THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

SOME BASIC PROPERTIES OF LEFT ALGEBRAIC GROUPS BI-ALGEBRAIC GROUPS AND ALGEBRAIC GROUPS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

ZENSHO NAKAO

Norman, Oklahoma

1977

SOME BASIC PROPERTIES OF LEFT ALGEBRAIC GROUPS

BI-ALGEBRAIC GROUPS AND ALGEBRAIC GROUPS

APPROVED BY

Andy Magid

Robert G. Moore

Walter S. Kelley

Leonard R. Rubin

Harold Huneke

DISSERTATION COMMITTEE

ACKNOWLEDGMENTS

I first wish to express my gratitude to Professor Andy Magid for his continuous assistance during the preparation of this thesis.

I also like to tender my appreciation to present members of the dissertation committee; Professors Harold Huneke, Walter Kelley, Robert Morris and Leonard Rubin, and also to past members; Professors John Brixey, Peter Leonard, T. K. Pan, W. T. Reid and Kirby Smith. I am especially thankful to Professor Morris who provided me with his lecture notes and an article of Richard Palais (to appear in American Journal of Mathematics) which are the essential references in this research.

Finally I want to express my gratefulness to my mother, Misa, for her quiet sacrifice, to my wife, Michiko, for her long patience and assistance in the preparation of the final manuscript, and to my two sons, Shinichiro and Seijiro, for their excellent health.

TABLE OF CONTENTS

SOME BASIC PROPERTIES OF LEFT ALGEBRAIC GROUPS BI-ALGEBRAIC GROUPS AND ALGEBRAIC GROUPS

INTRODUCTION

Left algebraic groups were first introduced into algebraic group theory by Magid [M1] . The concepts arose from study of the problem of covering spaces of affine algebraic groups: Suppose that $E \to G$ is a connected étale covering space of a connected affine algebraic group G over an algebraically closed field k . When does E admit a structure of algebraic group so that the covering projection is a homomorphism of algebraic groups? Magid found that E is in general a pro-affine variety which admits an abstract group structure in which the left translations by elements of E are all morphisms of the variety structure, and he called such a structure a left algebraic group.

This thesis continues the works of Magid on the theory of left algebraic groups. Also it will introduce bi-algebraic group structures in which both the left and right translations by group elements are all morphisms, which form an intermediate category between the category of left algebraic groups and that of algebraic groups. In chapter I we will quote some of the definitions and results from [M1] and from commutative algebras which are the ground works for the present investigations. In chapter II we will study left algebraic groups and bi-algebraic groups. Finally in chapter III we will ask and answer when a bi-algebraic group is an algebraic group.

Throughout the discussion k will denote the fixed algebraically closed ground field for varieties and algebraic groups (although we often repeat a statement to that effect for emphasis). All the algebras are assumed to be commutative and to have an identity element. If B is a subalgebra of A then it is assumed that the identity element of B coincides with that of A. Every algebra homomorphism is required to send the identity element of one algebra to that of another. All unadorned tensor products are taken over the fixed ground field k.

SOME BASIC PROPERTIES OF LEFT ALGEBRAIC GROUPS

BI-ALGEBRAIC GROUPS AND ALGEBRAIC GROUPS

CHAPTER I

ALMOST AFFINE ALGEBRAS AND VARIETIES

We begin by assembling the definitions of almost affine algebras and almost affine varieties and their basic properties from [M1] and other sources. Many additional properties of almost affine algebras and varieties will also be obtained.

DEFINITION 1.1. An algebra over an algebraically closed field k is called an almost affine algebra if it is an integral extension of an affine subalgebra, i.e., every element of the algebra satisfies some monic polynomial equation with coefficients lying in the affine subalgebra.

Almost affine algebras share the following properties with affine algebras which are essential for translation of algebraic properties into geometric ones.

PROPOSITION 1.2. (a) An almost affine algebra is a

Jacobson (or Hilbert) ring, i.e., all prime ideals are intersections of maximal ideals.

(b) Every quotient of an almost affine algebra by a maximal ideal is (isomorphic to) the ground field k .

(c) An almost affine algebra has finite Krull dimension, i.e., the supremum of the length n of ascending chain of prime ideals $P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$ of the algebra is finite. If the algebra is an integral domain, this dimension is the same as the transcendence degree over k of its quotient field.

(d) The inverse image of a maximal ideal under an algebra homomorphism of almost affine algebras is again a maximal ideal.

The results and definition (1.1) are all stated in [M1, pp. 254-255, (1.1) and (1.2)] .

The next proposition, also due to Magid [M1, p. 255, (1.5)] , shows that almost affine algebra structures are preserved under quotient formation and localization with respect to the multiplicatively closed subset $S = \{ s^n : n \geq 0 \}$ where s is a non-nilpotent element.

<u>PROPOSITION 1.3</u>. Let A be an almost affine algebra, I an ideal of A . Then A/I and A_S are also almost affine algebras.

We will next define an almost affine variety over an algebraically closed field k . It will be endowed with a group variety structure in the next chapter.

DEFINITION 1.4. An almost affine variety over k is a pair $(V,A(V))$ where V is a point set and $A(V)$ an almost affine algebra of k-valued functions on V , called the coordinate ring of V (or the algebra of morphic functions on V), such that the following conditions are satisfied:

(i) $A(V)$ separates the points of V , i.e., if x and y are elements of V and $x \neq y$ then there is a morphic function f on V , i.e., an element of $A(V)$, such that $f(x) \neq f(y)$;

(ii) Every k-algebra homomorphism $A(V) \to k$ is the evaluation at a point of V , i.e., every element of the set $\mathrm{Alg}_k(A(V),k)$ of all k-algebra homomorphisms from $A(V)$ to k sends f to $f(x)$ for some element x in V for any morphic function f in $A(V)$. We will denote a homomorphism which is the evaluation at a point x of V by x^o .

An almost affine variety $(V,A(V))$ will usually be denoted by the first entry V alone, and $A(V)$ will of course denote its coordinate ring.

By conditions (i) and (ii) there exists a canonical bijection from V onto $\mathrm{Alg}_k(A(V),k)$ (which can be regarded also as the maximal ideal space $\mathrm{Max}(A(V))$ of $A(V)$, i.e., the subspace of $\mathrm{Spec}(A(V))$ consisting of all maximal ideals of $A(V)$ by proposition (1.2 (b)) since the ground field k is algebraically closed) given by $x \to x^{o}$ because (i) says the map is injective and (ii) says it is surjective, and accordingly we will identify V with the set $\mathrm{Alg}_k(A(V),k)$ ($= \mathrm{Max}(A(V))$). We could make the bijection even homeomorphism after introducing topologies into our spaces, which we will do shortly.

<u>DEFINITION 1.5</u>. Let V and W be almost affine varieties over k. A map $\mu : V \to W$ is called a <u>morphism</u> (or a <u>morphic map</u>) if $A(W)\circ\mu \subseteq A(V)$, i.e., $f\circ\mu$ belongs to $A(V)$ for every f in $A(W)$.

If $\mu : V \to W$ is a morphism of almost affine varieties, then it induces an algebra homomorphism μ^* from $A(W)$ back to $A(V)$, called a <u>comorphism</u> associated to μ, given by composition with μ, i.e., $\mu^* : A(W) \to A(V)$ by $f \to f\circ\mu$.

If V is an almost affine variety, then $A(V)$ is reduced, i.e., it has no nilpotent elements. Suppose, conversely, that A is any reduced almost affine algebra over k. Then the pair $(\mathrm{Max}(A),A)$ is an almost affine variety over k. To prove the assertion let f be in A, then f can be regarded as a function $f : \mathrm{Max}(A) \to k$ by $M \to f + M$, i.e., f modulo the maximal ideal M in A/M ($\simeq k$) for every maximal ideal M of A by proposition

(1.2 (b)). Suppose M and N are distinct maximal ideals of A. Then there exists an element f of A with f in M but not in N, so that $f(M) = 0$ and $f(N) \neq 0$, i.e., A separates the points of $\mathrm{Max}(A)$.

Next assume that $\eta : A \to k$ is a k-algebra homomorphism. Then η is surjective and so its kernel is a maximal ideal of A, say M. It will be shown that $\eta = M^{o}$. The algebra homomorphism η factors through $A/M \cong k$, hence $\eta(f) = f + M = f(M)$ for all f in A, i.e., $\eta = M^{o}$. Hence the pair $(\mathrm{Max}(A), A)$ is indeed an almost affine variety. Thus a canonical almost affine variety is constructed from a reduced almost affine algebra.

Further suppose that $\eta : A \to B$ is an algebra homomorphism of almost affine algebras. Then η induces a morphism $\eta_* : \mathrm{Max}(B) \to \mathrm{Max}(A)$ of almost affine varieties. To prove the assertion, let M be an arbitrary element of $\mathrm{Max}(B)$. To define a morphism, it is enough to determine $(\eta_*(M))^{o}$ because of the canonical bijection between $\mathrm{Max}(A)$ and $\mathrm{Alg}_k(A,k)$. So let $(\eta_*(M))^{o} = M^{o}\circ\eta$, then η_* so defined is a morphism because $(\eta_*)^* = \eta$. Also if $\mu : V \to W$ is a morphism of almost affine varieties, then $(\mu^*)_* = \mu$, and if $\nu : W \to U$ is another morphism of almost affine varieties then so is $\nu\circ\mu : V \to U$ and $(\nu\circ\mu)^* = \mu^*\circ\nu^*$. Therefore, the following proposition is now established:

<u>PROPOSITION 1.6</u>. There is an anti-equivalence between the

category of reduced almost affine algebras and the category of almost affine varieties, the functor being $A \to \mathrm{Max}(A)$, $\eta \to \eta_*$ and the inverse functor being $V \to A(V)$, $\mu \to \mu^*$ [M1, p. 256, (1.7)] .

Since an almost affine algebra is a Jacobson ring by proposition (1.2 (a)) in which the radical of any ideal is the intersection of the maixmal ideals containing it, an important generalization of the usual affine Hilbert Nullstellensatz can be stated in the following form:

PROPOSITION 1.7 (Hilbert Nullstellensatz). Let $(V,A(V))$ be an almost affine variety, and let I be an ideal of $A(V)$. If a morphic function f vanishes on the annihilator of I in V, then some power of f lies in the ideal I.

If $(V,A(V))$ is an almost affine variety, we can introduce the weakest topology on V so that all morphic functions f in $A(V)$ are continuous where the ground field k carries the Zariski topology. This topology on V is also called the Zariski topology. Thus a basis for the closed sets of V are subsets in V of the form $\{ x : f(x) = 0$ for some f in $A(V) \}$, which will be denoted by $v(f)$. For f in $A(V)$ let V_f denote the complement of $v(f)$ in V, called a principal (or basic) open set. Then a basis for the Zariski topology on V is given by sets of the form V_f where f is in $A(V)$.

Since the points of V are in canonical bijection with the maximal ideals of $A(V)$ or equivalently with the elements of the set $Alg_k(A(V),k)$, the Zariski topology on V can be transported to $Max(A(V))$ or to $Alg_k(A(V),k)$, making the bijection homeomorphism. We note that the principal open sets in $Max(A(V))$ and $Alg_k(A(V),k)$ are then of the form $Max(A(V))_f = \{ M \text{ in } Max(A(V)) : M \text{ does not contain } f \}$, and $Alg_k(A(V),k)_f = \{ \eta \text{ in } Alg_k(A(V),k) : \eta(f) \neq 0 \}$, respectively, for some morphic function f in $A(V)$. We also note that if W is closed in V then W coincides with the annihilator in V of the annihilator of W in $A(V)$.

We continue to list properties of almost affine varieties analogous to those of affine varieties.

PROPOSITION 1.8. All closed subspaces and principal open subsets of an almost affine variety are again almost affine varieties [Ml, p. 257, (1.8)].

PROPOSITION 1.9. Let $\mu : V \to W$ be a morphism of almost affine varieties such that the associated comorphism $\mu^* : A(W) \to A(V)$ is surjective. Then the image $\mu(V)$ is an almost affine subvariety of W and $\mu : V \to \mu(V)$ is an isomorphism of almost affine varieties.

PROOF: The following proof is just a verbatim translation of the proof for affine varieties in [St, p. 3, Lemma] . It is first proved that $\mu(V)$ is the annihilator in W of the kernel of μ^*, so that $\mu(V)$ is closed in W hence a subvariety of W . Let v be an element of V . Then for all g in $\ker(\mu^*)$, $0 = \mu^*(g) = g\circ\mu$, hence $g(\mu(v)) = 0$ for all g in $\ker(\mu^*)$, i.e., $\mu(V)$ is contained in the annihilator of $\ker(\mu^*)$. To prove opposite inclusion, let w be in the annihilator of $\ker(\mu^*)$, i.e., $g(w) = 0$ for all g in $\ker(\mu^*)$. Regard w as an algebra homomorphism, i.e., as w^o in $\mathrm{Alg}_k(A(V),k)$. Then $w^o(g) = 0$ for all g in $\ker(\mu^*)$. Since the sequence

$$0 \to \ker(\mu^*) \to A(W) \to A(V) \to 0$$

is exact, we have $A(V) = A(W)/\ker(\mu^*)$, and so w^o maps $A(V)$ to k , consequently, there is some element v of V such that $w^o(h) = v^o(\mu^*(h))$ for all h of $A(W)$. But then $w^o(h) = (\mu(v))^o(h)$ for all h in $A(W)$, so $w = \mu(v)$, hence the annihilator of $\ker(\mu^*)$ is contained in $\mu(V)$, whence $\mu(V)$ is equal to the annihilator of $\ker(\mu^*)$ in W .

Now $A(W)|\mu(V)$ is isomorphic to $A(W)/I$, where I is the annihilator of $\mu(V)$ in $A(W)$, and I equals the radical of $\ker(\mu^*)$ by proposition (1.7) . But the radical of $\ker(\mu^*)$ equals $\ker(\mu^*)$ since $A(W)/\ker(\mu^*)$ is reduced (because $A(V)$ is reduced), hence $A(W)|\mu(V)$ is isomorphic to $A(W)/\ker(\mu^*) = A(V)$. Therefore, the pair $(\mu(V),A(W)|\mu(V))$ is an almost affine variety by proposition (1.8) , and isomorphic to V because $A(V)$ is isomorphic to $A(W)|\mu(V)$. Now our proof of proposition (1.9) is complete.

Next, the existence of direct product of finite or infinite family of almost affine varieties will be established.

PROPOSITION 1.10. If $\{ (V_i, A(V_i)) : i = 1,\ldots,n \}$ are almost affine varieties, then so is their direct product $(\Pi V_i, \otimes A(V_i))$.

PROOF: It suffices to prove the assertion when the indexing set is $\{ 1,2 \}$. The demonstration is divided into three steps.

(1) $A(V_1) \otimes A(V_2)$ is an almost affine algebra.

