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CHAPTER I

INTRODUCTION

Research for this paper began with an investigation of the various, seemingly disparate, classes of objects enumerated by the formula $s_k = \frac{1}{k}\binom{2k-1}{k-1}$, $k = 1,2,3,\ldots$. Many of the collections counted by this formula are essentially different (See Chapter IX.); hence it seemed worthwhile to try to unify these by means of a single algebraic structure. Because of the apparent complexity of structure exhibited by certain of the types of objects enumerated by s_k, the simplicity of the following result (a corollary to Theorems 3.1 and 4.16) was unexpected.

> If, for each positive integer k, S_k is a set containing $s_k = \frac{1}{k}\binom{2k-2}{k-1}$ elements, with $S_1 \cap S_j = \emptyset$ for $i \neq j$, and if $S = \bigcup_{i=1}^{\infty} S_i$, then a binary operation $*$ can be defined on S such that $\langle S,* \rangle$ is a free groupoid, freely generated by S_1.

The lack of structure for this unifying system was initially disappointing in light of the somewhat complex examples which illustrate it. Resulting from this, however, was the discovery that inherent within the free groupoid is much additional structure, definable in terms of the given operation (See Chapter VI.). Consideration of the various applications of the formula for s_k also led to Theorem 5.5, which yields an interesting alternate definition of the finitely generated free groupoid.

It seems appropriate to consider first a short history of the occurrences of the combinatorial formula given above since, as has been indicated, it is from within these examples that many of the following ideas came to life.

In this work a distinction is made between propositions and theorems. Those assertions which are considered more significant or which are referred to frequently are called theorems; others are called propositions.

CHAPTER II

HISTORY OF THE COMBINATORIAL FORMULA, $s_k = \frac{1}{k}\binom{2k-2}{k-1}$

In this brief survey of the different settings in which the given formula has occurred, an effort has been made to credit each problem to its original proposer or solver. However, it should be noted that many of these problems have been rediscovered many times throughout the literature, and a compilation of references to all of these has not seemed feasible. (See [6] for an attempt at this by W. G. Brown.)

Apparently the formula for s_k was first used to count the number of different ways in which a convex polygon of k+1 sides ($k \geq 2$) may be divided into triangles by the insertion of k-2 (non-intersecting) diagonals. This problem seems to have been posed originally by L. Euler who gave, without proof, the numbers $s_2, s_3, \ldots, s_8$ [34]. The proof was supplied by J. A. deSegner [33] in a paper first published in the Petersburg Transactions. E. Lucas [22] credits Segner with the recurrence relation $s_k = s_1 s_{k-1} + s_2 s_{k-2} + \cdot \cdot \cdot + s_{k-1} s_1$ for $k \geq 2$ where s_1 is defined to be 1. Both Brown [5], [6] and Lucas [22] attribute to Euler [15] the discovery of the formula $s_k = \frac{1}{k}\binom{2k-2}{k-1}$. In constructing these diagonalizations of a polygon it is assumed that the vertices, hence also the edges, are distinguishable. This distinction may be accomplished by identifying in some manner an arbitrary but fixed vertex or edge. As an example, the two different diagonalizations of a quadrilateral can be illustrated as in Figure 2.1.

Figure 2.1.

E. T. Bell [3] and E. Netto [26] credit E. Catalan [8] with first showing that s_k is the number of different products of k factors, taken in a fixed order, in a non-associative, non-commutative algebra. Catalan also noted the equivalence of this problem to that of Euler mentioned above. Lucas [22], pages 489-490, describes a simple one-to-one correspondence, between the set of dissections of an n-gon into triangles and the set of ways of associating n-1 factors, which he credits to a private communication from Élie Perrin.

I. M. H. Etherington [14] shows the result, similar to Catalan's, that s_k is the number of possible interpretations of the power x^k in a general non-commutative, non-associative algebra. A related question, solved by J. H. M. Wedderburn [36] in a slightly different form, is to find the number of possible different interpretations of x^k in a commutative, non-associative algebra.

The next application of the formula for s_k occurred in the problem of counting trees, i.e., connected planar graphs without circuits. (The reader unfamiliar with the basic definitions of elementary graph theory is referred to Chapters 1, 2 and 4 of [28].) Study and enumeration of trees began in 1857 with A. Cayley [9] who said that he was led to consideration of these structures by the researches of J. J. Sylvester concerning the changes of the independent variables in the differential calculus.

Following [21] we define a plane tree as a tree which is embedded in the plane (with no two distinct edges intersecting) and a planted plane tree as a plane tree in which a vertex of degree 1 has been distinguished as the root. Two planted plane trees are said to be map-isomorphic if there exists an orientation-preserving homeomorphism of the plane which maps one onto the other in such a manner that the roots correspond. In [21] it is shown that the number of non-map-isomorphic planted plane trees having $k \geq 2$ vertices is s_{k-1}. Cayley [10] demomstrated that s_{k-1} is also the number of non-may-isomorphic planted, trivalent (every vertex of degree 1 or 3), plane trees having 2k-2 vertices or k terminal vertices (vertices of degree 1). A one-to-one correspondence is given in [21] between the set of planted plane trees having k vertices and the set of planted trivalent plane trees having k terminal vertices.

W. J. Growney in [19] exhibits a one-to-one correspondence between the set of planted trivalent plane trees and the class B of binary sequences which is defined thus:

If $b = b_1 b_2 \ldots b_k$, $b_i \in \{0,1\}$, for $i = 1,2,\ldots,k$, then $b \in B$ if and only if

(i) $k \geq 2$,

(ii) $b_1 = 1$, and

(iii) for each $j \in \{2,3,\ldots,k-1\}$, the number of 1's in $b_2b_3\ldots b_j$ never exceeds the number of 0's, but in $b_1b_2\ldots b_k$ the number of 1's exceeds the number of 0's by 2.

Elements of B of a given length can also be enumerated by applying the results of W. A. Whitworth, given below. The number of sequences of length 2k is found to be s_k for $k = 1,2,\ldots$.

A one-to-one correspondence between the set of planted trivalent plane trees with k+1 terminals and the set of distinct ways of combining k factors, in a given order, in a non-associative, non-commutative algebra is given by Forder in [17].

In 1878 Whitworth [37] showed that s_{k+1} is the total number of routes from the origin (0,0) to the point (k,k) in the Cartesian plane which do not go above the line $y = x$ and which consist of k horizontal paces and k vertical paces, each one unit in length. Whitworth actually solved a generalization of the stated problem, by showing that the total number of such routes from (0,0) to (m,n), $m \geq n$, which do not cross the diagonal line is $\frac{m-n+1}{m+1}\binom{m+n}{n}$.

J. Bertrand [4] in 1887 published a solution to the following problem:

> Suppose two candidates P and Q obtain, respectively, p and q votes in an election ($p > q$). If the

> votes are counted in random order, what is the probability that the winning candidate holds the lead throughout the counting?

This is the classical "ballot problem" and s_{k+1} occurs in the special case $p = k+1$, $q = k$, k any non-negative integer, as the number of ballot sequences in which candidate Q is always behind his opponent P in the counting.

If we make the observation, in the problem of Whitworth, that s_{k+1} also counts the number of routes from $(0,0)$ to $(k+1,k)$ which do not touch the diagonal, then we may solve the ballot problem by using the result of Whitworth and by associating a horizontal unit with a vote for P and a vertical unit with a vote for Q. We then have a one-to-one correspondence between the set of admissible routes from $(0,0)$ to $(k+1,k)$ and the set of all ballot sequences of length $2k+1$ in which candidate P holds the lead throughout the counting.

One finds that s_k or s_{k+1} occurs in the solution to each of the following problems -- all similar in that each, like the ballot problem, can be solved immediately by appropriately applying the result of Whitworth.

(1) A man drinks in random order $k > 0$ glasses of wine and k glasses of water (all of equal volume); show that the odds are k to one against his never having drunk throughout the process more wine than water. (Educational Times, June, 1878, question 5669.) [37].

(2) In how many orders can k positive units and k negative units be arranged so that the sum to any number of terms may never be negative? [13] and [37].

(3) If we take the non-negative half of an ordinary number line and place a marker at 0, in how many ways may the marker make 2k moves, each one unit in length, which return it to its starting point? [13].

(4) Given k+1 letters of one kind and k letters of another kind, in how many ways can they be arranged so that, moving along the arrangement from one end to the other, the number of letters of the first kind passed over is greater than the number of the second kind at any instant? [18] and [20].

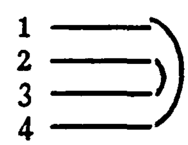
(5) In how many ways can one order 2k numbers, all different, with k in each of two horizontal rows, in such a manner that the numbers are always increasing from left to right and from top to bottom? (This is the "Problème des deux files de soldats" of Lucas [22], page 86.)

(6) For each natural number k, what is the number of sequences of 2k-1 consecutive positive integers which begin and end with 1? (This problem, and (7) also, is due to this author and while each may, out of this context, appear to be an interesting problem, in this setting it is trivial.)

(7) For each natural number k, what is the number of (strictly) increasing sequences $[a_1, a_2, \ldots, a_k]$ of k positive integers such that $a_1 = 1$ and $a_j \leq 2(j-1)$ for each $j \in \{2, 3, \ldots, k\}$?

In [24], T. Motzkin gives the following definition: Let F be a function of k arguments that is (i) almost symmetric (F becomes F or -F for every permutation of the arguments) and (ii) such that the $\binom{2k}{k}$ functions of any k from among 2k arguments $a_{11}, a_{12}, a_{21}, a_{22}, \ldots, a_{k1}, a_{k2}$ are independent. (An example of such an F is given in [23].) We then define the <u>function ratio</u> of the k pairs of arguments, in the given order as $G = \prod F(a_{1r_1}, \ldots, a_{kr_k})^{\lambda}$ where $\lambda = \sum r_j$ and the product extends over the 2k possible systems $r_1, \ldots, r_k$. Motzkin proceeds to show that the number of independent function ratios that can be formed of 2k arguments is s_{k+1}. In his proof he sets up a one-to-one correspondence between the set of independent function ratios for a function F of k arguments, satisfying (i) and (ii), and the set of ways of joining 2k distinguishable points on a circle by k non-intersecting chords. He then shows the number of ways of doing the latter to be $\frac{1}{k+1}\binom{2k}{k} = s_{k+1}$. A similar result is obtained by J. Touchard as a special case in [35].

As part of a somewhat confusing history of the formula for s_k, in [2] we find mention that s_k gives the number of rhyming schemes for stanzas of a certain length when suitaable conditions are imposed. This author was unable to find other references to this claim in the literature and has independently formulated the following:

> Given a stanza of k lines, we shall define a *rhyme scheme* as a partition of $\{1,2,\ldots,k\}$ into i subsets, $1 \leq i \leq k$, each subset being called a *rhyming set*. In the manner of G. Puttenham [30], each rhyme scheme may be represented pictorially by listing the labelled lines in order, e.g.,
>
> 1 ———
> 2 ———
> . . . .
> . . . .
> . . . .
> k-1 ———
> k ———
>
> and then connecting, by means of arcs drawn to the right of the lines, each line to the nearest line following it those number belongs to the same element of the partition. Call a rhyme scheme *planar* if this pictorial representation, called a *Puttenham diagram*, can be drawn without crossing of arcs.
>
> For example, if $k = 4$, the rhyme scheme $\{\{1,4\}, \{2,3\}\}$ with Puttenham diagram  is planar while $\{\{1,3\}, \{2,4\}\}$ is not planar. It will later be seen that the number of planar rhyme schemes for stanzas of k lines is s_{k+1}.

Information Theory also contains a class of objects -- in particular, a collection of binary codes -- enumerated by s_k. However, in order to make the result we wish to state meaningful, we need some preliminary definitions adapted from [1] and

[16].

Let $S = \{a_1,\ldots,a_q\}$ be any finite set of symbols. Then any mapping of the set of all finite sequences of elements of S into the set of finite sequences of elements of $X = \{0,1\}$ is called a _binary code_. S is the _source alphabet_ and X the _code alphabet_. A binary _block_ code is a code which maps each symbol of S into a fixed binary sequence. Images of the elements of S are called _code words_. A block code is said to be _non-singular_ if all the code words are distinct. If $X_i = x_{i_1} x_{i_2} \ldots x_{i_m}$ is a code word, then any sequence $x_{i_1} x_{i_2} \cdots x_{i_j}$, where $i \leq j \leq m$, is called a _prefix_ of the code word X_i. A non-singular binary code is said to be _instantaneous_ if no complete word of the code is a prefix of some other code word. We will call an instantaneous binary code C _complete_ if the set of code words of C is not properly contained in the set of code words of any other instantaneous binary code.

Fano [16], page 66, describes informally a one-to one correspondence between the collection of complete instantaneous binary codes having $k \geq 2$ words and the set of planted trivalent plane trees having k+1 terminal vertices. By virtue of the correspondence indicated by Fano, we are able to state that s_{k+1} is the number of distinct complete instantaneous binary codes of k words, $k \geq 2$. (Two binary codes will be considered distinct if there is a code word of one which is not a code word of the other.)

In Chapters VII and VIII we shall return to a number of these examples to illustrate the properties of a free groupoid with a single generator. In Chapter IX it is shown that, although similar, in some significant sense many of these examples are essentially different.

CHAPTER III

BASIC DEFINITIONS

Often following Bruck [7], pages 1-8. we adopt the following definitions. By a <u>binary operation</u> $*$, also called a <u>product</u>, on the (non-empty) set G, we mean a (single-valued) mapping $*$ from some (possibly empty) subset R_* of GxG into G. If $(a,b) \in R_*$, we shall write $*ab$ for the image of (a,b) under $*$ and if $c = *ab$, c will be referred to as <u>the product of a and b</u>. Use of the prefix notation $*ab$, rather than the more usual infix notation $a*b$, enables us to avoid the use of parentheses in associating products. For example, we have $**abc$ instead of $(a*b)*c$ and $*a*bc$ in place of $a*(b*c)$.

A <u>halfgroupoid</u> $\langle G,*\rangle$ is an algebraic system consisting of a non-empty set G and a binary operation $*$ on G. If $\langle G,*\rangle$ is a halfgroupoid, and H is a non-empty subset of G, then $\langle H,*_H\rangle$ is a <u>subhalfgroupoid</u> of $\langle G,*\rangle$ provided $\langle H,*_H\rangle$ is a halfgroupoid such that the domain R_{*_H} of $*_H$ is a subset of $R_* \cap (HxH)$ with $*ab = *_Hab$ whenever $(a,b) \in R_{*_H}$. Henceforth we shall denote the operation $*_H$ in a subhalfgroupoid simply by $*$.

A <u>groupoid</u> is a halfgroupoid $\langle G,*\rangle$ such that $*ab$ is defined and in G for all $a,b \in G$. A <u>subgroupoid</u> of a halfgroupoid $\langle G,*\rangle$ is a groupoid $\langle H,*\rangle$ which is a subhalfgroupoid of $\langle G,*\rangle$. For convenience, in the following we will usually refer to the groupoid or halfgroupoid $\langle G,*\rangle$ simply by G.

If H is a subhalfgroupoid of the halfgroupoid G then H is <u>closed in G</u> if $R_{*_H} = R_* \cap (H \times H)$. We shall say that H <u>generates</u> G if and only if the only closed subhalfgroupoid of G containing H as a subhalfgroupoid is G itself.

A <u>homomorphism</u> $\emptyset$ of a halfgroupoid G into(onto) a halfgroupoid K is a single-valued mapping of G into(onto) K such that $*_G ab = c$ in G implies $*_K \emptyset a \emptyset b = \emptyset c$ in K. (Here $\emptyset x$ denotes the image of $x \in G$ under $\emptyset$, while $*_G$ and $*_K$ denote the respective operations in G and K.) An <u>isomorphism</u> $\emptyset$ of G onto K is a one-to-one homomorphism of G onto K such that $\emptyset^{-1}$ is a homomorphism of K onto G.

