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COMBINED CONDUCTION AND RADIATION IN AN ABSORBING, EMITTING AND SCATTERING CYLINDRICAL MEDIUM USING THE FINITE ELEMENT METHOD

The University of Oklahoma PH.D. 1981

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THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

COMBINED CONDUCTION AND RADIATION IN AN ABSORBING, EMITTING AND SCATTERING CYLINDRICAL MEDIUM USING THE FINITE ELEMENT METHOD

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

ROLLIN FERNANDES

Norman, Oklahoma

1981

COMBINED CONDUCTION AND RADIATION IN AN

ABSORBING, EMITTING AND SCATTERING

CYLINDRICAL MEDIUM USING THE

FINITE ELEMENT METHOD

A DISSERTATION

APPROVED FOR THE SCHOOL OF AEROSPACE, MECHANICAL

AND NUCLEAR ENGINEERING

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ACKNOWLEDGEMENTS

I wish to express my sincere appreciation and gratitude to Dr. John Francis who served as director of this dissertation. His constant willingness to assist and the many long hours that he did, in fact, devote were of immense value in the completion of this dissertation.

I am honored that Dr. Tom Love has taken time to be on my committee. His knowledge and expertise is appreciated.

I also wish to thank Dr. Egle, Prof. Turkington and Dr. Eliason for consenting to serve on my committee.

Finally, I wish to express thanks to my wife, Bertha, and my family, especially my parents, for their support.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

NOMENCLATURE

C_p	specific heat at constant pressure
G	incident radiation or irradiation
I	radiative intensity
k	thermal conductivity
n	refractive index
N	conduction radiation parameter
q	radiative heat flux
q"	conductive plus radiative heat flux
r	radius
r_o	inner radius
R	outer radius
RAT	positive real number less than 1
s	coordinate along the ray
t	time
T	temperature
T_{oven}	temperature at the outside radius
T_{wire}	temperature at the inner radius
x	coordinate along the horizontal projection of the ray
α	thermal diffusivity and angle between a ray and the horizontal
β	extinction coefficient and angle
γ	local azimuthal angle

ε	emissivity
θ	outward drawn normal to the cylindrical surface
θ'	polar angle
κ	absorption coefficient
ν	frequency
ρ	density
σ	scattering coefficient
σ_s	Stefan-Boltzmann constant
τ	optical distance
Φ, ϕ	radiation potential
Ψ	azimuthal angle
ω	solid angle
$\omega_o = \frac{\sigma}{\beta}$	albedo

Nondimensionalized variables

$\eta =$	$\frac{G}{n^2 \sigma_s T^4_{reference}}$
$Q_r =$	$\frac{q}{n^2 \sigma_s T^4_{reference}}$
$N =$	$\frac{k\beta}{4n^2 \sigma_s T^4_{reference}}$
$\Psi =$	interpolating function and $\frac{q''}{k\beta T_{reference}}$
$\Phi, \phi =$	$\frac{\pi \ I}{n^2 \sigma_s T^4_{ref}}$
$r^* =$	βr
$r^*_o =$	βr_o

$R^* = \beta R$

$t^* = \dfrac{k\beta^2 t}{\rho C_p}$

$T^* = \dfrac{T}{T_{reference}}$

$x^* = \beta x$

$\tau = R^* - r_o^*$

Subscripts

ν	monochromatic
b	black body
e	element
r	radiative

Superscripts

*	dimensionless
1,2	node index

CHAPTER I

INTRODUCTION

With the advent of the recent energy crisis and rapidly spiraling energy prices, conservation of energy has become a primary concern. One of the many ways to reduce consumption of energy is the use of insulation to reduce heat transfer. The proper use of insulation materials requires an understanding of the properties of thermal insulation particularly, the value of thermal conductivity. However, the determination of conductivity of insulating materials is rendered difficult because of the combination of conduction, convection and radiation in the process of heat transfer. The combination of these three modes vastly increases the complexity of the analysis of any system, and, consequently, makes the modeling of the experimental procedure for the determination of thermal conductivity much more difficult.

Several methods for the measuring of thermal conductivity have been established, of which the "guarded hot-plate" method is the most common. The control system in the procedure is very sophisticated, and the data are taken under steady state conditions which require a long time. Moreover, to insure one dimensionality, the system must be fairly large in size. Besides, thermal conductivity measured by this method will not be accurate if semi-transparent materials are used. Only by using a combined conduction and radiation heat transfer model can the true values of thermal conductivity be found. Other geometries, e.g., spherical and cylindrical, have similar problems to the guarded hot-plate method.

Another method of determining thermal conductivity is the so-called "hot-wire method." Since it is a transient method, data can be obtained relatively quickly. Also the system can be constructed relatively small. Numerous investigators have studied the hot-wire method both experimentally and analytically [1,2,3]. A recent summary of the state of the art for the hot-wire thermal conductivity method was reported by Davis [4], and Shil'pran [5], comparing it with other methods.

Briefly, the hot-wire method [6,7,8] consists of embedding a small wire in the center of a rectangular parallelopiped of insulating material and applying a step flux to the wire. The wire temperature is then monitored, and from the temperature versus time relationship at large time, the thermal conductivity is determined. Thus the geometry of the hot-wire method is cylindrical.

All the studies mentioned earlier consider heat transfer by conduction only. Often other modes of heat transfer are also present. When values for thermal conductivity are required in semi-transparent insulating materials, the effect of heat transfer by radiation is also important. For many insulating materials the heat-transfer by convection is however, negligible. As a first step toward understanding combined conduction and radiation in a cylindrical medium with a given flux on one boundary (i.e. the hot-wire problem), the problem of combined conduction and radiation in a cylindrical medium with known temperatures on the boundaries should be investigated.

Combined conduction and radiation in a medium has been analyzed by numerous authors using various geometrics. Viskanta and Merriam [9] studied the problem for a spherical geometry using various boundary conditions. Fernandes, Francis and Reddy [10] , as well as Viskanta [11] have been among the many authors who have studied the problem of combined conduction and radiation for the planar geometry. Chang and Smith [12] have studied the transient and steady state problem using the Eddington approximation for a medium between coaxial cylinders, while Howell [13] solved the steady state problem using exchange factors. Saito et al. [18] investigated the hot wire method analytically and experimentally with combined conductive and radiative heat transfer. A number of solutions [14-17] were obtained considering radiative equilibrium for the cylindrical geometry. Of these only the solutions obtained by Perlmutter and Howell [17] can be considered exact.

The purpose of this study is to solve the problem of combined conduction and radiation in an absorbing, emitting and scattering medium of cylindrical geometry, and with known temperatures on the boundaries.

CHAPTER II

FORMULATION

The problem will be formulated using the procedure of Kestin [21] or Kuznetsov [22]. For the cylindrical geometry infinitely long in the z direction, the intensity I depends on the radius r, polar angle θ' and the azimuthal angle γ. The following notation will be used to specify the angles shown in figure 1:

$\beta \equiv$ horizontal projection of the acute angle between the direction of intensity and a line from the point of entry into the medium to the center of the wire

$\gamma \equiv$ horizontal projection of the angle measured clockwise from the radius to the direction of the intensity

$\theta \equiv$ angle between the direction of the intensity and the outward drawn normal to the surface

$\alpha \equiv$ angle between the ray and the horizontal

$\theta' \equiv$ polar angle

From figure 2

Upper Region $\qquad \theta' = \frac{\pi}{2} - \alpha$

$$d\omega = \sin\theta' \, d\theta' \, d\alpha$$

$$= - \cos\alpha d\alpha d\gamma$$

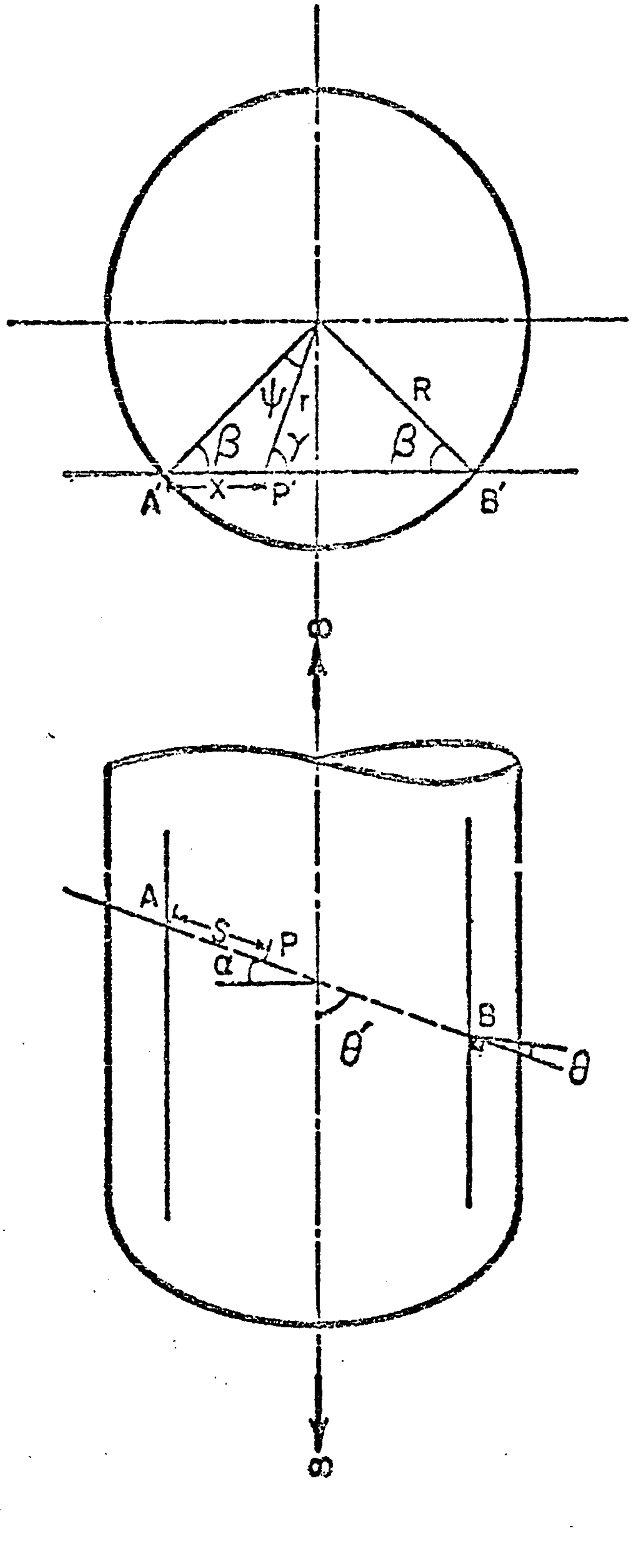

Figure 1. Coordinate System for Cylindrical Geometry.

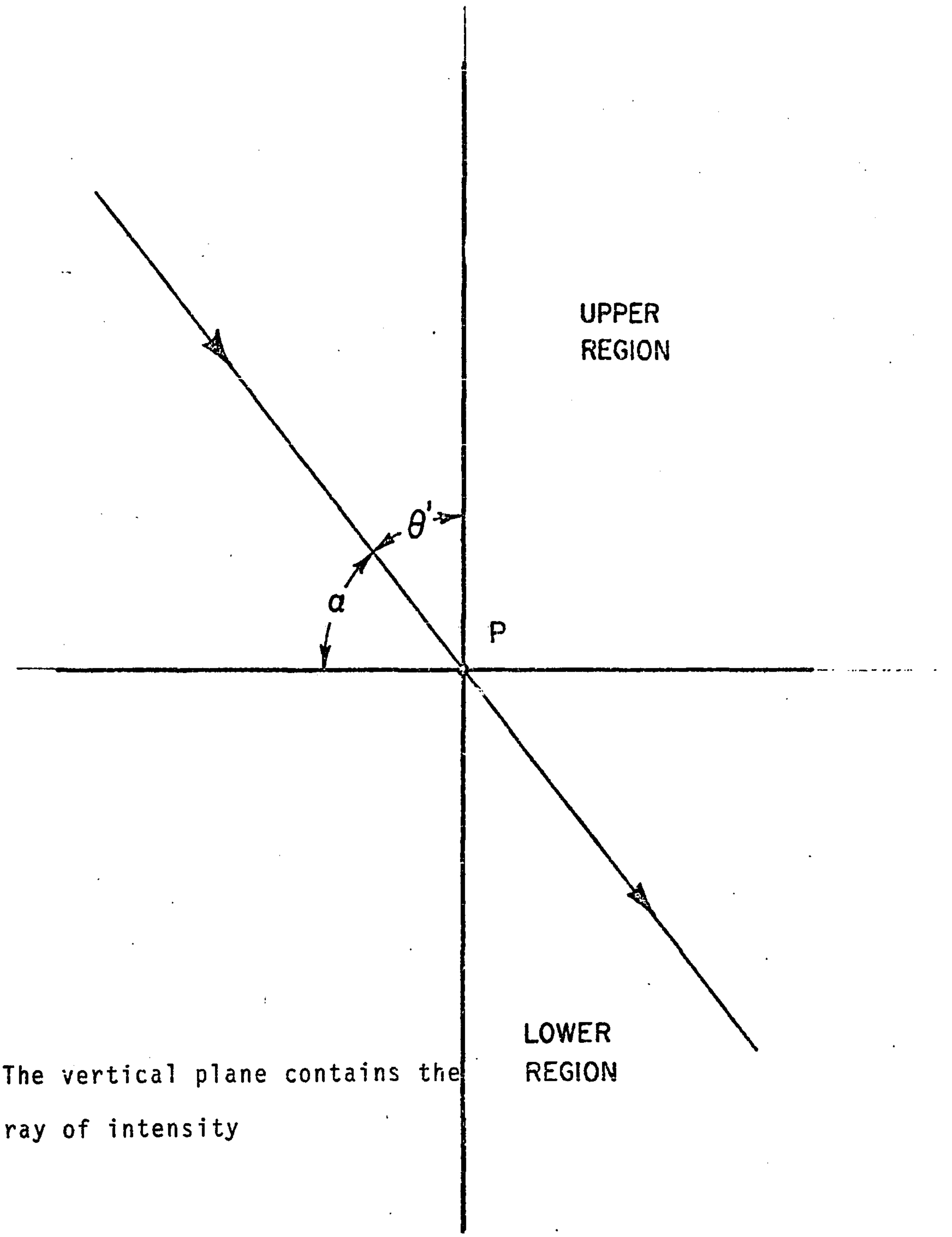

Figure 2. Upper and Lower Regions

As θ' varies from 0 to $\frac{\pi}{2}$, α varies from $\frac{\pi}{2}$ to 0

<u>Lower Region</u> $\theta' = \frac{\pi}{2} + \alpha$

$$d\omega = \sin\theta' \, d\theta' \, d\gamma$$

$$= \cos\alpha d\alpha d\gamma$$

As θ' varies from $\frac{\pi}{2}$ to π, α varies from 0 to $\frac{\pi}{2}$

The irradiation $G_\nu(r) = \int_\omega I \, d\omega$

$$= -\int_{\frac{\pi}{2}}^{0}\int_{0}^{2\pi} I_\nu \cos\alpha d\gamma d\alpha + \int_{0}^{\pi/2}\int_{0}^{2\pi} I_\nu \cos\alpha d\gamma d\alpha$$

$$= 2\int_{0}^{\pi/2}\int_{0}^{2\pi} I_\nu \cos\alpha d\gamma d\alpha \qquad (1)$$

From Figure 3 it can be shown that $\cos\theta'' = \cos\alpha' \cos\gamma'$. (2)

By dividing the annulus into four regions as in Figure 4 and using equation (2) we have in:

region I $\cos(\pi-\theta) = \cos\alpha \cos\gamma \quad 0 \leq \gamma < \frac{\pi}{2}$

region II $\cos\theta = \cos\alpha \cos(\pi-\gamma) \quad \frac{\pi}{2} \leq \gamma < \pi$

region III $\cos\theta = \cos\alpha \cos(\gamma-\pi) \quad \pi \leq \gamma < \frac{3\pi}{2}$

region IV $\cos(\pi-\theta) = \cos\alpha \cos(2\pi-\gamma) \quad \frac{3\pi}{2} \leq \gamma < 2\pi$

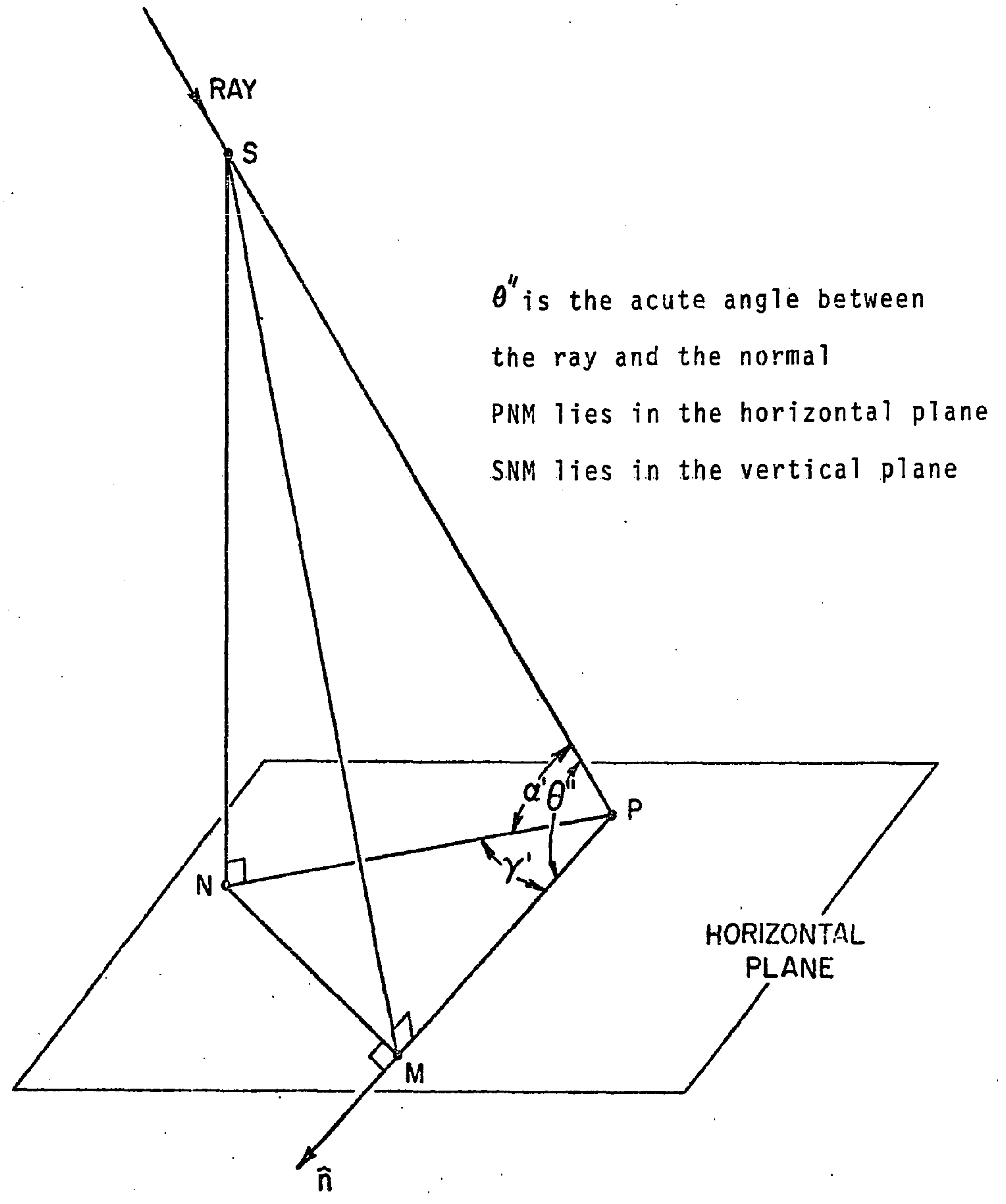

Figure 3. Angles

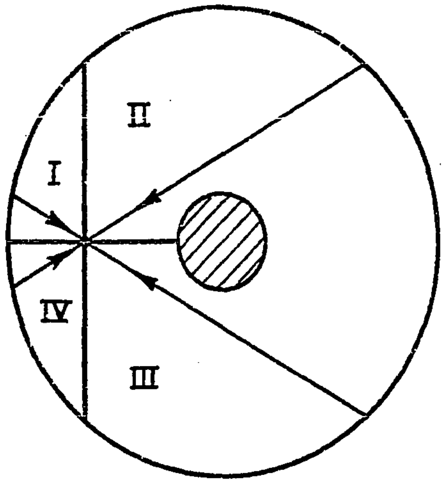

Figure 4. Regions

Thus in each region $\cos\theta = -\cos\alpha\,\cos\gamma$ (2a)

The net radiative flux directed radially outward is given by

$$q_\nu(r) = \int_\omega I_\nu \cos\theta \, d\omega$$

$$= \int_{\frac{\pi}{2}}^{0}\int_0^{2\pi} I_\nu \cos^2\alpha\cos\gamma\, d\gamma d\alpha - \int_0^{\pi/2}\int_0^{2\pi} I_\nu\cos^2\alpha\,\cos\gamma d\gamma d\alpha$$

$$= -2\int_0^{\frac{\pi}{2}}\int_0^{2\pi} I_\nu \cos^2\alpha\,\cos\gamma\, d\gamma d\alpha \qquad (3)$$

The transport equation using the coordinates shown in Figure 1 is given as

$$\frac{dI_\nu}{ds} = -\beta_\nu I_\nu + \kappa_\nu \dot{I}_\nu + \frac{\sigma_\nu}{4\pi} G_\nu \qquad (4)$$

where $\beta_\nu = \sigma_\nu + \kappa_\nu$

= extinction coefficient

$\sigma_\nu \equiv$ scattering coefficient

$\kappa_\nu \equiv$ absorption coefficient and $G_\nu \equiv$ irradiation or the incident intensity defined in equation (1)

For a medium with refractive index n we have from equation (4)

$$\frac{dI_\nu}{ds} = -\beta_\nu I_\nu + \kappa_\nu n^2 I_{b\nu} + \frac{\sigma_\nu}{4\pi} G_\nu \tag{5}$$

where $I_{b\nu}$ is the black body intensity or Planck's function

From Figure 1 it is seen that $ds = \frac{dx}{\cos\alpha}$ (6)

and $r^2 = x^2 + R^2 - 2xR\cos\beta$ (7)

Along a given ray β is constant and by taking the derivative of equation (7) we obtain

$$rdr = dx\,(x - R\cos\beta) \tag{8}$$

From Figure 1 we have

$$\cos\gamma = \frac{R\cos\beta - x}{r} \qquad 0 \leq \gamma < \frac{\pi}{2}$$

$$\cos(\pi-\gamma) = \frac{x - R\cos\beta}{r} \qquad \frac{\pi}{2} \leq \gamma < \pi$$

Using the above expressions along with equation (8) we obtain the result

$$\cos\gamma = -\frac{dr}{dx} \qquad 0 \leq \gamma < \pi \tag{9}$$

Noting from Figure 1 that $\frac{dx}{ds} = \cos\alpha$ and using equation (9), equation (5) becomes

$$\frac{dI_\nu}{dr} - \frac{\beta_\nu I_\nu}{\cos\gamma\cos\alpha} = \frac{-\kappa_\nu n^2 I_{b\nu}}{\cos\gamma\cos\alpha} - \frac{\sigma_\nu G_\nu}{4\pi\cos\gamma\cos\alpha} \tag{10}$$

By denoting I_ν^- as the intensity in the range $0 \leq \gamma < \frac{\pi}{2}$

and I_ν^+ as the intensity in the range $\frac{\pi}{2} \leq \gamma < \pi$, and further by expressing equation (10) for the range of γ between 0 and $\frac{\pi}{2}$ only we obtain the following transport equations

$$\frac{dI_\nu^-}{dr} - \frac{\beta_\nu \quad I_\nu^-}{\cos\gamma\cos\alpha} = \frac{-\kappa_\nu n^2 I_{b\nu}}{\cos\gamma\cos\alpha} - \frac{\sigma_\nu \; G_\nu}{4\pi\cos\gamma\cos\alpha} \quad 0 \leq \gamma < \frac{\pi}{2} \tag{11}$$

and

$$\frac{dI_\nu^+}{dr} + \frac{\beta_\nu I_\nu^+}{\cos\gamma\cos\alpha} = \frac{\kappa_\nu n^2 \quad I_{b\nu}}{\cos\gamma\cos\alpha} + \frac{\sigma_\nu G_\nu}{4\pi\cos\gamma\cos\alpha} \quad 0 \leq \gamma < \frac{\pi}{2} \tag{12}$$

It should be noted here that the directions I^+ and I^- are applicable only to a point located at some radius r. Using the procedure of Kesten [21] we will now show that equation (11) is applicable of $0 \leq x < 2R\cos\beta$ and equation (12) is applicable for $R\cos\beta < x \leq 2R\cos\beta$

From equation (7) we can show that

$$x - R\cos\beta = \pm\{r^2 - R^2\sin^2\beta\}^{\frac{1}{2}}$$

For $x>R\cos\beta$, $x - R\cos\beta = +\{r^2 - R^2\sin^2\beta\}^{\frac{1}{2}}$

and for $x < R\cos\beta$, $x - R\cos\beta = -\{r^2 - R^2\sin^2\beta\}^{\frac{1}{2}}$

Thus equation (8) becomes

$$dx = -\frac{r\,dr}{\{r^2 - R^2\sin^2\beta\}^{\frac{1}{2}}} \quad \text{for } 0 \leq x < R\cos\beta$$

and $$dx = \frac{r\,dr}{\{r^2 - R^2\sin^2\beta\}^{\frac{1}{2}}} \quad \text{for } R\cos\beta < x \leq 2R\cos\beta$$

Using the above relationships together with equation (6) we obtain equation (11) valid for $0 \leq x < R\cos\beta$, and equation (12) for $R\cos\beta < x \leq 2R\cos\beta$

Relationship between γ and β

From Figure 1 it is seen that

$$r \sin \gamma = R \sin \beta \text{ for } 0 \leq \gamma < \frac{\pi}{2}$$

$$r \sin (\pi - \gamma) = R \sin\beta \quad \text{for } \frac{\pi}{2} \leq \gamma < \pi$$

$$r \sin (\gamma - \pi) = R \sin\beta \quad \text{for } \leq \gamma < \frac{3\pi}{2}$$

and $$r \sin (2\pi - \gamma) = R \sin\beta \quad \text{for } \frac{3\pi}{2} \leq \gamma < 2\pi$$

Thus $$r \sin \gamma = R \sin\beta \quad \text{for } 0 \leq \gamma < \pi \tag{13}$$

and $$r \sin \gamma = - R \sin\beta \quad \text{for } \pi \leq \gamma < 2\pi$$

Using equation (13) we have $\cos\gamma = \{1 - \sin^2\gamma\}^{\frac{1}{2}}$

$$\{1 - (\frac{R}{r}\sin\beta)^2\}^{\frac{1}{2}} \tag{14}$$

Since $\cos\gamma$ is positive in the range $0 \leq \gamma < \frac{\pi}{2}$ we take the positive square root in equation (14)

Substituting equation (14) into equations (11) and (12) we have

$$\frac{dI_\nu^-}{dr} - \frac{\beta_\nu r \; I_\nu^-}{\{r^2 - R^2\sin^2\beta\}^{\frac{1}{2}}\cos\alpha} = \frac{-\kappa_\nu \; rn^2 \; I_{b\nu}}{\{r^2 - R^2\sin^2\beta\}^{\frac{1}{2}}\cos\alpha} - \frac{\sigma_\nu \; r \; G_\nu}{4\pi \; \{r^2 - R^2\sin^2\beta\}^{\frac{1}{2}}\cos\alpha} \tag{15}$$

$$\frac{dI_\nu^+}{dr} + \frac{\beta_\nu r \; I_\nu^+}{\{r^2 - R^2\sin^2\beta\}^{\frac{1}{2}}\cos\alpha} = \frac{\kappa_\nu r \; n^2 I_{b\nu}}{\{r^2 - R^2\sin^2\beta\}^{\frac{1}{2}}\cos\alpha}$$

$$+ \frac{\sigma_\nu \; r \; G_\nu}{4\pi\{r^2 - R^2\sin^2\beta\}^{\frac{1}{2}}\cos\alpha} \tag{16}$$

Equations (15) and (16) are applicable for $0 \leq \gamma < \frac{\pi}{2}$

Using the procedure of Gordaninejad [19] and Putcha [20] we divide the annular region, Figure 5, into one in which the rays intercept the inner cylinder, subscripted 1, and a region in which rays do not intercept the inner cylinder, subscripted 2.