First it is shown that $A_1 \otimes A(V_2)$ is integral over $A_1 \otimes A_2$, where A_i is an affine algebra over which $A(V_i)$ is integral. Let $x = \Sigma a_i \otimes b_i$ be an element of $A_1 \otimes A(V_2)$, where the sum is taken over $i = 1, \ldots, m$. Then because b_i is integral over A_2 , $A_2[b_1,\ldots,b_m]$ is a finitely generated A_2-module. So there is a surjective A_2-module homomorphism η from $(A_2)^z$ to $A_2[b_1,\ldots,b_m]$, where z is a positive integer. Then the map $1 \otimes \eta$ from $A_1 \otimes (A_2)^z$ to $A_1 \otimes (A_2[b_1,\ldots,b_m])$ is a surjective $A_1 \otimes A_2$-module homomorphism because $A_1 \otimes (A_2)^z = (A_1 \otimes A_2)^z$, which means that $A_1 \otimes (A_2[b_1,\ldots,b_m])$ is a finitely generated $A_1 \otimes A_2$-module. But $(A_1 \otimes A_2)[x] = (A_1 \otimes A_2)[\Sigma a_i \otimes b_i] \subseteq A_1 \otimes (A_2[b_1,\ldots,b_m])$. Hence x is integral over $A_1 \otimes A_2$.

A similar argument shows that $A(V_1) \otimes A(V_2)$ is integral over $A_1 \otimes A(V_2)$. By transitivity of integral dependence $A(V_1) \otimes A(V_2)$ is integral over $A_1 \otimes A_2$. Since $A_1 \otimes A_2$ is known to be an affine algebra [F, pp. 29-30, (1.30)] , we just proved that $A(V_1) \otimes A(V_2)$ is an almost affine algebra which is integral over the affine algebra $A_1 \otimes A_2$.

(2) $A(V_1) \otimes A(V_2)$ is reduced.

Let f be an element of $A(V_1) \otimes A(V_2)$. Then we can write

$$f = \Sigma g_i \otimes h_i ,$$

where h_i are assumed to be linearly independent over k . Let x be an arbitrary element of V_1 . Then we can regard x as an element of $\mathrm{Alg}_k(A(V_1) \otimes A(V_2), A(V_2))$ by

$$\Sigma a_i \otimes b_i \to \Sigma a_i(x) b_i ,$$

where $\Sigma a_i \otimes b_i$ is a typical element of $A(V_1) \otimes A(V_2)$. Now, suppose $f^n = 0$ for some positive integer n . Then $0 = x(f^n) = (x(f))^n$, so $x(f) = 0$ because $A(V_2)$ is reduced. But then $0 = \Sigma g_i(x) h_i$, which implies that $g_i(x) = 0$ for all i since h_i are linearly independent. Hence $g_i = 0$ because x is arbitrary, so we must have $f = 0$.

(3) The evaluation map $(\ )^o$ from $V_1 \times V_2$ to $\mathrm{Alg}_k(A(V_1) \otimes A(V_2), k)$ is a bijection.

It is a consequence of the general fact: If A , B

and C are algebras over k, then $\mathrm{Alg}_k(A \otimes B, C) = \mathrm{Alg}_k(A,C) \times \mathrm{Alg}_k(B,C)$. In our particular setting, we have that $\mathrm{Alg}_k(A(V_1) \otimes A(V_2),k) = \mathrm{Alg}_k(A(V_1),k) \times \mathrm{Alg}_k(A(V_2),k) = V_1 \times V_2$.

Assertions (1), (2) and (3) establish that the pair $(V_1 \times V_2, A(V_1) \otimes A(V_2))$ is an almost affine variety. Hence by induction the product $(\Pi V_i, \otimes A(V_i))$ is an almost affine variety. This completes the proof of proposition (1.10).

<u>LEMMA 1.11</u>. Let $\{ V_i : i \text{ from } I \}$ be a family of almost affine varieties, and let $\{ A_i : i \text{ from } I \}$ be the set of affine algebras such that for each i from I $A(V_i)$ is an integral extension of A_i. Then $\underset{I}{\otimes} A(V_i)$ is an almost affine algebra if and only if $\underset{I}{\otimes} A_i$ is an almost affine algebra.

<u>PROOF</u>: First suppose that $\underset{I}{\otimes} A_i$ is an almost affine algebra. By definition $\underset{I}{\otimes} A(V_i)$ is the direct limit of all the finite tensor products $\underset{F}{\otimes} A(V_i)$ (where F is a finite subset of I) which are all integral over $\underset{F}{\otimes} A_i$ as we can see from proposition (1.10), so $\underset{I}{\otimes} A(V_i)$ is an integral extension of $\underset{I}{\otimes} A_i$, which means that $\underset{I}{\otimes} A(V_i)$ itself is an almost affine algebra.

Next, assume that $\underset{I}{\otimes} A(V_i)$ is an almost affine algebra. We found above that $\underset{I}{\otimes} A(V_i)$ is integral over $\underset{I}{\otimes} A_i$. Now, we may regard each A_i as an integral extension of a polynomial ring B_i by the normalization lemma (See, for example, [H1, p. 8, (1.10)].) where B_i looks like

$k[X_1^i,\ldots,X_{n_i}^i]$. Then $\underset{I}{\otimes} B_i$ is isomorphic to $k[X_1^i,\ldots,X_{n_i}^i$: i from I]. But $\underset{I}{\otimes} A(V_i)$ must have finite Krull dimension by proposition (1.2 (c)), so that all except finitely many B_i must be k . Let $I_o = \{$ i from I : B_i is not k $\}$. Then I_o is a finite subset of I . So $\underset{I}{\otimes} B_i \simeq \underset{I_o}{\otimes} B_i$ which is an affine algebra. But $\underset{I}{\otimes} A_i$ is integral over $\underset{I}{\otimes} B_i$, hence it is an almost affine algebra.

REMARK 1.12. In lemma (1.11) $\underset{I}{\otimes} A_i$ is an almost affine algebra if and only if all but finitely many A_i are integral over k .

PROOF: An argument analogous to the one used in the proof of lemma (1.11) will work.

PROPOSITION 1.13. Let $\{$ $(V_i, A(V_i))$: i from I $\}$ be a family of almost affine varieties. Then the product $(\underset{I}{\Pi} V_i, \underset{I}{\otimes} A(V_i))$ is again an almost affine variety if and only if all but finitely many A_i are integral over the ground field k (where each A_i is that affine subalgebra of $A(V_i)$ such that $A(V_i)$ is integral over A_i).

PROOF: First assume that all but finitely many A_i are integral over k . We will divide the demonstration into three steps.

(1) $\underset{I}{\otimes} A(V_i)$ is an almost affine algebra.

This is lemma (1.11) combined with remark (1.12).

(2) $\underset{I}{\otimes} A(V_i)$ is reduced.

Let $x = (x_i)$ be an element of $\underset{I-\{j\}}{\Pi} V_i$, and regard it as an element of $\mathrm{Alg}_k(\underset{I}{\otimes} A(V_i), A(V_j))$. Suppose f is an element of $\underset{I}{\otimes} A(V_i)$. Then $f = \Sigma b_i$ (finite sum), where $b_i = \otimes \hat{b}_{in}$ (where all but finitely many $\hat{b}_{in}$ are 1). If $f^m = 0$, then $0 = x(f^m) = (x(f))^m$, so $x(f) = 0$ because $A(V_j)$ is reduced. But $0 = x(f) = \Sigma(\Pi \hat{b}_{in}(x_n)\hat{b}_{ij})$, so $\Pi \hat{b}_{in}(x_n) = 0$ for all i, for we could choose $\hat{b}_{ij}$ to be linearly independent over k. So for each i we get $\hat{b}_{in}(x_n) = 0$ for some n in I, which means that for each i, $\hat{b}_{in} = 0$ for some n from I because we can chooce x_n as an arbitrary element of V_n. Hence $f = 0$.

(3) The evaluation map $(\ )^o$ from $\underset{I}{\Pi} V_i$ to $\mathrm{Alg}_k(\underset{I}{\otimes} A(V_i), k)$ is a bijection.

The fact mentioned in the proof of proposition (1.10), (3), is still valid for infinite tensor product, i.e., if B_i and C are algebras over k, then $\mathrm{Alg}_k(\underset{I}{\otimes} B_i, C) = \underset{I}{\Pi}\mathrm{Alg}_k(B_i, C)$. From this fact it follows that $\mathrm{Alg}_k(\underset{I}{\otimes} A(V_i), k) = \underset{I}{\Pi}\mathrm{Alg}_k(A(V_i), k) = \underset{I}{\Pi} V_i$, which means that the evaluation map $(\ )^o : \underset{I}{\Pi} V_i \to \mathrm{Alg}_k(\underset{I}{\otimes} A(V_i), k)$ is a bijection.

The proof of the only if part is immediate from lemma (1.11) and remark (1.12). This completes the proof of proposition (1.13).

We remark here that the Zariski topology which we are using on the product topology is not the product topology of Zariski topologies as the following example shows: Consider the product variety $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers regarded as the 1-dimensional affine variety over itself. Let $\Delta = \{ (x,y) \text{ in } \mathbb{C} \times \mathbb{C} : x = y \}$ be the diagonal. Then the diagonal is closed in the product topology if and only if $\mathbb{C}$ is Hausdorff. But $\mathbb{C}$ is not Hausdorff in the Zariski topology, and the diagonal is clearly closed in $\mathbb{C}^2$ with respect to the Zariski topology.

We will show next that the product above is the appropriate categorical definition of direct product in the category of almost affine varieties.

PROPOSITION 1.14. Let $\{ V_i : i = 1,\dots,n \}$ and W be almost affine varieties. Then (i) injection (described in the proof below) $V_j \to \Pi V_i$ as well as the canonical projections $\Pi V_i \to V_j$, are morphisms of almost affine varieties, and furthermore, the canonical projections are all open maps; (ii) a family $W \to V_i$ of morphisms of almost affine varieties yields one and only one morphism of almost affine varieties $W \to \Pi V_i$ whose composites with the canonical projections are the given morphisms, i.e., ΠV_i is a universal object in the category of almost affine varieties.

PROOF: Again it suffices to prove the assertion when the indexing set is $\{1,2\}$.

(i) An injection, say, $V_1 \to V_1 \times V_2$ given by $x \to (x,y_o)$ where y_o is a fixed element of V_2 is a morphism because it is associated to the evaluation homomorphism $A(V_1) \otimes A(V_2) \to A(V_1)$ given by $f \otimes g \to g(y_o)f$. Similarly, the canonical projection $V_1 \times V_2 \to V_1$ defined by $(x,y) \to x$ is a morphism since it is associated to the inclusion algebra homomorphism $A(V_1) \to A(V_1) \otimes A(V_2)$ given by $f \to f \otimes 1$ for f in $A(V_1)$.

Now it will be shown that the canonical projections $\pi_i : V_1 \times V_2 \to V_i$ are open maps. We will only consider the map $\pi_1 : V_1 \times V_2 \to V_1$. It is enough to show that $\pi_1((V_1 \times V_2)_f)$ is open in V_1 for all f in $A(V_1) \otimes A(V_2)$. Now, $(V_1 \times V_2)_f = \{ (x,y) \text{ in } V_1 \times V_2 : f(x,y) \neq 0 \}$. Thus $\pi_1((V_1 \times V_2)_f) = \{ x \text{ in } V_1 : (x,y) \text{ in } V_1 \times V_2 \text{ and } f(x,y) \neq 0 \} = \bigcup \{ x : f_y(x) = f(x,y) \neq 0 \}$, where the union is taken over all y with (x,y) in $V_1 \times V_2$ and $f(x,y) \neq 0$. But f_y belongs to $A(V_1)$ for a fixed y in V_2 , so $\{ x : f_y(x) \neq 0 \}$ is open in V_1 , hence $\pi_1((V_1 \times V_2)_f)$ is open, and therefore π_1 is an open map.

(ii) Suppose that there is given a pair of morphisms of almost affine varieties $\mu_1 : W \to V_1$ and $\mu_2 : W \to V_2$. Then there are associated comorphisms $\mu_1^* : A(V_1) \to A(W)$ and $\mu_2^* : A(V_2) \to A(W)$, so that there is a unique algebra homomorphism $\mu_1^* \otimes \mu_2^* : A(V_1) \otimes A(V_2) \to A(W)$ given by $f \otimes g \to \mu_1^*(f)\mu_2^*(g)$ such that the following diagram is commutative:

$$\begin{tikzcd} & A(V_1) \otimes A(V_2) \arrow[d, "\mu_1^* \otimes \mu_2^*"] & \\ A(V_1) \arrow[ur, "\pi_1^*"] \arrow[r, "\mu_1^*"'] & A(W) & A(V_2) \arrow[ul, "\pi_2^*"'] \arrow[l, "\mu_2^*"] \end{tikzcd}$$

Hence the corresponding diagram of almost affine varieties

$$
\begin{tikzcd}
 & V_1 \times V_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
V_1 & & V_2 \\
 & W \arrow[ul, "\mu_1"] \arrow[ur, "\mu_2"'] \arrow[uu, "(\mu_1^* \otimes \mu_2^*)_*"'] &
\end{tikzcd}
$$

is also commutative. The morphism $(\mu_1^* \otimes \mu_2^*)_* : W \to V_1 \times V_2$ has evidently the required properties. This completes the proof of proposition (1.14).

<u>PROPOSITION 1.15</u>. Let $\{ V_i : i \text{ from } I \}$ and W be almost affine varieties such that the product $\prod_I V_i$ exists as an almost affine variety (as asserted in proposition (1.13)). Then (i) injections $V_{i_o} \to \prod_I V_i$ (as the ones given in the proof of proposition (1.12)) as well as the canonical projections $\prod_I V_i \to V_{i_o}$, are morphisms of almost affine varieties, and furthermore, the canonical projections are open maps in the Zariski topology; (ii) a family $W \to V_i$ of morphisms of almost affine varieties yields one and only one morphism of almost affine varieties $W \to \prod_I V_i$ whose composites with the canonical projections are the given morphisms.

<u>PROOF</u>: A little modification of the proof of proposition (1.14) above yields the results.

<u>PROPOSITION 1.16</u>. If V and W are irreducible (i.e., $A(V)$ and $A(W)$ are integral domains) almost affine varieties, then

so is their direct product $V \times W$.

PROOF: See [H1, p. 28, (4.2)] for an algebraic proof and [F, p. 30, (1.31)] for a topological one.

The following sequence of propositions give immediate consequences of the definitions of almost affine algebra and variety.

PROPOSITION 1.17. Let $\{ A_i ; i \text{ from } I \}$ be a direct system of almost affine algebras all integral over a fixed affine subalgebra A_o which is element-wise invariant under all transition maps. Then $\text{dir lim } A_i$ is an almost affine algebra.

