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A MULTI-OBJECTIVE OPTIMIZATION FRAMEWORK FOR A SCHEDULE-AWARE TRANSIT SYSTEM WITH DYNAMIC BUS ASSIGNMENT

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A MULTI-OBJECTIVE OPTIMIZATION FRAMEWORK FOR A SCHEDULE-AWARE TRANSIT SYSTEM WITH DYNAMIC BUS ASSIGNMENT

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Abstract

Traditional fixed-route systems provide structure and scalability but rely on static route assignments and homogeneous fleets, limiting their ability to respond to variations in passenger demand. In contrast, demand-responsive transportation (DRT) systems incorporate passenger-specific information and adapt routing decisions, but typically rely on smaller vehicles and fully flexible routing that is challenging to scale to large transit networks. This work proposes an intermediate approach that maintains the structure of fixed-route systems while introducing flexibility through dynamic bus-route assignments that adapt to passenger demand over time. The problem is formulated as a multi-objective mixed-integer linear program for a schedule-aware bus transportation system operating over a finite discrete-time planning horizon. A heterogeneous fleet of buses operates on predefined routes and is dynamically assigned across routes based on evolving passenger demand. The optimization problem minimizes a weighted combination of passenger travel time cost and bus operating cost. Passenger travel time cost is modeled asymmetrically to encourage punctual service.

The proposed system enables several key behaviors. By allowing dynamic assignment of buses to different routes, a limited fleet can effectively serve more routes than available vehicles, improving overall system coverage. This flexibility is complemented by coordinated transfer synchronization, where buses are dispatched in advance to facilitate seamless passenger transfers. At the same time, the model incorporates cost-aware utilization of a heterogeneous fleet, prioritizing lower-cost vehicles while still respecting operational requirements such as capacity. The system also prioritizes passengers with tighter deadlines, and responds systematically to objective function weighting, shifting between service-oriented and cost-efficient solutions. In high-demand scenar-

ios, capacity-aware allocation ensures that larger buses are assigned where they are most effective. Finally, the discrete-time formulation enables direct animation and visualization of system operations, providing clear and interpretable insight into system behavior.

To isolate the impact of dynamic route assignment, the proposed framework is compared against a fixed-route baseline that retains the key behaviors of the proposed system but restricts buses to fixed route assignments. Simulation results show that dynamic bus assignment translates into significant performance gains. Relative to the fixed-route baseline, passenger travel time cost is reduced by 67%, with only a 14% increase in operating cost. Overall, this work shows that meaningful improvements in transit performance can be achieved not by abandoning fixed-route structures, but by introducing targeted flexibility in how buses are assigned within them.

Chapter 1

Introduction

1.1 Background and Motivation

Public transit systems play an important role in urban mobility by providing high-capacity transportation in dense urban environments (Chen et al., 2024). In many cities, public transit operations are organized around fixed routes and fixed headways (Myungseob et al., 2019). Under this paradigm, passengers are required to adapt their travel schedules to the availability of service rather than the service adapting to passenger demand.

While fixed-route, fixed-headway operations simplify planning and dispatch, they often result in long passenger waiting times, unreliable arrival times, and poor alignment between service availability and passenger travel needs (Hernandez et al., 2025). As a result, some passengers choose to walk or rely on alternative, often more expensive modes of transportation, even in cities where public transit fares are low or fully subsidized (Vigil and Miklos, 2025).

Such shifts away from public transit contribute to declining ridership, which in turn places financial strain on transit agencies that rely on fare revenue to offset a substantial portion of operating costs. For example, the Chicago Transit Authority recovers approximately 41% of its operating costs through fare revenue (Zipper, 2023). Declines in passenger demand introduce budgetary pressures, often prompting service reductions aimed at controlling expenses (Zipper, 2023). These service cuts can further degrade service quality and reliability, reinforcing the cycle of ridership loss and financial instability.

Fixed-route, fixed-headway operating strategies become particularly challenging in resource-constrained environments. These strategies often favor a homogeneous fleet in which vehicles of identical capacity are permanently assigned to predefined routes (Walker, 2020). In resource-constrained environments, transit fleets are often limited in number and are heterogeneous, with vehicle capacities ranging from small passenger vehicles to full-size buses (Hall, 2025). In these contexts, variability in passenger demand makes static route assignments impractical, necessitating routing and dispatch decisions that adapt to changing passenger needs.

Demand-responsive transportation (DRT) has demonstrated measurable benefits when implemented on targeted segments of existing public transit networks. A recent study evaluating the implementation of DRT on two routes in the Asprela area of Porto, Portugal, reported a 135% decrease in the number of stops and an 81% decrease in the number of kilometers traveled while serving the same number of passengers (Melo et al., 2024). Additionally, research on flexible on-demand transit services has found that the increased responsiveness in DRT systems can significantly improve service adaptability and reduce passenger waiting times (Liyanage et al., 2024).

While DRT approaches demonstrate promising benefits, existing solutions are typically limited to homogeneous fleets in low-demand settings that provide near curb-to-curb service. This motivates the need for optimization frameworks that incorporate passenger-schedule-awareness from DRT while operating over heterogeneous fleets with predefined routes, without permanently assigning specific vehicles to specific routes.

1.2 Literature Review

1.2.1 Optimization Foundations for Routing and Scheduling

Dantzig and Ramser (1959) introduced one of the earliest mathematical formulations of the vehicle routing problem (VRP) by extending the traveling salesman problem (TSP) to account for multiple vehicles and capacity constraints.

Kwak and Schniederjans (1985) introduced goal programming approaches for transportation problems involving multiple shipping terminals. Their framework allowed decision-makers to explicitly specify desired supply and demand service levels as well as priority structures to resolve trade-offs when constraints prevent full satisfaction of all objectives.

Kara and Bektas (2003) proposed an integer linear programming (ILP) formulation for the generalized vehicle routing problem (GVRP), demonstrating how routing decisions can be modeled using binary variables while enforcing vehicle capacity and load-factor requirements.

These formulations illustrate the strength of linear programming (LP) in providing rigorous and exact representations of vehicle routing problems. However, as problem size grows, computational complexity increases rapidly. Existing literature has shown that this scalability challenge can be addressed through structured modeling and decomposition techniques. In particular, Fisher (1981) introduced the Lagrangian relaxation framework, showing that many LP formulations become difficult to solve due to a small number of constraints that create strong dependencies among decision variables. By relaxing these constraints and embedding them into the objective function through Lagrange multipliers, large problems could be decomposed into smaller, more manageable subproblems.

More recently, Li et al. (2025) proposed a data-driven constraint reduction approach that focuses on identifying inequality constraints that are active at the optimal solution. The authors showed that many mixed-integer linear programming (MILP) problems contain a large number of inactive constraints that do not influence the optimal solution. By learning to predict which inequality constraints are likely to be active at optimality, the proposed method reduces the effective constraint set, leading to smaller problem instances and improved computational performance.

Beyond constraint reduction and Lagrangian relaxation, linearization techniques play a key role in expanding the practical applicability of LP models. Asghari et al. (2022) provided a comprehensive survey of transformation and linearization methods for handling nonlinear terms commonly encountered in routing and scheduling problems. These methods address expressions such as products of binary and continuous variables, absolute value terms, and min-max operators. Through appropriate transformations, nonlinear cost structures and logical relationships can be embedded within a linear optimization framework.

Building on these optimization foundations, a substantial body of transportation research has applied MILP formulations to the planning and operation of fixed-route public transit systems. In such models, routes are typically assumed to be fixed, and the primary operational decisions involve timetable design, headway control, and fleet allocation, with the objective of reducing operating costs and improving passenger service levels.

1.2.2 Fixed-Route Bus Operations and Timetable Optimization

Ceder et al. (2001) and Chu et al. (2019) formulated fixed-route, fixed-headway MILP problems aimed at reducing passenger transfer waiting times as a major component of total passenger travel time. Their approaches coordinate departure times across multiple routes to promote simultaneous bus arrivals at transfer nodes, while adhering to operator-specified headway constraints.

Other studies have sought to balance passenger travel time with operator costs. Shang et al. (2019) developed a multi-depot vehicle scheduling model that incorporates passenger waiting costs alongside bus operating costs using a weighted-sum objective formulation. Their approach determines the minimum number of vehicles required to serve a given timetable, followed by timetable adjustments designed to reduce fleet size while constraining increases in passenger waiting cost.

Similarly, Liu et al. (2023) introduced a multi-objective fixed-route bus scheduling model that minimizes both operating costs and passenger travel times. Their work incorporates passenger perception by treating passenger discomfort from waiting as a cost. In particular, passenger waiting cost is modeled as a non-linear function of headway, such that longer intervals between consecutive buses incur disproportionately higher waiting penalties, reflecting increased passenger dissatisfaction as service becomes less frequent.

Tang et al. (2020) developed a mixed-integer optimization model for fixed bidirectional bus lines that explicitly incorporates passenger waiting time as a monetized cost alongside operator objectives such as fleet minimization. Time-dependent demand and variable running times are considered, allowing the model to account for the stochastic nature of traffic while balancing service quality and operating efficiency.

Other work has explored real-time, control-based strategies to improve service regularity in fixed-route systems. Lima et al. (2021) proposed a headway control strategy that incorporates bus load information to reduce passenger time spent onboard buses. In fixed-route operations, buses are typically scheduled to arrive at evenly spaced intervals. Delays to a leading bus can cause the following bus to catch up, resulting in bus bunching. Traditional headway control methods address this effect by holding buses at stops to restore regular spacing. In contrast, Lima et al. (2021) introduced a load-sensitive control policy in which passenger load is explicitly considered in holding decisions, such that heavily loaded buses are held less.

Jariyasunant et al. (2011) developed a mobile transit trip planning system that leverages real-time vehicle data to predict bus arrival times. Their work aimed to improve passenger experience by supporting more informed trip planning under uncertain service conditions.

Collectively, these approaches commonly assume homogeneous fleets, static route assignments and passenger demand represented in aggregate through origin-destination pairs, thereby limiting their ability to capture individualized demand patterns and dynamic fleet behavior.

1.2.3 Demand-Responsive and Flex-Route Transport

DRT systems differ from traditional fixed-route public transit by adapting routing and scheduling decisions in response to passenger requests, offering virtual stops and flexible routes in place of fixed timetables (Deka et al., 2023). For example, Qiu et al. (2014) studied a system in which vehicles follow a predefined base route but are permitted to deviate within a bounded distance to serve curb-to-curb requests. Passenger origins and destinations are used to generate virtual stops, subject to a maximum allowable walking distance for each passenger.

Similarly, Tong et al. (2017) proposed a DRT model for passenger-vehicle assignment and routing in a customized bus service. In their formulation, passengers submit pickup and drop-off requests that include preferred locations and time constraints. Requests with similar characteristics are clustered together, and new routes are generated to serve those groups. Service feasibility is limited by predefined maximum walking distances and passengers may be denied service if operational thresholds are not satisfied. In addition, vehicles are dispatched only if minimum load factor requirements are met.

Cao and Wang (2017) and Wang et al. (2023) develop optimization frameworks for customized buses with modular capacity, with objectives that balance total operating cost and passenger travel time. The use of modular vehicles allows system capacity to adapt to time-varying passenger demand. In contrast to studies that assume passengers always follow operator-assigned routes and transfers, both works incorporate transfer-related penalties and incentives, with Wang et al. (2023) explicitly modeling passenger willingness to transfer through a logit-based behavioral framework.

Bögl et al. (2015) addressed a flex-route school bus routing and scheduling problem focusing on transporting students from home to school while allowing students attending different schools to share buses. They decomposed the problem into bus stop selection, student-bus assignment, and route generation. The model constrains the number of transfers, walking distances and student arrival times, with the objective of minimizing total operating cost.

Beyond static formulations, several studies consider dynamic and stochastic demand requests. Schilde et al. (2011) studied a dynamic stochastic dial-a-ride problem motivated by patient transportation operated by the Austrian Red Cross. Patient travel requests arrive both in advance and in real-time, vehicles have limited capacity, request rejection is not allowed, and strict service constraints are imposed on detours and arrival time windows. Similarly, Cortés et al. (2009) formulated a hybrid adaptive predictive control approach for a dynamic pickup and delivery problem in which a fraction of passenger requests is unknown in advance and routing decisions are made in real-time. Their formulation presents system evolution using state-space variables and incorporates stochastic demand forecasts derived from historical data to anticipate future requests.

In multi-modal logistics literature, Alshabibi et al. (2025) considered a multi-objective MILP framework that minimizes total transportation cost, total delivery time, and greenhouse gas emissions. Their model allows mode switching between road and rail transport, and incorporates accident risk into routing decisions. SteadieSeifi et al. (2021) proposed a rolling-horizon model that repeatedly re-optimizes routing decisions as new information becomes available. While formulated in a multi-modal logistics context, the framework demonstrates how time-indexed planning and forecast-based demand information can be embedded within offline optimization models.

1.2.4 Alternative Modeling and Solution Approaches

Beyond fixed-route and DRT formulations, other literature has explored routing, scheduling, and assignment problems using learning-based methods. Bogyrbayeva et al. (2024) provided a comprehensive review of machine learning approaches for vehicle routing problems. They identified supervised learning, reinforcement learning and hybrid methods that integrate machine learning within heuristic solution frameworks.

Novoa and Storer (2009) modeled a vehicle routing problem with stochastic customer demands as a Markov decision process, where routing decisions for a single vehicle are made sequentially as demand is revealed. Their approach minimizes expected routing cost without predefining a complete route, with decisions at each stage determining whether to visit a customer or return to the depot.

Basso et al. (2022) formulated an electric vehicle routing problem with stochastic passenger demand using reinforcement learning. The objective was to minimize expected energy consumption while ensuring route feasibility by scheduling charging when necessary. Their approach learns the relationship between the battery state, routing decisions, and charging actions in order to decide whether a charging station should be visited to avoid battery depletion.

Zhao et al. (2020) addressed traffic prediction for scheduling applications by modeling road networks as connected graphs and learning how traffic conditions propagate through the network over time. Roads are represented as nodes connected according to their physical layout, and traffic speeds observed on each road are used to learn both how congestion spreads spatially and how it evolves over time.

These studies demonstrate the potential of learning-based and predictive methods for routing and scheduling under uncertainty. Unfortunately, many existing approaches focus on single, low-capacity vehicles, rely on extensive training, and differ from the operational structure of fixed-route public transit.

1.3 Problem Statement

This thesis considers the operation of a hypothetical public transit system over a finite discrete time planning horizon. The system consists of a single central depot and a heterogeneous fleet of buses that differ in passenger capacity. The fleet size is fixed, and all buses are assumed to travel at identical speeds without accounting for traffic effects. Possible bus routes are predefined and unidirectional, with all routes originating and ending at the central depot. Once a bus is assigned to a route and departs the depot, it is required to complete the entire route without deviation or early termination. All buses start at the central depot at the beginning of the planning horizon and must return there by its end.

Passengers are modeled as individual agents who are allowed to transfer between buses. Passenger demand information is assumed to be known at the beginning of the planning horizon. It includes each passenger's departure bus stand and desired departure time, as well as arrival bus stand and desired arrival time. Desired arrival times are required to be feasible, meaning that at least one feasible path through the transit network allows the passenger to travel from their origin to their destination within their desired time window. The system is designed to serve all passengers; virtual vehicles or demand-relaxation mechanisms commonly used in the literature to allow passenger rejection are not considered. Instead, unserved passengers incur a large penalty in the objective function. The amount of time it takes for a passenger to board or disembark a bus is neglected.

The central problem is to determine how a heterogeneous fleet of buses should be deployed across predefined routes, including when each bus should depart and which passengers it should serve. The goal is to jointly optimize operating efficiency and passenger service quality, while observing bus capacity limits and routing constraints.

1.4 Thesis Overview

To address this problem, a multi-objective mixed integer linear programming formulation is developed using a discrete time representation. The model focuses on operational-level scheduling and assignment decisions, rather than strategic route design and stochastic traffic modeling.

At each discrete time step, decision variables determine the route assignment and spatial progression of every bus, as well as the state of every passenger, either waiting at a bus stand or onboard a bus. This discrete temporal structure enables precise modeling of boarding, riding, transferring, and disembarking events while preserving logical consistency across time. The formulation is intentionally structured to support visualization and animation of system operations, enabling validation of physical feasibility and clear communication of results to planners.

All decision variables, except for slack variables, are binary and represent bus-to-route assignments, bus departure times, and passenger-to-bus assignments. Capacity limits are enforced at every time step and spatial movement is represented through discrete locations. The distance between adjacent discrete locations corresponds to the distance a bus travels during a single time step.

The objective function to be minimized is a weighted sum of passenger travel time cost and bus operating cost. Passenger travel time cost is defined as the deviation between each passenger's desired and actual arrival time. It is modeled using a piecewise-linear asymmetric structure that penalizes late arrivals and slightly rewards early arrivals. The nonlinear piecewise representation is reformulated using standard linearization techniques to preserve the MILP structure. Bus operating cost is proportional to the number of discrete time steps during which a bus is actively assigned to a route. Idle buses incur no operating cost.

The resulting MILP is solved using MATLAB's integer linear programming solver (The MathWorks, Inc., 2025). This solver employs a branch-and-bound algorithm with cutting-planes and linear programming relaxations to obtain near-optimal solutions subject to a specified optimality tolerance (The MathWorks, Inc., 2025). All optimization experiments are implemented in MATLAB (R2025b) and executed on a machine with 16 GB of RAM and a 2.8 GHz Intel Core i7 processor.

1.5 Thesis Contributions

This thesis develops a mathematical model for operating a fixed-route transit system in a flexible and demand-aware manner. Although the model is inspired by DRT systems, it is fundamentally different. In DRT systems such as those proposed by Qiu et al. (2014) and Tong et al. (2017), vehicles are allowed to freely deviate from routes in response to passenger requests. In this work, buses are not permitted to terminate or deviate from routes. However, they can be assigned to different predefined routes at different times based on changing demand. This preserves the structure of a traditional bus network while adding flexibility in how the fleet is used.

Unlike classical fixed-route systems such as those studied by Ceder et al. (2001) and Liu et al. (2023), where homogeneous buses are committed to a specific route through timetables or headway policies, buses in this framework are not permanently tied to specific routes or departure intervals. Instead, the model determines which bus operates on which route and at what time. Since the model operates on a heterogeneous fleet, bus-route assignments become a critical decision for managing capacity constraints and operating costs. When demand is low, buses may remain idle and incur no operating cost, whereas increased demand on a route can be addressed by assigning available buses to provide additional service.

Another key difference between this work and both the fixed-route and DRT literature is that passenger demand is modeled at the individual level. Unlike Chu et al. (2019) and Tong et al. (2017), in this work, passengers are not aggregated into origin-destination clusters. Instead, the model determines which specific bus or buses serve each passenger. These decisions are made by minimizing an objective function that incorporates each passenger's desired departure and arrival times. The asymmetric passenger travel time cost in the objective function, slightly rewards early arrivals and penalizes late arrivals, ensuring that passengers with tighter time constraints are prioritized.

Another important contribution of this thesis is how system behavior is modeled over time. The decision variables allow tracking of both buses and passengers at each discrete time step. The model keeps track of where each bus is, whether it is assigned to a route or idle, and whether each passenger is waiting at a bus stand or onboard a bus. Boarding, riding, transferring and disembarking are all explicitly modeled. This produces a realistic and consistent system evolution that stands in stark contrast to those proposed by Ceder et al. (2001), Chu et al. (2019), and Liu et al. (2023) which rely on a timetable or summary statistics. Because all decisions are indexed over time, the optimized solution can be precisely simulated and visualized. This makes it easier to validate feasibility and communicate operational plans to transit planners.

As will be detailed in subsequent chapters, the objective function balances passenger service quality and operating cost. It does so by representing passenger travel time cost and bus operating cost as a weighted sum to be jointly minimized. By adjusting the weighting between passenger travel time cost and operating cost, transit agencies can explore different trade-offs. Placing more emphasis on passenger travel time improves punctuality but may require more buses in service. Placing more emphasis on cost reduces fleet usage but may increase waiting time.

Overall, this thesis presents a flexible fixed-route transit framework that combines dynamic bus-route assignments, passenger-level scheduling, realistic time-based system modeling, and cost-service tradeoff analysis within a single optimization model.

1.6 Thesis Organization

The thesis is organized as follows. Chapter 2 presents the formulation of the proposed bus transportation model, including the constraints and objective function. Chapter 3 validates the constraint formulation through illustrative examples and case-based analysis. Chapter 4 presents simulation results and evaluates system performance. Chapter 5 concludes the thesis and discusses limitations and future work. References are provided at the end of the document.

Chapter 2

Bus Transportation System Model

In this chapter, the sets, indices, and parameters used throughout the thesis are formally defined, the decision variables are introduced, and the constraints governing passenger and bus movement within the discrete-time framework are presented. The objective function is subsequently defined and the complete problem formulation is provided at the end of the chapter.

2.1 Sets, Indices, and Parameters

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_{N_S}\}$ denote the set of N_S bus stands. For each $s \in \{1, 2, \ldots, N_S\}$, let S_s represent bus stand s, with S_1 denoting the only central bus depot.

Instead of explicitly defining pairwise distances between two bus stands, discrete bus locations are introduced to model spatial progression along routes. Let $\mathcal{L} = \{L_1, L_2, \ldots, L_{N_L}\}$ denote the set of N_L locations where $N_L \geq N_S$. For each $\ell \in \{1, 2, \ldots, N_L\}$, let L_ℓ represent discrete spatial position ℓ. The distance between two consecutive locations along a route corresponds to the distance a bus can travel within one discrete time step.

The indexing of locations is defined so that all bus stand locations appear first. Specifically, if there are N_S bus stands, then locations $L_1, L_2, \ldots, L_{N_S}$ correspond to bus stands $S_1, S_2, \ldots, S_{N_S}$, respectively. Any remaining locations that do not correspond to bus stands are assigned indices $L_{N_S+1}, L_{N_S+2} \ldots, L_{N_L}$ Accordingly, a mapping ϕ :

$\mathcal{S} \to \mathcal{L}$ is defined such that for each $s \in \{1, 2, \ldots, N_S\}$, $\phi(S_s) = L_{\ell_s}$, which means that location ℓ_s corresponds to bus stand s. The mapping is injective, ensuring that each bus stand is associated with a unique location, while some locations may not correspond to any bus stand.

Let $\mathcal{R} = \{R_0, R_1, R_2, \ldots, R_{N_R}\}$ denote the set of $N_R + 1$ bus routes, where R_0 is a special bus route to be defined shortly. For each $r \in \{0, 1, 2, \ldots, N_R\}$, let R_r be defined as an ordered sequence of locations

$$R_r = (R_r(1), R_r(2), \ldots, R_r(E_r)),$$

where E_r denotes the number of locations visited along route r and $R_r(k)$ denotes the location at the k^{th} position of route r. The locations $R_r(1), R_r(2), \ldots, R_r(E_r - 1)$ are all distinct, and the route is constrained to start and end at the central bus depot by enforcing

$$R_r(1) = R_r(E_r) = S_1.$$

The index $r = 0$ is used to represent an unassigned route or idle state. Therefore, if a bus is assigned to route 0, it means that the bus is not currently assigned to a route.

Let $\mathcal{B} = \{B_1, B_2, \ldots, B_{N_B}\}$ denote the set of N_B buses. For each $b \in \{1, 2, \ldots, N_B\}$, let B_b represent bus b which has capacity C_b. Each bus b is assumed to travel at a constant speed and incur an operating cost per unit time denoted as c_b.