Setting $r^* = \beta r$, $R^* = \beta R^*$, $F(r^*,\beta) = \{r^{*2} - R^{*2}\sin^2\beta\}^{\frac{1}{2}}$,

$\omega_0 = \frac{\sigma_\nu}{\beta_\nu}$, $1 - \omega_0 = \frac{\kappa_\nu}{\beta_\nu}$ and assuming gray and temperature

independent properties, we can rewrite equations (15) and (16) as

$$\frac{dI^-}{dr^*} - \frac{r^*}{F(r^*,\beta)\cos\alpha} I^- = -\frac{(1-\omega_0)\, r^*\, n^2 I_b}{F(r^*,\beta)\cos\alpha} - \frac{\omega_0\, r^*\, G}{4\pi F(r^*,\beta)\cos\alpha} \qquad (17)$$

and

$$\frac{dI^+}{dr^*} + \frac{r^*}{F(r^*,\beta)\cos\alpha} I^+ = \frac{(1-\omega_0)\, r^*\, n^2 I_b}{F(r^*,\beta)\cos\alpha} + \frac{\omega_0\, r^*\, G}{4\pi\, F(r^*,\beta)\cos\alpha} \qquad (18)$$

Using integrating factors we obtain from equation (17)

$$I_1^- e^{-\int \frac{r^*}{F(r^*,\beta)\cos\alpha} dr^*}\Bigg|_{R^*}^{r^*} = -\int_{R^*}^{r^*} \frac{\{(1-\omega_0) n^2 I_b(r') + \frac{\omega_0}{4\pi} G(r')\}\, r'}{F(r',\beta)\cos\alpha}\, e^{-\int \frac{r'\, dr'}{F(r',\beta)\cos\alpha}}\, dr'$$

$$\text{for } 0 \leq \beta < \sin^{-1}\frac{r_o^*}{R^*} \qquad (19)$$

Figure 5. Regions for Different Intensities.

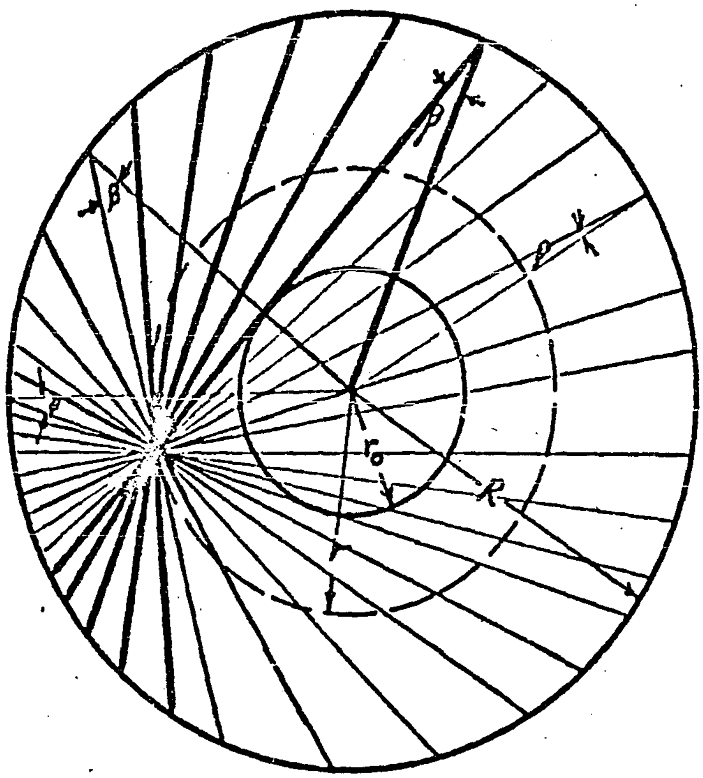

Table 1

Limits for the Regions

	γ	β	χ	I
1	$\sin^{-1}\frac{r_0}{r} > \gamma \ge 0$	$\sin^{-1}\frac{r_0}{R} > \beta \ge 0$	$R\cos\beta > \chi \ge 0$	I_1^-
2	$\frac{\pi}{2} > \gamma \ge \sin^{-1}\frac{r_0}{r}$	$\sin^{-1}\frac{r}{R} > \beta \ge \sin^{-1}\frac{r_0}{R}$	$R\cos\beta > \chi \ge 0$	I_2^-
3	$\pi - \sin^{-1}\frac{r_0}{r} \ge \gamma > \frac{\pi}{2}$	$\sin^{-1}\frac{r}{R} > \beta \ge \sin^{-1}\frac{r_0}{R}$	$2R\cos\beta \ge \chi > R\cos\beta$	I_2^+
4	$\pi \ge \gamma > \pi - \sin^{-1}\frac{r_0}{r}$	$\sin^{-1}\frac{r_0}{R} > \beta \ge 0$	$2R\cos\beta \ge \chi > R\cos\beta$	I_1^+

Since $\int \frac{r}{\{r^2-a^2\}^{\frac{1}{2}}} dr = \{r^2-a^2\}^{\frac{1}{2}}$ for a>o, and since $R^* \sin\beta > 0$

equation (19) can be rewritten as

$$I_1^-(r^*,\alpha,\beta)e^{-\frac{F(r^*,\beta)}{\cos\alpha}} - I^-(R^*)e^{\frac{-R^*\cos\beta}{\cos\alpha}}$$

$$= -\int_{R^*}^{r^*} \frac{\{(1-\omega_0)n^2 I_b(r')+\frac{\omega_0}{4\pi}G(r')\}\, r' e^{-\frac{F(r',\beta)}{\cos\alpha}}}{F(r',\beta)\ \cos\alpha}\, dr'$$

where at $r^*=r^*$ $I_1^-=I_1^-\ (r^*,\alpha,\beta)$

and at $r^*=R^*$ $I_1^-=I^-\ (R^*)$

$$I_1^-(r^*,\alpha,\beta) = e^{\frac{F(r^*,\beta)}{\cos\alpha}}\Big\{ I^-(R^*)\ e^{\frac{-R^*\cos\beta}{\cos\alpha}} + \int_{r^*}^{R^*} \frac{\{(1-\omega_0)n^2 I_b(r')+\frac{\omega_0}{4\pi}G(r')\}r'}{F(r',\beta)\ \cos\alpha}\, e^{-\frac{F(r',\beta)}{\cos\alpha}}\, dr' \Big\}$$

for $0 \leq \beta < \sin^{-1} \frac{r_o}{R}$ (20)

By a similar argument it is seen that

$$I_2^-(r^*,\alpha,\beta) = e^{\frac{F(r^*,\beta)}{\cos\alpha}}\Big\{ I^-(R^*)\ e^{\frac{-R^*\cos\beta}{\cos\alpha}} + \int_{r^*}^{R^*} \frac{\{(1-\omega_0)n^2 I_b(r')+\frac{\omega_0}{4\pi}G(r')\}r'}{F(r',\beta)\ \cos\alpha}\, e^{-\frac{F(r',\beta)}{\cos\alpha}}\, dr' \Big\}$$

for $\sin^{-1}\frac{r_o}{R} \le \beta < \sin^{-1}\frac{r}{R}$ (21)

Using integrating factors on equation (18) we obtain

$$I_1^+ \, e^{\int \frac{r^*}{F(r^*,\beta)\cos\alpha}dr^*}\Bigg|_{r_0^*}^{r^*} = \int_{r_0^*}^{r^*} \frac{\{(1-\omega_0)n^2 I_b(r') + \frac{\omega_0}{4\pi}G(r')\}r'}{F(r',\beta)\cos\alpha} \, e^{\int \frac{r'}{F(r',\beta)} \frac{dr'}{\cos\alpha}} \, dr'$$

for $0 \le \beta < \sin^{-1}\frac{r_0^*}{R^*}$

Using the boundary conditions

at $r^* = r^*$ $\quad I_1^+ = I_1^+(r^*,\alpha,\beta)$

and at $r^* = r_0^*$ $\quad I_1^+ = I_1^+(r_0^*)$

the above equation can be written as

$$I_1^+(r^*,\alpha,\beta) = e^{\frac{-F(r^*,\beta)}{\cos\alpha}} \Big\{ I_1^+(r_0^*)\, e^{\frac{F(r_0^*,\beta)}{\cos\alpha}} + \int_{r_0^*}^{r^*} \frac{\{(1-\omega_0)n^2 I_b(r') + \frac{\omega_0}{4\pi}G(r')\}r' e^{\frac{F(r',\beta)}{\cos\alpha}}}{F(r',\beta)\cos\alpha} dr' \Big\}$$

for $0 \le \beta < \sin^{-1}\frac{r_0^*}{R^*}$ (22)

To solve for I_2^+ we recognize that I^- is valid for $0 \le x < R\cos\beta$ and I^+ is valid for $R\cos\beta < x \le 2R\cos\beta$. Thus at $x=R\cos\beta$, i.e., $r^* = R^* \sin\beta$,

$$I_2^+(R^* \sin\beta) = I_2^-(R^* \sin\beta)$$

Using integrating factors on equation (18) we can show that

$$I_2^+(r^*,\alpha,\beta)\, e^{\frac{F(r^*,\beta)}{\cos\alpha}} - I_2^+(R^*\sin\beta) = \int_{R^*\sin\beta}^{r^*} \frac{\{(1-\omega_0)n^2 I_b(r') + \frac{\omega_0}{4\pi}G(r')\}r'\, e^{\frac{F(r',\beta)}{\cos\alpha}}}{F(r',\beta)\cos\alpha}\, dr' \tag{23}$$

for $\sin^{-1}\frac{r_o^*}{R^*} \le \beta < \sin^{-1}\frac{r^*}{R^*}$

From equation (21) it is seen that

$$I_2^+(R^* \sin\beta) = I_2^-(R^* \sin\beta) = I^-(R^*)\, e^{-\frac{R^*\cos\beta}{\cos\alpha}}$$

$$+ \int_{R^*\sin\beta}^{R^*} \frac{\{(1-\omega_0)n^2 I_b(r') + \frac{\omega_0}{4\pi}G(r')\}r'\, e^{-\frac{F(r',\beta)}{\cos\alpha}}}{F(r',\beta)\cos\alpha}\, dr'$$

Thus equation (23) becomes

$$I_2^+(r^*,\alpha,\beta) = e^{-\frac{F(r^*,\beta)}{\cos\alpha}} \Big\{ I^-(R^*)\, e^{-\frac{R^*\cos\beta}{\cos\alpha}}$$

$$+ \int_{R\sin\beta}^{R^*} \frac{\{(1-\omega_0)n^2 I_b(r') + \frac{\omega_0}{4\pi}G(r')\}r'\, e^{-\frac{F(r',\beta)}{\cos\alpha}}}{F(r',\beta)\cos\alpha}\, dr'$$

$$+ \int_{R^*\sin\beta}^{r^*} \frac{\{(1-\omega_0)n^2 I_b(r') + \frac{\omega_0}{4\pi}G(r')\}r'\, e^{\frac{F(r',\beta)}{\cos\alpha}}}{F(r',\beta)\cos\alpha}\, dr' \Big\}$$

for $\sin^{-1}\frac{r_o^*}{R^*} \le \beta < \sin^{-1}\frac{r^*}{R^*}$ (24)

Rewriting the expressions for I_1^-, I_1^-, I_2^+ and I_2^+ in

dimensionless terms $T^* = \frac{T}{T_{ref}}$, $\phi = \frac{\pi I}{n^2 \sigma T_{s\,ref}^4}$, $\eta = \frac{G}{n^2 \sigma T_{s\,ref}^4}$ we obtain

$$\phi_1^- (r^*,\alpha,\beta) = e^{\frac{F(r^*,\beta)}{\cos\alpha}} \{ \phi^- (R^*)\, e^{\frac{-R^*\cos\beta}{\cos\alpha}} + \frac{1}{\cos\alpha} \int_{r^*}^{R^*} \frac{\{(1-\omega_0) T^{*4}(r') + \frac{\omega_0}{4}\eta(r')\}}{F(r',\ )} r' e^{\frac{-F(r',\beta)}{\cos\alpha}} dr' \}$$

for $0 \leq \beta < \sin^{-1} \frac{r_0^*}{R^*}$ (25)

$\phi_2^- (r^*,\alpha,\beta)$ is similar to $\phi_1^- (r^*,\alpha,\beta)$ except that β lies in the range $\sin^{-1} \frac{r_0^*}{R^*} \leq \beta < \sin^{-1} \frac{r^*}{R^*}$

$$\phi_1^+ (r^*,\alpha,\beta) = e^{\frac{-F(r^*,\beta)}{\cos\alpha}} \{ \phi_1^+ (r_0^*)\, e^{\frac{F(r_0^*,\beta)}{\cos\alpha}} + \frac{1}{\cos\alpha} \int_{r_0^*}^{r^*} \frac{\{(1-\omega_0) T^{*4}(r') + \frac{\omega_0}{4}\eta(r')\}}{F(r',\beta)} r' e^{\frac{F(r',\beta)}{\cos\alpha}} dr' \}$$

for $0 \leq \beta < \sin^{-1} \frac{r_0^*}{R^*}$ (26)

$$\phi_2^+ (r^*,\alpha,\beta) = e^{\frac{-F(r^*,\beta)}{\cos\alpha}} \{ \phi^- (R^*)\, e^{\frac{-R^*\cos\beta}{\cos\alpha}} + \frac{1}{\cos\alpha} \int_{R^*\sin\beta}^{R^*} \frac{\{(1-\omega_0) T^{*4}(r') + \frac{\omega_0}{4}\eta(r')\}}{F(r',\beta)} r' e^{\frac{-F(r',\beta)}{\cos\alpha}} dr'$$

$$+\frac{1}{\cos\alpha}\int_{R^*\sin\beta}^{r^*}\frac{\{(1-\omega_0)T^{*4}(r')+\frac{\omega_0}{4}\eta(r')\}r'e^{\frac{F(r',\beta)}{\cos\alpha}}}{F(r',\ )}dr'\ \}$$

for $\sin^{-1}\frac{r_o^*}{R^*}\leq\beta<\sin^{-1}\frac{r^*}{R^*}$ (27)

Because of symmetry observed in figure 5 equation (1) can be written as

$$G(r^*) = 4\int_0^{\pi/2}\int_0^{\pi} I\cos\alpha d\gamma d\alpha$$

$$= 4\int_0^{\pi/2}\int_0^{\pi/2} I^-\cos\alpha d\gamma d\alpha + 4\int_0^{\pi/2}\int_{\pi/2}^{\pi} I^+\cos\alpha d\gamma d\alpha$$

By using the transformation $\gamma' = \pi - \gamma$ we can redefine

I^+ in the range $0\leq\gamma'<\frac{\pi}{2}$

$$G(r^*) = 4\int_0^{\pi/2}\int_0^{\pi/2}(I^++I^-)\cos\alpha d\gamma d\alpha$$

Since $\sin\gamma = \frac{R^*}{r^*}\sin\beta \qquad 0\leq\gamma\leq\frac{\pi}{2}$

$$d\gamma = \frac{R^*\cos\beta\, d\beta}{\{r^{*2}-R^{*2}\sin^2\beta\}^{\frac{1}{2}}}$$

and thus $G(r^*) = 4\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_o^*}{R^*}}\frac{[I_1^+(r^*,\alpha,\beta)+I_1^-(r^*,\alpha,\beta)]}{F(r^*,\beta)}R^*\cos\beta\cos\alpha d\beta d\alpha$

$$+4\int_{0}^{\pi/2}\int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \frac{[I_2^+(r^*,\alpha,\beta)+I_2^-(r^*,\alpha,\beta)]\ R^*\ \cos\beta\cos\alpha d\beta d\alpha}{F(r^*,\beta)} \tag{28}$$

By non dimensionalizing as before equation (28) becomes

$$\eta(r^*) = \frac{4R^*}{\pi}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r^*}{R^*}} \frac{\phi_1^-(r^*,\alpha,\beta)\ \cos\alpha\cos\beta}{F(r^*,\beta)}\ d\beta d\alpha$$

$$+\ \frac{4R^*}{\pi}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \frac{\phi_1^+(r^*,\alpha,\beta)\ \cos\alpha\cos\beta}{F(r^*,\beta)}\ d\beta d\alpha$$

$$+\ \frac{4R^*}{\pi}\int_0^{\pi/2}\int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \frac{\phi_2^+(r^*,\alpha,\beta)\ \cos\alpha\cos\beta}{F(r^*,\beta)}\ d\beta d\alpha \tag{29}$$

where ϕ_1^-, ϕ_1^+ and β_2^+ are as defined in equations (25) through (27).

$$\eta(r^*) = \frac{4R^*}{\pi}\ \phi^-(R^*)\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r^*}{R^*}} \frac{\cos\alpha\ \cos\beta}{F(r^*,\beta)}\ e^{\frac{F(r^*,\beta)-R^*\ \cos\beta}{\cos\alpha}}\ d\beta d\alpha$$

$$+\ \frac{4R^*}{\pi}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r^*}{R^*}}\int_{r^*}^{R^*} \frac{\cos\beta}{F(r^*,\beta)}\ \frac{\{(1-\omega_0)T^{*4}(r')+\frac{\omega_0}{4}\eta(r')\}\ r'}{F(r',\beta)}\ e^{\frac{F(r^*,\beta)-F(r',\beta)}{\cos\alpha}}\ dr'\ d\beta d\alpha$$

$$+ \frac{4R^*}{\pi} \phi_i^+ (r_o^*) \int_0^{\pi/2} \int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \frac{\cos\alpha \cos\beta}{F(r^*,\beta)} e^{\frac{F(r_o^*,\beta)-F(r^*,\beta)}{\cos\alpha}} d\beta d\alpha$$

$$+ \frac{4R^*}{\pi} \int_0^{\pi/2} \int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \int_{r_o^*}^{r^*} \frac{\cos\beta}{F(r^*,\beta)} \frac{\{(1-\omega_o)T^{*4}(r') + \frac{\omega_o}{4}\eta(r')\}}{F(r',\beta)} r' e^{\frac{F(r',\beta)-F(r^*,\beta)}{\cos\alpha}} dr' d\beta d\alpha$$

$$+ \frac{4R^*}{\pi} \phi^- (R^*) \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \frac{\cos\alpha \cos\beta}{F(r^*,\beta)} e^{-\frac{F(r^*,\beta)+R^*\cos\beta}{\cos\alpha}} d\beta d\alpha$$

$$+ \frac{4R^*}{\pi} \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \int_{R^*\sin\beta}^{R^*} \frac{\cos\beta}{F(r^*,\beta)} r' \frac{\{(1-\omega_o)T^{*4}(r') + \frac{\omega_o}{4}\eta(r')\}}{F(r',\beta)} e^{-\frac{F(r^*,\beta)+F(r',\beta)}{\cos\alpha}} dr' d\beta d\alpha$$

$$+ \frac{4R^*}{\pi} \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \int_{R^*\sin\beta}^{r^*} \frac{\cos\beta}{F(r^*,\beta)} r' \frac{\{(1-\omega_o)T^{*4}(r') + \frac{\omega_o}{4}\eta(r')\}}{F(r',\beta)} e^{\frac{F(r',\beta)-F(r^*,\beta)}{\cos\alpha}} dr' d\beta d\alpha \qquad (30)$$

Due to symmetry equation (3) can be written as

$$q(r^*) = -4 \int_0^{\pi/2} \int_0^{\pi} I \cos^2\alpha \cos\gamma \; d\gamma d\alpha$$

$$= -4 \int_0^{\pi/2} \int_0^{\pi/2} I^- \cos^2\alpha \cos\gamma d\gamma d\alpha - 4 \int_0^{\pi/2} \int_{\pi/2}^{\pi} I^+ \cos^2\alpha \cos\gamma d\gamma d\alpha$$

Using the transformation $\gamma' = \pi - \gamma$ the second term on the right hand side can be rewritten for γ in the range $0 \le \gamma < \frac{\pi}{2}$ By making use of equation (13) we can obtain the net radiative flux directed radially outward as

$$q(r^*) = \frac{4R^*}{r^*} \int_0^{\pi/2} \int_0^{\sin^{-1}\frac{r_o^*}{R^*}} (I_1^+ - I_1^-) \cos^2\alpha \cos\beta \; d\beta d\alpha$$

$$+ \frac{4R^*}{r^*} \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} (I_2^+ - I_2^-) \cos^2\alpha \cos\beta \; d\beta d\alpha \qquad (31)$$

Let $Q(r^*) = \frac{q(r^*)}{n^2 \sigma_s T_{ref}^4}$

The net dimensionless radiative flux directed radially outward is

$$Q(r^*) = \frac{4R^*}{\pi r^*} \int_0^{\pi/2} \int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \{\phi_1^+ (r^*, \alpha, \beta) - \phi_1^-(r^*, \alpha, \beta)\}$$

$$\cos^2\alpha \cos\beta \; d\beta d\alpha$$

$$+ \frac{4R^*}{\pi r^*} \int_{0}^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \{\phi_2^+(r^*,\alpha,\beta) - \phi_2^-(r^*,\alpha,\beta)\} \cos^2\alpha \cos\beta \, d\beta \, d\alpha \tag{32}$$

The energy equation can be written as

$$\rho C_p \frac{\partial T}{\partial t} + \nabla.(-k\nabla T + q) = 0$$

Thus $\rho C_p \frac{\partial T}{\partial t} - k\nabla^2 T = - \nabla.q$

For the cylindrical geometry we have

$$\rho C_p \frac{\partial T}{\partial t} = k\frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial T}{\partial r}) - \frac{1}{r} \frac{\partial}{\partial r} (rq)$$

$$= k\frac{\partial^2 T}{\partial r^2} + \frac{k}{r} \frac{\partial T}{\partial r} - \frac{\partial q}{\partial r} - \frac{q}{r}$$

Using the dimensionless parameters

$$r^* = \beta r, \; T^* = \frac{T}{T_{ref}}, \; Q_r = \frac{q}{n^2 \sigma_s T_{ref}^4}, \; N = \frac{k\beta}{4n^2 \sigma_s T_{ref}^4} \text{ and } t^* = \frac{k\beta^2 t}{\rho C_p}$$

the energy equation becomes

$$- \frac{\partial T^*}{\partial t^*} + \frac{\partial^2 T^*}{\partial r^{*2}} + \frac{1}{r^*} \frac{\partial T^*}{\partial r^*} = \frac{1}{4N} \frac{\partial Q_r}{\partial r^*} + \frac{1}{4N} \frac{Q_r}{r^*} \tag{33}$$

The conduction radiation parameter N dictates the proportion of radiative and conductive components of energy transfer. $N = 0$ implies heat transfer by radiation only, whereas $N = \infty$ implies heat transfer by conduction only.

