$,
- Concatenation of segments of type(b) and type(d) has index $m^{i_1 k+j_1} n^{i_2 k+j_2}$.

Hence we also have $D \subseteq \mathscr{I}(x_k)$. $\square$

Corollary 3.4.3.1. *$\mathscr{D}(\Gamma_1, V_1) = S_1[n] - \{1\}$ and $\mathscr{D}(\Gamma_k, V_k) = S_k[n] - \{p\}$ for $k \geq 2$.*

Proof. Since the set $\mathscr{I}(x_k)$ does not contain 1 for $k \geq 1$, by Proposition 3.4.1 we have $\mathscr{D}(\Gamma_k, V_k) = \mathscr{I}(x_k)$. $\square$

Corollary 3.4.3.2. *If the prime number p divides both m and d, then the lattices $\Gamma_k \in Aut(X_{dm,dn})$ are pairwise abstractly incommensurable for $k \geq 1$.*

Proof. Enumerate the elements of S_i in order for $i \geq 1$ and notice that each element divides the next one. Taking the ratio of successive elements we obtain the sequences $(m, m, m, \cdots)$ for S_1 and

$$\Big(\frac{m}{p}, \underbrace{m, m, \cdots, m}_{k-2 \text{ elements}}, pm, \frac{m}{p}, \underbrace{m, m, \cdots, m}_{k-2 \text{ elements}}, pm, \frac{m}{p}, \underbrace{m, m, \cdots, m}_{k-2 \text{ elements}}, \cdots\Big)$$

for S_k, $k \geq 2$. The tails of these ratio sequences are unchanged when passing from S_i to S_i/r for any $r \in \mathbb{N}$ because the values of $\gcd(r, m^j)$, and $\gcd(r, pm^{kj})$ stabilize as $j \to \infty$, all to the same number. The tail for S_1 will never agree with the tail for S_k as $p \neq 1$, so we get S_1 not equivalent to S_k for $k \geq 2$. Also, for $k, l \geq 2$, the tails for S_l and S_k will agree if and only if $l = k$. Using Lemma 2.10.1, we conclude that $S_l[n]$ is not equivalent to $S_k[n]$ for $k \neq l$. Furthermore, it's evident that $S_1[n]$ is equal to the set $(S_1[n] - \{1\})/n$, which is equivalent to $\mathscr{D}(\Gamma_1, V_1) = S_1[n] - \{1\}$. Similarly, $S_k[n]$ is equal to the set $(S_k[n] - \{p\})/n$, which is equivalent to $\mathscr{D}(\Gamma_k, V_k) = S_k[n] - \{p\}$. Therefore by Lemma 2.10.2 the depth profiles of Γ_l and Γ_k are not equivalent for $k \neq l$ and hence these groups are not abstractly commensurable.

$\square$

3.4.2 Incommensurable lattices in Case (II)

In this section, we provide examples of lattices in $\mathrm{Aut}(X_{dm,dn})$ that are abstractly incommensurable when m or n is 1, and such that there exists $p < d$ (not necessarily a prime), $m, n \neq p$ such that either $p \mid m$ or $p \mid n$. Without loss of generality, we can assume $m = 1$, $n \neq p$, and $p \mid n$. [For24] provides an example of incommensurable lattices in $\mathrm{Aut}(X_{d,dn})$ when $n \neq p$, using depth profile as a commensurability invariant (see Theorem 1.1.1(2)). Building on this, we will construct infinitely many such examples using the depth profile.

The lattice $\mathbf{\Delta_k}$ for $\mathbf{k \geq 2}$. Consider the group Δ_k defined by the directed labeled graph (D_k, δ_k) shown in Figure(3.7). The graph D_k consists of k vertices $v_1, v_2, \cdots v_k$ and $dk - p + 1$ directed edges. There are d directed edges from v_i to v_{i+1} for $1 \leq i \leq k-1$ and $d - p$ directed edges from v_k to v_1, each with initial label 1 and terminal label n. Additionally, there is a directed edge from v_k to v_1 with initial label p and terminal label np.

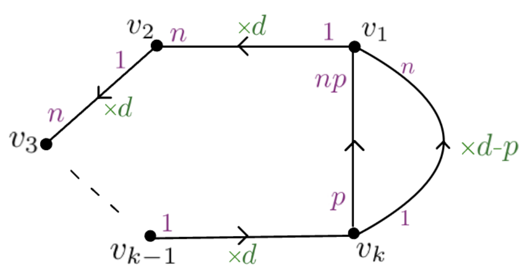


Figure 3.7: The graph D_k for $k \geq 2$, defining a lattice in $\mathrm{Aut}(X_{d,dn})$ for $p < d$.

Performing a sequence of collapse and slide moves on D_k for $k \geq 2$ (see figure 3.8 for an example with $k = 5$), we obtain the following:

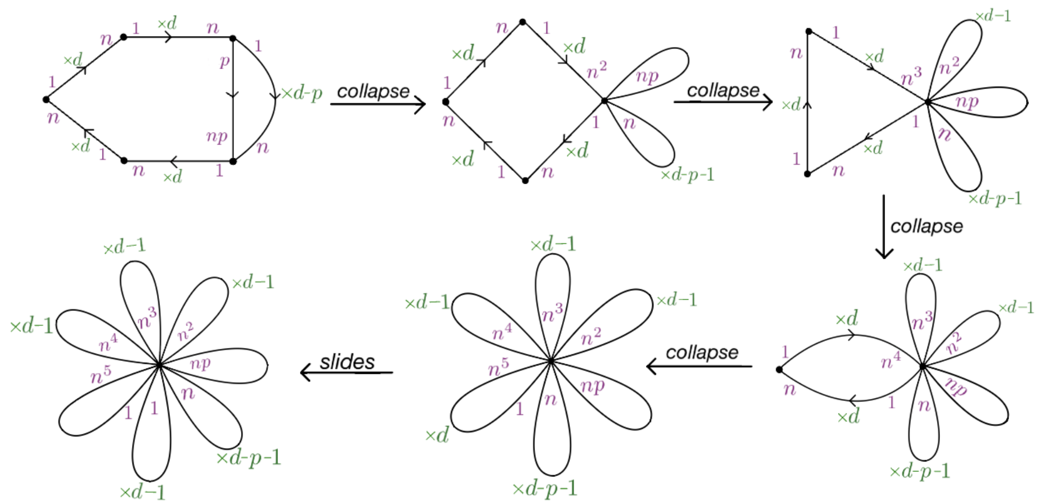


Figure 3.8: A sequence of collapse and slide moves on (Δ_5, δ_5) that gives a bouquet of circles with the fundamental group Δ_5. A number z inside the petal indicates that both ends of the petal are labeled z, while $\times y$ above a petal denotes the multiplicity of that petal.

$$\begin{aligned}\Delta_k &\cong BS(pn, pn) \vee \bigvee_{d} BS(1, n^k) \vee \bigvee_{d-p-1} BS(n, n) \vee \bigvee_{d-1} BS(n^2, n^2) \vee \bigvee_{d-1} BS(n^3, n^3) \\ &\quad \vee \cdots \bigvee_{d-1} BS(n^{k-1}, n^{k-1}) \\ &\cong BS(pn, pn) \vee BS(1, n^k) \vee \bigvee_{d-1} BS(1, 1) \vee \bigvee_{d-p-1} BS(n, n) \vee \bigvee_{d-1} BS(n^2, n^2) \vee \\ &\quad \bigvee_{d-1} BS(n^3, n^3) \cdots \bigvee_{d-1} BS(n^{k-1}, n^{k-1}).\end{aligned} \tag{3.9}$$

The following result from [For24] will be used to compute the depth profile of Δ_k:

Proposition 3.4.4. *[For24] Let $G = BS(1, N) \vee \bigvee_{i=1}^{r} BS(n_i, n_i)$ for some $r \geq 1$, $N > 1$, and n_i dividing N. Suppose the set $\{n_1, n_2, \cdots n_r\}$ is closed under taking* lcm *and contains* 1*. Then, for the vertex group* V*,*

$$\mathscr{D}(G, V) = \{n_i N^j : j \geq 0 \text{ and } 1 \leq i \leq r\}.$$

Proposition 3.4.5. *The set of groups* $\{\Gamma_1, \Delta_k : k \geq 2\}$ *defines pairwise incommensurable lattices in* $\mathrm{Aut}(X_{d,dn})$ *when* $n \geq d$*,* $gcd(n, d) = 1$*, and* n *has a divisor* $p < d$*.*

Proof. For $k \geq 2$, the group Δ_k defines a lattice in $\mathrm{Aut}(X_{d,dn})$ by Theorem 2.5.1. Let V_k be the vertex group of the GBS structure 3.9 for Δ_k. By Proposition 3.4.4, the depth profile of Δ_k is:

$$\mathscr{D}(\Delta_k, V_k) = \begin{cases} \{pn^{ki+1}, n^{ki}, n^{ki+1}, n^{ki+2}, \cdots n^{ki+k-1} : i \geq 0\} & \text{if } d > p+1 \\ \{pn^{ki+1}, n^{ki}, n^{ki+2}, \cdots n^{ki+k-1} : i \geq 0\} & \text{if } d = p+1 \end{cases} \tag{3.10}$$

In both cases, enumerating the elements of $\mathscr{D}(\Delta_k, V_k)$ in ascending order and computing the successive ratios we get the periodic sequence:

$$\begin{cases} n, p, \frac{n}{p}, \underbrace{n, \cdots, n}_{k-1 \text{ times}}, p, \frac{n}{p}, \underbrace{n, \cdots, n}_{k-1 \text{ times}} \cdots & \text{if } d > p+1 \\ pn, \frac{n}{p}, \underbrace{n, \cdots, n}_{k-2 \text{ times}}, pn, \frac{n}{p}, \underbrace{n, \cdots, n}_{k-2 \text{ times}} \cdots & \text{if } d = p+1. \end{cases}$$

By Corollary 3.4.3.1

$$\mathscr{D}(\Gamma_1, V_1) = \{n^i : i \geq 0\},$$

and the ratio of successive terms in $\mathscr{D}(\Gamma_1, V_1)$ is $(n, n, n, \cdots)$. As the tail of these ratio sequences never agrees which is a commensurability invariant as mentioned in Corollary 3.4.3.2, we conclude that any two groups in the set $\{\Gamma_1, \Delta_k : k \geq 2\}$ are incommensurable. $\square$

3.5 Revisiting the CRKZ invariant

In this section, we define a class of GBS groups and provide a necessary condition for two groups in this class to be abstractly commensurable. This condition will enable us to prove that the set of groups $\{\Gamma_1, \Delta_k : k \geq 2\}$ defines pairwise incommensurable lattices in $\mathrm{Aut}(X_{d,dn})$ when $n < d$ and $n \nmid d$. Furthermore, we also demonstrate that this condition is sufficient for a special subset of this class.

In [CRKZ21], the authors introduced a class of GBS groups and constructed an isomorphism invariant for groups in this class (see subsection 3.5.2 for details). We will refer to this invariant as the CRKZ invariant. Here we will give a new description of the CRKZ invariant for a larger class of GBS groups and prove the scaling property of the CRKZ invariant for finite index subgroups arising from topological covers (Theorem 3.5.4). It follows that the CRKZ invariant is also a complete commensurability invariant.

Fix an integer $l \geq 1$. Suppose G is a non-elementary GBS group whose image under the modular homomorphism $q : G \to \mathbb{Q}^*$ is generated by $1/n^{lL}$ for some $n \in \mathbb{N}$ and $L \in \mathbb{Z}$. Suppose G is represented by a labeled graph (A, λ), and for all edges $e \in E(A)$, $\lambda(e) = n^{i_e}$ for $i_e \geq 0$. To each GBS group in this form, we will associate a vector $\vec{X}^l(G) \in (\mathbb{N} \cup \{0\})^l$ well defined up to cyclic permutation. We call this vector the *length l CRKZ invariant* of G.

Definition 3.5.1. Suppose $v_0 \in V(A)$ is a fixed vertex.

1. A vertex $v \in V(A)$ has *level i* with respect to the base vertex v_0 if for any path $(e_1, e_2, \cdots e_r)$ from v_0 to v, $q_n^A(e_1, e_2, \cdots e_r) \equiv i \pmod{l}$.

2. An edge $e \in E(A)$ has *level i* with respect to base vertex v_0 if, for any path $(f_1, f_2, \cdots, f_s)$ from v_0 to $\partial_0(e)$, $\nu_n(\lambda(e)) + q_n^A(f_1, f_2, \cdots, f_s) \equiv i \pmod{l}$.

Let $V^i_{v_0}(A)$ (and $E^i_{v_0}(A)$) denote the set of vertices (and edges) that have level i with respect to the base vertex v_0. In particular,

$$V^i_{v_0}(A) = \{v \in V(A) : q^A_n(e_1, e_2, \cdots e_r) \equiv i \pmod{l}\}$$

$$E^i_{v_0}(A) = \{e \in E(A) : \nu_n(\lambda(e)) + q^A_n(f_1, f_2, \cdots, f_s) \equiv i \pmod{l}\}$$

where $(e_1, e_2, \cdots, e_r)$ is an edge path in A from v_0 to v, and $(f_1, f_2, \cdots, f_s)$ is an edge path in A from v_0 to $\partial_0(e)$. Note that $V^{i+kl}_{v_0}(A) = V^i_{v_0}(A)$ and $E^{i+kl}_{v_0}(A) = E^i_{v_0}(A)$ for all $1 \le i \le l$ and $k \in \mathbb{N}$.

Definitions 3.5.1 is independent of the choice of path from v_0 to v since any two such paths $(e_1, e_2, \cdots, e_r)$ and $(f_1, f_2, \cdots, f_s)$ in A determines a 1-cycle in $H_1(A)$ given by

$$(e_1, e_2, \cdots, e_r, \overline{f}_s, \overline{f}_{s-1}, \cdots, \overline{f}_1)$$

Therefore

$$\begin{aligned} q^A_n(e_1, e_2, \cdots, e_r) - q^A_n(f_1, f_2, \cdots, f_s) &= q^A_n(e_1, e_2, \cdots, e_r, \overline{f}_s, \overline{f}_{s-1} \cdots \overline{f}_1) \\ &\equiv 0 \pmod{l} \end{aligned}$$

implies $q^A_n(e_1, e_2, \cdots, e_r) \equiv q^A_n(f_1, f_2, \cdots, f_s) \pmod{l}$.

Remark 3.5.1. Edges e and $\overline{e}$ always have the same level, so $e \in E^i_{v_0}(A)$ if and only if $\overline{e} \in E^i_{v_0}(A)$. In particular, $|E^i_{v_0}(A)|$ is even.