Time is modeled in discrete steps indexed by $t \in \{0, 1, \ldots, N\}$, where N denotes the planning horizon. The index t acts as a counter and does not necessarily correspond to a physical time unit. Instead, let $\Delta t > 0$ denote the duration of one discrete time step in physical units (e.g., seconds or minutes). The physical time corresponding to time step t is therefore $t\Delta t$.

Let $\mathcal{P} = \{P_1, P_2, \ldots, P_{N_P}\}$ denote the set of N_P passengers. For each $i \in \{1, 2, \ldots, N_P\}$, let P_i represent passenger i. Each passenger i is associated with an origin-destination pair (S_{X_i}, S_{Y_i}), where $X_i, Y_i \in \{1, 2, \ldots, N_S\}$ and $X_i \neq Y_i$. This indicates that passenger i wishes to travel from bus stand S_{X_i} to bus stand S_{Y_i}.

Let $I_i \in \{0, 1, \ldots, N\}$ represent the time at which passenger i starts waiting at the origin bus stand S_{X_i}. In addition, let $J_i \in \{0, 1, \ldots, N\}$ represent the desired arrival time of passenger i at the destination bus stand S_{Y_i}, where $J_i > I_i$. The desired arrival time J_i must be feasible, meaning that there exists at least one sequence of bus movements connecting S_{X_i} to S_{Y_i} that can be completed in $(J_i - I_i)$ time steps. This sequence may involve transfers between different routes and buses.

Let $K_i \in \{1, 2, \ldots, N\}$, with $K_i > I_i$, denote the actual arrival time of passenger i at the destination bus stand S_{Y_i}. Therefore, $(K_i - J_i) > 0$ indicates arriving late whereas $(K_i - J_i) < 0$ indicates arriving early.

2.2 Decision Variables

All decision variables in the model are binary. The formulation includes three classes of decision variables, namely bus-route assignment variables, bus-location variables and passenger assignment variables. These variables are defined over the relevant index sets of buses, routes, locations, passengers, and time steps. Together, they describe the complete operational state of the system at each time step and enable the model to determine how buses and passengers evolve over time.

2.2.1 Bus-Route Assignment

The bus-route assignment variables determine whether a bus is active or idle and, if active, which route it is assigned to at each time step. The decision variable w_{brt} takes the value 1 if bus b is assigned to route r at time step t, and 0 otherwise. These variables allow the model to determine the allocation of the fleet over time, including when buses are deployed, assigned, or remain idle.

2.2.2 Bus-Location Assignment

The bus-location variables describe the position of each bus at each time step. The decision variable x_{blt} takes the value 1 if bus b is at location ℓ at time step t, and 0 otherwise. The evolution of bus location variables across successive time steps is governed by the assigned route and is enforced through location consistency and movement constraints introduced later in Section 2.4.2.

2.2.3 Passenger-Bus and Passenger-Location Assignment

Passenger assignment variables determine how passengers move through the system. The decision variable y_{ibt} takes the value 1 if passenger i is on bus b at time step t, and 0 otherwise. The decision variable z_{ist} is 1 if passenger i is at bus stand s at time step t, and 0 otherwise. Together, these variables allow the model to decide passenger-level actions, including waiting, boarding, riding, and disembarking.

2.2.4 Temporal Representation

To remove ambiguity in the ordering of events within a single discrete time step t, the notational conventions t^- and t^+ are introduced. These represent infinitesimal time instants immediately before and immediately after t, respectively. Each decision variable is defined at both t^- and t^+.

At t^-, buses and passengers are interpreted as having arrived at their respective locations. At t^+, buses are assigned to routes, while passengers may board, disembark buses, or remain at their current locations. Between t^+ and $(t+1)^-$, buses move along their assigned routes to the next location. Passengers only move during this interval if they are onboard a bus, otherwise they remain at their current location. These representations enforce a consistent sequence of events within each time step, and their relationships are formalized through the constraints introduced in subsequent sections.

2.3 Initial and Final Conditions

This section defines the initial and final conditions imposed on decision variables. These conditions ensure that all buses and passengers begin and end the planning horizon in a consistent and physically meaningful state.

At 0^-, which represents the beginning of the planning horizon, all buses are at the central depot. This is enforced by

$$x_{b10^-} = 1 \quad \forall b, \tag{2.1}$$

which places each bus at the central depot at the start of the planning horizon. In addition, all buses are assigned to the idle state ($r = 0$) at 0^-. This is enforced by

$$w_{b00^-} = 1 \quad \forall b, \tag{2.2}$$

which ensures that all buses are in the idle state at the beginning of the planning horizon.

At the end of the planning horizon, all buses are required to be at the central depot. This condition is imposed at N^+ by

$$x_{b1N^+} = 1 \quad \forall b, \tag{2.3}$$

which places each bus at the central depot at the end of the planning horizon. Furthermore, all buses are required to be in the idle state at N^+. This is enforced by

$$w_{b0N^+} = 1 \quad \forall b, \tag{2.4}$$

which ensures that no bus remains assigned to a route after the planning horizon ends.

Passenger entry into the system is also governed by initial conditions. The full set of passenger demand information, (X_i, Y_i, I_i, J_i), is assumed to be known for all passengers at the beginning of the planning horizon. However, passenger i can only be served from time step I_i onwards. For all time steps prior to I_i, passenger i is required to remain at their origin bus stand X_i. This initial condition is enforced by

$$z_{iX_it^-} = 1 \quad \forall i, \forall t \in \{0, 1, \ldots, I_i\}. \tag{2.5}$$

and

$$z_{iX_it^+} - z_{iX_it^-} = 0 \quad \forall i, \forall t \in \{0, 1, \ldots, I_i - 1\}. \tag{2.6}$$

Together, these constraints prevent passenger i from leaving bus stand X_i prior to their arrival at the bus stand.

Once passenger i reaches their destination bus stand Y_i, they are required to remain there for the remainder of the planning horizon. This ensures that passenger i cannot be on any bus after reaching their destination. This condition is enforced by

$$z_{iY_it^-} - z_{iY_it^+} \leq 0 \quad \forall i, \forall t. \tag{2.7}$$

These initial and final conditions ensure that all bus decision variables are well-defined at the boundaries of the planning horizon. They also ensure that passengers are only served after arriving at their origin bus stand and are no longer served once they have reached their destination bus stand.

2.4 Constraints

This section presents the set of constraints that define the feasible operation of the proposed system. These constraints govern bus-route assignments, bus-location assignments, and passenger-bus and passenger-location assignments over the planning horizon. Together, they define the admissible evolution of buses and passengers and enforce physically consistent system behavior. A detailed validation of the constraint set and its induced system behavior is provided in Chapter 3.

2.4.1 Bus-Route Assignment Constraints

This subsection introduces the constraints that govern when bus-route assignments may occur and how they evolve over time. In particular, the constraints enforce that each bus is assigned to exactly one route at any given time step, and that changes in route assignments only occur when a bus is at the central depot. They also ensure that once a bus is assigned to a route, it must complete that route before it can be assigned to another route.

Following completion of a route, represented by the bus being at the central depot at some t^-, the bus is assigned to the idle state at t^+ and remains assigned to the idle state at $(t+1)^-$. This enforces a dwell time requirement that ensures that route completion and subsequent route assignment decisions occur in distinct time steps.

Figure 2.1 provides a graphical illustration of permitted bus-route assignments between t^- and t^+. If a bus b is assigned to the idle state, and at the central depot at t^-, then at t^+, it may either remain idle or be assigned to any route $r \in \{1, 2, \ldots, N_R\}$, as

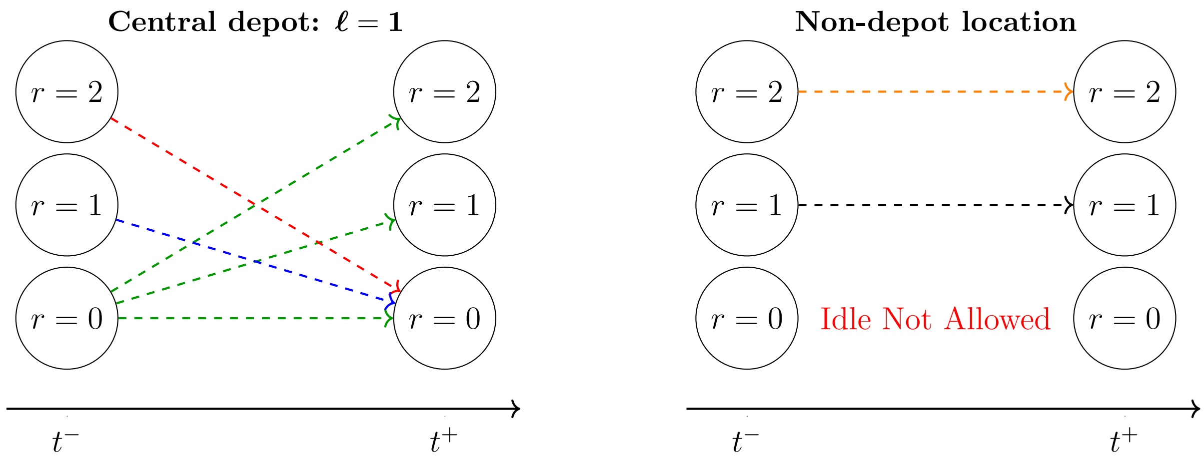


Figure 2.1: Bus-route assignments between t^- and t^+ for a system with $N_R = 2$. When the bus is at the central depot (left), route assignments may be updated. When the bus is at any other location (right), the route assignment remains fixed, and assignment to the idle state is prohibited.

indicated by the green dashed arrows. If a bus b assigned to a route $r \in \{1, 2, \ldots, N_R\}$ arrives at the central depot at t^-, it is assigned to the idle state at t^+, as indicated by the blue and red dashed arrows. When a bus b is not at the central depot and is assigned to some route $r \in \{1, 2, \ldots, N_R\}$ at t^-, it remains assigned to the same route r at t^+, as indicated by the orange and black dashed arrows. In this case assignment to the idle state is not permitted.

The route assignment rules described above are formalized through the following set of constraints on w_{brt}. For any discrete time step t, the route-assignment variables for each bus b satisfy

$$\sum_{r=0}^{N_R} w_{brt^-} = 1 \quad \forall b, \forall t, \tag{2.8}$$

at t^- and

$$\sum_{r=0}^{N_R} w_{brt^+} = 1 \quad \forall b, \forall t, \tag{2.9}$$

at t^+, where the summation is taken over all routes $r \in \{0, 1, \ldots, N_R\}$. These constraints enforce that each bus is assigned to exactly one route at both t^- and t^+.

To ensure bus route assignments can only change at the central depot,

$$w_{brt^+} \leq w_{brt^-} + x_{b1t^-} \quad \forall b, \forall r \neq 0, \forall t, \tag{2.10}$$

and

$$w_{brt^+} \geq w_{brt^-} - x_{b1t^-} \quad \forall b, \forall r \neq 0, \forall t \tag{2.11}$$

are imposed. When a bus is not at the central depot, i.e. $x_{b1t^-} = 0$, constraints (2.10) and (2.11) enforce equality between w_{brt^-} and w_{brt^+}. As a result, when a bus is not at the central depot, its assignment remains fixed until the route is completed.

Once a bus completes a route at the central depot, the following constraint

$$w_{brt^+} \leq 1 + w_{b0t^-} - x_{b1t^-} \quad \forall b, \forall r \neq 0, \forall t. \tag{2.12}$$

is imposed. When a bus is at the central depot and is not idle at t^-, i.e. $x_{b1t^-} = 1$ and $w_{b0t^-} = 0$, this constraint enforces

$$w_{brt^+} = 0 \quad \forall r \neq 0,$$

so no route $r \in \{1, 2, \ldots, N_R\}$ can be assigned at t^+. Together with (2.9), this assigns bus b to the idle state at t^+.

The temporal dependence between successive assignment decisions is captured by

$$w_{brt^-} - w_{br(t-1)^+} = 0 \quad \forall b, \forall r, \forall t \geq 1 \tag{2.13}$$

which makes the route assignment of each bus at t^- dependent on the assignment at $(t-1)^+$. This enforces consistency of route assignments across consecutive time steps by ensuring that the assignment at $(t-1)^+$ is carried forward to t^-. As a result, route assignment changes can only occur between t^- and t^+ within each time step.

2.4.2 Bus-Location Assignment Constraints

This subsection introduces the constraints that govern the assignment of buses to locations and the evolution of bus-location decision variables over time. The objective of these constraints is threefold. First, they ensure that each bus occupies exactly one location at each time step. Second, they enforce consistency in the bus's location between t^- and t^+. Finally, they ensure that bus movement through locations respects the ordered structure of the bus's assigned route.

At each discrete time step t, each bus b is assigned to exactly one location. This is enforced by

$$\sum_{\ell=1}^{N_L} x_{blt^-} = 1, \quad \forall b, \forall t \tag{2.14}$$

at t^-, and

$$\sum_{\ell=1}^{N_L} x_{blt^+} = 1, \quad \forall b, \forall t \tag{2.15}$$

at t^+, ensuring that each bus b is assigned to exactly one location at all time steps.

Since buses travel between t^+ and $(t+1)^-$, there should be no change in bus-location assignment between t^- and t^+. This requirement is enforced by

$$x_{blt^-} - x_{blt^+} = 0, \quad \forall b, \forall l, \forall t \tag{2.16}$$

which ensures that the location of bus b at t^- is identical to its location at t^+. This prevents artificial spatial transitions within a single time step.

The progression of each bus through locations is governed by the ordered structure of its assigned route. As defined in Section 2.1, each route r is defined as an ordered sequence of locations,

$$R_r = (R_r(1), R_r(2), ..., R_r(E_r))$$

where $k \in \{1, 2, \ldots, E_r\}$, denotes the position index along route r, and $R_r(k)$ denotes the location visited at the k-th position of that route.

If a bus b is assigned to route r at $(t-1)^+$, then its location at t^- must correspond to the next location along that route. This behavior is enforced through

$$x_{b,R_r(k),t^-} - x_{b,R_r(k-1),(t-1)^+} + w_{br(t-1)^+} \leq 1 \quad \forall b, \forall t, \forall r \neq 0, \forall k \geq 1 \tag{2.17}$$

and

$$-x_{b,R_r(k),t^-} + x_{b,R_r(k-1),(t-1)^+} + w_{br(t-1)^+} \leq 1 \quad \forall b, \forall t, \forall r \neq 0, \forall k \geq 1. \tag{2.18}$$

In this notation, $x_{b,R_r(k),t^-}$ follows the same indexing structure as x_{blt^-}, with the location index explicitly expressed as $R_r(k)$ to emphasize its dependence on the position index along route r. When $r = 0$, bus location assignment is enforced through

$$x_{b1(t+1)^-} - x_{b1t^+} + w_{b0t^+} \leq 1 \quad \forall b, \forall t \tag{2.19}$$

and

$$-x_{b1(t+1)^-} + x_{b1t^+} + w_{b0t^+} \leq 1 \quad \forall b, \forall t \tag{2.20}$$

These constraints are a special case of constraints (2.17) and (2.18) and they ensure that when bus b is assigned to the idle state at t^+, it remains at the central depot at $(t+1)^-$. As a result, the bus does not move between t^+ and $(t+1)^-$ when it is assigned to the idle state at t^+.

2.4.3 Passenger-Bus and Passenger-Location Assignment Constraints

This subsection introduces the constraints that govern passenger-bus and passenger-location assignments over the planning horizon. These constraints ensure that at each time step, each passenger is assigned either to a bus stand or to a bus. Passenger transitions between bus stands and buses are restricted to physically admissible events. Boarding may occur only when a passenger and bus are at the same bus stand and passengers may only board or disembark buses at bus stands. In addition, bus capacity constraints are enforced at each time step to ensure that passenger assignments remain feasible.

2.4.3.1 General Passenger Assignment Constraints

This subsection defines the constraints that determine whether each passenger is assigned to a bus stand or a bus and ensure that these assignments evolve consistently over time.

At each t^- and t^+, each passenger i is either at a bus stand s or on a bus b. This is enforced by

$$\sum_{b=1}^{N_B} y_{ibt^-} + \sum_{s=1}^{N_S} z_{ist^-} = 1 \quad \forall i, \forall t \tag{2.21}$$

and

$$\sum_{b=1}^{N_B} y_{ibt^+} + \sum_{s=1}^{N_S} z_{ist^+} = 1 \quad \forall i, \forall t. \tag{2.22}$$

These constraints ensure that each passenger cannot be simultaneously located on a bus and at a bus stand.

If passenger i at bus stand s at $(t-1)^+$, then the passenger must remain at that same bus stand at t^-. This is enforced by

$$z_{ist^-} - z_{is(t-1)^+} = 0 \quad \forall i, \forall s, \forall t, \tag{2.23}$$

which prevents changes in passenger location in the absence of a boarding event. Similarly, if passenger i is on bus b at $(t-1)^+$, then the passenger must remain on that same bus at t^-. This is enforced by

$$y_{ibt^-} - y_{ib(t-1)^+} = 0 \quad \forall i, \forall b, \forall t. \tag{2.24}$$

Together, these constraints ensure that passenger states are carried forward from $(t-1)^+$ to t^-, so that any change in passenger location or bus assignment occurs through bus movement, boarding, or disembarking events.

Bus capacity constraints are enforced at each discrete time step to ensure that passenger assignments remain feasible. For each bus b, the number of passengers on it cannot exceed its capacity C_b. This is enforced at t^- by

$$\sum_{i=1}^{N_P} y_{ibt^-} \leq C_b, \quad \forall b, \forall t, \tag{2.25}$$

and at t^+ by

$$\sum_{i=1}^{N_P} y_{ibt^+} \leq C_b, \quad \forall b, \forall t. \tag{2.26}$$

These constraints ensure that bus occupancy remains within capacity limits at all time steps.

2.4.3.2 Passenger-Bus Assignment Constraints

Passenger-bus assignment constraints are governed by three conditions. A passenger may board a bus if and only if both the passenger and the bus being boarded are at the same bus stand. A passenger already onboard a bus may disembark if and only if that bus is at a bus stand. Passengers are not permitted to transfer directly between buses within the same time step. Instead, a passenger must first disembark at a bus stand, remain there for at least one time step, and only then become eligible to board another bus at that bus stand.

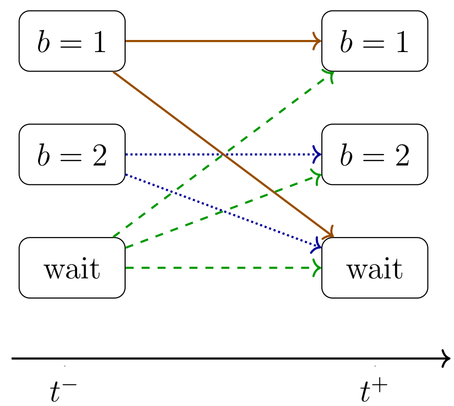


Figure 2.2: Illustration of passenger state transitions between t^- and t^+ for buses 1 and 2 at some bus stand s.

Figure 2.2 provides a graphical illustration of permitted passenger-bus assignments between t^- and t^+. Buses 1 and 2 are at some bus stand s at time step t. Horizontal arrows represent passengers remaining on the same bus or waiting at the same bus stand between t^- and t^+. Non-horizontal arrows from a bus at t^- to the waiting state at t^+ represent passengers disembarking, while non-horizontal transitions from the waiting state at t^- to a bus at t^+ represent passengers boarding. Direct transfers between buses within the same time step are *not* permitted. As shown in Figure 2.2, there are no arrows connecting buses 1 and 2, and a passenger who disembarks from a bus between t^- and t^+ enters the waiting state at t^+.

Constraints

$$y_{ibt^+} \leq y_{ibt^-} + 1 + \sum_{s=1}^{N_S} \frac{s}{N_S} x_{b\ell_s t^-} - \sum_{s=1}^{N_S} \frac{s}{N_S} z_{ist^-} \quad \forall b, \forall i, \forall t \tag{2.27}$$

and

$$y_{ibt^+} \leq y_{ibt^-} + 1 - \sum_{s=1}^{N_S} \frac{s}{N_S} x_{b\ell_s t^-} + \sum_{s=1}^{N_S} \frac{s}{N_S} z_{ist^-} \quad \forall b, \forall i, \forall t \tag{2.28}$$

ensure that passenger i, at bus stand s, may board bus b at t^+ only if the passenger was not on a bus at t^-, i.e. $y_{ibt^-} = 0$, and the bus being boarded was at bus stand s at t^-, i.e. $x_{b\ell_s t^-} = 1$.

Constraints

$$y_{ibt^+} \leq y_{ibt^-} + \sum_{s=1}^{N_s} x_{b\ell_s t^-} \quad \forall b, \forall i, \forall t \tag{2.29}$$

and

$$y_{ibt^+} \geq y_{ibt^-} - \sum_{s=1}^{N_s} x_{b\ell_s t^-} \leq 0 \quad \forall b, \forall i, \forall t \tag{2.30}$$

ensure that passenger i on bus b at t^- can disembark at t^+ if bus b is at a bus stand. If the bus is not at a bus stand, the passenger is forced to remain on the bus.

2.4.3.3 Passenger-Location Assignment Constraints

Passenger-location constraints ensure that all passenger location assignments remain physically admissible and that transitions between locations occur exclusively through bus-mediated events. If a passenger is at a bus stand, they can only leave that bus stand if a bus is present at that same bus stand. In the absence of a bus, the passenger must remain at that bus stand. Similarly, if a passenger is onboard a bus, they can only disembark at a bus stand s if the bus is at the same bus stand.

The first constraint enforces location persistence in the absence of a bus. If a passenger i is at bus stand s at t^- and no bus is present at that stand, then the passenger must remain at that bus stand at t^+. This is enforced by

$$z_{ist^+} \geq -\sum_{b=1}^{N_B} x_{bl\ell_s t^-} + z_{ist^-} \quad \forall s, \forall i, \forall t. \tag{2.31}$$

When no bus is present, the summation term evaluates to zero and the constraint forces $z_{ist^+} \geq z_{ist^-}$, forcing the z_{ist} assignment at t^- to equal that at t^+. When a bus is present, the inequality relaxes and permits the passenger to leave the bus stand.

The second constraint enforces passenger disembarkation behavior. A passenger i may be at bus stand s at t^+ only if they were already at that bus stand at t^- or onboard a bus that was at that bus stand at t^-. If the passenger is onboard a bus b that is not at bus stand s, the formulation prevents relocation to that stand. This condition is enforced by

$$z_{ist^+} \leq \frac{1}{2} x_{b\ell_s t^-} + \frac{1}{2} y_{ibt^-} + \sum_{\substack{b'=1 \\ b' \neq b}}^{N_B} y_{ib't^-} + z_{ist^-} \quad \forall b, \forall s, \forall i, \forall t. \tag{2.32}$$

The terms $x_{b\ell_s t^-}$ and y_{ibt^-} take the value 1 only when the passenger is on bus b and the bus is at bus stand s at t^-. If either term is zero, the passenger is prevented from disembarking at bus stand s

2.5 Objective Function

This thesis formulates a multi-objective optimization problem that minimizes bus operating cost and passenger travel time cost. Bus operating cost is determined by the number of time steps during which buses are active, i.e., not assigned to the idle state. Passenger travel time cost is determined by the deviation between passengers' actual and desired arrival times. The two objectives are combined using a weighted-sum approach, which allows a system planner to tune the relative importance of cost efficiency and service performance.