Taking the derivative of the terms in equation (32) we have

$$\frac{\partial Q_r}{\partial r^*} = - \frac{4R^*}{\pi r^{*2}} \int_{0}^{\pi/2} \int_{0}^{\sin^{-1}\frac{r_o^*}{R^*}} \{\phi_1^+ - \phi_1^-\} \cos^2\alpha \cos\beta \, d\beta \, d\alpha$$

$$- \frac{4R^*}{\pi r^{*2}} \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \{\phi_2^+ - \phi_2\} \cos^2\alpha \cos\beta \, d\beta d\alpha$$

$$+ \frac{4R^*}{\pi r^*} \int_0^{\pi/2} \int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \{\frac{\partial\phi_1^+}{\partial r^*} - \frac{\partial\phi_1^-}{\partial r^*}\} \cos^2\alpha \cos\beta \, d\beta d\alpha$$

$$+ \frac{4R^*}{\pi r^*} \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \{\frac{\partial\phi_2^+}{\partial r^*} - \frac{\partial\phi_2^-}{\partial r^*}\} \cos^2\alpha \cos\beta \, d\beta d\alpha$$

$$+ \frac{4R^*}{\pi r^*} \int_0^{\pi/2} \{\phi_2^+(r^*,\alpha, \sin^{-1}\frac{r^*}{R^*}) - \phi_2^-(r^*,\alpha, \sin^{-1}\frac{r^*}{R^*})\}$$

$$\cos^2\alpha \cos(\sin^{-1}\frac{r^*}{R^*}) \, d(\sin^{-1}\frac{r^*}{R^*}) \, d\alpha$$

Now $\phi_2^+(r^*,\alpha, \sin^{-1}\frac{r^*}{R^*}) = \phi_2^-(r^*,\alpha, \sin^{-1}\frac{r^*}{R^*})$

Thus $$\frac{1}{4} \frac{\partial Q_r}{\partial r^*} + \frac{1}{4} \frac{Q_r}{r^*} = \frac{R^*}{\pi r^*} \int_0^{\pi/2} \int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \{\frac{\partial\phi_1^+}{\partial r^*} - \frac{\partial\phi_1^-}{\partial r^*}\} \cos^2\alpha \cos\beta \, d\beta d\alpha$$

$$+ \frac{R^*}{\pi r^*} \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \{\frac{\partial\phi_2^+}{\partial r^*} - \frac{\partial\phi_2^-}{\partial r^*}\} \cos^2\alpha \cos\beta \, d\beta d\alpha$$

$$= -\frac{R^*}{\pi r^*}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r^*}{R^*}} \frac{\partial\phi_1^-}{\partial r^*}\cos^2\alpha\ \cos\beta\ d\beta d\alpha$$

$$+\frac{R^*}{\pi r^*}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \frac{\partial\phi_1^+}{\partial r^*}\cos^2\alpha\ \cos\beta\ d\beta d\alpha$$

$$+\frac{R^*}{\pi r^*}\int_0^{\pi/2}\int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \frac{\partial\phi_2^+}{\partial r^*}\cos^2\alpha\ \cos\beta\ d\beta d\alpha \qquad (34)$$

From equation (25) we notice that

$$\frac{\partial\phi_1^-}{\partial r^*} = \frac{\partial\phi_2^-}{\partial r^*} = \frac{r^*}{F(r^*,\beta)\cos\alpha}\, e^{\frac{F(r^*,\beta)}{\cos\alpha}} \Big\{\phi^-(R^*)\, e^{-R^*\frac{\cos\beta}{\cos\alpha}}$$

$$+\frac{1}{\cos\alpha}\int_{r^*}^{R^*} \frac{\{(1-\omega_o)T^{*4}(r')+\frac{\omega_o}{4}\eta(r')\}r'}{F(r',\beta)}\, e^{-\frac{F(r',\beta)}{\cos\alpha}}\, dr'\Big\}$$

$$+\, e^{\frac{F(r^*,\beta)}{\cos\alpha}}\Big\{-\frac{1}{\cos\alpha}\,\frac{\{(1-\omega_o)T^{*4}(r^*)+\frac{\omega_o}{4}\eta(r^*)\}r^*}{F(r^*,\beta)}\, e^{-\frac{F(r^*,\beta)}{\cos\alpha}}\Big\}$$

$$= \frac{r^*}{F(r^*,\beta)\cos\alpha}\Big\{\phi^-(R^*)\, e^{\frac{F(r^*,\beta)-R^*\cos\beta}{\cos\alpha}}$$

$$+\frac{1}{\cos\alpha}\int_{r^*}^{R^*}\frac{\{(1-\omega_o)T^{*4}(r')+\frac{\omega_o}{4}\eta(r')\}r'}{F(r',\beta)}\, e^{\frac{F(r^*,\beta)-F(r',\beta)}{\cos\alpha}}\, dr'$$

$$- (1-\omega_0)T^{*4}(r^*) - \frac{\omega_0}{4}\eta(r^*)\} \qquad (35)$$

Similarly we can show that

$$\frac{\partial \phi_1^+}{\partial r^*} = \frac{r^*}{F(r^*,\beta)\cos\alpha}\{(1-\omega_0)T^{*4}(r^*) + \frac{\omega_0}{4}\eta(r^*) - \Phi_1^+(r_0^*)\, e^{\frac{F(r_0^*,\beta)-F(r^*,\beta)}{\cos\alpha}}$$

$$- \frac{1}{\cos\alpha}\int_{r_0^*}^{r^*} \{\frac{(1-\omega_0)T^{*4}(r')+\frac{\omega_0}{4}\eta(r')\}r'}{F(r',\beta)}\, e^{\frac{F(r',\beta)-F(r^*,\beta)}{\cos\alpha}}\, dr'\} \qquad (36)$$

and

$$\frac{\partial \phi_2^+}{\partial r^*} = \frac{r^*}{F(r^*,\beta)\cos\alpha}\{(1-\omega_0)T^{*4}(r^*) + \frac{\omega_0}{4}\eta(r^*) - \phi^-(R^*)\, e^{-\frac{F(r^*,\beta)+R^*\cos\beta}{\cos\alpha}}$$

$$- \frac{1}{\cos\alpha}\int_{R^*\sin\beta}^{R^*} \{\frac{(1-\omega_0)T^{*4}(r')+\frac{\omega_0}{4}\eta(r')\}r'}{F(r',\beta)}\, e^{-\frac{F(r^*,\beta)+F(r',\beta)}{\cos\alpha}}\, dr'$$

$$- \frac{1}{\cos\alpha}\int_{R^*\sin\beta}^{r^*} \{\frac{(1-\omega_0)T^{*4}(r')+\frac{\omega_0}{4}\eta(r')\}r'}{F(r',\beta)}\, e^{\frac{F(r',\beta)-F(r^*,\beta)}{\cos\alpha}}\, dr'\} \qquad (37)$$

Substituting equations (35), (36) and (37) into equation (34) we obtain

$$\frac{1}{4}\frac{\partial Q_r}{\partial r^*} + \frac{1}{4}\frac{Q_r}{r^*} = \frac{2R^*}{\pi}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r^*}{R^*}} \frac{\cos\alpha\cos\beta\{(1-\omega_0)T^{*4}(r^*)+\frac{\omega_0}{4}\eta(r^*)\}\, d\beta d\alpha}{F(r^*,\beta)}$$

$$-\frac{\eta}{4}$$

$$= (1-\omega_0)\, T^{*4}(r^*) + \frac{\omega_0}{4}\eta(r^*) - \frac{\eta(r^*)}{4}$$

$$= (1-\omega_0)\ \{T^{*4}(r^*) - \frac{n(r^*)}{4}\}$$

Using the above relation in equation (33) we have

$$-\frac{\partial T^*}{\partial t^*} + \frac{\partial^2 T^*}{\partial r^{*2}} + \frac{1}{r^*}\ \frac{\partial T^*}{\partial r^*} = \frac{(1-\omega_0)}{N}\ \{T^{*4}(r^*) - \frac{n(r^*)}{4}\} \tag{38}$$

Boundary Conditions

Most insulating materials have surfaces which appear rough when compared to wavelengths in which heat transfer by radiation dominates. As a result we can associate an effective diffuse reflectance with the two surfaces of our cylindrical geometry. Thus, assuming diffuse surfaces of reflectances ρ_1 and ρ_2, the boundary conditions $I_1^-(R^*)$ or $\Phi^-(R^*)$, and $I_1^+(r_0^*)$ or $\Phi_1^+(r_0^*)$ can be evaluated in the following manner.

The net radiative flux at the oven boundary moving radially outward can be obtained from equation (31) as

$$q(R^*) = 4\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_0^*}{R^*}} \{I_1^+(R^*,\alpha,\beta) - I^-(R^*)\}\ \cos^2\alpha\ \cos\beta\ d\beta d\alpha$$

$$+\ 4\int_0^{\pi/2}\int_{\sin^{-1}\frac{r_0^*}{R^*}}^{\pi/2} \{I_2^+(R^*,\alpha,\beta) - I^-(R^*)\}\ \cos^2\alpha\ \cos\beta\ d\beta d\alpha$$

The net radiative flux at the oven boundary moving radially inward is given by

$$-q(R^*) = 4\int_0^{\pi/2}\int_0^{\pi/2} I^-\ (R^*)\ \cos^2\alpha\ \cos\beta\ d\beta d\alpha$$

$$- 4 \int_0^{\pi/2} \int_0^{\sin^{-1}\frac{r_o^*}{R^*}} I_1^+(R^*,\alpha,\beta) \cos^2\alpha \cos\beta \, d\beta d\alpha$$

$$- 4 \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\pi/2} I_2^+(R^*,\alpha,\beta) \cos^2\alpha \cos\beta \, d\beta d\alpha \qquad (39)$$

The first term on the RHS is the radiant energy leaving and thus the remaining terms denote the incident energy. Since the surface radiosity may be expressed as

$\pi I^-(R^*) = (1-\rho_1)$ Emitted flux $+ \rho_1$ Incident flux, and since the first term on the RHS of equation (39) can be simplified to $\pi I^-(R^*)$ we obtain the following boundary condition at the oven/material surface

$$\pi I^-(R^*) = (1-\rho_1) \pi n^2 I_{oven} + 4\rho_1 \int_0^{\pi/2} \int_0^{\sin^{-1}\frac{r_o^*}{R^*}} I_1^+(R^*,\alpha,\beta) \cos^2\alpha \cos\beta \, d\beta d\alpha$$

$$+ 4\rho_1 \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\pi/2} I_2^+(R^*,\alpha,\beta) \cos^2\alpha \cos\beta \, d\beta d\alpha$$

Using dimensionless parameters as before, the above equation becomes

$$\Phi^-(R^*) = (1-\rho_1) T^{*4}_{oven} + \frac{4\rho_1}{\pi} \int_0^{\pi/2} \int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \phi_1^+(R^*,\alpha,\beta) \cos^2\alpha \cos\beta \, d\beta d\alpha$$

$$+ \frac{4\rho_1}{\pi} \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\pi/2} \phi_2^+(R^*,\alpha,\beta) \cos^2\alpha \cos\beta \, d\beta d\alpha \qquad (40)$$

At the wire/material boundary the net radiative flux outward is

$$q(r_0^*) = \frac{4R^*}{r_0^*}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_0^*}{R^*}}\{\ I_1^+(r_0^*) - I_1^-(r_0^*,\alpha,\beta)\ \}\ \cos^2\alpha\ \cos\beta\ d\beta d\alpha$$

$$= \pi I_1^+(r_0^*) - \frac{4R^*}{r_0^*}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_0^*}{R^*}} I_1^-(r_0^*,\alpha,\beta)\ \cos^2\alpha\ \cos\beta\ d\beta d\alpha$$

Using an analysis similar to the one for the other boundary we obtain

$$\phi_1^+(r_0^*) = (1-\rho_2)\ T^{*4}_{\text{wire}} + \frac{4R^*\rho_2}{\pi r_0^*}\int_0^{\pi_2}\int_0^{\sin^{-1}\frac{r_0^*}{R^*}}\phi_1^-(r_0^*,\alpha,\beta)\ \cos^2\alpha\ \cos\beta\ d\beta d\alpha \tag{41}$$

Substituting equations (25), (26) and (27) into equations (40) and (41) we obtain the following

$$\phi^-(R^*) = (1-\rho_1)T^{*4}_{\text{oven}} + \frac{4\rho_1}{\pi}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_0^*}{R^*}} \cos^2\alpha\ \cos\beta\ e^{\frac{1}{\cos\alpha}\{F(r_0^*,\beta)-R^*\cos\beta\ \}}\ \phi_1^+(r_0^*)\,d\beta d\alpha$$

$$+ \frac{4\rho_1}{\pi}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_0^*}{R^*}}\int_{r_0^*}^{R^*} \frac{\cos\alpha\ \cos\beta\{(1-\omega_0)T^{*4}(r') + \frac{\omega_0}{4}\eta(r')\}r'}{F(r',\beta)}\ e^{\frac{1}{\cos\alpha}\{F(r',\beta)-R^*\cos\beta\}}\ dr'\ d\beta d\alpha$$

$$+\frac{4\rho_1}{\pi}\int_0^{\pi/2}\int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\pi/2}\cos^2\alpha\cos\beta\, e^{-\frac{2R^*\cos\beta}{\cos\alpha}}\,\Phi^-(R^*)\,d\beta d\alpha$$

$$+\frac{4\rho_1}{\pi}\int_0^{\pi/2}\int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\pi/2}\int_{R^*\sin\beta}^{R^*}\cos\alpha\cos\beta\frac{\{(1-\omega_o)T^{*4}(r')+\frac{\omega_o}{4}\eta(r')\}r'}{F(r',\beta)}\, e^{\frac{-R^*\cos\beta}{\cos\alpha}}\left\{e^{\frac{F(r',\beta)}{\cos\alpha}}+e^{\frac{-F(r',\beta)}{\cos\alpha}}\right\}dr'\,d\beta d\alpha \tag{42}$$

and

$$\Phi_1^+(r_o^*)=(1-\rho_2)T^{*4}_{wire}+\frac{4R^*\rho_2}{\pi\, r_o^*}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_o^*}{R^*}}\cos^2\alpha\cos\beta\, e^{\frac{1}{\cos\alpha}\{F(r_o^*,\beta)-R^*\cos\beta\}}\,\Phi^-(R^*)\,d\beta d\alpha$$

$$+\frac{4R^*\rho_2}{\pi r_o^*}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_o^*}{R^*}}\int_{r_o^*}^{R^*}\cos\alpha\frac{\cos\beta\{(1-\omega_o)T^{*4}(r')+\frac{\omega_o}{4}\eta(r')\}r'}{F(r',\beta)}\, e^{\frac{1}{\cos\alpha}\{F(r_o^*,\beta)-F(r',\beta)\}}\,dr'\,d\beta d\alpha \tag{43}$$

Equations (38) and (30) along with the boundary conditions given by equations (42) and (43) are an exact formulation of the problem for diffuse boundary surfaces. One of the methods that can be used to solve this problem is the Galerkin finite element method.

Heat Flux

The total heat flux $q'' = -k\frac{dT}{dr} + q$

Let $Q_r = \frac{q}{n^2\sigma_s T^4_{ref}}$

Thus $q'' = -k\beta T_{ref} \frac{dT^*}{dr^*} + n^2\sigma_s T^4_{ref} Q_r$

Let $\Psi = \frac{q''}{\kappa\beta T_{ref}}$

Thus $\Psi = -\frac{dT^*}{dr^*} + \frac{1}{4N} Q_r$

Now Q_r is given by

$$Q_r = -\frac{4R^*}{\pi r^*}\int_0^{\pi/2}\int_0^{\pi} \phi \cos^2\alpha \cos\beta \, d\beta d\alpha$$

$$\text{Thus at the wire } Q_{r_o} = \frac{4R^*}{\pi r^*_o}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r^*_o}{R^*}} \cos^2\alpha \cos\beta \{\phi_1^+(r^*_o,\alpha,\beta)-\phi_1^-(r^*_o,\alpha,\beta)\} \, d\beta d\alpha$$

Using equations (25) and (26) and simplifying we obtain

$$Q_{r_o} = \Phi_1^+(r^*_o) - \frac{4R^*}{\pi r^*_o}\Phi^-(R^*)\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r^*_o}{R^*}} \cos^2\alpha \cos\beta \, e^{\frac{1}{\cos\alpha}\{F(r^*_o,\beta) - R^*\cos\beta\}} \, d\beta d\alpha$$

$$- \frac{4R^*}{\pi r^*_o}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r^*_o}{R^*}}\int_{r^*_o}^{R^*} r' \frac{\cos\alpha \cos\beta\{(1-\omega_o)T^{*4}+\frac{\omega}{4}\Omega\eta\}}{F(r',\beta)} e^{\frac{1}{\cos\alpha}\{F(r^*_o,\beta)-F(r',\beta)\}} \, dr' \, d\beta d\alpha$$

The total heat flux at the inner wire is

$$\Psi\Big|_{r=r_o} = -\frac{dT^*}{dr^*}\Big|_{r=r_o} + \frac{1}{4N} Q_{r_0}$$

CHAPTER III

FINITE ELEMENT FORMULATION

The finite element procedure is to divide the radial domain r_o^* to R^* into a finite number of elements. The unknown functions $T^*(r^*)$ and $\eta(r^*)$ are approximated over each element in the form

$$T_e(r^*) = \sum_{i=1}^{2} \psi_e^{(i)}(r^*)\ T_e^{(i)} \tag{44}$$

$$\text{and}\quad \eta(r^*) = \sum_{i=1}^{2} \psi_e^{(i)}(r^*)\ \eta_e^{(i)}$$

where $\psi_e^{(i)}$ are linear interpolating functions, the subscript e applies to an element e, and the superscript i denotes the node index. Linear interpolating are chosen for this problem since they represent the simplest functions which satisfy the conditions for completeness and compatibility. The expressions for $\psi_e^{(1)}$ and $\psi_e^{(2)}$ are given as

$$\psi_e^{(1)}(r^*) = \frac{r_e^{*(2)} - r^*}{r_e^{*(2)} - r_e^{*(1)}} \tag{45}$$

$$\text{and}\quad \psi_e^{(2)}(r^*) = \frac{r^* - r_e^{*(1)}}{r_e^{*(2)}\ (r_e^{*(1)}}$$

where $r_e^{*(1)}$ represents the location of the first node of element e and $r_e^{*(2)}$ represents the location of the second node of element e.

The integral terms integrated with respect to r^* and r' are represented by a summation of integrals over the individual elements in the radial domain r_o^* to R^*. Substitution of the assumed form of the unknown functions, i.e. equation (44), into the governing equations, multiplication by the interpolating functions $\Psi_e^{(j)}(r^*)$, j=1,2, and integrating over the individual elements constitutes the Galerkin formulation. The energy equation (38) contains a conduction term involving second derivatives. Applying the Galerkin procedure to this term and integrating by parts we obtain

$$\int_{r_e^{*(1)}}^{r_e^{*(2)}} \frac{\partial^2 T^*}{\partial r^{*2}} \Psi_e^{(j)}(r^*)\, dr^* = -\int_{r_e^{*(1)}}^{r_e^{*(2)}} \frac{\partial T^*}{\partial r^*} \frac{d\Psi_e^{(j)}(r^*)}{dr^*}\, dr^*$$

$$+ \frac{\partial T^*}{\partial r^*} \Psi_e^{(j)}(r^*) \Bigg|_{r_e^{*(1)}}^{r_e^{*(2)}}$$

Using equations (44) and (45) we can show that the above equation becomes

$$\int_{r_e^{*(1)}}^{r_e^{*(2)}} \frac{\partial^2 T^*}{\partial r^{*2}} \Psi_e^{(j)}(r^*)\, dr^* = -\int_{r_e^{*(1)}}^{r_e^{*(2)}} \frac{d}{dr^*} \sum_{i=1}^{2} \Psi_e^{(i)}(r^*) T_e^{*(i)} \frac{d}{dr^*} \Psi_e^{(j)}(r^*)\, dr^*$$

$$+ (-1)^j \frac{\partial T^*}{\partial r^*} \Bigg|_{r_e^{(2)}} \qquad (46)$$

Applying the Galerkin formulation to the energy equation (38) we thus obtain

$$- \int_{r_e^{*(1)}}^{r_e^{*2}} \sum_{i=1}^{2} \frac{\partial T^*}{\partial t^*} \Psi_e^{(i)}(r^*) \Psi_e^{(j)}(r^*) - \int_{r_e^{*(1)}}^{r_e^{*(2)}} \sum_{i=1}^{2} \frac{d}{dr^*} \Psi_e^{(i)}(r^*) T_e^{*(i)} \frac{d}{dr^*} \Psi_e^{(j)}(r^*)\, dr^*$$

$$+ \int_{r_e^{*(1)}}^{r_e^{*2}} \frac{1}{r^*} \sum_{i=1}^{2} \frac{d}{dr^*} \Psi_e^{(i)}(r^*)\, T_e^{*(i)}\, \Psi_e^{(j)}(r^*)\, dr^*$$

$$- \frac{(1-\omega_0)}{N} \int_{r_e^{*(1)}}^{r_e^{*(2)}} \{ \sum_{i=1}^{2} \Psi_e^{(i)}(r^*)\, T_e^{*(i)} \}^4\, \Psi_e^{(j)}(r^*)\, dr^*$$

$$+ \frac{(1-\omega_0)}{4N} \int_{r_e^{*(1)}}^{r_e^{*(2)}} \sum_{i=1}^{2} \Psi_e^{(i)}(r^*)\, \eta_e^{(i)}\, \Psi_e^{(j)}(r^*)\, dr^* + (-1)^j \frac{\partial T^*}{\partial r^*}\bigg|_{r_e^{(j)}} = 0 \qquad (47)$$

Before applying the Galerkin formulation to the expression for the irradiation $\eta(r^*)$ given by equation (30), we must rewrite $\Phi^-(R^*)$ and $\Phi_I^+(r_o^*)$ given by equations (42) and (43) in terms of the unknown variables $T^*(r^*)$ and $\eta(r^*)$. After substituting for $\Phi^-(R^*)$ and $\Phi_I^+(r_o^*)$ into equation (30) we then apply the Galerkin formulation and after solving for the unknowns $T^*(r^*)$ and $\eta(r^*)$ we can then evaluate $\Phi^-(R^*)$ and $\Phi_I^+(r_o^*)$ and thus obtain the flux. A mathematically equivalent method of evaluating $T^*(r^*)$, $\eta(r^*)$, $\Phi^-(R^*)$ and $\Phi_I^+(r_o^*)$ would be to treat equations (30), (42) and (43) as simultaneous equations together with equation (38) and apply the Galerkin technique to each of the four equations. It is understood that $\Phi^-(R^*)$

and $\Phi_1^+(r_o^*)$ are boundary value constants and as such taking their variations within each element is physically meaningless. However, the solutions obtained for $\Phi^-(R^*)$ and $\Phi_1^+(r_o^*)$ should be constant for each node and the formulation is mathematically equivalent to the conventional technique. This 'new' method has the advantages of less computational time and ease of equation and programming managebility. It's disadvantage lies in requiring a bit more computer storage. It should be noted that the author used both methods and obtained identical results.