We say that a halfgroupoid G is <u>free over</u> a subhalfgroupoid H of G if for every groupoid K and homomorphism $\emptyset$ of H into K, $\emptyset$ can be extended to a homomorphism of G into K. G is <u>freely generated</u> by H if G is both free over and generated by H. We note that <u>every</u> halfgroupoid H freely generates a groupoid G, unique to within isomorphism ([17], page 4, Theorem 1.1.).

If G is a groupoid for which there exists a non-empty subset B of G which freely generates G, then G is called <u>free</u> and B is a <u>free basis</u> of G. Existence of free groupoids is assured by the theorem of Bruck stated above since any non-empty set may be regarded as a halfgroupoid with no products defined.

The following (from [7], page 4) is a useful necessary and sufficient condition that a groupoid G be freely generated by a given subhalfgroupoid H of G: if $a \in G$, $a \notin H$, then $a = *bc$ for one and only one ordered pair b,c in G.

If $a = *bc$ in a halfgroupoid G, then we say that b and c <u>divide a in G</u>. An element of G is <u>prime in G</u> if it has no divisors in G. A (finite or infinite) sequence $[a_i]$ of elements $a_1, a_2, \ldots$ of G is a <u>divisor chain</u> of G if each member a_{i+1} divides the preceding member a_i in G. A divisor chain $[a_i]$ is <u>finite over</u> a subhalfgroupoid H of G if there exists an integer k such that $a_i \in H$ whenever $i > k$; the chain has <u>length n over H</u> if n is the least such integer. A groupoid G is free if and only if every divisor chain in G is finite; moreover if G is free then G has one and only one free basis, namely the set of all primes in G ([7], page 6). The <u>rank</u> of a free groupoid G is the cardinal number of its unique free basis; if this is finite, G is said to be <u>finitely generated</u>.

If G is any countably infinite set, any finite subset B of G can be selected and a product $*$ defined in G so that G is a free groupoid with basis B. Although this is a consequence of the aforementioned theorem of Bruck, a constructive proof is given below.

<u>Proposition 3.1</u>. If $G = \{g_1, g_2, \ldots\}$ is a countably infinite set of distinct elements and if $B = \{g_1, g_2, \ldots, g_n\}$, for an arbitrary but fixed positive integer n, then a product $*$ can be defined in G so that $\langle G,*\rangle$ is a free groupoid with free basis B.

<u>Proof</u>: Since G is in one-to-one correspondence with the set J_+ of positive integers, it is without loss of generality

that we assume G to be totally well-ordered in the manner given. A total well-ordering can be induced on GxG by defining $(g_i,g_j) \leq (g_k,g_m)$ if and only if $i+j \leq k+m$ with $i \leq k$ if $i+j = k+m$. If we define $*$: GxG $\longrightarrow$ G by (i) $*(g_1,g_1) = g_{n+1}$ and (ii) if (g_i,g_j) has predecessor (g_k,g_m) in the ordering, then $*(g_i,g_j) = g_r$ if and only if $*(g_k,g_m) = g_{r-1}$, then $*$ is seen to be an injective mapping with range G - B. The desired conclusion follows. $\square$

CHAPTER IV

PRELIMINARY RESULTS AND COUNTING THEOREMS

For any object y, let $\langle Y,*\rangle$ be a free groupoid with basis $\{y\}$. An immediate consequence of the preceding results on free groupoids is:

if $z \in Y$ then either $z = y$ or there exist unique $z_1, z_2 \in Y$ such that $z = *z_1z_2$.

In order to avoid cumbersome terminology and notation, we will in the following consider * both as a mapping and the name for the mapping. Similarly y will be treated both as an object and a symbol denoting the object. This identification of name and object will be referred to as the canonical identification and, in each case, context should make the usage clear.

Definition 4.1. Let λ be the mapping, from Y to the set J_+ of positive integers, defined by

(4.1.1) $\lambda(y) = 1$ and

(4.1.2) if $z = *z_1z_2$, $z_1,z_2 \in Y$, then $\lambda(z) = \lambda(z_1) + \lambda(z_2)$.

$\lambda(z)$ is called the length of z. Since factorization in Y is unique and every divisor chain of Y is finite and terminates with y, λ is a well-defined mapping whose domain is indeed Y.

Proposition 4.2. The range of λ is J_+.

Proof: By induction on $\lambda(z)$. By the preceding definition, 1 belongs to the range of λ. Suppose n belongs to the range of λ for some $n \in J_+$. Choose $z \in Y$ such that $\lambda(z) = n$. Then

$*zy \in Y$ and $\lambda(*zy) = \lambda(z) + \lambda(y) = n + 1$. []

Corollary 4.2a. For $z \in Y$, $\lambda(z) = 1$ if and only if $z = y$.

If $z = z_1z_2\cdots z_k$ is a string of k symbols ($k \in J_+$), each of which is either * or y, then for each $j \in \{1,2,\ldots,k\}$ let $N(j,*,z)$ be the number of occurrences of * in the initial j symbols of z and let $N(j,y,z)$ be the number of occurrences of y among the first j symbols of z. We see that $N(j,*,z) + N(j,y,z) = j$, for $j = 1,2,\ldots,k$. It is now possible to state the following characterization of the members of Y.

Theorem 4.3. If $z \in Y$, then z is the concatenation of $2\lambda(z)-1$ symbols, each of which is either * or y. Moreover, for each $j \in \{1,2,\ldots,2\lambda(z)-2\}$ we have $N(j,y,z) \leq N(j,*,z)$ while $N(2\lambda(z)-1,y,z) = N(2\lambda(z)-1,*,z) + 1$. Conversely, if $z = z_1z_2\cdots z_k$ with $z_i \in \{*,y\}$ for each $i \in \{1,2,\ldots,k\}$, and if for each $j \in \{1,2,\ldots,k-1\}$ we have $N(j,y,z) \leq N(j,*,z)$ with $N(k,y,z) = N(k,*,z) + 1$, then $z \in Y$.

Proof: Let $z \in Y$. We prove the first assertion by induction on $\lambda(z)$. If $\lambda(z) = 1$, then $z = y$ and all of the stated conditions are satisfied for, in that case, we have $k = 1$, $N(k,y,z) = 1$ and $N(k,*,z) = 0$. If $\lambda(z) > 1$ then $z \neq y$ and there exist unique $z_1,z_2 \in Y$ such that $z = *z_1z_2$. Since $\lambda(z_1) < \lambda(z)$, $\lambda(z_2) < \lambda(z)$, we may employ the induction hypothesis for z_1 and z_2. It is readily seen that if the conclusions hold for z_1 and z_2 then each also holds for $*z_1z_2$.

Conversely, let $z = z_1z_2\cdots z_k$, with $z_i \in \{*,y\}$ for each $i \in \{1,2,\ldots,k\}$ and suppose that for each $j \in$

$\{1,2,\ldots,k-1\}$ we have $N(j,y,z) \leq N(j,*,z)$ with $N(k,y,z) = N(k,*,z) + 1$. This last equation together with $N(k,y,z) + N(k,*,z) = k$ implies k is odd. We will prove the desired result by induction on $k' = (k+1)/2$. If $k' = 1$ then $k = 1$ and since $N(k,y,z) = 1$, $N(k,*,z) = 0$, it must be that $z = y \in Y$. If $k' > 1$ then $k > 1$ and $\{1,2,\ldots,k-1\} \neq \emptyset$. Let j be the least member of $\{1,2,\ldots,k-1\}$ such that $N(j,*,z) = N(j,y,z)$; existence of such a j is assured by the hypothesis. Also by virtue of the stated condition, we know $z_1 = *$. Consider $z' = z_2z_3\cdots z_j$ and $z'' = z_{j+1}z_{j+2}\cdots z_k$. Since z satisfied the given specifications, so must z' and z''. By the induction hypothesis they are in Y. Thus $z = *z'z''$ is also in Y. []

Hence we have that a string of k symbols ($k \in J_+$) is an element of Y if and only if k is odd, each symbol is either $*$ or y and, reading from left to right, the number of y's encountered never exceeds the number of $*$'s until the final symbol is reached, and the total number of y's exceeds the total number of $*$'s by one. In the following, when we write an expression such as $z = z_1z_2\cdots z_k$ for $z \in Y$ we will also mean that $z_i \in \{*,y\}$ for each $i \in \{1,2,\ldots,k\}$ and that $k = 2\lambda(z) - 1$, as above.

<u>Corollary 4.3a</u>. If $z = z_1z_2\cdots z_k \in Y$, then k is odd and $z_k = y$. If $k > 1$, then $z_1 = *$ and $z_{k-1} = y$.

<u>Corollary 4.3b</u>. If $z = z_1z_2\cdots z_k \in Y$, where $k > 1$, then there exists a $j \in \{1,2,\ldots,k-2\}$ such that $z_j = *$, $z_{j+1} = z_{j+2} = y$.

<u>Corollary 4.3c</u>. If $z = z_1z_2\cdots z_k \in Y$, then $N(k,y,z) = \lambda(z)$.

Example 4.4. Theorem 4.3 and its corollaries give a convenient way of recognizing and naming elements of Y and of finding their length. For example, if $z_1 = *y*y*yy$, $z_2 = *yyy$, $z_3 = ***yyyy$, we see that $z_1, z_3 \in Y$, $z_2 \notin Y$ and $\lambda(z_1) = \lambda(z_3) = 4$.

The following concept of successor will be useful later (Chapter V) in obtaining a new characterization of a finitely generated free groupoid.

Definition 4.5. The element $*yy$ of Y will be called the successor of y in Y and if $z = *z_1z_2 \in Y$, $z_1, z_2 \in Y$, then a successor of z in Y is any element of Y of the form $*z_1'z_2$ or $*z_1z_2'$ where z_1' is a successor of z_1 in Y and z_2' is a successor of z_2 in Y. If z' is a successor of z in Y we will call z a predecessor of z'.

Examples of successors and predecessors are deferred until their existence is established and a method is obtained (Proposition 4.8) for finding all the successors and predecessors of a given element of Y.

Proposition 4.6. If $z \in Y$, then z has a successor in Y.

Proof: By induction on $\lambda(z)$. If $\lambda(z) = 1$, then $z = y$ and $*yy \in Y$ is a successor of z. Suppose that $\lambda(z) = n$, $n > 1$, and that the proposition is true for all $\bar{z} \in Y$ with $\lambda(\bar{z}) < n$. Since $z \neq y$ there exist $z_1, z_2 \in Y$ such that $z = *z_1z_2$. Now $\lambda(z_1) < \lambda(z)$ and, by the induction hypothesis, z_1 has a successor z_1' in Y. Then $z' = *z_1'z_2$ is in Y and is a successor of z. $\square$

Proposition 4.7. For $z \in Y$, if z' is a successor of z in Y then $\lambda(z') = \lambda(z) + 1$.

Proof: By induction on $\lambda(z)$. $\square$

Proposition 4.8. If $z = z_1 z_2 \ldots z_k \in Y$ then, for each $j \in \{1,2,\ldots,k\}$ such that $z_j = y$, the element $z' = z_1 \ldots z_{j-1} {*}y z_j z_{j+1} \ldots z_k$ is a successor of z in Y.

Proof: By induction on $\lambda(z)$. If $\lambda(z) = 1$, then $z = y$ and $z' = {*}yy$ is a successor of z in Y. Suppose then z to be such that $\lambda(z) = n$, $n > 1$ and that the proposition is true for all elements of Y having length less than n. There exist unique $\bar{z}_1, \bar{z}_2 \in Y$ such that $z = {*}\bar{z}_1 \bar{z}_2$ with $\lambda(\bar{z}_1) < n$, $\lambda(\bar{z}_2) < n$. By applying the induction hypothesis and the definition of successor, we reach the desired conclusion. $\square$

Corollary 4.8a. If $z \in Y$, $z \neq y$, then z has a predecessor in Y.

Proof: If $z = z_1 z_2 \ldots z_k$ (where $z_i \in \{*,y\}$ for each $i \in \{1,2,\ldots,k\}$) then, by Corollary 4.3b, there must exist a j in $\{1,2,\ldots,k-2\}$ such that $z_j = *$, $z_{j+1} = z_{j+2} = y$. By the preceeding proposition, z is a successor of $\bar{z} = z_1 \ldots z_{j-1} z_{j+2} \ldots z_k$. $\square$

Corollary 4.8b. If $z \in Y$ has $\lambda(z) = n$, then z has exactly n distinct successors.

Proof: By Proposition 4.8 and Corollary 4.3c, z has at least n successors. That this is the exact number follows from the definitions of successor and length, by mathematical induction. $\square$

Example 4.9. Applying Proposition 4.8 and Corollary 4.8b, we find the list of successors of *y*yy in Y to be: **yy*yy, *y**yyy, and *y*y*yy.

Example 4.10. Although Corollary 4.8a indicates that if $z \in Y$, $z \neq y$, then z has a predecessor in Y, we do not have a formula for the number of predecessors of z. Elements such as *yy, **yyy, ***yyyy and ****yyyyy have unique predecessors. However **yy*yy has 2 predecessors, ***yy*yy*yy has 3 predecessors, ****yy*yy*yy*yy has 4 predecessors and, in general, an element of Y of the form $\underbrace{**\ldots*}_{k-1\ *\text{'s}}\underbrace{*yy*yy\ \ldots\ *yy}_{k\ *yy\text{'s}}$ has k predecessors for each positive integer k.

If we denote by S_z the set of successors of z in Y, then we can state

Proposition 4.11. For $z_1, z_2 \in Y$, if $z_1 \neq z_2$, then $S_{z_1} \neq S_{z_2}$.

Proof: The conclusion clearly holds if $\lambda(z_1) \neq \lambda(z_2)$ for then S_{z_1} and S_{z_2} have different numbers of elements. If $\lambda(z_1) = \lambda(z_2) = 1$, then $z_1 = z_2 = y$ and the hypothesis cannot be satisfied. Suppose then that $z_1 \neq z_2$ and $\lambda(z_1) = \lambda(z_2) > 1$. There exist unique $z_{11}, z_{12}, z_{21}, z_{22} \in Y$ such that $z_1 = {*}z_{11}z_{12}$, $z_2 = {*}z_{21}z_{22}$. It must be that $z_{11} \neq z_{21}$ or $z_{12} \neq z_{22}$; without loss of generality we suppose the former. Now $S_{z_1} = \{{*}z_{11}'z_{12}\colon z_{11}' \in S_{z_{11}}\} \cup \{{*}z_{11}z_{12}'\colon z_{12}' \in S_{z_{12}}\}$ and $S_{z_2} = \{{*}z_{21}'z_{22}\colon z_{21}' \in S_{z_{21}}\} \cup \{{*}z_{21}z_{22}'\colon z_{22}' \in S_{z_{22}}\}$. Since $z_{11} \neq z_{21}$, $\{{*}z_{11}z_{12}'\colon z_{12}' \in S_{z_{12}}\} \cap \{{*}z_{21}z_{22}'\colon z_{22}' \in S_{z_{22}}\} = \emptyset$. If we suppose that $S_{z_1} = S_{z_2}$, then for each $\bar{z}_{22} \in S_{z_{22}}$, the preceding sentence implies $*z_{21}\bar{z}_{22} \in \{*z_{11}'z_{12} : z_{11}' \in S_{z_{11}}\}$. Hence there exists $\bar{z}_{11}$ in $S_{z_{11}}$ such that $*z_{21}\bar{z}_{22} = *\bar{z}_{11}z_{12}$, and thus $z_{21} \in S_{z_{11}}$. Similarly if $\bar{z}_{12} \in S_{z_{12}}$, then the empty intersection implies there exists $\bar{z}_{21} \in S_{z_{21}}$ such that $*z_{11}\bar{z}_{12} = *\bar{z}_{21}z_{22}$ and thus $z_{11} \in S_{z_{21}}$. But not both of the statements $z_{11} \in S_{z_{21}}$ and $z_{21} \in S_{z_{11}}$ can be true. This contradiction implies $S_{z_1} \neq S_{z_2}$. []

Thus we see that an element of Y is uniquely determined by its set of successors. Given any successor set S_z, we may find z by finding the set of predecessors for each element of S_z and by taking their intersection.