2.5.1 Bus Operating Cost

A bus incurs operating cost at each time step in which it is assigned to a route $r \neq 0$. Thus, operating cost is proportional to the number of discrete time steps during which a bus is active. For a given bus b, the total number of active time steps is given by

$$(N+1) - \sum_{t=0}^{N} w_{b0t^-}.$$

Given that each bus has an operating cost per unit time, c_b, the total operating cost for N_B buses is given by

$$\sum_{b=1}^{N_B} c_b \left(N + 1 - \sum_{t=0}^{N} w_{b0t^-} \right). \tag{2.33}$$

Expanding this expression yields

$$\sum_{b=1}^{N_B} c_b(N+1) - \sum_{b=1}^{N_B} \sum_{t=0}^{N} c_b w_{b0t^-}.$$

Since $\sum_{b=1}^{N_B} c_b(N+1)$ does not depend on the decision variables, it does not affect optimality. Therefore the bus operating cost can be expressed as

$$-\sum_{b=1}^{N_B} \sum_{t=0}^{N} c_b w_{b0t^-}. \tag{2.34}$$

2.5.2 Passenger Travel Time Cost

Passenger travel time cost is defined as the deviation between each passenger's actual arrival time and desired arrival time. For each passenger i, let W_i denote the travel time cost given by

$$W_i = K_i - J_i.$$

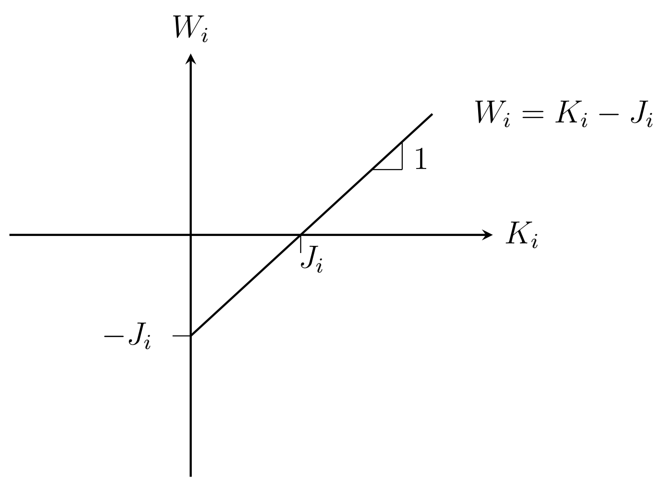


Figure 2.3: Symmetric passenger travel time cost, $W_i = K_i - J_i$, illustrating equal weighting of early and late arrivals.

The relationship between W_i, K_i and J_i is illustrated in Figure 2.3. When $K_i < J_i$, the passenger travel time cost is negative, which means the formulation rewards early arrival. When $K_i > J_i$ the travel time cost is positive, which means late arrival is penalized.

This symmetric formulation can lead to behavior that does not accurately reflect passenger service quality. Consider passengers 1 and 2, with desired arrival time steps $J_1 = 3$ and $J_2 = 6$. Suppose a feasible solution delivers passenger 1 at $K_1 = 6$ and passenger 2 at $K_2 = 3$. In this case, passenger 1 arrives three time steps late, while passenger 2 arrives three time steps early. The resulting passenger travel time costs are $W_1 = 3$ and $W_2 = -3$, giving a total passenger travel time cost of zero. This is identical to the cost obtained when both passengers arrive exactly at their desired time steps.

This example shows that the symmetric formulation does not accurately reflect service quality. Significant lateness for one passenger can be offset by equivalent earliness for another even though late arrivals are typically more undesirable in practice. To address this limitation, a modified passenger travel time cost is introduced in Figure 2.4.

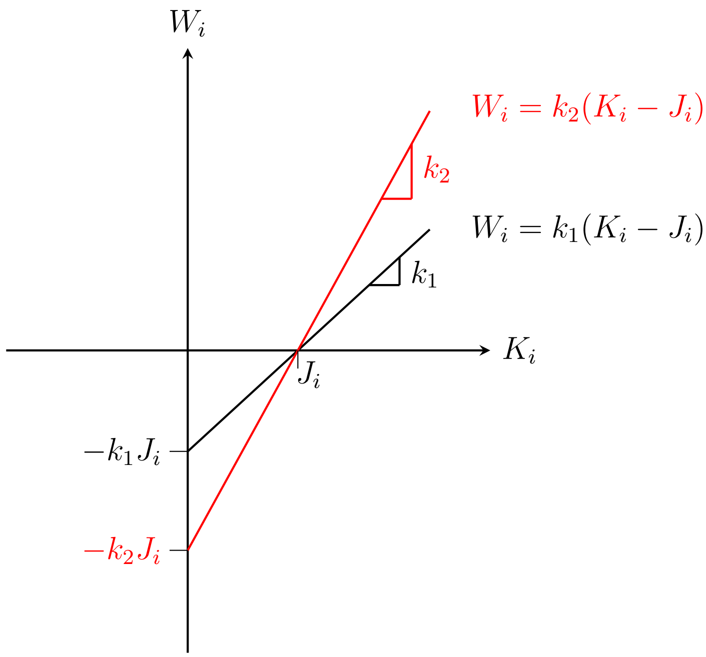


Figure 2.4: Piecewise-linear passenger travel time cost demonstrating asymmetric weighting between early and late arrivals.

In this revised passenger travel time cost, the cost for each passenger i is defined as

$$W_i = \max\left\{k_1(K_i - J_i),\ k_2(K_i - J_i)\right\},$$

where k_1 and k_2 are nonnegative weighting parameters satisfying $k_2 > k_1$. This formulation produces a piecewise-linear cost function. When $K_i < J_i$, $(K_i - J_i)$ is negative and the maximum selects $W_i = k_1(K_i - J_i)$ resulting in a smaller reward for early arrival. When $K_i > J_i$, $(K_i - J_i)$ is positive, and the maximum selects $W_i = k_2(K_i - J_i)$ resulting in a larger penalty for late arrival. As a result, early arrivals remain partially rewarded, but they cannot fully offset lateness of the same magnitude across passengers.

The passenger's arrival time K_i is computed using the variable $z_{iY_it^+}$. As shown in (2.7), once passenger i arrives at their destination bus stand Y_i, they remain there for the remainder of the planning horizon. Therefore, K_i can be calculated by

$$K_i = (N+1) - \sum_{t=0}^{N} z_{iY_it^+}.$$

Substituting this expression into the piecewise-linear passenger travel time cost gives

$$W_i = \max\left\{k_1\left((N+1) - \sum_{t=0}^{N} z_{iY_it^+} - J_i\right),\ k_2\left((N+1) - \sum_{t=0}^{N} z_{iY_it^+} - J_i\right)\right\}.$$

The total travel time objective therefore becomes

$$\sum_{i=1}^{N_P} \max\left\{k_1\left(N+1 - \sum_{t=0}^{N} z_{iY_it^+} - J_i\right),\ k_2\left(N+1 - \sum_{t=0}^{N} z_{iY_it^+} - J_i\right)\right\}. \tag{2.35}$$

To preserve linearity, slack variables β_i are introduced to represent the passenger travel time cost. The objective becomes

$$\sum_{i=1}^{N_P} \beta_i$$

subject to

$$-k_1 \sum_{t=0}^{N} z_{iY_it^+} - \beta_i \le -k_1(N+1-J_i), \quad \forall i \tag{2.36}$$

$$-k_2 \sum_{t=0}^{N} z_{iY_it^+} - \beta_i \le -k_2(N+1-J_i), \quad \forall i. \tag{2.37}$$

This linear reformulation preserves the MILP structure of the overall optimization problem.

2.6 Problem Formulation

The objective of the proposed system is to jointly minimize bus operating cost and passenger travel time cost. A weighted sum approach is used to balance these two objectives. The resulting optimization problem is given by

$$\min\ \omega_1 \left(-\sum_{b=1}^{N_B} \sum_{t=0}^{N} c_b w_{b0t^-} \right) + \omega_2 \sum_{i=1}^{N_P} \beta_i$$

subject to

Initial and final constraints (2.1)–(2.7),

Bus-route assignment constraints (2.8)–(2.13),

Bus-location assignment constraints (2.14)–(2.18),

Passenger-bus and passenger-location assignment constraints (2.21)–(2.32),

Passenger travel time linearization constraints (2.36)–(2.37).

Here, $\omega_1, \omega_2 \geq 0$ are user-selected weights controlling the trade-off between operational efficiency and service quality.

This completes the formulation of the proposed optimization problem. The model is expressed as an MILP that determines bus-route assignments, bus-location assignments, and passenger-bus and passenger-location assignments over the planning horizon. The following chapter verifies the constraints by examining system behavior through illustrative examples and case-based analysis.

Chapter 3

Verification of Constraints

This chapter verifies that the constraints introduced in Chapter 2 enforce admissible system evolution and prevent infeasible assignments. While the previous chapter defined the constraints governing bus and passenger assignments, their collective behavior is examined through case-based analysis and illustrative examples.

3.1 Bus-Route Assignment Constraints

This section analyzes the behavior of the bus-route assignment constraints using the example route network shown in Figure 3.1. In particular, it shows that a bus can be freely assigned to any route at the start of the planning horizon, completes a selected route without deviation or premature termination, and is forced into the idle state upon route completion. The analysis also demonstrates that these properties are preserved in networks with overlapping routes, where ambiguous assignments could arise.

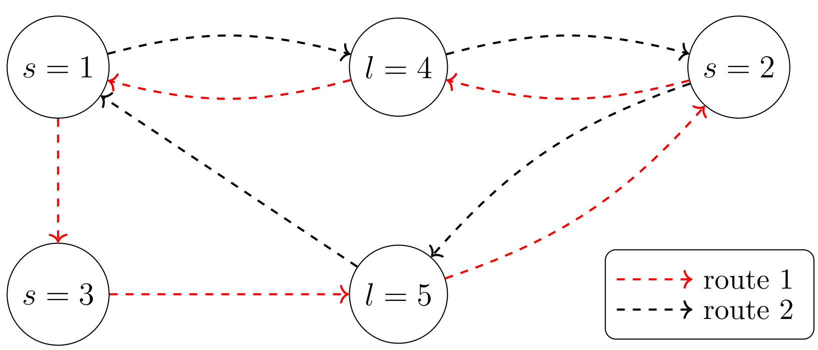


Figure 3.1: Example route network with routes 1 and 2 sharing multiple locations.

The example shown in Figure 3.1 considers a single bus and two candidate routes, routes 1 and 2. The route network consists of five locations, $\mathcal{L} = \{L_1, L_2, L_3, L_4, L_5\}$, of which three are designated as bus stands, $\mathcal{S} = \{S_1, S_2, S_3\}$. Using $\phi : \mathcal{S} \to \mathcal{L}$, the bus stands correspond to locations $\phi(S_1) = L_1$, $\phi(S_2) = L_2$, and $\phi(S_3) = L_3$.

The two bus routes are defined as

$$R_1 = (S_1, S_3, L_5, S_2, L_4, S_1)\,,$$

$$R_2 = (S_1, L_4, S_2, L_5, S_1)\,,$$

and the planning horizon is set to $N = 10$ time steps.

The initial conditions in Section 2.3 imply that, at 0^-, bus 1 is at the central depot, so that $x_{110^-} = 1$. In addition, the bus is initially idle, so $w_{100^-} = 1$. As summarized in Table 3.1, constraint (2.8) makes $w_{110^-} = 0$ and $w_{120^-} = 0$, since

$$w_{100^-} + w_{110^-} + w_{120^-} = 1.$$

Route	**Inputs**			**Constraints on w_{brt^+}**			**Output**
0	x_{110^-} 1	w_{100^-} 1		(2.10)	(2.11)	(2.12)	w_{110^+} 0 or 1
1	x_{110^-} 1	w_{100^-} 1	w_{110^-} 0	(2.10) $w_{110^+} \leq 1$	(2.11) $w_{110^+} \geq -1$	(2.12) $w_{110^+} \leq 1$	w_{110^+} 0 or 1
2	x_{110^-} 1	w_{100^-} 1	w_{120^-} 0	(2.10) $w_{120^+} \leq 1$	(2.11) $w_{120^+} \geq -1$	(2.12) $w_{120^+} \leq 1$	w_{120^+} 0 or 1

Table 3.1: Admissible bus–route assignments for bus 1 at the central depot at 0^+.

Given these decision variable values at 0^-, constraints (2.10), (2.11), and (2.12) define the admissible assignments at 0^+ for all $r \neq 0$. As shown in Table 3.1, these constraints allow w_{110^+} and w_{120^+} to take values in $\{0, 1\}$. Constraint (2.9) then ensures that exactly one bus-route decision variable equals 1 at 0^+, with all others equal to 0.

As a result, at the beginning of the planning horizon, when bus 1 is at the central depot, the constraints allow assignment to any route $r \in \{0, 1, 2\}$. This matches the intended modeling objective of full bus-route assignment flexibility at the central depot at 0^+.

Assume the optimization model assigns bus 1 to route 2 at 0^+, yielding $w_{120^+} = 1$. Constraint (2.9) then makes $w_{100^+} = 0$ and $w_{110^+} = 0$. Since constraint (2.13) enforces that bus-route assignments at $(t-1)^+$ match those at t^-, $w_{101^-} = 0, w_{111^-} = 0$, and $w_{121^-} = 1$, as summarized in Table 3.2. Because the bus advances by one location along route 2 between 0^+ and 1^-, it is now at location 2 at 1^-, so $x_{111^-} = 0$.

Route	**Inputs**			**Constraints on $\boldsymbol{w_{brt^+}}$**			**Output**
0	x_{111^-} 0	w_{101^-} 0		(2.10)	(2.11)	(2.12)	w_{111^+} 0
1	x_{111^-} 0	w_{101^-} 0	w_{110^-} 0	(2.10) $w_{111^+} \leq 0$	(2.11) $w_{111^+} \geq 0$	(2.12) $w_{111^+} \leq 1$	w_{111^+} 0
2	x_{111^-} 0	w_{101^-} 0	w_{120^-} 1	(2.10) $w_{121^+} \leq 1$	(2.11) $w_{121^+} \geq 1$	(2.12) $w_{121^+} \leq 1$	w_{121^+} 1

Table 3.2: Admissible bus-route assignments for bus 1 at 1^+ after being assigned to route 2 at 0^+.

As summarized in Table 3.2, constraints (2.10), (2.11), and (2.12) make $w_{111^+} = 0$ and $w_{121^+} = 1$. Constraint (2.9) makes $w_{101^+} = 0$, since the bus can be assigned to only one route. As a result, even though routes 1 and 2 share location 2, the constraints prevent termination or deviation and ensure that bus 1 continues on route 2.

The constraints enforce continued assignment of bus 1 to route 2 until it returns to the central depot at $t = 4$, so that $x_{114^-} = 1$. At 4^-, the bus-route assignment variables satisfy $w_{104^-} = 0$, $w_{114^-} = 0$, and $w_{124^-} = 1$.

At 4^+, the bus relinquishes its assignment to route 2 and is assigned to the idle state. As summarized in Table 3.2, constraints (2.10), (2.11), and (2.12) enforce $w_{114^+} = 0$ and $w_{124^+} = 0$, while allowing $w_{104^+} \in \{0, 1\}$. Constraint (2.9) then makes $w_{104^+} = 1$,

Route	**Inputs**			**Constraints on w_{brt^+}**			**Output**
0	x_{114^-} 0	w_{104^-} 0		(2.10)	(2.11)	(2.12)	w_{114^+} 1
1	x_{114^-} 1	w_{104^-} 0	w_{114^-} 0	(2.10) $w_{114^+} \leq 1$	(2.11) $w_{114^+} \geq -1$	(2.12) $w_{114^+} \leq 0$	w_{114^+} 0
2	x_{114^-} 1	w_{104^-} 0	w_{124^-} 1	(2.10) $w_{124^+} \leq 2$	(2.11) $w_{124^+} \geq 0$	(2.12) $w_{124^+} \leq 0$	w_{124^+} 0

Table 3.3: Admissible bus-route assignment outcomes for bus 1 at route completion ($t = 4$), illustrating enforced transition to the idle condition.

forcing the bus to be assigned to the idle state. Since constraint (2.13) enforces equality between bus-route assignments at $(t-1)^+$ and those at t^-, the bus will also be assigned to the idle state at 5^-, enforcing the dwell period described in Chapter 2. Once the bus is assigned to the idle state at 5^-, the constraints behave as in Table 3.1, allowing assignment to any route $r \in \{0, 1, 2\}$.

This example demonstrates that the bus-route assignment constraints enforce consistent and admissible behavior throughout the planning horizon. When the bus is at the central depot, it can be assigned to any route only at the start of the planning horizon or after one time step in the idle state. Once assigned, it is required to complete the route without deviation or premature termination. Upon completion, the bus is forced into the idle state and remains there for one time step before it can be assigned to another route. This confirms that the constraints correctly govern assignment decisions and prevent infeasible assignments.

3.2 Bus-Location Assignment Constraints

This section verifies that the bus-location constraints enforce ordered progression along an assigned route. Using the example network in Figure 3.1 in Section 3.1, it is shown that once a route is selected, the bus moves through the sequence of locations in the correct order. The analysis also shows that when the bus enters the idle state at the central depot, it remains at the depot for the required dwell period.

Location	**Inputs**		**Constraints on x_{blt^+}**		**Output**
1	x_{110^+}	w_{120^+}	(2.17)	(2.18)	x_{121^-}
	1	1	$x_{121^-} \leq 1$	$x_{101^-} \geq 1$	1

Table 3.4: Admissible bus-location assignments for bus 1 at 1^- after being assigned to route 2 at 0^+ as in Section 3.1.

Consider Figure 3.1 and the scenario described in Section 3.1. As in Section 3.1, assume bus 1 is assigned to route 2 at 0^+, so $w_{120^+} = 1$, as summarized in Table 3.4. From the initial conditions, bus 1 is assigned to the central depot at 0^-, so $x_{110^-} = 0$. Constraint (2.16) then makes $x_{110^+} = 1$, as shown in Table 3.4, since it ensures that the bus does not move between t^- and t^+.

Because the bus is assigned to route 2 at 0^+, constraints (2.17) and (2.18) require that its location at 1^- corresponds to the next location in route 2. Since $R_2(2) = L_2$, these constraints force $x_{121^-} = 1$, as shown in Table 3.4. Once the bus is assigned to location 2 at 1^-, constraint (2.14) prevents the bus from being assigned to multiple locations simultaneously.

Location	**Inputs**		**Constraints on x_{blt^+}**		**Output**
1	x_{114^+}	w_{104^+}	(2.19)	(2.20)	x_{115^-}
	1	1	$x_{115^-} \leq 1$	$x_{115^-} \geq 1$	1

Table 3.5: Admissible bus-location assignments for bus 1 at 5^- following assignment to the idle state at 4^+.

Constraints (2.17) and (2.18) continue to enforce ordered progression along route 2 until the bus is assigned to the idle state at 4^+. Constraints (2.19) and (2.20) then enforce that the bus remains at the central depot at 5^-. As shown in Table 3.5, this gives $x_{115^-} = 1$. Therefore, when the bus is at the central depot and assigned to the idle state at t^+, it remains at the central depot at $(t+1)^-$, enforcing location consistency during the dwell period.

This example demonstrates that the bus-location constraints enforce ordered progression along an assigned route. Once a route is selected, the bus follows the prescribed sequence of locations without skipping or revisiting any location. Upon returning to the central depot and entering the idle state, the constraints ensure that the bus remains at the depot for the required dwell period.

3.3 Passenger-Bus Assignment Constraints

This section verifies that the passenger-bus assignment constraints correctly determine when a passenger can be assigned to a bus. The constraints must permit boarding only when a passenger and bus are at the same bus stand, prohibit passengers from transferring between buses within a single time step, and allow disembarkment only when the passenger's assigned bus is at a bus stand. The analysis considers all possible passenger-bus configurations and confirms that the constraints eliminate infeasible assignments while preserving all admissible ones.

Figure 3.2 summarizes the variables required to evaluate the passenger-bus assignment constraints, including the passenger-location, passenger-bus, and bus-location assignments at t^-. In this figure, each column on the horizontal b-axis represents a bus whereas each row on the vertical s-axis represents a bus stand. The vertical gray column in the third quadrant shows z_{ist^-} assignments for passenger i at t^- across all bus stands $s \in \{1, 2, \ldots, N_S\}$. An entry equal to one along the s-axis indicates that the passenger is at that bus stand. By constraint (2.21), all remaining entries along the s-axis would equal zero.

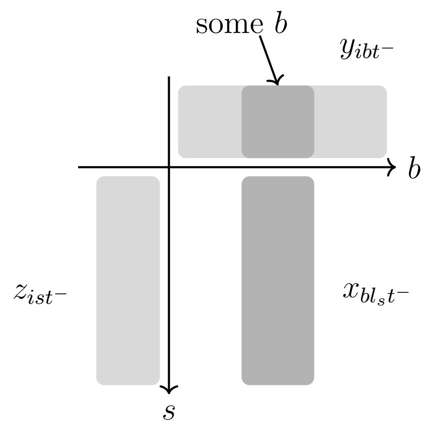


Figure 3.2: Case-analysis template illustrating passenger-location, passenger-bus, and bus-location variables used to evaluate y_{ibt^+} assignments.

The horizontal gray row in the first quadrant shows y_{ibt^-} assignments for passenger i across all buses $b \in \{1, 2, \ldots, N_B\}$. An entry equal to one along the b-axis indicates that passenger i is onboard a bus at t^-. The dark gray box pinpoints the bus b being focused on when evaluating feasible y_{ibt^+} assignments. The corresponding vertical column in the fourth quadrant represents the bus-location variables, $x_{b\ell_s t}$, for that bus b, indicating whether it is at a bus stand $s \in \{1, 2, \ldots, N_S\}$.

Constraint (2.21) ensures that passenger i occupies exactly one state at t^-. If the passenger is at a bus stand s at t^-, they cannot simultaneously be onboard any bus or at another bus stand. If the passenger is onboard bus b at t^-, they cannot simultaneously be at any bus stand or on another bus $b' \neq b$.

For bus b, constraint (2.14) ensures that the sum of the entries in the dark gray column of the fourth quadrant, is either zero or one across all bus stands. A sum equal to one indicates that the bus is at a bus stand. A sum equal to zero indicates that the bus is currently at a location that is not a bus stand. This distinction is critical, because boarding and disembarking can only occur when the bus is at a bus stand. If the bus is at a location that is not a bus stand, passenger boarding or disembarkment is not permitted.

At any time step t, there exist seven logically distinct cases describing the relationship between passenger i, their location, bus b, and its location. These cases can be grouped into three categories based on the passenger's state at t^-.

3.3.1 Category I: Passenger at Bus Stand s

The first category consists of three cases shown in Figure 3.3, in which passenger i is at bus stand s at t^-. These cases demonstrate that passenger i can be on bus b at t^+ only if bus b is present at the same bus stand as the passenger at t^-. If the passenger and bus are not at the same bus stand at t^-, the constraints ensure that the passenger cannot be on that bus at t^+.