Applying the Galerkin formulation to equation (42) we obtain

$$\left\{1 - \frac{4\rho_1}{\pi}\int_0^{\pi/2}\int_{\sin^{-1}r_o^*}^{\pi/2} \cos^2\alpha \cos\beta \, e^{-2R^*\frac{\cos\beta}{\cos\alpha}} d\beta d\alpha\right\}\int_{r_e^{*(1)}}^{r_e^{*(2)}} \sum_{i=1}^{2} \Psi_e^{(i)}(r^*)\,\Psi_e^{(j)}(r^*)\Phi_e^{-(i)}(R^*)\,dr^*$$

$$= (1-\rho_1)T^{*4}_{oven}\int_{r_e^{*(1)}}^{r_e^{*(2)}} \Psi_e^{(j)}(r^*)\,dr^*$$

$$+ \frac{4\rho_1}{\pi}\int_{r_e^{*(1)}}^{r_e^{*(2)}}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \cos^2\alpha\cos\beta\, e^{\frac{1}{\cos\alpha}\{F(r_o^*,\beta)-R^*\cos\beta\}} \sum_{i=1}^{2}\psi_e^{(i)}(r^*)\,\psi_e^{(j)}(r^*)\,\Phi_{1e}^{+(i)}(r_o^*)\, dr^*\, d\beta d\alpha$$

$$+ \frac{4\rho_1(1-\omega_0)}{\pi}\int_{r_e^{*(1)}}^{r_e^{*(2)}}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \sum_{f=1}^{NEM}\int_{r_f'^{(1)}}^{r_f'^{(2)}} M_1F\{(r',\beta)-R^*\cos\beta\}\Psi_e^{(j)}(r^*)\,dr'\,d\beta d\alpha dr^*$$

$$+ \frac{\rho_1\omega_0}{\pi}\int_{r_e^{*(1)}}^{r_e^{*(2)}}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \sum_{f=1}^{NEM}\int_{r_f'^{(1)}}^{r_f'^{(1)}} M_2\{F(r',\beta)-R^*\cos\beta\}\Psi_e^{(j)}(r^*)\,dr'\,d\beta d\alpha dr^*$$

$$+ \frac{4\rho_1}{\pi}(1-\omega_o) \int_{r_e^{*(1)}}^{r_e^{*(2)}} \int_{o}^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\pi/2} \sum_{\substack{f=\text{appropriate}\\ \text{element}}}^{NEM} \int_{r_f'^{(1)}}^{r_f'^{(2)}} \{M_1\{F(r',\beta)-R^*\cos\beta\}+M_1\{-F(r',\beta)-R^*\cos\beta\}\}$$

$$\psi_e^{(j)}(r^*)\ dr'\ d\beta\ d\alpha\ dr^*$$

$$+ \frac{\rho_1\omega_o}{\pi} \int_{r_e^{*(1)}}^{r_e^{*(2)}} \int_{o}^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\pi/2} \sum_{\substack{f=\text{appropriate}\\ \text{element}}}^{NEM} \int_{r_f'^{(1)}}^{r_f'^{(2)}} \{M_2\{F(r',\beta)-R^*\cos\beta\}+M_2\{-F(r',\beta)-R^*\cos\beta\}$$

$$\psi_e^{(j)}(r^*)\ dr'\ d\beta\ d\alpha\ dr^* \qquad (48)$$

and from equation (43) we obtain

$$\int_{r_e^{*(1)}}^{r_e^{*(2)}} \sum_{i=1}^{2} \psi_e^{(i)}(r^*)\ \psi_e^{(j)}(r^*)\ \Phi_{1_e}^{+(i)}(r_o^*) = (1-\rho_2)T^{*4}_{wire} \int_{r_e^{*(1)}}^{r_e^{*(2)}} \psi_e^{(j)}(r^*)\ dr^*$$

$$+ \frac{4R^*\rho_2}{\pi r_o^*} \int_{r_e^{*(1)}}^{r_e^{*(2)}} \int_{o}^{\pi_2} \int_{o}^{\sin^{-1}\frac{r_o^*}{R^*}} \cos^2\alpha\ \cos\beta\ e^{\frac{1}{\cos\alpha}\{F(r_o^*,\beta)-R^*\cos\beta\}}$$

$$\sum_{i=1}^{2} \psi_e^{(i)}(r^*)\ \psi_e^{(j)}(r^*)\ \Phi_e^{-}(R^*)\,d\beta d\alpha dr^*$$

$$+\frac{4R^*\rho_2\ (1-\omega_0)}{\pi\, r_o^*}\int_{r_e^{*(1)}}^{r_e^{*(2)}}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_o^*}{R^*}}\sum_{f=1}^{NEM}\int_{r_f'^{(1)}}^{r_f'^{(2)}} M_1\{\ F(r_o^*,\beta)-F(r',\beta)\}\Psi_e^{(j)}(r^*)\ dr'\ d\beta d\alpha\ dr^*$$

$$+\frac{R^*\rho_2\omega_0}{\pi r_o^*}\int_{r_e^{*(1)}}^{r_e^{*(2)}}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_o^*}{R^*}}\sum_{f=1}^{NEM}\int_{r_f'^{(1)}}^{r_f'^{(2)}} M_2\{\ F(r_o^*,\beta)-F(r',\beta)\}\ \Psi_e^{(j)}(r^*)\ dr'\ d\beta d\alpha\ dr^* \qquad (49)$$

while from equation (30) we obtain

$$\int_{r_e^{*(1)}}^{r_e^{*(2)}}\sum_{i=1}^{2}\eta_e^{(i)}\ \Psi_e^{(i)}(r^*)\Psi_e^{(j)}(r^*)\ dr^*$$

$$=\frac{4R^*}{\pi}\int_{r_e^{*(1)}}^{r_e^{*(2)}}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r^*}{R^*}} M_3\{F(r^*,\beta)-R^*\cos\beta\}\sum_{i=1}^{2}\Phi_e^{-(i)}(R^*)\ \Psi_e^{(i)}(r^*)\Psi_e^{(j)}(r^*)\ d\beta\ d\alpha\ dr^*$$

$$+\frac{4R^*}{\pi}\int_{r_e^{*(1)}}^{r_e^{*(2)}}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r_o^*}{R^*}} M_3\{F(r_o^*,\beta)-F(r^*,\beta)\}\sum_{i=1}^{2}\Phi_{1e}^{+(i)}(r_o^*)\ \Psi_e^{(i)}(r^*)\ \Psi_e^{(j)}(r^*)\ d\beta\ d\alpha\ dr^*$$

$$+\frac{4R^*}{\pi}\int_{r_e^{*(1)}}^{r_e^{*(2)}}\int_0^{\pi/2}\int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} M_3\{-F(r^*,\beta)-R^*\cos\beta\}\sum_{i=1}^{2}\Phi_e^{-(i)}(R^*)\ \Psi_e^{(i)}(r^*)\ \Psi_e^{(j)}(r^*)\ d\beta\ d\alpha\ dr^*$$

$$+\frac{4R^*(1-\omega_0)}{\pi}\int_{r_e^{*(1)}}^{r_e^{*(2)}}\int_0^{\pi/2}\int_0^{\sin^{-1}\frac{r^*}{R^*}}\sum_{f=e}^{NEM}\int_{r_f'^{(1)}}^{r_f'^{(2)}} M_4\{F(r^*,\beta)-F(r',\beta)\}\ \Psi_e^{(j)}(r^*)\ dr'\ d\beta\ d\alpha\ dr^*$$

$$+ \frac{R\omega_o}{\pi} \int_{r_e^{*(1)}}^{r_e^{*(2)}} \int_0^{\pi/2} \int_0^{\sin^{-1}\frac{r^*}{R^*}} \sum_{f=e}^{NEM} \int_{r_f^{\prime(1)}}^{r_f^{\prime(2)}} M_5\{F(r^*,\beta)-F(r',\beta)\}\ \Psi_e^{(j)}(r^*)\ dr'\ d\beta d\alpha\ dr^*$$

$$+ \frac{4R^*(1-\omega_o)}{\pi} \int_{r_e^{*(1)}}^{r_e^{*(2)}} \int_0^{\pi/2} \int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \sum_{f=1}^{e} \int_{r_f^{\prime(1)}}^{r_f^{\prime(2)}} M_4\{F(r',\beta)-F(r^*,\beta)\}\ \Psi_e^{(j)}(r^*)\ dr'\ d\beta d\alpha\ dr^*$$

$$+ \frac{R^*\omega_o}{\pi} \int_{r_e^{*(1)}}^{r_e^{*(2)}} \int_0^{\pi/2} \int_0^{\sin^{-1}\frac{r_o^*}{R^*}} \sum_{f=1}^{e} \int_{r_f^{\prime(1)}}^{r_f^{\prime(2)}} M_5\{F(r',\beta)-F(r^*,\beta)\}\ \Psi_e^{(j)}(r^*)\ dr'\ d\beta d\alpha\ dr^*$$

$$+ \frac{4R^*(1-\omega_o)}{\pi} \int_{r_e^{*(1)}}^{r_e^{*(2)}} \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \sum_{\substack{f=\text{appropriate}\\ \text{element}}}^{NEM} \int_{r_f^{\prime(1)}}^{r_f^{\prime(2)}} M_4\{-F(r^*,\beta)-F(r',\beta)\}\ \Psi_e^{(j)}(r^*)\ dr'\ d\beta d\alpha\ dr^*$$

$$+ \frac{R^*\omega_o}{\pi} \int_{r_e^{*(1)}}^{r_e^{*(2)}} \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \sum_{\substack{f=\text{appropriate}\\ \text{element}}}^{NEM} \int_{r_f^{\prime(1)}}^{r_f^{\prime(2)}} M_5\{-F(r^*,\beta)-F(r',\beta)\}\ \Psi_e^{(j)}(r^*)\ dr'\ d\beta d\alpha\ dr^*$$

$$+ \frac{4R^*(1-\omega_o)}{\pi} \int_{r_e^{*(1)}}^{r_e^{*(2)}} \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \sum_{\substack{f=\text{appropriate}\\ \text{element}}}^{e} \int_{r_f^{\prime(1)}}^{r_f^{\prime(2)}} M_4\{F(r',\beta)-F(r^*,\beta)\}\ \Psi_e^{(j)}(r^*)\ dr'\ d\beta d\alpha\ dr^*$$

$$+ \frac{R^*\omega_o}{\pi} \int_{r_e^{*(1)}}^{r_e^{*(2)}} \int_0^{\pi/2} \int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \sum_{\substack{f=\text{appropriate}\\ \text{element}}}^{e} \int_{r_f'^{(1)}}^{r_f'^{(2)}} M_5\{F(r',\beta)-F(r^*,\beta)\}\, \Psi_e^{(j)}(r^*)\, dr'\ d\beta d\alpha\ dr^* \tag{50}$$

where

$$M_1\{F(r',\beta)-R^*\cos\beta\} = \frac{\cos\alpha\ \cos\beta}{F(r',\beta)}\{\sum_{i=1}^{2} T_f^{*(i)}\ \Psi_f^{(i)}(r')\}^4\ r' e^{\frac{1}{\cos\alpha}\{F(r',\beta)-R^*\cos\beta\}},$$

$$M_2\{F(r',\beta)-R^*\cos\beta\} = \frac{\cos\alpha\ \cos\beta}{F(r',\beta)}\{\sum_{i=1}^{2} \eta_f^{(i)}\ \Psi_f^{(i)}(r')\}\ r'\ e^{\frac{1}{\cos\alpha}\{F(r',\beta)-R^*\cos\beta\}},$$

$$M_3\{F(r^*,\beta)-R^*\cos\beta\} = \frac{\cos\alpha\ \cos\beta}{F(r^*,\beta)}\ e^{\frac{1}{\cos\alpha}\{F(r^*,\beta)-R^*\cos\beta\}},$$

$$M_4\{F(r^*,\beta)-F(r',\beta)\} = \frac{\cos\beta}{F(r^*,\beta)}\ \frac{r'}{F(r',\beta)}\{\sum_{i=1}^{2} T_f^{*(i)}\ \Psi_f^{(i)}(r')\}^4\ e^{\frac{1}{\cos\alpha}\{F(r^*,\beta)-F(r',\beta)\}},$$

$$M_5\{F(r^*,\beta)-F(r',\beta)\} = \frac{\cos\beta}{F(r^*,\beta)}\ \frac{r'}{F(r',\beta)}\sum_{i=1}^{2} \eta_f^{(i)}\ \Psi_f^{(i)}(r')\ e^{\frac{1}{\cos\alpha}\{F(r^*,\beta)-F(r',\beta)\}},$$

NEM = number of elements in the domain r_o^* to R^*, and the arguments of M_1, M_2, M_3, M_4 and M_5 correspond to the powers of the exponential integrals.

Some terms in equations (48) and (50) which are integrated with respect to r' do not have limits which coincide with the nodes of the elements. An example of such terms are those whose lower limits are $R^*\sin\beta$. These integrals are treated in the following manner.

Let $R^*\sin\beta$ lie within some element e as shown below in figure 6

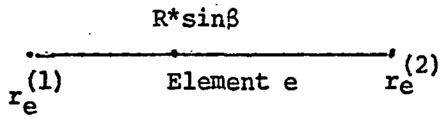

Figure 6. Location for Variable Limits.

we can prove that
$$\int_{R^*\sin\beta}^{r_e^{(2)}} \eta\, dr = \int_{R^*\sin\beta}^{r_e^{(2)}} \{\eta_e^{(1)}\,\Psi_e^{(1)}(r) + \eta_e^{(2)}\Psi_e^{(2)}(r)\}\, dr \tag{51}$$

where the superscripts (1) and (2) are the nodes of element e, and $\Psi_e^{(i)}(r)$, i=1,2 are the interpolating functions for element e. In order to satisfy continuity in the solution for η we must have $\eta(R^*\sin\beta)$ as the value of η at $r=R^*\sin\beta$. Thus

$$\int_{R^*\sin\beta}^{r_e^{(2)}} \eta\, dr \doteq \int_{R^*\sin\beta}^{r_e^{(2)}} \{\eta_e^{(var)}\,\Psi_e^{(var)}(r) + \eta_e^{(2)}\,\Psi_e^{(2)}(r)\}\, dr \tag{52}$$

where var is the location of $R^*\sin\beta$

and $$\Psi_e^{(var)}(r) = \frac{r_e^{(2)} - r}{r_e^{(2)} - R^*\sin\beta}$$

$$\Psi_e^{(2)}(r) = \frac{r - R^*\sin\beta}{r_e^{(2)} - R^*\sin\beta}$$

Now $\eta(R^*\sin\beta) = \eta_e^{(var)}$

$$= \frac{r_e^{(2)} - R^* \sin\beta}{r_e^{(2)} - r_e^{(1)}} \eta_e^{(1)} + \frac{R^* \sin\beta - r_e^{(1)}}{r_e^{(2)} - r_e^{(1)}} \eta_e^{(2)}$$

Thus from equation (52) we obtain

$$\int_{R^* \sin\beta}^{r_e^{(2)}} \eta \, dr \doteq \int_{R^* \sin\beta}^{r_e^{(2)}} \{[\frac{r_e^{(2)} - R^* \sin\beta}{r_e^{(2)} - r_e^{(1)}} \eta_e^{(1)} + \frac{R^* \sin\beta - r_e^{(1)}}{r_e^{(2)} - r_e^{(1)}} \eta_e^{(2)}][\frac{r_e^{(2)} - r}{r_e^{(2)} - R^* \sin\beta}]$$

$$+ \eta_e^{(2)} \frac{r - R^* \sin\beta}{r_e^{(2)} - R^* \sin\beta}]\} \, dr$$

$$= \int_{R^* \sin\beta}^{r_e^{(2)}} \{ \frac{r_e^{(2)} - r}{r_e^{(2)} - r_e^{(1)}} \eta_e^{(1)} + \frac{r - r_e^{(1)}}{r_e^{(2)} - r_e^{(1)}} \eta_e^{(2)} \} \, dr$$

$$= \int_{R^* \sin\beta}^{r_e^{(2)}} \sum_{i=1}^{2} \eta_e^{(i)} \Psi_e^{(i)}(r) \, dr$$

and equation (51) has been proved.

Now that the finite element approximations have been made and the Galerkin procedure used, the next step involves the assembly of equations. Setting $j = 1,2$ the flux term

$(-1)^j \left. \frac{\partial T^*}{\partial r^*} \right|_{r_e^{(j)}}$ in equation (47) reduces to $- \left. \frac{\partial T^*}{\partial r^*} \right|_{r_e^{(1)}}$ for j=1, and $\left.\frac{\partial T^*}{\partial r^*}\right|_{r_e^{(2)}}$ for j=2. Thus from equations (47), (48), (49) and (50) we see that for each element e we have 8 equations and 10 unknowns, i.e. T_e^*, η_e, $\Phi_e^-(R^*)$, $\Phi_{1_e}^+(r_o^*)$ and $\frac{dT^*}{dr^*}$ which are unknown at both nodes 1 and 2 of each element.

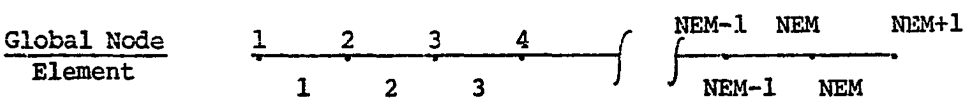

Figure 7. - Element and Node Locations.

Thus for NEM elements we have (8)(NEM) equations and (10)(NEM) unknowns. But the elements are connected (Figure 7) such that the second node of an element is shared by the first node of the right side element. Thus

$$\left.\frac{\partial T^*}{\partial r^*}\right|_{r_e^{(2)}} = \left.\frac{\partial T^*}{\partial r^*}\right|_{r_{e+1}^{(1)}}$$ since $r_e^{(2)}$ and $r_{e+1}^{(1)}$ are the same node. In order to reduce the number of unknowns, the equation corresponding to the second node, i.e. j=2 of element e, is added to the equation corresponding to the first node, i.e., j=1 of element e+1. The only equations which remain unaffected are the equations for the first node of element 1 and the equation for the second node of element NEM. Adding the equations in this manner eliminates the conductive flux unknowns at the interior nodes. Thus we have (4)(NEM+1) equations with (4)(NEM+1) + 2 unknowns, the unknowns being T^*, η, $\Phi^-(R^*)$ and $\Phi^+(r_o^*)$ at each node 1,2,3, ... NEM+1 plus the conductive heat flux $\frac{\partial T^*}{\partial r^*}$ at global node 1 and at

node NEM+1. The prescribed boundary conditions are

$$\text{at} \quad r^* = r_o^* \qquad T^* = T^*_{wire}$$

$$\text{and} \quad \text{at } r^* = R^* \qquad T^* = T^*_{oven} \tag{53}$$

can now be solved by any suitable method.

Formulation of the Transient Term

One of the methods to evaluate the transient term in equation (47) is the Crank-Nicolson scheme. Denoting the present time step as n and the next time step as n+1 we have

$$\{T^*\}_{n+1} = \{T^*\}_n + \frac{\Delta t}{2}\{\dot{T}^*\}_{n+1} + \frac{\Delta t}{2}\{\dot{T}^*\}_n \tag{54}$$

where Δt is the time step. Adding equation (47) in the manner described in the previous section we obtain a set of simultaneous equations of the form

$$\underset{MXM}{[G]}\underset{MX1}{\{\dot{T}^*\}} + \underset{MXM}{[A_1]}\underset{MX1}{\{T^*\}} + \underset{MXM}{[B_1]}\underset{MX1}{\{\eta\}} + \underset{MXM}{[C_1]}\underset{MX1}{\{T^{*4}\}} + \underset{MXM}{[D_1]}\underset{MX1}{\left\{\frac{\partial T^*}{\partial r^*}\right\}} = 0 \tag{55}$$

where M = NEM+1

= number of nodes in the domain

Multiplying equation (54) by [G] we obtain

$$[G]\{\dot{T}^*\}_{n+1} = \frac{2}{\Delta t}[G]\{T^*\}_{n+1} - \frac{2}{\Delta t}[G]\{T^*\}_n - [G]\{\dot{T}^*\}_n \tag{56}$$

By evaluating equation (55) at time n+1 and making use of (56) we have

$$[\frac{2}{\Delta t}\,[G] + A_1]\,\{T^*\}_{n+1} + [C_1]\,\{T^{*4}\}_{n+1} + [B_1]\,\{\eta\}_{n+1}$$

$$+ [D_1]\,\{\frac{\partial T^*}{\partial r^*}\}_{n+1} = [G]\,\{\dot{T}^*\}_n + \frac{2}{\Delta t}\,[G]\,\{T^*\}_n \qquad (57)$$

The initial condition is given as at $t^* = 0 \quad T^* = T_{initial}$ Equations (56) and (57) together with the equations obtained from (48), (49) and (50) along with the prescribed boundary conditions (53) can now be solved for T^*, η, $\bar{\Phi}(R^*)$ and $\Phi_1^+(r_0^*)$ by any suitable method.

A computer program was written to solve both the steady state and transient problems. Since the equations are non linear an iterative procedure was used. A linear solution for T^* was assumed and thereby T^{*4} was evaluated. The resulting equations were solved and the new value of T^* compared with the initial guess. If convergence to within 0.001 was not obtained for T^*, the system of equations was solved again updating T^*, and hence T^{*4}, as the initial guess. After T^* converged the results were printed.

CHAPTER IV

NUMERICAL APPROXIMATIONS TO THE INTEGRALS

Before obtaining a solution to the problem given a particular set of parameters, a number of integrals need to be evaluated. One method of evaluating integrals consists of employing a Gauss quadrature. The Gauss quadrature has been used to evaluate all the integrals in the study. Some of the integrals have one or more singularities at the limits. The integrals themselves exist but the integrals approach infinity at one or more limits. While evaluating the integrals it was found that they were highly sensitive to any change in the quadrature used for the variable β, but not sensitive to the change in quadrature of the other variables, namely r^*, r' and α. Thus in order to improve the answers, the limits on β were divided into two equal parts and a six point quadrature applied over each part. A four point quardature was used for the integrals with respect to r^*, r' and α. On comparing solutions for the radiative equilibrium case (N=0) with those obtained by the Monte Carlo technique [17] good results were obtained for the cases involving black surfaces i.e. $\varepsilon_1 = \varepsilon_2 = 1$. But the results for surface emissivities other than 1 were extremely poor. The difficulty arises in the singularities of the integrals. By observing the integrals in the expressions for the irradiation η and the boundary terms $\bar{\Phi}(R^*)$ and $\Phi_1^+(r_o^*)$ we recognize that some integrands are singular at the limits of β or r' or both. Since the integrals were sensitive to the quadrature on β they were approximated by the largest value of the exponential term in the integrand near the singularity in β. Thus any approximations with respect to singularities in r' were ignored. It should be noted that because the Gauss quadrature is used the integrands do not approach infinity since the function is never evaluated at the limits. The integrals were evaluated in the following manner

$$\frac{4R^*}{\pi}\int_{0}^{\pi/2}\int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{r^*}{B^*}} \frac{\Phi^{-}(R^*)\cos\alpha\cos\beta}{\{r^{*2}-R^{*2}\sin^2\beta\}^{\frac{1}{2}}}\, e^{\frac{1}{\cos\alpha}\{\sqrt{r^{*2}-R^{*2}\sin^2\beta}-R^*\cos\beta\}}\, d\beta d\alpha$$

$$=\frac{4R^*}{\pi}\int_{0}^{\pi/2}\int_{\sin^{-1}\frac{r_o^*}{R^*}}^{\sin^{-1}\frac{RAT\ r^*}{R^*}} \frac{\Phi^{-}(R^*)\cos\alpha\cos\beta}{\{r^{*2}-R^{*2}\sin^2\beta\}^{\frac{1}{2}}}\, e^{\frac{1}{\cos\alpha}\{\sqrt{r^{*2}-R^{*2}\sin^2\beta}-R^*\cos\beta\}}\, d\beta d\alpha$$

$$+\frac{4R^*}{\pi}\int_{0}^{\pi/2}\int_{\sin^{-1}\frac{RAT\ r^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \frac{\Phi^{-}(R^*)\cos\alpha\cos\beta}{\{r^{*2}-R^{*2}\sin^2\beta\}^{\frac{1}{2}}}\, e^{\frac{1}{\cos\alpha}\{\sqrt{r^{*2}-R^{*2}\sin^2\beta}-R^*\cos\beta\}} \tag{58}$$

where RAT represents some positive real number close to but less than 1. The singularity in the first term on the right hand side of equation (58) has now been eliminated. The exponential term of the integrand in the second integral is approximated by its value at the upper limit and reduces to

$$e^{-\frac{\{R^{*2}-r^{*2}\}^{\frac{1}{2}}}{\cos\alpha}} \qquad \text{since} \quad \beta = \sin^{-1}\frac{r^*}{R^*}$$

Thus the second integral is equal to

$$\frac{4R^*}{\pi}\int_0^{\pi/2}\int_{\sin^{-1}\mathrm{RAT}\,\frac{r^*}{R^*}}^{\sin^{-1}\frac{r^*}{R^*}} \frac{\Phi^-(R^*)\cos\alpha\cos\beta}{\{r^{*2}-R^{*2}\sin^2\beta\}^{\frac{1}{2}}}\, e^{-\frac{\{R^{*2}-r^{*2}\}^{\frac{1}{2}}}{\cos\alpha}}\, d\beta d\alpha$$

$$= \frac{4}{\pi}\{\frac{\pi}{2} - \sin^{-1}(\mathrm{RAT})\}\int_0^{\pi/2} \cos\alpha\; e^{-\frac{\{R^{*2}-r^{*2}\}^{\frac{1}{2}}}{\cos\alpha}}\, \Phi^-(R^*)\, d\alpha$$

The Galerkin technique can now be applied to the integral above resulting in

$$\frac{4}{\pi}\{\frac{\pi}{2} - \sin^{-1}(\mathrm{RAT})\}\int_{r_e^{(1)}}^{r_e^{(2)}}\int_0^{\pi/2} \cos\alpha\; e^{-\frac{\{R^{*2}-r^{*2}\}^{\frac{1}{2}}}{\cos\alpha}} \sum_{i=1}^{2} \Phi^{-(i)}(R^*)\;\Psi_e^{(i)}(r^*)\;\Psi_e^{(j)}(r^*)\; d\alpha dr^*$$

The other integrals with singularities in β are handled in a similar manner. The value of RAT was chosen such that the radiative equilibrium solution for non black surfaces closely approximated those obtained by the Monte Carlo method [17]. This criteria was selected because, to the author's knowledge, the Monte Carlo solution is the only known exact solution available for comparison. Setting RAT = 0.97 gave adequate results.

CHAPTER V

RESULTS

Known exact solutions for the problem of combined conduction and radiation in the cylindrical geometry exist only for the extreme cases, i.e. conduction only or radiative equilibrium. Pure conduction is a trivial problem and the results can be readily verified by consideration of the cases where $\omega_0 = 1$.

Radiative equilibrium solutions were compared with those obtained by the Monte Carlo technique [17]. Figure 8 shows results obtained by the finite element and Monte Carlo methods for various surface emissivities. In figure 9 the two methods were compared for black surfaces and different radius ratios and optical depths. The radial domain was divided into 20 equal elements. The results compare well with those obtained by the Monte Carlo technique.