Definition 4.12. If $z = z_1z_2\cdots z_k \in Y$ (with $z_i \in \{*,y\}$ for each $i \in \{1,2,\ldots,k\}$) then $z_{j,r} = z_jz_{j+1}\cdots z_{j+r}$, where $j \in \{1,2,\ldots,k\}$, $r \in \{0,1,\ldots,k-j\}$, will be called a component of z if and only if $z_{j,r} \in Y$.

Proposition 4.13. If $z = z_1z_2\cdots z_k \in Y$, then for each $j \in \{1,2,\ldots,k\}$ there exists a unique $r \in \{0,1,\ldots,k-j\}$ such that $z_jz_{j+1}\cdots z_{j+r}$ is a component of z. This string will be called the jth component of z and denoted $\bar{z}_j$.

Proof: By induction on $\lambda(z)$. Since the assertion is trivially true for $z = y$, suppose that $z \in Y$ has $\lambda(z) > 1$ and that the proposition is true for all z' having $\lambda(z') < \lambda(z)$. Since there exist unique t_1,t_2 in Y such that $z = *t_1t_2$, the conclusion holds for $j = 1$. For each $j \in \{2,3,\ldots,2\lambda(z) - 1\}$ existence of an $r \in \{0,1,\ldots,2\lambda(z) - 1 - j\}$, such that $z_jz_{j+1}\cdots z_{j+r}$ is a component of z, is assured by the induction hypothesis, applied to t_1 and t_2. The uniqueness of this component follows from Theorem 4.3. []

Proposition 4.14. If $z = z_1 z_2 \cdots z_k \in Y$ and if $j \in \{1,2,\ldots,k\}$ is chosen so that $z_j = y$, then either $z_{j-1} = *$ or there exists a unique component $\bar{z}_m$ of z such that $z_{m-1} = *$ and $z = z_1 \cdots z_{m-1} \bar{z}_m z_j \cdots z_k$.

Proof: By induction on $\lambda(z)$. []

If we let n_k be the number of elements in the set $N_k = \{z : z \in Y, \lambda(z) = k\}$, for each positive integer k, we can prove the following theorems, anticipated in a preceding chapter.

Proposition 4.15. $n_1 = 1$, and for $k > 1$, $n_k = \sum_{i=1}^{k-1} n_i n_{k-i}$.

Proof: Since $N_1 = \{y\}$, we have $n_1 = 1$. If $k > 1$, then n_k is also the number of elements in $Z_k = \{(z_1,z_2) : z_1,z_2 \in Y, \lambda(z_1) + \lambda(z_2) = k\}$. By an elementary counting principle, the number of pairs in Z_k with first term having length i and second term having length k-i is $n_k n_{k-i}$; hence the total number of elements in Z_k is, as desired, $\sum n_i n_{k-i}$. []

Theorem 4.16. $n_k = \frac{1}{k}\binom{2k-2}{k-1}$, for each positive integer k.

Proof: We will show that $n_k = \frac{4k-6}{k}\, n_{k-1}$ for $k > 1$. Since $n_1 = 1$ and $\frac{1}{k}\binom{2k-2}{k-1} = \frac{4k-6}{k}\left[\frac{1}{k-1}\binom{2k-4}{k-2}\right]$, the truth of the desired formula follows by mathematical induction.

Suppose $z = z_1 z_2 \cdots z_t \in N_{k-1}$ (where $k > 1$ and $z_i \in \{*,y\}$ for each $i \in \{1,2,\ldots,t\}$). By Theorem 4.3 the elements $z' = z_1 \cdots z_{i-1} * y \bar{z}_i z_{i+r+1} \cdots z_t$ and $z' = z_1 \cdots z_{i-1} * \bar{z}_i y z_{i+r+1} \cdots z_t$ are in Y and, by Corollary 4.3c, they are in N_k.

By $\bar{z}_i = z_i \cdots z_{i+r}$ we mean, of course, the ith component of z; we will refer to z' and z", respectively, as the left and right expansions of the ith component of z. Since, by Proposition 4.13, z has $t = 2(k-1) - 1 = 2k - 3$ components, each element of N_{k-1} yields $4k-6$ such elements of N_k.

On the other hand, if we can show that each element of N_k is obtained k times as a left or right expansion of some component of some element of N_{k-1}, then we will have the relationship $k \cdot n_k = (4k-6)\, n_{k-1}$, as desired. Suppose $z = z_1 z_2 \cdots z_t \in N_k$ ($k > 1$) with $z_i \in \{*, y\}$ for each $i \in \{1,2,\ldots,t\}$. By Proposition 4.14, if $j \in \{1,2,\ldots,t\}$ is chosen so that $z_j = y$, then either $z = z_1 \cdots z_{j-2} {*} y z_{j+1} \cdots z_t$ or there exists a unique component $\bar{z}_m$ of z such that $z = z_1 \cdots z_{m-2} {*} \bar{z}_m y z_{j+1} \cdots z_t$. Hence z is the left expansion of the (j-1)st component of $z' = z_1 \cdots z_{j-2} z_{j+1} \cdots z_t \in N_{k-1}$ or z is the right expansion of the (m-1)st component of $z'' = z_1 \cdots z_{m-2} \bar{z}_m z_{j+1} \cdots z_t$. Since $N(t,y,z) = k$, there are k choices of j for which $z_j = y$. Hence each $z \in N_k$ is obtained exactly k times by left and right expansions of components of elements of N_{k-1}. []

<u>Corollary 4.16a</u>. If $n_1 = 1$, then $\sum_{i=1}^{k-1} n_i n_{k-i} = \frac{1}{k}\binom{2k-2}{k-1}$, for $k = 2,3,4,\ldots$.

<u>Example 4.17</u>. If $k = 5$ then $n_k = 5$ and $N_k = \{$***yyyy,**y*yyy, **yy*yy,*y**yyy, *y*y*yy$\}$.

Proposition 4.15 suggests a procedure for finding all elements of Y of length $k > 1$, namely by finding all products $*z_1 z_2$ where $\lambda(z_1) + \lambda(z_2) = k$. We consider below a

means of generating the elements of Y of a given length without a list of all the elements of shorter lengths. To accomplish this we will set up a one-to-one correspondence between the elements of N_k and a class of sequences of positive integers of length k.

Let $z \in Y$, $\lambda(z) = k > 0$, and suppose $z = z_1z_2\cdots z_{2k-1}$ where each $z_i \in \{*,y\}$. Then z determines a (strictly) increasing sequence $[a_1,a_2,\ldots,a_k]$ of positive integers where $z_{2k-a_1}, z_{2k-a_2}, \ldots, z_{2k-a_k}$ are the k occurrences of y in z, reading from right to left.

<u>Proposition 4.18</u>. $a_1 = 1$ and $a_i \leq 2(i-1)$ for $i = 2,\ldots,k$.

<u>Proof:</u> Suppose $a_i > 2(i-1)$ for some $i \in \{2,\ldots,k\}$. This tells us that the ith y in z, reading from right to left, occurs to the left of a string of length at least 2(i-1). Since this string contains exactly i-1 y's it must contain at least i-1 *'s. For this to be so is a violation of Theorem 4.3. By Corollary 4.3a, $a_1 = 1$. $\square$

<u>Proposition 4.19</u>. If $[a_1,a_2,\ldots,a_k]$ is an increasing sequence of k positive integers with $a_1 = 1$ and with $a_i \leq 2(i-1)$ for $i = 2,3,\ldots,k$, then there exists and element $z = z_1z_2\cdots z_{2k-1}$ of Y of length k such that $k_{2k-a_1}, z_{2k-a_2}, \ldots, z_{2k-a_k}$ are the k occurrences of y in z.

<u>Proof:</u> Any sequence of k y's and k-1 *'s such that the positions $2k-a_1,\ldots,2k-a_k$ of the y's satisfy $a_1 = 1$, $a_i \leq 2(i-1)$ for $i > 1$ satisfies the conditions of Theorem 4.3 and hence is in Y. $\square$

Example 4.20. The element *y*yy of Y determines the sequence $[1,2,4]$ while the sequence $[1,2,4,6]$ determines *y*y*yy $\in$ Y.

Propositions 4.18 and 4.19 have established a one-to-one correspondence between Y and $P = \bigcup_{k=1}^{\infty} P_k$ where P_k is the collection of the sequences described above which have length k. More particularly we have demonstrated a one-to-one correspondence between N_k and P_k.

It is now possible to define and ordering on P_k in the following way. If $a = [a_1,\ldots,a_k]$ and $b = [b_1,\ldots,b_k]$ are in P_k then $a \leq b$ if and only if there is an i, $1 \leq i \leq k$, such that $a_i \leq b_i$ and $a_j = b_j$ for $j = 1,2,\ldots,i-1$. Propositions 4.21 and 4.22 follow immediately.

Proposition 4.21. The relation $\leq$ defines a linear order for P_k. That is

(i) for all $a \in P_k$, $a \leq a$,

(ii) if $a \leq b$ and $b \leq a$, then $a = b$,

(iii) if $a \leq b$ and $b \leq c$, then $a \leq c$,

(iv) for any pair a,b in P_k either $a \leq b$ or $b \leq a$.

Proposition 4.22. If $a \in P_k$, then $[1,2,\ldots,k] \leq a \leq [1,2,4,\ldots,2k-2]$.

Since P_k is finite, we know from Proposition 4.21 that if $a \in P_k$, $a \neq [1,2,4,\ldots,2k-2]$, then there exists a unique element I(a) of P_k such that $a \leq I(a)$, $a \neq I(a)$ and if $b \in P_k$, $b \neq a$ and $a \leq b \leq I(a)$, then $b = I(a)$. Call I(a) the immediate successor of a. It is possible to state

Proposition 4.23. Let $a = [a_1,\ldots,a_k] \in P_k$ and let $j \in \{1,2,\ldots,k\}$ be the greatest integer for which $a_j < 2j-2$. Then

$[a_1,\ldots,a_{j-1},a_j+1,\ a_j+2,\ldots,a_j+k-j+1] = I(a)$.

Proposition 4.24. Let $a = [1,2,\ldots,k]$. If $I^m(a) = I\,(I^{m-1}(a))$ for $m>1$, then a, $I(a)$, $I^2(a),\ldots,\ I^{n_k-1}(a)$ are all distinct and form the complete list of elements of P_k.

Now let F be the mapping defined on P which associates with each $a = [a_1,a_2,\ldots,a_k]$ in P_k the element $z = z_1z_2\ldots z_{2k-1}$ of N_k such that a_{2k-a_1}, . . . , z_{2k-a_k} are the k occurrences of y in z. As a direct consequence of all that we have developed following Theorem 4.16, we have

Theorem 4.25. Let $a = [1,2,\ldots,k] \in P_k$. Then $F(a)$, $F(I(a))$, . . . , $F(I^{n_k-1}(a))$ are all distinct and form the complete list of elements of N_k.

Example 4.26. For k = 5 the complete ordered list of the 14 elements of P_k and the corresponding list of elements of N_k are given below.

[1,2,3,4,5]	****yyyyy
[1,2,3,4,6]	***y*yyyy
[1,2,3,4,7]	**y**yyyy
[1,2,3,4,8]	*y***yyyy
[1,2,3,5,6]	***yy*yyy
[1,2,3,5,7]	**y*y*yyy
[1,2,3,5,8]	*y**y*yyy
[1,2,3,6,7]	**yy**yyy
[1,2,3,6,8]	*y*y**yyy
[1,2,4,5,6]	***yyy*yy
[1,2,4,5,7]	**y*yy*yy
[1,2,4,5,8]	*y**yy*yy
[1,2,4,6,7]	**yy*y*yy
[1,2,4,6,8]	*y*y*y*yy

Little change is required in most of the preceding to adapt it to the more general case of the finitely generated free groupoid of rank n, for any positive integer n. For example, if $B = \{y_1, y_2, \ldots, y_n\}$ is any finite collection of distinct objects and if $\langle Y', * \rangle$ is a free groupoid with basis B, we can state the following -- the omitted proofs in all cases being similar to those presented earlier.

<u>Definition 4.1'</u>. Let λ be the mapping from Y' to J_+ defined by

(4.1.1') $\lambda(z) = 1$ if and only if z is prime in Y',

(4.1.2') if $z = *z_1z_2$, $z_1, z_2 \in Y'$, then $\lambda(z) = \lambda(z_1) + \lambda(z_2)$.

$\lambda(z)$ is called the <u>length</u> of z.

As in our work with $\langle Y, * \rangle$, we make use of the canonical identification of objects with their names and state the following characterization of elements in Y'.

<u>Theorem 4.3'</u>. If $z \in Y'$, then z is a string of $2\lambda(z)-1$ symbols, each of which is in $\{*\} \cup B$. Moreover, if $N(j,B,z)$ is the number of occurrences of an element of B among the first j symbols of z and $N(j,*,z)$ is the number of occurrences of $*$ in the initial j symbols of z, we have $N(j,B,z) \leq N(j,*,z)$ for each $j \in \{1,2,\ldots,2\lambda(z)-2\}$, while $N(2\lambda(z)-1,B,z) = N(2\lambda(z)-1,*,z) + 1$. Conversely, if $z = z_1z_2\cdots z_k$ with $z_i \in \{*\} \cup B$ for each $i \in \{1,2,\ldots,k\}$, and if for each $j \in \{1,2,\ldots,k-1\}$ we have $N(j,B,z) \leq N(j,*,z)$ with $N(k,B,z) = N(k,*,z) + 1$, then $z \in Y'$.

<u>Definition 4.5'</u>. Any element in $S_{y_i} = \{*y_jy_i : y_j \in B\}$, $y_i \in B$, will be called a <u>successor</u> of y_i in Y'. Also if $z = *z_1z_2$, $z_1, z_2 \in Y'$, then a <u>successor</u> of z in Y' is any element of Y'

of the form $*z_1'z_2$ or $*z_1z_2'$ where z_1' is a successor of z_1 and z_2' is a successor of z_2. If z' is a successor of z in Y' we will call z a <u>predecessor</u> of z'.

<u>Proposition 4.6'</u>. If $z \in Y'$, then z has a successor in Y'.

<u>Proposition 4.7'</u>. For $z \in Y'$, if z' is a successor of z in Y', then $\lambda(z') = \lambda(z) + 1$.

<u>Proposition 4.8'</u>. If $z = z_1z_2\cdots z_k \in Y'$ and if $j \in \{1,2,\ldots,k\}$ is chosen so that $z_j = y_i \in B$, then $z' = z_1\cdots z_{j-1}*y_mz_jz_{j+1}\cdots z_k$ is a successor of z for any $y_m \in B$.

<u>Corollary 4.8a'</u>. If $z \in Y - B$, then z has a predecessor in Y'.

<u>Corollary 4.8b'</u>. If $z \in Y'$ and $\lambda(z) = k$, then z has exactly $n\cdot k$ successors in Y', where n is the rank of Y'.

<u>Example 4.9'</u>. By Theorem 4.3' we recognize that, if Y' is a free groupoid of rank 3 with basis $B = \{y_1,y_2,y_3\}$, then $*y_1*y_2y_3$ is in Y'. Applying Proposition 4.8' and Corollary 4.8b', we find the 9 successors of $*y_1*y_2y_3$ in Y' to be $**y_1y_1*y_2y_3$, $**y_2y_1*y_2y_3$, $**y_3y_1*y_2y_3$, $*y_1**y_1y_2y_3$, $*y_1**y_2y_2y_3$, $*y_1**y_3y_2y_3$, $*y_1*y_2*y_1y_3$, $*y_1*y_2*y_2y_3$, and $*y_1*y_2*y_3y_3$.

If we denote by S_z' the set of successors of z in Y', then it is possible to state

<u>Proposition 4.11'</u>. For $z_1,z_2 \in Y'$, if $z_1 \neq z_2$, then $S_{z_1}' \neq S_{z_2}'$.

<u>Definition 4.12'</u>. If $z = z_1z_2\cdots z_k \in Y'$ with each $z_i \in \{*\} \cup B$, then $z_{j,r} = z_jz_{j+1}\cdots z_{j+r}$, where $j \in \{1,2,\ldots,k\}$, $r \in \{0,1,\ldots,k-j\}$, will be called a <u>component</u> of z if and only if $z_{j,r} \in Y'$.