Definition 3.5.2. Let (A, λ) be a labeled graph. For a fixed integer $l \ge 1$ and $v_0 \in V(A)$, define a vector $\vec{X}^l_{v_0}(A)$ in $(\mathbb{N} \cap \{0\})^l$ as

$$\vec{X}^l_{v_0}(A) = \left(|E^0_{v_0}(A)| - 2|V^0_{v_0}(A)|, |E^1_{v_0}(A)| - 2|V^1_{v_0}(A)|, \cdots, |E^{l-1}_{v_0}(A)| - 2|V^{l-1}_{v_0}(A)|\right).$$

The next lemma shows that these vectors are independent of the choice base point, up to cyclic permutation.

Lemma 3.5.2. *Let v_0 and w_0 be two vertices in A. Then the vector $\vec{X}^l_{w_0}(A)$ is a cyclic permutation of $\vec{X}^l_{v_0}(A)$. In particular, $\sigma^{i_0}\vec{X}^l_{v_0}(A) = \vec{X}^l_{w_0}(A)$, where $\sigma = (l, l-1, \cdots, 1)$ is the cyclic permutation of $\{1, 2, \cdots, l\}$ and $i_0 = q^A_n(e_1, e_2, \cdots, e_r)$ for an edge path $(e_1, e_2, \cdots, e_r)$ from v_0 to w_0.*

Proof. For an edge path $(f_1, f_2, \cdots, f_s)$ from w_0 to v, $(e_1, e_2, \cdots, e_r, f_1, f_2, \cdots, f_s)$ is a path from v_0 to v with

$$\begin{aligned} q^A_n(e_1, e_2, \cdots, e_r, f_1, f_2, \cdots, f_s) &= q^A_n(e_1, e_2, \cdots, e_r) + q^A_n(f_1, f_2, \cdots, f_s) \\ &\equiv i_0 + q^A_n(f_1, f_2, \cdots, f_s) \pmod{l} \end{aligned}$$

Therefore, we have $V^i_{w_0}(A) = V^{i+i_0}_{v_0}(A)$ and $E^i_{w_0}(A) = E^{i+i_0}_{v_0}(A)$. Together with the facts $|V^{i+kl}_{v_0}(A)| = |V^i_{v_0}(A)|$ and $|E^{i+kl}_{v_0}(A)| = |E^i_{v_0}(A)|$ for all $1 \le i \le l$ and $k \in \mathbb{Z}$ we get,

$$\begin{aligned} \sigma^{i_0}\left(\vec{X}^l_{v_0}(A)\right) &= \sigma^{i_0}\big(|E^0_{v_0}(A)| - 2|V^0_{v_0}(A)|, |E^1_{v_0}(A)| - 2|V^1_{v_0}(A)|, \cdots, \\ &\qquad |E^{l-1}_{v_0}(A)| - 2|V^{l-1}_{v_0}(A)|\big) \\ &= \big(|E^{i_0}_{v_0}(A) - 2|V^{i_0}_{v_0}(A)|, |E^{i_0+1}_{v_0}(A)| - 2|V^{i_0+1}_{v_0}(A)|, \cdots, \\ &\qquad |E^{i_0+l-1}_{v_0}(A)| - 2|V^{i_0+l-1}_{v_0}(A)|\big) \\ &= \big(|E^0_{w_0}(A)| - 2|V^0_{w_0}(A)|, |E^1_{w_0}(A)| - 2|V^1_{w_0}(A)|, \cdots, \\ &\qquad |E^{l-1}_{w_0}(A)| - 2|V^{l-1}_{w_0}(A)| \\ &= \vec{X}^l_{w_0}(A). \end{aligned}$$

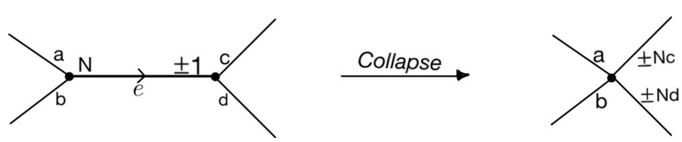
$\square$

We will denote any element in the subset $\{\vec{X}^l_{v_0}(A) : v_0 \in V(A)\} \subset (\mathbb{N} \cup \{0\})^l$ as $\vec{X}^l(A)$. In the view of Lemma 3.5.2, $\vec{X}^l(A)$ is a well-defined vector in $(\mathbb{N} \cup \{0\})^l$ up to cyclic permutation.

Any two splittings of a non-elementary GBS group are in the same deformation space. This means that these graphs of groups are related via a sequence of expansion and collapse moves. Thus, by the next lemma, for a non-elementary GBS group G, we can associate a vector $\vec{X}^l(G)$ well defined up to cyclic permutation. Sometime we will refer $\vec{X}^l(G)$ as $\vec{X}^l(A)$.

Lemma 3.5.3. *If two labeled graphs A and A' are in the same deformation space then $\vec{X}^l(A)$ is a cyclic permutation of $\vec{X}^l(A')$.*

Proof. It suffices to show that if A' is obtained from A via a collapse move, then $\vec{X}^l(A')$ is a cyclic permutation of $\vec{X}^l(A)$.



Suppose A' is obtained from A by collapsing an edge e with $\lambda(e) = N$, and $\lambda(\bar{e}) = 1$. Let us denote the image of a vertex $v \in A$ via collapse move by $v' \in A'$. Choose $\partial_0(e)$ and $(\partial_0(e))'$ to be the base vertex in A and A', respectively. If $\nu_n(N) \equiv i_0 \pmod{l}$, then the vertex $\partial_1(e)$ has level i_0, and edges $e, \bar{e}$ also have level i_0. Therefore

$$\left|V^i(A')\right| = \begin{cases} |V^i(A)| - 1, & \text{if } i \equiv i_0 \pmod{l} \\ |V^i(A)|, & \text{if } i \not\equiv i_0 \pmod{l}, \end{cases}$$

and

$$\left|E^i(A')\right| = \begin{cases} |E^i(A)| - 2, & \text{if } i \equiv i_0 \pmod{l} \\ |E^i(A)|, & \text{if } i \not\equiv i_0 \pmod{l}. \end{cases}$$

Hence $|E^{i_0}(A')| - 2|V^{i_0}(A')| = |E^{i_0}(A)| - 2 - 2(|V^{i_0}(A)| - 1) = |E^{i_0}(A)| - 2|V^{i_0}(A)|$, and $\vec{X}^l(A') = \vec{X}^l(A)$ follows from the definition.

□

Theorem 3.5.4. *Let G be a GBS group defined by labeled graph (A, λ) without a proper plateau such that for all $e \in E(A)$, $\lambda(e) = n^{i_e}$ for some $i_e \geq 0$ and $q(G) = \langle (1/n^{lL}) \rangle_{\mathbb{Q}^*}$. Then for every index d subgroup $H \leq G$, $\vec{X}^l(H)$ is a cyclic permutation of $d\vec{X}^l(G)$.*

Proof. Since A does not contain a proper p–plateau, the GBS group H is represented by a labeled graph $\tilde{A}$ for some d-sheeted topological covering $\pi : \tilde{A} \to A$, by Proposition 2.9.2. Fix a base vertex $v_0 \in A$ and $\tilde{v_0} \in \pi^{-1}(v_0) \subseteq V(\tilde{A})$.

For any $\tilde{v} \in V(\tilde{A})$ and a directed edge path $(\tilde{e}_1, \tilde{e}_2, \cdots, \tilde{e}_r)$ from $\tilde{v_0}$ to $\tilde{v}$, the path $(\pi(\tilde{e}_1), \pi(\tilde{e}_2), \cdots, \pi(\tilde{e}_r))$ is directed edge path in A from v_0 to $\pi(\tilde{v})$ with

$$q_n^{\tilde{A}}(\tilde{e}_1, \tilde{e}_2, \cdots, \tilde{e}_r) = q_n^A(\pi(\tilde{e}_1), \pi(\tilde{e}_2), \cdots, \pi(\tilde{e}_r)).$$

Therefore, for $\tilde{v} \in \pi^{-1}(v)$ and $v \in V^i(A)$, we have $\tilde{v} \in V^i(\tilde{A})$. Similarly, for $\tilde{e} \in \pi^{-1}(e)$ and $e \in E^i(A)$, we have $\tilde{e} \in E^i(\tilde{A})$. Thus, $|E^i(\tilde{A})| = d|E^i(A)|$ and $|V^i(\tilde{A})| = d|V^i(A)|$. Now $\vec{X}^l_{\tilde{v}_0}(\tilde{A}) = d\vec{X}^l_{v_0}(A)$ follows from the definition of $\vec{X}^l$. □

Corollary 3.5.4.1. *Suppose G_i are the GBS group represented by the directed labeled graphs (A_i, λ_i) without proper p–plateau, for $i = 1, 2$. If G_1 is commensurable to G_2, then $c_1\vec{X}^l(G_1)$ is a cyclic permutation of $c_2\vec{X}^l(G_2)$ for some $c_1, c_2 \in \mathbb{N}$.*

Proof. Suppose G_1 is commensurable to G_2, then for some finite index subgroup $H_i \leq G_i$, H_1 is isomorphic to H_2. Let H_i be represented by the labeled graph B_i

with the covering map $\pi_i : B_i \to A_i$. Then B_1 and B_2 are related via a sequence of expansion and collapse moves. Therefore, $\vec{X}^l(B_1)$ is a cyclic permutation of $\vec{X}^l(B_2)$. Also, $[G_i : H_i]\vec{X}^l(G_i) = \vec{X}^l(B_i) = \vec{X}^l(H_i)$ by Proposition 3.5.4.1. Thus, if G_2 is commensurable to G_2, then $c_1\vec{X}^l(G_1)$ is a cyclic permutation of $c_2\vec{X}^l(G_2)$ for $c_i = [G_i : H_i]$. □

3.5.1 Incommensurable lattices in Case (III)

Next, we prove that any two lattices in $\{\Gamma_1, \Delta_k : k \geq 2\}$ are incommensurable when $p = n$.

Recall the lattice Γ_1 defined by Figure 3.3 for $m = 1$. By collapsing the edge e_d, and performing $d-1$ slide moves, we find that

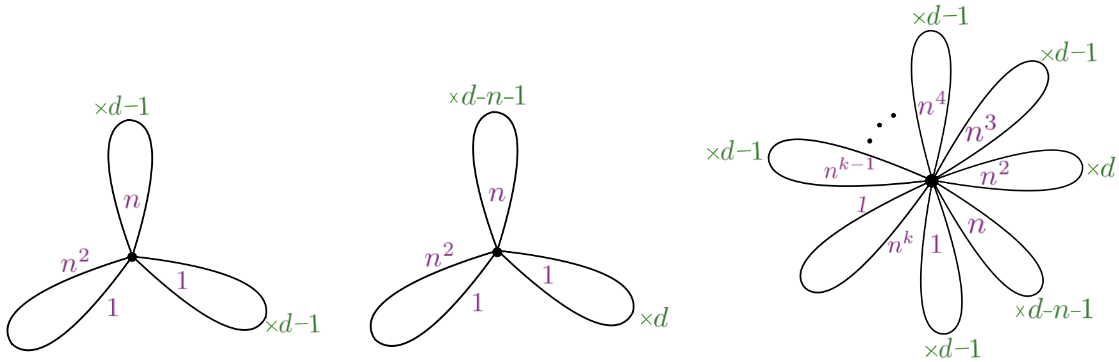
$$\begin{aligned}\Gamma_1 &\cong \bigvee_d BS(1, n^2) \vee \bigvee_{d-1} BS(n, n) \\ &\cong BS(1, n^2) \vee \bigvee_{d-1} BS(1, 1) \vee \bigvee_{d-1} BS(n, n)\end{aligned}$$

Also, the group Δ_k for $n = p$ from (3.9) is

$$\begin{aligned}\Delta_k \cong\ & BS(n^2, n^2) \vee BS(1, n^k) \vee \bigvee_{d-1} BS(1, 1) \vee \bigvee_{d-n-1} BS(n, n) \vee \bigvee_{d-1} BS(n^2, n^2) \vee \\ & \bigvee_{d-1} BS(n^3, n^3) \qquad \cdots \bigvee_{d-1} BS(n^{k-1}, n^{k-1}).\end{aligned}$$

Therefore, Γ_1 and Δ_k are the fundamental groups of the labeled graphs which are bouquets of circles. These bouquets consist of one loop labeled 1 and n^k, along with additional loops labeled n^i for $0 \leq i \leq k-1$, as shown in Figure 3.9.

Proposition 3.5.5. *The set of groups $\{\Gamma_1, \Delta_k : k \geq 2\}$ define pairwise incommensurable torsion-free uniform lattices in* $\mathrm{Aut}(X_{d,dn})$*, when $n < d$ and* $\gcd(n, d) = 1$.



(a) (B_1', μ_1') (b) (D_2', δ_2') (c) (D_k', δ_k'), for $k \geq 3$

Figure 3.9: Labeled graphs representing the groups Γ_1 and Δ_k for $k \geq 2$.

Proof. To show that any two groups in $\{\Gamma_1, \Delta_k : k \geq 2\}$ are incommensurable, we will demonstrate that they contain incommensurable finite index subgroups.

To obtain an index l subgroup $\Delta_{k,l}$ in Δ_k, unwind the loop in the graph (D_k', δ_k') labeled 1 and n^k into a circle of length l (see Figure 3.10 for an example). Next, collapse all but one of the edges labeled 1 and n^k to obtain a bouquet of circles with edges labeled n^i for $0 \leq i \leq kl$. The fundamental group of this labeled graph is $\Delta_{k,l}$. Similarly, unwinding the loop labeled 1 and n^2 in (B_1', μ_1') into the circle of length l gives index l subgroup in $\Gamma_{1,l}$ in Γ_1. It is straightforward to see that the modular homomorphism of $\Delta_{k,l}$ is generated by $\frac{1}{n^{kl}}$, and the vector $\vec{X}^{kl}(\Delta_{k,l})$ is as follows:

1. $\vec{X}^{2l}(\Delta_{2,l}) = 2(d, d-n-1, d, d-n-1, \cdots, d, d-n-1) \in \mathbb{N}^{2l}$

2. $\vec{X}^{kl}(\Delta_{k,l}) = 2(d-1, d-n-1, d, \underbrace{d-1, \cdots, d-1}_{k-3 \text{ elements}}, \cdots d-1, d-n-1, d,$ $\underbrace{d-1, \cdots, d-1}_{k-3 \text{ elements}})$ in $\mathbb{N}^{kl}$ for $k \geq 3$.