Combined conduction and radiation solution were compared with those obtained by Chang [12]. Although Chang's solution is based on an Eddington's first approximation model of radiation heat transfer, it is one of the few known solutions that exist for this problem even though scattering is omitted. The steady state results are compared in figure 13, while the transient case is compared in figure 15. The radial domain was divided into 19 elements of which 12 closest to the wire were of length 0.025 and the remaining 7 were of length 0.10.

Figure 10 shows the effect of the conduction radiation parameter N on the temperature distribution. N equal to infinity implies heat transfer by conduction only and thus we expect a log curve. N = 0 implies heat transfer by radiation only and we would expect temperature slip at the boundary.

Figure 11 shows the effect of scattering on the temperature distribution. For $\omega_0 = 1$, the medium does not absorb but only scatters radiation, and thus we would expect a log curve. For other values of ω_0 the medium absorbs incident radiation.

Figure 12 shows the effect of surface emissivities on the irradiation η, while figure 14 shows their effects on the temperature distribution. It is seen that at the hot surface, i.e. $r^* = r_o^*$, the irradiation decreases as the emissivity decreases since less radiation is emitted, while the initial slope of the temperature increases with decrease in emissivity.

Figure 16 shows the effect of scattering on the transient temperature distribution. Nineteen elements were used of sizes 0.025 and 0.1 and the time step chosen was 0.001. This time step was selected to avoid any temperature fluctuations at early times.

Tables 2, 3 and 4 present results for the conductive, radiative and total heat flux at the wire. It is seen that decreasing N increases the conductive and total heat flux. Decreasing the surface emissivities reduces the radiative heat flux. The conductive fluxes obtained in these tables are solved as part of the system of equations rather than being calculated after the temperature profiles are obtained.

TABLE 2

Heat Transfer Rates for Surface Emissivities of 0.1

$r_o^*=1.0$ $R^*=2.0$ $T^*_{wire}=1.0$ $T^*_{oven}=0.1$ $\varepsilon_1=\varepsilon_2=0.1$

ω_o	$\left.\frac{dR^*}{dr^*}\right\|_{r^*=r_o^*}$	$\left.Q_r\right\|_{r^*=r_o^*}$	Ψ
	N=1.0		
0	-1.343	0.076	1.362
0.5	-1.323	0.079	1.343
1.0	-1.298	0.062	1.314
	N=0.5		
0	-1.388	0.076	1.426
0.5	-1.348	0.079	1.387
1.0	-1.298	0.062	1.329
	N=0.1		
0	-1.731	0.076	1.921
0.5	-1.541	0.079	1.738
1.0	-1.298	0.062	1.454

TABLE 3

Heat Transfer Rates for Surface Emissivities of 0.5

$T^*_{wire} = 1.0$ $T^*_{oven} = 0.1$ $\varepsilon_1 = \varepsilon_2 = 0.5$ $r^*_o = 1.0$ $R^* = 2.0$

ω_o	$dT^*/dr^* \big\|_{r=r^*_o}$	$Q_r \big\|_{r=r^*_o}$	Ψ
	N=1.0		
0.0	-1.315	0.380	1.410
0.5	-1.302	0.384	1.398
1.0	-1.298	0.340	1.383
	N=0.5		
0.0	-1.333	0.379	1.523
0.5	-1.307	0.383	1.498
1.0	-1.298	0.340	1.468
	N=0.1		
0.0	-1.499	0.372	2.430
0.5	-1.363	0.378	2.308
1.0	-1.298	0.340	2.148

TABLE 4

Heat Transfer Rates for Surface Emissivities of 1.0

$T^*_{wire} = 1.0$ $T^*_{oven} = 0.1$ $\varepsilon_1 = \varepsilon_2 = 1.0$ $r^*_o = 1.0$ $R^* = 2.0$

ω_o	$\left.\frac{dT^*}{dr^*}\right\|_{r=r^*_o}$	$\left.Q_r\right\|_{r=r^*_o}$	Ψ
	N=1.0		
0	-1.284	0.777	1.479
0.5	-1.283	0.767	1.475
1.0	-1.298	0.702	1.474
	N=0.5		
0	-1.274	0.773	1.660
0.5	-1.270	0.765	1.653
1.0	-1.298	0.702	1.649
	N=0.1		
0	-1.273	0.748	3.144
0.5	-1.212	0.751	3.088
1.0	-1.298	0.702	3.053

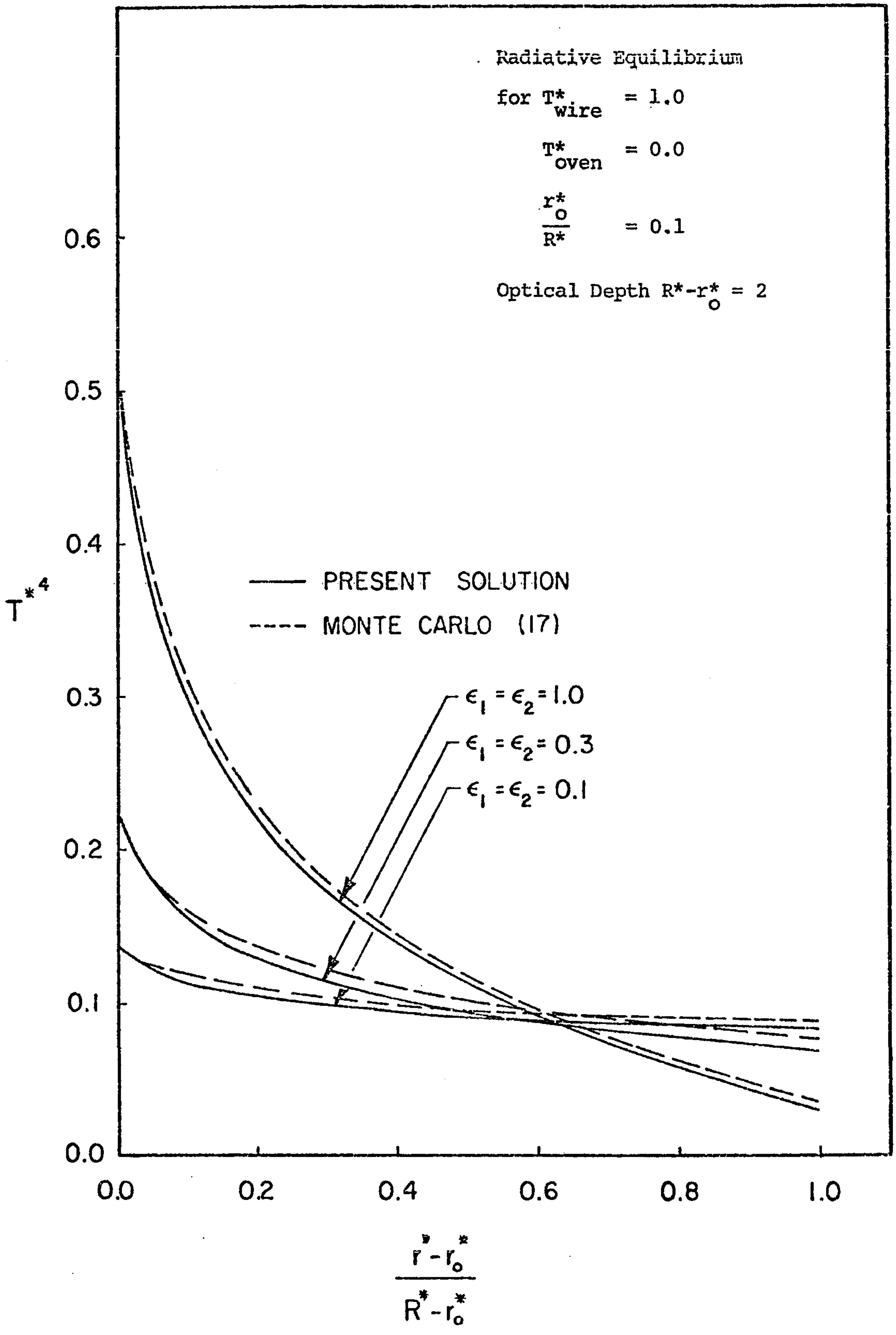

Figure 8. Radiative Equilibrium.

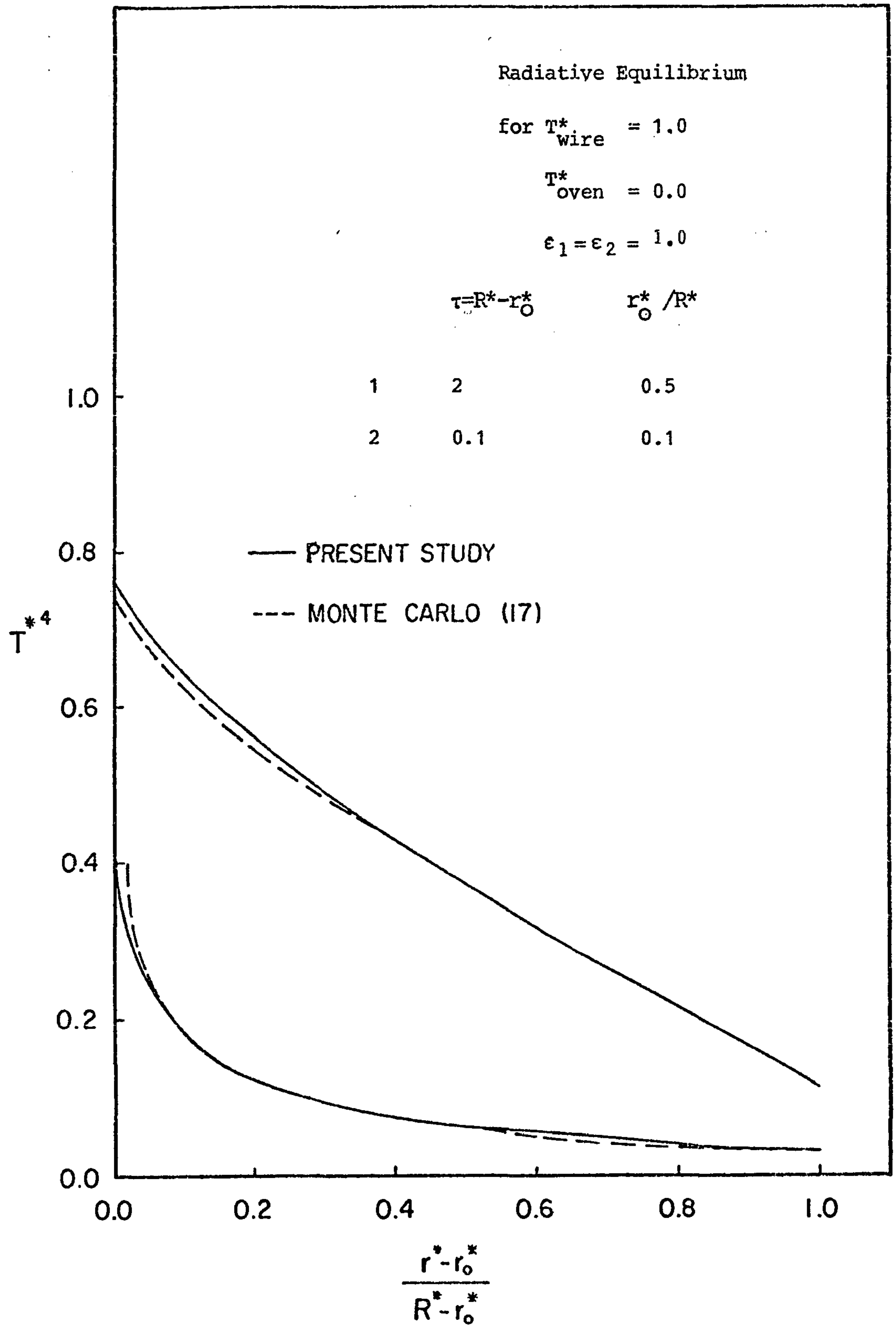

Figure 9. Radiative Equilibrium.

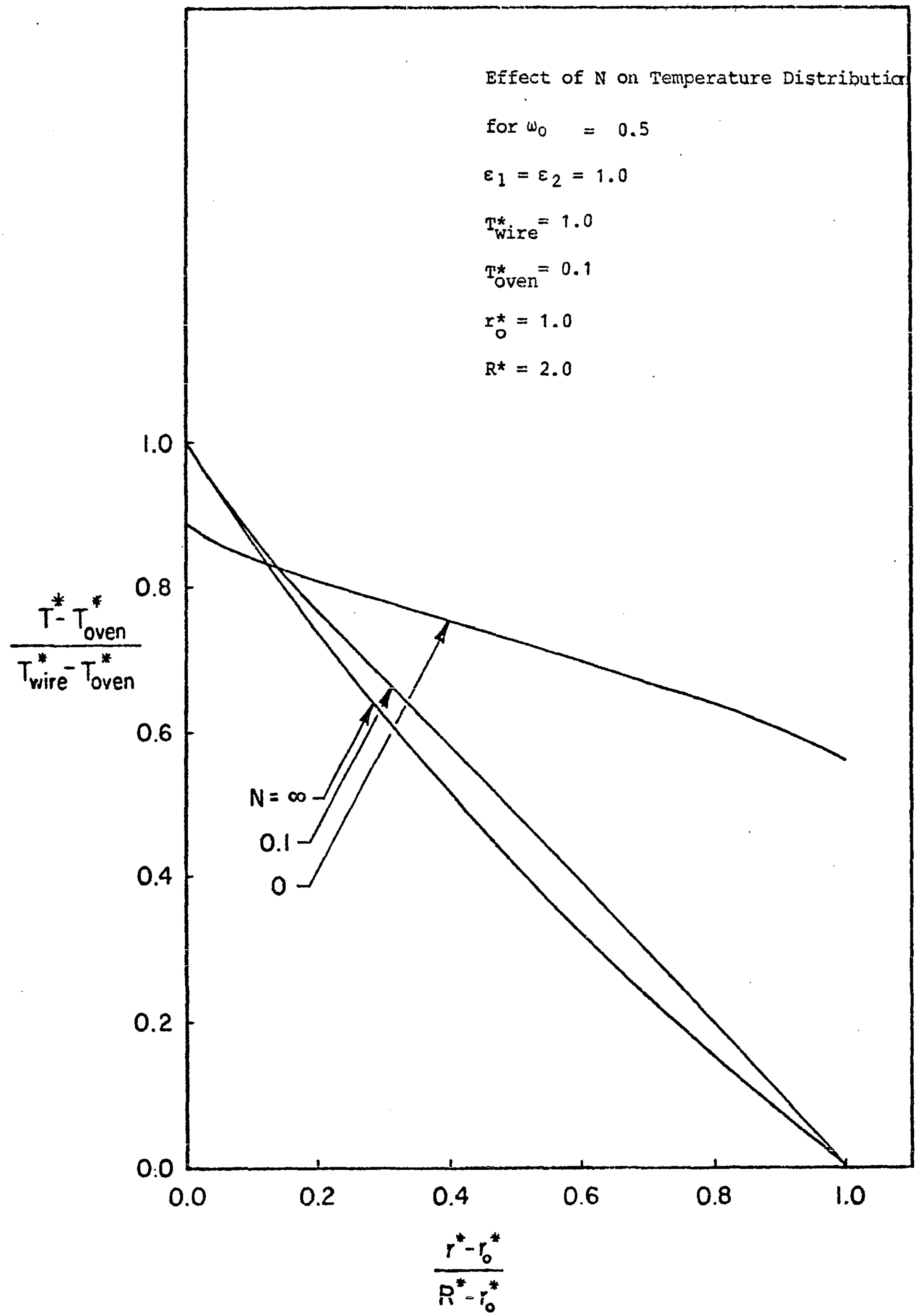

Figure 10. Effect of N on Temperature Distribution.

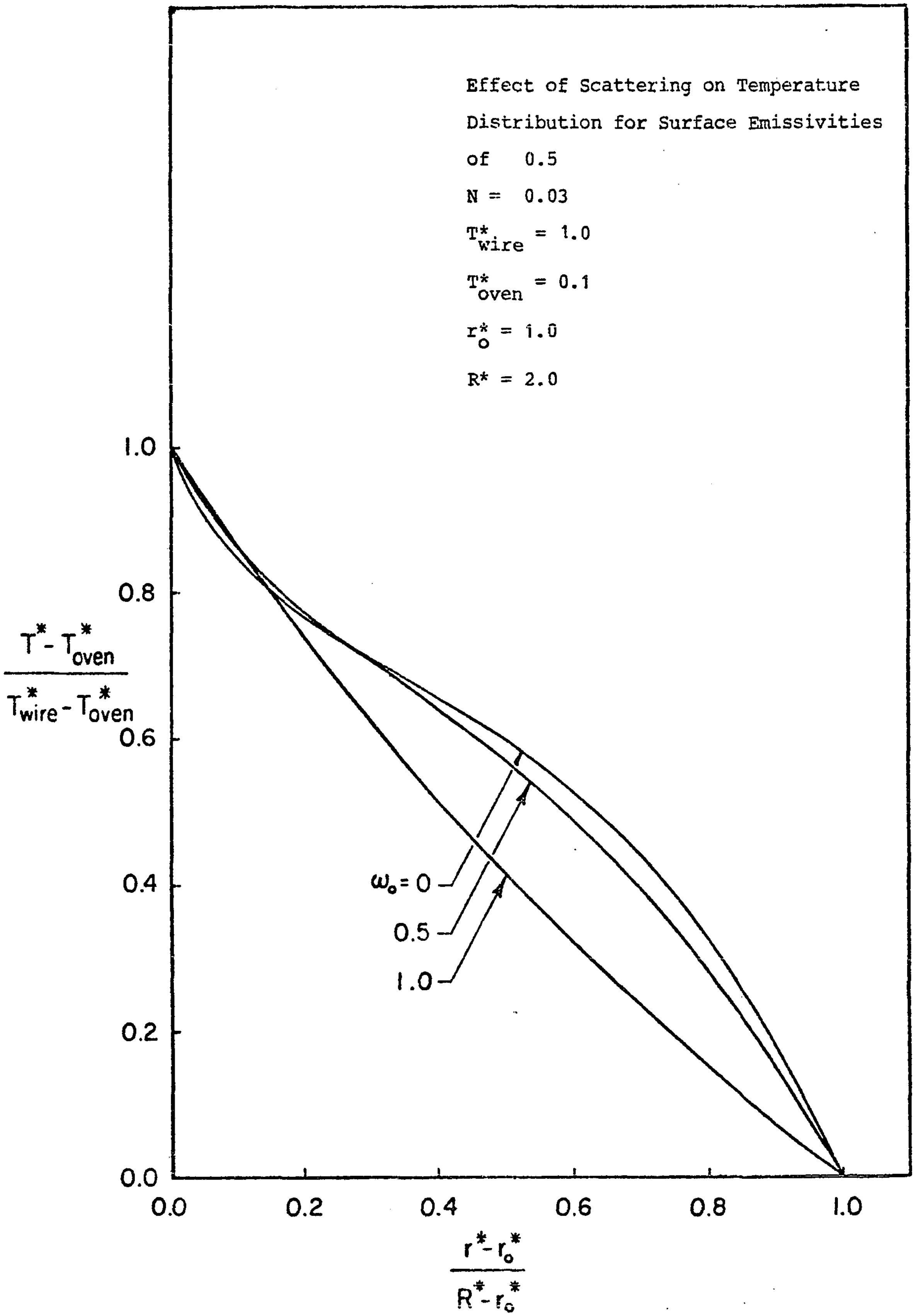

Figure 11. Effect of Scattering on Temperature Distribution for Surface Emissivities.

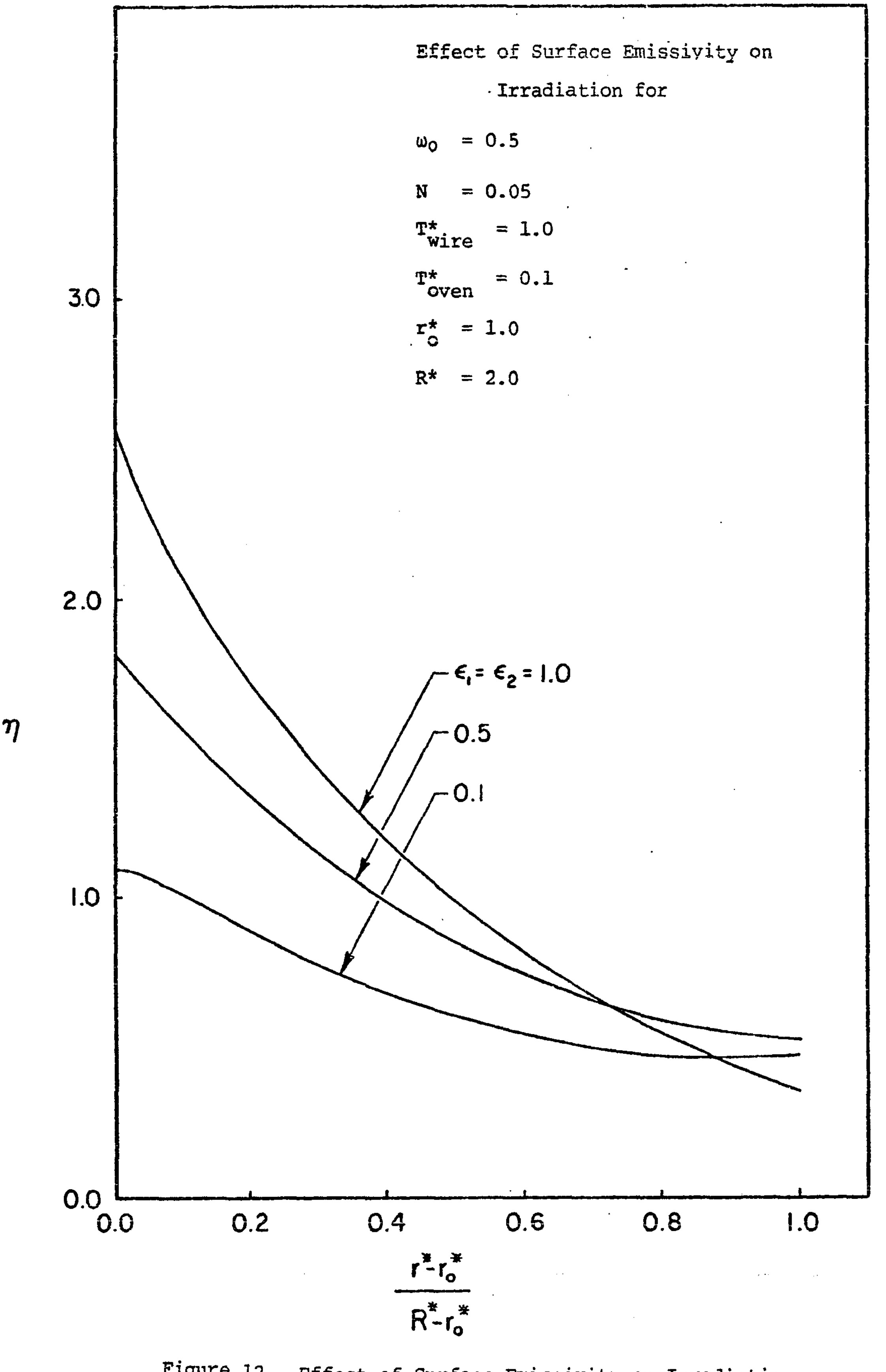

Figure 12. Effect of Surface Emissivity on Irradiation.

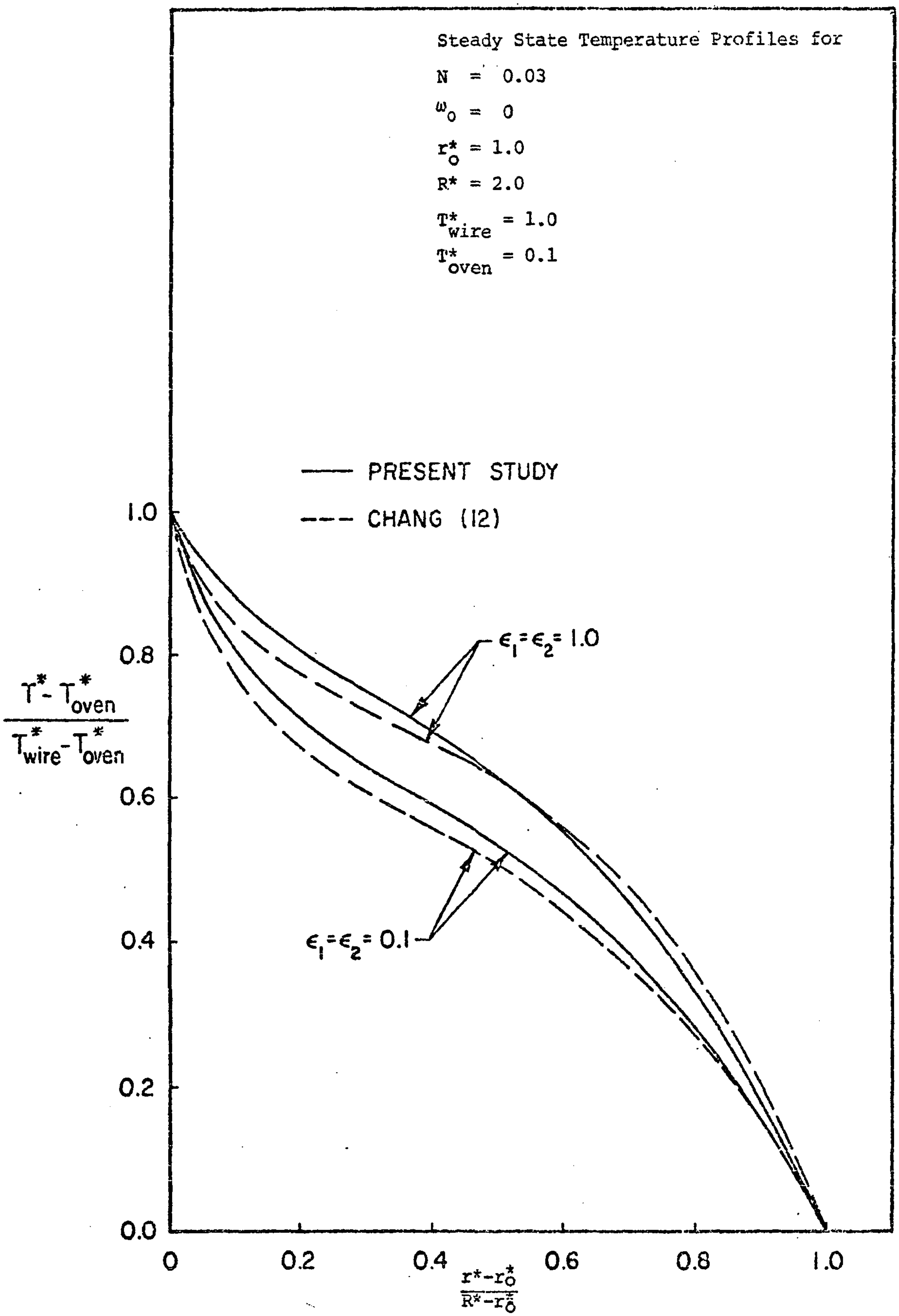

Figure 13. Steady State Temperature Profiles.