<u>Proposition 4.13'</u>. With z as in the preceding definition, then for each $j \in \{1,2,\ldots,k\}$ there exists a unique $r \in$

$\{0,1,\ldots,k-j\}$ such that $z_j z_{j+1} \cdots z_{j+r}$ is a component of z. This string is called __the jth component of z__ and denoted $\bar{z}_j'$.

__Proposition 4.14'.__ For z as in Definition 4.13', if $j \in \{1,2,\ldots,k\}$ is chosen so that $z_j \in B$, then either $z_{j-1} = *$ or there exists a unique factor $\bar{z}_m'$ of z such that $z_{m-1} = *$ and $z = z_1 \cdots z_{m-1} \bar{z}_m' z_j \cdots z_k$.

With n_k' as the number of elements in the set $N_k' = \{z : z \in Y', \lambda(z) = k\}$, for each positive integer k, one can state the following counting theorems, both provable in a manner similar to their earlier counterparts.

__Proposition 4.15'.__ If $n \in J_+$ is the rank of the free groupoid Y', then $n_1' = n$ and for $k > 1$, $n_k' = \sum_{i=1}^{k-1} n_i' n_{k-i}'$.

__Theorem 4.16'.__ $n_k' = \frac{(n)^k}{k}\binom{2k-2}{k-1}$ for each positive integer k.

__Corollary 4.16a'.__ If $n_1' = n$, then $\sum_{i=1}^{k-1} n_i' n_{k-i}' = \frac{(n)^k}{k}\binom{2k-2}{k-1}$, for $k = 2,3,\ldots$.

CHAPTER V

A NEW CHARACTERIZATION OF FREE GROUPOIDS OF FINITE RANK

The successor concept, defined in Chapter IV, now provides for an alternate characterization of the finitely generated free groupoid.

For any non-empty set X, let P(X) denote the power set of X and let $P_k(X)$ be that subset of P(X) which contains all the k-element subsets of X. Then let A be the union of a countably infinite sequence $\{A_i\}$ of non-empty, pairwise disjoint sets subject to the following conditions.

(S-1) A_1 is in one-to-one correspondence with $\{1,2,\ldots,n\}$ for some natural number n.

(S-2) There exists an injective mapping $S\colon A \longrightarrow P(A)$ such that if $a \in A_k$, then $S(a) \in P_{nk}(A_{k+1})$ and $\bigcup_{a\in A_k} S(a) = A_{k+1}$.

S will be called a <u>successor mapping</u> and, for each $a \in A$, S(a) is <u>the successor set of a</u>. We note in passing the immediate consequence of (S-1) and (S-2) that A_i is a finite set, for each positive integer i.

In the free groupoid $<Y', *>$ of Chapter IV, if we define $S'\colon Y' \longrightarrow P(Y')$ by $S'(z) = S'_z$ for each z in Y', then $\{N'_i\}$ is a countable collection of non-empty, pairwise disjoint sets for which (S-1) and (S-2) are satisfied.

If, further, there exists a relation R_* with domain AxA and range containing $A - A_1$, compatible with the successor mapping as prescribed by the conditions below, then it can be shown (Proposition 5.4) that R_* defines a single-valued mapping $*_A$ of AxA into A. In fact, $<A,*_A>$ is (Theorem 5.5) a free groupoid of rank n with free basis A_1. Writing $*_A ab = c$ to mean $((a,b),c) \in R_*$, the conditions are, for a,b,c,d in A:

(S-3) $c = *_A ab$ if and only if $\{*_A a'b : a' \in S(a)\} \cup \{*_A ab' : b' \in S(b)\} = S(c)$.

(S-4) If $\{*_A a'b : a' \in S(a)\} \cap \{*_A cd' : d' \in S(d)\} \neq \emptyset$, then $c \in S(a)$ and $b \in S(d)$.

(S-5) If $a \in A_j$ then $\{*_A a'b : a' \in S(a)\}$ and $\{*_A ba' : a' \in S(a)\}$ each contain $j \cdot n$ distinct elements, for any b in A.

(S-6) If $a,b \in A_j$ with $a \neq b$, then $\{*_A a'c : a' \in S(a)\} \neq \{*_A b'd : b' \in S(b)\}$ and $\{*_A ca' : a' \in S(a)\} \neq \{*_A db' : b' \in S(b)\}$ for any $c,d \in A_i$, for any positive integer i.

Proposition 5.1. With S' as defined above, for each $i \in J_+$, the operation * in the free groupoid $<Y',*>$ of rank n satisfies the compatibility conditions (S-3) - (S-6).

Proof: (S-3) is a direct consequence of the definition of successor; (S-4) and (S-6) hold since * is an injective mapping; (S-5) follows from Corollary 4.8b' and the injectivity of *. $\square$

Proposition 5.2. For $a,b \in A$, $\{*_A a'b : a' \in S(a)\} \cap \{*_A ab' : b' \in S(b)\} = \emptyset$.

Proof: If the above intersection is non-empty, then (S-4) tells us that $a \in S(a)$ and $b \in S(b)$, both of which are impossible. $\square$

Proposition 5.3. If $a \in A_i$, $b \in A_j$, then $*_A ab \in A_{i+j}$.

Proof: If $a \in A_i$, $b \in A_j$, then $\{*_A a'b : a' \in S(a)\}$ contains $n \cdot i$ distinct elements and $\{*_A ab' : b' \in S(b)\}$ contains $n \cdot j$ distinct elements. Since they are disjoint, their union contains $ni + nj = n(i+j)$ elements. But this union is $S(*_A ab)$, and the definition of the successor mapping requires us to conclude that $*_A ab \in A_{i+j}$. $\square$

Corollary 5.3a. The range of $*_A$ is $A - A_1$.

Proposition 5.4. For $a,b,c,d \in A$, $*_A ab = *_A cd$ implies $a = c$ and $b = d$.

Proof: If $*_A ab = *_A cd$ then $\{*_A a'b : a' \in S(a)\} \cup \{*_A ab' : b' \in S(b)\} = \{*_A c'd : c' \in S(c)\} \cup \{*_A cd' : d' \in S(d)\}$. Suppose $a \in A_i$, $b \in A_j$. If $\{*_A a'b : a' \in S(a)\} \cap \{*_A cd' : d' \in S(d)\} \neq \emptyset$ then $c \in S(a)$ and $b \in S(d)$, hence $c \in A_{i+1}$, $d \in S_{j-1}$. Thus $\{*_A c'd : c' \in S(c)\}$ contains $n(i+1)$ elements, at least n of which must lie in $\{*_A ab' : b' \in S(b)\}$ since $\{*_A a'b : a' \in S(a)\}$ contains only ni elements. But if $\{*_A c'd : c' \in S(c)\} \cap \{*_A ab' : b' \in S(b)\} \neq \emptyset$ then $a \in S(c)$ and $d \in S(b)$, which is a contradiction. Thus we conclude $\{*_A a'b : a' \in S(a)\} \cap \{*_A cd' : d' \in S(d)\} = \emptyset$. Similar reasoning leads us also to conclude that $\{*_A ab' : b' \in S(b)\} \cap \{*_A c'd : c' \in S(c)\} = \emptyset$. Hence $\{*_A a'b : a' \in S(a)\} = \{*_A c'd : d' \in S(d)\}$ and $\{*_A ab' : b' \in S(b)\} = \{*_A cd' : d' \in S(d)\}$ from which it follows that $c \in A_i$, $d \in A_j$. (S-6) then implies that $a = c$ and $b = d$. $\square$

Recalling that the number of elements in $A_1 \subset A$ was specified as n, we can state, as a direct consequence of all the preceding

Theorem 5.5. $\langle A, *_A \rangle$ is a free groupoid of rank n.

Corollary 5.5a. The number of elements in A_k is $\frac{(n)^k}{k}\binom{2k-2}{k-1}$ for each natural number k.

CHAPTER VI

STRUCTURE INHERENT IN THE FINITELY GENERATED FREE GROUPOID

It has often been of interest in mathematics to consider the problem of embedding a certain algebraic system in other more restricted systems, i.e., in those satisfying additional properties. In the following we deal with an opposite problem, namely, the possibility of finding certain more restricted systems within a given system. For simplicity of exposition we will consider only the free groupoid $\langle Y,*\rangle$ with a single generator; however, many of the definitions and results may be generalized in an obvious manner to any finitely generated free groupoid.

For any $z \in Y$, $z \neq y$, we know that there exist unique $z_1, z_2 \in Y$ such that $z = *z_1z_2$. Henceforth z_1 and z_2 will be referred to, respectively, as the <u>left principal factor</u> of z, abbreviated LPFz, and the <u>right principal factor</u> of z, RPFz. This terminology aids us in defining the following additional operations in Y.

<u>Definition 6.1</u>. Let $*_e, *_i : Y\mathrm{x}Y \longrightarrow Y$ be the mappings, with domain YxY and range Y, defined in terms of the mapping * by

$$(6.1.1)\quad \begin{matrix} *_e z_1 z_2 = *\mathrm{LPF}z_1\mathrm{RPF}z_2 \\ *_i z_1 z_2 = *\mathrm{LPF}z_2\mathrm{RPF}z_1 \end{matrix} \qquad \text{for } z_1, z_2 \in Y - \{y\}.$$

$$(6.1.2)\quad *_x zy = *_x yz = y \quad \text{for any } z \in Y,\ x \in \{e,i\}.$$

$*_e z_1 z_2$ and $*_i z_1 z_2$ will be called, respectively, the <u>external product</u> and <u>internal product</u> of z_1 and z_2.

Several properties of $*_e$ and $*_i$ which are immediate consequences of the definition are contained in

<u>Proposition 6.2</u>. For $z_1, z_2, z_3 \in Y$

(i) $*_e z_1 z_2 = *_i z_2 z_1$.

(ii) $*_x *_x z_1 z_2 z_3 = *_x z_1 *_x z_2 z_3$ for $x \in \{e,i\}$; i.e., $*_e$ and $*_i$ are associative in Y.

(iii) If $z_2 \neq y$ then $*_x *_x z_1 z_2 z_1 = z_1$ for $x \in \{e,i\}$.

(iv) $*_x z_1 *_w z_2 z_3 = *_x z_1 z_2$
$*_x *_w z_1 z_2 z_3 = *_x z_2 z_3$ for $x,w \in \{e,i\}$, $x \neq w$.

<u>Corollary 6.2a</u>. $\langle Y, *_e \rangle$ and $\langle Y, *_i \rangle$ are semigroups.

We will return to explore further the properties of $*_e$ and $*_i$ after defining a unary operation in Y called reflection. For that purpose let $z = z_1 z_2 \cdots z_k$ be an arbitrary but fixed element of $Y - \{y\}$ with, as before, each $z_i \in \{*,y\}$. Further let $a_1 < a_2 < \ldots < a_j$ be such that $j = (k-1)/2$ and $z_{a_1}, z_{a_2}, \ldots, z_{a_j}$ are the j occurrences of $*$ in z. Define $f_1(z), f_2(z), \ldots, f_j(z)$ in the following manner. $f_i(z)$ is that string obtained from z by interchanging the left principal factor and the right principal factor of the ith component $\bar{z}_i$ of z (See Proposition 4.13.), and it will be called the <u>ith partial reflection</u> of z. We observe that $f_i(z) \in Y$ for $i = 1,2,\ldots,k$ since the condition imposed by Theorem 4.3 is preserved by performing the stated operation on z. Also, clearly, $\lambda(f_i(z)) = \lambda(z)$. Each f_i is thus a mapping with domain and range each equal to $\{z : z \in Y \text{ and } \lambda(z) > i\}$. It is now possible to state

<u>Definition 6.3</u>. For each $z \in Y$ the <u>reflection</u> of z, denoted

z^r, is defined by

(6.3.1) $z^r = y$ if $z = y$.

(6.3.2) If $\lambda(z) = k > 1$, then $z^r = f_1 \circ f_2 \circ \cdots \circ f_{k-1}(z)$,

where $\circ$ denotes composition of mappings.

By induction we have $z^r \in Y$.

Example 6.4. If $z = {**}y{*}yyy$, then $f_1(z) = {*}y{*}y{*}yy$, $f_2(z) = {***}yyyy$, $f_3(z) = z$ and $z^r = {*}y{**}yyy$.

Since, for each $i \in J_+$, f_i is bijective (In fact, $f_i^2(z) = z$ for all z in the domain of f_i.) it follows that

Proposition 6.5. For each $z \in Y$ there exists a unique $z' \in Y$ such that $(z')^r = z$.

It is not, in general, true that $f_i \circ f_j(z) = f_j \circ f_i(z)$ since, for example, $f_1 \circ f_3({**}yy{**}yyy) = {**}y{*}yy{*}yy$ while $f_3 \circ f_1({**}yy{**}yyy) = {***}yyy{*}yy$. We can, however, establish the following result.

Proposition 6.6. For $z \in Y$, if $\lambda(z) = k > 1$, then $f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1(z) = f_1 \circ f_2 \circ \cdots \circ f_{k-1}(z) = z^r$.

Proof: By mathematical induction. Since the proposition is trivially true for $k = 2$, suppose that $k > 2$ and that the result is true for $1, 2, \ldots, k-1$. For $z \in Y$ having $\lambda(z) = k$, suppose that $z_1 = \text{LPF}z$, $z_2 = \text{RPF}z$. If $k_1 = \lambda(z_1)$ and $k_2 = \lambda(z_2)$ then $k_1 + k_2 = k$ and we have

$$\begin{aligned} f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1(z) &= f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1({*}z_1 z_2) \\ &= f_{k-1} \circ f_{k-2} \circ \cdots \circ f_2({*}z_2 z_1) \\ &= {*}f_{k_2-1} \circ \cdots \circ f_1(z_2) f_{k_1-1} \circ \cdots \circ f_1(z_1) \\ &= {*}f_1 \circ \cdots \circ f_{k_2-1}(z_2) f_1 \circ \cdots \circ f_{k_1-1}(z_1) \end{aligned}$$

$= f_1[*f_1 \circ \cdots \circ f_{k_1-1}(z_1) f_1 \circ \cdots \circ f_{k_2-1}(z_2)]$

$= f_1 \circ f_2 \circ \cdots \circ f_{k-1}(*z_1 z_2)$

$= f_1 \circ f_2 \circ \cdots \circ f_{k-1}(z)$

$= z^r$. ▯

Corollary 6.6a. For any $z \in Y$, $(z^r)^r = z$.

Corollary 6.6b. For all $z,t \in Y$, (i) $(*zt)^r = *t^r z^r$, (ii) $(*_e zt)^r = *_e t^r z^r$, and (iii) $(*_i zt)^r = *_i t^r z^r$.

Proposition 6.7. If in the free groupoid $<Y,*>$ is defined a unary operation ' subject to

(i) $y' = y$, and

(ii) $(*zt)' = *t'z'$ fpr a;; $z,t \in Y$,

then $z' = z^r$ for all z in Y.

Proof: By induction on $\lambda(z)$. The case for $\lambda(z) = 1$ is established by (i) above. Suppose then that $z \in Y$ is such that $\lambda(z) = k > 1$. If $z_1 = \text{LPF}z$, $z_2 = \text{RPF}z$, then $\lambda(z_1) < k$, $\lambda(z_2) < k$ and we have $z' = (*z_1 z_2)' = *z_2' z_1' = *z_2^r z_1^r = (*z_1 z_2)^r = z^r$. ▯

Definition 6.8. For $z \in Y$, if $z^r = z$, z will be called symmetric in Y.

The next proposition is a direct consequence of Corollaries 6.7a and b. From it Propositions 6.10 and 6.11 follow in turn.

Proposition 6.9. For each z in Y, $*zz^r$, $*_e zz^r$ and $*_i zz^r$ are symmetric in Y.

Proposition 6.10. Given $z \in Y$ with $z_1 = \text{LPF}z$, $z_2 = \text{RPF}z$, z is symmetric if and only if $z_1^r = z_2$.

Proposition 6.11. There exists a z in Y with $z^r = z$ and

$\lambda(z) = k$ if and only if $k = 1$ or k is even.