Meanwhile, the modular homomorphism of $\Gamma_{1,l}$ is generated by $\frac{1}{n^{2l}}$, and

$$\vec{X}^{2l}(\Delta_{1,l}) = 2(d-1, d-1, \cdots, d-1, d-1) \in \mathbb{N}^{2l}.$$

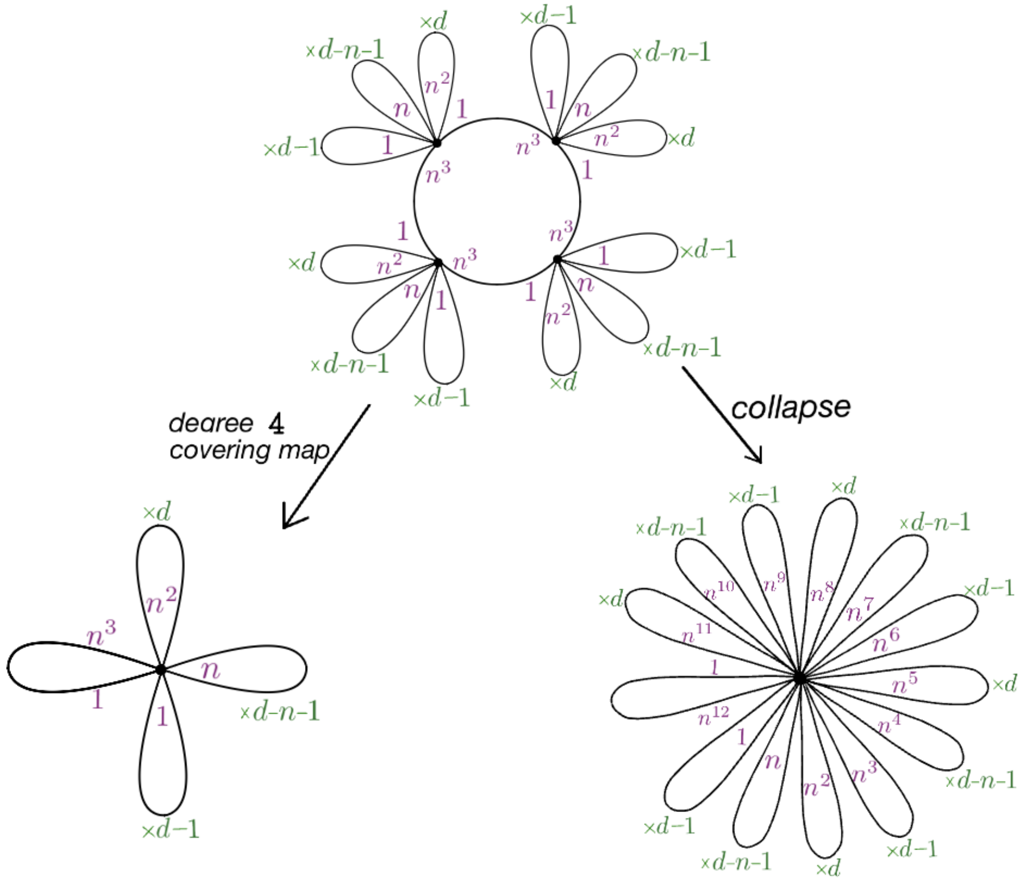


Figure 3.10: The labeled graph on the right represents an index 4 subgroup $\Delta_{3,4}$ in Δ_3.

Since all entries of $\vec{X}^{2l}(\Gamma_{1,l})$ are equal, $\vec{X}^{2l}(\Gamma_{1,l})$ and any cyclic permutation of $\vec{X}^{2l}(\Delta_{l,2})$ are linearly independent. Therefore, by Corollary 3.5.4.1, $\Gamma_{1,l}$ and $\Delta_{l,2}$ are not commensurable for all $l \geq 2$.

For $k, l \geq 3$, $\vec{X}^{kl}(\Delta_{k,l})$ has $k-2$ consecutive equal terms, whereas $\vec{X}^{lk}(\Delta_{l,k})$ has $l-2$ consecutive equal terms. Hence, $\vec{X}^{kl}(\Delta_{k,l})$ and any cyclic permutation of $\vec{X}^{kl}(\Delta_{l,k})$ are linearly independent, for $k, l \geq 3$. By the similar argument $\vec{X}^{2k}(\Delta_{2,k})$ and any cyclic permutation of $\vec{X}^{2k}(\Delta_{k,2})$ are linearly independent for $k \geq 3$. This proves that for $k, l \geq 2$, $\Delta_{k,l}$ and $\Delta_{l,k}$ are commensurable if and only if $k = l$. Hence the same is true for Δ_k and Δ_l. $\square$

3.5.2 Commensurability criterion for some GBS groups

We conclude this section by giving a solution for the commensurability problem for a subclass of GBS groups denoted by $\mathcal{C}_{n,l}$. This class of GBS group was introduced in [CRKZ21]. For every $l \geq 1$, $n \geq 2$, $k \geq 2$, $0 \leq a_1, a_2, \cdots, a_{k-1} \leq l-1$, denote by $A = A(n, l; a_1, a_2, \cdots, a_{k-1})$ the following labeled graph: it is bouquet of circles $e_1, \cdots, e_k$ with $\lambda(e_1) = 1$, $\lambda(\bar{e}_1) = n^l$, and $\lambda(e_i) = \lambda(\bar{e}_i) = n^{a_{i-1}}$ for $2 \leq i \leq k$. The GBS group for this labeled graph is given by

$$BS(1, n^l) \vee BS(n^{a_1}, n^{a_1}) \vee BS(n^{a_2}, n^{a_2}) \vee \cdots \vee BS(n^{a_{k-1}}, n^{a_{k-1}}).$$

The solution to the isomorphism problem for groups in $\mathcal{C}_{n,l}$ is given in of [CRKZ21, Theorem 5.1].

Remark 3.5.6. The vector associated in [CRKZ21] to a GBS group G in $\mathcal{C}_{n,l}$ is $\frac{1}{2}\vec{X}^l(G)$.

Theorem 3.5.7. *[CRKZ21, Theorem 5.1] Let G_1 and G_2 are two GBS groups in $\mathcal{C}_{n,l}$. Suppose G_1 is the GBS group defined by a labeled graph $A_1 = A(n, l; a_1, a_2, \cdots, a_{k_1-1})$, and G_2 be the GBS group represented by a labeled graph $A_2 = A(n, l; b_1, b_2, \cdots, b_{k_2-1})$. Then G_1 is isomorphic to G_2 if and only if*

1. *$k_1 = k_2$*
2. *$\vec{X}^l(A_1)$ is a cyclic permutation of $\vec{X}^l(A_2)$.*

The following result provides a necessary and sufficient condition for two GBS groups in $\mathcal{C}_{n,l}$ to be commensurable.

Theorem 3.5.8. *Let G_1 and G_2 are two GBS groups in $\mathcal{C}_{n,l}$. Suppose G_1 is the GBS group defined by a labeled graph $A_1 = A(n, l; a_1, a_2, \cdots, a_{k_1-1})$, and G_2 be the GBS group represented by a labeled graph $A_2 = A(n, l; b_1, b_2, \cdots, b_{k_2-1})$. Then G_1 is commensurable to G_2 if and only if $c_1\vec{X}^l(A_1)$ is a cyclic permutation of $c_2\vec{X}^l(A_2)$ for some $c_1, c_2 \in \mathbb{N}$.*

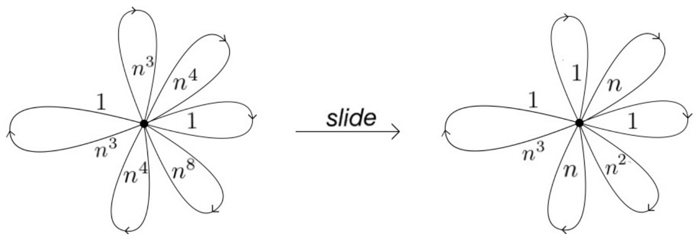


Figure 3.11: Slide moves on the bouquet of circles with $l = 3$. Here $x^0 = 2$, $x^1 = 2$, and $x^2 = 1$.

Proof. The "only if" direction follows from Corollary 3.5.4.1. For the converse, by applying an induction move, we can assume that $c_1\vec{X}^l(A_1) = c_2\vec{X}^l(A_2)$. Perform slide moves on labeled graphs A_i for $i = 1, 2$ (see Figure3.11 for an example) to obtain the bouquet of circles whose GBS group which is isomorphic to G_i is the following:

$$BS(1, n^l) \vee \bigvee_{x_i^0} BS(1, 1) \vee \bigvee_{x_i^1} BS(n, n) \vee \cdots \vee \bigvee_{x_i^{l-1}} BS(n^{l-1}, n^{l-1})$$

Therefore, the graph A_i is in the same deformation space as the graph A_i', which is defined by a bouquet of circles with x_i^j circles each with label n^j on both ends for each j, and one circle with label 1 on one end and n^l on the other end. One can see that $\vec{X}^l(A_i) = \vec{X}^l(G_i) = 2(x_i^0, x_i^1, \cdots, x_i^{l-1})$.

Suppose $j_0 \leq l - 1$ is the smallest number such that $x_i^{j_0} \neq 0$, and $x_i^j = 0$ for all $0 \leq j < j_0$. Let B_i' denote a c_i sheeted topological covering of A_i' that unwinds a loop in A_i' with labels n^{j_0} on both ends, into a cycle of length c_i (Figure 3.12 illustrate

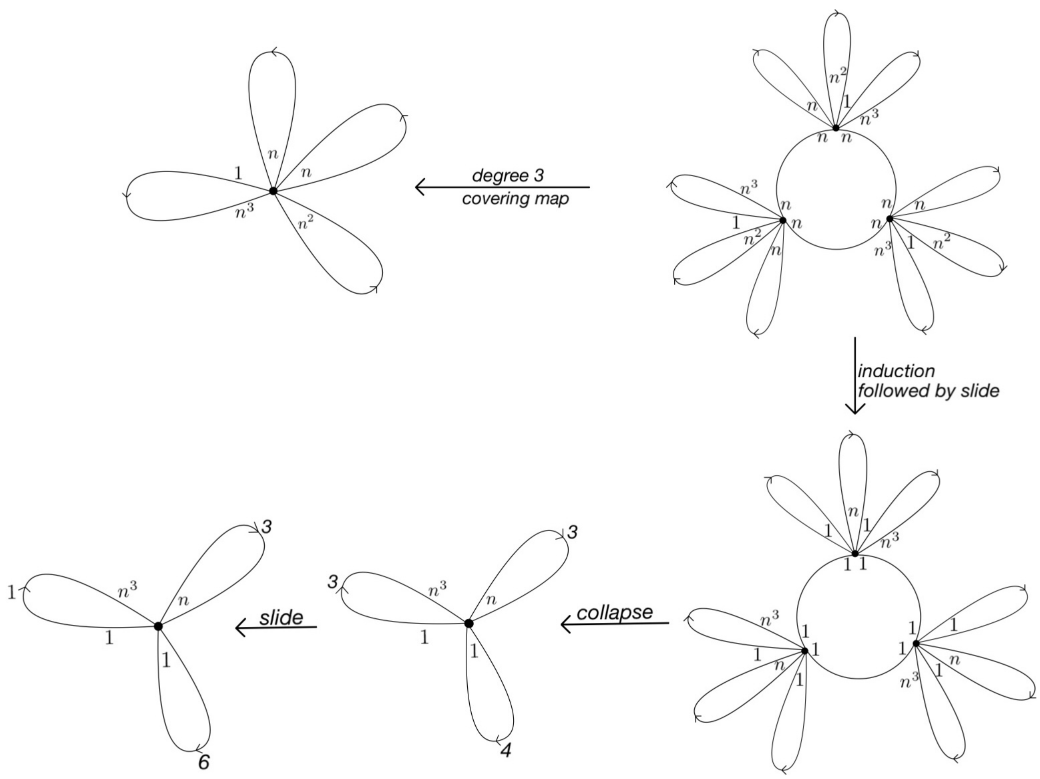


Figure 3.12: Example illustrating Theorem 3.5.8 for $l = 3$, $x^0 = 0$, $x^1 = 2$, $x^2 = 1$, $j_0 = 1$ and $c = 3$. Here a number z inside the petal means both ends of the petal have label z, and a number above a petal represents the multiplicity of that petal.

an example of this). We can apply an induction move to each vertex of B'_i to make the labels on the cycle n^l. By performing slide moves to each vertex of the resulting labeled graph, we can get all labels on the cycle to be 1. Now, by applying $c_i - 1$ collapse moves, we obtain a bouquet of circles. We can adjust the petal labels by applying slide moves, resulting in $c_1 x_1^j = c_2 x_2^j$ petals with labels n^{j-j_0+1} on both ends for $0 \leq j \leq l-1$, along with one circle having a label 1 on one end and n^l on the other end.

Thus, the labeled graphs B'_1 and B'_2 are related by slide moves to the same labeled graph. Consequently, they represent isomorphic finite index subgroups of G_1 and G_2, respectively. This completes the proof of the theorem.

□

3.6 Reduced graphs with no proper plateau

In the following section, we will give examples of incommensurable lattices in $\mathrm{Aut}(X_{dm,dn})$, when $\gcd(m,d) = \gcd(n,d) = 1$ for $m,n > 1$, and there is a prime $q < d$ which divides either m or n. Without loss of generality, we can assume q divides m (hence $\gcd(m,q) = q$ and $\gcd(d,q) = 1$).

Let D be the largest number dividing $d - q$ which is coprime to both m and n. Let $d - q = ID$.

The lattice $\mathbf{\Lambda_l}$ for $\mathbf{l \geq 2}$. Consider the lattice Λ_l defined by the directed labeled graph (L_l, λ_l) given in Figure (3.13). It is a graph with l vertices $v_1, v_2, \cdots, v_l$ and $d(l-1) + D + 1$ directed edges. There are d directed edges from v_i to v_{i+1} for $1 \leq i \leq k-1$ with initial label m and terminal label n. One directed edge from v_l to v_1 with initial label qm, and terminal label qn. Lastly, there are D edges from v_l to v_1 with initial label Im, and terminal label In.

Recall the graph (B_1, μ_1) defined by Figure 3.3 and its fundamental group Γ_1. Since $m,n > 1$, the labeled graphs (B_1, μ_1) and (L_l, λ_l) are reduced for all $l \geq 1$. The next two lemmas imply that these labeled graphs do not contain any proper p–plateau. Hence, the finite index subgroups of Γ_1 and Λ_l are given by topological coverings of B_1 and L_l for $l \geq 2$. By analyzing these covers, we will show that any two groups in $\{\Gamma_1,\, \Lambda_l : l \geq 2\}$ are abstractly incommensurable.

Remark 3.6.1. For the labeled graph (A, λ) the following holds which can also be found in [Lev15]:

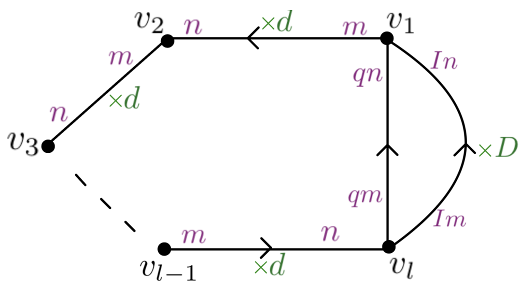


Figure 3.13: Graph B_l for $l \geq 2$, defining lattices in $\mathrm{Aut}(X_{dm,dn})$ when $gcd(m,d) = gcd(n,d) = 1$ and a prime $q < d$ divides m.