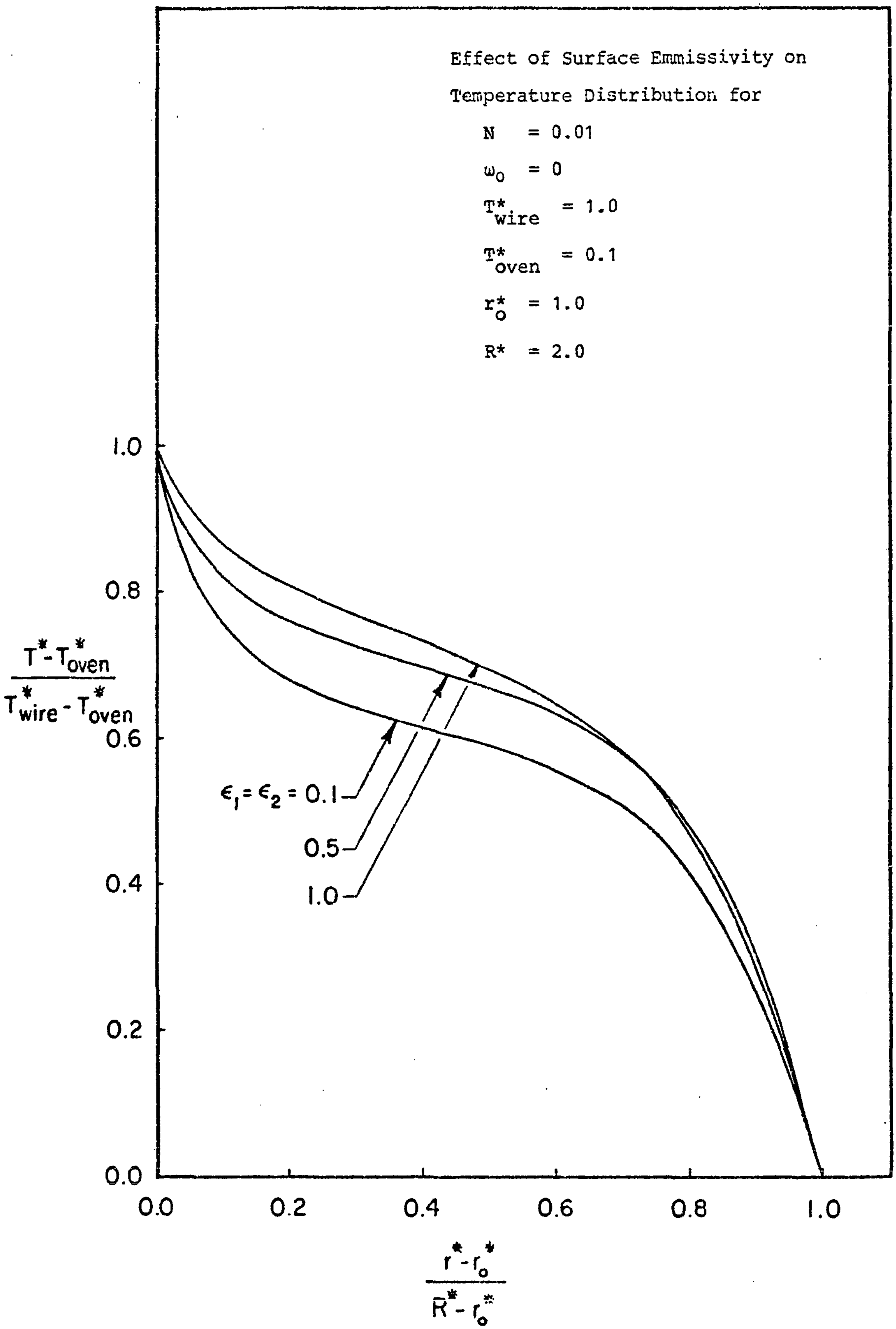

Figure 14. Effect of Surface Emissivity on Temperature Distribution.

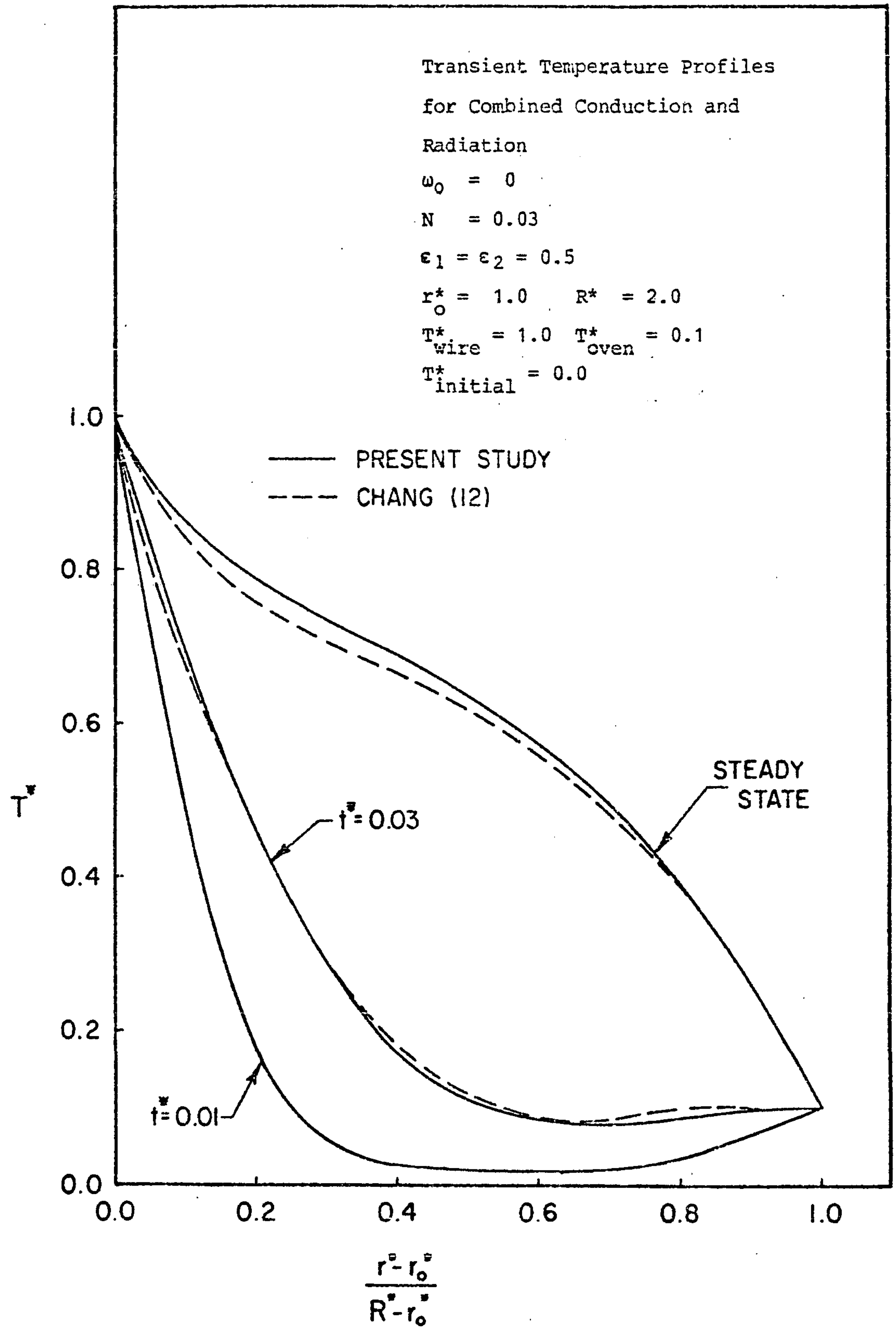

Figure 15. Transient Temperature Profiles for Combined Conduction and Radiation.

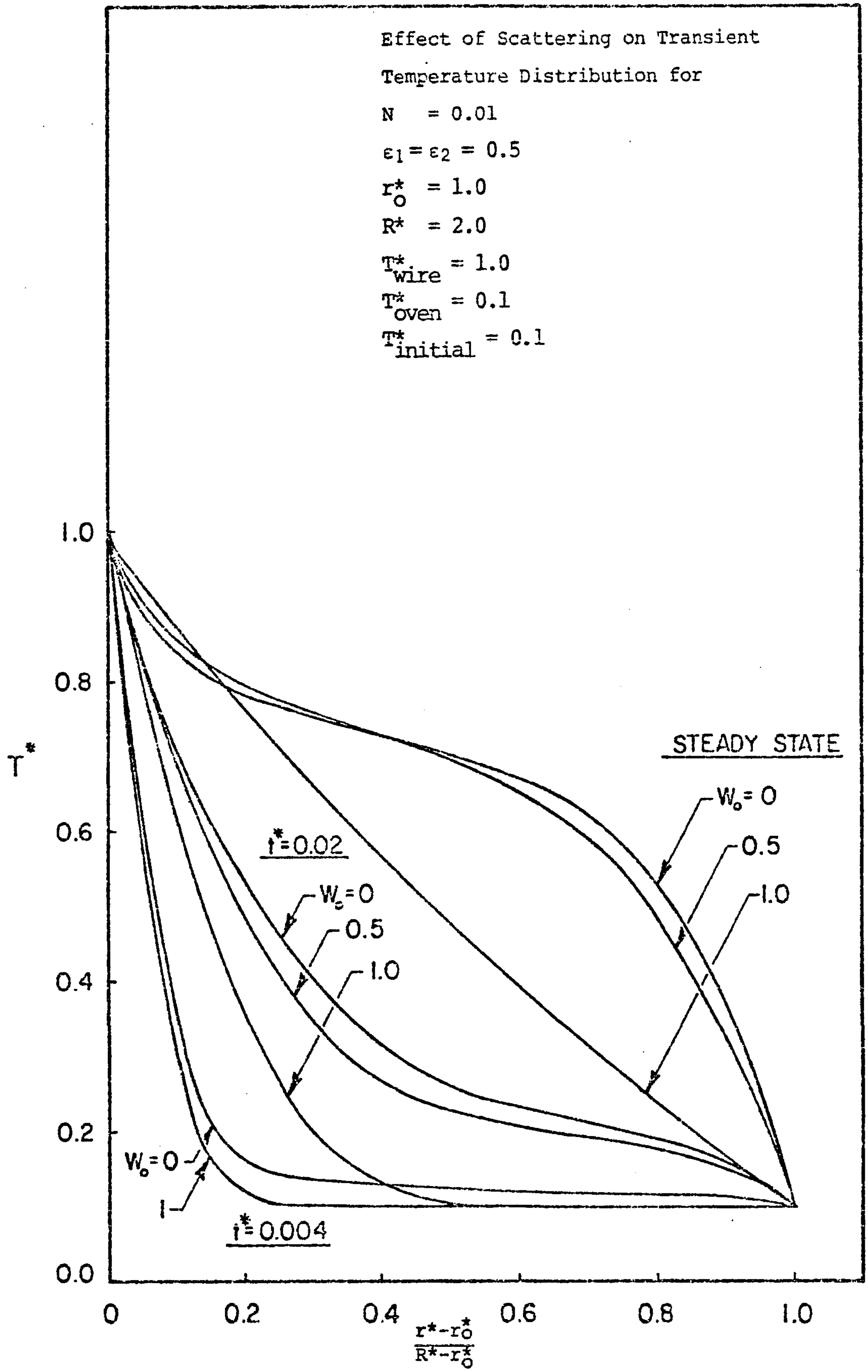

Figure 16. Effect of Scattering on Transient Temperature Distribution.

CHAPTER VI

CONCLUSIONS

This study has solved the problem of combined conduction and radiation in an absorbing, emitting and scattering medium of cylindrical geometry bounded by grey surfaces at known temperatures. The solutions compare well with known exact solutions. The author believes that the finite element technique is best suited for the problem because of the ease with which integrals with variable limits can be handled and the accuracy with which the conductive flux can be evaluated for the steady state case. Another advantage lies in the fact that the elements do not have to be of equal length and the programming was such that very few changes needed to be made to change the length of each element or to increase the number of elements. Nineteen or twenty elements were found to be sufficient for accuracy.

A drawback of the study was the inaccuracy of the Gauss quadrature when evaluating integrals with singularities. This problem will be present no matter what method is used to solve the governing equations. This study made use of the techniques outlined in Chapter IV to obtain good results. Another method could involve the use of a more suitable quadrature. It should be noted that the inability of the Gauss quadrature to properly evaluate the integrals with singularities had very little effect on the temperature distribution, but had a large effect on the irradiation η. Thus whenever the solution for η was crucial, i.e. radiative equilibrium cases ($T^{*4} = \frac{\eta}{4}$), poor results would be obtained unless the integrals were evaluated properly. However, if the solution for η is not crucial to the problem, i.e. most combined conduction and radiation problems, the solution for the temperature distribution is fairly accurate.