If v_i is the number of elements in $V_i = \{ z : z \in Y, \lambda(z) = 2i, z^r = z\}$, for $i \in J_+$, then as a corollary to Theorems 4.16 and 4.17 we have

Proposition 6.12. $v_1 = 1$ and $v_i = \sum_{k=1}^{i-1} v_k v_{i-k} = \frac{1}{i}\binom{2i-2}{i-1}$ for $i \geq 2$.

Although in Chapter IV we defined a (linear) ordering in Y for the purpose of generating all elements of Y with length k, this ordering in not compatible with the other structure defined in Y in a meaningful way. We consider below a partial ordering which satisfies a number of compatibility conditions (See Propositions 6.14, 6.15, 6.18, 6.19.).

Definition 6.13. Let $R_{\leq} \subset Y \times Y$ be the relation in Y satisfying:

For $z,z' \in Y$, $(z,z') \in R_{\leq}$ if and only if

(6.13.1) $\lambda(z) \leq \lambda(z')$ and

(6.13.2) if $z = z_1 z_2 \cdots z_k$ with each $z_i \in \{*,y\}$, then $z' = z_1' z_2' \cdots z_k'$ with each $z_i' \in \{*\} \cup Y$ and $z_i' = *$ if and only if $z_i = *$.

If $(z,z') \in R_{\leq}$, write $z \leq z'$.

A consequence of Proposition 4.13 is that if (6.13.2) can be satisfied for a pair z,z' then the expression $z_1' z_2' \cdots z_k'$ for z' is unique. Also we note that if $z \leq z'$ and $\lambda(z) = \lambda(z')$ then $z = z'$.

Proposition 6.14. For $z,z' \in Y$, if $z \leq z'$ then $z^r \leq (z')^r$.

Proof: Since $z \leq z'$, $LPFz \leq LPFz'$ and $RPFz \leq RPFz'$. Now $z^r = (*LPFzRPFz)^r = *(RPFz)^r(LPFz)^r$, and the desired result follows by mathematical induction. $\square$

The observation that $z \leq z'$ if and only if $\mathrm{LPF}z \leq \mathrm{LPF}z'$ and $\mathrm{RPF}z \leq \mathrm{RPF}z'$ is often, as above, useful in proofs by mathematical induction. In addition it enables us to state

Proposition 6.15. For $z_1, z_2, z_3, z_4 \in Y$ the following conditions are equivalent:

(i) $z_1 \leq z_2$ and $z_3 \leq z_4$

(ii) $*z_1z_3 \leq *z_2z_4$

(iii) $*_e z_1z_3 \leq *_e z_2z_4$ and $*_i z_1z_3 \leq *_i z_2z_4$

Example 6.16. If $z = {*}{*}y{*}yyy$ and $z' = {*}{*}{*}yy{*}{*}yyy{*}y{*}{*}yyy$ then $z \leq z'$ with $z_1' = *$, $z_2' = *$, $z_3' = {*}yy$, $z_4' = *$, $z_5' = {*}yy$, $z_6' = y$, $z_7' = {*}y{*}{*}yyy$. However, if $z'' = {*}{*}yy{*}y{*}{*}yyy$, then $z \not\leq z''$.

As an immediate consequence of Definition 6.13 we have

Proposition 6.17. The relation $\leq$ defines a partial order for Y; i.e., for all $z, z', z'' \in Y$

(i) $z \leq z$,

(ii) if $z \leq z'$ and $z' \leq z$ then $z = z'$,

(iii) if $z \leq z'$ and $z' \leq z''$ then $z \leq z''$.

The pair z, z'' of Example 6.16 illustrates that $\leq$ does not linearly order Y.

Proposition 6.18. For each $z \in Y$ and each successor z_s of z, $z < z_s$. Furthermore, if $z \leq z'$ in Y with z_s' any successor of z' then $z < z_s'$. Finally, if $z \leq z'$ and if z_s is any successor of z, there exists a successor z_s' of z' such that $z_s \leq z_s'$.

Proof: The first two assertions are direct consequences of Definition 6.13. The remaining statement will be proved by mathematical induction on $\lambda(z)$. If $\lambda(z) = 1$, then $z = y$ and the unique successor, $*yy$, of z precedes the successor of any element z' of Y, $z' \neq y$. Suppose then that z and z' are such that $z \leq z'$, $\lambda(z) = k > 1$ and that the proposition is true for all pairs z, z' such that $\lambda(z) < k$. Let $z_1 = LPFz$, $z_2 = RPFz$, $z_1' = LPFz'$ and $z_2' = RPFz'$. Now $z_1 \leq z_1'$, $z_2 \leq z_2'$ and, by the induction hypothesis, if $z_{1,s}$ and $z_{2,s}$ are any successors of z_1 and z_2, respectively, there exist successors $z_{1,s}'$, $z_{2,s}'$ of z_1' and z_2' such that $z_{1,s} \leq z_{1,s}'$, $z_{2,s} \leq z_{2,s}'$. The desired conslusion follows from the definition of successor. []

If $z' \in Y$ is a successor of z in Y let us say that z' may be obtained from z by an expansion of z. More precisely, if $z = z_1 z_2 \ldots z_k$ with $z_j = y$ for $j \in \{1,2,\ldots,k\}$, and if $m = N(j,y,z)$ then the successor $z' = z_1 \cdots z_{j-1} {*y} z_j z_{j+1} \cdots z_k$ of z will be called *the mth expansion of z* and denoted $E_m(z)$. This aids us in stating

Proposition 6.19. For any z, $z' \in Y$, $z < z'$ if and only if z' may be obtained from z by $\lambda(z') - \lambda(z)$ consecutive expansions, i.e., if and only if whenever $p = \lambda(z') - \lambda(z)$ there exist $m_i \in \{1,2,\ldots,\lambda(z)+i-1\}$ such that $z' = E_{m_p} \circ \cdots \circ E_{m_2}(z) \circ E_{m_1}(z)$.

Proof: When the stated condition holds, the fact that $z < z'$ follows from Proposition 6.18 and the transitivity of $<$, by mathematical induction.

Conversely, suppose $z < z'$. If $\lambda(z') - \lambda(z) = 1$ then condition (6.13.2) implies that z' is a successor of z. Let z and z' be such that $\lambda(z') - \lambda(z) = k > 1$ and suppose the proposition bo be true for all z, z' with $0 < \lambda(z') - \lambda(z) < k$. If $z = y$ then, since every $z' \neq y$ has a predecessor in Y, there exists a sequence $z', z_2, \ldots, z_{k-1}, y$ such that each, after the first, is a predecessor of the preceding, establishing the proposition in this case. If $\lambda(z) > 1$ then LPFz and RPFz are defined with LPFz' and RPFz' obtainable from them by successive expansions. Appropriate changes in the subscripts yields a corresponding expansion for z = *LPFzRPFz which yields z'. $\square$

<u>Proposition 6.20</u>. Every pair $z, z' \in Y$ has an infimum, $\inf(z,z')$, and a supremum, $\sup(z,z')$, in Y.

<u>Proof</u>: If $z = y$ then $\inf(z,z') = y$ and $\sup(z,z') = z'$ for any $z' \in Y$ with a similar result if $z' = y$. Thus the proposition is true if $\lambda(z) = 1$ or $\lambda(z') = 1$ and this establishes a basis for a proof by mathematical induction on $\lambda(z) + \lambda(z')$. Suppose that $z \neq y$ and $z' \neq y$ are such that $\lambda(z) + \lambda(z') = k > 3$ and that infima and suprema exist for all pairs of elements of Y the sum of whose lengths is less than k. By the induction hypothesis inf(LPFz,LPFz'), inf(RPFz,RPFz'), sup(LPFz,LPFz') and sup(RPFz,RPFz') all exist in Y. Denoting these elements, repsectively, by r_1, r_2, t_1 and t_2, consider $r = {*}r_1r_2$ and $t = {*}t_1t_2$. Since $r \leq z$ if and only if $r_1 \leq$ LPFz and $r_2 \leq$ RPFz, and $z \leq t$ if and only if LPFz $\leq t_1$ and

RPFz $\leq t_2$, we have that $r = \inf(z,z')$ and $t = \sup(z,z')$. []

<u>Theorem 6.21</u>. $<Y,\leq>$ is a distributive lattice.

<u>Proof</u>: Since $<Y,\leq>$ is a lattice if and only if Y is partially ordered by $\leq$ and for each pair $z,z' \in Y$, $\inf(z,z')$ and $\sup(z,z')$ exist in Y, from Propositions 6.17 and 6.20 we conclude that this is true. To prove that the lattice is distributive, i.e., that for z,z',z'' in Y the formulas

$$\inf(z,\sup(z',z'')) = \sup(\inf(z,z'),\inf(z,z''))$$

$$\sup(z,\inf(z',z'')) = \inf(\sup(z,z'),\sup(z,z''))$$

hold, we make use of the following result, found in [12].

> A lattice is distributive if and only if it does not contain a sublattice isomorphic to either of the 5-element lattices of Figure 6.1.

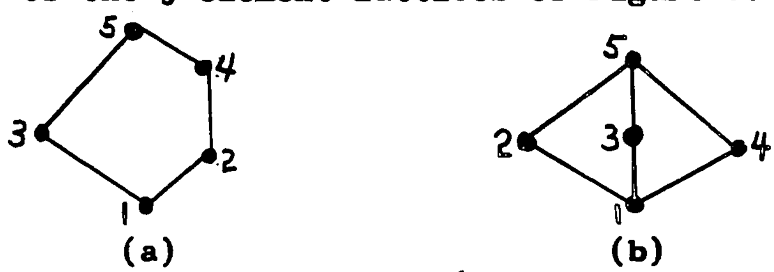


Figure 6.1.

A sublattice isomorphic to the figure shown above in (a) cannot occur in Y for if $z_1,\ldots,z_5$ are the elements of Y corresponding to the points labelled 1,...,5 and if $\lambda(z_1) = k$ then, since z_3 and z_2 are successors of z_1, z_4 is a successor of z_2, and z_5 is a successor of z_3 and z_4, we have $\lambda(z_5) = k+2$ and $\lambda(z_5) = k+3$ which is, of course, impossible.

Similarly, if $z_1,\ldots,z_5$ are the elements of Y corresponding to the points labelled 1,...,5 in Figure 6.1(b) then z_2, z_3 and z_4 are successors of z_1 while z_5 is a successor of z_2, z_3 and z_4. Thus if $z_1 = z_{11}z_{12}\cdots z_{1n}$ with each $z_{1j} \in \{*,y\}$, then there exist $n_1,n_2,n_3 \in \{2,3,\ldots,n\}$, all different, such that $z_{1n_i} = y$ and $z_{1+i} = z_1\cdots z_{1n_i-1}{*}yz_{1n_i}z_{1n_i+1}\cdots \cdot z_{1n}$ for $i = 1,2,3$. It is clearly impossible to transform z_2, z_3 and z_4 into the same element z_5 of Y by performing a single expansion on each. Thus $\langle Y,\leq\rangle$ contains no sublattice isomorphic to the figure in (b) and is distributive. $\square$

We now turn to consideration of a different type of structure within Y. Suppose $z = z_1z_2\cdots z_k$ and t are each elements of Y with each $z_i \in \{*,y\}$. Choose $p \in \{1,2,\ldots,k\}$ such that $N(p,y,z) = 1$. Define operations $*^1$ and $*^2$ in Y by $*^1zt = *z_1\cdots z_{p-1}tz_{p+1}\cdots z_ky$ and $*^2zt = *yz_1\cdots z_{k-1}t$. Alternately, $*^1zt = *z^ty$, where z^t is the result of substituting t for the initial y in z, and $*^2zt = *yz_t$, where z_t is the result of substituting t for the final y in z. Since the condition of Theorem 4.3 is preserved by these operations their images are indeed elements of Y.

For each $k \in J_+$ let $\bar{y}^k$ denote that element of Y consisting of k-1 *'s followed by k y's, and let $\bar{y}_k$ denote the element of Y consisting of k-1 repetitions of *y followed by a final y. If we define $Y_1 = \bigcup_{k=1}^{\infty}\{\bar{y}^k\}$ and $Y_2 = \bigcup_{k=1}^{\infty}\{\bar{y}_k\}$, it follows that

<u>Proposition 6.22</u>. The operation $*^i$ is commutative and associative in Y_i, $i = 1,2$.

With the notions of freeness and generators defined for semigroups in a manner analogous to that for groupoids, we have

Corollary 6.22a. $<Y_1,*^1>$ and $<Y_2,*^2>$ are commutative semigroups, each freely generated by y.

Proposition 6.23. Y_1 and Y_2 are each totally well-ordered by $\leq$.

Proof: For each positive integer k there exists a unique element z_j of Y_j, $j \in \{1,2\}$, such that $\lambda(z_j) = k$. The total well-ordering of Y_j follows from that of J_+ since, for $z,z' \in Y_j$, $z \leq z'$ if and only if $\lambda(z) \leq \lambda(z')$. $\square$

Proposition 6.24. $Y_1 = \{z^r\colon z \in Y_2\}$, $Y_2 = \{z^r\colon z \in Y_1\}$.

Proof: By induction on $\lambda(z)$. $\square$

The subsets Y_1 and Y_2 are also significant for the operations $*_e$ and $*_i$, as the following proposition indicates.

Proposition 6.25. For each $j \in \{1,2\}$ and each $x \in \{e,i\}$ the system $<Y_j,*_x>$ is a semigroup, a subsemigroup of $<Y,*_x>$.

Proof: The conclusion follows readily from the observation that, for $z_{11},z_{12} \in Y_1 - \{y\}$, $*_e z_{11}z_{12} = z_{11}$ and $*_i z_{11}z_{12} = z_{12}$ and, for $z_{21},z_{22} \in Y_2 - \{y\}$, $*_e z_{21}z_{22} = z_{22}$ and $*_i z_{21}z_{22} = z_{21}$. $\square$

After presenting, in Chapter VII, several examples of finitely generated free groupoids we will return, in Chapter VIII, to illustrate these internal structures with an example.

CHAPTER VII

EXAMPLES

By virtue of Proposition 3.1, an operation can be defined on any countably infinite set in such a way that the set together with the operation is a free groupoid with any chosen element of the set as generator. The primary purpose of this chapter is to show that, for the examples of Chapter II, there exists a "natural" operation for which the set becomes a free groupoid with the so-called "simplest" element of the collection as generator and such that the elements of length k are exactly those enumerated by the formula $\frac{1}{k}\binom{2k-2}{k-1}$, $k \in J_+$. Some examples of successors also are given. To avoid at least some of the tedium, later examples are abbreviated and proofs are omitted. The chapter concludes with examples of free groupoids with more than one generator.

DIAGONALIZATIONS OF POLYGONS

For $k \geq 3$ let d_k be a convex polygon with n vertices labelled, in clockwise order, using the natural numbers 1,2, ...,n, starting with an arbitrary but fixed vertex. Let D_n be the set of distinct plane figures obtainable from d_n by dividing it into n-2 triangles by the insertion of n-3 (non-intersecting) diagonals. (Two triangulations of d_n will be considered distinct if there does not exist an orientation-preserving homeomorphism of the plane which maps one onto the other in such a manner that the vertices labelled 1 correspond.)

Let D_2 be the set containing the labelled segment 1•——•2. Form $D = \bigcup_{i\geq 2}^{\infty} D_i$. We can define a mapping $*_D$: $D \times D \longrightarrow D$ in the following way:

Choose $a,b \in D$ and suppose that the labels of a are $1,2,\ldots,j$ and those of b are $1,2,\ldots,k$. Form a plane figure c' by superimposing vertex 1 of b on vertex j of a in such a manner that (i) the two figures a and b enclose no common area and their boundaries have only the named vertices in common and (ii) vertex k of b may be connected to vertex 1 of a by a straight line segment which does not intersect either a or b except at the named vertices. Insertion in c' of the segment described in (ii) and relabelling of the vertices which originally belonged to b by the replacement $i \longleftarrow j + (i-1)$ yields an element c of D. Define the image under $*_D$ of the ordered pair (a,b) as c; symbolically, we write $*_D ab = c$ or, when convenient, $a *_D b = c$.