PROOF: Let x be an element of $\text{dir lim } A_i$. Then there is an almost affine algebra A_i and x_i with $x = \eta_i(x)$ where η_i is an algebra homomorphism in the definition of $\text{dir lim } A_i$. Since x_i is integral over A_o there exists a monic polynomial $P(X)$ in $A_o[X]$ with $P(x_i) = 0$. Then $0 = \hat{P}(x)$ where $\hat{P}(X)$ is a monic polynomial over $\eta_i(A_o)$ with coefficients the images of the coefficients of $P(X)$, so that X is integral over $\eta_i(A_o)$ which is inside $\text{dir lim } \eta_i(A_o) \simeq A_o$ (because $\eta_i(A_o) \simeq A_o$).

COROLLARY 1.18. Let $\{ V_i ; i \text{ from } I \}$ be projective system of almost affine varieties with all $A(V_i)$ satisfying the same conditions as in proposition (1.17). Then $\text{proj lim } V_i$ is also an almost affine variety.

PROOF: From the projective system $\{ V_i : i \text{ from } I \}$ we obtain a direct system $\{ A(V_i) : i \text{ from } I \}$ of almost affine algebras. So by proposition (1.17) above, dir lim $A(V_i)$ is an almost affine algebra, and it is reduced. Then $\mathrm{Max}(\text{dir lim } A(V_i))$ is an almost affine variety, and $\mathrm{Max}(\text{dir lim } A(V_i)) = \mathrm{Alg}_k(\text{dir lim } A(V_i),k) =$ proj lim $\mathrm{Alg}_k(A(V_i),k) =$ proj lim V_i . Hence the pair (proj lim V_i, dir lim $A(V_i)$) is an almost affine variety. This completes the proof of proposition (1.18).

PROPOSITION 1.19. Let $\{ V_i : i \text{ from } I \}$ be a projective system of almost affine varieties and V be an almost affine variety which is the projective limit of the system $\{ V_i : i \text{ from } I \}$. Then every morphic function f on V factors through V_i as $f = f_i \circ \mu_i$ for some i in I where μ_i is a morphism of almost affine varieties defining V as the projective limit of V_i , and f_i is a morphic function on V_i .

PROOF: Since $A(V) = \text{dir lim } A(V_i)$, $f = \mu_i^*(f_i)$ for some i in I . So $f = f_i \circ \mu_i$.

PROPOSITION 1.20. Every almost affine variety is a pro-affine variety, i.e., is a projective limit of affine varieties.

PROOF: Let V be an almost affine variety. Since the algebra $A(V)$ of morphic functions on V is an almost affine algebra, it is integral over an affine subalgebra A_o , say, so that given any finite subset E of $A(V)$, the algebra $A_o[E]$ is a finitely generated A_o-module, hence a finitely generated A_o-algebra. Then $A(V) = \text{dir lim } A_o[E]$ (where the limit is taken over all the finite subsets E of $A(V)$). Using the anti-equivalence of the category of almost affine varieties and the categoty of reduced almost affine algebras, we get a projective system $\{ \text{Max}(A_o[E]) : E$ is a finite subset of $A(V) \}$ of almost affine varieties. Now $\text{proj lim Max}(A_o[E]) = \text{proj lim Alg}_k(A_o[E],k) = \text{Alg}_k(\text{dir lim } A_o[E],k) = \text{Alg}_k(A(V),k) = V$, which proves that V is the projective limit of affine varieties $\text{Max}(A_o[E])$.

We conclude the chapter with examples of almost affine varieties.

<u>EXAMPLES 1.21</u>. (1) All affine varieties.

(2) Let k be a field of characteristic $p \neq 0$, and let

$$A = k[X_1,\ldots,X_n,\ldots]/I ,$$

where I is the ideal generated by $\{ X_{i+1}^p - X_i : i = 1,\ldots n,\ldots \}$. Then A is an integral domain almost affine algebra which is not affine. Let $V = \text{Max}(A)$. Then $V = \{ (x_i) \text{ in } k^\infty : x_{i+1}^p = x_i \}$, and we get an almost affine variety (V,A) which is not affine.

Regard k as the 1-dimensional affine variety, and define an affine variety $V_i = k$, for $i = 1,\ldots,n,\ldots$, and a transition morphism $\mu_i : V_{i+1} \to V_i$ by $x \to x^p$. Then $V = \text{proj lim } V_i$, i.e., V is a projective limit of copies of 1-dimensional affine variety k .

Consider the projection morphism $\pi : V \to V_1 = k$ given by $(x_i) \to x_1$. Then π is a bijection because k is of characteristic p (but not an isomorphism; in fact none of the bijective transition morphisms μ_i is an isomorphism).

In a similar fashion, we can construct such an almost affine variety over a field of characteristic zero, but the resulting variety is no longer in bijection with the ground field regarded as a 1-dimensional affine variety.

(3) Let V be a connected affine variety over k of characteristic $p \neq 0$ with positive dimension, and let A be the separable closure of $A(V)$. Then A is an almost affine algebra which is never affine, and $\mathrm{Max}(A)$ is an almost affine variety. See [M1, p. 271, (3.7) and (3.8)] and also [M2] for definition and proof.

CHAPTER II

LEFT ALGEBRAIC GROUPS AND BI-ALGEBRAIC GROUPS

We will define a left algebraic group and bi-algebraic group as a type of group varieties by imposing a group structure on an almost affine variety. Left algebraic groups were originally defined by Magid [M1], and his work is the starting point of our investigation in this chapter.

DEFINITION 2.1. A *left algebraic group* over an algebraically closed field k is an almost affine variety G which possesses a structure of abstract group such that for each fixed element x of G the left translation $\lambda_x : G \to G$ given by $y \to xy$ is a morphism of the almost affine variety structure. A *homomorphism* of left algebraic groups over k is a morphism of almost affine varieties which is at the same time a homomorphism of the underlying abstract group structures.

A *bi-algebraic group* over k is a left algebraic group G in which the right translation $\rho_x : G \to G$ given by $y \to yx$ is also a morphism of the almost affine variety structure for every element x of G .

We will often denote $(\lambda_x)^*(f)$ by $f\cdot x$ and $(\rho_x)^*(f)$ by $x\cdot f$ for a morphic function f in $A(G)$.

The question whether a bi-algebraic group is a pro-affine algebraic group will be taken up in the next chapter. We note that if G is an affine algebraic group we require that its algebra of morphic (or polynomial) functions $A(G)$ be a fully stable subalgebra (i.e., a Hopf subalgebra) of the Hopf algebra of all representative functions on G . Refer, for example, to the definition of an affine algebraic group given by Hochschild in [H1, pp. 14-25, chapters 2-3] . In the definition of a left algebraic group, however, we do not require $A(G)$ to be a subalgebra of the above mentioned Hopf algebra. In fact. Magid gave an example of a left algebraic group G such that there are non-representative morphic functions in $A(G)$. See [M1, p. 271, (3.6.4)] . Magid also adopted the following definition: A left algebraic group G is called a representative left algebraic group if (i) $A(G)$ is a subalgebra of the Hopf algebra of all representative functions on G , and (ii) $A(G)$ is an affine algebra [M3, p. 1, (1.1)] .

DEFINITION 2.2. Let H be an abstract subgroup of a left algebraic (or bi-algebraic) group G . Then H is called a left algebraic (or bi-algebraic) subgroup of G if H is a closed subvariety of G and is a left algebraic (or bi-algebraic) group by itself. A morphic function on H is just the restriction of a morphic function on G to H, so that the algebra of morphic functions on H is $A(G)/I$, where I is the annihilator ideal of H in $A(G)$.

We will first prove that the direct product of finite or infinite family of left algebraic (or bi-algebraic) groups exists in the category of left algebraic (or bi-algebraic) groups.

PROPOSITION 2.3. If $\{ G_i : i = 1,\dots,n \}$ are left algebraic (or bi-algebraic) groups, then so is their direct product ΠG_i.

PROOF: Because of proposition (1.10) it suffices to show that ΠG_i carries a left algebraic (or bi-algebraic) group structure. We only consider the left algebraic case. The bi-algebraic case can be handled similarly. It may be assumed that the indexing set is $\{ 1,2 \}$. To prove that all left translations on $G_1 \times G_2$ are morphisms, let (x_1,x_2) and (y_1,y_2) be elements of $G_1 \times G_2$ and h of $A(G_1 \times G_2) = A(G_1) \otimes A(G_2)$. Then h is in the form $\Sigma f_i \otimes g_i$ (the sum is taken over finite i) where f_i is in $A(G_1)$ and g_i in $A(G_2)$ and so $[\, h\cdot(x_1,x_2)](y_1,y_2) =$ $[\, (\Sigma f_i \otimes g_i)\cdot(x_1,y_2)](y_1,y_2) = (\Sigma f_i \otimes g_i)(x_1y_1,x_2y_2) =$ $\Sigma f_i(x_1y_1)g_i(x_2y_2) = \Sigma(f_i\cdot x_1)(y_1)(g_i\cdot x_2)(y_2) =$ $[\Sigma(f_i\cdot x_1) \otimes (g_i\cdot x_2)](y_1,y_2)$, so that $h\cdot(x_1,x_2) =$ $\Sigma(f_i\cdot x_1) \otimes (g_i\cdot x_2)$ and hence $h\cdot(x_1,x_2)$ belongs to

$A(G_1) \otimes A(G_2)$ because $f_i \cdot x_1$ is in $A(G_1)$ and $g_i \cdot x_2$ in $A(G_2)$ for all i, x_1 in G_1 and x_2 in G_2. Hence all left translations are morphisms. This completes the proof of proposition (2.3).

PROPOSITION 2.4. Let $\{ G_i : i \text{ from } I \}$ be an arbitrary family of left algebraic (or bi-algebraic) groups such that $\bigotimes_I A(G_i)$ is an almost affine algebra. Then the direct product $\prod_I G_i$ is a left algebraic (or bi-algebraic) group.

PROOF: Again we consider only the left algebraic case. By proposition (1.13) we just need to show that $\prod_I G_i$ carries a left algebraic group structure. Let x and y be taken from $\prod_I G_i$. To prove that all left translations are morphisms, it is enough to consider the case when f is a generator of $\bigotimes_I A(G_i)$, that is, f is in the form $\bigotimes_I f_i$, where $f_i = 1$ for all but finitely many i. Now $(f \cdot x)(y) = f(xy) = (\bigotimes_I f_i)((x_j y_j))$, where $x = (x_i)$ and $y = (y_i)$. Then $(\bigotimes_I f_i)((x_j y_j)) = \prod f_i(x_i y_i)$ (the product now is taken over finite i, and f_i are those $f_i \neq 1$) $= \prod (f_i \cdot x_i)(y_i) = (\bigotimes_I f_i \cdot x_i)(y)$, i.e., $f \cdot x = \bigotimes_I f_i \cdot x_i$ which belongs to $\bigotimes_I A(G_i)$ because $f_i \cdot x_i$ belongs to $A(G_i)$. Hence all left translations on $\prod_I G_i$ are morphisms. The proof of proposition (2.4) is now complete.

As in propositions (1.14) and (1.15) we can show that the product above is a universal object in the category of left algebraic (or bi-algebraic) groups.

PROPOSITION 2.5. Let $\{ G_i : i = 1,\ldots,n \}$ and H be left algebraic (or bi-algebraic) groups. Then (i) the canonical injections $G_{i_o} \to \Pi G_i$, as well as the canonical projections $\Pi G_i \to G_{i_o}$ are homomorphisms of left algebraic (or bi-algebraic) groups, and the canonical projections are all open homomorphisms; (ii) a family $H \to G_i$ of homomorphisms of left algebraic (or bi-algebraic) groups yields one and only one homomórphism of left algebraic (or bi-algebraic) groups $H \to \Pi G_i$ whose composites with the canonical projections are the given homomorphisms.

PROOF: (i) The proof is similar to that of proposition (1.14(i)).

(ii) Suppose there is given a family of homomorphisms of left algebraic (or bi-algebraic) groups $\mu_i : H \to G_i$. Then it yields a homomorphism $\mu : H \to \Pi G_i$ given by $\mu(x) = (\mu_i(x))$ such that its composite with the i-th projection is μ_i. This homomorphism must be unique because the i-th coordinate of the image of any point x of H under μ must coincide with $\mu_i(x)$ because of the prescribed property of μ. This completes the proof of proposition (2.5).

PROPOSITION 2.6. Let $\{ G_i : i \text{ from } I \}$ and H be left algebraic (or bi-algebraic) groups such that the direct product $\prod_I G_i$ exists in the category of left algebraic (or bi-algebraic) groups. Then (i) the canonical injections $G_{i_o} \to \prod_I G_i$, as well as the canonical projections

$\prod_I G_i \to G_i$, are homomorphisms of left algebraic (or bi-algebraic) groups, and furthermore, the canonical projections are open homomorphisms; (ii) a family $H \to G_i$ of homomorphisms of left algebraic (or bi-algebraic) groups yields one and only one homomorphism of left algebraic (or bi-algebraic) groups $H \to \prod_I G_i$ whose composites with the canonical projections are the given homomorphisms.

PROOF: (i) A slight modification of the proof of proposition (1.14) will yield the result.

(ii) The proof of proposition (2.5(ii)) above extends easily to the infinite case.

We will next prove that the category of left algebraic (or bi-aglebraic) groups is closed under the projective limit.

PROPOSITION 2.7. Let $\{ G_i : i \text{ from } I \}$ be a projective system of left algebraic (or bi-algebraic) groups over k such that $\otimes_I A(G_i)$ is an almost affine algebra. Then $\text{proj lim } G_i$ is also a left algebraic (or bi-algebraic) group over k.

PROOF: The abstract subgroup $\text{proj lim } G_i$ of $\prod_I G_i$ is closed (in the Zariski topology) since it is the annihilator of the ideal generated by the set $\{ f_i \circ \mu_{ij} \circ \pi_j - f_i \circ \pi_i : i \leq j \}$ where f_i are morphic functions on G_i, $\mu_{ij} : G_j \to G_i$ homomorphisms in the projective system $\{ G_i : i \text{ from } I \}$ and $\pi_j : \prod_I G_i \to G_j$ canonical projection homomorphisms. Since $\prod_I G_i$ is a left algebraic (or bi-algebraic) group by proposition (2.4), proj lim G_i is also a left algebraic (or bi-algebraic) group with $(\otimes_I A(G_i))/J$ as its algebra of morphic functions where J is the annihilator ideal of proj lim G_i in $\otimes_I A(G_i)$. This completes the proof of proposition (2.7).

The projective limit of affine left algebraic groups need not be a left algebraic group in the sense of definition (2.1), although it is a certain type of pro-affine left algebraic groups. Furthermore, Magid proved that there are left algebraic groups which are not the projective limit of affine left algebraic groups [M1, p. 271, (3.6.3)]. Compare the last assertion with proposition (1.20) which states that every almost affine variety is the projective limit of affine varieties.

We now give examples of left algebraic or bi-algebraic subgroups.