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APPENDIX

```
//FE        JOB ########,'FERNANDOES              ',CLASS=J,TYPRUN=COPY        JOE
/*JOBPARM D=NEL,F=9002
/*ROUTE  PRINT LOCAL
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(4),W(4),R(21),T(21),GSTIF(84,85),C3(20,20,5),ETA(21)
      DIMENSION D4(20,20,5),C2(20,20,2,5),D1(20,2,5),GSAVE(84,85)
      DIMENSION GM(21,21),TOLD(21,1),TOTOLD(21,1),TNEW(21,1)
      DIMENSION TOTNEW(21,1),RI(21,1),RHS(21)
      DIMENSION Y(6),V(6)
      F1(RP,BETA)=DSQRT(RP**2-(RL**2)*(DSIN(BETA)**2))
      F2(ALP,BETA)=(DCOS(ALP)**2)*DCOS(BETA)/DEXP(2.0*RL*DCOS(BETA)/
     1DCOS(ALP))
      F3(RP,BETA,ALP)=DCOS(ALP)*DCOS(BETA)*RP/F1(RP,BETA)*
     1(DEXP((F1(RP,BETA)-RL*DCOS(BETA))/DCOS(ALP))
     2+1.0/DEXP((F1(RP,BETA)+RL*DCOS(BETA))/DCOS(ALP)))
      SF1(RP)=((-1)**I)*(RP-FF+DFLOAT(I-1)*(FF-EE))/(FF-EE)
      SF2(RR)=((-1)**J)*(RR-HH+DFLOAT(J-1)*(HH-GG))/(HH-GG)
      SF3(RR)=((-1)**I)*(RR-HH+DFLOAT(I-1)*(HH-GG))/(HH-GG)
      SSF1(RP)=(FF-RP)/(FF-EE)
      SSF2(RP)=(RP-EE)/(FF-EE)
      F4(ALP,BETA)=(DCOS(ALP)**2)*DCOS(BETA)*DEXP((F1(RS,BETA)-RL*
     1DCOS(BETA))/DCOS(ALP))
      F5(RP,BETA,ALP)=DCOS(ALP)*DCOS(BETA)*RP/F1(RP,BETA)*DEXP((F1(RS,
     1BETA)-F1(RP,BETA))/DCOS(ALP))
      F6(RR,BETA,ALP)=DCOS(ALP)*DCOS(BETA)/F1(RR,BETA)*DEXP((F1(RR,BETA)
     1-RL*DCOS(BETA))/DCOS(ALP))
      F7(RR,BETA,ALP)=DCOS(ALP)*DCOS(BETA)/F1(RR,BETA)*DEXP((F1(RS,BETA)
     1-F1(RR,BETA))/DCOS(ALP))
      F8(RR,BETA,ALP)=DCOS(ALP)*DCOS(BETA)/F1(RR,BETA)/DEXP((F1(RR,BETA)
     1+RL*DCOS(BETA))/DCOS(ALP))
      F9(RR,RP,BETA,ALP)=DCOS(BETA)/F1(RR,BETA)*RP/F1(RP,BETA)*
     1DEXP((F1(RR,BETA)-F1(RP,BETA))/DCOS(ALP))
      F10(RR,RP,BETA,ALP)=DCOS(BETA)/F1(RR,BETA)*RP/F1(RP,BETA)*
     1DEXP((F1(RP,BETA)-F1(RR,BETA))/DCOS(ALP))
      F11(RR,RP,BETA,ALP)=DCOS(BETA)/F1(RR,BETA)*RP/F1(RP,BETA)/
     1DEXP((F1(RP,BETA)+F1(RR,BETA))/DCOS(ALP))
      F12(RR)=SF2(RR)*((-1)**I)/(HH-GG)/RR
      SSF3(RR)=(HH-RR)/(HH-GG)
      SSF4(RR)=(RR-GG)/(HH-GG)
      F13(RP,BETA,ALP)=DCOS(ALP)*DCOS(BETA)*RP/F1(RP,BETA)
     1*DEXP((F1(RP,BETA)-RL*DCOS(BETA))/DCOS(ALP))
      F20(RR,ALP)=DCOS(ALP)/DEXP(DSQRT(RL**2-RR**2)/DCOS(ALP))
      F21(RR,RP,ALP)=RP/DSQRT(RP**2-RR**2)/DEXP(DSQRT(RP**2-RR**2)/
     1DCOS(ALP))
      F22(RR,ALP)=DCOS(ALP)*DEXP((RATY*RR-DSQRT(RL**2-(RAT*RR)**2))
     1/DCOS(ALP))
      F23(RR,RP,ALP)=RP/DSQRT(RP**2-(RAT*RR)**2)*DEXP((RATY*RR
     1-DSQRT(RP**2-(RAT*RR)**2))/DCOS(ALP))
      F24(RR,RP,ALP)=RP/DSQRT(RP**2-(RAT*RR)**2)/DEXP((RATY*RR
     1-DSQRT(RP**2-(RAT*RR)**2))/DCOS(ALP))
      DATA Y/-0.2386191861,0.2386191861,-0.6612093865,0.6612093865,
     1-0.9324695142,0.9324695142/
      DATA V/2*0.4679139346,2*0.360761573,2*0.1713244924/
      DATA X/-0.339981044,0.339981044,-0.861136312,0.861136312/
      DATA W/2*0.652145155,2*0.347854845/
      NIP=1
      NN=21
      RS=1.0
      RL=2.0
      DO 490 I=1,NN
  490 R(I)=RS+DFLOAT(I-1)*(RL-RS)/(NN-1)
      NQUAD=4
      PI=3.1415927
```

```
      AA=0.0
      BB=PI/2.0
      NEM=NN-1
      AN=0.5
      WO=0.0
      RHO1=0.0
      RHO2=0.0
      PROP=0.5
      TWIRE=1.0
      TOVEN=0.1
      ITYPE=0
      TINIT=0.1
      DELT=0.001
C
C   EVALUATE THE INTEGRALS FOR THE PHI MINUS(R OUTER) EQUATION
C
      DO 12 I=1,NEM
      DO 12 J=1,NEM
      DO 12 K=1,5
      C3(I,J,K)=0.0
      D4(I,J,K)=0.0
      DO 12 L=1,2
      D1(I,L,K)=0.0
      C2(I,J,L,K)=0.0
   12 CONTINUE
      CC=DARSIN(RS/RL)
      DD=PI/2.0
      TLMT=DD
       DD=CC+(DD-CC)*PRCP
      SUM=0.0
      DO 1010 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      A=(BB-AA)/2.0
      B=(BB+AA)/2.0
      DO 1 I=1,NQUAD
      DO 1 J=1,6
    1 SUM=SUM+W(I)*V(J)*F2(X(I)*A+B,Y(J)*C+D)*C*A
      CC=DD
      DD=TLMT
 1010 CONTINUE
      DENOM=1.0-4.0*RHO1*SUM/PI
      NR=NN*4
      NC=NR+1
      DO 1000 I=1,NR
      DO 1000 J=1,NC
 1000 GSTIF(I,J)=0.0
      IF(RHO1.EQ.0.0)GO TO 600
      IF(WO.EQ.0.0)GO TO 601
      DO 1020 I=1,2
      CC=DARSIN(RS/RL)
      DD=PI/2.0
      TLMT=DD
       DD=CC+(DD-CC)*PRCP
      DO 1020 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      DO 6 M=1,6
      SUM=0.0
      EE=RL*DSIN(Y(M)*C+D)
      IF(EE.GE.R(1).AND.EE.LE.R(NN))GO TO 5
      WRITE(6,510)EE
  510 FORMAT(1X,'VALUE OF EE EXCEEDS THE LIMITS',E20.6)
    5 CONTINUE
      DO 3 ICHECK=1,NN
      IF(EE.LE.R(ICHECK))GO TO 4
```

```
    3 CONTINUE
    4 FF=R(ICHECK)
      E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      EE=R(ICHECK-1)
      FF=R(ICHECK)
      DO 2 L=1,NQUAD
      DO 2 N=1,NQUAD
      SUM=SUM+W(L)*V(M)*W(N)*A*C*E*F3(X(N)*E+F,Y(M)*C+D,X(L)*A+B)
     1*SF1(X(N)*E+F)
    2 CONTINUE
      NELEM=ICHECK-1
      SUM=-SUM*RHO1*WO/PI/DENOM
      DO 100 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      GSTIF(IL+2*NN,NELEM+NN+I-1)=GSTIF(IL+2*NN,NELEM+NN+I-1)+SUM
     1*(HH-GG)/2.0
      GSTIF(IL+2*NN+1,NELEM+NN+I-1)=GSTIF(IL+2*NN+1,NELEM+NN+I-1)+SUM
     1*(HH-GG)/2.0
  100 CONTINUE
      IF(ICHECK.EQ.NN)GO TO 6
      ICHECK=ICHECK+1
      EE=R(ICHECK-1)
      SUM=0.0
      GO TO 4
    6 CONTINUE
      CC=DD
      DD=TLMT
 1020 CONTINUE
  601 CONTINUE
      IF(WO.EQ.1.0)GO TO 600
      CC=DARSIN(RS/RL)
      DD=PI/2.0
      TLMT=DD
       DD=CC+(DD-CC)*PRCP
      DO 1030 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      DO 7 M=1,6
      EE=RL*DSIN(Y(M)*C+D)
      DO 8 ICHECK=1,NN
      IF(EE.LE.R(ICHECK))GO TO 9
    8 CONTINUE
    9 FF=R(ICHECK)
      NELEM=ICHECK-1
      E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      EE=R(ICHECK-1)
      FF=R(ICHECK)
      DO 10 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      DO 10 L=1,NQUAD
      DO 10 N=1,NQUAD
      FAKE=W(L)*V(M)*W(N)*A*C*E*F3(X(N)*E+F,Y(M)*C+D,X(L)*A+B)
     1*4.0*RHO1*(1.0-WO)/PI*(HH-GG)/2.0/DENOM
      C3(IL,NELEM,1)=C3(IL,NELEM,1)+FAKE*(SSF1(X(N)*E+F)**4)
      C3(IL,NELEM,2)=C3(IL,NELEM,2)+FAKE*(SSF1(X(N)*E+F)**3)
     1*SSF2(X(N)*E+F)
      C3(IL,NELEM,3)=C3(IL,NELEM,3)+FAKE*(SSF1(X(N)*E+F)**2)
     1*(SSF2(X(N)*E+F)**2)
      C3(IL,NELEM,4)=C3(IL,NELEM,4)+FAKE*SSF1(X(N)*E+F)
     1*(SSF2(X(N)*E+F)**3)
      C3(IL,NELEM,5)=C3(IL,NELEM,5)+FAKE*(SSF2(X(N)*E+F)**4)
   10 CONTINUE
```

```
      IF(ICHECK.EQ.NN)GO TO 7
      ICHECK=ICHECK+1
      EE=R(ICHECK-1)
      GO TO 9
    7 CONTINUE
      CC=DD
      DD=TLMT
 1030 CONTINUE
  600 CONTINUE
      DO 11 J=1,2
      DO 11 I=1,2
      DO 11 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      IF((I+J).EQ.3)GO TO 13
      GSTIF(NN*2+IL+J-1,NN*2+IL+I-1)=GSTIF(NN*2+IL+J-1,NN*2+IL+I-1)
     1+(HH-GG)/3.0
      GO TO 11
   13 GSTIF(NN*2+IL+J-1,NN*2+IL+I-1)=GSTIF(NN*2+IL+J-1,NN*2+IL+I-1)
     1+(HH-GG)/6.0
   11 CONTINUE
      DO 112 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      GSTIF(NN*2+IL,NC)=GSTIF(NN*2+IL,NC)+(HH-GG)/2.0*(1.0-RHO1)/DENOM
     1*(TOVEN**4)
      GSTIF(NN*2+IL+1,NC)=GSTIF(NN*2+IL+1,NC)+(HH-GG)/2.0*(1.0-RHO1)
     1/DENOM*(TOVEN**4)
  112 CONTINUE
      IF(RHO1.EQ.0.0)GO TO 2602
      CC=0.0
      DD=DARSIN(RS/RL)
      TLMT=DD
      DD=CC+(DD-CC)*PROP
      DO 2000 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      IF(WO.EQ.0.0)GO TC 604
      DO 300 I=1,2
      DO 300 NELEM=1,NEM
      EE=R(NELEM)
      FF=R(NELEM+1)
      E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      SUM=0.0
      DO 301 L=1,NQUAD
      DO 301 M=1,6
      DO 301 N=1,NQUAD
  301 SUM=SUM+W(L)*V(M)*W(N)*A*C*E*F13(X(N)*E+F,Y(M)*C+D,X(L)*A+B)
     1*SF1(X(N)*E+F)
      SUM=-SUM*WC*RHC1/PI/DENOM
      DO 300 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      GSTIF(NN*2+IL,NN+NELEM+I-1)=GSTIF(NN*2+IL,NN+NELEM+I-1)
     1+SUM*(HH-GG)/2.0
      GSTIF(NN*2+IL+1,NN+NELEM+I-1)=GSTIF(NN*2+IL+1,NN+NELEM+I-1)
     1+SUM*(HH-GG)/2.0
  300 CONTINUE
  604 CONTINUE
      IF(WO.EQ.1.0)GO TC 602
      DO 302 NELEM=1,NEM
      EE=R(NELEM)
      FF=R(NELEM+1)
      E=(FF-EE)/2.0
      F=(FF+EE)/2.0
```

```
      DO 302 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      DO 302 L=1,NQUAD
      DO 302 M=1,6
      DO 302 N=1,NQUAD
      FAKE=W(L)*V(M)*W(N)*A*C*E*F13(X(N)*E+F,Y(M)*C+D,X(L)*A+B)
     1*4.0*RHO1*(1.0-WO)/PI*(HH-GG)/2.0/DENOM
      C3(IL,NELEM,1)=C3(IL,NELEM,1)+FAKE*(SSF1(X(N)*E+F)**4)
      C3(IL,NELEM,2)=C3(IL,NELEM,2)+FAKE*(SSF1(X(N)*E+F)**3)
     1*SSF2(X(N)*E+F)
      C3(IL,NELEM,3)=C3(IL,NELEM,3)+FAKE*(SSF1(X(N)*E+F)**2)
     1*(SSF2(X(N)*E+F)**2)
      C3(IL,NELEM,4)=C3(IL,NELEM,4)+FAKE*SSF1(X(N)*E+F)
     1*(SSF2(X(N)*E+F)**3)
      C3(IL,NELEM,5)=C3(IL,NELEM,5)+FAKE*(SSF2(X(N)*E+F)**4)
  302 CONTINUE
  602 CONTINUE
      CC=DD
      DD=TLMT
 2000 CONTINUE
 2602 CONTINUE
C
C     INTEGRALS FOR THE EQUATION FOR PHI ONE PLUS(R INNER)
C
      CC=0.0
      DD=DARSIN(RS/RL)
      TLMT=DD
      DD=CC+(DD-CC)*FRCP
      SUM=0.0
      DO 2010 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      DO 14 L=1,NCUAD
      DO 14 M=1,6
   14 SUM=SUM+F4(X(L)*A+B,Y(M)*C+D)*W(L)*V(M)*A*C
      CC=DD
      DD=TLMT
 2010 CONTINUE
      SUM1=-SUM*4.0*RHO1/PI/DENOM
      SUM=-SUM*4.0*RL*RHO2/(PI*RS)
      DO 15 J=1,2
      DO 15 I=1,2
      DO 15 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      IF((I+J).EQ.3)GO TO 16
      GSTIF(NN*3+IL+J-1,NN*3+IL+I-1)=GSTIF(NN*3+IL+J-1,NN*3+IL+I-1)
     1+(HH-GG)/3.0
      GSTIF(NN*3+IL+J-1,NN*2+IL+I-1)=GSTIF(NN*3+IL+J-1,NN*2+IL+I-1)
     1+SUM*(HH-GG)/3.0
      GSTIF(NN*2+IL+J-1,NN*3+IL+I-1)=GSTIF(NN*2+IL+J-1,NN*3+IL+I-1)
     1+SUM1*(HH-GG)/3.0
      GO TO 15
   16 GSTIF(NN*3+IL+J-1,NN*2+IL+I-1)=GSTIF(NN*3+IL+J-1,NN*2+IL+I-1)
     1+SUM*(HH-GG)/6.0
      GSTIF(NN*3+IL+J-1,NN*3+IL+I-1)=GSTIF(NN*3+IL+J-1,NN*3+IL+I-1)
     1+(HH-GG)/6.0
      GSTIF(NN*2+IL+J-1,NN*3+IL+I-1)=GSTIF(NN*2+IL+J-1,NN*3+IL+I-1)
     1+SUM1*(HH-GG)/6.0
   15 CONTINUE
      IF(RHO2.EQ.0.0)GO TO 2603
      CC=0.0
      DD=DARSIN(RS/RL)
      TLMT=DD
      DD=CC+(DD-CC)*FRCP
```

```
      DO 2020 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      IF(WO.EQ.0.0)GO TO 605
      DO 17 I=1,2
      DO 17 NELEM=1,NEM
      EE=R(NELEM)
      FF=R(NELEM+1)
      E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      SUM=0.0
      DO 18 L=1,NQUAD
      DO 18 M=1,6
      DO 18 N=1,NQUAD
   18 SUM=SUM+F5(X(N)*E+F,Y(M)*C+D,X(L)*A+B)*SF1(X(N)*E+F)*W(L)*
     1V(M)*W(N)*A*C*E
      SUM=-SUM*WO*RL*RHC2/(PI*RS)
      DO 19 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      GSTIF(NN*3+IL,NN+NELEM+I-1)=GSTIF(NN*3+IL,NN+NELEM+I-1)
     1+SUM*(HH-GG)/2.0
      GSTIF(NN*3+IL+1,NN+NELEM+I-1)=GSTIF(NN*3+IL+1,NN+NELEM+I-1)
     1+SUM*(HH-GG)/2.0
   19 CONTINUE
   17 CONTINUE
  605 CONTINUE
      IF(WO.EQ.1.0)GO TC 603
      DO 20 NELEM=1,NEM
      EE=R(NELEM)
      FF=R(NELEM+1)
      E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      DO 20 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      DO 20 L=1,NQUAD
      DO 20 M=1,6
      DO 20 N=1,NQUAD
      FAKE=W(L)*V(M)*W(N)*A*C*E*F5(X(N)*E+F,Y(M)*C+C,X(L)*A+B)
     1*4.0*(1.0-WO)*RL*RHO2/(PI*RS)*(HH-GG)/2.0
      D4(IL,NELEM,1)=D4(IL,NELEM,1)+FAKE*(SSF1(X(N)*E+F)**4)
      D4(IL,NELEM,2)=D4(IL,NELEM,2)+FAKE*(SSF1(X(N)*E+F)**3)
     1*(SSF2(X(N)*E+F))
      D4(IL,NELEM,3)=D4(IL,NELEM,3)+FAKE*(SSF1(X(N)*E+F)**2)
     1*(SSF2(X(N)*E+F)**2)
      D4(IL,NELEM,4)=D4(IL,NELEM,4)+FAKE*(SSF1(X(N)*E+F))
     1*(SSF2(X(N)*E+F)**3)
      D4(IL,NELEM,5)=D4(IL,NELEM,5)+FAKE*(SSF2(X(N)*E+F)**4)
   20 CONTINUE
  603 CONTINUE
      CC=DD
      DD=TLMT
 2020 CONTINUE
 2603 CONTINUE
      DO 21 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      GSTIF(NN*3+IL,NC)=GSTIF(NN*3+IL,NC)+(1.0-RHO2)*(TWIRE**4)
     1*(HH-GG)/2.0
      GSTIF(NN*3+IL+1,NC)=GSTIF(NN*3+IL+1,NC)+(1.0-RHO2)*(TWIRE**4)
     1*(HH-GG)/2.0
   21 CONTINUE
C
C     INTEGRALS FOR ETA
C
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      DO 22 J=1,2
      DO 22 I=1,2
      DO 22 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      IF((I+J).EQ.3)GO TO 23
      GSTIF(NN+IL+J-1,NN+IL+I-1)=GSTIF(NN+IL+J-1,NN+IL+I-1)+(HH-GG)/3.0
      GO TO 22
   23 GSTIF(NN+IL+J-1,NN+IL+I-1)=GSTIF(NN+IL+J-1,NN+IL+I-1)+(HH-GG)/6.0
   22 CONTINUE
      RAT=0.97
      RATY=DSQRT(1.0D0-RAT**2)
      VAL=PI/2.0-DARSIN(RAT)
      NEXC=1
      DO 3000 J=1,2
      DO 3000 I=1,2
      DO 3000 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
       G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      SUM=0.0
      SUM1=0.0
      DO 4000 L=1,NQUAD
      DO 4000 K=1,NQUAD
      SUM=SUM+W(L)*W(K)*A*G*F20(X(K)*G+H,X(L)*A+B)
     1*SF2(X(K)*G+H)*SF3(X(K)*G+H)
      IF(IL.LE.NEXC)GO TO 4000
      SUM1=SUM1+W(L)*W(K)*A*G*F22(X(K)*G+H,X(L)*A+B)
     1*SF2(X(K)*G+H)*SF3(X(K)*G+H)
 4000 CONTINUE
      SUM=-SUM*4.0*VAL/PI
      SUM1=-SUM1*4.0*VAL/PI
      GSTIF(NN+IL+J-1,NN*2+IL+I-1)=GSTIF(NN+IL+J-1,NN*2+IL+I-1)+SUM
     1+SUM1
 3000 CONTINUE
      DO 24 J=1,2
      DO 24 I=1,2
      DO 24 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      SUM=0.0
      DO 1040 K=1,NQUAD
      CC=0.0
      RAT=0.97
      IF(IL.LE.NEXC)RAT=1.0
      DD=DARSIN((X(K)*G+H)/RL*RAT)
      TLMT=DD
      DD=CC+(DD-CC)*FROP
      DO 1040 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      DO 25 L=1,NQUAD
      DO 25 M=1,6
      SUM=SUM+W(L)*V(M)*W(K)*A*C*G*F6(X(K)*G+H,Y(M)*C+D,X(L)*A+B)
     1*SF2(X(K)*G+H)*SF3(X(K)*G+H)
   25 CONTINUE
      CC=DD
      DD=TLMT
 1040 CONTINUE
      SUM=-SUM*4.0*RL/PI
      GSTIF(NN+IL+J-1,NN*2+IL+I-1)=GSTIF(NN+IL+J-1,NN*2+IL+I-1)+SUM
   24 CONTINUE
      IF(WO.EQ.0.0)GO TO 615
```

```
      DO 5000 J=1,2
      DO 5000 I=1,2
      DO 5000 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      DO 5000 K=1,NQUAD
      ICHECK=IL+1
      EE=X(K)*G+H
 5020 FF=R(ICHECK)
      E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      NELEM=ICHECK-1
      EE=R(NELEM)
      SUM=0.0
      SUM1=0.0
      DO 5030 L=1,NQUAD
      DO 5030 N=1,NQUAD
      SUM=SUM+W(K)*W(L)*W(N)*A*G*E*F21(X(K)*G+H,X(N)*E+F,X(L)*A+B)
     1*SF1(X(N)*E+F)*SF2(X(K)*G+H)
      IF(IL.LE.NEXC)GO TO 5030
      SUM1=SUM1+W(K)*W(L)*W(N)*A*G*E*F23(X(K)*G+H,X(N)*E+F,X(L)*A+B)
     1*SF1(X(N)*E+F)*SF2(X(K)*G+H)
 5030 CONTINUE
      SUM=-SUM*WO*VAL/PI
      SUM1=-SUM1*WO*VAL/PI
      GSTIF(NN+IL+J-1,NN+NELEM+I-1)=GSTIF(NN+IL+J-1,NN+NELEM+I-1)+SUM
     1+SUM1
      IF(ICHECK.EQ.NN)GO TO 5000
      ICHECK=ICHECK+1
      EE=R(ICHECK-1)
      GO TO 5020
 5000 CONTINUE
  615 CONTINUE
      IF(WO.EQ.1.0)GO TO 616
      DO 5050 J=1,2
      DO 5050 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      DO 5050 K=1,NQUAD
      ICHECK=IL+1
      EE=X(K)*G+H
 5060 FF=R(ICHECK)
      E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      NELEM=ICHECK-1
      EE=R(NELEM)
      DO 5070 L=1,NQUAD
      DO 5070 N=1,NQUAD
      IF(IL.LE.NEXC)GO TO 7050
      EXC=1.0
      GO TO 7060
 7050 EXC=0.0
 7060 CONTINUE
      FAKE=W(K)*W(L)*W(N)*A*G*E*SF2(X(K)*G+H)*4.0*VAL*(1.0-WO)/PI
     1*(F21(X(K)*G+H,X(N)*E+F,X(L)*A+B)+F23(X(K)*G+H,X(N)*E+F,X(L)*A+B)
     2*EXC)
      C2(IL,NELEM,J,1)=C2(IL,NELEM,J,1)+FAKE*(SSF1(X(N)*E+F)**4)
      C2(IL,NELEM,J,2)=C2(IL,NELEM,J,2)+FAKE*(SSF1(X(N)*E+F)**3)
     1*(SSF2(X(N)*E+F))
      C2(IL,NELEM,J,3)=C2(IL,NELEM,J,3)+FAKE*(SSF1(X(N)*E+F)**2)
     1*(SSF2(X(N)*E+F)**2)
      C2(IL,NELEM,J,4)=C2(IL,NELEM,J,4)+FAKE*(SSF1(X(N)*E+F))
```

```
     1*(SSF2(X(N)*E+F)**3)
      C2(IL,NELEM,J,5)=C2(IL,NELEM,J,5)+FAKE*(SSF2(X(N)*E+F)**4)
 5070 CONTINUE
      IF(ICHECK.EQ.NN)GC TO 5050
      ICHECK=ICHECK+1
      EE=R(ICHECK-1)
      GO TO 5060
 5050 CONTINUE
  616 CONTINUE
      CC=0.0
      DD=DARSIN(RS/RL)
      TLMT=DD
      DD=CC+(DD-CC)*PROP
      DO 2030 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      DO 26 J=1,2
      DO 26 I=1,2
      DO 26 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      SUM=0.0
      DO 27 K=1,NCUAD
      DO 27 L=1,NQUAD
      DO 27 M=1,6
      SUM=SUM+W(L)*V(M)*W(K)*A*C*G*F7(X(K)*G+H,Y(M)*C+D,X(L)*A+B)
     1*SF2(X(K)*G+H)*SF3(X(K)*G+H)
   27 CONTINUE
      SUM=-SUM*4.C*RL/PI
      GSTIF(NN+IL+J-1,NN*3+IL+I-1)=GSTIF(NN+IL+J-1,NN*3+IL+I-1)+SUM
   26 CONTINUE
      CC=DD
      DD=TLMT
 2030 CONTINUE
      DO 28 J=1,2
      DO 28 I=1,2
      DO 28 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      SUM=0.0
      DO 1050 K=1,NQUAD
      CC=DARSIN(RS/RL)
      DD=DARSIN((X(K)*G+H)/RL*RAT)
      TLMT=DD
      DD=CC+(DD-CC)*PROP
      DO 1050 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      DO 29 L=1,NQUAD
      DO 29 M=1,6
      SUM=SUM+W(L)*V(M)*W(K)*A*C*G*F8(X(K)*G+H,Y(M)*C+D,X(L)*A+B)
     1*SF2(X(K)*G+H)*SF3(X(K)*G+H)
   29 CONTINUE
      CC=DD
      DD=TLMT
 1050 CONTINUE
      SUM=-SUM*4.0*RL/PI
      GSTIF(NN+IL+J-1,NN*2+IL+I-1)=GSTIF(NN+IL+J-1,NN*2+IL+I-1)+SUM
   28 CONTINUE
      CC=0.0
      IF(WO.EQ.0.0)GO TC 606
      DO 30 J=1,2
```

```
      DO 30 I=1,2
      DO 30 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      DO 30 K=1,NQUAD
      DD=DARSIN((X(K)*G+H)/RL*RAT)
      CC=0.0
      TLMT=DD
      DD=CC+(DD-CC)*PROP
      DO 1060 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      ICHECK=IL+1
      EE=X(K)*G+H
   32 FF=R(ICHECK)
      E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      NELEM=ICHECK-1
      EE=R(NELEM)
      SUM=0.0
      DO 31 L=1,NQUAD
      DO 31 M=1,6
      DO 31 N=1,NQUAD
      SUM=SUM+W(K)*W(L)*V(M)*W(N)*A*C*G*E*F9(X(K)*G+H,X(N)*E+F,
     1Y(M)*C+D,X(L)*A+B)*SF1(X(N)*E+F)*SF2(X(K)*G+H)
   31 CONTINUE
      SUM=-SUM*WC*RL/PI
      GSTIF(NN+IL+J-1,NN+NELEM+I-1)=GSTIF(NN+IL+J-1,NN+NELEM+I-1)+SUM
      IF(ICHECK.EQ.NN)GO TO 1070
      ICHECK=ICHECK+1
      EE=R(ICHECK-1)
      GO TO 32
 1070 CC=DD
      DD=TLMT
 1060 CONTINUE
   30 CONTINUE
  606 CONTINUE
      IF(WO.EQ.1.0)GO TO 607
      DO 33 J=1,2
      DO 33 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      DO 33 K=1,NQUAD
      DD=DARSIN((X(K)*G+H)/RL*RAT)
      CC=0.0
      TLMT=DD
      DD=CC+(DD-CC)*PROP
      DO 1080 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      ICHECK=IL+1
      EE=X(K)*G+H
   34 FF=R(ICHECK)
      E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      NELEM=ICHECK-1
      EE=R(NELEM)
      DO 35 L=1,NQUAD
      DO 35 M=1,6
      DO 35 N=1,NQUAD
      FAKE=W(K)*W(L)*V(M)*W(N)*A*C*G*E*F9(X(K)*G+H,X(N)*E+F,
     1Y(M)*C+D,X(L)*A+B)*SF2(X(K)*G+H)*4.0*RL*(1.0-WC)/PI
```

```
      C2(IL,NELEM,J,1)=C2(IL,NELEM,J,1)+FAKE*(SSF1(X(N)*E+F)**4)