The first case, illustrated in Figure 3.3(a), considers a scenario in which passenger i is at some bus stand s, so that $z_{ist^-} = 1$. Bus b, for which assignment of passenger i at t^+ is being evaluated, is not located at bus stand s, which implies that $x_{b\ell_s t^-} = 0$. In fact, bus b is not at any bus stand but is instead at a bus location $\ell \notin \{1, 2, \ldots, N_S\}$, which implies that

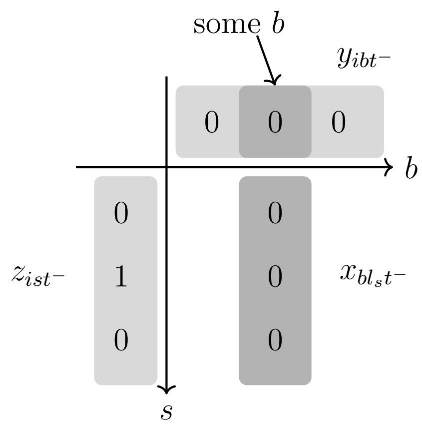
$$\sum_{s=1}^{N_S} x_{b\ell_s t^-} = 0,$$

as shown in Table 3.6.

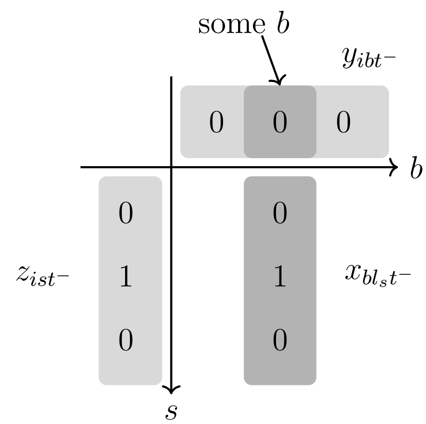
Inputs				**Constraints on y_{ibt^+}**				**Output**
z_{ist^-}	$x_{b\ell_s t^-}$	$\sum_{s=1}^{N_S} x_{b\ell_s t^-}$	y_{ibt^-}	(2.27)	(2.28)	(2.29)	(2.30)	y_{ibt^+}
1	0	0	0	$\leq 1 - \frac{s}{N_S}$	$\leq 1 + \frac{s}{N_S}$	≤ 0	≥ 0	0

Table 3.6: Case 1: Passenger at a bus stand while bus b is not at any bus stand.

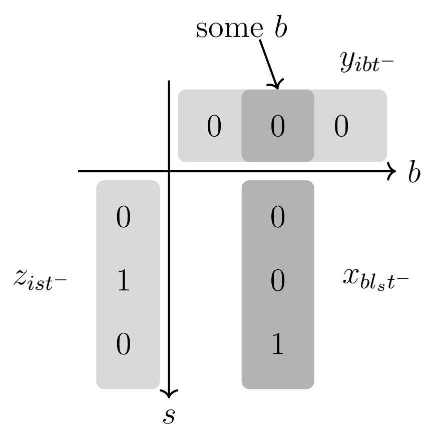
From Table 3.6, constraint (2.29) restricts y_{ibt^+} to values less than or equal to 0. Since y_{ibt^+} is binary, this forces $y_{ibt^+} = 0$. As a result, because bus b is not at the same bus stand as passenger i, the constraints prohibit assignment of passenger i to bus b at t^+.



(a) Case 1



(b) Case 2



(c) Case 3

Figure 3.3: Category I configurations: Passenger-bus, passenger-location and bus-location variables at t^- when the passenger is at a bus stand s. Subfigure (a) corresponds to Case 1, in which bus b is not at any bus stand; (b) corresponds to Case 2, in which bus b is at the same bus stand as the passenger; and (c) corresponds to Case 3, in which bus b and the passenger are at different bus stands.

The second case, shown in Figure 3.3(b), considers a scenario in which passenger i is at some bus stand s, so that $z_{ist^-} = 1$, and bus b is at the same bus stand, implying that $x_{b\ell_st^-} = 1$. It follows that

$$\sum_{s=1}^{N_S} x_{b\ell_st^-} = 1,$$

as shown in Table 3.7.

Inputs				Constraints on y_{ibt^+}				Output
z_{ist^-}	$x_{b\ell_s t^-}$	$\sum_{s=1}^{N_S} x_{b\ell_s t^-}$	y_{ibt^-}	(2.27)	(2.28)	(2.29)	(2.30)	y_{ibt^+}
1	1	1	0	≤ 1	≤ 1	≤ 1	≥ -1	0 or 1

Table 3.7: Case 2: Evaluation of passenger-bus constraints when both passenger i and bus b are at the same bus stand s at t^-.

From Table 3.7, the constraints do not impose a restriction that fixes the value of y_{ibt^+}. The resulting bounds allow y_{ibt^+} to take any value within $[-1, 1]$. Since y_{ibt^+} is binary, the feasible values reduce to $\{0, 1\}$. Therefore, when passenger i and bus b are at the same bus stand at t^-, passenger i may be on bus b at t^+, but the assignment is not mandatory.

The final case in Category I, illustrated in Figure 3.3(c), considers a scenario in which passenger i is at bus stand s at t^-, so that $z_{ist^-} = 1$, while bus b is at a different bus stand $s' \neq s$. In this configuration, $x_{b\ell_s t^-} = 0$ for bus stand s, but

$$\sum_{s=1}^{N_S} x_{b\ell_s t^-} = 1,$$

since the bus is at some bus stand s', as shown in Figure 3.3(c).

Inputs				Constraints on y_{ibt^+}				Output
z_{ist^-}	$x_{b\ell_s t^-}$	$\sum_{s=1}^{N_S} x_{b\ell_s t^-}$	y_{ibt^-}	(2.27)	(2.28)	(2.29)	(2.30)	y_{ibt^+}
1	0	1	0	$\leq 1 + \frac{s'}{N_S} - \frac{s}{N_S}$	$\leq 1 - \frac{s'}{N_S} + \frac{s}{N_S}$	≤ 1	≥ -1	0

Table 3.8: Case 3: Evaluation of passenger-bus constraints when passenger i is at bus stand s while bus b is at bus stand $s' \neq s$ at t^-.

From Table 3.8, constraints (2.29) and (2.30) permit y_{ibt^+} to lie within the interval $[-1, 1]$. Thus, for Case 3, the restriction on y_{ibt^+} is determined by constraints (2.27) and (2.28). These two constraints impose the upper bounds,

$$y_{ibt^+} \leq 1 + \frac{s'}{N_S} - \frac{s}{N_S},$$

$$y_{ibt^+} \leq 1 - \frac{s'}{N_S} + \frac{s}{N_S}.$$

Because $s' \neq s$, exactly one of the two quantities

$$\frac{s'}{N_S} - \frac{s}{N_S} \quad \text{and} \quad \frac{s}{N_S} - \frac{s'}{N_S}$$

is strictly positive, while the other is strictly negative. If $s' > s$, then

$$1 - \frac{s'}{N_S} + \frac{s}{N_S} < 1$$

so constraint (2.28) yields an upper bound strictly less than 1. If $s' < s$, then

$$1 + \frac{s'}{N_S} - \frac{s}{N_S} < 1$$

so constraint (2.27) yields an upper bound strictly less than 1.

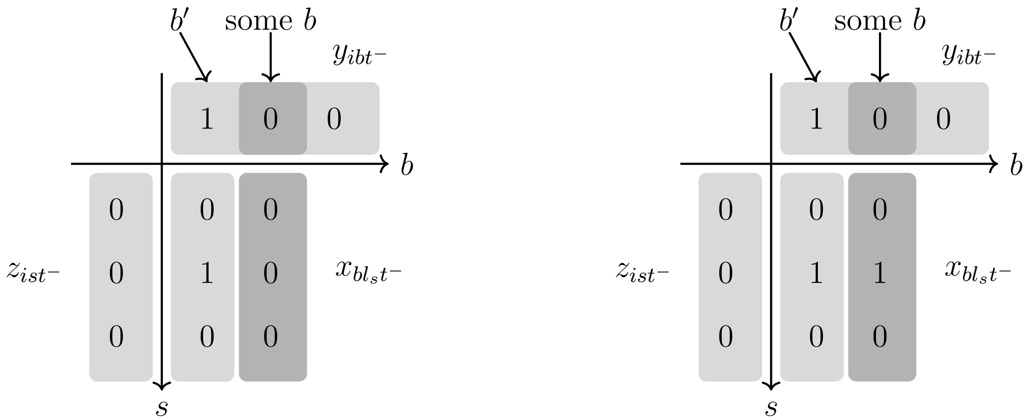
Therefore, in both cases, whether $s' > s$ or $s' < s$, either (2.28) or (2.27) enforces an upper bound strictly less than 1, implying that $y_{ibt^+} < 1$. Since y_{ibt^+} is binary, it follows that $y_{ibt^+} = 0$. Thus, although both the passenger and the bus are at bus stands, boarding is not permitted because they are not at the same bus stand.

Collectively, the three cases in Category I demonstrate that when passenger i is at bus stand s at t^-, they can only be on bus b at t^+ if bus b is also at bus stand s. In all other spatial configurations, the constraints enforce $y_{ibt^+} = 0$, ensuring that the passenger cannot board a bus that is not at the same bus stand as them.

3.3.2 Category II: Passenger Onboard Another Bus b'

The second category, shown in Figure 3.4, includes two cases in which passenger i is on bus $b' \neq b$ at t^-. These cases establish that the constraints prohibit passenger i

from transferring from bus b' to bus b at t^+. The analysis evaluates the passenger-bus decision variables under the different cases and shows that they enforce $y_{ibt^+} = 0$. As a result, the constraints prohibit the passenger from transferring from bus b' to bus b within the same time step.



(a) Case 4

(b) Case 5

Figure 3.4: Category II configurations: Passenger-bus, passenger-location and bus-location assignments at t^- when passenger i is onboard bus $b' \neq b$ at bus stand s. Subfigure (a) corresponds to Case 4, in which bus b is not at any bus stand and (b) corresponds to Case 5, in which bus b is at bus stand s.

Inputs				**Constraints on $\boldsymbol{y_{ibt^+}}$**				**Output**
z_{ist^-}	$x_{b\ell_st^-}$	$\sum_{s=1}^{N_S} x_{b\ell_st^-}$	y_{ibt^-}	(2.27)	(2.28)	(2.29)	(2.30)	y_{ibt^+}
0	0	0	0	≤ 1	≤ 1	≤ 0	≥ 0	0

Table 3.9: Case 4: Evaluation of passenger-bus constraints when passenger i is onboard bus $b' \neq b$ and bus b is at location $\ell \notin \mathcal{S}$.

Case 4, shown in Figure 3.4(a), considers a scenario in which passenger i is on bus $b' \neq b$ at t^-, so that $y_{ib't^-} = 1$, and bus b' is at bus stand s. Constraint (2.21) ensures that $y_{ibt^-} = 0$, since the passenger cannot be simultaneously onboard both bus b' and bus b. Furthermore, bus b is not at a bus stand, but is instead at location

$\ell \notin \{1, 2, \ldots, N_S\}$, so that $x_{b\ell_s t^-} = 0$ and

$$\sum_{s=1}^{N_S} x_{b\ell_s t^-} = 0,$$

as shown in Table 3.9.

From Table 3.9, constraint (2.29) enforces that $y_{ibt^+} \leq 0$. Since, y_{ibt^+} is binary, it follows that $y_{ibt^+} = 0$. Therefore, if passenger i is onboard bus $b' \neq b$ at t^-, assignment to bus b at t^+ is not feasible when bus b is not at a bus stand.

In Case 5, the passenger is onboard bus $b' \neq b$, so that $y_{ib't^-} = 1$. By constraint (2.21), this implies that $y_{ibt^-} = 0$. The distinction between Case 4 and Case 5 is that, in the latter, bus b is at a bus stand. In particular, both buses b and b' are at the same bus stand s. As a result, $x_{b\ell_s t^-} = 1$ and

$$\sum_{s=1}^{N_S} x_{b\ell_s t^-} = 1,$$

as shown in Table 3.10.

Inputs				**Constraints on y_{ibt^+}**				**Output**
z_{ist^-}	$x_{b\ell_s t^-}$	$\sum_{s=1}^{N_S} x_{b\ell_s t^-}$	y_{ibt^-}	(2.27)	(2.28)	(2.29)	(2.30)	y_{ibt^+}
0	1	1	0	$\leq 1 + \frac{s}{N_S}$	$\leq 1 - \frac{s}{N_S}$	≤ 1	≥ -1	0

Table 3.10: Case 5: Evaluation of passenger-bus constraints when passenger i is onboard bus $b' \neq b$ and bus b is at bus stand s.

From Table 3.10, constraint (2.28) enforces

$$y_{ibt^+} \leq 1 - \frac{s}{N_S}.$$

Since s takes values in $\{1, \ldots, N_S\}$, it follows that $1 - \frac{s}{N_S}$ is strictly less than 1. Therefore, constraint (2.28) makes $y_{ibt^+} = 0$. As a result, if passenger i is onboard bus

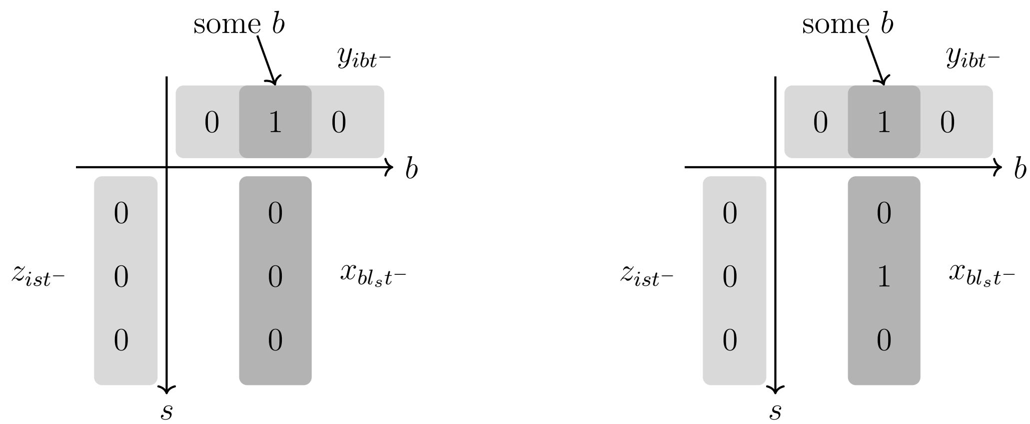
$b' \neq b$ at t^-, assignment to bus b at t^+ is not feasible, even when bus b is at bus stand s.

Although Figure 3.4(b) illustrates the case in which buses b and b' are located at the same bus stand, the result is not limited to this configuration. Constraint (2.28) does not depend on the specific bus-location decision variable $x_{b\ell_s t^-}$, but rather on the aggregate term $\sum_{s=1}^{N_S} x_{b\ell_s t^-}$, which is 1 whenever bus b is at any bus stand at t^-. Therefore, the same conclusion holds for all configurations in which bus b is at a bus stand.

Together, Cases 4 and 5 show that the constraints prohibit direct transfer between buses within the same time step. This ensures a consistent and well-defined evolution of passenger state throughout the planning horizon.

3.3.3 Category III: Passenger Onboard Bus b

The third and final category consists of two cases in which passenger i is onboard bus b at t^-. These cases establish that disembarking bus b is permitted only when bus b is at some bus stand $s \in \{1, 2, \ldots, N_S\}$. If bus b is at a location that is not a bus stand, the constraints force passenger i to remain onboard bus b. A passenger is said to have disembarked from bus b at t^+ if $y_{ibt^-} = 1$ and $y_{ibt^+} = 0$.



(a) Case 6 (b) Case 7

Figure 3.5: Category III configurations: Representation of passenger-bus, passenger-location and bus-location variables at t^- when passenger i is onboard bus b. Subfigure (a) corresponds to Case 6, in which bus b is not at any bus stand and (b) corresponds to Case 7, in which bus b is at bus stand s.

Case 6 considers the configuration in which passenger i is onboard bus b at t^-, so that $y_{ibt^-} = 1$. By constraint (2.21), this implies that $z_{ist^-} = 0$ for all s. Thus, bus b is not at any bus stand but is instead at some $l \notin \{1, 2, \ldots, N_S\}$, yielding $x_{b\ell_st^-} = 0$ and

$$\sum_{s=1}^{N_S} x_{b\ell_st^-} = 0,$$

as shown in Table 3.11.

Inputs				**Constraints on $\boldsymbol{y_{ibt^+}}$**				**Output**
z_{ist^-}	$x_{b\ell_st^-}$	$\sum_{s=1}^{N_S} x_{b\ell_st^-}$	y_{ibt^-}	(2.27)	(2.28)	(2.29)	(2.30)	y_{ibt^+}
0	0	0	1	≤ 2	≤ 2	≤ 1	≥ 1	1

Table 3.11: Case 6: Evaluation of passenger-bus constraints when passenger i is onboard bus b that is not at any bus stand.

From Table 3.11, constraint (2.30) makes $y_{ibt^+} \geq 1$. This implies that when bus b is not at a bus stand, passenger i is not permitted to disembark the bus. The constraints force the passenger to remain onboard until the bus reaches a bus stand.

In Case 7, passenger i is onboard bus b at t^- and bus b is at bus stand s, yielding $x_{b\ell_s t^-} = 1$ and

$$\sum_{s=1}^{N_S} x_{b\ell_s t^-} = 1,$$

as shown in Table 3.12.

Inputs				Constraints on y_{ibt^+}				Output
z_{ist^-}	$x_{b\ell_s t^-}$	$\sum_{s=1}^{N_S} x_{b\ell_s t^-}$	y_{ibt^-}	(2.27)	(2.28)	(2.29)	(2.30)	y_{ibt^+}
0	1	1	1	$\leq 2 + \frac{s}{N_S}$	$\leq 2 - \frac{s}{N_S}$	≤ 2	≥ 0	0 or 1

Table 3.12: Case 7: Evaluation of passenger-bus constraints when passenger i is on board bus b at bus stand s.

From Table 3.12, constraint (2.30) gives the lower bound, $y_{ibt^+} \geq 0$. The remaining constraints enforce upper bounds. Constraint (2.29) yields $y_{ibt^+} \leq 2$. For constraints (2.27) and (2.28), since $s \in \{1, 2, \ldots, N_S\}$,

$$0 < \frac{s}{N_S} \leq 1.$$

Therefore, the tightest upper bound that constraint (2.27) can enforce is $y_{ibt^+} \leq 2$, whereas the tightest upper bound constraint (2.28) can enforce is $y_{ibt^+} \leq 1$. Together, the four constraints enforce $0 \leq y_{ibt^+} \leq 1$. This implies that when a passenger is onboard bus b and the bus is at some bus stand s, disembarking is feasible, but not mandatory.

Collectively, the seven cases across the three categories exhaust all logically distinct passenger-bus configurations at t^-. The analysis demonstrates that the four passenger-bus assignment constraints precisely characterize admissible y_{ibt^+} assignments. That is, boarding is permitted only when both the bus and passenger are at the same bus stand, direct bus-to-bus transfer is prohibited within a single discrete time step, and disembarking is allowed only when the bus is at a bus stand.

3.4 Passenger-Location Assignment Constraints

This section verifies that the passenger-location assignment constraints correctly enforce passenger-location assignments. The constraints must permit disembarkment at a bus stand only when a passenger's assigned bus is at that bus stand, permit departure from a bus stand only when a bus is present at that bus stand, and prevent passenger movement to another bus stand without a valid bus-mediated movement. The verification considers all possible passenger-location, passenger-bus, and bus-location configurations and confirms that the constraints eliminate infeasible assignments while preserving all admissible ones.

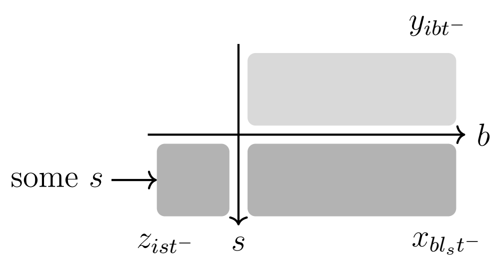


Figure 3.6: Case-analysis template illustrating passenger-location, passenger-bus, and bus-location assignments at t^-.

Figure 3.6 summarizes the passenger-location, passenger-bus, and bus-location assignments at t^-. The horizontal light gray row in the first quadrant shows y_{ibt^-} for passenger i across all buses $b \in \{1, 2, \ldots, N_B\}$. Since constraint (2.21) enforces that a passenger may be on at most one bus at any time step, the sum of the values in this row satisfies

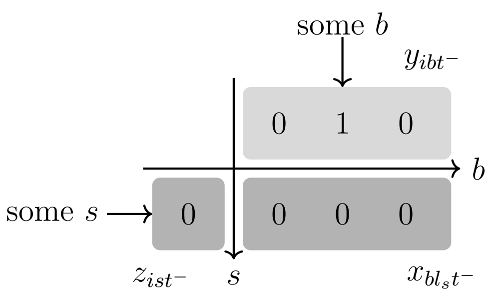
$$\sum_{b=1}^{N_B} y_{ibt^-} \leq 1.$$

The dark gray block in the third quadrant shows the value of z_{ist^-}, while the horizontal dark gray row in the fourth quadrant shows the values of $x_{b\ell_st^-}$ for each bus.

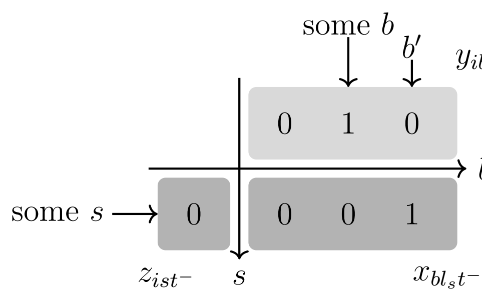
At any t^-, seven distinct cases characterize the relationship between passenger i, bus b, its location, and a candidate bus stand s. The cases are grouped into three categories based on whether the passenger is onboard bus b, at bus stand s, or at another bus stand $s' \neq s$.

3.4.1 Category I: Passenger Onboard Bus b

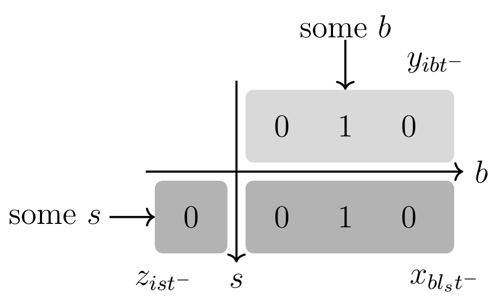
The first category, shown in Figure 3.7, consists of three cases in which passenger i is on bus b at t^-. These cases demonstrate that disembarkment at bus stand s, formally described as $z_{ist^+} = 1$ given $y_{ibt^-} = 1$, is feasible only when bus b is at bus stand s. If the passenger is on bus b, and bus b is not at bus stand s at t^-, the constraints enforce $z_{ist^+} = 0$.



(a) Case 1



(b) Case 2



(c) Case 3

Figure 3.7: Category I configurations: Passenger-bus, passenger-location and bus-location variables at t^- when passenger i is onboard bus b. Subfigure (a) corresponds to Case 1, in which bus b is not at bus stand s; (b) corresponds to Case 2, in which bus b is not at bus stand s, but there is a $b' \neq b$ at bus stand s; and (c) corresponds to Case 3, in which bus b is at bus stand s.