<u>EXAMPLES 2.8</u>. (1) Let G be a left algebraic group. Then (i) the connected component G_1 of the identity element 1 of G is a left algebraic subgroup of G; (ii) if G is a bi-algebraic group then G_1 is bi-algebraic and normal, and if G is affine with morphic inversion then G_1 bi-algebraic and normal of finite index, and G_1 is the only connected left algebraic subgroup of finite index in G.

<u>PROOF</u>: (i) Since G_1 is a closed subset of G, we only need to show that G_1 is an abstract subroup of G. Now, if x is an arbitrary element of G_1 then xG_1 is a connected component of G containing x because the left translation by x is an automorphism of G, hence $xG_1 = G_1$. Since x is arbitrary, xy belongs to G_1 for all elements x and y of G_1. Also $x^{-1}G_1$ is a connected component of G containing 1, where $(\ )^{-1} : G \to G$ is the inversion of the abstract group structure, hence $x^{-1}G_1 = G_1$, so that x^{-1} belongs to G_1 for every element x in G_1. Therefore, G_1 is an abstract subgroup of G, hence a left algebraic subgroup of G.

(ii) If G is a bi-algebraic group, then clearly G_1 is a bi-algebraic subgroup of G. Also $x^{-1}G_1x$ is a connected component of G containing 1 for every element x of G, so that $x^{-1}G_1x = G_1$ for all x in G. Hence G_1 is normal in G. If G is affine with morphic inversion, then, by [F, pp. 44-45, (2.2)], G can be written as the finite disjoint union of cosets $\{x_1G_1, \ldots, x_nG_1\}$, which are also irreducible components of G. Hence G_1 is a normal (as we saw above) bi-algebraic subgroup of finite index. Next, let H be any connected left algebraic subgroup of finite index in G. Write G as the disjoint and finite union of cosets $\{y_1H, \ldots, y_mH\}$. Then $G_1 = G_1 \cap (\bigcup y_iH) = \bigcup(G_1 \cap y_iH)$, so that $G_1 = G_1 \cap y_jH$ for some j because the right hand side is the disjoint union of closed subsets of G . So we must have $G_1 \subseteq y_jH$, hence $G_1 = y_jH$ since y_jH is connected. Then y_j is an element of G_1 , and so is y_j^{-1} . Hence $H = y_j^{-1}G_1 = G_1$.

(2) Let G be a bi-algebraic group. Then the following are all bi-algebraic subgroups of G : (i) the center Z of G ; (ii) the centralizer C_y of an element y of G ; (iii) the normalizer N_E of a subset E of G ; (iv) the stabilizer $S_E = \{ x \text{ in } G ; xE = E \}$ of a subset E of G .

PROOF: Since Z, C_y, N_E , and S_E are all abstract subgroups of G , we only need to prove that they are closed subsets of G . But it follows from the fact that (i) $Z = \{ x \text{ in } G : (f\cdot y - y\cdot f)(x) = 0$ for all y in G and f in $A(G) \}$, (ii) $C_y = \{ x \text{ in } G : (f\cdot y - y\cdot f)(x) = 0$ for all f in $A(G) \}$, (iii) $N_E = \{ x \text{ in } G : (f\cdot E - E\cdot f)(x) = 0$ for all f in $A(G) \}$, (iv) $S_E = \{ x \text{ in } G : (E\cdot f - f(E))(x) = 0$ for all f in $A(G) \}$, which are the annihilators in G of the ideals of $A(G)$ generated by $\{ f\cdot y - y\cdot f : y$ in G and f in $A(G) \}$, $\{ f\cdot y - y\cdot f : f$ in $A(G) \}$, $\{ f\cdot E - E\cdot f : f$ in $A(G) \}$, and $\{ E\cdot f - f(E) : f$ in $A(G) \}$, respectively, because $f\cdot y$, $y\cdot f$, $f\cdot E$, $E\cdot f$ are all elements or subsets of $A(G)$ for every y in G and f in $A(G)$ since G is a bi-algebraic group.

(3) Let G be a bi-algebraic group, and let E be any subset of morphic functions in $A(G)$. Then the set of

(left) E-invariant elements in G, ${}^{E}G = \{ x$ in $G : f \cdot x = f$ for all f in $E \}$ is a bi-algebraic subgroup of G.

PROOF: It is first shown that ${}^{E}G$ is an abstract subgroup of G. Suppose that x and y are elements of ${}^{E}G$. Then $f \cdot x = f$ and $f \cdot y = f$ for all f in E. So $f \cdot (xy) = (f \cdot x) \cdot y = f \cdot y = f$ for all f in E. Clearly 1 is in ${}^{E}G$. Now, for a fixed element x of G, the map $G \to G$ given by $y \to x^{-1}y$ is a bijection, hence $G = \{ x^{-1}y : y$ in $G \}$. Suppose x is an element of ${}^{E}G$. Then $f \cdot x = f$ for all f in E. Therefore, for any f in E, $(f \cdot x)(x^{-1}y) = f(x^{-1}y)$ for all y in G, whence $f(y) = (f \cdot x^{-1})(y)$ for all y in G, which means that for all f in E, $f = f \cdot x^{-1}$, hence x^{-1} belongs to ${}^{E}G$, so that ${}^{E}G$ is an abstract subgroup of G.

Next it is established that ${}^{E}G$ is a closed subset of G. But it is a consequence of the fact that ${}^{E}G = \{ x$ in $G : (f \cdot x)(y) = f(y)$ for all f in E and y in $G \} = \{ x$ in $G : (y \cdot f)(x) = f(y)$ for all f in E and y in $G \} = \{ x$ in $G : (y \cdot f - f(y))(x) = 0$ for all f in E and y in $G \}$, which is the annihilator in G of the ideal in $A(G)$ generated by the set $\{ y \cdot f - f(y) : f$ in E and y in $G \}$.

(4) Let G be a bi-algebraic group, and let E be a G-stable subset of $A(G)$. Then the set of E-invariant elements of G, ${}^{E}G^{E} = \{ x$ in $G : f \cdot x = f = x \cdot f$ for all f in $E \}$ is a normal bi-algebraic subgroup of G.

PROOF: Let $H = {}^{E}G^{E}$. By a similar reasoning as in the proof of the example above, we can show that H is a bi-algebraic subgroup of G . So we only prove that H is normal in G . Let x be any element of H . Then $f\cdot x = f = x\cdot f$ for all f in E . Now, $f\cdot(yxy^{-1}) = ((f\cdot y)\cdot x)\cdot y^{-1} = (f\cdot y)\cdot y^{-1} = f\cdot(yy^{-1}) = f$ for all y in G and f in E . Similarly, $(yxy^{-1})\cdot f = f$ for all y in G and f in E . Therefore, yHy^{-1} is a subset of H , hence H is a normal subgroup of G .

(5) Let G be a bi-algebraic group such that the group inversion is a morphism (hence homeomorphism), and let H be an abstract subgroup of H . Then (i) the closure $Cl(H)$ of H is a bi-algebraic subgroup of G , and (ii) if H is normal in G then so is $Cl(H)$.

PROOF: (i) We first show that if x is in $Cl(H)$ and $f|H = 0$ (where f is from $A(G)$) then $f\cdot x|H = 0$. Let y be in H . Then $(f\cdot x)(y) = f(xy) = (y\cdot f)(x)$. So it is enough to prove that $(y\cdot f)|H = 0$. But if z is from H then $(y\cdot f)(z) = f(zy) = 0$.

Next we show that $Cl(H)$ is closed under the multiplication of G . Let x and y be from $Cl(H)$, and let f be from $A(G)$ such that $f|H = 0$. Then $f(xy) = (f\cdot x)(y) = 0$ by the result above because $f\cdot x|H = 0$.

Finally we will show that $Cl(H)$ is closed under the inversion of G . Since $(\ )^{-1} : G \to G$ is a homeomorphism, $(Cl(H))^{-1} = Cl((H)^{-1}) = Cl(H)$. Hence $Cl(H)$ is a bi-algebraic subgroup of G .

(ii) Let x be from G and y from $Cl(H)$, and let f be from $A(G)$ with $f|H = 0$. Then $f(xyx^{-1}) = (x^{-1}fx)(y) = 0$ because $(x^{-1}fx)|H = 0$ due to the normality of H in G . Hence xyx^{-1} belongs to $Cl(H)$, so $Cl(H)$ is also normal in G .

We next list a few examples of left algebraic groups.

<u>EXAMPLES 2.9</u>. (1) All affine (hence all linear) algebraic groups and their closed subgroups.

(2) The invertible elements of any finite dimensional associative algebra (which is in fact an affine algebraic group). For more information on linear algebraic groups consult [B].

(3) Magid showed that a left algebraic group structure arises from an étale covering of an affine algebraic group. See [M1, p. 260, (2.3) and (2.4)] .

(4) Hochschild and Mostow proved that every complex analytic group having a faithful representation carries a left algebraic group structure. Refer to [HM2, p. 118, (3.1)] .

Magid also proved that a left algebraic group structure is given by semi-direct product of algebraic groups as is shown in the follwoing lemma. Before we state the result and its corollary as an additional example of left algebraic groups we will recall the necessary definition: Let H and

K be left algebraic groups over k, and $s : K \to \mathrm{Aut}(H)$ a homomorphism of abstract groups, where $\mathrm{Aut}(H)$ denotes the group of left algebraic group automorphisms of H. Then the semi-direct product $H \cdot_s K$ is the group whose underlying set is the cartesian product $H \times K$ and whose group multiplication is given by

$$(x_1, x_2)(y_1, y_2) = (x_1 s(x_2)(y_1), x_2 y_2) .$$

Now the following result is taken from [M1, p. 267, (3.1)] .

LEMMA 2.10. The semi-direct product $H \cdot_s K$ defined above is a left algebraic group.

We will obtain further examples of left algebraic groups.

EXAMPLES 2.11. (1) This is an example of an affine left algebraic group which is not an affine bi-algebraic group (and hence is not an affine algebraic group). Refer also to [HM2, p. 128] and [M1, p. 267, (3.2)] .

Define a multiplication by a fixed non-zero complex number from left $: \mathbb{C} \to \mathbb{C}$ by $x \to ax$, which is a morphism from $\mathbb{C}$ into $\mathbb{C}$ (where $\mathbb{C}$ is regarded as the 1-dimensional affine additive group), and is clearly an automorphism of $\mathbb{C}$. Let $s : \mathbb{C} \to \mathrm{Aut}(\mathbb{C})$ be defined by $s(x)(y) = \exp(x)y$ for all x and y from $\mathbb{C}$, i.e., $s(x)$ is an algebraic group automorphism given by left structure is also preserved under a homomorphism whose associated comorphism is surjective.

<u>PROPOSITION 2.12</u>. Let $\mu : G \to H$ be a homomorphism of left algebraic (or bi-algebraic) groups such that μ^* is surjective. Then $\mu(G)$ is a left algebraic (or bi-algebraic) subgroup of H and $\mu : G \to \mu(G)$ is an isomorphism of left algebraic (or bi-algebraic) groups.

<u>PROPOSITION 2.13</u>. Let G be an irreducible left algebraic (or bi-algebraic) group and H an algebraic group, and let $\mu : G \to H$ be a homomorphism. Then $\mu(G)$ is an algebraic subgroup of H .

<u>PROOF</u>: The image $\mu(G)$ is a subgroup of H which contains a nonempty open subset of $\mathrm{Cl}(\mu(G))$ [M1, p. 257, (1.9)] . By example (2.8(5)), $\mathrm{Cl}(\mu(G))$ is an irreducible algebraic group. Since $\mu(G)$ contains an open subset of $\mathrm{Cl}(\mu(G))$, we must have $\mu(G) = \mathrm{Cl}(\mu(G))$. Hence $\mu(G)$ is algebraic.

We note that in proposition (2.13) if G is affine then we do not need to have G irreducible in view of example (2.8(1)). We will use this fact later in chapter III.

<u>COROLLARY 2.14</u>. If μ^* is surjective in proposition (2.13), then G is an algebraic group.

<u>PROOF</u>: Combine proposition (2.13) with proposition (2.12), and the result follows.

integral.

(1) The subalgebra A_o^H of H-invariant elements of A_o, { f in A_o : $f \cdot x = f$ for every element x of H }, is an affine algebra.

A_o is an affine algebra, so we can write $A_o = k[f_1,\ldots,f_n]$, where f_i are generators of A_o. We may assume that A_o is H-stable because we can replace A_o by $k[f_i \cdot x : i = 1,\ldots,n$ and x from $H]$ which is, by assumption (i), an affine subalgebra of A(G) and is H-stable. Therefore, we may even assume that $f_i \cdot x$ all belong to $\{ f_1,\ldots,f_n \}$ for every element x of H and all $i = 1,\ldots,n$. Then H restricted to A_o is a subgroup of S_n, the symmetric group on n symbols, so that H acts as a finite automorphism group of the affine algebra A_o. Hence A_o^H is a finitely generated algebra [F, pp. 186-187, (5.50)].

(2) The algebra $A(G)^H = \{ f$ in $A(G) : f \cdot x = f$ for all elements x of $H \}$ of H-invariant elements of A(G) is an almost affine algebra.

To prove the assertion, we establish that $A(G)^H$ is an integral extension of A_o^H which was shown to be an affine algebra above. Let f be an arbitrary element of $A(G)^H$. Since f is integral over A_o, there exists a monic polynomial P(X) in $A_o[X]$ with $P(f) = 0$. By assumption (i), the set of H-translates of f, $\{ f \cdot x : x$ in $H \}$ is finite, so that we can form a product $\hat{P}(X) = \Pi P(X) \cdot x$ where $P(X) \cdot x$

are polynomials in X whose coefficients are the x-translates of those of $P(X)$, and the product is taken over all the elements x of H . Then $\hat{P}(f) = 0$ since $P(X)$ appears among the factors of $\hat{P}(X)$. We also have $\hat{P}(X)\cdot x = \hat{P}(X)$ for all elements x of H , so that $\hat{P}(X)$ belongs to $A_o^H[X]$. Since $\hat{P}(X)$ is a monic polynomial, it is a polynomial for integral dependence of f over A_o^H .

(3) $A(G)$ is integral over $A(G)^H$.

Let $A_o = k[f_1,\ldots,f_n]$ as before. Then each generator f_i is integral over A_o^H because f_i satisfies a polynomial $P_i(X) = \Pi(X - f_i\cdot x)$ which is a monic polynomial with coefficients in A_o^H where the product is taken over the elements x of H . So A_o is integral over A_o^H , whence $A(G)$ is integral over A_o^H .

(4) Every element of $\mathrm{Alg}_k(A(G)^H,k)$ is the restriction to $A(G)^H$ of an element of $\mathrm{Alg}_k(A(G),k)$.