      C2(IL,NELEM,J,2)=C2(IL,NELEM,J,2)+FAKE*(SSF1(X(N)*E+F)**3)
     1*(SSF2(X(N)*E+F))
      C2(IL,NELEM,J,3)=C2(IL,NELEM,J,3)+FAKE*(SSF1(X(N)*E+F)**2)
     1*(SSF2(X(N)*E+F)**2)
      C2(IL,NELEM,J,4)=C2(IL,NELEM,J,4)+FAKE*(SSF1(X(N)*E+F))
     1*(SSF2(X(N)*E+F)**3)
      C2(IL,NELEM,J,5)=C2(IL,NELEM,J,5)+FAKE*(SSF2(X(N)*E+F)**4)
   35 CONTINUE
      IF(ICHECK.EQ.NN)GC TC 1090
      ICHECK=ICHECK+1
      EE=R(ICHECK-1)
      GO TO 34
 1090 CC=DD
      DD=TLMT
 1080 CONTINUE
   33 CONTINUE
  607 CONTINUE
      CC=0.0
      DD=DARSIN(RS/RL)
      TLMT=DD
      DD=CC+(DD-CC)*PROP
      DO 2040 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      IF(WO.EQ.0.0)GO TC 608
      DO 36 J=1,2
      DO 36 I=1,2
      DO 36 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      DO 36 K=1,NGUAD
      UL=X(K)*G+H
      EE=RS
      DO 36 NELEM=1,IL
      IF(NELEM.EQ.IL)GO TO 39
      FF=R(NELEM+1)
      GO TO 40
   39 FF=UL
   40 E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      FF=R(NELEM+1)
      SUM=0.0
      DO 41 L=1,NGUAD
      DO 41 M=1,6
      DO 41 N=1,NQUAD
      SUM=SUM+W(K)*W(L)*V(M)*W(N)*A*C*G*E*F10(X(K)*G+H,X(N)*E+F,
     1Y(M)*C+D,X(L)*A+B)*SF1(X(N)*E+F)*SF2(X(K)*G+H)
   41 CONTINUE
      SUM=-SUM*WO*RL/PI
      GSTIF(NN+IL+J-1,NN+NELEM+I-1)=GSTIF(NN+IL+J-1,NN+NELEM+I-1)+SUM
      IF(NELEM.EQ.IL)GO TO 36
      EE=R(NELEM+1)
   36 CONTINUE
  608 CONTINUE
      IF(WO.EQ.1.0)GO TC 609
      DO 42 J=1,2
      DO 42 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      DO 42 K=1,NGUAD
      UL=X(K)*G+H
```

```
      EE=RS
      DO 42 NELEM=1,IL
      IF(NELEM.EQ.IL)GO TO 43
      FF=R(NELEM+1)
      GO TO 44
   43 FF=UL
   44 E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      FF=R(NELEM+1)
      DO 45 L=1,NCUAD
      DO 45 M=1,6
      DO 45 N=1,NCUAD
      FAKE=W(K)*W(L)*V(M)*W(N)*A*C*G*E*F10(X(K)*G+H,X(N)*E+F,
     1Y(M)*C+D,X(L)*A+B)*SF2(X(K)*G+H)*4.0*RL*(1.0-WC)/PI
      C2(IL,NELEM,J,1)=C2(IL,NELEM,J,1)+FAKE*(SSF1(X(N)*E+F)**4)
      C2(IL,NELEM,J,2)=C2(IL,NELEM,J,2)+FAKE*(SSF1(X(N)*E+F)**3)
     1*(SSF2(X(N)*E+F))
      C2(IL,NELEM,J,3)=C2(IL,NELEM,J,3)+FAKE*(SSF1(X(N)*E+F)**2)
     1*(SSF2(X(N)*E+F)**2)
      C2(IL,NELEM,J,4)=C2(IL,NELEM,J,4)+FAKE*(SSF1(X(N)*E+F))
     1*(SSF2(X(N)*E+F)**3)
      C2(IL,NELEM,J,5)=C2(IL,NELEM,J,5)+FAKE*(SSF2(X(N)*E+F)**4)
   45 CONTINUE
      IF(NELEM.EQ.IL)GO TO 42
      EE=R(NELEM+1)
   42 CONTINUE
  609 CONTINUE
      CC=DD
      DD=TLMT
 2040 CONTINUE
      CC=DARSIN(RS/RL)
      IF(WO.EQ.0.0)GO TO 610
      DO 46 J=1,2
      DO 46 I=1,2
      DO 46 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      DO 46 K=1,NQUAD
      RAT=0.97
      IF(IL.LE.NEXC)RAT=1.0
      DD=DARSIN((X(K)*G+H)/RL*RAT)
      CC=DARSIN(RS/RL)
      TLMT=DD
      DD=CC+(DD-CC)*PROP
      DO 1100 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      DO 1110 M=1,6
      EE=RL*DSIN(Y(M)*C+D)
      IF(EE.GE.R(1).AND.EE.LE.R(NN))GO TO 49
      WRITE(6,515)EE
  515 FORMAT(1X,'THE VALUE OF EE EXCEEDS THE LIMITS IN ETA EQN',E20.6)
   49 CONTINUE
      DO 50 ICHECK=1,NN
      IF(EE.LE.R(ICHECK))GO TO 51
   50 CONTINUE
   51 FF=R(ICHECK)
      E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      EE=R(ICHECK-1)
      FF=R(ICHECK)
      SUM=0.0
      DO 52 L=1,NCUAD
      DO 52 N=1,NQUAD
```

```
      SUM=SUM+W(K)*W(L)*V(M)*W(N)*A*C*G*E*F11(X(K)*G+H,X(N)*E+F,
     1Y(M)*C+D,X(L)*A+B)*SF1(X(N)*E+F)*SF2(X(K)*G+H)
   52 CONTINUE
      SUM=-SUM*WO*RL/PI
      NELEM=ICHECK-1
      GSTIF(NN+IL+J-1,NN+NELEM+I-1)=GSTIF(NN+IL+J-1,NN+NELEM+I-1)+SUM
      IF(ICHECK.EG.NN)GC TC 1110
      ICHECK=ICHECK+1
      EE=R(ICHECK-1)
      GO TO 51
 1110 CONTINUE
      CC=DD
      DD=TLMT
 1100 CONTINUE
   46 CONTINUE
  610 CONTINUE
      IF(WO.EQ.1.0)GO TC 611
      DO 53 J=1,2
      DO 53 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      DO 53 K=1,NQUAD
      RAT=0.97
      IF(IL.LE.NEXC)RAT=1.0
      DD=DARSIN((X(K)*G+H)/RL*RAT)
      CC=DARSIN(RS/RL)
      TLMT=DD
      DD=CC+(DD-CC)*PROP
      DO 1120 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      DO 1130 M=1,6
      EE=RL*DSIN(Y(M)*C+D)
      DO 54 ICHECK=1,NN
      IF(EE.LE.R(ICHECK))GO TO 55
   54 CONTINUE
   55 FF=R(ICHECK)
      NELEM=ICHECK-1
      E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      EE=R(ICHECK-1)
      FF=R(ICHECK)
      DO 56 L=1,NQUAD
      DO 56 N=1,NQUAD
      FAKE=W(K)*W(L)*V(M)*W(N)*A*C*G*E*F11(X(K)*G+H,X(N)*E+F,
     1Y(M)*C+D,X(L)*A+B)*SF2(X(K)*G+H)*4.0*RL*(1.0-WO)/PI
      C2(IL,NELEM,J,1)=C2(IL,NELEM,J,1)+FAKE*(SSF1(X(N)*E+F)**4)
      C2(IL,NELEM,J,2)=C2(IL,NELEM,J,2)+FAKE*(SSF1(X(N)*E+F)**3)
     1*(SSF2(X(N)*E+F))
      C2(IL,NELEM,J,3)=C2(IL,NELEM,J,3)+FAKE*(SSF1(X(N)*E+F)**2)
     1*(SSF2(X(N)*E+F)**2)
      C2(IL,NELEM,J,4)=C2(IL,NELEM,J,4)+FAKE*(SSF1(X(N)*E+F))
     1*(SSF2(X(N)*E+F)**3)
      C2(IL,NELEM,J,5)=C2(IL,NELEM,J,5)+FAKE*(SSF2(X(N)*E+F)**4)
   56 CONTINUE
      IF(ICHECK.EQ.NN)GO TO 1130
      ICHECK=ICHECK+1
      EE=R(ICHECK-1)
      GO TO 55
 1130 CONTINUE
      CC=DD
      DD=TLMT
 1120 CONTINUE
   53 CONTINUE
```

```
611 CONTINUE
    CC=DARSIN(RS/RL)
    IF(WO.EQ.0.0)GO TO 612
    DO 57 J=1,2
    DO 57 I=1,2
    DO 57 IL=1,NEM
    GG=R(IL)
    HH=R(IL+1)
    G=(HH-GG)/2.0
    H=(HH+GG)/2.0
    DO 57 K=1,NCUAD
    UL=X(K)*G+H
    DD=DARSIN((X(K)*G+H)/RL)
    CC=DARSIN(RS/RL)
    TLMT=DD
    DD=CC+(DD-CC)*PROP
    DO 1140 MS=1,2
    C=(DD-CC)/2.0
    D=(DD+CC)/2.0
    DO 1150 M=1,6
    EE=RL*DSIN(Y(M)*C+D)
    DO 59 ICHECK=1,NN
    IF(EE.LE.R(ICHECK))GO TO 60
 59 CONTINUE
 60 LLOW=ICHECK-1
    DO 61 ICH=1,NN
    IF(UL.LE.R(ICH))GC TO 62
 61 CONTINUE
 62 LUP=ICH-1
    DO 1150 NELEM=LLCW,LUP
    IF(NELEM.EQ.LLCW.CR.NELEM.EQ.LUP)GO TO 63
    EE=R(NELEM)
    FF=R(NELEM+1)
    GO TO 64
 63 CONTINUE
    IF(NELEM.EQ.LLOW.AND.NELEM.EQ.LUP)GC TO 65
    IF(NELEM.EQ.LLOW)GO TO 66
    EE=R(NELEM)
    GO TO 65
 66 FF=R(NELEM+1)
    GO TO 64
 65 FF=UL
 64 E=(FF-EE)/2.0
    F=(FF+EE)/2.0
    EE=R(NELEM)
    FF=R(NELEM+1)
    SUM=0.0
    DO 58 L=1,NQUAD
    DO 58 N=1,NCUAD
    SUM=SUM+W(K)*W(L)*V(M)*W(N)*A*C*G*E*F10(X(K)*G+H,X(N)*E+F,
   1Y(M)*C+D,X(L)*A+B)*SF1(X(N)*E+F)*SF2(X(K)*G+H)
 58 CONTINUE
    SUM=-SUM*WO*RL/PI
    GSTIF(NN+IL+J-1,NN+NELEM+I-1)=GSTIF(NN+IL+J-1,NN+NELEM+I-1)+SUM
1150 CONTINUE
    CC=DD
    DD=TLMT
1140 CONTINUE
 57 CONTINUE
612 CONTINUE
    IF(WO.EQ.1.0)GO TO 613
    DO 67 J=1,2
    DO 67 IL=1,NEM
    GG=R(IL)
    HH=R(IL+1)
    G=(HH-GG)/2.0
```

```
      H=(HH+GG)/2.0
      DO 67 K=1,NQUAD
      UL=X(K)*G+H
      DD=DARSIN((X(K)*G+H)/RL)
      CC=DARSIN(RS/RL)
      TLMT=DD
      DD=CC+(DD-CC)*FRCP
      DO 1160 MS=1,2
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      DO 1170 M=1,6
      EE=RL*DSIN(Y(M)*C+D)
      DO 68 ICHECK=1,NN
      IF(EE.LE.R(ICHECK))GO TO 69
   68 CONTINUE
   69 LLOW=ICHECK-1
      DO 70 ICH=1,NN
      IF(UL.LE.R(ICH))GC TO 71
   70 CONTINUE
   71 LUP=ICH-1
      DO 1170 NELEM=LLOW,LUP
      IF(NELEM.EQ.LLCW.OR.NELEM.EQ.LUP)GO TO 72
      EE=R(NELEM)
      FF=R(NELEM+1)
      GO TO 73
   72 CONTINUE
      IF(NELEM.EQ.LLOW.AND.NELEM.EQ.LUP)GO TO 74
      IF(NELEM.EQ.LLCW)GO TO 75
      EE=R(NELEM)
      GO TO 74
   75 FF=R(NELEM+1)
      GO TO 73
   74 FF=UL
   73 E=(FF-EE)/2.0
      F=(FF+EE)/2.0
      EE=R(NELEM)
      FF=R(NELEM+1)
      DO 1170 L=1,NQUAD
      DO 1170 N=1,NQUAD
      FAKE=W(K)*W(L)*V(M)*W(N)*A*C*G*E*F10(X(K)*G+H,X(N)*E+F,
     1Y(M)*C+D,X(L)*A+B)*SF2(X(K)*G+H)*4.0*RL*(1.0-WG)/PI
      C2(IL,NELEM,J,1)=C2(IL,NELEM,J,1)+FAKE*(SSF1(X(N)*E+F)**4)
      C2(IL,NELEM,J,2)=C2(IL,NELEM,J,2)+FAKE*(SSF1(X(N)*E+F)**3)
     1*(SSF2(X(N)*E+F))
      C2(IL,NELEM,J,3)=C2(IL,NELEM,J,3)+FAKE*(SSF1(X(N)*E+F)**2)
     1*(SSF2(X(N)*E+F)**2)
      C2(IL,NELEM,J,4)=C2(IL,NELEM,J,4)+FAKE*(SSF1(X(N)*E+F))
     1*(SSF2(X(N)*E+F)**3)
      C2(IL,NELEM,J,5)=C2(IL,NELEM,J,5)+FAKE*(SSF2(X(N)*E+F)**4)
 1170 CONTINUE
      CC=DD
      DD=TLMT
 1160 CONTINUE
   67 CONTINUE
  613 CONTINUE
C
C   INTEGRALS FOR THE ENERGY EQUATION
C
      DO 400 I=1,NN
      DO 400 J=1,NN
  400 GM(I,J)=0.0
      DO 76 J=1,2
      DO 76 I=1,2
      DO 76 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
```

```
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      IF((I+J).EQ.3)GO TO 77
      GSTIF(IL+J-1,NN+IL+I-1)=GSTIF(IL+J-1,NN+IL+I-1)
     1+(HH-GG)/3.0*(1.0-WC)/(4.0*AN)
      GSTIF(IL+J-1,IL+I-1)=GSTIF(IL+J-1,IL+I-1)-(1.0/(HH-GG))
     1-2.0/DELT*(HH-GG)/3.0*DFLOAT(ITYPE)
      GM(IL+J-1,IL+I-1)=GM(IL+J-1,IL+I-1)-(HH-GG)/3.0
      GO TO 78
   77 GSTIF(IL+J-1,IL+I-1)=GSTIF(IL+J-1,IL+I-1)+(1.0/(HH-GG))
     1-2.0/DELT*(HH-GG)/6.0*DFLOAT(ITYPE)
      GSTIF(IL+J-1,NN+IL+I-1)=GSTIF(IL+J-1,NN+IL+I-1)
     1+(HH-GG)/6.0*(1.0-WO)/(4.0*AN)
      GM(IL+J-1,IL+I-1)=GM(IL+J-1,IL+I-1)-(HH-GG)/6.C
   78 SUM=0.0
      DO 790 K=1,NQUAD
      SUM=SUM+W(K)*G*F12(X(K)*G+H)
  790 CONTINUE
      GSTIF(IL+J-1,IL+I-1)=GSTIF(IL+J-1,IL+I-1)+SUM
   76 CONTINUE
      DO 79 J=1,2
      DO 79 IL=1,NEM
      GG=R(IL)
      HH=R(IL+1)
      G=(HH-GG)/2.0
      H=(HH+GG)/2.0
      DO 79 K=1,NQUAD
      FAKE=W(K)*G*SF2(X(K)*G+H)*(1.0-WO)/AN
      D1(IL,J,1)=D1(IL,J,1)+FAKE*(SSF3(X(K)*G+H)**4)
      D1(IL,J,2)=D1(IL,J,2)+FAKE*(SSF3(X(K)*G+H)**3)
     1*(SSF4(X(K)*G+H))
      D1(IL,J,3)=D1(IL,J,3)+FAKE*(SSF3(X(K)*G+H)**2)
     1*(SSF4(X(K)*G+H)**2)
      D1(IL,J,4)=D1(IL,J,4)+FAKE*(SSF3(X(K)*G+H))
     1*(SSF4(X(K)*G+H)**3)
      D1(IL,J,5)=D1(IL,J,5)+FAKE*(SSF4(X(K)*G+H)**4)
   79 CONTINUE
      WRITE(6,500)
      WRITE(6,520)WC,RHC1,RHO2,RS,RL,TWIRE,TOVEN,AN
  520 FORMAT(1X,'WO=',F4.1,3X,'RHO1=',F4.1,3X,'RHO2=',F4.1,3X,
     1'RS=',F4.1,3X,'RL=',F4.1,3X,'TWIRE=',F4.1,3X,'TOVEN=',F4.1,3X,
     2'N=',F4.2)
C
C   SAVE STIFFNESS MATRIX
C
      DO 80 I=1,NR
      DO 80 J=1,NC
   80 GSAVE(I,J)=GSTIF(I,J)
C
C   SOLUTION SOLVER STARTING WITH LINEAR TEMPERATURE PROFILE
C
      IF(ITYPE.EQ.1)GO TO 401
      DO 81 I=1,NN
      T(I)=TWIRE+(TOVEN-TWIRE)/(RL-RS)*(R(I)-RS)
   81 CONTINUE
      GO TO 86
  401 T(1)=TWIRE
      T(NN)=TCVEN
      DO 402 I=2,NEM
  402 T(I)=TINIT
      DO 351 I=1,NN
      TOLD(I,1)=TINIT
  351 TDTOLD(I,1)=0.0
      DO 350 I=2,NEM
  350 TNEW(I,1)=TCLD(I,1)
      TNEW(1,1)=TWIRE
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```
      TNEW(NN,1)=TOVEN
      NEST=0
  353 NEST=NEST+1
      IF(NEST.GT.800)GO TO 410
      CALL UPDTDT(DELT,GM,TNEW,TOLD,TDTOLD,TDTNEW,NN,NEM)
      DO 352 I=1,NN
      TOLD(I,1)=TNEW(I,1)
  352 TDTOLD(I,1)=TDTNEW(I,1)
      CALL MATMLT(GM,NN,NN,TDTOLD,1,RI)
      DO 355 I=1,NN
  355 RHS(I)=RI(I,1)
      CALL MATMLT(GM,NN,NN,TOLD,1,RI)
      DO 356 I=1,NN
  356 RHS(I)=RHS(I)+(2.0/DELT)*RI(I,1)
      IF(NEST.GT.1)GO TO 85
      GO TO 86
   85 CONTINUE
      DO 89 I=2,NEM
   89 T(I)=GSTIF(I,NC)
      T(1)=TWIRE
      T(NN)=TOVEN
      DO 87 I=1,NR
      DO 87 J=1,NC
   87 GSTIF(I,J)=0.0
      DO 88 I=1,NR
      DO 88 J=1,NC
   88 GSTIF(I,J)=GSAVE(I,J)
   86 CONTINUE
      IF(ITYPE.EQ.0)GO TO 404
      DO 403 I=1,NN
  403 GSTIF(I,NC)=GSTIF(I,NC)+RHS(I)
  404 CONTINUE
      DO 82 L=1,NEM
      DO 82 K=1,NEM
      THET1=T(K)
      THET2=T(K+1)
      THET12=THET1**2
      THET13=THET1**3
      THET14=THET1**4
      THET22=THET2**2
      THET23=THET2**3
      THET24=THET2**4
      GSTIF(NN*2+L,NC)=GSTIF(NN*2+L,NC)+THET14*C3(L,K,1)+4.0*THET13
     1*THET2*C3(L,K,2)+6.0*THET12*THET22*C3(L,K,3)+4.0 *THET1*THET23
     1*C3(L,K,4)+THET24*C3(L,K,5)
      GSTIF(NN*2+L+1,NC)=GSTIF(NN*2+L+1,NC)+THET14*C3(L,K,1)+4.0*THET13
     1*THET2*C3(L,K,2)+6.0*THET12*THET22*C3(L,K,3)+4.0 *THET1*THET23
     1*C3(L,K,4)+THET24*C3(L,K,5)
      GSTIF(NN*3+L,NC)=GSTIF(NN*3+L,NC)+THET14*D4(L,K,1)+4.0*THET13
     1*THET2*D4(L,K,2)+6.0*THET12*THET22*D4(L,K,3)+4.0 *THET1*THET23
     1*D4(L,K,4)+THET24*D4(L,K,5)
      GSTIF(NN*3+L+1,NC)=GSTIF(NN*3+L+1,NC)+THET14*D4(L,K,1)+4.0*THET13
     1*THET2*D4(L,K,2)+6.0*THET12*THET22*D4(L,K,3)+4.0 *THET1*THET23
     1*D4(L,K,4)+THET24*D4(L,K,5)
      DO 82 J=1,2
      GSTIF(NN+L+J-1,NC)=GSTIF(NN+L+J-1,NC)+THET14*C2(L,K,J,1)
     1+4.0*THET13*THET2*C2(L,K,J,2)+6.0*THET12*THET22*C2(L,K,J,3)
     2+4.0*THET1*THET23*C2(L,K,J,4)+THET24*C2(L,K,J,5)
   82 CONTINUE
      DO 83 L=1,NEM
      THET1=T(L)
      THET2=T(L+1)
      THET12=THET1**2
      THET13=THET1**3
      THET14=THET1**4
      THET22=THET2**2
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      THET23=THET2**3
      THET24=THET2**4
      DO 83 J=1,2
      GSTIF(L+J-1,NC)=GSTIF(L+J-1,NC)+THET14*D1(L,J,1)+4.0*THET13*THET2
     1*D1(L,J,2)+6.0*THET12*THET22*D1(L,J,3)+4.0*THET1*THET23*D1(L,J,4)
     2+THET24*D1(L,J,5)
   83 CONTINUE
      CALL SOLVE(GSTIF,NR,NC,TWIRE,TOVEN,NN)
      DO 84 I=2,NEM
      THETA=GSTIF(I,NC)
      IF(DABS(THETA-T(I)).GT.0.001)GO TO 85
   84 CONTINUE
      IF(ITYPE.EQ.0)GO TO 420
      TNEW(1,1)=TWIRE
      TNEW(NN,1)=TOVEN
      DO 354 I=2,NEM
  354 TNEW(I,1)=GSTIF(I,NC)
      IF(NEST.EQ.1.OR.NEST.EQ.2)GO TO 360
      IF(NEST.EQ.3.OR.NEST.EQ.4)GO TO 360
      IF(NEST.EQ.10.OR.NEST.EQ.20)GO TO 360
      IF(NEST.EQ.30.OR.NEST.EQ.120)GO TO 360
      IF(NEST.EQ.200.OR.NEST.EQ.300)GO TO 360
      IF(NEST.EQ.400.OR.NEST.EQ.500)GO TO 360
      IF(NEST.EQ.550.OR.NEST.EQ.600)GO TO 360
      IF(NEST.EQ.650.OR.NEST.EQ.700)GO TO 360
      IF(NEST.EQ.750.OR.NEST.EQ.800)GO TO 360
      GO TO 353
  360 TIME=NEST*DELT
      WRITE(6,500)
      WRITE(6,530)TIME
  530 FORMAT(1X,'AT TIME =',F7.5)
  420 CONTINUE
      IF(ITYPE.EQ.1)GO TO 405
      DTHETA=GSTIF(1,NC)
      GO TO 406
  405 DTHETA=(GSTIF(2,NC)-TWIRE)/(R(2)-R(1))
  406 CONTINUE
      DO 90 I=2,NEM
   90 T(I)=GSTIF(I,NC)
      T(1)=TWIRE
      T(NN)=TOVEN
      DO 91 I=1,NN
   91 ETA(I)=GSTIF(I+NN,NC)
      BVWIRE=GSTIF(NN*3+1,NC)
      BVOVEN=GSTIF(NN*2+1,NC)
      CALL RESULT(X,W,T,ETA,RS,RL,BVWIRE,BVOVEN,R,WO,NN,AN,DTHETA,Q,FLX,
     1NIP)
      WRITE(6,500)
  500 FORMAT(1H1)
      WRITE(6,501)(GSTIF(I,NC),I=1,NR)
  501 FORMAT(1X,E20.6)
      WRITE(6,525)DTHETA,Q,FLX
  525 FORMAT(1X,3F10.3)
      IF(ITYPE.EQ.0)GO TO 410
      GO TO 353
  410 CONTINUE
      STOP
      END
      SUBROUTINE SOLVE(GSTIF,NR,NC,TWIRE,TOVEN,NN)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION GSTIF(NR,NC)
      DATA EPS/1.0D-10/
      DO 50 I=1,NR
      GSTIF(I,NC)=GSTIF(I,NC)-GSTIF(I,1)*TWIRE-GSTIF(I,NN)*TOVEN
   50 CONTINUE
      GSTIF(1,1)=-1.0
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      GSTIF(NN,NN)=1.0
      DO 61 J=1,NR
      IF(J.EQ.1)GO TO 62
      GSTIF(J,1)=0.0
   62 CONTINUE
      IF(J.EQ.NN)GO TO 61
      GSTIF(J,NN)=0.0
   61 CONTINUE
      DO 9 K=1,NR
      IF(DABS(GSTIF(K,K)).GT.EPS)GO TO 5
      WRITE(6,202)
  202 FORMAT(1X,'SMALL PIVOT - MATRIX MAY BE SINGULAR')
      GO TO 1
    5 KP1=K+1
      DO 6 J=KP1,NC
    6 GSTIF(K,J)=GSTIF(K,J)/GSTIF(K,K)
      GSTIF(K,K)=1.0D0
      DO 9 I=1,NR
      IF(I.EQ.K.OR.GSTIF(I,K).EQ.0.0)GO TO 9
      DO 8 J=KP1,NC
    8 GSTIF(I,J)=GSTIF(I,J)-GSTIF(I,K)*GSTIF(K,J)
      GSTIF(I,K)=0.0D0
    9 CONTINUE
    1 CONTINUE
      RETURN
      END
      SUBROUTINE RESULT(X,W,T,ETA,RS,RL,BVWIRE,BVOVEN,R,WO,NN,AN,DTHETA,
     1Q,FLX,NIP)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(4),W(4),T(21),ETA(21),R(21),P(4,6,21),FR(21),ZI(21)
      DIMENSION Y(6),V(6)
      F16(RP,BETA)=DSQRT(RP**2-(RL**2)*(DSIN(BETA)**2))
      F14(BETA,ALP)=(DCOS(ALP)**2)*DCOS(BETA)*DEXP((F16(RS,BETA)-
     1RL*DCOS(BETA))/DCOS(ALP))
      F15(RP,BETA,ALP)=DCOS(ALP)*DCOS(BETA)*RP/F16(RP,BETA)*
     1DEXP((F16(RS,BETA)-F16(RP,BETA))/DCOS(ALP))
      DATA Y/-0.2386191861,0.2386191861,-0.6612093865,0.6612093865,
     1-0.9324695142,0.9324695142/
      DATA V/2*0.4679139346,2*0.360761573,2*0.1713244924/
      NQUAD=4
      PI=3.1415927
      AA=0.0
      BB=PI/2.0
      CC=0.0
      DD=DARSIN(RS/RL)
      A=(BB-AA)/2.0
      B=(BB+AA)/2.0
      C=(DD-CC)/2.0
      D=(DD+CC)/2.0
      SUM=0.0
      DO 10 L=1,NQUAD
      DO 10 M=1,6
   10 SUM=SUM+W(L)*V(M)*A*C*F14(Y(M)*C+D,X(L)*A+B)
      SUM=-SUM*4.0*RL*BVOVEN/(PI*RS)
      SUM1=0.0
      DO 20 L=1,NQUAD
      DO 20 M=1,6
      DO 20 N=1,NN
   20 P(L,M,N)=W(L)*V(M)*A*C*F15(R(N),Y(M)*C+D,X(L)*A+B)
     1*((1.0-WO)*(T(N)**4)+WO*ETA(N)/4.0)
      H=R(2)-R(1)
      H1=R(NN)-R(NN-1)
      NL=NN-NIP+1
      IF(NIP.EQ.1)GO TO 45
      DO 35 N=1,NIP
      FR(N)=0.0
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      DO 35 L=1,NQUAD
      DO 35 M=1,6
   35 FR(N)=FR(N)+P(L,M,N)
      CALL DQSF(H,FR,ZI,NIP)
      SUM1=ZI(NIP)
   45 K=1
      DO 40 N=NIP,NN
      FR(K)=0.0
      DO 30 L=1,NCUAD
      DO 30 M=1,6
   30 FR(K)=FR(K)+P(L,M,N)
   40 K=K+1
      CALL DQSF(H1,FR,ZI,NL)
      SUM1=SUM1+ZI(NL)
      SUM1=-SUM1*4.C*RL/(PI*RS)
      Q=BVWIRE+SUM+SUM1
      FLX=-DTHETA+Q/(4.0*AN)
      RETURN
      END
      SUBROUTINE UPDTDT(DELT,G,TNEW,TOLD,TDTOLD,TDTNEW,NN,NEM)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION G(NN,NN),TNEW(21,1),TOLD(21,1),TDTOLD(21,1),TDTNEW(21,1)
      DIMENSION R(21,1),RRR(21,22)
      DATA EPS/1.0D-10/
      NL=NN+1
      DO 50 I=1,NN
      DO 50 J=1,NN
   50 RRR(I,J)=G(I,J)
      DO 49 I=1,NN
   49 RRR(I,NL)=0.0
      CALL MATMLT(G,NN,NN,TNEW,1,R)
      DO 51 I=1,NN
   51 RRR(I,NL)=RRR(I,NL)+(2.0/DELT)*R(I,1)
      CALL MATMLT(G,NN,NN,TOLD,1,R)
      DO 52 I=1,NN
   52 RRR(I,NL)=RRR(I,NL)-(2.0/DELT)*R(I,1)
      CALL MATMLT(G,NN,NN,TDTOLD,1,R)
      DO 53 I=1,NN
   53 RRR(I,NL)=RRR(I,NL)-R(I,1)
      DO 9 K=1,NN
      IF(DABS(RRR(K,K)).GT.EPS)GC TO 5
      WRITE(6,202)
  202 FORMAT(5X,'SUBROUTINE UPDTDT-PIVOTSMALL-MATRIX SINGULAR')
      GO TO 1
    5 KP1=K+1
      DO 6 J=KP1,NL
    6 RRR(K,J)=RRR(K,J)/RRR(K,K)
      RRR(K,K)=1.0
      DO 9 I=1,NN
      IF(I.EQ.K.OR.RRR(I,K).EQ.0.0)GO TO 9
      DO 8 J=KP1,NL
    8 RRR(I,J)=RRR(I,J)-RRR(I,K)*RRR(K,J)
      RRR(I,K)=0.0
    9 CONTINUE
      DO 55 I=1,NN
   55 TDTNEW(I,1)=RRR(I,NL)
    1 CONTINUE
      RETURN
      END
      SUBROUTINE MATMLT(A,M,N,B,L,C)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(21,21),B(21,1),C(21,1)
C     SUBROUTINE MULTIPLIES A(M,N) BY B(N,L) TC GIVE C(M,L)
      DO 10 I=1,M
      DO 10 J=1,L
      C(I,J)=0.0
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      DO 10 K=1,N
   10 C(I,J)=C(I,J)+A(I,K)*B(K,J)
      RETURN
      END
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------ JES2 JOB STATISTICS ------

1,321 CARDS READ

0 SYSOUT PRINT RECORDS

0 SYSOUT PUNCH RECORDS

0.00 MINUTES EXECUTION TIME