The first case, shown in Figure 3.7(a), considers the configuration in which passenger i is on bus b at t^-, so that $y_{ibt^-} = 1$, while bus b is not at bus stand s, yielding $x_{b\ell_s t^-} = 0$. Furthermore, no other bus $b' \neq b$ is at bus stand s, implying that

$$\sum_{b=1}^{N_B} x_{b\ell_s t^-} = 0,$$

as summarized in Table 3.13.

Inputs					**Constraints on z_{ist^+}**		**Output**
z_{ist^-}	y_{ibt^-}	$x_{b\ell_s t^-}$	$\sum_{b=1}^{N_B} x_{b\ell_s t^-}$	$\sum_{\substack{b'=1 \\ b'\neq b}}^{N_B} y_{ib't^-}$	(2.31)	(2.32)	z_{ist^+}
0	1	0	0	0	≥ 0	$\leq \frac{1}{2}$	0

Table 3.13: Case 1: Evaluation of passenger-location constraints when passenger i is onboard bus b and bus b is not at bus stand s.

From Table 3.13, constraint (2.32) yields $z_{ist^+} \leq \frac{1}{2}$, which implies that $z_{ist^+} = 0$. Therefore, if passenger i is on bus b at t^-, bus b is not at bus stand s, and no other bus $b' \neq b$ is at bus stand s, the constraints prohibit passenger i from disembarking at bus stand s.

The second case, illustrated in Figure 3.7(b), is similar to Case 1 in that passenger i is on bus b, so that $y_{ibt^-} = 1$, while bus b is not at bus stand s, yielding $x_{b\ell_s t^-} = 0$. However, in this case, another bus $b' \neq b$ is at bus stand s. As a result,

$$\sum_{b=1}^{N_B} x_{b\ell_s t^-} = 1,$$

as summarized in Table 3.14.

Similar to Case 1, Table 3.14 shows that constraint (2.32) still yields $z_{ist^+} \leq \frac{1}{2}$, which implies that $z_{ist^+} = 0$. As a result, the constraints successfully enforce that passenger i, on bus b at t^-, may not disembark at bus stand s at t^+, when bus b is not at bus stand s at t^-.

Inputs					Constraints on z_{ist^+}		Output
z_{ist^-}	y_{ibt^-}	$x_{b\ell_s t^-}$	$\sum_{b=1}^{N_B} x_{b\ell_s t^-}$	$\sum_{\substack{b'=1 \\ b' \neq b}}^{N_B} y_{ib't^-}$	(2.31)	(2.32)	z_{ist^+}
0	1	0	1	0	≥ -1	$\leq \frac{1}{2}$	0

Table 3.14: Case 2: Evaluation of passenger-location constraints when passenger i is onboard bus b, bus b is not at bus stand s, and another bus $b' \neq b$ is at bus stand s.

Case 2 admits multiple configurations. In particular, up to $N_B - 1$ buses, i.e., all buses except b, may be at bus stand s. In such configurations,

$$\sum_{b=1}^{N_B} x_{b\ell_s t^-} \leq N_B - 1.$$

This affects only constraint (2.31) which becomes $z_{ist^+} \geq -(N_B - 1)$. Since the right-hand side is non-positive, this lower bound does not restrict z_{ist^+}. Constraint (2.32), however, remains unchanged and continues to enforce $z_{ist^+} = 0$.

Thus, regardless of how many other buses are present at s, the conclusion remains identical to Case 1. If passenger i is on bus b and bus b is not at bus stand s at t^-, the constraints prohibit passenger i from disembarking at bus stand s at t^+.

Case 3, illustrated in Figure 3.7(c), considers the configuration in which passenger i is on bus b and bus b is at bus stand s. As summarized in Table 3.15, this corresponds to $y_{ibt^-} = 1$, $x_{b\ell_s t^-} = 1$, and

$$\sum_{b=1}^{N_B} x_{b\ell_s t^-} = 1.$$

Inputs					Constraints on z_{ist^+}		Output
z_{ist^-}	y_{ibt^-}	$x_{b\ell_s t^-}$	$\sum_{b=1}^{N_B} x_{b\ell_s t^-}$	$\sum_{\substack{b'=1 \\ b' \neq b}}^{N_B} y_{ib't^-}$	(2.31)	(2.32)	z_{ist^+}
0	1	1	1	0	≥ -1	≤ 1	0 or 1

Table 3.15: Case 3: Evaluation of passenger-location constraints when passenger i is onboard bus b, and bus b is at bus stand s.

As expected, constraints (2.31) and (2.32) allow z_{ist^+} to take any value in $\{0, 1\}$.

This means that passenger i can either remain on bus b or disembark at bus stand s.

As in Case 2, multiple configurations are possible. Additional buses may also be at bus stand s, increasing $\sum_{b=1}^{N_B} x_{b\ell_s t^-}$. This affects constraint (2.31), which yields $z_{ist^+} \geq -(N_B)$ when all buses are at bus stand s. Since z_{ist^+} is binary, any non-positive lower bound does not restrict its value, and $z_{ist^+} \in \{0, 1\}$ remains feasible.

Together, the three cases in Category I demonstrate that when a passenger is on bus b, constraints (2.31) and (2.32) permit disembarkment at bus stand s if and only if bus b is also at bus stand s. In all other configurations where passenger i is on bus b but the bus is not at bus stand s, disembarkment at bus stand s is prohibited.

3.4.2 Category II: Passenger at Bus Stand s

The second category, illustrated in Figure 3.8, presents two cases in which passenger i is at bus stand s at t^-, so that $z_{ist^-} = 1$. By constraint, (2.21), this implies that $y_{ibt^-} = 0$ for all buses. These cases demonstrate that passenger i is allowed to leave bus stand s only when a bus is at that bus stand.

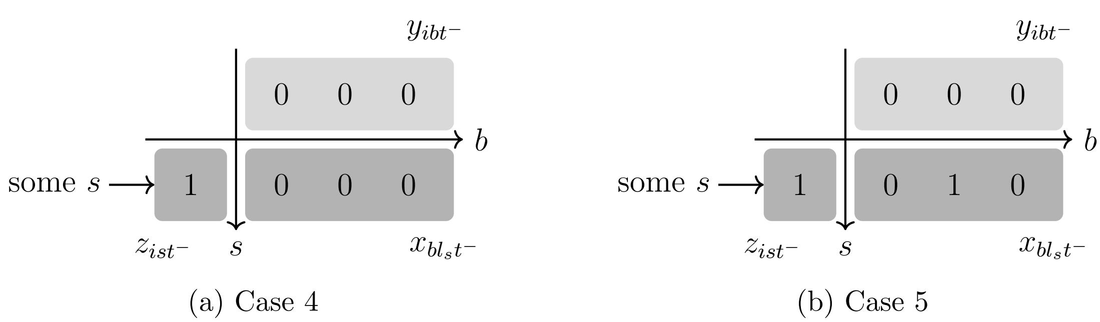


Figure 3.8: Category II configurations: Representation of passenger-bus, passenger-location and bus-location variables at t^- when passenger i is at bus stand s. Subfigure (a) corresponds to Case 4, in which passenger i is at bus stand s, and there is no bus at bus stand s (b) corresponds to Case 5, in which passenger i is at bus stand s, and there is a bus at bus stand s.

Case 4, shown in Figure 3.8(a), considers the configuration in which passenger i is at bus stand s, bus b is not at bus stand s so that $x_{b\ell_s t^-} = 0$, and there are no other buses at bus stand s so that

$$\sum_{b=1}^{N_B} x_{b\ell_s t^-} = 0,$$

as summarized in Table 3.16.

Inputs					**Constraints on z_{ist^+}**		**Output**
z_{ist^-}	y_{ibt^-}	$x_{b\ell_s t^-}$	$\sum_{b=1}^{N_B} x_{b\ell_s t^-}$	$\sum_{\substack{b'=1 \\ b'\neq b}}^{N_B} y_{ib't^-}$	(2.31)	(2.32)	z_{ist^+}
1	0	0	0	0	≥ 1	≤ 1	1

Table 3.16: Case 4: Evaluation of passenger-location constraints when passenger i is at bus stand s and there is no bus at bus stand s.

From Table 3.16, constraint (2.31) makes $z_{ist^+} = 1$. Thus, when passenger i is at a bus stand s and no bus is present at bus stand s, the passenger must remain at bus stand s. This ensures that passengers cannot teleport from one bus stand to another and can only move through bus-mediated movements.

Case 5, illustrated in Figure 3.8(b), considers the configuration in which passenger i is at bus stand s and bus b is also at bus stand s. In this case $x_{b\ell_s t^-} = 1$ and

$$\sum_{b=1}^{N_B} x_{b\ell_s t^-} = 1.$$

Inputs					**Constraints on z_{ist^+}**		**Output**
z_{ist^-}	y_{ibt^-}	$x_{b\ell_s t^-}$	$\sum_{b=1}^{N_B} x_{b\ell_s t^-}$	$\sum_{\substack{b'=1 \\ b'\neq b}}^{N_B} y_{ib't^-}$	(2.31)	(2.32)	z_{ist^+}
1	0	1	1	0	≥ 0	$\leq 1\frac{1}{2}$	0 or 1

Table 3.17: Case 5: Evaluation of passenger-location constraints when passenger i is at bus stand s and there is a bus b at bus stand s.

As summarized in Table 3.17, neither constraint is binding, and therefore $z_{ist^+} \in \{0, 1\}$. Passenger i can remain at the bus stand or board the bus.

Multiple configurations of Case 5 are possible. Additional buses may also be present at bus stand s, increasing $\sum_{b=1}^{N_B} x_{b\ell_s t^-}$. This affects only the lower bound enforced by constraint (2.31),

$$z_{ist^+} \geq 1 - \sum_{b=1}^{N_B} x_{b\ell_s t^-}.$$

If all buses are present at bus stand s, the lower bound becomes $z_{ist^+} \geq 1 - N_B$, which is non-binding for a binary variable. Constraint (2.32) continues to allow $z_{ist^+} \leq 1\frac{1}{2}$.

Collectively, the two cases in Category II demonstrate that passengers cannot leave their current bus stand when there is no bus for them to board at that bus stand. When passenger i is at bus stand s and one or more buses are also at bus stand s at t^-, passenger i can remain at bus stand s or board a bus at t^+. When passenger i is at bus stand s and no bus is present at s at t^-, constraints (2.31) and (2.32) ensure that the passenger does not leave the bus stand at t^+.

3.4.3 Category III: Passenger at Another Bus Stand s'

The final category, illustrated in Figure 3.9, consists of two cases in which passenger i is at another bus stand $s' \neq s$ at t^-. In this configuration, $z_{ist^-} = 0$ and by constraint (2.21), $y_{ibt^-} = 0$. These cases demonstrate that passenger i at bus stand $s' \neq s$ at t^- cannot suddenly appear at bus stand s at t^+ since buses do not move between t^- and t^+.

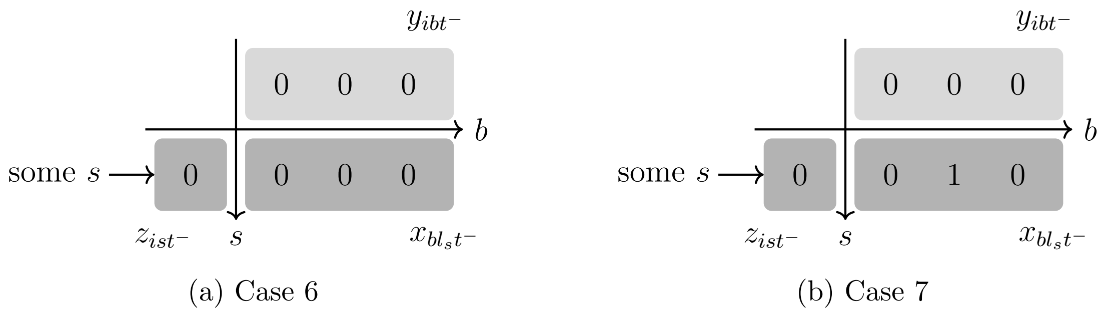


Figure 3.9: Category III configurations: Representation of passenger-bus, passenger-location and bus-location variables at t^- when passenger i is at bus stand $s' \neq s$. Subfigure (a) corresponds to Case 6, in which no bus is at bus stand s and (b) corresponds to Case 7, in which a bus is at bus stand s.

Case 6, shown in Figure 3.9(a), demonstrates the configuration in which passenger i is at some bus stand $s' \neq s$ and no bus is at bus stand s at t^-. The corresponding values are summarized in Table 3.18

Inputs					**Constraints on z_{ist^+}**		**Output**
z_{ist^-}	y_{ibt^-}	$x_{b\ell_st^-}$	$\sum_{b=1}^{N_B} x_{b\ell_st^-}$	$\sum_{\substack{b'=1 \\ b'\neq b}}^{N_B} y_{ib't^-}$	(2.31)	(2.32)	z_{ist^+}
0	0	0	0	0	≥ 0	≤ 0	0

Table 3.18: Case 6: Evaluation of passenger-location constraints when passenger i is at another bus stand $s' \neq s$ and there is no bus at bus stand s.

From Table 3.18, constraint (2.32) enforces that passenger i at bus stand $s' \neq s$ at t^- does not jump to bus stand s at t^+. Since passengers can only move through bus-mediated movements and buses do not move between t^- and t^+, if a passenger is at $s' \neq s$ at t^-, they must remain at $s' \neq s$ at t^+.

Case 7, shown in Figure 3.9(b), is slightly different, but demonstrates a similar result. Passenger i is still at bus stand $s' \neq s$, but now there is a bus at bus stand s. This configuration yields the variables summarized in Table 3.19.

As expected the constraints do not allow passenger i to jump to bus stand s, because constraint (2.32) makes $z_{ist^+} = 0$. Even if all the buses were at bus stand s,

Inputs					Constraints on z_{ist^+}		Output
z_{ist^-}	y_{ibt^-}	$x_{b\ell_s t^-}$	$\sum_{b=1}^{N_B} x_{b\ell_s t^-}$	$\sum_{\substack{b'=1 \\ b'\neq b}}^{N_B} y_{ib't^-}$	(2.31)	(2.32)	z_{ist^+}
0	0	1	1	0	≥ -1	$\leq \frac{1}{2}$	0

Table 3.19: Case 7: Evaluation of passenger-location constraints when passenger i is at another bus stand $s' \neq s$ and there is a bus at bus stand s.

that would only affect constraint (2.31), making the lower bound increasingly negative, while constraint (2.32) continues to enforce $z_{ist^+} = 0$.

Collectively, the cases in Category III demonstrate that a passenger at bus stand $s' \neq s$ at t^- cannot jump to bus stand s at t^+. Passenger location changes remain strictly mediated by bus movements and since buses do not move between t^- and t^+, passengers cannot move from bus stand $s' \neq s$ at t^- to bus stand s at t^+.

This chapter verified the constraint formulation by demonstrating that the constraints collectively enforce admissible system behavior and eliminate infeasible assignments. Through case-based analysis and illustrative examples, it has been shown that the constraints correctly govern bus-route assignments, bus-location evolution, passenger-bus, and passenger-location assignments. With the validity of the formulation established, the next chapter applies the model to simulation scenarios in order to examine system performance and quantify the impact of the proposed approach.

Chapter 4

Simulation Results

This chapter presents and discusses the results obtained from the implementation of the proposed optimization model across several simulation scenarios. The first seven scenarios are deliberately constructed to clearly illustrate key behaviors of the model, including transfer synchronization, dynamic assignment of buses across routes, sensitivity to passengers' desired arrival times, utilization of a heterogeneous fleet, responsiveness to objective function weights, and adherence to bus capacity constraints.

After establishing these qualitative insights, the final section isolates the impact of dynamic bus assignment by performing a quantitative comparison against a fixed-route assignment baseline. The comparison evaluates passenger travel time cost, percentage of passengers arriving late, and the number of route assignments completed by buses. This sections considers larger demand instances to assess the performance of the proposed framework under more operationally challenging conditions.

4.1 Simulation Setup

All simulations were conducted using the route network illustrated in Figure 4.1. The network was intentionally designed with overlapping route segments, transfer nodes, and varying route lengths. These features create a setting in which scheduling becomes non-trivial, requiring the model to reason about bus timing, sequencing, and route assignment, as well as passenger-bus assignments. Together, these interacting decisions significantly impact overall system performance.

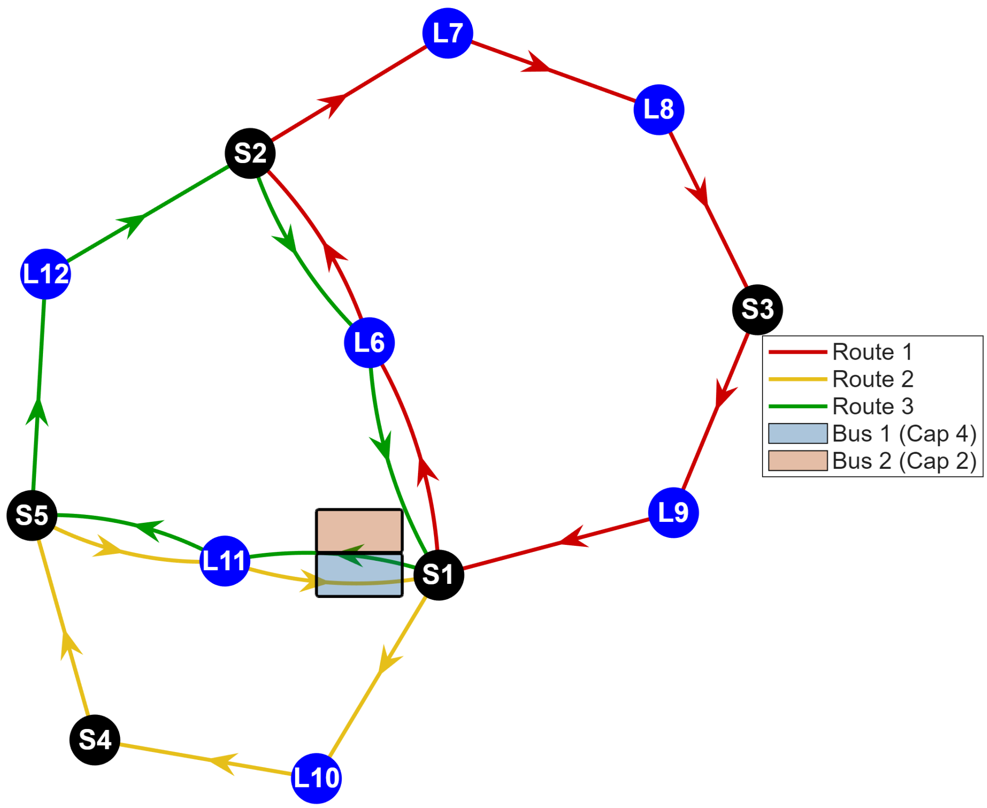


Figure 4.1: Simulation network consisting of five bus stands, twelve locations, and three routes with overlapping segments. The network supports passenger transfers and dynamic allocation of two heterogeneous buses.

The network is defined over a set of bus stands $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$ with $N_S = 5$, and a set of locations $\mathcal{L} = \{L_1, L_2, \dots, L_{12}\}$ with $N_L = 12$. The system operates over routes $\mathcal{R} = \{R_0, R_1, R_2, R_3\}$ which are given by

$$R_1 = (S_1, L_6, S_2, L_7, L_8, S_3, L_9, S_1), \quad E_1 = 8$$

$$R_2 = (S_1, L_{10}, S_4, S_5, L_{11}, S_1), \quad E_2 = 6$$

$$R_3 = (S_1, L_{11}, S_5, L_{12}, S_2, L_6, S_1), \quad E_3 = 7.$$

Among the defined routes, route R_1 is the longest and route R_2 is the shortest. As defined in Section 2.1, assigning a bus to route R_0 indicates that the bus is not assigned to any route. Furthermore, based on the definition of locations in Section 2.1, each edge in a route is traversed in one discrete time step.

The system considers a heterogeneous fleet with buses that differ in both capacity and operating cost. Unless otherwise stated, for the simulation results referenced in Section 4.2, bus B_1 has capacity $C_1 = 4$ and operating cost $c_1 = 1$, while bus B_2 has capacity $C_2 = 2$ and operating cost $c_2 = 0.5$, which is half that of the larger bus.

The planning horizon is set to $N = 20$ discrete time steps. Unless otherwise stated, passenger demand data is generated synthetically. Passenger origins and destinations are drawn from $\mathcal{S}$, while desired arrival times are drawn from the interval $[0, 20]$. This setup produces a diverse range of routing and scheduling scenarios, including cases requiring transfers and cases with competing demand across multiple routes.

The objective function minimizes a weighted sum of passenger travel time cost and bus operating cost. In all scenarios except one, passenger travel time cost is assigned a weight of 0.8, while bus operating cost is assigned a weight of 0.2. Alternative weighting approaches, including Criterion Impact Loss (CILOS) (Ayan et al., 2023) and Method based on the Removal Effects of Criteria (MEREC) (Keshavarz-Ghorabaee et al., 2021), were explored during model development. However, a fixed operator-specified weighting scheme was ultimately adopted to provide a clear and controlled illustration of the effects of prioritizing one objective over the other.

An asymmetric cost structure is applied to passenger arrival times to prioritize punctuality. Each time step of early arrival is rewarded by 1 unit, while each time step of late arrival incurs a penalty of 2 units. This asymmetry is introduced to encourage the model to deliver passengers before their desired arrival times while preventing scenarios in which improvements in early arrivals mask poor performance in late arrivals.

All optimization problems were solved using MATLAB (R2025b)'s integer linear programming solver (The MathWorks, Inc., 2025). The solver was configured with a relative optimality gap tolerance of 10^{-4} which determines how close the obtained solution must be to the best known bound before termination. All computations were performed on a Dell Inspiron 7706 with 16 GB of RAM and a 2.8 GHz Intel Core i7 processor.

4.2 Key System Behaviors

This section presents eight simulation scenarios designed to illustrate key behaviors of the proposed model and to verify that the resulting solutions remain consistent with fundamental physical and operational constraints. The scenarios are intentionally constructed with a small number of passengers to allow for clear interpretation of system dynamics. Each scenario is accompanied by animations that enable direct observations of the system evolution and resulting decisions.

Due to space limitations, only selected time steps that capture key events are presented. Complete time-step animations for all presented scenarios are available in the GitHub repository (Mugwadi, 2026).

4.2.1 Scenario 1: Nominal System Operation

This scenario considers a system with one passenger, $i = 1$, and one bus, $b = 1$. The bus has capacity $C_1 = 4$ and operating cost $c_1 = 1$. The passenger's demand parameters are given in Table 4.1 and these include the origin bus stand, destination bus stand, arrival time at origin, and desired arrival time at the destination bus stand. The passenger originates at bus stand 2 at time 0 and desires to arrive at bus stand 3 by time 10. This setting provides a minimal environment in which the fundamental mechanics of the model can be cleanly observed.

Passenger i	X_i	Y_i	I_i	J_i
1	2	3	0	10

Table 4.1: Passenger demand parameters for Scenario 1, specifying origin X_i, destination Y_i, arrival time at the origin bus stand I_i, and desired arrival time at the destination bus stand J_i.

Figure 4.3 and 4.4 show selected time steps from the simulation results that illustrate the operation of the system. In Figure 4.3(a), the initial conditions at 0^- are satisfied. The bus is at the central depot, consistent with the initial conditions imposed on x_{b10^-}. In addition, the bus is in the idle state, as enforced by the initial conditions on w_{b00^-}. The route assignment shown in Figure 4.2 confirms that the bus is in the idle state at the start of the planning horizon.