1. For a vertex $v \in V(A)$, $\{v\}$ is a p–plateau if and only if p divides every label at v.

2. Let $P \subseteq A$ be a p–plateau and $e \in E(A)$. Then the following holds:

   a) If $e \in E(P)$, then $p \nmid \lambda(e), \lambda(\overline{e})$.

   b) If $\partial_0(e), \partial_1(e) \in V(P)$ but $e \notin E(P)$, then $p \mid \lambda(e), \lambda(\overline{e})$.

   c) If $\partial_0(e) \in P$ and $\partial_1(e) \notin V(P)$, then $p \mid \lambda(e)$.

   d) If $\partial_0(e), \partial_1(e) \notin P$, then there is no restriction.

3. If $P \subseteq A$ is a p–plateau for prime p not dividing any label of A, then $P = A$.

Lemma 3.6.2. (B_1, μ_1) *does not contain a proper plateau.*

Proof. $\{u\}, \{v\} \subset B_1$ are not p–plateaus for any prime number p by Remark 3.6.1(1) and the fact that $\gcd(m,n) = 1$.

If $e \in E(P)$ for some p–plateau $P \subseteq B_1$, then by Remark 3.6.1(2a), $p \nmid \mu_1(e), \mu_1(\overline{e})$. Therefore p does not divide m or n and by Remark, 3.6.1(3) $P = B_1$. □

Lemma 3.6.3. (L_l, λ_l) *does not contain a proper plateau for* $l \geq 2$.

Proof. Let $P \subseteq L_l$ be a p–plateau for a prime number p. If p is coprime to all m, n, q, I, then by Remark 3.6.1(3), $P = L_l$, hence P is not a proper p–plateau. We will show that if p divides any of the numbers m, n, q, I, then P is a null graph.

If $p \mid q$, then $p = q$ and $q \mid m$. Since $\gcd(m, n) = 1$ and $q \mid m$, we have $q \nmid n$. We claim that $q \nmid I$. Assuming $q \mid I$, we will have the following sequence of implications, contradicting the fact that $\gcd(d, q) = 1$;

$$q \mid I \implies q \mid DI \implies q \mid d - q \implies q \mid d.$$

$P \neq \{v_i\}$ for $1 \leq i \leq l$ by Remark 3.6.1(1) and the fact that $q \nmid I, n$. By Remark 3.6.1 (2a), P doesn't contain any edge since $q \mid m$. Therefore P is a null graph.

If $p \mid m$, then $p \nmid n$ as $\gcd(m, n) = 1$. If $p = q$ then we have already shown that L_l does not contain proper q-plateau. If $p \neq q$, then $P \neq \{v_i\}$ by Remark 3.6.1(1) and the fact that $p \nmid n, q$. Moreover, the edges are not contained in P as $p \mid m$.

If $p \mid n$, then $p \nmid m$ and $p \nmid q$. $P \neq \{u\}, \{v\}$ due to Remark 3.6.1(1) and the fact that $p \nmid m, q$. The edges are not contained in P as $p \mid n$.

We have seen that L_l does not contain a proper p–plateau for p dividing m, n or q. Therefore, assume $p \nmid m, n, q$. Recall that D was chosen to be the largest number such that $D \mid (d - q)$ and $\gcd(D, n) = \gcd(D, m) = 1$. If $p \mid I$, then $d - q = DI$ implies $pD \mid (d - q)$. Also, $\gcd(D, n) = \gcd(D, m) = 1 = \gcd(p, m) = \gcd(n, p)$ implies $\gcd(pD, n) = \gcd(pD, m) = 1$, contradicting the choice of D. □

Proposition 3.6.4. *For $i \in \{1, 2\}$, let G_i be the GBS groups represented by a labeled graphs (A_i, λ_i) with no proper plateau and such that $\lambda(e_i) \neq \pm 1$ for any $e_i \in E(A_i)$. Furthermore, suppose $q(G_i) \cap \mathbb{Z} = \{1\}$. Then if G_1 and G_2 are commensurable, then $\frac{|V(A_1)|}{|V(A_2)|} = \frac{|E(A_1)|}{|E(A_2)|}$.*

Proof. Since G_1 and G_2 are commensurable, there exist admissible covers $(\tilde{A}_i, \tilde{\lambda}_i)$ of (A_i, λ_i) representing isomorphic GBS groups.

Since (A_i, λ_i) does not contain a proper plateau, by Proposition 2.9.2, $\tilde{A}_i$ is a topological cover of A_i of some degree d_i. Therefore,

$$|V(\tilde{A}_i)| = d_i|V(A_i)| \text{ and } |E(\tilde{A}_i)| = d_i|E(A_i)|. \tag{3.11}$$

Since $\lambda(e_i) \neq \pm 1$ for any $e_i \in E(A_i)$, and $\tilde{A}_i$ is a topological cover of A_i, we have $\lambda(\tilde{e}_i) \neq \pm 1$ for any $\tilde{e}_i \in E(\tilde{A}_i)$. Therefore, $\tilde{A}_i$ is a reduced graph. By Proposition 2.7.1, $(\tilde{A}_1, \tilde{\lambda}_1)$ and $(\tilde{A}_2, \tilde{\lambda}_2)$ represent isomorphic GBS groups if and only if they are related by slide moves. Since slide moves do not change the numbers of vertices and edges in a graph, we have

$$|V(\tilde{A}_1)| = |V(\tilde{A}_2)| \text{ and } |E(\tilde{A}_1)| = |E(\tilde{A}_2)|. \tag{3.12}$$

From equations (3.12) and (3.11), we get the following equality;

$$\frac{|V(A_1)|}{|V(A_2)|} = \frac{d_2}{d_1} = \frac{|E(A_1)|}{|E(A_2)|}.$$

□

Corollary 3.6.4.1. *The GBS groups Γ_1 and Λ_l are not abstractly commensurable for $l \geq 2$.*

Proof. Note that B_1 and L_l is reduced graph for all $l \geq 2$ as $m, n > 1$. Also, since $\gcd(m, n) = 1$, it follows that, $q(\Gamma_k) \cap \mathbb{Z} = \{1\}$.

Now, assume Γ_1 and Γ_l are commensurable groups for $l \geq 2$. By Proposition 3.6.4 we have

$$\frac{|V(B_1)|}{|V(L_l)|} = \frac{|E(B_1)|}{|E(L_l)|}$$

which is equivalent to

$$\frac{2}{l} = \frac{2d}{d(l-1) + D + 1}$$

Rearranging and simplifying this equation yields $d = D + 1$. Since (L_l, λ_l) is a uniform lattice in $\mathrm{Aut}(X_{dm,dn})$, by Proposition 2.5.2 and the fact that $I \geq 1$, we get the contradiction

$$\begin{aligned} dm &= \sum_{e \in E_0^+(v_l)} \mu_l(e) \\ &= qm + mDI \\ &\geq qm + mD \\ &= qm + (d-1)m \\ &\geq 2m + (d-1)m \\ &= (d+1)m \end{aligned}$$

where the last inequality follows from the fact that q is a prime number, and hence $q \geq 2$. ☐

Corollary 3.6.4.2. *The GBS groups Λ_k and Λ_l are not abstractly commensurable for $k, l \geq 2$ and $k \neq l$.*

Proof. Assume Λ_k and Λ_l are commensurable groups. Then, by Proposition 3.6.4, we have the following statements:

$$\frac{|V(L_l)|}{|V(L_k)|} = \frac{|E(L_l)|}{|E(L_k)|}$$

$$\begin{aligned} \frac{l}{k} &= \frac{d(l-1) + D + 1}{d(k-1) + D + 1} \\ l(k-1)d + l(D+1)) &= k(l-1)d + k(D+1)) \\ l(D - d + 1) &= k(D - d + 1). \end{aligned}$$

Since $d \neq D + 1$ (by the same argument as in the proof of 3.6.4.1), it follows that $k = l$. This completes the proof of the corollary.

□

Finally, we conclude this chapter by giving necessary and sufficient conditions for the cell complex $X_{dm,dn}$ to satisfy Leighton's Property.

Theorem 3.6.5. *The Baumslag-Solitar complex $X_{dm,dn}$ has the Leighton property if and only if m and n have no divisor less than or equal to d.*

Proof. When m or n has a divisor less than or equal to d, sections 3.4, 3.5, and 3.6 provide examples of incommensurable lattices in $\mathrm{Aut}(X_{dm,dn})$. This proves the forward direction of the Theorem. The converse direction is proved in Theorem 3.3.8 □

Chapter 4

Lattices in Generalized Baumslag-Solitar Complexes

In [For24], the author shows that every torsion-free uniform lattice in $\mathrm{Aut}(X_{m,n})$ is a generalized Baumslag-Solitar group, that is the fundamental group of a graph of infinite cyclic groups. A natural next question is: which groups (possibly with torsion) can occur as uniform lattices in $\mathrm{Aut}(X_{m,n})$? Theorem 4.3.1 answers this by proving that any uniform lattice in $\mathrm{Aut}(X_{(m,n)})$ must be the fundamental group of a graph of 2-ended groups. More generally, if one considers the automorphism group of an arbitrary GBS complex, $\mathrm{Aut}(X_{(A,\lambda)})$, then every uniform lattice in it is likewise the fundamental group of a graph of 2-ended groups.

Conversely, given any graph of 2–ended group G, one can produce (possibly after passing to a finite quotient) a uniform lattice in the automorphism group of a suitable "almost GBS" complex. In some cases, this almost GBS complex is in fact an honest GBS complex.

In the next section, we introduce the notion of an almost GBS complex and give a couple of conditions under which it is actually a GBS complex.

4.1 Double-labeled graphs and almost GBS complexes

Definition 4.1.1. A *double-labeled graph* (A, l, ρ) is a finite graph A together with two label functions $l, \rho : E(A) \longrightarrow \mathbb{N}$.

For a double-labeled graph (A, l, ρ), we construct a 2-dimensional cell complex $X_{(A,l,\rho)}$, referred to as an *almost GBS complex*. This complex is homeomorphic to the standard GBS complex associated with the labeled graph $(A, l\rho)$; that is,

$$X_{(A,l,\rho)} \cong X_{(A,l\rho)} \cong T_{(A,l\rho)} \times \mathbb{R},$$

where $(l\rho)(e) = l(e)\rho(e)$. Let $\pi : X_{(A,l,\rho)} \longrightarrow T_{(A,l\rho)}$ denote the natural projection map. As in the standard case, the complex $X_{(A,l,\rho)}$ is constructed from two types of subspaces: branching lines $\pi^{-1}(v)$ for each $v \in V(T_{(A,l\rho)})$, and strips $\pi^{-1}(e)$ for each $e \in E(T_{(A,l\rho)})$. Each strip $\pi^{-1}(e)$ is tiled as a

$$\Big(\frac{l(f)}{\gcd(l(f), l(\overline{f}))}, \frac{l(\overline{f}))}{\gcd(l(f), l(\overline{f}))}\Big)\text{-}\textit{strip,}$$

where $f = q(e) \in E(A)$ is the image of e under the projection of $T_{(A,l\rho)}$ onto A. Let $v \in V(T_{(A,l\rho)})$ and suppose $q(v) \in V(A)$ is its image. For every edge $f \in E_0(q(v))$, there are $l(f)\rho(f)$ edges $f_1, f_2, \cdots f_{l(f)\rho(f)} \in E(T_{(A,l\rho)})$ with initial vertex v that map to f under q. These correspond to the $l(f)\rho(f)$ strips $\pi^{-1}(f_1), \pi^{-1}(f_2), \cdots \pi^{-1}(f_{l(f)\rho(f)})$ attached to $\pi^{-1}(v)$ in $k = \frac{l(f)}{\gcd(l(f), l(\overline{f}))}$ distinct ways. In the complex $X_{(A,l,\rho)}$, these strips are joined to $\pi^{-1}(v)$ along the coset $i + k\mathbb{Z}$ for all $i = 1, 2, \cdots l(f)\rho(f)$. Once again, the neighborhood of each branching line, together with the Bass-Serre tree $T_{(A,l\rho)}$, completely determines the complex $X_{(A,l,\rho)}$.

In general, $X_{(A,l,\rho)}$ need not arise from the labeled graph $(A, l\rho)$. The lemma below provides sufficient conditions under which the complex $X_{(A,l,\rho)}$ is indeed the GBS complex of $(A, l\rho)$.

Lemma 4.1.1. *Let (A, l, ρ) be a double-labeled graph such that* $\gcd(l(f), l(\overline{f})) = 1$ *and $\rho(f) = \rho(\overline{f})$ for every $f \in E(A)$. Then the almost GBS complex associated with (A, l, ρ) coincides with the standard GBS complex $X_{(A,l\rho)}$ associated with the labeled graph $(A, l\rho)$.*

Proof. Consider a strip $\pi^{-1}(e)$ in $(A, l\rho)$ with $q(e) = f \in E(A)$. This strip is tiled as a (m, n)-strip, where

$$(m, n) = \Big(\frac{\rho(f)l(f)}{\gcd(\rho(f)l(f), \rho(\overline{f})l(\overline{f}))}, \frac{\rho(\overline{f})l(\overline{f})}{\gcd(\rho(f)l(f), \rho(\overline{f})l(\overline{f}))}\Big).$$

Since $\rho(f) = \rho(\overline{f})$, we factor it out to get

$$(m, n) = \Big(\frac{l(f)}{\gcd(l(f), l(\overline{f}))}, \frac{l(\overline{f})}{\gcd(l(f), l(\overline{f}))}\Big)$$

This shows that the tiling in the strip of $X_{(A,l\rho)}$ exactly matches the one in $X_{(A,l,\rho)}$. Therefore, the two complexes are identical in their cell structure. □

More generally, an almost GBS complex $X_{(A,l,\rho)}$ may coincide with the GBS complex of a different labeled graph than $(A, l\rho)$. The lemma below gives sufficient conditions for exactly this situation.

Lemma 4.1.2. *Let (A, l, ρ) be a double-labeled graph such that $\rho(e) = \rho(f)$ whenever $\partial_0(e) = \partial_0(f)$. Then the almost GBS complex $X_{(A,l,\rho)}$ is actually a GBS complex.*

Proof. For each $v \in V(A)$, define $\rho(v) = \rho(e)$ for any $e \in E_0(v)$. Let N be the least common multiple of all the values $\rho(v)$ as v ranges over $V(A)$. We construct a labeled graph (B, λ) along with a projection map $\pi : B \longrightarrow A$ as follows. Begin with $A \times \{1, 2, \cdots N\}$. For each vertex $v \in V(A)$, partition the set of vertices (v, i) into blocks of size $\rho(v)$, and identify each block to a single vertex in B. For instance, group and identify $(v, 1), (v, 2), \cdots (v, \rho(n))$ into one vertex, then $(v, \rho(n) + 1), (v, \rho(n) + 2), \cdots (v, 2\rho(n))$ into another, and so on. Repeat this process for each vertex $v \in V(A)$. The resulting graph B comes with a natural projection π onto A. Next, define a label function $\lambda(e, i) = l(\pi(e, i))$ for each edge $(e, i) \in E(B)$. This makes (B, λ) a labeled graph. Moreover, this defines an admissible branched cover of the labeled graph $(A, l\rho)$, with the degree function $d : V(B) \sqcup E(B) \longrightarrow \mathbb{N}$ given by

$$d(v, i) = \rho(v), \text{ and } d(e, i) = 1.$$

Consequently, the Bass-Serre tree of (B, λ) coincides with that of $(A, l\rho)$, and the GBS complex associated with (B, λ) is identical to the almost GBS complex $X_{(A,l,\rho)}$. $\square$

4.2 Two-ended groups

Let G be a finitely generated group. Consider the Cayley graph $\Delta(G, S)$ of G with respect to a generating set S. The ends of G are defined as the number of ends of $\Delta(G, S)$, and this is independent of the choice of generating set S. By a theorem of Hopf, a finitely generated group has 0, 1, 2, or ∞ many ends.