To prove the assertion, it is enough to show that every algebra homomorphism from $A(G)^H$ to k has an extension to $A(G)$. Let η be an algebra homomorphism from $A(G)^H$ to k . Then its kernel is a maximal ideal of $A(G)^H$. Since $A(G)$ is integral over $A(G)^H$ as shown above, there exists a maximal ideal M of $A(G)$ lying over $\ker(\eta)$, i.e., $A(G)^H \cap M = \ker(\eta)$. Define an algebra homomorphism $\hat{\eta}$ from $A(G)$ to k by

$$\hat{\eta}(f) = f + M ,$$

for any f in $A(G)$. Let g be an arbitrary element of $A(G)^H$. Then $\hat{\eta}(g) = g + M = g + \ker(\eta)$, which shows that $\hat{\eta}$ is an extension of η to $A(G)$.

(5) The evaluation map from G/H into $\mathrm{Alg}_k(A(G)^H,k)$ is a bijection.

The evaluation map $(\ )^o$ from G/H in $\mathrm{Alg}_k(A(G)^H,k)$ is given by

$$(xH)^o(f) = f(x) = x^o(f) \ ,$$

for every element x of G and f of $A(G)^H$. The map is well-defined, and surjective since every element of $\mathrm{Alg}_k(A(G)^H,k)$ is in the form x^o for some element x of G by result (4). To show that the map is injective, let $zH \neq wH$. Then, by assumption (ii), there exists a morphic function f in $A(G)$ such that $f|zH = 0$ and $f|wH = 1$. Since the set of H-translates of f , $\{ f\cdot x : x \text{ in } H \}$ is finite by hypothesis (i), we can form a product $g = \Pi f\cdot x$, where the product is taken over all x in H . Then $g\cdot y = g$ for all elements y of H , i.e., g belongs to $A(G)^H$, and also $g(zH) = 0$ and $g(wH) = 1$,i.e., $(zH)^0 \neq (wH)^o$ on $A(G)^H$, hence the evaluation map $(\ )^o$ is injective.

(6) Summarizing results (1) through (5), we have an almost affine variety $(G/H,A(G)^H)$. It only remains to show that G/H carries the structure of left algebraic group. Let f be in $A(G)^H$ and x in G . Then

$(f\cdot x)\cdot y = f\cdot(xy) = f\cdot(y'x) = (f\cdot y')\cdot x = f\cdot x$, since H is normal so that y' belongs to H , and f is H-invariant. Therefore, all left translations on G/H are morphisms. This completes the proof of proposition (2.15).

The following proposition shows the universal propeaty of a quotient left algebraic group.

PROPOSITION 2.16 Let H be a properly normal left algebraic subgroup of a left algebraic group G . Then (i) the canonical projection π from G to G/H is a homomorphism; (ii) if π is any homomorphism of left algebraic groups from G into G' whose kernel contains H then there is one and only one homomorphism $\hat{\mu}$ from G/H to G' such that $\hat{\mu}\circ\pi = \mu$; (iii) G/H is unique up to isomorphism.

PROOF: (i) Let f be an element of $A(G/H) = A(G)^H$. Then for all elements x of G , $f\circ\pi(x) = f(xH) = (f\cdot H)(x) = f(x)$, i.e., $f\circ\pi$ is a morphic function on G . Hence π is a homomorphism.

(ii) There is a unique abstract group homomorphism $\hat{\mu} : G/H \to G'$ with $\hat{\mu}\circ\pi = \mu$. Therefore, it is enough to show that $\hat{\mu}$ is a morphism. Let f be a morphic function in $A(G')$, x an element of H and y of G . Then $((f\circ\hat{\mu})\cdot x)(y) = f(\hat{\mu}(xy)) = f(\hat{\mu}(y)) = (f\circ\hat{\mu})(y)$, i.e., $(f\circ\hat{\mu})\cdot x = f\circ\hat{\mu}$ for all elements x of H , so that $f\circ\hat{\mu}$ is an element of $A(G/H) = A(G)^H$. Hence $\hat{\mu}$ is a homomorphism.

(iii) This is an immediate consequence of result (ii) above.

PROPOSITION 2.17. In propositon (2.16), assume further that the ground field is of characteristic zero, G and G' are irreducible, and that μ is surjective with $\ker(\mu) = H$. Then $\hat{\mu} : G/H \to G'$ is an isomorphism of left algebraic groups.

PROOF: The homomorphism $\hat{\mu} : G/H \to G'$ is a bijective homomorphism of left algebraic groups, and G/H is also an irreducible left algebraic group. Then by [M1, p. 257, (1.10)], $A(G/H) = A(G)^H$ is purely inseparable algebraic over the quotient field of the image of $A(G')$ under the associated comorphism $(\hat{\mu})^* : A(G') \to A(G/H)$ (which can be regarded as embedding because $\hat{\mu}$ is surjective). But then $(\hat{\mu})^*$ must be surjective since the characteristic of k is zero. Hence by proposition (2.12) $\hat{\mu}$ must be an isomorphism.

The following propositions give examples of properly normal left algebraic subgroups.

PROPOSITION 2.18. Let H and K be normal left algebraic subgroups of a left algebraic group G with $G = H \times K$. Then G/H is also a left algebraic group, and it is isomorphic to K.

PROOF: The assertion is proved in three steps.

(1) $A(G)^H$ $(= (A(H) \otimes A(K))^H)$ is isomorphic to $A(K)$.

semi-direct product with respect to a certain homomorphism $s : K \to \mathrm{Aut}(H)$. Then H is a properly normal left algebraic subgroup of G , and G/H is isomorphic to K as a left algebraic group.

PROOF: We are identifying H with $H \times \{ 1 \}$, so that we can view H as a normal left algebraic subgroup of $G = H \cdot_s K$ under the multiplication in the semi-direct product (although K cannot be normal in G). Now the rest of the proof is identical to that of proposition (2.18).

PROPOSITION 2.20. Let H be normal finite left algebraic subgroup of a left algebraic group G . Then the abstract factor group G/H has a left algebraic group structure with $A(G)^H$ as its algebra of morphic functions.

PROOF: First H is locally finite on $A(G)$, since H is finite. Next it is shown that $A(G)$ separates the class of cosets of H . Let $xH \neq yH$, and let $H = \{ h_1,\ldots,h_n\}$. Then $xH = \{ xh_1,\ldots,xh_n \}$ and $yH = \{ yh_1,\ldots,yh_n \}$. Now regard G as $\mathrm{Max}(A(G))$, and then $xh_1,\ldots,xh_n,yh_1,\ldots,yh_n$ are all distinct maximal ideals of $A(G)$. We assert that for each $i = 1,\ldots,n$ there exists a morphic function f_i on G such that f_i is not in xh_i but is in all the others, hence in their intersection (hence in their product). If this is not the case then every element in the product of all the other maximal ideals xh_j and yh_j belong to the maximal ideal xh_i , so that xh_i contains the product, hence contains one of the maximal ideals in the product (since xh_i

corresponds a finite group H_f (as defined below) which is a homomorphic image of H .

For a morphic function f on G , define a relation $\sim_f$ over H by $x \sim_f y$ if and only if $g \cdot x = g \cdot y$ for every element g of the finite set $\{ f \cdot x : x \text{ is in } H \}$. Then $\sim_f$ is an equivalence relation on H . To show this, let x , y and z be elements of H . It is clear that $x \sim_f x$ and that $x \sim_f y$ implies $y \sim_f x$. Suppose next that $x \sim_f y$ and $y \sim_f z$, and let g be any element of the set $\{ f \cdot x : x \text{ from } H \}$. Then $g \cdot x = g \cdot y = g \cdot z$, i.e., $x \sim_f z$. Now let $H_f = H/\sim_f$. The set H_f is made into an abstract group. Let $[x]$ denote the equivalence class containing x , and define the multiplication on H_f by

$$[x]\ [y] = [xy] \ .$$

First we will show that the product is well-defined. Suppose that $[x] = [x']$ and $[y] = [y']$. Then for every g in the set $\{ f \cdot x : x \text{ is in } H \}$, $g \cdot (xy) = (g \cdot x) \cdot y = (g \cdot x) \cdot y' = (g \cdot x') \cdot y' = g \cdot (x'y')$ because $g \cdot x$ is also in the set $\{ f \cdot x : x \text{ is in } H \}$. Hence $[xy] = [x'y']$. With this multiplication H_f is an abstract group with the identity element $[1]$ and $[x^{-1}]$ being the inverse of $[x]$.

Next we claim that H_f is a finite group. Now, $x \not\sim_f y$ if and only if there is some g in the set $\{ f \cdot x : x \text{ is in } H \}$ with $g \cdot x \neq g \cdot y$. But there are only finitely many distinct $g \cdot x$, so only finitely many distinct equivalence classes, whence the group H_f is a finite (hence left algebraic) group.

(ii) For each morphic function f on H there exists a normal subgroup H^f of H of finite index such that f belongs to $A(H)^{H^f}$.

By result (i) above we get a finite group H_f which is a homomorphic image of H, hence a normal subgroup H^f of finite index in H such that $H_f = H/H^f$. We claim that f lies in $A(H)^{H^f}$. Let y be an element of H^f. Then $y \sim_f 1$, and hence $g \cdot y = g$ for all g in the set $\{ f \cdot x :$ x in $H \}$. In particular, $f \cdot y = f$. Since y is arbitrary in H^f, f belongs to $A(H)^{H^f}$.

(iii) There exists an injective homomorphism from H into the pro-finite left algebraic group $\hat{H}$.

For each i from I, we have a homomorphism from H onto H/H_i. These together give a homomorphism $\mu : H \to \prod_I H/H_i$. The image of H under μ is contained in $\hat{H}$ by definition of $\hat{H}$. We will show that μ is an injection. Let x be an element of H with $x \neq 1$. Then there is a morphic function f on H with $f \cdot x \neq f$ (because H is itself an almost affine variety). By result (ii) there exists a subgroup H^f of finite index in H with f lying in $A(H)^{H^f}$. The image of x in H/H^f takes f to $f \cdot x$ (f now regarded as a function on H/H^f), so is not the identity element in H/H^f. Hence the image of x in $\hat{H}$ under μ cannot be the identity element, so μ is an injection.

Now $Cl(\mu(H))$ in $\hat{H}$ (with respect to the induced topology on $\hat{H}$ from the product topology of $\prod_I H/H_i$) is a closed topological subgroup of $\hat{H}$ (where $\hat{H}$ is now regarded as a closed topological subgroup of $\prod_I H/H_i$ because $\hat{H}$ is also closed with respect to the product topology on $\prod_I H/h_i$) so $Cl(\mu(H))$ is a pro-finite topological group. (See [Hi] for more details on pro-finite topological groups.) This completes the proof of proposition (2.21).

We will show next that certain pro-finite left algebraic subgroups are locally finite.

<u>PROPOSITION 2.22</u>. Let H be a normal pro-finite left algebraic subgroup of a left algebraic group G such that only finitely many coordinate rings $A(xH)$ of the closed subvarieties xH are distinct (although they are all isomorphic). Then H is locally finite on $A(G)$.

<u>PROOF</u>: It must be shown that the set of H-translates of g in $A(G)$, $\{ g \cdot x : x \text{ in } H \}$, is a finite set. Since $H = \text{proj lim } H_i$ as a left algebraic group where H_i are all finite left algebraic groups, given a morphic function g on G, we get, by proposition (1.19), $(g \cdot y|H)(w) = ((g|H) \cdot y)(w) = ((g_i o \mu_i) \cdot y)(w)$, where μ_i are homomorphisms in the definition of $H = \text{proj lim } H_i$, g_i is an element of $A(H_i)$ for some i, and y and w from H. But $((g_i o \mu_i) \cdot y)(w) = (g_i o \mu_i)(yw) = g_i(\mu_i(yw)) = g_i(\mu_i(y)\mu_i(w)) =$

$(g_i \cdot \mu_i(y)) \circ \mu_i(w)$, so that $g \cdot y|H = (g_i \cdot \mu_i(y)) \circ \mu_i$, hence $\{ g \cdot y|H : y \text{ in } H \} = \{ (g_i \cdot \mu_i(y)) \circ \mu_i : y \text{ in } H \}$ where the set on the right hand side is a finite set because the set $\{ \mu_i(y) : y \text{ in } H \}$ (which is a subset of H_i) is a finite set. Then it follows that the set $\{ g \cdot y|xH : y \text{ in } H \}$ is also a finite set because $A(xH)$ is isomorphic to $A(H)$.

Now $G = \bigcup xH$, where the union is taken over all x in G , and xH are closed sets in G , so that if g is in $A(G)$ and y in H then $g \cdot y$ is completely determined by its restriction $g \cdot y|xH$. But the set $\{ g \cdot y|xH : y$ in $H \}$, as we saw above, is a finite set, and there are only finitely many distinct such algebras $A(xH)$ when x ranges over G because of the given hypothesis, so that the set $\{ g \cdot y : y \text{ in } H \}$ is a finite set. Hence H is locally finite on $A(G)$. This completes the proof of proposition (2.22).

The following example shows that locally finiteness is not a necessary condition for a given normal subgroup to be properly normal: Let $G = \mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. Then by proposition (2.18) $H = \mathbb{C} \times \{0\}$ is a properly normal left algebraic subgroup of G (such that $G/H \simeq \mathbb{C}$) . Take $f = X + Y$ from $A(G) = k[X,Y]$. Then for any x in H , $f \cdot x = X + Y + x = f + x$, which forms an infinite set when x ranges over H (although its linear span over k is finite dimensional), so is not locally finite.

CHAPTER III

BI-ALGEBRAIC GROUPS AND ALGEBRAIC GROUPS

In this chapter we investigate when a bi-algebraic group is an algebraic group. More specifically, the problem can be stated as follows: Suppose that G is a bi-algebraic group over an algebraically closed field k. (1) Is the group multiplication $G \times G \to G$ a morphism? (2) Is the group inversion $G \to G$ also a morphism? (3) Are questions (1) and (2) independent of each other?

The set of problems was first formulated in the category of affine bi-algebraic groups by Palais [P, §10], and he himself obtained many significant results. Later, Magid [M4] studied the same problems in the same (that is, affine) setting, and proved that the answer to (1) is in the affirmative (if the ground field k is infinite and perfect, which always holds in our case since k is assumed to be algebraically closed).

We will examine the problems in a different manner, and obtain similar results. We will also study the problems in a more general setting where the given bi-algebraic group is almost affine as a variety. What we are trying to achieve is an algebraic analogue of Montgomery's theorem [Mo] which states that every complete metric separable bi-topological group is a topological group.

We rely heavily on the results of Palais [P] for definitions and techniques of proof in obtaining the first several propositions which extend the affine case to the almost affine case.

We begin with a definition and a general lemma: A sequence $\{ f_n \}$ of morphic functions on a variety V (affine or almost affine) is said to be <u>point-wise eventually zero</u> if for each x in V there exists a positive integer N_x such that $f_n(x) = 0$ when n is greater than N_x. The sequence $\{ f_n \}$ is said to be <u>eventually zero</u> if $f_n = 0$ for all sufficiently large n.