The passenger appears in the system at their origin bus stand at 0^- since $I_1 = 0$. Passengers are visually represented in the animation according to their state. A passenger enters the system at their specified time I_i, appearing as a green circle labeled with index i. The passenger remains green, until arriving at their destination at time K_i, at which point they will be shown in red for one time instance to indicate successful delivery. After this, the passenger is removed from the animation.

Figure 4.3(b) shows the system at 0^+. The bus remains at the central depot, consistent with the event ordering defined in Section 2.2. Recall that buses arrive at their locations at t^- and move between t^+ and $(t+1)^-$. As a result, the bus does not change position within the same time step, i.e., between 0^- and 0^+.

Since bus 1 is at the central depot, it is eligible to be assigned to a route. As shown in Figure 4.2, the bus is assigned to route 1 at 0^+. In Figure 4.3(c), the bus begins traversing this route and moves from the central depot to location 6.

At 2^-, shown in Figure 4.3(d), bus 1 arrives at the passenger's origin bus stand. The passenger boards bus 1 at 2^+, as shown in Figure 4.3(e), since both passenger 1 and bus 1 are at the same bus stand. Passenger 1 remains on the bus until 5^-, shown in Figure 4.3(f), when the bus reaches their destination bus stand. At 5^+, shown in Figure 4.4(a), the passenger disembarks, indicated by the change in color from green to red, before being removed from the animation in Figure 4.4(b).

The bus continues traversing route 1 and returns to the central depot at time step 7^-, as shown in Figure 4.4(c), where it remains for the rest of the planning horizon as shown in Figure 4.2. Throughout this process, the bus does not deviate from the assigned route or terminate service prematurely. At all time steps the bus occupies exactly one location, and its movement respects the ordered structure of the route.

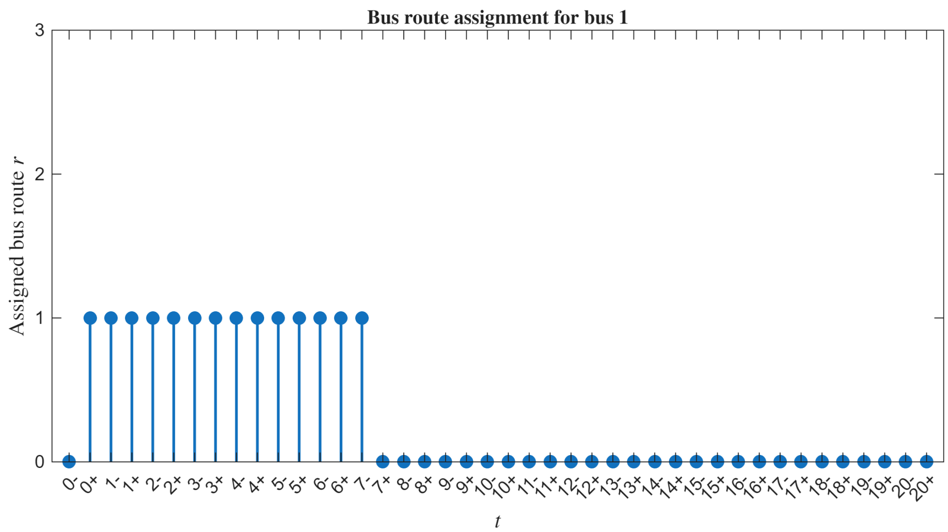
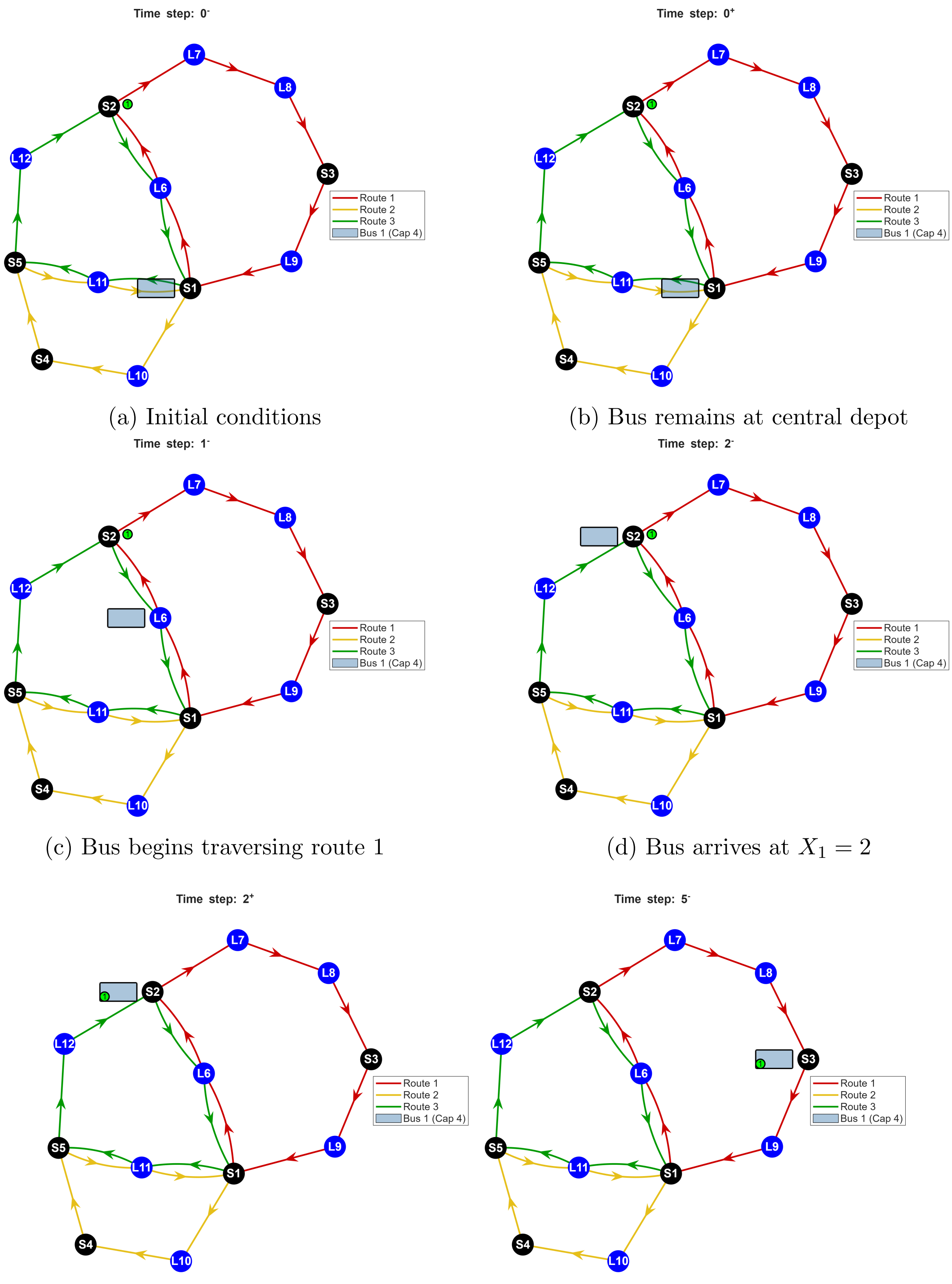


Figure 4.2: Bus route assignments for bus 1 over the planning horizon, showing assignment to route 1 and transition to the idle state after completing the route and returning to the central depot.



(a) Initial conditions

(b) Bus remains at central depot

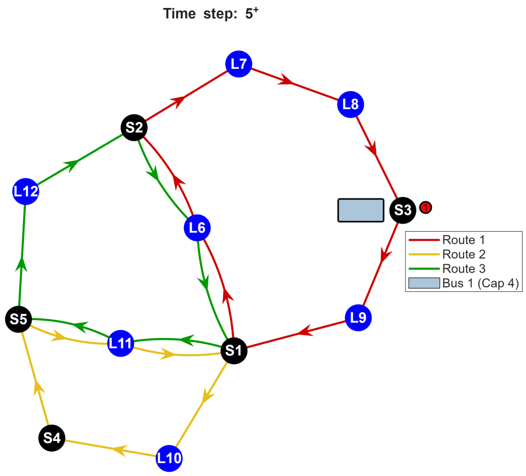
(c) Bus begins traversing route 1

(d) Bus arrives at $X_1 = 2$

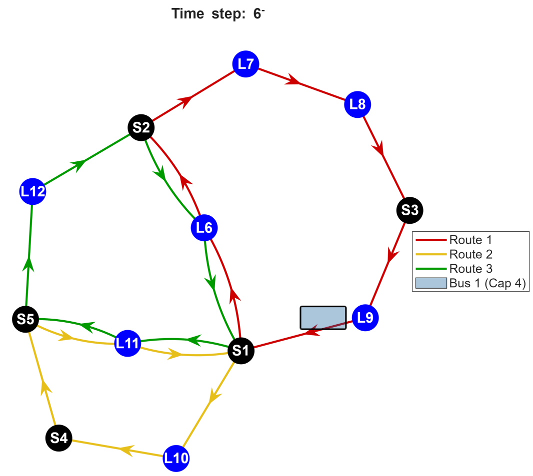
(e) Passenger boards bus

(f) Bus reaches $Y_1 = 3$

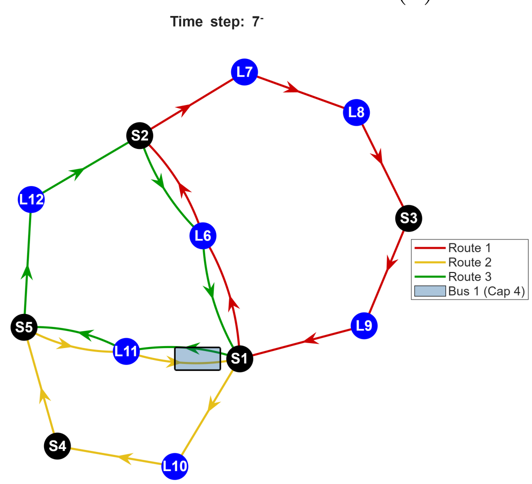
Figure 4.3: Selected time steps illustrating system operation in Scenario 1.



(a) Passenger disembarks bus



(b) Bus continues along route 1

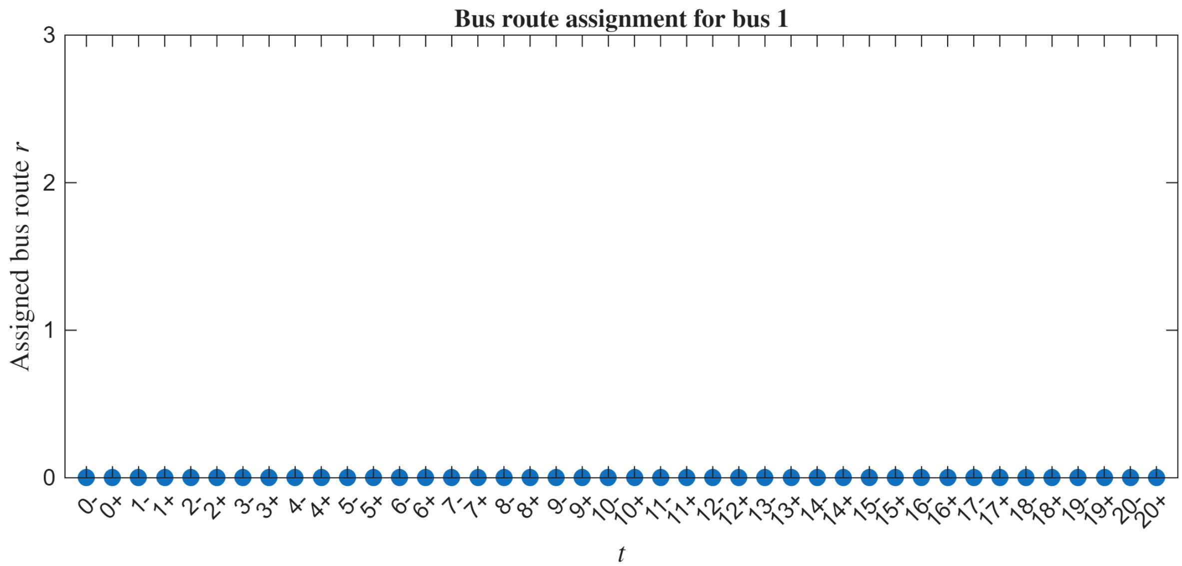


(c) Bus returns to the central depot

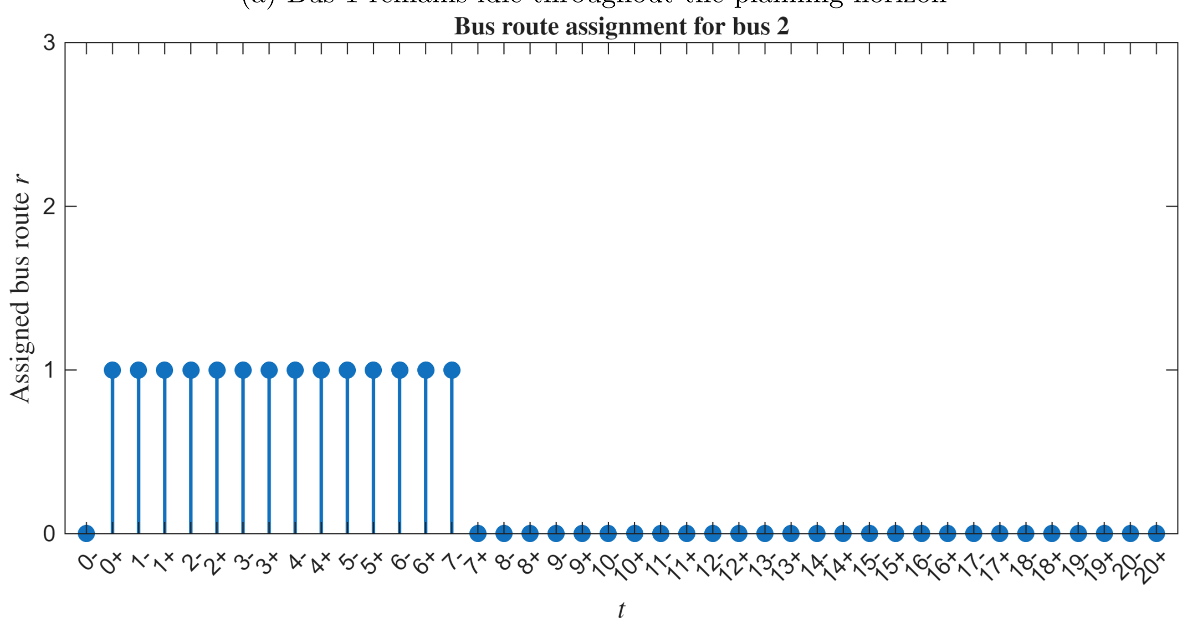
Figure 4.4: Selected time steps illustrating system operation in Scenario 1 (continued).

4.2.2 Scenario 2: Heterogeneous Fleet Utilization

This scenario considers the same passenger demand as Scenario 1, with identical origin, destination, and timing parameters. The system is extended by introducing a second bus $b = 2$, in addition to the original bus $b = 1$. Bus 2 has capacity $C_2 = 2$ and operating cost $c_2 = 0.5$, which is lower than that of bus 1. This setup is used to evaluate how the system allocates service in the presence of a heterogeneous fleet. In particular, the scenario examines whether the system selects the lower-cost bus when both buses are capable of serving the passenger.

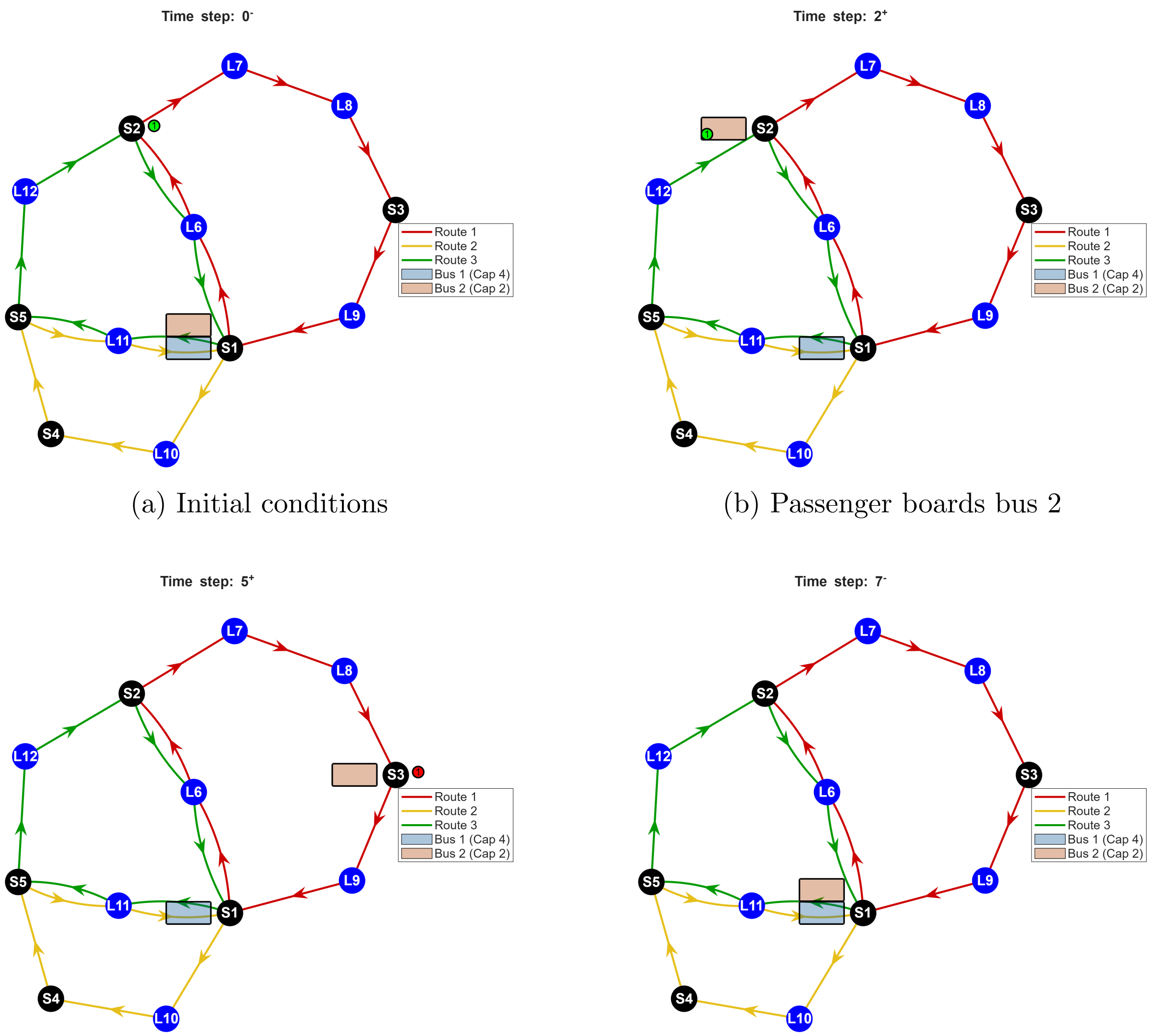


(a) Bus 1 remains idle throughout the planning horizon



(b) Bus 2 serves the passenger before returning to the idle state

Figure 4.5: Route assignments for buses 1 and 2, illustrating the utilization of the lower-cost bus for passenger service.



(a) Initial conditions

(b) Passenger boards bus 2

(c) Passenger disembarks bus 2

(d) Bus 2 returns to the central depot

Figure 4.6: Selected time steps illustrating system operation in Scenario 2, demonstrating allocation of service to the lower-cost bus while the other bus remains idle.

As shown in Figure 4.5(a), bus 1 remains idle throughout the planning horizon, while bus 2 is assigned to route 1 from 0^+ to 7^-, as shown in Figure 4.5(b). In contrast to Scenario 1, where the passenger is served by bus 1, the introduction of a second bus with lower operating costs results in bus 2 being selected to serve the passenger. This assignment is reflected in the simulation frames shown in Figure 4.6, where passenger 1 boards bus 2 while bus 1 remains idle at the central depot. This behavior is consistent with the objective function, as bus 2 has a lower operating cost. As a result, when presented with a heterogeneous fleet, the system allocates service in a manner that minimizes operating cost.

4.2.3 Scenario 3: Transfer Synchronization

This scenario considers the same bus configuration as Scenario 2, with two buses and their corresponding capacity and operating cost parameters. A single passenger is again considered, however, the origin bus stand is modified as shown in Table 4.2. The passenger now originates at bus stand 5 and like Scenario 2, they wish to travel to bus stand 3, arriving at the origin at time step 0 and desiring arrival at their destination by time step 10.

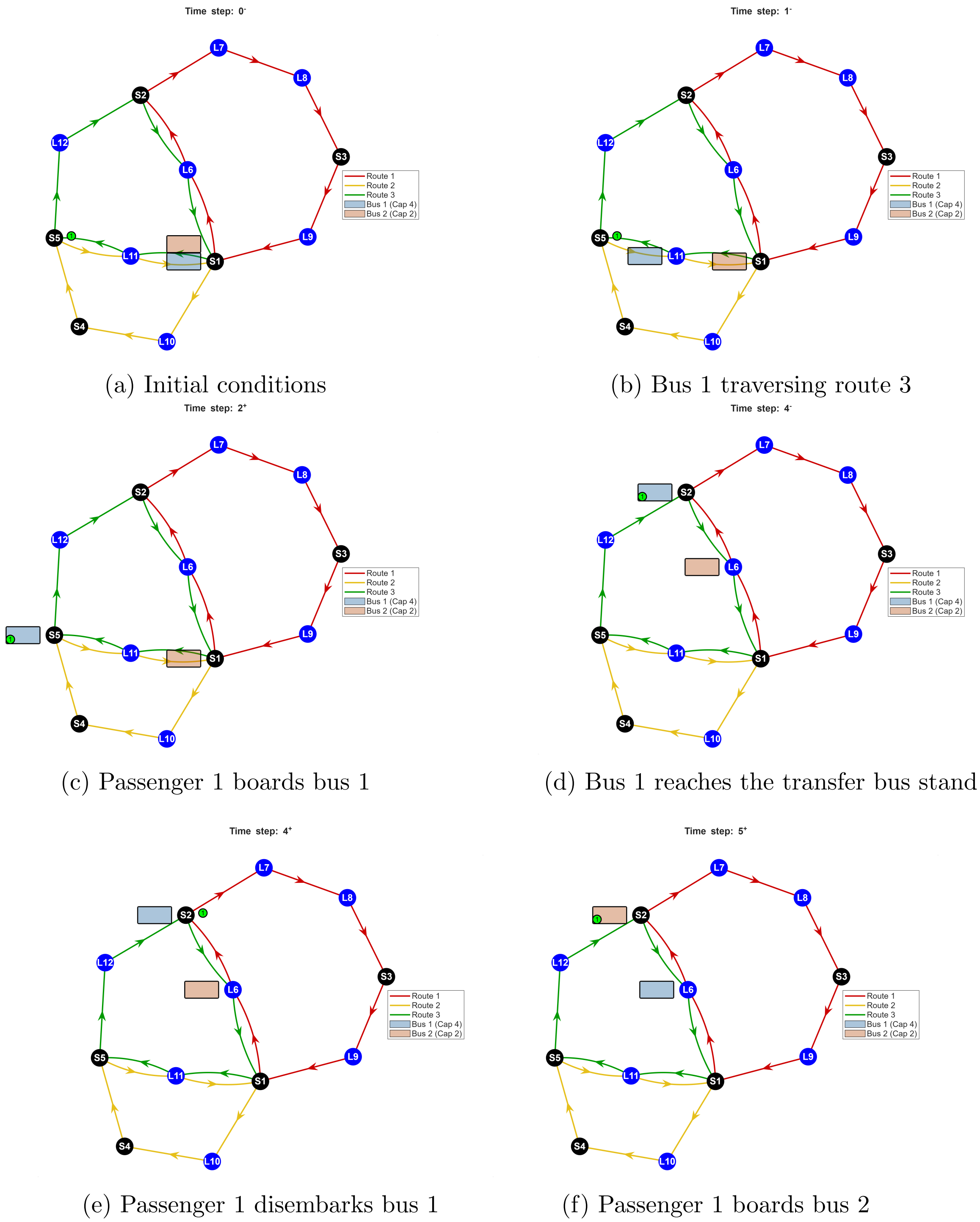
Under this configuration, the passenger cannot be served by a single route. As a result, a transfer between routes is required. This scenario is designed to evaluate how the system coordinates transfers to ensure successful passenger delivery.