The following theorem from [Wal79] classifies the set of 2-ended groups.

Theorem 4.2.1. ([Wal79], Theorem 5.12) *The following conditions on a finitely generated group G are equivalent:*

1. *G is 2-ended group,*

2. *G is a either a finite by cyclic group $G = K*_K \cong K \rtimes \mathbb{Z}$ or $G = A *_K B$ for a finite group K with $[A : K] = [B : K] = 2$,*

3. G has a finite normal subgroup with quotient $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$.

Remark 4.2.2. For a 2-ended group G, the finite normal subgroup appearing in statement (3) in Theorem 4.2.1 is the same subgroup K in statement (2). We refer to this subgroup as the "canonical subgroup" of G. This subgroup consists of all the torsion elements of G that act on the Cayley graph $\Delta(G, S)$ by preserving each end. H is also the kernal of the action of G on its Bass-Serre tree.

Lemma 4.2.3. *Let $G = K \rtimes \langle t \rangle$ be a finite-by-cyclic group for a finite group K and an infinite cyclic group generated by t. Then, for any other decomposition $G \cong F \rtimes \langle s \rangle$, with F finite and s infinite order, we have*

$$K = F \text{ and } s = kt^{\pm 1} \text{ for some } k \in K$$

Proof. Suppose $\phi : F \rtimes \langle s \rangle \longrightarrow\ K \rtimes \langle t \rangle$ is an isomorphism of finite-by-cyclic groups. Then it is trivial that ϕ restricts to an isomorphism $\phi|_F : F\ F \longrightarrow K$. This induces an isomorphism between the quotient groups:

$$\overline{\phi} : (F \rtimes \langle s \rangle)/F \longrightarrow (K \rtimes \langle t \rangle)/K.$$

Since both quotients are infinite cyclic groups, we must have $\overline{\phi}(sF) = \phi(s)K = t^{\pm 1}K$. Thus, $\phi(s) = t^{\pm 1}k'$ for some $k' \in K$. Alternatively, $\phi(s) = kt^{\pm 1}$ for some $k \in K$, since $\langle t \rangle$ acts on K by conjugation. $\square$

Remark 4.2.4. Let $G = K \rtimes \langle t \rangle$ be a finite-by-cyclic group, and let $H = F \rtimes \langle kt^l \rangle$ be a finite-by-cyclic subgroup of G, where K and F are finite groups, and $k \in K$, and $l \in \mathbb{Z}$. Then $[G : H] = |l|[K : F]$. This follows from the observation that if $\{x_1, x_2, \cdots, x_n\}$ is a complete set of coset representatives of F in K, then a complete set of coset representative of H in G is given by

$$\{t^i x_j : 0 \leq i < |l|, 1 \leq j \leq n\}.$$

Let G be a graph of orientable 2-ended groups. Then we can assign two label functions $l, \rho : E(A) \longrightarrow \mathbb{N}$ such that if $e \in E_0(v)$ and $G_v = K_v \rtimes \mathbb{Z}$ and $G_e = K_e \rtimes \mathbb{Z}$ then, $\rho(e) = [K_v : K_e]$ and $l = \frac{[G_v:G_e]}{\rho(e)}$.

We conclude this section by showing that the fundamental group of a non-orientable 2-ended group contains, as a finite-index subgroup, a fundamental group of an orientable 2-ended group. To prove this, we use the following theorem from [For02].

Theorem 4.2.5. *[For02] Let G be a group, and let X and Y be two cocompact G-trees. Then the following conditions are equivalent;*

1. *X and Y have the same elliptic subgroups.*
2. *There is a coarsely equivariant quasi-isometry $\psi : X \longrightarrow Y$.*

Lemma 4.2.6. *Suppose G is the fundamental group of a graph of 2-ended groups. Then there exists a subgroup $G_0 \leq G$ of index 1 or 2 which admits a structure of a graph of orientable 2-ended groups.*

Proof. Let B be the underlying graph for the graph of groups structure on G. For each vertex group G_v, let $\mathbb{R}_v$ denote the Bass-Serre tree associated to G_v, which is homeomorphic to $\mathbb{R}$. For $g \in G_v$, let $\overline{g} : \mathbb{R}_v \longrightarrow \mathbb{R}_v$ denote the action of g on $\mathbb{R}_v$. This action induces a bijection on the space of ends, $\text{Ends}(\mathbb{R}_v)$, i.e. $\overline{g} : \text{Ends}(\mathbb{R}_v) \longrightarrow \text{Ends}(\mathbb{R}_v)$. We define a group homomorphism $\phi_v : G_v \longrightarrow \mathbb{Z}_2$ by setting $\phi_v(g) = 0$ if $\overline{g}$ acts trivially on $\text{Ends}(\mathbb{R}_v)$, and $\phi_v(g) = 1$ otherwise. Similarly, for each edge group G_e, define a homomorphism $\phi_e : G_e \longrightarrow \mathbb{Z}_2$ in the same manner. Now consider an edge $e \in E_0(v)$ and the injective map $f_e : G_e \longrightarrow G_v$. We claim that the following diagram 4.1 commutes, that is, $\phi_v \circ f_e = \phi_e$. In other words, whether an element of G_e preserves or reverses ends agrees with its image in G_v. This compatibility ensures that we can glue the homomorphism ϕ_i together to obtain a global homomorphisms $\phi : G \longrightarrow \mathbb{Z}_2$. The kernel of this map is then a well-defined subgroup of index 1 or 2 in G.

$$\begin{tikzcd} G_e \arrow[rr, "f_e"] \arrow[dr, "\phi_e"'] & & G_v \arrow[dl, "\phi_v"] \\ & \mathbb{Z}_2 & \end{tikzcd} \tag{4.1}$$

Since $f_e(G_e)$, which is isomorphic to G_e, is a subgroup of G_v, it follows that G_e also acts on $\mathbb{R}_v$. Therefore, $\mathbb{R}_e$ and $\mathbb{R}_v$ are two cocompact G_e-trees. We claim that $\mathbb{R}_v$ and $\mathbb{R}_e$ have the same elliptic subgroups.

Suppose $H \leq G_e$ is an elliptic subgroup for the action of G_e on $\mathbb{R}_e$. Then there exists a vertex $v_e \in V(\mathbb{R}_e)$ such that $hv_e = v_e$ for all $h \in H$, i..e. $H \subset \text{stab}_{G_e}(v_e)$. Since G_e is a 2-ended group, all elliptic elements have finite order. Therefore, each $h \in H$ acts on $\mathbb{R}_v$ as an elliptic element as well (i.e., it can not act by a nontrivial translation). Thus, each element of H fixes some vertex of $\mathbb{R}_v$. Suppose for $h_1, h_2 \in H$ we have $h_i \in \text{stab}_{G_e}(v_i)$ for some vertices $v_1, v_2 \in V(\mathbb{R}_v)$. If $v_1 \neq v_2$, then the product $h_1 h_2$ would act via nontrivial translation of $\mathbb{R}_v$, which contradicts the fact that each element of H fixes some vertex of $\mathbb{R}_v$. Therefore, $v_1 = v_2$, and $H \leq \text{stab}_{G_e}(v_1)$; hence we conclude that H is an elliptic subgroup for the action on $\mathbb{R}_v$ as well. A similar argument shows that elliptic subgroups for the action on G_e on $\mathbb{R}_v$ are also elliptic subgroups for the action on G_e on $\mathbb{R}_e$.

By Theorem 4.2.5, there exists a coarsely G_e-equivariant quasi isometry $\psi : \mathbb{R}_e \longrightarrow \mathbb{R}_v$ which makes the following diagram commute:

$$\begin{tikzcd} \mathbb{R}_e \arrow[r, "\bar{g}"] \arrow[d, "\psi"'] & \mathbb{R}_e \arrow[d, "\psi"] \\ \mathbb{R}_v \arrow[r, "\overline{f_e(g)}"] & \mathbb{R}_v \end{tikzcd}$$

which in turn includes the following commutative diagram on the ends:

ψ induces homeomorphism on Ends($\mathbb{R}_e$) on the ends, making the following diagram commute,

$$\begin{tikzcd} \mathrm{Ends}(\mathbb{R}_e) \arrow[r, "\bar{g}"] \arrow[d, "\psi"'] & \mathrm{Ends}(\mathbb{R}_e) \arrow[d, "\psi"] \\ \mathrm{Ends}(\mathbb{R}_v) \arrow[r, "\overline{f_e(g)}"] & \mathrm{Ends}(\mathbb{R}_v) \end{tikzcd}$$

This ensures that the diagram in 4.1 commutes, so we get a well-defined map $\phi : G \longrightarrow \mathbb{Z}_2$, whose kernel, denoted G_0, is a subgroup of G. By the subgroup theorem, G_0 admits a graph of groups decomposition with a finite underlying graph. The vertex groups are isomorphic to $G_0 \cap gG_vg^{-1}$, and edge groups are isomorphic to $G_0 \cap gG_eg^{-1}$ for $g \in G$. Let $(B, \mathcal{Z})$ be the graph of space associated to the given graph of groups decomposition of G, with total space Z_B, and let $\tilde{Z}_B$ denote its universal cover. Each conjugate gG_vg^{-1} acts on a vertex space $\tilde{Z}_v$ in $\tilde{Z}_B$, which is 2-ended space corresponding to the universal cover of Z_v. Therefore, the subgroup $G_0 \cap gG_vg^{-1}$ also acts on $\tilde{Z}_v$ and is therefore a 2-ended group. Moreover, $G_0 \cap gG_vg^{-1}$ consists precisely of those elements in gG_vg^{-1} that preserve the two ends of $\tilde{Z}_v$. Similarly each $G_0 \cap gG_eg^{-1}$ is also orientable 2-ended group. Thus, G_0 is the fundamental group of a graph of an orientable 2-ended groups. □

4.2.1 Bass-Serre tree for 2-ended groups

The Bass-Serre tree for a 2-ended group is homeomorphic to a line. Here, we describe in detail how to construct the tree.

Orientable 2-ended groups- Consider an orientable 2-ended group given by an HNN extension of the form $G = K*_K$. To distinguish between vertex and edge groups, we denote the vertex group as $K_v = K$ and the edge group as $K_e = K$. According to Theorem 3.14 in [Wal79], the Bass-Serre tree of G has:

1. Vertices corresponding to cosets of the vertex group gK_v. The stabilizer of such a vertex under the action of G is gK_vg^{-1}.

2. Edges corresponding to cosets of the edge group gK_e. The stabilizer of such an edge under the action of G is gK_eg^{-1}. Each edge gK_e connects the vertices gK_v and gtK_v, where t is the stable letter in the group G.

Let $\mathcal{Z}$ be the graph of space structure for G, with edge space Z_e, and vertex space Z_v, total space Z, and universal cover $\tilde{Z}$ with covering map $p : \tilde{Z} \longrightarrow Z$. Then $\tilde{Z}$ consists of universal covers of vertex space Z_v and edge space Z_e. By Lemma 3.12 in [Wal79], there is a natural correspondence between the cosets gK_v and the connected components of $p^{-1}(Z_v)$, which consists of copies of universal cover of Z_v. Similarly, there is a natural correspondence between the cosets gK_e and the connected components of $p^{-1}(Z_e)$, which consists of universal cover of Z_e. Denote the component of $p^{-1}(Z_e)$ corresponding to coset gK_e by $\tilde{Z}_{gK_e}$ and the component of $p^{-1}(Z_v)$ corresponding to coset gK_v by $\tilde{Z}_{gK_v}$.

Now, fix a basepoint $z_v \in Z_v$, and a preimage $\tilde{z}_v \in \tilde{Z}$. Consider the orbit $G\tilde{z}_v$. Two points $g_1\tilde{z}_v$ and $g_2\tilde{z}_v$ lie in the same component $\tilde{Z}_{gK_v}$ if and only if there exists some element $k \in \mathrm{stab}(gK_v) = gK_vg^{-1} = K_v = K$ such that $g_2\tilde{z}_v = kg_1\tilde{z}_v$. Thus, the vertices in the Bass-Serre tree can also be described as equivalence classes of the relation on $G\tilde{z}_v$, where the equivalence relation is defined by:

$$g_1\tilde{z}_v \sim g_2\tilde{z}_v \text{ if and only if } g_1\tilde{z}_v = kg_2\tilde{z}_v \text{ for some } k \in K.$$

In this way, vertices in the Bass-Serre tree are the equivalence classes $[g_1\tilde{z}_v]$, and the edges connect $[g_1\tilde{z}_v]$ to $[g_1t\tilde{z}_v]$.

Non-orientable 2-ended groups- Consider a non-orientable 2-ended group $G = A *_K B$, expressed as an amalgamated free product. According to Theorem 3.14 in [Wal79], the Bass-Serre tree of G has:

1. Vertices corresponding to cosets gA and gB, where $g \in G$. The stabilizers of these vertices under the action of G are gAg^{-1} and gBg^{-1}, respectively.

2. Edges corresponding to cosets gK of the edge group. The stabilizer of such an edge under the action of G is gKg^{-1}. Each edge gK connects the vertices gA and gB.

We provide another way to construct the Bass-Serre tree. Let $\mathcal{Z}$ be the graph of spaces structure for G, consisting of an edge space Z_K, vertex spaces Z_A and Z_B, and total space Z. Let $\tilde{Z}$ denote the universal cover of Z, with covering map $p : \tilde{Z} \longrightarrow Z$. Then $\tilde{Z}$ contains of universal covers of the vertex spaces Z_A, Z_B, and the edge space Z_K. By Lemma 3.12 in [**?**], there is a natural correspondence between the cosets gK and the connected components of $p^{-1}(Z_K)$, which consists of copies of the universal cover of Z_K. Similarly, there is a natural correspondence between the cosets gA and gB and the connected components of $p^{-1}(Z_A)$ and $p^{-1}(Z_B)$, respectively. Denote the component of $p^{-1}(Z_K)$ corresponding to coset gK by $\tilde{Z}_{gK}$ and the components of $p^{-1}(Z_A)$ and $p^{-1}(Z_B)$ corresponding to cosets gA and gB by $\tilde{Z}_{gA}$ and $\tilde{Z}_{gB}$, respectively.