Recall also the following: If X and Y are sets, and $f : X \times Y \to k$ is a function, then f^x and f_y both denote functions $Y \to k$ and $X \to k$ given by $f^x(y) = f(x,y)$ and $f_y(x) = f(x,y)$, respectively, for any element (x,y) of $X \times Y$. (We are simply fixing one of the two variables at a time.)

<u>LEMMA 3.1</u>. Let X be any set and let B be a finite dimensional subspace of the vector space k^X of all k-valued functions on X. Then $\{ x^o : x \text{ from } X \}$ spans the dual space B^* of B, so in particular there exist $\{ x_1, \ldots, x_n \}$ in X such that $\{ (x_1)^o, \ldots, (x_n)^o \}$ is a basis for

B^* . Choose such $\{ x_1,\ldots,x_n \}$ and let $\{ \xi_1 \ldots,\xi_n \}$ be the dual basis for $B^{**} = B$. If Y is any set and $f : X \times Y \to k$ is any function such that f_y lies in B for all y in Y then f can be expressed as $\Sigma\xi_i \otimes (f^{x_i})$.

PROOF: See [P, (3.1)].

COROLLARY 3.2. If $\{ f_n \}$ is a sequence of morphic functions on a variety (affine or almost affine) which is point-wise eventually zero but is not eventually zero, then there is a subsequence of $\{ f_n \}$ whose elements are linearly independant.

PROOF: See [P, (3.3)].

The following proposition is an easy extension of [P, (7.1) and (7.4)] to the almost affine case where the original setting was affine.

PROPOSITION 3.3. If V is an irreducible variety (affine or almost affine), then the following are all equivalent:

(1) If V is decomposed as the countable union of subsets $\{ S_n \}$ then at least one S_n is dense in V .

(2) V cannot be decomposed as the countable union of proper closed subvarieties.

(3) V cannot be decomposed as the countable increasing union of proper closed subvarieties.

(4) If V is the increasing union of a sequence of closed subvarieties $\{ V_n \}$ whose union is V then some

V_n is equal to V.

(5) V is of the second category relative to the Zariski topology.

(6) If V is the increasing union of a sequence of subsets $\{ S_n \}$ then some S_n is dense in V.

(7) If a sequence $\{ f_n \}$ of morphic functions on V is point-wise eventually zero then it is eventually zero.

PROOF: In this entire proof the union is taken over a countable collection of sets or varieties.

(1) $\rightarrow$ (2): Suppose that $V = \bigcup V_i$ where V_i are proper closed subvarieties. Then $V = \bigcup \mathrm{Cl}(V_i)$. But $\mathrm{Cl}(V_i) = V_i \neq V$ for any index i, which contradicts (1).

(2) $\rightarrow$ (3): Obvious.

(3) $\rightarrow$ (4): Clear.

(4) $\rightarrow$ (6): Obvious.

(6) $\rightarrow$ (1): Suppose $V = \bigcup S_n$, and assume that no $\mathrm{Cl}(S_n)$ equals V. Let $T_1 = S_1$, and $T_n = S_1 \cup \ldots \cup S_n$, $n = 1, \ldots.$ Then $T_n \subset T_{n+1}$ for all $n = 1, \ldots,$ and also $V = \bigcup T_n$. So $V = \mathrm{Cl}(T_n)$ for some index n. Then $V = \mathrm{Cl}(S_1 \cup \ldots \cup S_n) = \mathrm{Cl}(S_1) \cup \ldots \cup \mathrm{Cl}(S_n)$, which is impossible since V is irreducible.

Thus we have proved that (1), (2), (3), (4) and (6) are all equivalent.

(2) $\rightarrow$ (5): Let $V = \bigcup V_n$, where V_n are closed. Then each V_n is a closed subvariety. So $V = \mathrm{Cl}(V_n) = V_n$ for some V_n. Since V has non-empty interior, so does

V_n .

(5) $\rightarrow$ (2): Suppose $V = \bigcup V_n$, where V_n are closed subvarieties. Then some V_n has non-empty interior U . But $Cl(U) = V$ because V is irreducible, so $V = Cl(U) = Cl(V_n) = V_n$.

Now we have that all (1) through (6) are equivalent. So we show the equivalence between (3) and (7), our proof will be complete.

(3) $\rightarrow$ (7): Let $\{ f_n \}$ be a point-wise eventually zero sequence of morphic functions on V . Set $V_n = \bigcap f_k^{-1}(0)$ (where the intersection is taken over $k > n$. Then V_n is a closed subvariety, and $V_n \subset V_{n+1}$, $n = 1,\ldots$. We claim that $V = \bigcup V_n$. To prove this let x be an element of V , then $f_n(x) = 0$ for sufficiently large n . Say $f_k(x) = 0$ whenever $k > n_o$ then x must lie in V_{n_o} , hence the claim is established. Now by hypothesis we must have $V = V_n$ for some index n . Then we have $f_k(x) = 0$ whenever $k > n$ for all x in V, i.e., $f_k = 0$ whenever $k > n$, i.e., $\{ f_n \}$ is eventually zero.

(7) $\rightarrow$ (3): Suppose (3) does not hold. Then we can write $V = \bigcup V_n$, where V_n are closed subvarieties and $V_n \subset V_{n+1}$ for $n = 1,\ldots$. Pick a morphic function $f_n \neq 0$ on V so that $f_n | V_n = 0$. Now, any x in V lies in all V_n for sufficiently large n , so $f_n(x) = 0$ for such n , i.e., the sequence $\{ f_n \}$ is point-wise eventually zero. But $\{ f_n \}$ is not eventually zero, for all f_n are

dimensional subspace of $A(V)$. Let B be the linear span of $\{ f_y : y$ from $S \}$, which is, a fortiori, a finite dimensional subspace of k^X . Apply lemma (3.1) to $f|V \times S : V \times V \to k$, and we can choose $\{ x_1,\ldots,x_n \}$ from V and a basis $\{ \xi_1,\ldots,\xi_n \}$ for B such that

$$f|V \times S = \Sigma \xi_i \otimes (f^{x_i}|S) ,$$

because $(f|V \times S)^{x_i} = f^{x_i}|S$. Let $g = \Sigma \xi_i \otimes f^{x_i}$. Then g is a morphic function on $V \times V$ (for ξ_i and f^{x_i} are morphic functions on V). We claim that $f = g$. Clearly $f^x|S = g^x|S$ by definition of g . But S is dense in V , so $f^x = g^x$ for all x in V . Hence $f(x,y) = f^x(y) = g^x(y) = g(x,y)$ for all (x,y) in $V \times V$, i.e., $f = g$ as claimed, which implies that f is also a morphic function on $V \times V$ since g is.

Next suppose that $f : V \times V \to k$ is a morphic function. Then we can put $f = \Sigma g_i \otimes h_i$, where g_i and h_i are elements of $A(V)$. Now let (x,y) be an element of $V \times V$. Then $f^x(y) = f(x,y) = \Sigma g_i(x)h_i(y)$, so $f^x = \Sigma g_i(x)h_i$, i.e., f^x lies in the span of h_i , hence the span of $\{ f^x : x$ from $V \}$ is finite dimensional subspace of $A(V)$. (We can argue similarly for the span of $\{ f_y : y$ from $V \}$.)

Palais [P, (9.3)] showed that every affine variety over an uncountable algebraically closed field is algebraically of the second category. In the following sequence of propositions we will generalize the result and others into almost affine varieties.

PROPOSITION 3.6. Let V be an irreducible almost affine variety over k such that $\dim_k A(V)$ (dimension over k of $A(V)$) is countable. Then V is algebraically of the second category if and only if every separately morphic function on $V \times V \to k$ is a morphic function.

PROOF: The proof requires only a slight modification of [P, (8.2) and (8.3)]. First suppose that V is algebraically of the second category, and that $f : V \times V \to k$ is a separately morphic function. By hypothesis we can write $A(V)$ as a countable increasing union of finite dimensional subspaces A_i of dimension i. Let S_i be the subset of V consisting of y such that $f_y : V \to k$ belongs to A_i. Then V is a countable increasing union of S_i, hence some S_i is dense in V because V is algebraically of the second category. So it follows from proposition (3.5) that f is a morphic function on $V \times V$.

Next suppose that V is not algebraically of the second category. We will show that there exists a separately morphic function $f : V \times V \to k$ which is not a morphic function. By definition we can find a sequence $\{ f_n \}$ of morphic functions on V which is point-wise eventually zero but is not eventually zero. By corollary (3.2) we can assume that the f_n are linearly independent. Then $f(x,y) = \Sigma f_n(x) f_n(y)$ $(n = 1, \ldots)$ is a well-defined separately morphic function on $V \times V$. For each positive integer d we will show that there exist $y_1,\ldots,y_d$ in V such that $f_{y_1},\ldots,f_{y_d}$ are linearly independent which by lemma (3.5) implies that f is not a morphic function on $V \times V$. Since $f_y = \Sigma f_n(y) f_n$ and the f_n are linearly independent in $A(V)$ it will suffice to choose $y_1,\ldots,y_d$ in V so that the sequence $(f_1(y_i),\ldots,f_n(y_i),\ldots)$ are linearly independent in k^∞ . In particular it will suffice to choose them so that $(f_1(y_i),\ldots,f_d(y_i))$ are basis for k^d . Let B be the d-dimensional subspace of $A(V)$ spanned by $f_1,\ldots,f_d$ and choose by lemma (3.1) $y_1,\ldots,$ y_d in V such that $\{ (y_1)^o,\ldots,(y_d)^o \}$ is a basis for B^* . If $\{ \psi_1,\ldots,\psi_d \}$ is the basis for B^* dual to $f_1,$ $\ldots,f_d$ then $(f_1(y_i),\ldots,f_d(y_i))$ is just the vector components of $(y_i)^o$ with respect to the basis $\{ \psi_1,\ldots,\psi_d \}$. This completes the proof of proposition (3.6).

<u>PROPOSITION 3.7</u>. Let V be an almost affine variety over k which cannot be decomposed into a union of $\dim_k A(V)$ or less proper closed closed subvarieties. Then every separately morphic function $f : V \times V \to k$ is a morphic function.

<u>PROOF</u>: We can always write $A(V) = \bigcup A_i$ where $\dim_k A_i$ is finite and the union is taken over I with $\mathrm{card}(I) =$ $\dim_k A(V)$. Let $S_i = \{ x$ in $V : f_x$ belongs to $A_i \}$. Then $V = \bigcup S_i$ where the union is again over I , for if x is in V then f_x is in $A(V)$ because f is a separa-

tely morphic function, so that f_x lies in some A_i . Hence x is in S_i , so $V = \bigcup S_i$ as required.

Now $V = \bigcup S_i \subseteq \bigcup Cl(S_i) \subseteq V$, so $V = \bigcup Cl(S_i)$. Then by assumption some $Cl(S_i)$ equals V . Thus we have found a dense subset $S = S_i$ of V such that for all x in S morphic functions f_x lie in a fixed finite dimensional subspace of $A(V)$, namely, A_i . So by lemma (3.5), f is a morphic function on $V \times V$.

<u>PROPOSITION 3.8</u>. Suppose that V is an irreducible affine variety over k such that the following conditions are all satisfied:

(i) If W is a closed subvariety of V of dimension $\dim V - 1$, then W is not an I-union (where card(I) is infinite) of proper closed subvarieties;

(ii) V itself ia an I-union of proper closed subvarieties.

Then V has at most card(I) distinct irreducible closed subvarieties of dimension $\dim V - 1$.

<u>PROOF</u>: By hypothesis (ii) we can put

$$V = \bigcup V_i ,$$

where the union is taken over I . We may assume that the varieties V_i are all irreducible.

Now suppose that Y is an irreducible closed subvariety of V of dimension $\dim V - 1$. Then

$$Y = \bigcup (Y \cap V_i) ,$$

where the union is taken over I . But then by hypothesis (i) $Y = Y \cap V_i$ for some i from I . So $Y \subseteq V_i$, hence $\dim Y \leq \dim V_i$. On the other hand, V_i is a proper closed subvariety of the irreducible variety V , so that $\dim V_i \leq \dim V - 1 = \dim Y$. Hence $\dim Y = \dim V_i$, so $Y = V_i$ because V_i is irreducible. This result means that there are at most card(I) distinct such Y . This completes the proof of proposition (3.8).

PROPOSITION 3.9. Let V be as in proposition (3.8), and let f be a polynomial function on V . Then the set $E = \{ \alpha$ in $k : \dim f^{-1}(\alpha) > \dim V - 2 \}$ has cardinality at most card(I) .

PROOF: If $\dim f^{-1}(\alpha) = \dim V$, then $f^{-1}(\alpha) = V$. So f is a constant function on V , and hence no $\beta \neq \alpha$ belongs to E . If $\dim f^{-1}(\alpha) = \dim V - 1$ then one of its irreducible components is one of card(I) distinct irreducible subvarieties of dimension $\dim V - 1$ by proposition (3.8) . Since $f^{-1}(\beta) \cap f^{-1}(\alpha) = \emptyset$ when $\beta \neq \alpha$, the set $S = \{ \alpha$ in $k : \dim f^{-1}(\alpha) = \dim V - 1 \}$ has cardinality at most card(I) . Hence $\text{card}(E) \leq \text{card}(S) + 1 \leq \text{card}(I)$, since card(I) is infinite. So the proof is now complete.

PROPOSITION 3.10. No affine variety V over k such that card(k) is uncountable can be written as a union of less than card(k) proper closed subvarieties.

PROOF: It is enough to consider the case when the given variety is irreducible. If there is an irreducible affine variety which can be decomposed as an I-union of proper closed subvarieties where card(I) < card(k) , pick one of the smallest dimension, say Y . So we have

$$Y = \bigcup Y_i ,$$

where the union is taken over I . As before we may assume that Y_i are all irreducible. If dim Y = 0 , then dim Y_i = 0 because $Y_i \neq \emptyset$, so Y_i is a point, so that card(Y) = card(I) which is infinite. On the other hand if dim Y = 0 then Y itself is a point, which is a contradiction since we just found that card(Y) is infinite. Hence dim Y must be at least 1 , so that there is a non-constant surjective polynomial function f on Y ; we can pick such a polynomial function because Y can be mapped surjectively onto an affine space k^n by the normalization lemma (because A(Y) is an integral extension of an affine polynomial ring $k[X_1,\ldots,X_n]$ where n = dim Y) and the composition of the surjection with one of the projection maps yields the function f we want. Since k is algebraically closed, dim $f^{-1}(\alpha)$ = dim Y - 1 for all α in k , so that $S = \{ \alpha$ in k : dim $f^{-1}(\alpha)$ = dim Y - 1 $\} = k$.

Now the variety Y satisfies all the hypotheses (i) and (ii) in proposition (3.8), so, by proposition (3.9), we have $\mathrm{card}(E) \leq \mathrm{card}(I)$, using the notation of the proposition. But we have just proved that $S = k$, so that $\mathrm{card}(I) \geq \mathrm{card}(k)$ (since $S \subset E$) which contradicts the hypothesis $\mathrm{card}(I) < \mathrm{card}(k)$. This completes the proof of proposition (3.10).