Passenger i	X_i	Y_i	I_i	J_i
1	5	3	0	10

Table 4.2: Passenger demand parameters for Scenario 3, specifying origin bus stand X_i, destination bus stand Y_i, arrival time at the origin bus stand I_i, and desired arrival time at the destination bus stand J_i.

As shown in Figure 4.7(a), passenger 1 arrives at their origin bus stand at 0^-. By 1^-, shown in Figure 4.7(b), bus 1 is at location 11 along route 3, which includes the passenger's origin bus stand. At 2^+, the passenger boards bus 1, as shown in Figure 4.7(c), and by 4^-, bus 1 arrives at the transfer bus stand, as shown in Figure 4.7(d). At this time, bus 2 has already begun traversing route 1 toward the same transfer bus stand.

At 4^+, shown in Figure 4.7(e), passenger 1 disembarks bus 1 and subsequently boards bus 2 at 5^+, as shown in Figure 4.7(f). Bus 1 continues along its route toward the



(a) Initial conditions

(b) Bus 1 traversing route 3

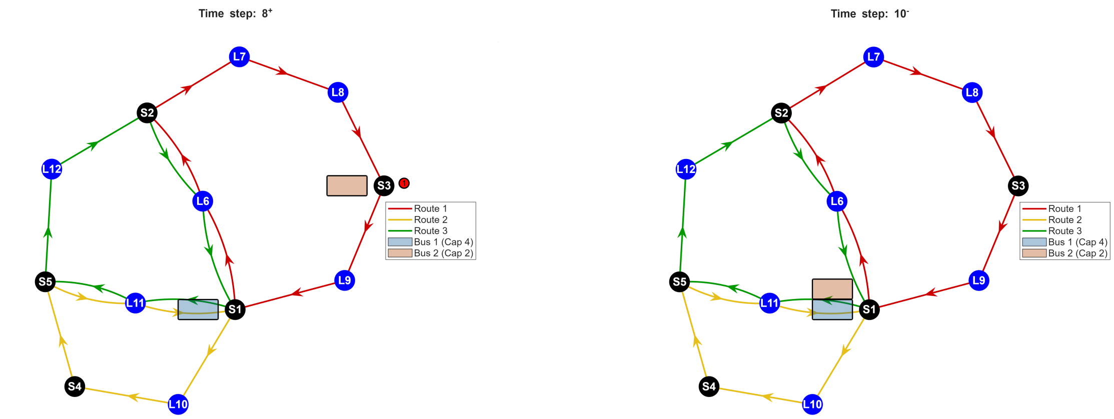
(c) Passenger 1 boards bus 1

(d) Bus 1 reaches the transfer bus stand

(e) Passenger 1 disembarks bus 1

(f) Passenger 1 boards bus 2

Figure 4.7: Selected time steps illustrating transfer synchronization in Scenario 3, where the passenger transfers between buses to reach the destination.



(a) Passenger 1 disembarks bus 2

(b) Bus 2 returns to central depot

Figure 4.8: Selected time steps illustrating transfer synchronization in Scenario 3, where the passenger transfers between buses to reach the destination (continued).

central depot. Bus 2 proceeds along route 1 and delivers passenger 1 to the destination bus stand, where the passenger disembarks at 7^+, as shown in Figure 4.8(a). By 10^-, bus 2 returns to the central depot, completing its route, as shown in Figure 4.8(b).

This scenario demonstrates that the system correctly synchronizes transfers between buses. Bus 2 is dispatched in advance of the passenger's arrival at the transfer bus stand, ensuring that the passenger can transfer without unnecessary delay. This behavior arises from the objective function, which incentivizes early arrival and leads the system to coordinate transfers so that the passenger reaches their destination before the desired time J_i.

It is also observed that the two buses do not arrive at the transfer bus stand at the same time. This is a result of the model design, which prevents passengers from transferring between buses within the same time step, thereby avoiding ambiguity in passenger state transitions.

Finally, the assignment of buses reflects the interaction between route structure and operating cost. Bus 1 serves the shorter route, while the lower-cost bus is assigned to the longer route. This indicates that the system balances operating cost and routing requirements when coordinating transfers.

4.2.4 Scenario 4: Multi-Route Service with Limited Fleet

This scenario is designed to demonstrate how allowing dynamic assignment of buses across routes enables two buses to effectively serve three routes. The same bus configuration as in Scenario 3 is maintained, with two buses and their corresponding capacity and operating cost parameters. Passenger 1 and its demand parameters are also unchanged.

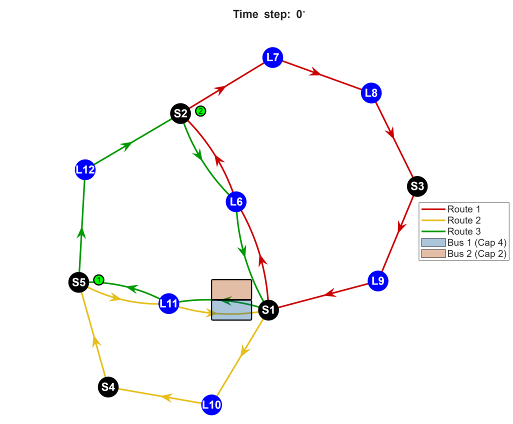
A second passenger is introduced, with demand parameters shown in Table 4.3. Passenger 2 intends to travel from bus stand 2 to bus stand 4, arriving at the origin at time step 0 and desiring arrival at the destination bus stand by time step 10. This demand configuration creates additional load on the system and requires coordination across multiple routes.

Passenger i	X_i	Y_i	I_i	J_i
1	5	3	0	10
2	2	4	0	10

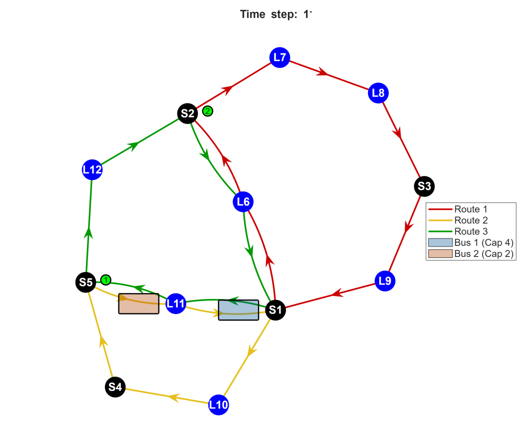
Table 4.3: Passenger demand parameters for Scenario 4, specifying origin bus stand X_i, destination bus stand Y_i, arrival time at the origin bus stand I_i, and desired arrival time at the destination bus stand J_i.

As shown in Figure 4.9(a), both passengers appear at their respective origin bus stands at 0^-. By 1^-, bus 2 is at location 11 along route 3, moving toward the origin bus stand of passenger 1, as shown in Figure 4.9(b). At 2^+, passenger 1 boards bus 2, as shown in Figure 4.9(c), and bus 2 continues along route 3.

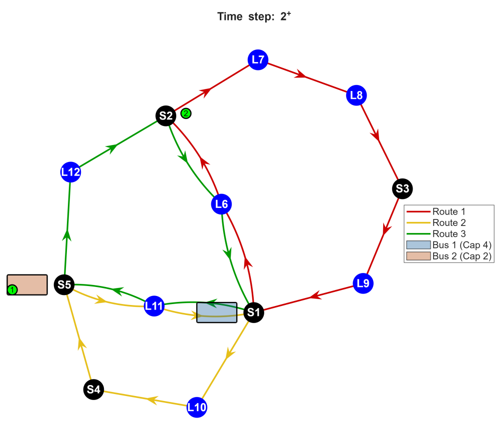
At 4^+, shown in Figure 4.9(d), passenger 1 disembarks bus 2 at the transfer bus stand. At this time, bus 1 has already begun moving toward the transfer bus stand along route 1. At 5^+, passenger 1 boards bus 1, as shown in Figure 4.9(e), while bus 2 continues toward the central depot carrying passenger 2. Passenger 2 disembarks at



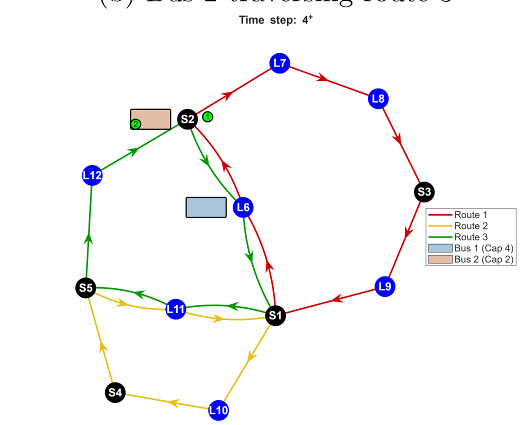
(a) Initial conditions



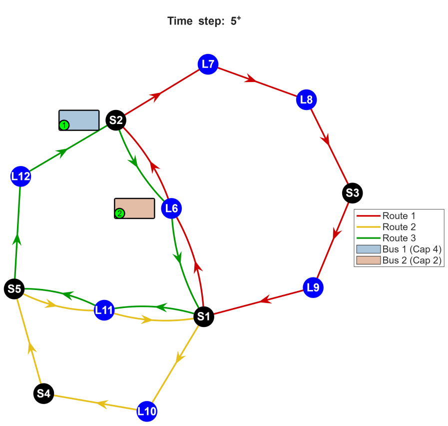
(b) Bus 2 traversing route 3



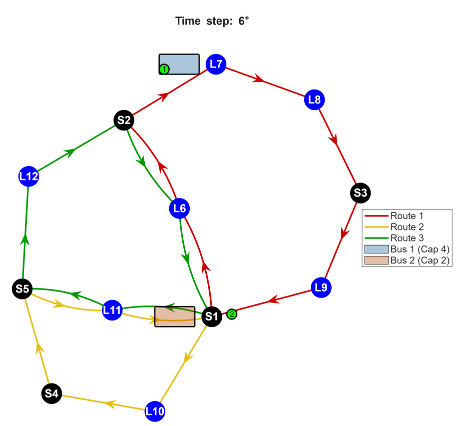
(c) Passenger 1 boards bus 2



(d) Passenger 1 disembarks bus 2, while passenger 2 boards bus 2

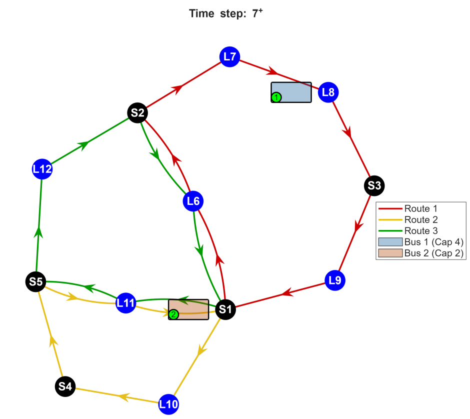


(e) Passenger 1 boards bus 1

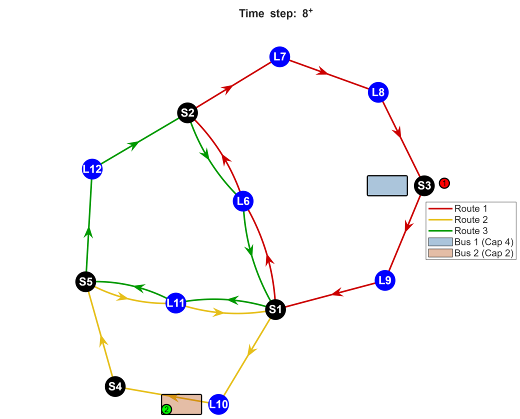


(f) Passenger 2 disembarks bus 2

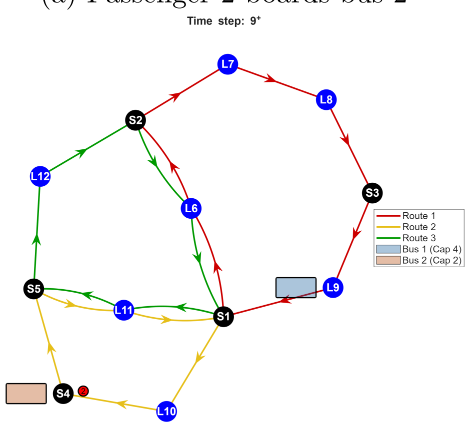
Figure 4.9: Selected time steps illustrating dynamic assignment of buses across routes in Scenario 4, where two buses are coordinated to serve multiple routes.



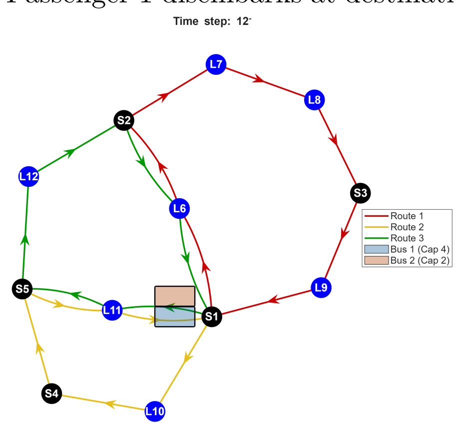
(a) Passenger 2 boards bus 2



(b) Passenger 1 disembarks at destination



(c) Passenger 2 disembarks at destination



(d) Both buses at central depot

Figure 4.10: Selected time steps illustrating dynamic assignment of buses across routes in Scenario 4, where two buses are coordinated to serve multiple routes (continued).

the central depot at 6^+, as shown in Figure 4.9(f), after which bus 2 transitions to the idle state upon completing its route.

At 7^+, passenger 2 boards bus 2 again, as shown in Figure 4.10(a). By 8^-, shown in Figure 4.10(b), bus 2 is traversing route 2, while passenger 1 disembarks at their destination. Passenger 2 is then delivered to their destination at 9^+, as shown in Figure 4.10(c). By 12^-, both buses are back at the central depot, as shown in Figure 4.10(d).

This scenario demonstrates that dynamic assignment enables a limited fleet to effectively serve multiple routes by reusing available buses. Bus 2 is initially assigned to route 3, and is later assigned to route 2, while bus 1 serves route 1.

This assignment also reflects the operating cost structure in the objective function. Bus 2, which has a lower operating cost, is initially assigned to the shorter route, while bus 1 is assigned to the longer route. This creates an opportunity for bus 2 to be assigned to an additional route while bus 1 remains in service. Ultimately, the lower-cost bus is utilized for a longer duration that the higher-cost bus, demonstrating the model balances route structure and operating cost when assigning buses.

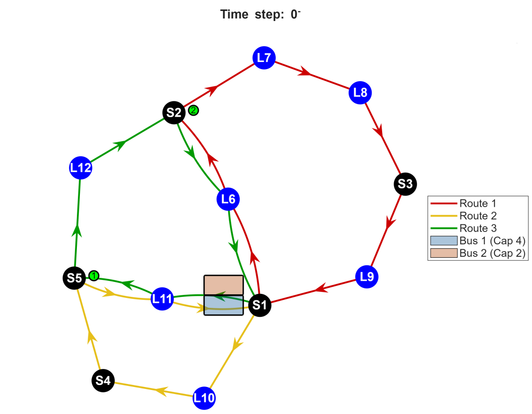
4.2.5 Scenario 5: Prioritization Based on Arrival Deadlines

This scenario is designed to evaluate whether the system prioritizes passengers with tighter time constraints when assigning buses. The same bus configuration and passenger set as Scenario 4 are considered with both passengers retaining their origin bus stands and arrival times at the origin. Passenger 1 also maintains the same destination and desired arrival time at the destination. As shown in Table 4.4, passenger 2's demand is modified so that both passengers have a common destination at bus stand 3. In addition, passenger 2 desires to arrive by time step 5, which is earlier than the desired arrival time of passenger 1.

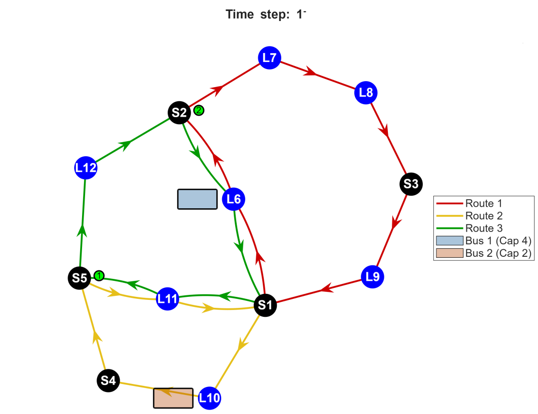
Passenger i	$\boldsymbol{X_i}$	$\boldsymbol{Y_i}$	$\boldsymbol{I_i}$	$\boldsymbol{J_i}$
1	5	3	0	10
2	2	3	0	5

Table 4.4: Passenger demand parameters for Scenario 5, specifying origin bus stand X_i, destination bus stand Y_i, arrival time at the origin bus stand I_i, and desired arrival time at the destination bus stand J_i.

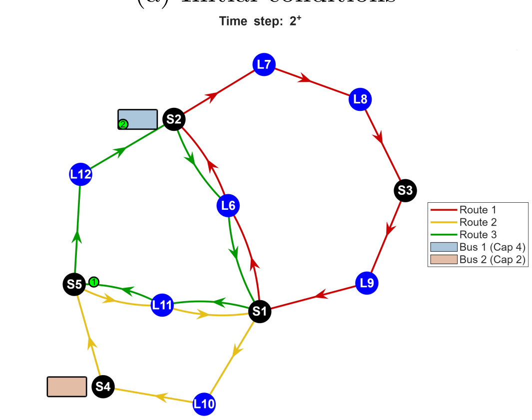
The passengers appear at their origin bus stands at 0^-, as shown in Figure 4.11(a). By 1^-, bus 2 is at location 10 along route 2, while bus 1 is at location 6 along route 1, as shown in Figure 4.11(b). At 2^+, passenger 2 boards bus 1, as shown in Figure 4.11(c),



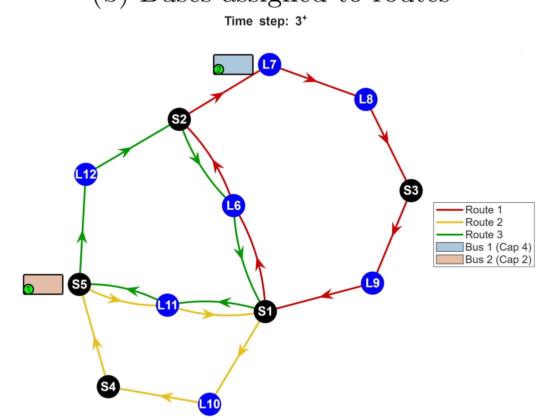
(a) Initial conditions



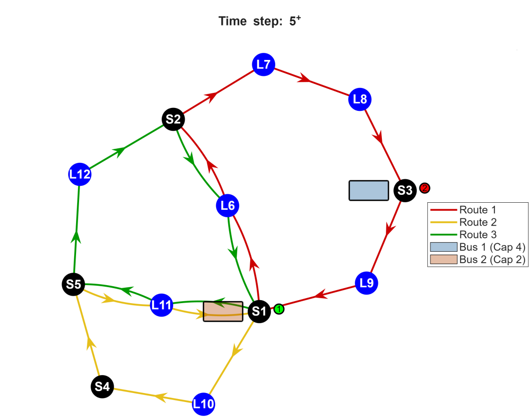
(b) Buses assigned to routes



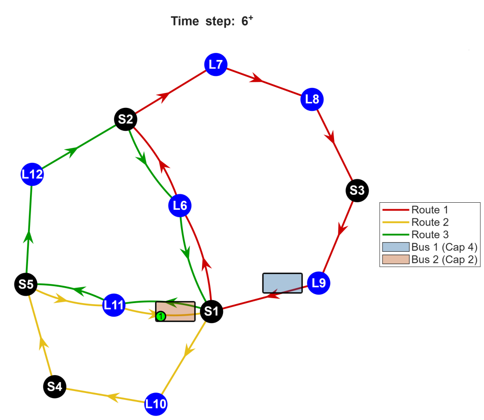
(c) Passenger 2 boards bus 1



(d) Passenger 1 boards bus 2

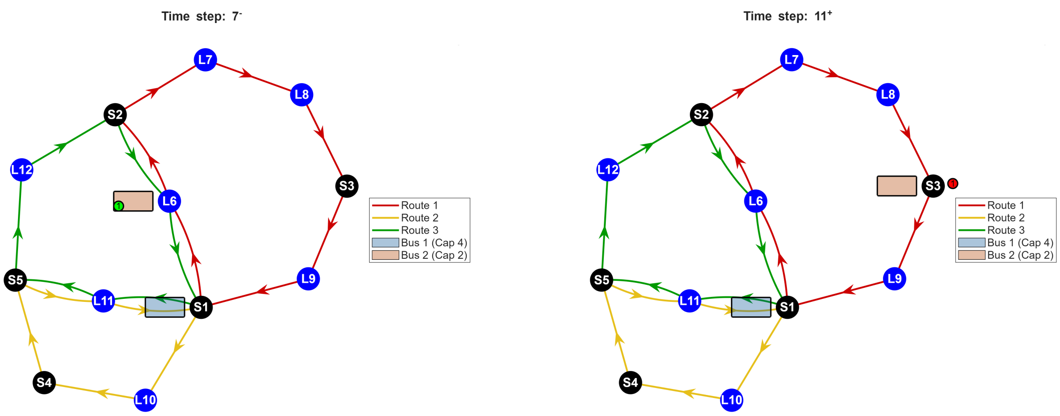


(e) Passenger 2 arrives on time



(f) Passenger 1 transfer at depot

Figure 4.11: Selected time steps illustrating system operation in Scenario 5, highlighting prioritization of the passenger with the tighter arrival deadline.



(a) Bus 2 traverses route 1 (b) Passenger 1 arrives at destination

Figure 4.12: Selected time steps illustrating system operation in Scenario 5, highlighting prioritization of the passenger with the tighter arrival deadline. (continued).

while passenger 1 boards bus 2 at 3^+, as shown in Figure 4.11(d).