Now, fix a path $\gamma : [0,1] \longrightarrow Z$ with $\gamma(0) = z_A \in Z_A$, $\gamma(1) = z_B \in Z_B$, and $\gamma(t) \notin Z_A \cup Z_B$ for all $t \in (0,1)$. Choose a lift $\tilde{z}_A \in \tilde{Z}_A \subset \tilde{Z}$ of z_A, and let $\tilde{\gamma}_{\tilde{z}_A}$ be the lift of γ based at $\tilde{z}_A$. Let $\tilde{z}_B = \tilde{\gamma}_{\tilde{z}_A}(1)$ which is a lift of z_B. Note that $g\tilde{z}_A \in \tilde{Z}_{gA}$ and $g\tilde{z}_B \in \tilde{Z}_{gB}$.

Define an equivalence relation in $G\tilde{z}_A$, declaring $\tilde{z}_1 \sim \tilde{z}_2$ if and only if $\tilde{z}_1$ and $\tilde{z}_2$ are in the same componant of $p^{-1}(Z_A)$, that is, $\tilde{z}_1, \tilde{z}_2 \in \tilde{Z}_{gA}$ for some $g \in G$. This means that $\tilde{z}_2 = gag^{-1}\tilde{z}_1$ for some $a \in A$. Then the equivalence classes correspond bijectively to the connected components of $p^{-1}(Z_A)$, hence to the cosets of A in G. Similarly, define an equivalence relation on $G\tilde{z}_B$ by declaring any two elements $\tilde{z}_1, \tilde{z}_2 \in G\tilde{z}_B$ are equivalent if they are in the same component of $p^{-1}(Z_B)$. These equivalence classes correspond to the connected components of $p^{-1}(Z_B)$. Finally, for $\tilde{z}_1 \in G\tilde{z}_A$ and a lift $\tilde{\gamma}_{\tilde{z}_1}$ of γ based at $\tilde{z}_1$, the equivalence class $[\tilde{z}_1]$ is connected to $[\tilde{\gamma}_{\tilde{z}_1}(1)]$ via an edge.

4.3 Structure of lattices in a generalized GBS complex

Now, we are ready to prove the main result of this chapter.

Theorem 4.3.1. *If G is the fundamental group of a graph of 2-ended groups, then a quotient of G by a finite normal subgroup is a uniform lattice in the automorphism group of an almost GBS complex. Conversely, every uniform lattice in the automorphism group of a GBS complex is the fundamental group of a graph of 2-ended groups. Furthermore, every uniform lattice in the orientation-preserving automorphisms of a GBS complex is the fundamental group of a graph of orientable 2-ended groups.*

Proof. Suppose G is a uniform lattice in the automorphism group of a GBS complex $X_{(A,\lambda)}$. Then G acts on $X_{(A,\lambda)}$ cocompactly and discretely. This action induces an action on the associated Bass-Serre tree $T_{(A,\lambda)}$. To eliminate edge inversion in the action of G on $T_{(A,\lambda)}$, we subdivide each edge of the tree once, resulting in a new tree. The quotient of this action is a finite graph B, where the vertex and edge groups correspond to the stabilizers of horizontal lines and strips in $X_{(A,\lambda)}$. Since G acts on $X_{(A,\lambda)}$ discretely, these stabilizers are 2-ended groups. Therefore, G is the fundamental group of a graph of 2-ended groups. Furthermore, if the action of G on $X_{(A,\lambda)}$ is orientation-preserving, then the stabilizers of the vertices and edges are orientable two-ended groups. Hence, in this case, G is the fundamental group of a graph of orientable 2-ended groups.

Case (I)- We first show that if G is the fundamental group of a graph B of orientable 2-ended groups, then G is a lattice in the automorphism group of a GBS complex. Let $(B, \mathcal{Z})$ be a graph of spaces for this graph of group decomposition, and Z_B be the corresponding total space. Let T denote the Bass-Serre tree for G, and let $\tilde{Z}_B$ be the universal cover of Z_B. For each $g \in G$, we denote the deck transformation induced by g by $\overline{g} : \tilde{Z}_B \longrightarrow \tilde{Z}_B$.

For each vertex $v \in V(B)$, let Z_v be the vertex space in $(B, \mathcal{Z})$, and choose a basepoint $z_v \in Z_v$, with a lift $\tilde{z}_v \in \tilde{Z}_B$. Similarly, for each edge $e \in E(B)$, let Z_e be the corresponding edge space, with basepoint $z_e \in Z_e$, and a lift $\tilde{z}_e \in \tilde{Z}_B$. For each $e \in E(B)$ and its initial vertex $\partial_0(e) = v$ in B, fix a path $\gamma : [0, 1] \longrightarrow Z_B$ whose image is contained inside the mapping cylinder $(Z_v \sqcup Z_e \times [0, 1])/\phi_e(z) \sim (z, 0)$ such that $\gamma(0) = z_e$ and $\gamma(1) = z_v$ (recall Z_e is identified with $Z_e \times \{1\}$ inside Z_B). We denote all such paths by γ, suppressing their dependence on e and v for simplicity.

Define an equivalence relation on the orbit space $G\tilde{z}_v$ by declaring $g_1\tilde{z}_v \sim g_2\tilde{z}_v$ if:

- $g_1\tilde{z}_v$ and $g_2\tilde{z}_v$ are in the same vertex space $\tilde{Z}_{\tilde{v}}$ in the tree of space whose total space is $\tilde{Z}_B$. Note that $\tilde{v}$ is a lift of $v \in V(B)$ in the Bass-Serre tree T, and
- $g_1\tilde{z}_v = k_{\tilde{v}} g_2 \tilde{z}_v$ for some element $k_{\tilde{v}}$ in the canonical subgroup $K_{\tilde{v}}$ of the stabilizer of $\tilde{Z}_{\tilde{v}}$ (which coincides with the stabilizer of $\tilde{v}$).

It is straightforward to verify that this defines an equivalence relation on $G\tilde{z}_v$.

Construct a cell complex X where the vertex set $V(X)$ contains the equivalence classes $G\tilde{z}_v/\sim$, taken over all vertices $v \in V(B)$. These vertices are referred to as *black vertices.*

For a fixed vertex $v \in V(B)$, connect two vertices $[g_1\tilde{z}_v] \neq [g_2\tilde{z}_v]$ by an edge in X if:

- $g_1\tilde{z}_v$ and $g_2\tilde{z}_v$ lie in the same vertex space $\tilde{Z}_{\tilde{v}}$, and
- There exists a stable letter $t_{\tilde{v}}$ in the infinte cyclic group in the stabilizer of $\tilde{Z}_{\tilde{v}}$ such that $g_2\tilde{z}_v = t_{\tilde{v}}g_1\tilde{z}_v$. (By Lemma 4.2.3, and the fact that the canonical subgroup is normal, any other choice of stable letter yields an element equivalent to $t_{\tilde{v}}g_1\tilde{z}_v$ under the defined equivalence relation.) These will be called *horizontal edges.*

To ensure that the horizontal edges are well-defined under the equivalence relation, suppose $[g_1\tilde{z}_v] = [g_1'\tilde{z}_v]$, implying $g_1\tilde{z}_v = k_{\tilde{v}}g_1'\tilde{z}_v$ for some $k_{\tilde{v}} \in K_{\tilde{v}}$. Then,

$$g_2\tilde{z}_v = t_{\tilde{v}}g_1\tilde{z}_v = t_{\tilde{v}}k_{\tilde{v}}g_1'\tilde{z}_v = k_{\tilde{v}}'t_{\tilde{v}}g_1'\tilde{z}_v$$

where the last equality follows from the fact that the canonical subgroup $K_{\tilde{v}}$ is normal in the stabilizer of $\tilde{Z}_{\tilde{v}}$. Consequently, $[g_1'\tilde{z}_v]$ is also connected to $[t_{\tilde{v}}g_1'\tilde{z}_v] = [g_2\tilde{z}_v]$ by an horizontal edge. Similarly, we define equivalence relation on the orbit $G\tilde{z}_e$ for $e \in E(B)$ by declaring two elements $g_1\tilde{z}_e$ and $g_2\tilde{z}_e$, lying in an edge space $\tilde{Z}_{\tilde{e}} \subset \tilde{Z}_B$ (where $\tilde{e}$ is a lift of e in the Bass-Serre tree T), to be equivalent if

$$g_2\tilde{z}_e = k_{\tilde{e}}g_1\tilde{z}_e$$

for some element $k_{\tilde{e}}$ in the canonical subgroup $K_{\tilde{e}}$ of the stabilizer of $\tilde{Z}_{\tilde{e}}$ (i.e. the stabilizer of $\tilde{e} \in E(T)$). The equivalence classes under this relation form additional vertices in X, for each $e \in E(B)$. These vertices are referred to as *white vertices.* Edges between these equivalence classes are defined similarly as before: two distinct vertices $[g_1\tilde{z}_e] \neq [g_1\tilde{z}_e]$ are connected by an edge if there exists a stable letter $t_{\tilde{e}}$ (i.e. a generator of the infinite cyclic subgroup in the stabilizer of $\tilde{Z}_{\tilde{e}}$) such that

$$g\tilde{z}_e = t_{\tilde{e}}g_1\tilde{z}_e.$$

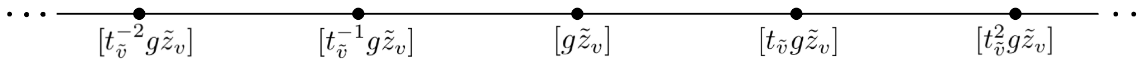


Figure 4.1: $L_{\tilde{v}}$

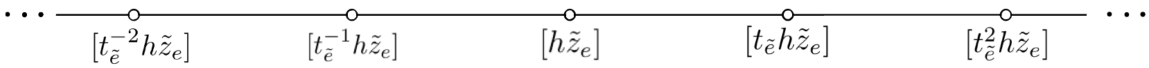


Figure 4.2: $L_{\tilde{e}}$

Up to this point, we have constructed a 1-dimensional cell complex isomorphic to $\mathbb{R}$ for each vertex and edge space in $\tilde{Z}_B$ (equivalently, for each vertex and edge in the Bass-Serre tree T). We denote the lines formed for each vertex $\tilde{v} \in V(T)$ by $L_{\tilde{v}}$, and those formed for each edge $\tilde{e} \in E(T)$ by $L_{\tilde{e}}$. See Figures 4.1 and 4.2.

Next, for each edge $\tilde{e} \in E(T)$ with initial vertex $\partial_0(\tilde{e}) = \tilde{v}$ in T, we define edges connecting the vertices in $L_{\tilde{e}}$ to vertices in $L_{\tilde{v}}$, which we refer to as vertical half-edges. For the vertex $[g_1\tilde{z}_e]$ in $L_{\tilde{e}}$ and the vertex $[g_2\tilde{z}_v]$ in $L_{\tilde{v}}$, add an edge in X if:

1. $g_1\tilde{z}_e$ lies in the edge space $\tilde{X}_{\tilde{e}}$ and $g_2\tilde{z}_v$ lies in the vertex space $\tilde{X}_{\tilde{v}}$, with $\tilde{e}$ adjacent to $\tilde{v}$ in T, and

2. For the path $\gamma : [0,1] \longrightarrow Z_B$ from z_e to z_v, the lift $\tilde{\gamma}_{g_1\tilde{z}_e}$ starting at $g\tilde{z}_e$ satisfies $[\tilde{\gamma}_{g_1\tilde{z}_e}(1)] = [g_2\tilde{z}_v]$.

To verify that the vertical half-edges are well-defined, suppose $[g_1\tilde{z}_e] = [g_1'\tilde{z}_e]$, implying $g_1'\tilde{z}_e = k_{\tilde{e}} g_1\tilde{z}_e$ for some $k_{\tilde{e}} \in K_{\tilde{e}}$. Let $\tilde{\gamma}_{k_{\tilde{e}} g_1\tilde{z}_e}$ be a lift of γ starting at $k_{\tilde{e}} g_1\tilde{z}_e$. By the property of covering maps, we have $\tilde{\gamma}_{k_{\tilde{e}} g_2\tilde{z}_v} = \overline{k}_{\tilde{e}} \circ \tilde{\gamma}_{g_1\tilde{z}_e}$, where $\overline{k}_{\tilde{e}}$ denotes the deck transformation corresponding to $k_{\tilde{e}}$. Since $\tilde{v}$ is adjacent to $\tilde{e}$, the stabilizer of $\tilde{Z}_{\tilde{e}}$ is contained in the stabilizer of $\tilde{Z}_{\tilde{v}}$, so $k_{\tilde{e}}$ is also an element in $K_{\tilde{v}}$. Therefore,

$$[\tilde{\gamma}_{k_{\tilde{e}} g_1\tilde{z}_e}(1)] = [\overline{k}_{\tilde{e}} \circ \tilde{\gamma}_{g_1\tilde{z}_e}(1)] = [\overline{k}_{\tilde{e}}(g_2\tilde{z}_v)] = [k_{\tilde{e}} g_2\tilde{z}_v].$$

Hence, $[g_1'\tilde{z}_e] = [k_{\tilde{e}}g_1\tilde{z}_e]$ is also connected to $[k_{\tilde{e}}g_2\tilde{z}_v] = [g_2\tilde{z}_v]$ by a vertical half-edge, showing that vertical half-edges are well defined.

We have constructed a connected cell complex X, and now we prove that the group G acts on X simultaneously. For a vertex $[g_1\tilde{z}_v] \in V(X)$, define $g[g_1\tilde{z}_v] = [gg_1\tilde{z}_v]$, similarly for vertices $[g_1\tilde{z}_e] \in V(X)$.

1. G preserves the vertices of X: Suppose $g \in G$, and let $[g_1\tilde{z}_v] = [g_2\tilde{z}_v]$, where $g_1\tilde{z}_v$, and $g_2\tilde{z}_v$ lie in the vertex space $\tilde{Z}_{\tilde{v}}$, and $g_2\tilde{z}_v = k_{\tilde{v}}g_1\tilde{z}_v$ for $k_{\tilde{v}} \in K_{\tilde{v}}$. Then $gg_1\tilde{z}_v$ and $gg_2\tilde{z}_v$ lie in the same vertex space $\tilde{Z}_{\tilde{w}}$, where $\tilde{w} = g\tilde{v}$ under the action of G on the Bass-Serre tree. Since the finite-order element $k_{\tilde{v}}$ stabilizes $\tilde{v}$, its conjugate $gk_{\tilde{v}}g^{-1}$ is a finite-order element in the stabilizer of $\tilde{w}$, and thus in the finite subgroup $K_{\tilde{w}} \subset Stab(\tilde{w})$. We compute:

$$gg_2\tilde{z}_v = gk_{\tilde{v}}g_1\tilde{z}_0 = gk_{\tilde{v}}g^{-1}gg_1\tilde{z}_v = (gk_{\tilde{v}}g^{-1})gg_1\tilde{z}_v$$

which shows that $[gg_2\tilde{z}_v] = [gg_1\tilde{z}_v]$ in $L_{\tilde{w}}$. Therefore, the action of G on black vertices is well-defined. An identical argument shows that G also acts on the white vertices of X.