We remark that proposition (3.10) above says that every affine variety over an uncountable field is algebraically of the second category, which is [P, (9.3)] as stated before.

Now we extend proposition (3.10) to almost affine varieties.

PROPOSITION 3.11. No irreducible almost affine variety V over k (which is assumed to be uncountable) can be written as a union of less than $\mathrm{card}(k)$ proper closed subvarieties.

PROOF: $A(V)$ is an integral extension of an affine subalgebra of $A(V)$, say A_0. Let V_0 be the irreducible affine variety $\mathrm{Max}(A_0)$. Then there is a surjective morphism $\mu : V \to V_0$ which corresponds to the inclusion $A_0 \hookrightarrow A(V)$.

We claim that the morphism is a closed map. Let C be a closed set in V. Then there is an ideal J in $A(V)$ such that we can consider C as the set of all maximal ideals of $A(V)$ containing the ideal J. The map μ acts on the points in C by sending M to $M \cap A_0$, so evidently $\mu(C)$ contains the set of all maximal ideals of

A_o containing $J \cap A_o$. So we will show the opposite inclusion. Let m be a maximal ideal of A_o containing $J \cap A_o$. Then there corresponds to m a maximal ideal m' of $A_o/(A_o \cap J)$. Since A/J is integral over $A_o/(A_o \cap J)$ (by proposition (1.3)), there exists a maximal ideal of A/J lying over m' , which in turn implies that there is a maximal ideal M of $A(V)$ containing J so that M goes to m under μ . Hence $\mu(C)$ is the set of all maximal ideals of A_o containing $J \cap A_o$, so $\mu(C)$ is a closed set in V_o corresponding to the radical ideal $J \cap A_o$.

Now suppose to the contrary that V can be written as an I-union of proper closed subvarieties where card(I) is less than card(k) . Then $V_o = \mu(V) = \mu(\bigcup V_i) = \bigcup \mu(V_i)$. So we must have $\mu(V_i) = V_o$ for some i from I by our hypothesis that card(I) is less than card(k) because $\mu(V_i)$ are all closed subvarieties of the affine variety V_o (by proposition (3.10)).

Since $A(V_i) = A(V)/J$ where J is the ideal of morphic functions vanishing on V_i , it is integral over an affine algebra $A_o/(J \cap A_o)$, so that we get another surjective morphism from V_i onto an affine variety $W = \mathrm{Max}(A_o/(J \cap A_o))$. The commutative diagram of algebras;

$$\begin{tikzcd} A(V) \arrow[r] & A(V)/J \\ A_o \arrow[u, hook] \arrow[r] & A_o/(J \cap A_o) \arrow[u, hook] \end{tikzcd}$$

yields the commutative diagram of varieties;

$$\begin{tikzcd} V \arrow[d, "\mu"'] & V_i \arrow[l, hook'] \arrow[d] \arrow[dl, dashed, "\mu"'] \\ V_o & W \arrow[l, hook', "\sim"'] \end{tikzcd}$$

in which all morphisms in the lower triangle are now surjective. Then the lower horizontal morphism is an isomorphism by proposition (1.9) because it is surjective and the corresponding algebra homomorphism is also surjective, which means that $A_o \cong A_o/(A_o \cap J)$, hence $A_o \cap J = 0$.

We claim next that $J = 0$. Let f be an arbitrary morphic function in J . Consider the following commutative diagram of affine algebras:

$$\begin{tikzcd} A_o[f] \arrow[r] & A_o[f]/(J \cap A_o[f]) \\ A_o \arrow[u, hook] \arrow[r, "\sim"] & A_o/((J \cap A_o[f]) \cap A_o) \arrow[u, hook] \end{tikzcd}$$

Since $A_o \cap J = 0$, we must have $((J \cap A_o[f]) \cap A_o) = 0$, so that the lower horizontal homomorphism is an isomorphism. The corresponding commutative diagram of affine varieties is as follows;

$$\begin{tikzcd} Y \arrow[d] & Y_o \arrow[l, hook'] \arrow[d] \\ V_o & X_o \arrow[l, hook', "\sim"'] \end{tikzcd}$$

where $Y = \mathrm{Max}(A_o[f])$, $Y_o = \mathrm{Max}(A_o[f]/(J \cap A_o[f]))$, and

REMARK 3.13. If we assume that the ground field k is uncountable and the variety is affine in corollary (3.12) above, then we always have $\dim_k A(V) < \mathrm{card}(k)$ because $\dim_k A(V)$ is countable. So every separately polynomial function $V \times V \to k$ is a polynomial function, which is again [P, (9.3)].

Now, we will specialize corollary (3.12) and remark (3.13) to the group multiplication of a bi-algebraic group and show that we can be less stringent on cardinality assumption due to the bi-algebraic group structure.

Let G be a bi-algebraic group (affine or irreducible almost affine) and let f be a morphic function on G. Then we will denote by $G\cdot f$, $f\cdot G$, and $G\cdot f\cdot G$ the linear spans (which are subspaces of $A(G)$) of $\{ x\cdot f : x \text{ from } G \}$, $\{ f\cdot x : x \text{ from } G \}$, and $\{ x\cdot f\cdot y : x \text{ and } y \text{ are from } G \}$, respectively.

PROPOSITION 3.14. If G is a bi-algebraic group (affine or irreducible almost affine) over an uncountable field k, then the following are all equivalent:

(1) $\dim_k f\cdot G < \mathrm{card}(k)$ for every f in $A(G)$.

(2) $\dim_k G\cdot f$ = finite for every f in $A(G)$.

(3) $\dim_k G\cdot f < \mathrm{card}(k)$ for every f in $A(G)$.

(4) $\dim_k f\cdot G$ = finite for every f in $A(G)$.

(5) $\dim_k G\cdot f\cdot G$ = finite for every f in $A(G)$.

(6) $\dim_k G\cdot f\cdot G < \mathrm{card}(k)$ for every f in $A(G)$.

$y_j \cdot f$ are all in $A(G)$. Evaluating the function at x in V_F , we get

$$\begin{aligned}
&(y\cdot f - \Sigma g_i(y)(y_i\cdot f))(x) = \\
&f(xy) - \Sigma g_i(y) f(xy_i) = \\
&(f\cdot x)(y) - \Sigma (f\cdot x)(y_i) g_i(y) = \\
&(f\cdot x)(y) - \Sigma b_i(x) g_i(y) = \\
&(f\cdot x - \Sigma b_i(x) g_i)(y) = 0 ,
\end{aligned}$$

where the sum is taken over F , which means that the morphic function $y\cdot f - \Sigma g_i(y)(y_i\cdot f)$ vanishes on a dense subset V_F of G , so it is an identically zero function on G , that is,

$$y\cdot f = \Sigma g_i(y)(y_i\cdot f) ,$$

where the sum is taken over F . Hence the span $G\cdot f$ of the left translates of f , $\{ y\cdot f : y$ from $G \}$, has finite dimension $\leq \mathrm{card}(F)$ with a generator $\{ y_i\cdot f :$ i from $F \}$.

(2) → (3): Obvious.

(3) → (4): Modify the proof of the first implication; (1) → (2).

(4) → (1): Obvious.

(4) → (5): Clear because $y\cdot f\cdot x = y\cdot(f\cdot x) = (y\cdot f)\cdot x$, and (2) and (4) are equivalent.

(5) → (6): Obvious.

(6) → (1): Clear because $f\cdot x = (1\cdot f)\cdot x = 1\cdot f\cdot x$.

(4) $\to$ (7): Suppose that $\mu : G \times G \to G$ is a group multiplication. Let $\{ f_1,\ldots,f_n \}$ be a basis of $f\cdot G$ such that we can choose y_j from G with $f_i(y_j) = \delta_{ij}$ (by lemma (3.1)), so that, for all x in G, $f\cdot x = \Sigma a_i(x) f_i$, where $a_i : G \to k$ are some functions. Now, $a_j(x) = (f\cdot x)(y_j) = (y_j\cdot f)(x)$, so $y_j\cdot f = a_j$, which means that all a_j are in $A(G)$. But $(f\cdot x)(y) = \Sigma a_i(x) f_i(y)$, so $f(xy) = \Sigma a_i(x) f_i(y)$, that is, $(f\circ\mu)(x,y) = (\Sigma a_i \otimes f_i)(x,y)$, hence $f\circ\mu = \Sigma a_i \otimes f_i$ belongs to $A(G) \otimes A(G)$, which means that μ is a morphism.

(7) $\to$ (8): We divide our demonstration into several steps.

(i) $A(G)$ is a bi-stable subalgebra of the Hopf algebra of all representative functions on G .

We must show that for any f in $A(G)$ the linear span $f\cdot G$ of the set $\{ f\cdot x : x$ in $G \}$ is a finite dimensional subspace of $A(G)$. Since the group multiplication $\mu : G \times G \to G$ is a morphism, which makes the coordinate ring $A(G)$ a bi-stable bi-algebra with the corresponding comorphism as its comultiplication μ^* : $A(G) \to A(G) \otimes A(G)$. Let $\mu^* = \gamma$. If f is an element of $A(G)$, let $\gamma(f) = \Sigma g_i \otimes h_i$, $i = 1,\ldots,n$, where g_i are chosen to be linearly independent over k . Then $f\cdot x = \Sigma g_i(x) h_i$ and so $f\cdot G$ is a finite dimensional with a basis $\{ h_i \}$.

(ii) G is isomorphic to the group of all proper algebra automorphisms of $A(G)$ under the representation of G in the latter group.

We first recall some facts about proper algebra automorphisms. An algebra automorphism σ of $A(G)$ is <u>proper</u> if and only if $\gamma\circ\sigma = (1_A \otimes \sigma)\circ\gamma$. But in our case σ is proper if and only if it commutes with any right translation given by $g \to g\cdot x$. To prove this, look at the left hand side in the defining equation. If f is an element of $A(G)$, and x and y of G , then $\gamma(\sigma(f))(x,y) = \Sigma g_i(x)h_i(y)$ (where we set $\gamma(\sigma(f)) = \Sigma g_i \otimes h_i$) $= \sigma(f)(xy) = (\sigma(f)\cdot x)(y)$. Next consider the right hand side of the equation. $((1_A \otimes \sigma)\circ\gamma(f))(x,y) = \Sigma a_i(x)(\sigma(b_i))(y)$ (where we assume that $\gamma(f) = \Sigma a_i \otimes b_i$) $= (\sigma(\Sigma a_i(x)b_i))(y) = (\sigma(f\cdot x))(y)$. Since y is arbitrary, we obtain that σ is a proper automorphism if and only if $\sigma(f)\cdot x = \sigma(f\cdot x)$ for all f in $A(G)$ and x in G , i.e., $(L_x)^*\circ\sigma = \sigma\circ(L_x)^*$ for all x in G .

We now show that the representation of G in the group of all proper algebra automorphisms of $A(G)$: $x \to (1_A \otimes x)\circ\gamma$, is an isomorphism. We note that $(1_A \otimes x)\circ\gamma$ is nothing but $(R_x)^*$, for if f is from $A(G)$ and y from G , then $((1_A \otimes x)\circ\gamma(f))(y) = \Sigma g_i(y)h_i(x)$ (where we let $\gamma(f) = \Sigma g_i \otimes h_i$) $= f(yx) = (x\cdot f)(y)$, hence $(1_A \otimes x)\circ\gamma(f) = x\cdot f$. So it is immediate from this fact that $(1_A \otimes x)\circ\gamma$ is indeed a proper algebra automorphism $A(G)$ because $(R_x)^*\circ(L_y)^* = (L_y)^*\circ(R_x)^*$ for all x and y from G . The map is clearly injective. To show that it is also surjective, let σ be any proper algebra automorphism of $A(G)$. Then $\varepsilon\circ\sigma$ is an algebra homomorphism $A(G) \to k$ (where $\varepsilon : A(G) \to k$ is given by $f \to f(1)$) , so it is in G (now regarded as $\mathrm{Alg}_k(A(G),k)$) . If we show that $(1_A \otimes (\varepsilon\circ\sigma))\circ\gamma = \sigma$ we are done. Now let f be an element of $A(G)$ and x of G . Then $((1_A \otimes (\varepsilon\circ\sigma))\circ\gamma(f))(x) = \Sigma g_i(x)(\sigma(h_i))(1)$ (where we let $\gamma(f) = \Sigma g_i \otimes h_i$) $= \sigma(f\cdot x)(1)$ (by the same argument as we used before) $= (\sigma(f)\cdot x)(1)$ (because σ is proper) $= \sigma(f)(x)$. Hence we have that $1_A \otimes (\varepsilon\circ\sigma)\circ\gamma = \sigma$ as required.

(iii) G is isomorphic to some pro-affine algebraic group.

Let B be the smallest Hopf algebra containing the bi-algebra $A(G)$. Then $\mathrm{Alg}_k(B,k)$ is a pro-affine algebraic group.

Hochschild and Mostow proved that $\mathrm{Alg}_k(B,k)$ is isomorphic with the group of all proper algebra automorphisms of $A(G)$ [HM1, p, 1129]: They showed that every proper algebra automorphism σ of $A(G)$ can be mapped to a unique extension of $\varepsilon\circ\sigma$ (in $\mathrm{Alg}_k(A(G),k) = G$) to $\mathrm{Alg}_k(B,k)$. Hence every element of G can be regarded also as an element of $\mathrm{Alg}_k(B,k)$, so we must have $G = \mathrm{Alg}_k(A(G),k) = \mathrm{Alg}_k(B,k)$ as we wished. We remark that they were interested to show only that the group of all proper algebra automorphisms of $A(G)$ is isomorphic with

$Alg_k(B,k)$. In fact, in their context, G is not isomorphic with the group of all proper automorphisms of $A(G)$ because they did not assume that the pair $(G,A(G))$ is a pro-affine algebraic variety but that they only assumed that $A(G)$ is a bi-stable algebra of representative functions on G .

(8) → (9): Immediate from the proof of the above implication; (7) → (8), (iii).

(9) → (4): Nothing to prove; it is (i) in the implication; (7) → (8). This completes the proof of proposition (3.14)

We should remark that the crucial point in the above discussion is the first implication; $\dim_k f \cdot G < \text{card}(k)$ implies $\dim_k G \cdot f = \text{finite}$, in whose proof we used proposition (3.11). All others are consequences of it. Although proposition (3.14) supplies necessary and sufficient conditions for an affine bi-algebraic group (over an uncountable field) to be algebraic, we will obtain the same through different way by explicitly exhibiting a morphic representation of an affine bi-algebraic group (over any field) in some general linear group.