For example, we have

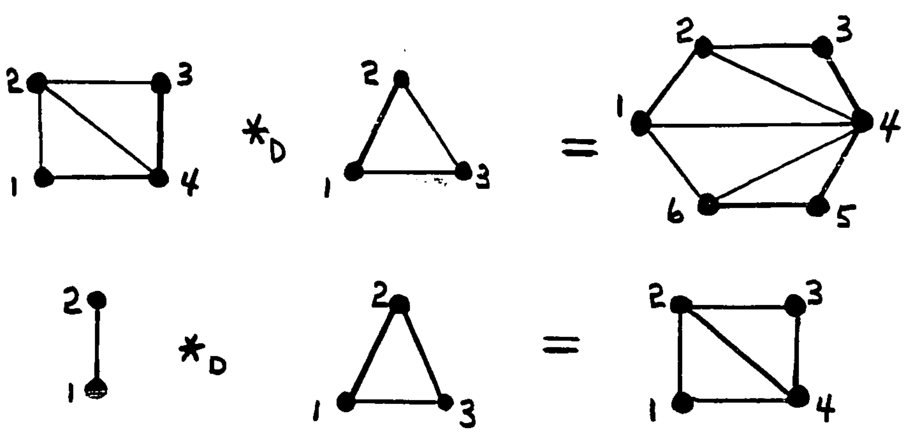


Figure 7.1.

Proposition 7.1. $\langle D, *_D\rangle$ is a free groupoid with generator 1•——•2.

Proof: What we must show is that for every $x \in D - \{1•——•2\}$ there exist unique $x_1, x_2 \in D$ such that $x = *_D\, x_1 x_2$. Suppose then that $x \in D$ and that the vertices of x are labelled $1, 2, \ldots, n$ $(n \geq 3)$. Let j, $1 < j < n$, be the third vertex of that triangle whose other two vertices are 1 and n. Let x_1 be the figure obtained from x by the deletion of the vertices $j+1, \ldots, m$ and all segments incident to these vertices. Clearly $x_1 \in D$. Similarly let x_2 be the figure obtained from x by the deletion of the vertices $1, 2, \ldots, j+1$ and by relabelling the remaining vertices by the replacement $i \longleftarrow i-j+1$. We also have $x_2 \in D$ and from the definition of $*_D$ it follows that $*_D x_1 x_2 = x$. The uniqueness of x_1 and x_2 follows from the uniqueness of the orientation of x and the uniqueness of the vertex labelled 1 in x. $\square$

If we define $\lambda_D\colon D \longrightarrow J_+$ by setting $\lambda_D(x)$ equal to 1 plus the number of triangles in x (See Figure 6.2) then $\lambda_D(1•——•2) = 1$ and if $x = *_D x_1 x_2$ it follows from the definition of $*_D$ that $\lambda_D(x) = \lambda_D(x_1) + \lambda_D(x_2)$. (Alternately, $\lambda_D(x) = n$ if and only if $x \in D_{n+1}$.) As a consequence of Theorem 4.16 we have

Corollary 7.1a. The number of distinct ways of dividing a convex polygon of n sides into $n-2$ triangles by the insertion of $n-3$ (non-intersecting) diagonals is $\frac{1}{n-1}\binom{2n-4}{n-2}$.

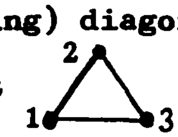
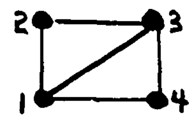
If we select  as the successor of 1•——•2 then we can define the successors of any x in D with $\lambda(x) > 1$

Figure 7.2.

by: x' is a successor of x in D if and only if x' can be obtained from x by (i) choosing any edge e of x with labels i and i+1, $1 \leq i \leq \lambda_D(x) + 1$, (ii) constructing a triangle with base e exterior to x, and (iii) labelling the new vertex i+1 and relabelling all vertices numbered i+1,i+2,..., $\lambda(x)+1$ by the replacement $j \longleftarrow j+1$. Clearly each $x \in D$ has $\lambda_D(x)$ successors and each $x \neq 1$•——•2 is a successor. The successors of



in D are

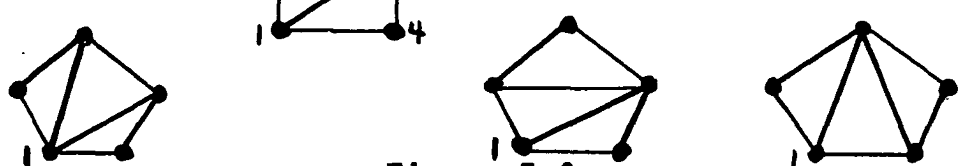


Figure 7.3.

<u>Corollary 7.2b</u>. If $x \in D$ has $\lambda_D(x) > 1$ then x' is a successor of $x = *_D x_1 x_2$ if and only if $x' = *_D x_1' x_2$ for some successor x_1' of x_1 or $x' = *_D x_1 x_2'$ for some successor x_2' of x_2.

<u>Proof</u>: Both the necessity and the sufficiency are direct consequences of the definitions of successor in D and the mapping $*_D$. ▯

BALLOT SEQUENCES

Given n+1 copies of the letter p and n copies of the letter q, let B_n be the collection of concatenations of these 2n+1 letters possessing the property that, moving along each string from left to right the number of p's passed over is always greater than the number of q's. Form $B = \bigcup_{n=0}^{\infty} B_n$ and define a mapping $*_B$: $B \times B \longrightarrow B$ in the following way. If $r,s \in B$, then $*_B rs = rsq$; i.e., the image of the ordered pair (r,s) under $*_B$ is the string of p's and q's obtained by concatenating the symbols of r, the symbols of s and the single symbol q, in that order. Clearly $*_B rs \in B$ and, in fact,

<u>Proposition 7.2</u>. $\langle B, *_B \rangle$ is a free groupoid with generator p.

In this and succeeding examples the straightforward proofs of the propositions will be omitted since, as in the preceding example, they are indeed direct consequences of the definitions.

If for $x \in B$ we define $\lambda_B(x)$ to be the number of occurrences of p in x (i.e., $\lambda_B(x) = n$ if and only if $x \in B_{n-1}$) then λ_B: $B \longrightarrow J_+$ satisfies $\lambda_B(p) = 1$ and $\lambda_B(x) = \lambda_B(x_1) + \lambda_B(x_2)$ whenever $x = *_B x_1 x_2 \in B$. We may thus apply Theorem 4.16 to obtain

<u>Corollary 7.2a</u>. The number of elements of B_n is $\frac{1}{n+1}\binom{2n}{n}$, $n = 0,1,2,\ldots$.

Choose ppq as the successor of p in B and, in general, for $x \in B$, call x' a successor of x if and only if x' may be obtained from x by replacement of some p in x by ppq. It is seen that each x in B has $\lambda_B(x)$ successors and if $x \neq p$ then x is a successor of some element. Furthermore

Corollary 7.2b. If $x \in B$ has $\lambda_B(x) > 1$ then x' is a successor of $x = *_B x_1 x_2$ if and only if $x' = *_B x_1' x_2$ for some successor x_1' of x_1 or $x' = *_B x_1 x_2'$ for some successor x_2' of x_2.

SEQUENCES OF CONSECUTIVE INTEGERS

For each positive integer k let S_k denote the collection of sequences of consecutive positive integers which begin and end with 1 and which have length 2k-1. Set $S = \bigcup_{k=1}^{\infty} S_k$. We can define a mapping $*_S\colon S \times S \longrightarrow S$ in the following way. If $a,b \in S$ then $*_S ab$ is the sequence formed by adding 1 to each member of a and then prefixing the resulting sequence with 1 and following this result by the elements of b, in their given order. We see that $*_S ab$ is indeed an element of S and it follows in a straightforward manner that

Proposition 7.3. $\langle S, *_S \rangle$ is a free groupoid generated by the sequence [1].

Defining $\lambda_S\colon S \longrightarrow J_+$ by $\lambda_S(x) = k$ if and only if $x \in S_k$, the usual properties of length follow, and we can also deduce

Corollary 7.3a. The number of elements of S_k is $\frac{1}{k}\binom{2k-2}{k-1}$, $k \in J_+$.

The successor of [1] is [1,2,1] and if $x = [x_1,\ldots,x_{2k-1}] \in S$, $k \geq 2$ then any sequence of the form $x' = [x_1,\ldots,x_i,x_i+1,x_i,x_{i+1},\ldots,x_{2k-1}]$ is a successor of x if and only if either $i = 2k-1$ or $x_i = x_{i+1}+1$. The often repeated properties of successors hold here as well.

INCREASING SEQUENCES OF INTEGERS

For each positive integer k let P_k denote the collection of sequences $[a_1,\ldots,a_k]$ of k positive integers satisfying $a_1 = 1$ and $a_j \leq 2(j-1)$ for $j = 2,\ldots,k$; form $P = \bigcup_{i=1}^{\infty} P_i$. Define a mapping $*_P$: $P \times P \longrightarrow P$ in the following manner. For $a,b \in P$, $*_P ab = [r,s]$ where r is such that $[r] = a$ and $s = b_1+(2k-1),b_2+(2k-1),\ldots,b_j+(2k-1)$ if and only if $b = [b_1,b_2,\ldots,b_j]$ and k is the length of a. It is easily seen that $*_P ab \in P$ and, in fact,

<u>Proposition 7.4</u>. $\langle P,*_P\rangle$ is a free groupoid generated by [1].

The statement $\lambda_P(x) = k$ if and only if $x \in P_k$ defines a mapping λ_P: $P \longrightarrow J_+$ which obeys the previously stated properties of length. Using λ_P we can show that the number of elements in P_k is $\frac{1}{k}\binom{2k-2}{k-1}$ for each positive integer k.

The element x' of P is said to be a successor of $x = [x_1,\ldots,x_n] \in P$ if and only if there exists an $i \in \{1,2,\ldots,n\}$ and a positive integer q such that $[x_1,\ldots,x_i,q,x_{i+1},\ldots,x_n] = x'$. For example, the unique successor of [1] is [1,2] and the three successors of [1,2,4] are [1,2,3,4], [1,2,4,5] and [1,2,4,6]. The familiar results may again be stated. In addition it may be shown that if F: $P \longrightarrow Y$ is the mapping defined following Proposition 4.24 then, for $a \in P$, $F(S_a) = S_{F(a)}$, where S_a and $S_{F(a)}$ are the sets of successors of a and F(a) in P and Y respectively.

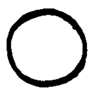
INSERTING CHORDS IN CIRCLES

For $n \geq 0$ let c_n be a circle with 2n distinguished points on it labelled, in clockwise order, using the natural numbers $1,2,\ldots,2n$, starting the labelling with an arbitrary but fixed distinguished point. Let C_n be the set of distinct plane figures obtainable from c_n by connecting the $2n$ points in pairs by non-intersecting chords. (Two such figures will be considered distinct if there does not exist an orientation-preserving homeomorphism of the plane which maps one onto the other in such a manner that the points labelled 1 correspond.) Form $C = \bigcup_{n=0}^{\infty} C_n$ and define a mapping $*_C$: $C \times C \longrightarrow C$ in the following way:

Let $a,b \in C$ where a has $2i$ distinguished points and b has $2j$ distinguished points. Consider the circle $c_{2i+2j+2}$ having $2i+2j+2$ distinguished points. Join points labelled 1 and $2i+2$ by a chord. For $k_1,k_2 \in \{2,..,2i+1\}$, join points labelled k_1 and k_2 by a chord in $c_{2i+2j+2}$ if and only if the points k_1-1 and k_2-1 were joined by a chord in a. For $n_1,n_2 \in \{2i+3,2i+4,\ldots,2i+2j+2\}$ join points labelled n_1 and n_2 by a chord in $c_{2i+2j+2}$ if and only if the points labelled n_1-2i-2 and n_2-2i-2 were connected by a chord in b. This procedure yields an element of C which we define to be the image of (a,b) under $*_C$ and denote by $*_C ab$.

<u>Proposition 7.5</u>. $<C,*_C>$ is a free groupoid generated by $\bigcirc$, the circle with no distinguished points.

If for $x \in C$ we define $\lambda_C(x)$ to be 1 plus the number of chords in x (i.e., $\lambda_C(x) = n+1$ if and only if $x \in C_n$) we may again deduce the combinatorial formula $\frac{1}{n+1}\binom{2n}{n}$ for the number of elements in C_n, $n = 0,1,2,\ldots$.

The successor of  in C is 1  2, and for x in C such that $\lambda_C(x) = k > 1$ a successor of x is any element of C obtainable by one of the following procedures:

(i) by adding a pair of distinguished points to the circumference of x between the points labelled 1 and 2k-2, labelling these 2k-1 and 2k in clockwise order and connecting them by a chord;

(ii) if i and j are the labels for some chord, $i < j$, then by adding a distinguished point to the circumference of x between i and i+1 and another, in a clockwise direction from the first, between j and j-1, by connecting these by a chord and then by relabelling, preserving the vertex labelled 1 and maintaining the clockwise order.

The usual results for successors hold.

RHYME SCHEMES

For each positive integer k, let R_k be the collection of all planar rhyme schemes for stanzas of k lines, and let R_0 contain the single element $\{\emptyset\}$. Form $R = \bigcup_{i=0}^{\infty} R_i$. We may define a mapping $*_R$: $R \times R \longrightarrow R$ in this manner: For $a,b \in R$

if $a \in R_0$, $b \in R_0$, then $*_R ab = \{\{1\}\}$;

if $a \in R_0$ and b is a partition of $\{1,2,\ldots,j\}$, then $*_R ab = \{\{1\}\} \cup c$ where c is the partition of $\{2,\ldots,j+1\}$ obtained by replacing each element x of an element of b by x+1;

if $b \in R_0$ and a is a partition of $\{1,2,\ldots,i\}$, then

$*_R ab$ is the partition of $\{1,2,\ldots,i+1\}$ obtained by inserting $i+1$ in that element of a containing 1;

if a is a partition of $\{1,2,\ldots,i\}$ and b is a partition of $\{1,2,\ldots,j\}$ then $*_R ab$ is the partition of $\{1,2,\ldots,i+j+1\}$ obtained by taking $a' \cup b'$ where a' is the partition of $\{1,2,\ldots,i+1\}$ obtained by inserting $i+1$ in that element of a which contains 1 and b' is that partition of $\{i+2,\ldots i+j\}$ formed by adding $i+1$ to each element of an element of b.

As before, we have

<u>Proposition 7.6</u>. $\langle R,*_R\rangle$ is a free groupoid generated by R_0.

If $\lambda_R(x) = k$ whenever $x \in R_{k-1}$, then λ_R satisfies the familiar properties of a length mapping and Theorem 4.16 again enables us to conclude that the number of elements of R_{k-1} is $\frac{1}{k}\binom{2k-2}{k-1}$ for each positive integer k.

As is necessary, the successor of $\{\emptyset\}$ is $\{\{1\}\}$. However, for $x \in R_k$ with $k > 0$ it is easier to describe the k+1 successors of x in terms of the Puttenham diagram. These successors are rhyme schemes whose diagrams are obtainable from that of x by the following procedure:

(i) insert a new line before line 1, after line k or between lines i and i+1 for some $i \in \{1,2,\ldots,k-1\}$;

(ii) if the line following the new line is the first line in a rhyming set, enlarge the rhyming set to contain the new line -- otherwise leave the new line unrhymed;

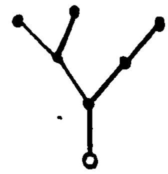
(iii) renumber the k+1 lines to include the new line.

PLANTED PLANE TREES

For each positive integer n, let T_n be the collection of non-map-isomorphic planted plane trees having n+1 vertices (n besides the root) and form $T = \bigcup_{n=1}^{\infty} T_n$. Let us call the edge incident to the root vertex the <u>root edge</u>. We may then define a mapping $*_T$: $T \times T \longrightarrow T$ in the following manner. For $r,s \in T$, $*_T rs$ is that element of T whose root is the root of r and which is obtained by identifying the root vertex of s with the non-root vertex of the root edge of r and positioning r and s in the plane in a manner such that (i) no edges of r and s intersect except at the named vertex and (ii) the root edge of s lies in a counterclockwise direction from the root edge of r. If we denote the root vertex by a hollow dot and other vertices by solid dots, then we may state

<u>Proposition 7.7</u>. $\langle T, *_T \rangle$ is a free groupoid, generated by $\circ\!\!-\!\!\!-\!\!\bullet$.