2. G preserves horizontal edges: Suppose $[g_1\tilde{z}_v]$ and $[g_2\tilde{z}_v]$ are connected by a horizontal edge in $L_{\tilde{v}}$, which means that $[g_1\tilde{z}_v] = [t_{\tilde{v}}g_2\tilde{z}_v]$ for a stable letter $t_{\tilde{v}}$ in the infinite cyclic group in $Stab(\tilde{v})$. Let $g \in G$, and suppose $g\tilde{v} = \tilde{w}$ in the Bass-Serre tree T. Then $gt_{\tilde{v}}g^{-1}$ is a stable letter in the infinite cyclic subgroup $Stab(\tilde{w})$, and both $gg_1\tilde{z}_v$ and $gg_2\tilde{z}_v$ lie in the vertex space $\tilde{Z}_{\tilde{w}}$. Since the action of G on $G\tilde{z}_v/\sim$ is well defined, we compute:

$$[gg_1\tilde{z}_v] = [gt_vg_2\tilde{z}_v] = [(gt_vg^{-1})gg_2\tilde{z}_v],$$

showing that $[gg_1\tilde{z}_v]$ and $[gg_2\tilde{z}_v]$ are connected by an horizontal edge in $L_{\tilde{w}}$, as required. A similar argument shows that horizontal edges in $L_{\tilde{e}}$ are mapped to horizontal edges in $L_{g\tilde{e}}$ under the action of $g \in G$.

3. G preserves vertical half-edges: For $g_1\tilde{z}_e$ in $\tilde{Z}_{\tilde{e}}$, let $\tilde{\gamma}_{g_1\tilde{z}_e}$ be the lift of the path γ starting from $g_1\tilde{z}_e$ then we know there is a vertical half-edge between $[g_1\tilde{z}_e]$ and $[\gamma_{g_1\tilde{z}_e}(1)]$. Note that $[\gamma_{g_1\tilde{z}_e}(1)]$ is a vertex in $L_{\partial_0(\tilde{e})}$. Suppose for $g \in G$, and $g\tilde{e} = \tilde{f}$ then $g\partial_0(\tilde{e}) = \partial_0(\tilde{f})$ in the Bass-Serre tree T. Also, the lift of γ starting from $gg_1\tilde{z}_e$ is $\overline{g} \circ \gamma_{g_1\tilde{z}_e}$ and $\overline{g} \circ \gamma_{g_1\tilde{z}_e}(1) = g\gamma_{g_1\tilde{z}_e}(1)$. Therefore $[gg_1\tilde{z}_e]$, which is a vertex in $L_{\partial_0(\tilde{e})}$, is connected by an vertical half-edge to $[g\gamma_{g_1\tilde{z}_e}(1)]$, which is a vertex in $L_{\partial_0(\tilde{f})}$.

Suppose $\tilde{e}$ is an edge in T, and $g\tilde{z}_e$ is an element in the edge space $\tilde{Z}_{\tilde{e}}$ and $\tilde{\gamma}_{g\tilde{x}_e}(1)$ is an element in the vertex space $\tilde{Z}_{\tilde{v}}$ for the initial vertex $\partial_0(\tilde{e}) = \tilde{v}$. We demonstrate that the combinatorial distance between $[\tilde{\gamma}_{g\tilde{x}_e}(1)]$ and $[\tilde{\gamma}_{t_{\tilde{e}}g\tilde{x}_e}(1)]$ in $L_{\tilde{v}}$ is equal to the combinatorial distance between $[\tilde{\gamma}_{t_{\tilde{e}}g\tilde{x}_e}(1)]$ and $[\tilde{\gamma}_{t_{\tilde{e}}^2g\tilde{x}_e}(1)]$ in $L_{\tilde{v}}$. This proves that for every $\tilde{e} \in E(T)$ we can assign a positive integer $l(\tilde{e})$ to it. Suppose the distance between $[\tilde{\gamma}_{g\tilde{x}_e}(1)]$ and $[\tilde{\gamma}_{t_{\tilde{e}}g\tilde{x}_e}(1)]$ is l, so that $[\tilde{\gamma}_{t_{\tilde{e}}g\tilde{x}_e}(1)] = [t_{\tilde{v}}^l\tilde{\gamma}_{g\tilde{x}_e}(1)]$. Since $\tilde{\gamma}_{t_{\tilde{e}}g\tilde{x}_e} = \overline{t}_{\tilde{e}} \circ \tilde{\gamma}_{g\tilde{x}_e}$, we have $\tilde{\gamma}_{t_{\tilde{e}}g\tilde{x}_e}(1) = t_{\tilde{e}}\tilde{\gamma}_{g\tilde{x}_e}(1)$, and thus $[t_{\tilde{e}}\tilde{\gamma}_{g\tilde{x}_e}(1)] = [\tilde{\gamma}_{t_{\tilde{e}}g\tilde{x}_e}(1)] = [t_{\tilde{v}}^l\tilde{\gamma}_{g\tilde{x}_e}(1)]$. This implies that $t_{\tilde{e}} = k_{\tilde{v}}t_{\tilde{v}}^l$ for some finite order element $k_{\tilde{v}}$ in the stabilizer of $\tilde{v}$. Consequently, $\tilde{\gamma}_{t_{\tilde{e}}^2g\tilde{x}_e}(1) = t_{\tilde{e}}\tilde{\gamma}_{t_{\tilde{e}}g\tilde{x}_e}(1) = t_{\tilde{v}}^l\tilde{\gamma}_{t_{\tilde{e}}g\tilde{x}_e}(1)$, showing that the combinatorial distance between $\tilde{\gamma}_{t_{\tilde{e}}^2g\tilde{x}_e}(1)$ and $\tilde{\gamma}_{t_{\tilde{e}}g\tilde{x}_e}(1)$ is also l, as claimed.

Suppose $g\tilde{e} = \tilde{f}$ in T. We claim that $l(\tilde{e}) = l(\tilde{f})$. This descends to a well-defined label function on the edges of the quotient graph B. Let $g_1\tilde{z}_e$ lie in $\tilde{Z}_{\tilde{e}}$; then $gg_1\tilde{z}_e$ lies in $\tilde{Z}_{\tilde{f}}$. Also, if $t_{\tilde{e}}$ is a stable letter in the stabilizer of $\tilde{e}$ then $gt_{\tilde{e}}g^{-1}$ is a stable letter in the stabilizer of $\tilde{f}$. Let $\tilde{\gamma}_{g_1\tilde{z}_e}$ be the lift of γ starting from $g_1\tilde{z}_e$, then $\overline{g} \circ \tilde{\gamma}_{g_1\tilde{z}_e}$ is a lift of γ starting from $gg_1\tilde{z}_e$. Suppose $l(\tilde{e}) = l$; then

$$[\tilde{\gamma}_{g_1\tilde{z}_e}(1)] = [t_{\tilde{v}}^l \tilde{\gamma}_{t_{\tilde{e}}g_1\tilde{z}_e}(1)],$$

where $\tilde{v} = \partial_0(\tilde{e})$. Then

$$[\tilde{\gamma}_{gg_1\tilde{z}_e}(1)] = [g\tilde{\gamma}_{g_1\tilde{z}_e}(1)] = [gt_{\tilde{v}}^l \tilde{\gamma}_{t_{\tilde{e}}g_1\tilde{z}_e}(1)] = [gt_{\tilde{v}}^l g^{-1} g\tilde{\gamma}_{t_{\tilde{e}}g_1\tilde{z}_e}(1)] = [(gt_{\tilde{v}}g^{-1})^l \tilde{\gamma}_{gt_{\tilde{e}}g_1\tilde{z}_e}(1)].$$

This shows that the combinatorial distance between $[\tilde{\gamma}_{gg_1\tilde{z}_e}(1)]$ and $[\tilde{\gamma}_{gt_{\tilde{e}}g_1\tilde{z}_e}(1)]$ in $L_{g\tilde{v}}$ is also l in $L_{g\tilde{v}}$. Hence, l defines a label for each edge in the graph B. Define $\rho : E(B) \longrightarrow \mathbb{N}$ as $\rho(e) = [K_e : K_{\partial_0(v)}]$, where K_e is the canonical subgroup in the edge group G_e and $K_{\partial_0(v)}$ is the canonical subgroup in the edge group $G_{\partial_0(v)}$. Then (B, l, ρ) is a double-labeled graph.

Claim 1- For any $v \in V(B)$ and $e \in E_0(v)$, a lift $\tilde{v} \in V(T)$ of v, and a vertex $[g\tilde{z}_v] \in L_{\tilde{v}}$, there are exactly $\rho(e)$ vertical half-edges incident to $[g\tilde{z}_v]$, each connecting to a vertex in some $L_{\tilde{e}_i}$, where $\tilde{e}_1, \tilde{e}_2, \cdots \tilde{e}_{\rho(e)} \in E_0(\tilde{v})$ are distinct lifts of e.

Consider a lift $\tilde{\overline{\gamma}}_{g\tilde{z}_v}$ of the inverse path $\overline{\gamma}$, starting from $g\tilde{z}_v$. Suppose that $\tilde{\overline{\gamma}}_{g\tilde{z}_v}(1) \in \tilde{Z}_{\tilde{e}_1}$, for some lift $\tilde{e}_1 \in E_0(\tilde{v})$ of e. Since $[K_{\tilde{v}} : K_{\tilde{e}_1}] = [K_v : K_e] = \rho(e)$, we can find distinct coset representatives $k_1, k_2, \cdots, k_{\rho(e)} \in K_{\tilde{v}}$ such that $k_1 \in K_{\tilde{e}_1}$ and $k_2, \cdots k_{\rho(e)} \in K_{\tilde{v}} - K_{\tilde{e}_1}$ for $i > 1$, and the cosets $k_i K_{\tilde{e}_1}$ are all distinct. By Bass-Serre theory, the edges $\tilde{e}_1 = k_1\tilde{e}_1, k_2\tilde{e}_1, \cdots, k_{\rho(e)}\tilde{e}_1$ are distinct lifts of in e in $E_0(\tilde{v})$. Moreover, any $k \in K_{\tilde{v}}$ can be written as $k = k_i k_{\tilde{e}_1}$ for some $i \in \{1, \cdots \rho(e)\}$ and $k_{\tilde{e}_1} \in K_{\tilde{e}_1}$. Consider the lift $\tilde{\overline{\gamma}}_{kg\tilde{z}_v}$ of $\overline{\gamma}$ starting at $kg\tilde{z}_v$. Then

$$\tilde{\overline{\gamma}}_{kg\tilde{z}_v}(1) = \tilde{\overline{\gamma}}_{k_i k_{\tilde{e}_1} g\tilde{z}_v}(1) = k_i k_{\tilde{e}_1} \tilde{\overline{\gamma}}_{g\tilde{z}_v}(1),$$

which implies

$$[\tilde{\overline{\gamma}}_{kg\tilde{z}_v}(1)] = [k_i k_{\tilde{e}_1} \tilde{\overline{\gamma}}_{g\tilde{z}_v}(1)] = k_i[k_{\tilde{e}_1} \tilde{\overline{\gamma}}_{g\tilde{z}_v}(1)] = k_i[\tilde{\overline{\gamma}}_{g\tilde{z}_v}(1)],$$

since $k_{\tilde{e}_1} \in K_{\tilde{e}_1}$ and $\tilde{\tilde{\gamma}}_{g\tilde{z}_v}(1) \in \tilde{Z}_{\tilde{e}_1}$. Therefore, the vertex $[g\tilde{z}_v] = [kg\tilde{z}_v]$ is connected by vertical half-edge to the vertex $k_i[\tilde{\tilde{\gamma}}_{g\tilde{z}_v}(1)] \in V(L_{\tilde{e}_i})$. This shows that there are exactly $\rho(e)$ such vertical half-edges emanating from $[g\tilde{z}_v]$, one for each lift $k_i\tilde{e}_1$ of e, as required.

By Remark 4.2.4 $[G_e : G_{\partial_0(v)}] = l(e)[K_e : K_{\partial_0(v)}] = l(e)\rho(e)$. Therefore, T is the Bass-Serre tree associated with the labeled graph $(B, l\rho)$. Now, in the complex X, delete all edges between white vertices and ignore the white vertices entirely. The resulting complex is homeomorphic to $T \times \mathbb{R}$, where the branching lines are $L_{\tilde{v}}$ for all $\tilde{v} \in V(T)$. Each strip is tiled by an $(l(e), l(\overline{e}))$-strip. We further subdivide the cells so that each strip becomes a

$$\Big(\frac{l(e)}{\gcd(l(e), l(\overline{e}))}, \frac{l(\overline{e})}{\gcd(l(e), l(\overline{e}))}\Big)\text{-strip.}$$

By claim 1, it follows that the new complex which we denote by $\overline{X}$ is the almost GBS complex $X_{(B,l,\rho)}$. We have already shown that G acts on X. Since this action preserves $\sqcup_{\tilde{e}} L_{\tilde{e}}$, we also get a well defined action of G on $\overline{X}$.

If $g \in G$ fixes any vertex $[g_1\tilde{z}_v]$, that is, $[gg_1\tilde{z}_v] = [g_1\tilde{z}_v]$, then $g = k_{\tilde{v}}$ for some finite-order element $k_{\tilde{v}}$ in the stabilizer of $\tilde{v}$. Since the torsion subgroup of the stabilizer of $\tilde{v}$ is finite, it follows that the stabilizer of the vertex $[g_1\tilde{z}_v]$ is also finite. Therefore, the action of G on $\overline{X}$ is discrete. Moreover, because the action of G on $\tilde{Z}_B$ is cocompact, the action of G on $\overline{X}$ is also cocompact. Hence, the quotient of G by the kernel of the action on $\overline{X}$ (which is a finite normal subgroup) is a uniform lattice in $\mathrm{Aut}(\overline{X})$.

General case- Suppose G is the fundamental group of the graph of 2-ended groups. Let $(B, \mathcal{Z})$ be a graph of spaces structure for G, where the associated graph of groups decomposition satisfies the property that each vertex space Z_v and edge space Z_e itself carries a graph of space structure for the graph of group decomposition of 2-ended groups G_v and G_e, respectively. If $G_v = A_v *_{K_v} B_v$, then the vertex space Z_v is built of vertex spaces Z_{A_v}, Z_{B_v}, and an edge space Z_{K_v}. Similarly, if $G_e = A_e *_{K_e} B_e$, then the edge space Z_e is consists of Z_{A_e}, Z_{B_e} and Z_{K_e}. In the case where $G_v = K_v *_{K_v}$ (or $G_e = K_e *_{K_e}$), then the vertex and edge spaces within Z_v (or Z_e) are homeomorphic to Z_{K_v} (or Z_{K_e}), respectively. Denote the total spaces for the graph of spaces $(B, \mathcal{Z})$ by Z_B. Let $\tilde{Z}_B$ be the universal cover of Z_B, with covering map $p : \tilde{Z}_B \longrightarrow Z_B$. Let T be the Bass-Serre tree corresponding to the decomposition of G.