<u>PROPOSITION 3.15</u>. Every affine bi-algebraic group is an affine algebraic group if and only if $\dim_k G \cdot f \cdot G$ is finite.
<u>PROOF</u>: The only if part is clear. (See proposition (3.18) where an explicit proof is written out.) Now suppose that

$\dim_k G\cdot f\cdot G$ is finite and so $\dim_k G\cdot f$ is finite. We divide our proof into several steps.

(1) There is an injective homomorphism from G into a general linear group.

Write $A(G) = k[f_1,\ldots,f_n]$, and let V be the vector subspace over k of $A(G)$ spanned by the set of left translates of f_i ,

$$\{x\cdot f_i : x \text{ in } G \text{ and } i = 1,\ldots,n\} .$$

We know that V is finite dimensional because $\dim_k G\cdot f_i$ is finite for all $i = 1,\ldots,n$.

Note that V is stable under $(\rho_x)^* : A(G) \to A(G)$ which is given by $f \to x\cdot f$ for all x in G and that V generates $A(G)$ as an algebra because the generators f_i are in V . So we can define a map $\rho : G \to gl(V)$ (the algebra of linear endomorphisms of V) by $\rho(x) = (\rho_x)^*|V$ such that $\rho(G)$ lies in $GL(V)$. For (f,f') in $V \times V^*$ let $X_{f,f'}$ be the matrix coefficient defined by

$$X_{f,f'}(T) = f'(T(f))$$

for all T in $gl(V)$. Then $gl(V)$ is an affine algebraic variety with coordinate ring $A(gl(V)) = k[X_{f,f'} : (f,f') \text{ in } V \times V^*]$. Also let D be a polynomial function in $A(gl(V))$ given by $D(T) =$ determinant of T , and let $A(GL(V)) = (A(gl(V)))[1/D]$. Then the pair $(GL(V),A(GL(V)))$ is an affine algebraic group. (See [Bo] or [St] for more value zero, it must be a unit, so that $1/(D\circ\rho)$ is itself a polynomial function. Clearly $\rho : G \to GL(V)$ is injective, hence ρ is a morphic injection of G into the general linear group $GL(V)$.

(2) Every homomorphic image of an affine bi-algebraic group in an affine algebraic group is an affine algebraic group.

This is just proposition (2.13).

Now we see by (1) and (2) that the image $\rho(G)$ in $GL(V)$ is an affine algebraic group. In order to show that G is itself an affine algebraic group we must show that G is isomorphic to its image $\rho(G)$. For this we need the next and final step.

(3) The comorphism $\rho^* : A(GL(V)) \to A(G)$ is surjective.

Let $\{ g_i : i = 1,\dots,m \}$ be a basis of V $(\subset A(G))$ and let $\{ \xi_i : i = 1,\dots,m \}$ be the dual basis for V^* . It is enough to show that each basis element g_i has a pre-image in $A(GL(V))$. Now for all T in $GL(V)$ we can write

$$T(g_i) = \Sigma \xi_j(T(g_i))g_j \ .$$

Define a polynomial function in $A(GL(V))$ by setting

$$h_i = \Sigma g_j(1) X_{g_i,\xi_j} \ ,$$

where 1 is the identity element of G . Then for any T

in $GL(V)$

$$h_i(T) = \Sigma g_j(1) X_{g_i, \xi_j}(T) =$$
$$\Sigma g_j(1) \xi_j(T(g_i)) =$$
$$T(g_i)(1) \ .$$

But $\rho^*(h_i)(x) = (h_i \circ \rho)(x) = h_i(\rho(x)) = h_i((\rho_x)^*) = ((\rho_x)^*(g_i))(1) = (x \cdot g_i)(1) = g_i(x)$, i.e., $\rho^*(h_i) = g_i$. Thus $\rho^*(A(GL(V)))$ contains V , and hence equals $A(G)$ because V generates $A(G)$. Hence ρ^* is a surjective algebra homomorphism.

Now applying proposition (2.14) we get that $\rho : G \to \rho(G)$ is an isomorphism of an affine bi-algebraic group G with an affine algebraic group $\rho(G)$. This means that G itself must be an affine algebraic group. This completes the proof of proposition (3.15).

A use of proposition (3.14) in connection with proposition (3.15) is the fact that $\dim_k G \cdot f \cdot G < \mathrm{card}(k)$ implies $\dim_k G \cdot f \cdot G$ is finite, thus making the sufficient condition less stringent when the ground field is uncountable. From this observation we can obtain the following:

COROLLARY 3.16. Every affine bi-algebraic group over an uncountable field is an algebraic group.

PROOF: Clear because $\dim_k G \cdot f \cdot G \leq \dim_k A(G)$ which is always countable.

COROLLARY 3.17. An affine bi-algebra which is a coordinate rings of an affine bi-algebraic group over an uncountable field always possesses an antipode, i.e., is a Hopf algebra.

We learn form proposition (3.15) or corollary (3.16) that every proposition generalizing the properties of affine algebraic groups into those of representative bi-algebraic groups is just a tautology.

Next we will obtain the same necessary and sufficient condition for an irreducible almost affine bi-algebraic group (over any field) to be a pro-affine algebraic group as in prposition (3.14), reducing the almost affine case to the affine case (i.e., proposition (3.15)) and taking the projective limit. Of course the result is evident due to proposition (3.14) if the ground field is uncountable, but we are proving it directly from proposition (3.15).

PROPOSITION 3.18. An irreducible almost affine bi-algebraic group over k is a pro-affine algebraic group if and only if for every morphic function f in $A(G)$ the linear span $G\cdot f\cdot G$ of $\{ x\cdot f\cdot y : x$ and y from $G \}$ is finite dimensional.

PROOF: First suppose that G is an algebraic group. The consequence is clear in view of the argument used in proposition (3.14), but we prove it directly for completeness. Then the group multiplication $G \times G \to G$ induces the comultiplication $A(G) \to A(G) \otimes A(G)$ which makes $A(G)$ a Hopf algebra. Denote the comultiplication by γ , and let $\gamma(f) = \Sigma g_i \otimes h_i$ (where g_i and h_i are linearly independent) for a morphic function f in $A(G)$. Then $f(xy) = \Sigma g_i(x)h_i(y)$, so $(y\cdot f)(x) = \Sigma h_i(y)g_i(x)$. Hence

$$(1) \qquad y\cdot f = \Sigma h_i(y)g_i ,$$

which is in a finite dimensional subspace of $A(G)$ generated by $\{ g_1,\ldots,g_n \}$ $(\subset G\cdot f\cdot G$ because we can solve system (1) above for g_i in terms of $y_j\cdot f$ choosing y_j so that the matrix $(h_i(y_j))$ is non-singular). But $f(xzy) = \Sigma g_i(x)h_i(zy) = \Sigma g_i(x)(y\cdot h_i)(z)$, so $(x\cdot f\cdot y)(z) = \Sigma g_i(x)(y\cdot h_i)(z)$. Hence,

$$(2) \qquad x\cdot f\cdot y = \Sigma g_i(x)(y\cdot h_i)$$

which is again in a finite demensional subspace of $A(G)$ spanned by $\{ y\cdot h_i : y$ form G and $i = 1,\ldots,n \}$ $(\subset G\cdot f\cdot G)$ since $\{ y\cdot h_i : y$ from $G \}$ spans a finite dimensional subspace of $A(G)$ for each i by the result above. We have thus established that the span $G\cdot f\cdot G$ of $\{ x\cdot f\cdot y : x$ and y from $G \}$ has finite dimension.

Next suppose that $\dim_k G\cdot f\cdot G$ is finite for each morphic function f in $A(G)$. Let $B_f = A_o[G\cdot f\cdot G]$ be the subalgebra of $A(G)$ containing A_o (which is assumed to be the affine subalgebra over which $A(G)$ is integral) generated by $G\cdot f\cdot G$. Then the subalgebra B_f is an affine algebra because A_o is affine and $\dim_k G\cdot f\cdot G$ is finite.

Furthermore, B_f is bi-stable under translations by the elements of G , and is integral over A_o .

Let $H_f = \{ x$ in $G : g \cdot x = g$ for all g in $B_f \}$. Then H_f is a normal subgroup of G , and is closed because B_f is bi-stable and $H_f = \{ x$ in $G :$ $(y \cdot g - g(y))(x) = 0$ for all g in B_f and y in $G \}$ which is the annihilator in G of the ideal of $A(G)$ generated by $\{ y \cdot g - g(y) : g$ in B_f and y in $G \}$.

Now, let $G_f = G/H_f$ and define a map π : $G_f \to \mathrm{Max}(B_f)$ given by $xH_f \to m_x$ = the maximal ideal of morphic functions in B_f vanishing at x (which is the same as the intersection of the maximal ideal of morphic functions in $A(G)$ vanishing at x with B_f) . Then the map π is well-defined, and is surjective because $A(G)$ is integral over B_f .

We will show that π is bijective so that we can identify G_f with $\mathrm{Max}(B_f)$ as bi-algebraic groups with the same coordinate ring B_f . It only remains to show that π is injective. We will first prove that $m_x = m_y$ if and only if $m_{(x^{-1}y)} = m_1$. Now, if $m_x = m_y$ and f an element of $m_{(x^{-1}y)}$, then $0 = f(x^{-1}y) = (f \cdot x^{-1})(y)$, so $f \cdot x^{-1}$ lies in m_y and hence $f \cdot x^{-1}$ lies in m_x so that $0 = (f \cdot x^{-1})(x) = f(1)$, i.e., $m_{(x^{-1}y)} \subseteq m_1$. On the other hand if $m_x = m_y$ and f is in m_1 then $0 = f(1) =$ $f(x^{-1}x) = (f \cdot x^{-1})(x)$, so that $f \cdot x^{-1}$ lies in m_x ($= m_y$) , hence $0 = (f \cdot x^{-1})(y) = f(x^{-1}y)$, so f lies in $m_{(x^{-1}y)}$.

Thus $m_1 \subseteq m_{(x^{-1}y)}$. Hence $m_1 = m_{(x^{-1}y)}$. Conversely suppose that $m_{(x^{-1}y)} = m_1$. Then $f(x) = 0$ if and only if $(f\cdot x)(1) = 0$ if and only if $(f\cdot x)(x^{-1}y) = 0$ if and only if $f(y) = 0$. Hence $m_x = m_y$.

Now, to show that π is injective we must show that $m_x = m_y$ implies $x^{-1}y$ lies in H_f . But by the result we proved above it is enough to show that $m_x = m_1$ implies x lies in H_f . Recall that x is in H_f if and only if $(y\cdot g - g(y))(x) = 0$ for all g in B_f and y in G . Suppose $m_x = m_1$, and g is an element of B_f , y of G . Then $(y\cdot g - g(y))(1) = g(y) - g(y) = 0$, so $0 = (y\cdot g - g(y))(x)$. Hence x is in H_f .

Thus the pair (G_f, B_f) is an affine variety. Moreover, G_f acts on B_f morphically from right and left because G is a bi-algebraic group, so that G_f is itself an affine bi-algebraic group. But $\dim_k G_f\cdot g\cdot G_f$ is finite for all g in B_f by definition of B_f , hence G_f is an affine algebraic group by proposition (3.15).

Now, we have $A(G) = \text{dir lim } B_f$, hence $G = \text{proj lim } G_f$, which means that G is now a pro-affine algebraic group. This completes the proof of proposition (3.18).

We should note again that because of proposition (3.14) we can replace the condition $\dim_k G\cdot f\cdot G$ is finite by the less restrictive one $\dim_k G\cdot f\cdot G < \text{card}(k)$ or even by $\dim_k G\cdot f < \text{card}(k)$ as in the affine case (i.e., proposition (3.15)) if the ground field k is uncountable.

We established by proposition (3.14) that if the ground field is sufficiently large every bi-algebraic group (affine or irreducible almost affine) is an algebraic group.

Although it appears that an almost affine algebraic group whose coordinate ring has countable dimension as a vector space over k (when k is also countable) is affine, the following example shows it is not so.

<u>EXAMPLE 3.19</u>. Refer back to example (2.11(2)). Here we are assuming that the ground field k is countable in that setting. Then G is a countable almost affine algebraic group with $\dim_k A(G)$ being countable. But G is not affine.

In conclusion, we summarize what is obtained: First, the answers to (1) and (2) of the problems of Palais stated at the outset of the chapter are both in the affirmative if and only if $\dim_k G\cdot f\cdot G$ is finite (or $\dim_k G\cdot f < \mathrm{card}(k)$ when $\mathrm{card}(k)$ is uncountable) for every morphic function f on G; and secondly, (2) always follows from (1). However, there still remains the question whether or not $\dim_k G\cdot f\cdot G$ is finite (or $\dim_k G\cdot f < \mathrm{card}(k)$ when $\mathrm{card}(k)$ is uncountable) for any irreducible almost affine bi-algebraic group, although it is always the case for every affine bi-algebraic group due to the result in [M4].

BIBLIOGRAPHY

[BM] Bochner, S. and Martin, W., Several Complex Variables, Princeton University Press, Princeton, 1948.

[B] Borel, A., Linear Algebraic Groups, W. A. Benjamin, Inc., New York, 1969.

[F] Fogarty, J., Invariant Theory, W. A. Benjamin, Inc., New York, 1969.

[Hi] Higgins, P., Introduction to Topological Groups, London Math. Soc. Lecture Note Series 15, Cambridge University Press, Cambridge, 1974.

[H1] Hochschild, G., Introduction to Affine Algebraic Groups, Holden-Day, Inc., San Francisco, 1971.

[H2] _____________, Coverings of Pro-affine Algebraic Groups, Pacific J. Math. 35 (1970), 399-415.

[HM1] Hochschild, G. and Mostow, G., Pro-affine Algebraic Groups, Amer. J. Math. 91 (1969), 1127-1140.

[HM2] ____________________________, On the Algebra of Representative Functions of an Analytic Group. Amer. J. Math. 83 (1961), 111-136.

[K] Kaplansky, I., Commutative Rings, The University of Chicago Press, Chicago, 1970.

[M1] Magid, A., Left Algebraic Groups, J. of Algebra 35 (1975), 253-272.

[M2] ________, The Separable Galois Theory of Commutative Rings, Marcel Dekker, Inc., New York, 1974.

[M3] ________, Analytic Left Algebraic Groups, Amer. J. Math., to appear.

[M4] ________, Separately Algebraic Group Laws, Amer. J. Math., to appear.

[Mo] Montgomery, D., Continuity in Topological Groups, Bulletin Amer. Math. Soc. 42 (1936), 879-882.

[Mu] Mumford, D., Introduction to Algebraic Geometry, Preliminary Version of Chapters I-III, Dept. of Math., Harvard University, Cambridge.

[P] Palais, R., Some Analogues of Hartogs' Theorem in an Algebraic Setting, Amer. J. Math., to appear.

[St] Steinberg, R., Conjugacy Classes of Algebraic Groups, Lecture Notes in Math. 366, Springer-Verlag, Berlin, 1974.

[Sw] Sweedler, M., Hopf Algebras, W. A. Benjamin, Inc., New York, 1969.