For $t \in T$, $\lambda_T(t)$ may be defined as the number of non-root vertices or, equivalently, as the number of edges in t, and we have once again that the number of elements of T with k+1 vertices is $\frac{1}{k}\binom{2k-2}{k-1}$ for all $k \in J_+$. A successor of t in T is any tree obtained from t by attaching an additional edge e to any vertex v in a position such that if the new edge is rotated in the plane in a counterclockwise direction about v, no other edge is encountered before the edge which leads to the root (See Figure 7.4.). The usual results for successors may be stated.

The six successors of  in T are

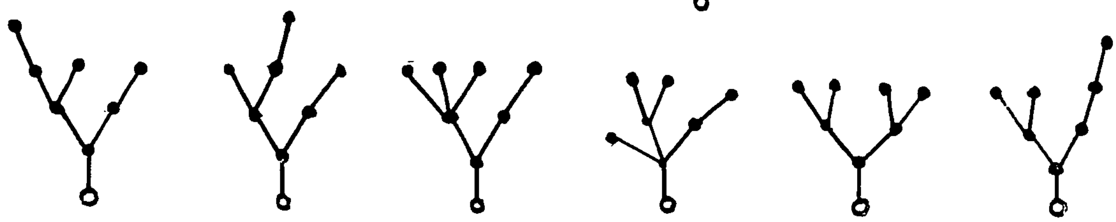


Figure 7.4.

PLANTED TRIVALENT PLANE TREES

For each natural number n, let W_n be the collection of all non-map-isomorphic planted trivalent plane trees in which there are n non-root vertices of degree 1. Form $W = \bigcup_{n=1}^{\infty} W_n$ and define a mapping $*_W$: $W \times W \longrightarrow W$ in the following way: for $r, s \in W$, $*_W rs$ is that element of W obtained by identifying the root vertex of r with the root vertex of s and positioning r and s in the plane in a manner such that (i) no edges of r and s intersect except at the point of identification and (ii) the root edge of s lies in a counterclockwise direction from the root edge of r, and then by joining a third edge at the common point, in a counterclockwise direction from the root edge of s, its free end being designated as the root of the new tree. (See Figure 8.1 for an example.).

<u>Proposition 7.8</u>. $< W, *_W >$ is a free groupoid, generated by .

If $\lambda_W(t)$ is defined as the number of non-root vertices of $t \in W$ which have degree 1, then λ_W satisfies the expected properties of length and we conclude again that the number of elements in W_k is $\frac{1}{k}\binom{2k-2}{k-1}$ for each natural number k.

A successor of t in W is any planted trivalent plane tree obtainable from t by attaching two edges at some terminal (non-root) vertex of t. Each element of W_n clearly has n successors and the other results for successors follow directly.

COMPLETE INSTANTANEOUS BINARY CODES

For each integer $k > 1$, let F_k be the collection of distinct sets of code words for all complete instantaneous binary codes having k words. Let F_1 contain the empty set of code words and form $F = \bigcup_{k=1}^{\infty} F_k$. We can define a mapping $*_F$: $F \times F \longrightarrow F$ as follows: for $a,b \in F-F_1$, $*_F ab$ is the union of the set of code words obtained by prefixing 0 to each element of a and 1 to each element of b. If $a \in F_1$ then $*_F ab = \{0\} \cup b'$ where b' is the set of code words obtained by prefixing 1 to each element of b. Similarly, if $b \in F_1$, $*_F ab = a' \cup \{1\}$ where a' is the result of prefixing 0 to each element of a. Thus, for $a,b \in F_1$, $*_F ab = \{0,1\}$ and is the smallest non-empty set of code words.

<u>Proposition 7.9</u>. $\langle F, *_F \rangle$ is a free groupoid, generated by F_1.

For a code c in F, if λ_F: $F \longrightarrow J_+$ is defined by $\lambda_F(c) = k$ if and only if $c \in F_k$, then λ_F satisfies the usual properties of length and the number of codes in F having k words is $\frac{1}{k}\binom{2k-2}{k-1}$, $k = 2,3,\ldots$.

The set $\{0,1\}$ of code words is the necessary successor of the empty code and for a code c in F having $\lambda_F(c) > 1$, c' is a successor of c in F if and only if all but two of the $\lambda_F(c) + 1$ code words of c' are code words of c and if these other words are obtainable from the remaining word of c by suffixing it with 0 and 1, respectively. The often repeated results for successors follow as expected.

FREE GROUPOIDS HAVING MORE THAN ONE GENERATOR

Several of the preceding examples may meaningfully be generalized to free groupoids having more than one generator. For example, suppose $Q = \{ 1, 2, \ldots, n \}$ and T^n is the set of all n-labelled planted plane trees having two or more vertices where, for t and t' in T^n, $t \neq t'$ if and only if either t and t' are non-map-isomorphic or they are map-isomorphic with some pair of corresponding edges labelled differently. Define $*_{T^n} : T^n x T^n \longrightarrow T^n$ in the following manner (as $*_T$ was defined earlier). For $r,s \in T^n$, $*_{T^n}rs$ is that element of T^n whose root is the root of r and which is obtained by identifying the root vertex of s with the noo-root vertex of the root edge of r and positioning r and s in the plane in a manner such that (i) no edges of r and s intersect except at the named vertex and (ii) the root edge of s lies in a counter-clockwise direction from the root edge of r (See Figure 7.5.).

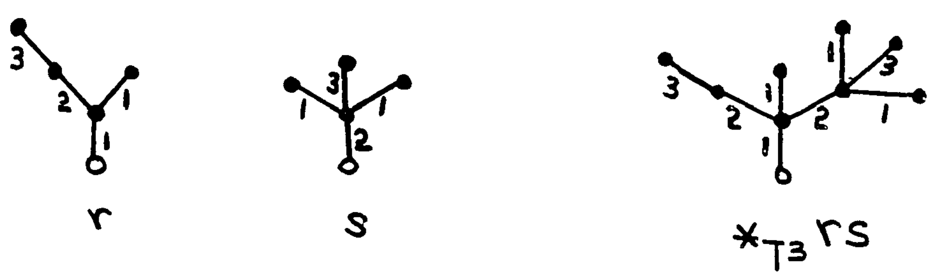


Figure 7.5.

Proposition 7.10. $\langle T^n, *_{T^n} \rangle$ is a free groupoid, freely generated by Q.

With $\lambda_{T^n}(t)$ as the number of non-root vertices of $t \in T$, we have

Corollary 7.10a. The number of elements t of T^n having $\lambda_{T^n}(t) = k$ is $\frac{(n)^k}{k}\binom{2k-2}{k-1}$.

CHAPTER VIII

STRUCTURE WITHIN THE GROUPOID OF PLANTED TRIVALENT PLANE TREES

Using the free groupoid $<W,*_W>$ whose element set contains all planted trivalent plane trees having two or more vertices of degree one, we can rather easily illustrate the operations and relations of Chapter VI. For convenience we will use the same symbols for the operations and relations in W as for their counterparts in Y introduced in the earlier chapter.

Figure 8.1 illustrates the operations $*_W$, $*_e$, $*_i$ and ^{r} for selected members of W.

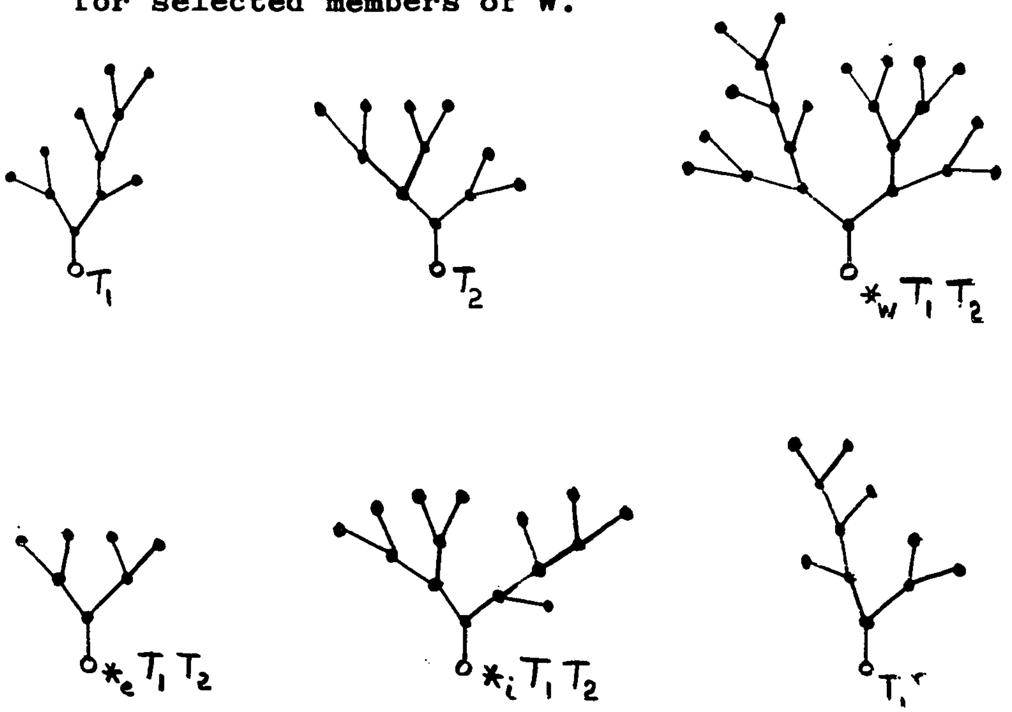


Figure 8.1.

The ordering defined in Y may be interpreted in W by saying that for $T,T' \in W$, $T \leq T'$ if and only if $T = T'$ or if T may be obtained from T' by a finite number of applications of the following "pruning" process. If e_1 and e_2 are edges of T' which meet at vertex v and which each have an end which is a vertex of degree 1, then the removal of $(e_i \cup e_2) - \{v\}$ from T' is called a <u>pruning</u> of T'. Figure 8.2 illustrates this ordering by indicating inf(T,T') and sup(T,T') for a pair T,T' of W.

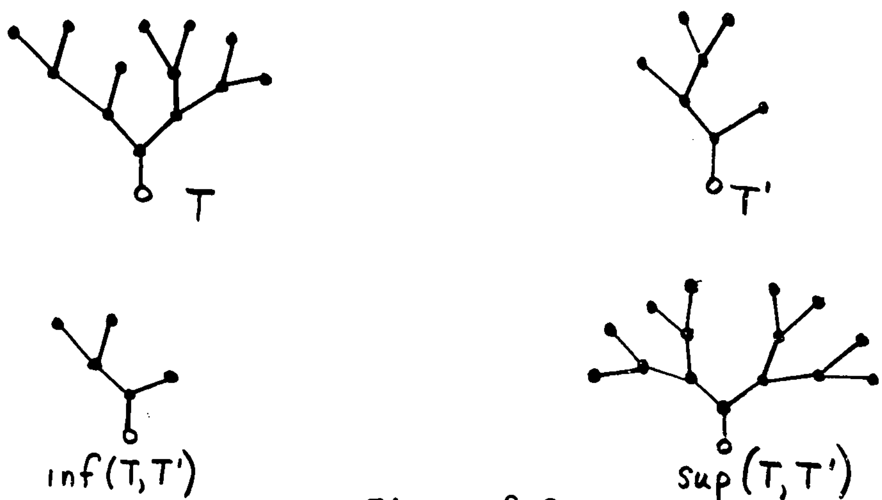


Figure 8.2.

For T ∈ W, call an edge e the <u>first terminal edge</u> of T if one of the ends of e is the first terminal vertex of T encountered by moving in a clockwise direction around T, starting at the root. An edge f of T is the <u>last terminal edge</u> of T if one of the ends of f is the first terminal vertex of T encountered by moving in a counterclockwise direction around T, again starting at the root. From $T_1, T_2 \in W$ we may obtain $*^1 T_1 T_2$ as follows:

($*^1$-1) Identify the root edge of T_2 with the first terminal edge of T_1, forming a new element T' of W.

($*^1$-2) Attach to the root of T' two new edges, designating the free end of the one which lies in a clockwise direction from the other as the new root. The resulting tree is $*^1T_1T_2$.

To obtain $*^2T_1T_2$:

($*^2$-1) Identify the root edge of T_2 with the last terminal edge of T_1, forming a new element T" of W.

($*^2$-2) Attach to the root of T" two new edges, designating the free end of the one which lies in a counter-clockwise direction from the other as the new root. The resulting tree is $*^2T_1T_2$.

Figure 8.3 gives examples of the application of these operations.

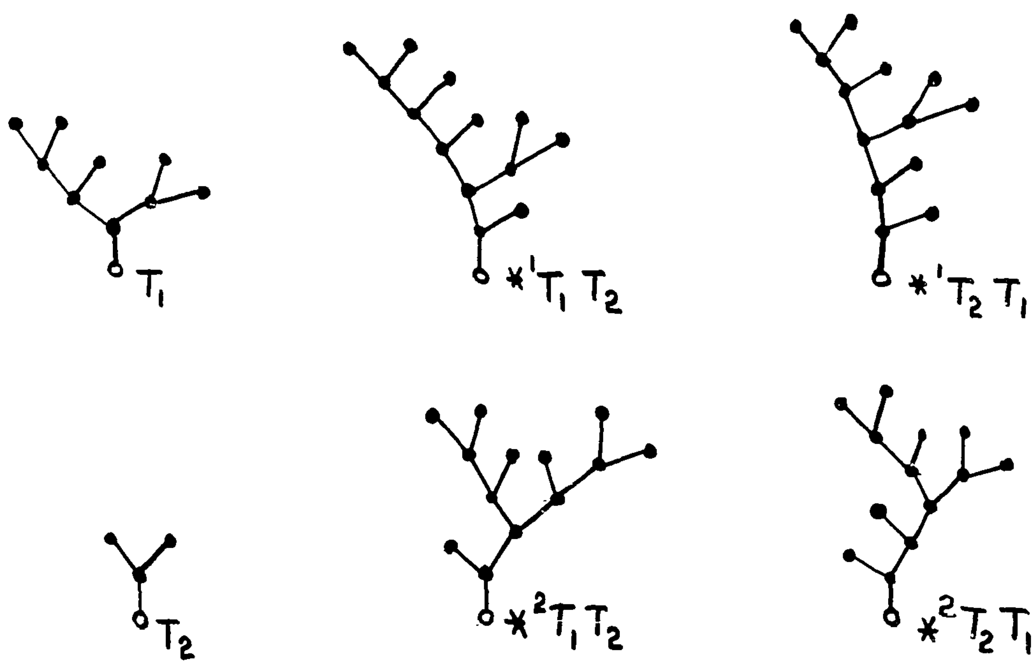


Figure 8.3.

For a vertex v of degree 3 in $T \in W$ let $e_{v,1}$, $e_{v,2}$ and $e_{v,3}$ be the edges meeting at v, labelled in a clockwise direction around v, starting with the edge nearest the root of T. If W_1 and W_2 are the subsets of W corresponding to Y_1 and Y_2 of Chapter VI, then the elements of W_1 and W_2 may be described thus:

$T \in W_1$ if and only if, for each vertex v of degree 3 in T, $e_{v,3}$ is a terminal edge;

$T \in W_2$ if and only if, for each vertex v of degree 3 in T, $e_{v,2}$ is a terminal edge.

Elements of W_1 and W_2 together with examples of the operations $*^1$ and $*^2$ in W_1 and W_2, respectively, are found in Figure 8.4.