For each orientable 2-ended group G_v (or G_e), fix a basepoint $z_v \in Z_{K_v} \subset Z_v$ (or $z_e \in Z_{K_e} \subset Z_e$), and a lift $\tilde{z}_v \in \tilde{Z}_v$ of z_v (or a lift $\tilde{z}_e \in \tilde{Z}_e$ of z_e). For each non-orientable 2-ended group $G_v = A_v *_{K_v} B_v$, fix basepints z_{A_v}, and z_{B_v}, together with a path $\gamma_v : [0,1] \longrightarrow Z_v$ such that $\gamma_v(0) = z_{A_v}$, $\gamma_v(1) = z_{B_v}$ and $\gamma_v(t) \notin Z_{A_v} \cup Z_{B_v}$ for $t \in (0,1)$, as in Subsection 4.2.1. Also, fix lifts $\tilde{z}_{A_v}$, and $\tilde{z}_{B_v}$ such that the lift $\tilde{\gamma}_{\tilde{z}_{A_v}}$ of γ_v based at $\tilde{z}_{A_v}$, satisfies $\tilde{\gamma}_{\tilde{z}_{A_v}}(1) = \tilde{z}_{v_B}$. Carry out the analogous construction for each non-orientable edge group G_e. Now, for an edge $e \in E(B)$ such that G_e is an orientable 2-ended group and $\partial_0(e) = v$, fix a path $\gamma : [0,1] \longrightarrow Z_B$ whose image is contained inside the mapping cylinder $M_e = (Z_v \sqcup Z_e \times [0,1])/\phi_e(z) \sim (z,0)$ such that $\gamma(0) = z_e$, and $\gamma(1) = \tilde{z}_v$ if G_v is orientable, or $\gamma(1) = \tilde{z}_{A_v}$ if G_v is non-orientable.

Now we build a cell complex X with two types of vertices, black vertices, which correspond to the vertex groups G_v, and white vertices, which correspond to the edge groups G_e.

When G_v and G_e are orientable groups, the vertex corresponding to them are constructed exactly as in Case(I) where G is graph of orientable 2-ended groups. The same construction also applies to the edges between them, thereby producing horizontal lines $L_{\tilde{v}}$ and $L_{\tilde{e}}$, along with edges connecting them whenever $\mathrm{stab}(\tilde{v})$ and stab($\tilde{e}$) are orientable 2-ended groups. When G_e is orientable and G_v is non-orientable with $\partial_0(e) = v$, construct vertical-half edges from $L_{\tilde{e}}$ to $L_{\tilde{v}}$ by lifting the path γ, which starts at z_e, and ends at z_{A_v} and lies entirely inside the mapping cylinder M_e.

When G_v is a non-orientable 2-ended group, define an equivalence relation on the orbit set $G\tilde{z}_{A_v}$, by declaring $g_1\tilde{z}_{A_v} \sim g_2\tilde{z}_{A_v}$ if and only if they lie in the same component of $p^{-1}(Z_{A_v})$. Similarly, define an equivalence relation on $G\tilde{z}_{B_v}$ by declaring $g_1\tilde{z}_{B_v} \sim g_2\tilde{z}_{B_v}$ if and only if they lie in the same component of $p^{-1}(Z_{B_v})$. The equivalence classes under these relations will be black vertices of X. We connect two black vertices $[g_1\tilde{z}_{A_v}]$ and $[g_2\tilde{z}_{B_v}]$ by a horizontal edge if the lift $\tilde{\gamma}_{v(g_1\tilde{z}_{A_v})}$ of the path γ_v based at $g_1\tilde{z}_{A_v}$ satisfies $[\tilde{\gamma}_{v(g_1\tilde{z}_{A_v})}(1)] = [g_2\tilde{z}_{B_v}]$. This also implies that $g_1\tilde{z}_{A_v}$ and $g_2\tilde{z}_{B_v}$ lie within the same vertex space $\tilde{Z}_{\tilde{v}} \subset \tilde{Z}_B$, for some $\tilde{v} \in V(T)$ mapping to v. This construction defines a bipartite graph, whose components are naturally identified with the Bass-Serre trees of the 2-ended non-orientable groups stab($\tilde{v}$), where $\tilde{v} \in V(T)$ maps to v (see Section 4.2.1). Since the vertices of this graph correspond to cosets gA_v and gB_v, the group G acts naturally on these Bass–Serre tree components. An analogous construction applies to non-orientable edge group G_e to obtain additional white vertices in X and edges between them. Now we construct vertical half-edges connecting vertices in $L_{\tilde{e}}$ to vertices in $L_{\tilde{v}}$ whenever $\partial_0(\tilde{e}) = \tilde{v}$ in T. When stab($\tilde{e}$) is an orientable group, we connect a vertex $[g\tilde{z}_e]$ in $L_{\tilde{e}}$ to the vertex $[\tilde{\gamma}_{g\tilde{z}_e}(1)]$ in $L_{\tilde{v}}$, where γ is the fixed path inside the mapping cylinder M_e for the image e of $\tilde{e}$ in B. As in case (I), we can check that any vertical half-edge emanating from $[g\tilde{z}_e]$ and landing in a vertex of $L_{\tilde{v}}$ is unique.

Now assume that stab($\tilde{e}$) is a non-orientable 2-ended group. Then stab($\tilde{v}$) is also a non-orientable 2-ended group. A vertex $[g_1\tilde{z}_{A_e}]$ in $L_{\tilde{e}}$ corresponds to a component of $p^{-1}(Z_{A_e})$, and hence each element of stabilizer of this component acts on $L_{\tilde{e}}$ by reflection accros the vertex $[g_1\tilde{z}_{A_e}]$. Suppose $g \in$ stab($\tilde{e}$) acts on the Bass-Serre tree $L_{\tilde{e}}$

by reflection across a unique vertex. Then g is a torsion element and also acts on the Bass–Serre tree $L_{\tilde{v}}$ by reflection across a unique vertex. We connect the corresponding vertices by a vertical half-edge. If $g_1, g_2 \in \text{stab}(\tilde{e})$ both fix the same vertex $w \in L_{\tilde{e}}$, then $g_1 g_2$ also fixes w and hence has finite order. On the other hand, if g_1 fixes a vertex $w_1 \in V(L_{\tilde{v}})$, and g_2 fixes a different vertex $w_2 \in V(L_{\tilde{v}})$, with $w_1 \neq w_2$, then $g_1 g_2$ acts on $L_{\tilde{v}}$ as a nontrivial translation, implying that $g_1 g_2$ has infinite order, which is a contradiction. Therefore, each vertex in $L_{\tilde{e}}$ is connected to a unique vertex in $L_{\tilde{v}}$.

We claim that we can choose paths inside the mapping cylinder M_e to also define these vertical edges. For that, pick two vertical edges ϵ_A and ϵ_B from $L_{\tilde{e}}$ to $L_{\tilde{v}}$, starting from two vertices $[g_1 \tilde{z}_{A_e}]$ and $[g_1 \tilde{z}_{B_e}]$, respectively. Choose a path $\hat{\gamma}_{eA}$ from $g_1 \tilde{z}_{A_e}$ to a representative of the end vertex of ϵ_A and a path $\hat{\gamma}_{eB}$ from $g_2 \tilde{z}_{B_e}$ to a representative of the end vertex of ϵ_B. Let $p(\hat{\gamma}_{eA}) = \gamma_{eA}$ and $p(\hat{\gamma}_{eB}) = \gamma_{eB}$. Every translation of $\hat{\gamma}_{eA}$ is a lift of γ_{eA}. The equivalence class of endpoints $[\hat{\gamma}_{eA}(0)] = [g_1 \tilde{z}_{A_e}]$ and $[\hat{\gamma}_{eA}(1)]$ are connected by vertical half edges, as there exist an element of $h \in G$ which acts on $L_{\tilde{e}}$ and $L_{\tilde{v}}$by reflection across $[\hat{\gamma}_{eA}(0)]$ and $[\hat{\gamma}_{eA}(1)]$, respectively. Hence, for every lift $\tilde{\gamma}_{eA} = \overline{g} \circ \hat{\gamma}_{eA}$ for some deck transformation $\overline{g}$, the representative of endpoints $[\overline{g} \circ \hat{\gamma}_{eA}(0)]$ and $[\overline{g} \circ \hat{\gamma}_{eA}(1)]$ are also connected by a vertical half-edge, since they are fixed by the conjugate $ghg^{-1} \in G$. This shows that for every lift $\tilde{e}$ of e and $\tilde{v}$ of v, the lines $L_{\tilde{e}}$ and $L_{\tilde{v}}$, together with all vertical half-edges between them, form a $(1, l(e))$-strip for some positive integer $l(e)$.

Once again, in the resulting complex X, delete all white vertices and edges between them to obtain a cell complex $\overline{X}$. Subdivide each cell in $\overline{X}$, if necessary, to get an almost GBS complex structure on $\overline{X}$. By construction, there is a natural action of G on $\overline{X}$. Once again, by quotienting out by the kernel of this action, which is a finite subgroup, we conclude that the quotient of G by a finite normal subgroup is a uniform lattice in the automorphism group of $\overline{X}$.

□

Corollary 4.3.1.1. *Let G be the fundamental group of a graph of orientable 2-ended groups over a graph B, such that for every $v \in V(B)$, we have $[K_v : K_e] = [K_v : K_f]$ for all $e, f \in E_0(v)$. Then a quotient of G by a finite normal subgroup is a uniform lattice in the automorphism group of a GBS complex.*

Proof. Since $\rho(e) = [K_v : K_e] = [K_v : K_f] = \rho(f)$ for all $e, f \in E_0(v)$, the hypothesis of Lemma 4.1.2 is met. Therefore, the almost GBS complex $X_{(B,l,\rho)}$ (constructed in the proof of Theorem 4.3.1) is in fact a GBS complex. □

4.4 A Non–Virtually Torsion-Free Lattice in $\mathrm{Aut}(X_{8,16})$

In light of Theorem 4.3.1, we observe that many lattices in $\mathrm{Aut}(X_{m,n})$ contain torsion elements. We now construct a uniform lattice in $\mathrm{Aut}(X_{8,16})$ that is not virtually torsion-free, thereby answering the question 1.0.2 in the negative.

Example 4.4.1. Consider the fundamental group of a graph of groups whose underlying graph A consists of a single vertex v and a single edge loop. The edge group is $\mathbb{Z} = \langle \alpha \rangle$, and the vertex group is the finite-by-cyclic group $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}$, given by the presentation:

$$\langle a, b, s : a^2 = b^2 = [a, b] = 1, sas^{-1} = b, sbs^{-1} = a \rangle.$$

The injective homomorhism $f_e : \mathbb{Z} \longrightarrow (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}$ is defined by $f_e(\alpha) = as^2$, and the injective homomorhism $f_{\bar{e}} : \mathbb{Z} \longrightarrow (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}$ is defined by $f_{\bar{e}}(\alpha) = s^4$. Then, the fundamental group Γ of the corresponding graph of groups has the presentation:

$$\Gamma = \langle a, b, s, t : a^2 = b^2 = [a, b] = 1, sas^{-1} = b, sbs^{-1} = a, tas^2t^{-1} = s^4 \rangle.$$

We claim that this group defines a lattice in $\mathrm{Aut}(X_{8,16})$. Furthermore, we show that this lattice is not virtually torsion-free.

To prove that the group is not virtually torsion-free, suppose there is a finite group Q and a homomorphism $\phi : \Gamma \longrightarrow Q$ with torsion-free kernel. Consider the element $\phi(as^2) \in \Gamma$. Since Q is finite, the image $\phi(as^2)$ has finite order; let this order be $n = \mathrm{ord}(\phi(as^2))$. The relation $tas^2t^{-1} = s^4$ in Γ implies that $\phi(t)\phi(as^2)\phi(t)^{-1} = \phi(s^4)$, so $\phi(as^2)$ and $\phi(s^4)$ are conjugate in Q. Therefore, they have the same order, and hence $\mathrm{ord}(\phi(s^4)) = n$. Note that in Γ,

$$(at^2)^2 = at^2at^2 = atbt^3 = ata^2t^4 = t^4.$$

This implies:

$$\mathrm{ord}(t^4) = \mathrm{ord}((at^2)^2) = \frac{\mathrm{ord}(at^2)}{\gcd(\mathrm{ord}(at^2), 2)} = \frac{n}{\gcd(n, 2)}.$$

Since $\mathrm{ord}(t^4) = n$, we must have $\gcd(n, 2) = 1$, i.e., n is odd. This implies that in Γ we have:

$$(as^2)^n = as^{2n}$$

Applying ϕ, and using $\phi((as^2)^n) = 1$, it follows that:

$$\phi(a) = \phi(s^{-2n}).$$

Now, applying ϕ to the relation $sas^{-1} = b$ we get,

$$\phi(b) = \phi(s)\phi(a)\phi(s)^{-1} = \phi(s)\phi(s^{-2n})\phi(s)^{-1} = \phi(s^{-2n}).$$

Hence

$$\phi(ab) = \phi(s^{-4n}) = 1.$$

This implies that $ab \in \ker(\phi)$ is a nontrivial torsion element, contradicting the assumption that $\ker(\phi)$ is torsion-free. Therefore, such a homomorphism ϕ can not exist. Thus Γ is virtually torsion-free.

In the terminology of the proof of Theorem 4.3.1, we have $l(e) = 2$, $l(\overline{e}) = 4$, and $\rho(e) = \rho(\overline{e}) = 4$, and Γ acts on the almost GBS complex $X_{(A,l,\rho)}$. By Lemma 4.1.2, this cell complex is also the GBS complex associated with the labeled graph shown in Figure 4.3, and is therefore isomorphic to $X_{8,16}$. Since the action of Γ on the Bass-Serre tree $T_{(A,l\rho)}$ has a trivial kernel, the induced action of Γ on $X_{8,16}$ also has a trivial kernel. Thus, $\Gamma \leq \mathrm{Aut}(X_{8,16})$ defines a uniform lattice that is not virtually torsion free.

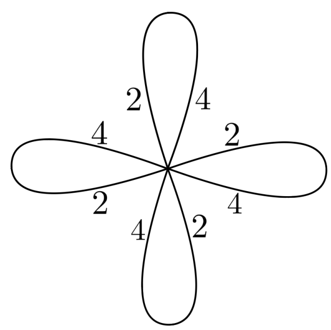


Figure 4.3: A labeled graph whose GBS complex is $X_{8,16}$

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