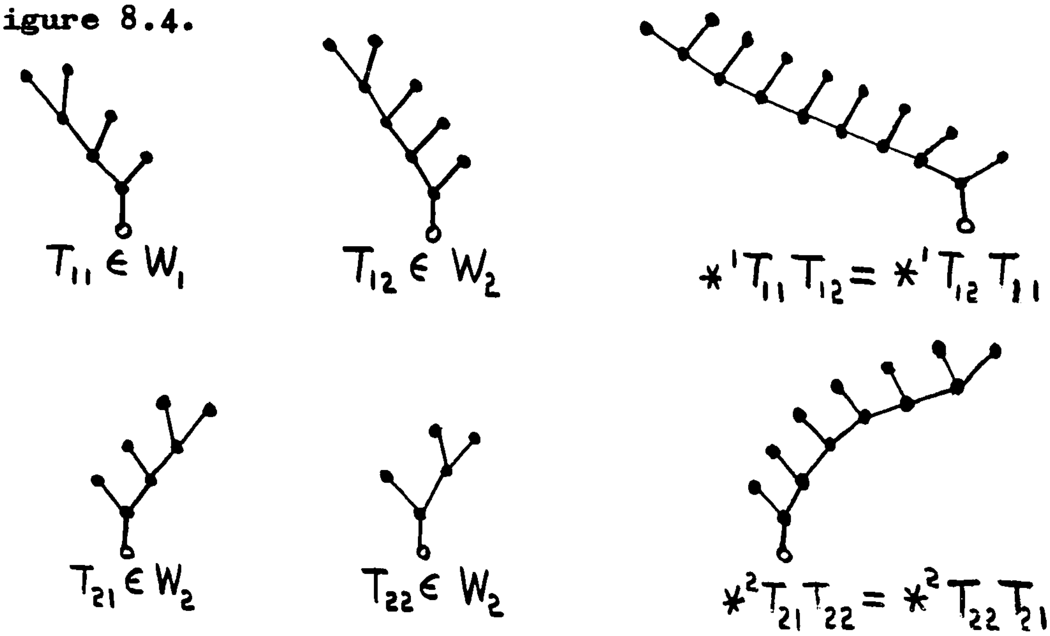


Figure 8.4.

CHAPTER IX

EQUIVALENCE AND ENUMERATIONS

The purpose of this chapter is to show that, for each of the examples in Chapter VII, it is possible to define reasonably natural notions of equivalence in such a manner that the enumeration of equivalence classes is different in all cases. If we denote by $\bar{D}$, $\bar{B}, \ldots$, $\bar{W}$ and $\bar{F}$ the collections of equivalence classes obtained by partitioning D, B,..., F using Definitions 9.1 - 9.9 below, then an operation $*_{\bar{X}}$ can be defined in $\bar{X}$, for each $X \in \{D,B,\ldots,F\}$, by

$$*_{\bar{X}}\,\bar{a}\bar{b} = \overline{*_X ab} \quad \text{for } \bar{a},\bar{b} \in \bar{X}$$

such that no two of the groupoids so obtained are isomorphic.

<u>Definition 9.1</u>. If $a,b \in D$ then a is equivalent to b if and only if b can be obtained from a by a cyclic permutation of the labels of the vertices of a.

<u>Definition 9.2</u>. For $a,b \in B$, a is equivalent to B if and only if a and b are identical strings of symbols.

<u>Definition 9.3</u>. If $a,b \in S$ then a is equivalent to b if and only if there exists a permutation of the elements of a which yields b.

<u>Definition 9.4</u>. For elements a and b of P, a is equivalent to b if and only if the sum of the integers composing a is equal to the sum of those composing b.

<u>Definition 9.5</u>. For $a,b \in C$, a is equivalent to b if and only if there exists a homeomorphism of the plane which maps

a onto b.

Definition 9.6. For elements a and b of R, a is equivalent to b if and only if there exists a natural number n such that both a and b are partitions of $\{1,2,\ldots,n\}$ and if there exists a permutation $\emptyset$ of the integers $1,2,\ldots,n$ such that replacement of each element x of an element of a by $\emptyset(x)$ yields b.

Definition 9.7. If $a,b \in T$ then a is equivalent to b if and only if there exists a one-to-one mapping of the vertices of a onto the vertices of b which preserves the degree of each vertex, preserves adjacencies, and preserves the root.

Definition 9.8. For $a,b \in W$, a is equivalent to b if and only if there exists a one-to-one mapping of the vertices of a onto the vertices of b which preserves the degree of each vertex, preserves adjacencies, and preserves the root (Compare Definition 9.7.).

Definition 9.9. If $a,b \in F$ then a is equivalent to b if and only if b can be obtained from a by interchanging the binary digits (i.e., replacing 1 by 0 and 0 by 1) in each code word of a.

In Figures 9.1 - 9.9 the 5 elements of length 4 are given for D, B,..., and F and the equivalence classes determined by the above definitions are indicated. Although this information is insufficient to distinguish differences among all the classes, the diagrams also serve to illustrate the various notions of equivalence.

The 5 elements of D_5, partitioned into 1 equivalence class:

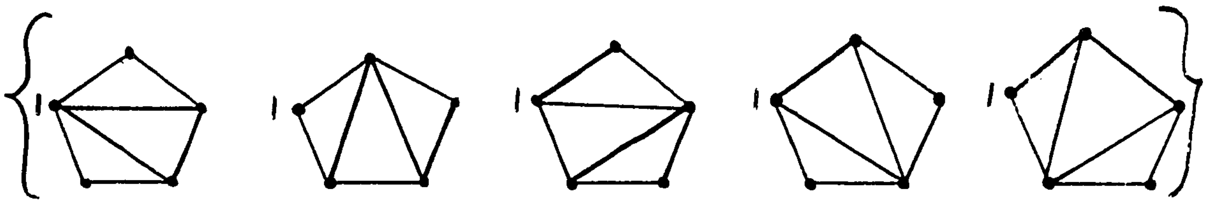


Figure 9.1.

The elements of B_3, partitioned into 5 classes:

{pppqqq} {pppqpqq} {pppqqpq} {ppqppqq} {ppqpqpq}

Figure 9.2.

The elements of S_4, partitioned into 4 classes:

{[1,2,3,4,3,2,1]} {[1,2,3,2,3,2,1]} {[1,2,1,2,1,2,1]}

{[1,2,1,2,3,2,1], [1,2,3,2,1,2,1]}

Figure 9.3.

The elements of P_4, partitioned into 4 classes:

{[1,2,3,4]} {[1,2,3,5]} {[1,2,3,6], [1,2,4,5]} {[1,2,4,6]}

Figure 9.4.

The elements of C_3, partitioned into 2 classes:

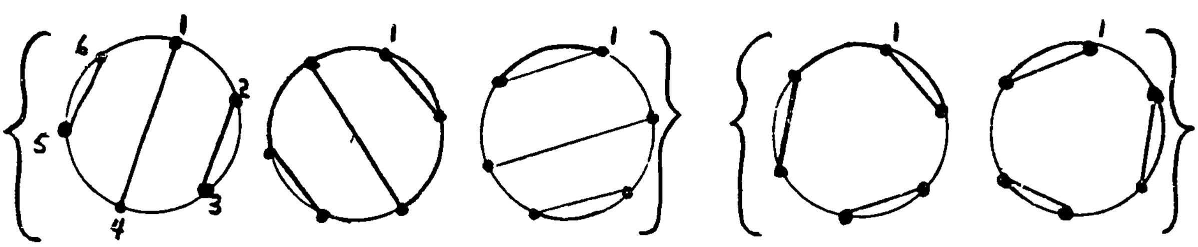


Figure 9.5.

The elements of R_3, partitioned into 3 classes:

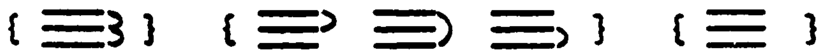


Figure 9.6.

The elements of T_4, partitioned into 4 classes:

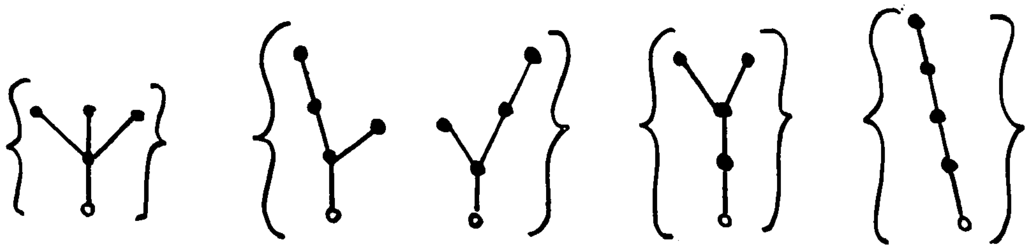


Figure 9.7.

The elements of W_4, partitioned into 2 classes:

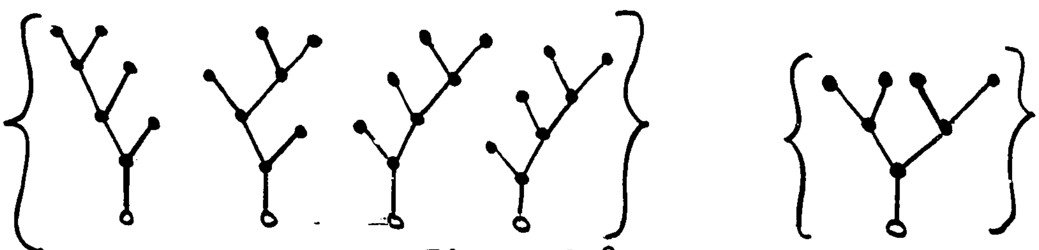


Figure 9.8.

The elements of F_4, partitioned into 3 classes:

$$\left\{\begin{Bmatrix}00\\01\\10\\11\end{Bmatrix}\right\} \quad \left\{\begin{Bmatrix}000\\001\\01\\1\end{Bmatrix} \begin{Bmatrix}0\\10\\110\\111\end{Bmatrix}\right\} \quad \left\{\begin{Bmatrix}00\\010\\011\\1\end{Bmatrix} \begin{Bmatrix}0\\100\\101\\11\end{Bmatrix}\right\}$$

Figure 9.9.

To distinguish among those groupoids for which the elements of length four are divided into the same number of classes, we mention the following:

(i) The 14 elements of S_5, P_5 and T_5 are partitioned into 8, 7 and 9 classes, respectively.

(ii) For C and W the elements of lengths 1,2,...,8 all are partitioned into the same number of equivalence classes. However the 5720 elements of length 9 in C are partitioned into 47 classes while the corresponding elements in W are partitioned into 46 classes.

(iii) The 14 elements of R_4 are partitioned into 5 classes while the 14 elements of F_5 are partitioned into 7 classes.

To complete our investigation of these examples, we will enumerate the equivalence classes in each case. Because these results are not directly related to the central theme of this paper only the conclusions are stated, proofs being omitted. For the results not this author's, however, references to the literature are given.

For $X \in \{D,B,S,\ldots,F\}$ let $\bar{x}_k$ denote the number of equivalence classes into which the elements of length k in X are divided.

Proposition 9.10 (Brown, [5]). $\bar{d}_k = \sum_{s|k} \phi(s) D_{s,k}$, $k \geq 3$, where ϕ is the Euler function (i.e., $\phi(s)$ is the number of positive integers which are less than or equal to s and are relatively prime to s) and $D_{s,k} = 0$ except for the following:

$$D_{1,k} = \frac{2(2k-5)!}{(k-3)!(k-1)!},$$

$$D_{2,2p} = \frac{2(2p-3)!}{(p-1)!(p-2)!}, \quad p \geq 2$$

$$D_{3,3p} = \frac{(2p-1)!}{(p-1)!(p-1)!}.$$

For example, we have $\bar{d}_3 = \bar{d}_4 = \bar{d}_5 = 1$, $\bar{d}_6 = 4$.

Since the equivalence relation defined on B is identity, there is no additional enumeration problem for $\bar{B}$.

The following result for S may be proved by mathematical induction.

<u>Proposition 9.11</u>. $\bar{s}_1 = 1$ and $\bar{s}_k = 2^{k-2}$ for $k = 2,3,4,\ldots$.

It is not difficult to show that if q is a natural number satisfying $k(k+1)/2 \leq q \leq k^2-k+1$ then there exists an element $r = [r_1,\ldots,r_k] \in P_k$ such that $\sum_{i=1}^{k} r_i = q$. From this it follows that

<u>Proposition 9.12</u>. $\bar{p}_k = (1/2)k^2 - (3/2)k + 2$ for $k = 1\ 2,\ldots$.

With each element c of C_k we can associate a tree in the following manner: place a vertex in each of the k+1 regions of c and join two vertices v_1 and v_2 by a line segment if and only if the regions containing v_1 and v_2 have a boundary chord in common. If we consider trees t_1 and t_2 distinct if and only if there does not exist a one-to-one mapping of the vertices of t_1 onto the vertices of t_2 which preserves the degree of each vertex and preserves adjacencies, then it is readily seen that c and c' belong to different elements of $\bar{C}_k$ if and only if they correspond to distinct trees. These trees were first enumerated by Cayley [11], and Otter [29] has shown that the number of trees with n vertices is the coefficient of x^n in $c(x) = r(x) -\frac{1}{2}r^2(x) + \frac{1}{2}r(x^2)$ where $r(x) = \sum r_n x^n$, and the r_i are given by the formula $r(x) = x(1-x)^{-r_1}(1-x^2)^{-r_2}\ldots$.

<u>Proposition 9.13</u>. $\bar{c}_k$ is the coefficient of x^{k+1} in the preceding expression for $c(x)$.

For example, we have (from [31])

i	1	2	3	4	5	6	7	8	9
c_i	1	1	2	5	14	42	132	429	1430
$\bar{c}_i$	1	1	1	2	3	6	11	23	47

There is a one-to-one correspondence between the elements of $\bar{R}_k$ and (unordered) partitions of k. Thus we can state

<u>Proposition 9.14 ([31], page 117)</u>. $\bar{r}_0 = 1$ and $\bar{r}_n = \bar{r}_{n-1} + \bar{r}_{n-2} - \bar{r}_{n-5} - \bar{r}_{n-7} + \dots + (-1)^{k-1}\bar{r}_{n-\frac{(3k^2-k)}{2}} + (-1)^{k-1}\bar{r}_{n-\frac{(3k^2+k)}{2}} + \dots$, using only non-negative subscripts.

For example, we have $\bar{r}_1 = 1$, $\bar{r}_2 = 2$, $\bar{r}_3 = 3$, $\bar{r}_4 = 5$, $\bar{r}_5 = 7$.

<u>Proposition 9.15. ([31], page 127)</u>.

$$\bar{t}_n = \sum \binom{\bar{t}_1+k_1-1}{k_1}\binom{\bar{t}_2+k_2-1}{k_2} \cdots \binom{\bar{t}_{n-1}+k_{n-1}-1}{k_{n-1}} \text{ for } n = 2,3,\dots$$

where the summation is taken over all $k_1,\dots,k_{n-1}$ such that $k_1 + 2k_2 + \dots + (n-1)k_{n-1} = n-1$, and $\bar{t}_1 = 1$.

We have, for example, $\bar{t}_1 = 1$, $\bar{t}_2 = 1$, $\bar{t}_3 = 2$, $\bar{t}_4 = 4$, $\bar{t}_5 = 9$, $\bar{t}_6 = 20$, $\bar{t}_7 = 48$.

<u>Proposition 9.16 ([32],[36])</u>. $\bar{w}_1 = 1$ and for $n > 1$,

$$\bar{w}_n = \sum_{s=1}^{[n/2]} \bar{w}_s\bar{w}_{n-s}$$ with a special modification to be made for any term $\bar{w}_{n/2}\bar{w}_{n/2}$ for even n, viz., $\bar{w}_{n/2}(\bar{w}_{n/2}+1)/2$.

For example, $\bar{w}_1 = 1$, $\bar{w}_2 = 1$, $\bar{w}_3 = 1$, $\bar{w}_4 = 2$, $\bar{w}_5 = 3$, $\bar{w}_6 = 6$,

$\bar{w}_7 = 11$, $\bar{w}_8 = 23$, $\bar{w}_9 = 46$.

In F we observe that sets of code words c and c' are equivalent if and only if $c^r = c'$, where c^r is the reflection of c. This aids us in establishing

Proposition 9.17. For $k \in J_+$, $\bar{f}_k = \frac{1}{2k}\binom{2k-2}{k-1} + x_k$ where $x_k = \frac{1}{2k}\binom{k-2}{k/2-1}$, if k is even and $x_k = 0$, otherwise.

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