Bus 1 then delivers passenger 2 to the destination bus stand at the desired arrival time, as shown in Figure 4.11(e), while passenger 1 disembarks at the central depot and awaits transfer. At 6+, passenger 1 boards bus 2, as shown in Figure 4.11(f), and by 7^-, bus 2 is at location 6 along route 1, as shown in Figure 4.12(a). Passenger 1 then disembarks bus 2 at 11^+, one time step after their desired arrival time, as shown in Figure 4.12(b).

Because desired arrival times are incorporated into the objective function, the system prioritizes passengers with tighter deadlines. In this scenario, passenger 2 is served directly to ensure on-time arrival, at the expense of additional routing and a minor delay for passenger 1.

An alternative assignment that minimizes travel time uniformly without considering desired arrival times, would assign bus 1 to route 2 to deliver passenger 1 to bus stand 2, after which bus 2 would be assigned to route 1 to serve both passengers and deliver them to their destination. This alternative would require fewer bus-route assignments and reduce overall travel time, compared to the solution shown in Scenario 5. However, it would result in passenger 2 missing their desired arrival time by three time steps and passenger 1 being early by 2 time steps.

The selected solution shown in Scenario 5 ensures that passenger 2 meets their deadline, while passenger 1 arrives 1 time step late. This demonstrates that the system is passenger schedule-aware and prioritizes passengers with tighter deadlines over uniform minimization of travel time.

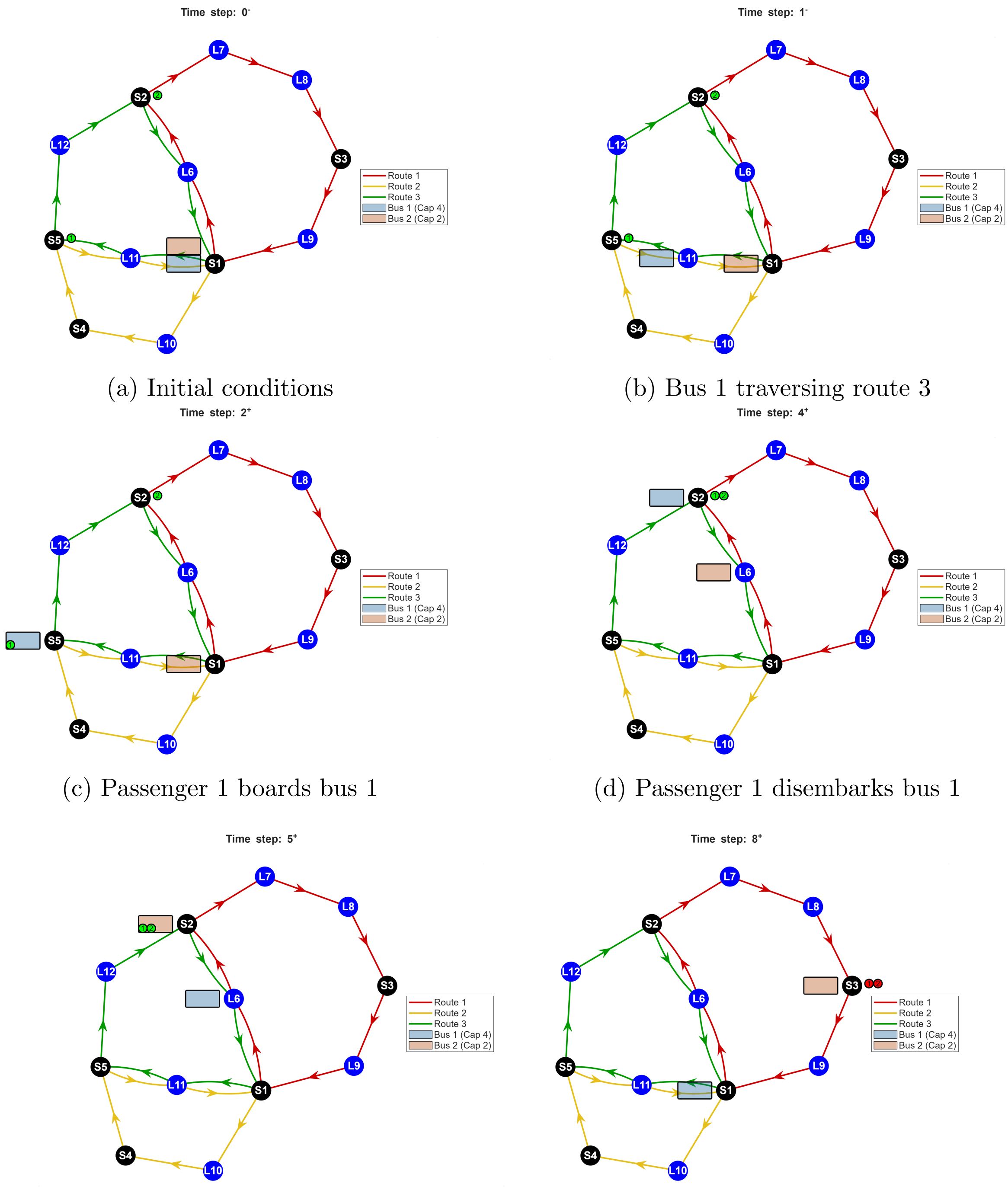
4.2.6 Scenario 6: Sensitivity to Changes in Objective Function Weights

This scenario considers the same bus configuration, passenger set, and demand parameters as in Scenario 5. The objective is to show how changing the objective function weights affects the resulting solution. In particular, the weight on bus operating cost is increased, while the weight on passenger travel time cost is reduced until a solution different from that in Scenario 5 is obtained.

By increasing the weight on bus operating cost to 0.38 and decreasing the weight on passenger travel time cost to 0.62, the system produces the solution shown in Figure 4.13, which reduces the number of times buses are assigned to active routes. This comes at the expense of missing passenger desired arrival times, demonstrating that increasing the emphasis on bus operating cost leads the model to favor more cost-efficient routing at the expense of service quality.

Both passengers appear at their origin bus stands at 0^-, as shown in Figure 4.13(a). By 1^-, bus 1 is at location 11 along route 3, moving toward passenger 1, as shown in Figure 4.13(b). At 2^+, passenger 1 boards bus 1 , as shown in Figure 4.13(c).

Bus 1 continues along route 3 and delivers passenger 1 at bus stand 2 at 4^+, as shown in Figure 4.13(d), while bus 2 is moving toward bus stand 2 to serve the passengers. At



(a) Initial conditions

(b) Bus 1 traversing route 3

(c) Passenger 1 boards bus 1

(d) Passenger 1 disembarks bus 1

(e) Passengers board bus 2

(f) Passengers delivered to destination

Figure 4.13: Selected time steps illustrating system operation in Scenario 6, highlighting the impact of increased emphasis on bus operating cost.

5^+, passengers 1 and 2 board bus 2, as shown in Figure 4.13(e), and are subsequently delivered to their destination at 8^+, as shown in Figure 4.13(f).

In this solution, both passengers are served using fewer coordinated routing decisions compared to Scenario 5. However, this comes at the expense of service quality. Passenger 2 is delivered three time steps late, while passenger 1 arrives two time steps early, in contrast to Scenario 5 where passenger 2 is delivered on time and passenger 1 arrives one time step late.

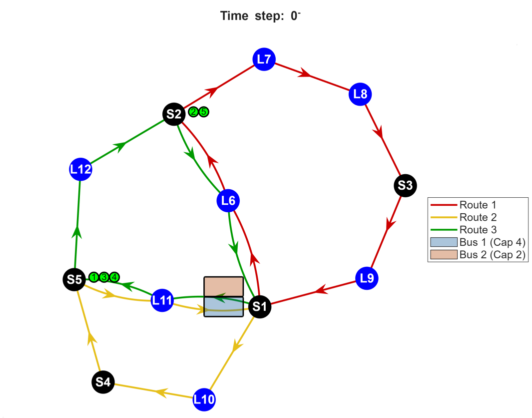
This outcome is consistent with the alternative solution discussed in Scenario 5, which minimizes routing effort but does not prioritize passenger schedules. By increasing the weight on bus operating cost, the model becomes less sensitive to meeting desired arrival times and instead favors solutions that reduce routing activity.

4.2.7 Scenario 7: Passenger Assignment Under Capacity Constraints

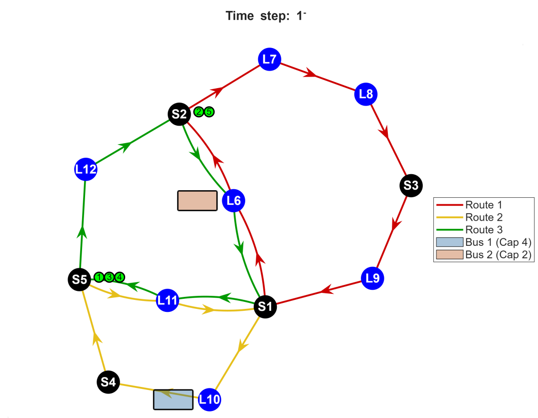
This scenario considers the same bus configuration as Scenario 6, but increases the total number of passengers to five. The objective function weights are restored to their baseline values, with a weight of 0.2 on bus operating cost and 0.8 on passenger travel time cost, consistent with Scenarios 1-5.

Passengers 1 and 2 retain their origin and destination bus stands, as well as their timing parameters at the origin and destination bus stands, as shown in Table 4.5. Passengers 3 and 4 are assigned the same origin, destination, and timing parameters as passenger 1, while passenger 5 shares the same parameters as passenger 2. This configuration creates overlapping demand to a common destination, resulting in passenger demand that exceeds the capacity of the buses.

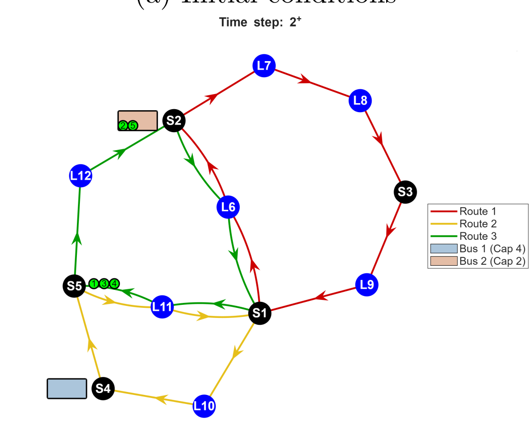
As shown in Figure 4.14(a), all passengers appear at their respective origin bus



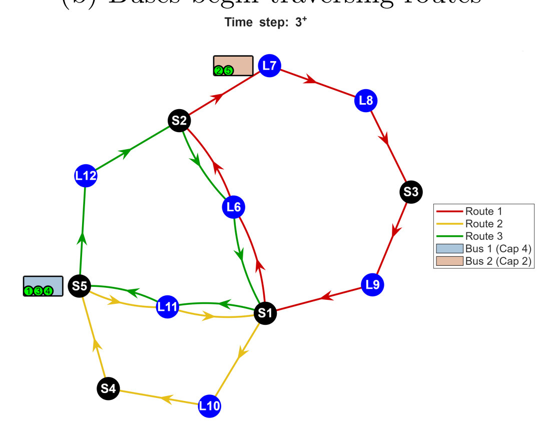
(a) Initial conditions



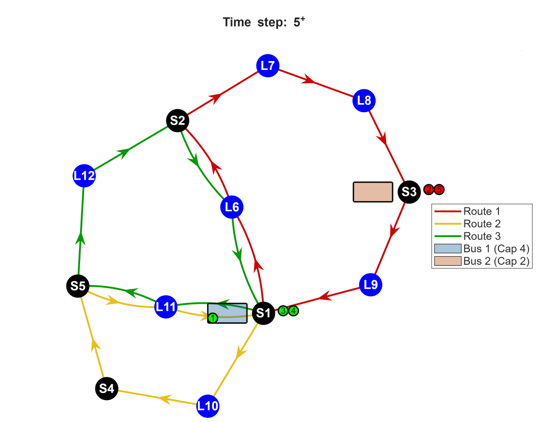
(b) Buses begin traversing routes



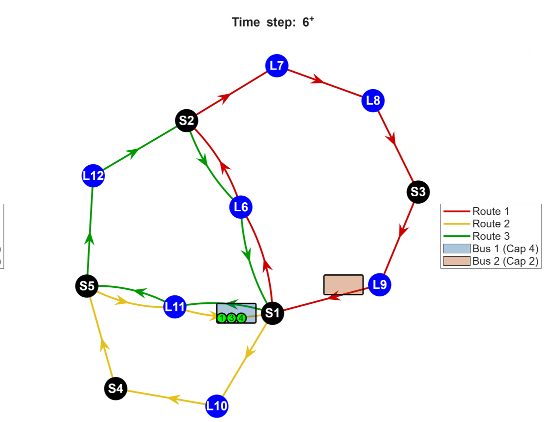
(c) Passengers 2 and 5 board bus 2



(d) Passengers 1, 3, and 4 board bus 1

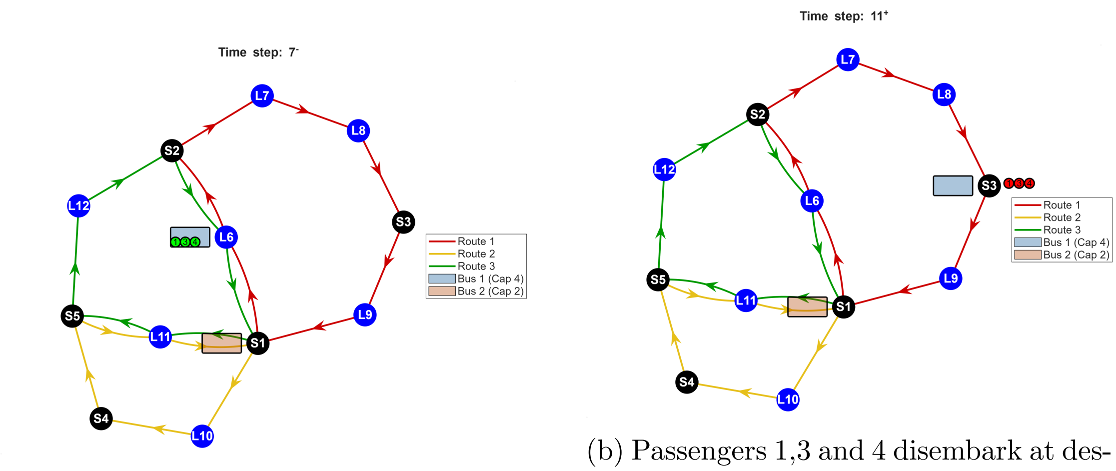


(e) Passengers 2 and 5 disembarks at destination



(f) Passengers 3 and 4 reboard after idle step

Figure 4.14: Selected time steps illustrating system operation in Scenario 7, highlighting capacity-constrained passenger assignment.



(a) Bus 1 traverses route 1

(b) Passengers 1,3 and 4 disembark at destination

Figure 4.15: Selected time steps illustrating system operation in Scenario 7 (continued).

Passenger i	X_i	Y_i	I_i	J_i
1	5	3	0	10
2	2	3	0	5
3	5	3	0	10
4	5	3	0	10
5	2	3	0	5

Table 4.5: Passenger demand parameters for Scenario 7, specifying origin bus stand X_i, destination bus stand Y_i, arrival time at the origin bus stand I_i, and desired arrival time at the destination bus stand J_i.

stands at 0^-. Three passengers are at bus stand 5, which exceeds the capacity of bus 2, while two passengers are at bus stand 2. Based on this, bus 1, which has higher capacity, is assigned to serve the route with higher demand, while bus 2 is assigned to serve the route with lower demand. As shown in Figure 4.14(b), bus 1 begins to move along route 2 to serve passengers 1,3, and 4, while bus 2 is assigned to route 2 to serve passengers 2 and 5.

At 2^+, passengers 2 and 5 board bus 2, as shown in Figure 4.14(c), and at 3^+, passengers 1,3 and 5 board bus 1, as shown in Figure 4.14(d). By 5^+, passengers 2 and 5 disembark bus 2 at their destination, as shown in Figure 4.14(e). At the same time, passengers 3 and 4 temporarily disembark at the central depot as bus 1 transitions to the idle state, which is required upon completing a route. At 6^+, these passengers board bus 1, as shown in Figure 4.14(f). By 7^-, bus 1 is traversing route 1 and passengers 1,3 and 4 are delivered to their destination by 11^+, as shown in Figure 4.15(b).

This scenario demonstrates that the system allocates buses in response to capacity constraints. Higher-capacity buses are assigned to routes with higher passenger demand, while lower-capacity buses are assigned to routes with lower demand. By allowing dynamic assignment across routes, the system is able to dynamically match available capacity to demand, ensuring that capacity constraints are satisfied while maintaining efficient operation.

This set of scenarios demonstrates that the proposed model captures a range of key system behaviors under varying operating conditions. The results show that the system is capable of dynamically assigning buses across routes, synchronizing transfers, and utilizing a heterogeneous fleet by assigning lower-cost buses where appropriate. In addition, the model effectively serves multiple routes with a limited fleet by reusing buses across different routes.

The system also prioritizes passengers with tighter arrival constraints and responds to changes in objective function weights, shifting between prioritizing service quality and reducing operating cost. Furthermore, it adapts to constraints such as vehicle capacity by assigning higher-capacity buses to routes with greater demand while maintaining feasible passenger assignments. Overall, these results highlight the flexibility of the model in balancing competing objectives while maintaining physically consistent system behavior.

4.3 Comparison With a Fixed-Route System

This section isolates the contribution of dynamic bus assignment by comparing the proposed model with a fixed-route system. The objective is to determine whether allowing dynamic assignment of buses across routes leads to improvements in service quality and operating efficiency.

The route network remains the same as that used in Section 4.2. The simulations are conducted using two buses with higher capacities relative to previous scenarios to accommodate the larger number of passengers considered in this section. Bus 1 has capacity $C_1 = 8$ and operating cost per unit time $c_1 = 1$ while bus 2 has capacity $C_2 = 4$ and operating cost per unit time $c_2 = 0.5$. The objective function weights are maintained as in Scenario 7, with passenger travel time cost weighted at 0.8 and bus operating cost at 0.2.

Five simulations are conducted, one with 5 passengers, two with 10 passengers and two with 15 passengers. Passenger demand is generated using a structured sampling procedure designed to produce randomized but physically consistent trip requests. Origin bus stands are selected according to a predefined probability distribution that places higher likelihood on the central depot and bus stand 2, creating concentrated demand patterns. Destinations are selected to ensure that origin and destination bus stands are not the same, with an additional preference toward the central depot whenever it is available.

For each passenger, the arrival time at the origin I_i is randomly generated within the planning horizon. The desired arrival time J_i is constructed by first computing the minimum travel time between the passenger's origin and destination, and then adding a randomly selected slack. In this setup, the slack can take values up to six time steps beyond the minimum feasible arrival time. Randomness in the generated passenger demand is controlled through a seeded random number generator, ensuring reproducibility across simulations.

For each experiment, simulations are conducted using both the proposed system and a fixed-route system. The fixed-route system is obtained by restricting buses to predetermined routes, removing the ability to dynamically assign buses across routes. All other system behaviors are retained, including transfer synchronization, deadline-based passenger prioritization, and decisions on when buses remain idle or begin serving routes.

In the fixed-route configuration, bus 1 is restricted to route 1 and the idle state, while bus 2 is restricted to route 2 and the idle state. Bus 1 is assigned to route 1 due to its higher capacity and the concentration of passenger demand along that route. Route 3 is excluded, as its segments overlap with those of routes 1 and 2, allowing passengers to be served without explicitly assigning buses to that route.

The comparison is carried out in three stages. First, results are examined at the individual passenger level across all simulations. For each passenger, the achieved arrival time is compared with the desired arrival time under both the proposed model and the fixed-route system. This allows identification of whether each passenger arrives early, on time, or late, as well as whether the passenger is served at all. From this, the percentage of passengers arriving early or late is computed for each system.

Next, aggregate performance metrics are evaluated for each simulation. In particular, total passenger travel time cost and total bus operating cost are compared between the proposed model and the fixed-route system to determine which approach provides better overall performance. Finally, animation results for the scenario with five passengers are presented to provide additional insight into the system behavior and to help explain the observed differences in results between the two approaches.

4.3.1 Passenger-Level Comparison

In this subsection, the performance of the proposed system is compared with that of the fixed-route system at the passenger level by examining the proportion of passengers who arrive early or late across simulations. Detailed results for individual passengers are presented in Table 4.8 for Simulations 1-3, and Table 4.9 for Simulations 4 and 5. For each passenger, the origin, destination, arrival time at origin, and desired arrival time at the destination are provided, along with the achieved arrival time under both systems, and the corresponding early or late deviation.

To summarize these results, Table 4.6 reports the total number of passengers arriving late in each simulation together with the total delay. The total delay for each simulation is computed by summing $(K_i - J_i)$ over all passengers who arrive late. For passenger 8 in Simulation 4, who is not delivered, the passenger is counted as late, but the delay is not included in the total delay values shown in Table 4.6.

Table 4.7 presents the corresponding results for early arrivals, including the total early arrival time, computed by summing up $(J_i - K_i)$ over all passengers who arrive early in each simulation. Passengers who arrive exactly at their desired time J_i are counted as early, as shown in Table 4.8 and Table 4.9.

		Proposed System		**Fixed-Route System**	
Simulation	$\boldsymbol{N_P}$	**Late**	**Total Delay**	**Late**	**Total Delay**
1	5	2	4	2	7
2	10	4	11	5	16
3	10	1	1	2	7
4	15	8	24	9	31
5	15	4	13	6	25
Total	**55**	19	53	24	86

Table 4.6: Summary of late passenger counts and total delay across simulations for the proposed and fixed-route systems.

The results in Table 4.6 and Table 4.7 show that the proposed model consistently

		Proposed System		Fixed-Route System	
Simulation	N_P	**Early**	**Total Early Time**	**Early**	**Total Early Time**
1	5	3	10	3	6
2	10	6	13	5	5
3	10	9	14	8	8
4	15	7	6	6	3
5	15	11	18	9	14
Total	**55**	36	61	31	36

Table 4.7: Summary of early passenger counts and total early arrival time across simulations for the proposed and fixed-route systems.

improves passenger-level outcomes relative to the fixed-route system. Across all simulations, the number of late passengers is reduced from 24 to 19, corresponding to an approximate 21% decrease. More significantly, the total delay experienced by late passengers is reduced from 86 to 53 time steps, representing a reduction of approximately 38%. This indicates that the proposed model not only decreases the number of late arrivals, but also reduces the severity of delays when they occur.

These improvements are observed consistently across simulations. For example, in Simulation 3, total delay is reduced from 7 to 1 time step, while in Simulation 5, delay is reduced from 25 to 13 time steps. Even in cases where the number of late passengers is similar, such as Simulation 4, the proposed model achieves a lower total delay, indicating more efficient routing and improved alignment with passenger schedules.

In addition, Table 4.7 shows that the number of passengers arriving early increases from 31 to 36, corresponding to an increase from 56.4% to 65.5% of all passengers. More importantly, the total early arrival time increases from 36 to 61 time steps, representing an approximate 69% increase. While the increase in the number of early passengers is modest, the magnitude of early arrivals is substantially higher under the proposed model, indicating that passengers not only arrive early more frequently, but also significantly earlier. Overall, the proposed model improves both the frequency and severity of passenger outcomes, reducing delays while enabling significantly earlier arrivals.

Simulation 1: $N_P = 5$										
					Proposed System			**Fixed-Route System**		
Passenger $\boldsymbol{i}$	$\boldsymbol{X_i}$	$\boldsymbol{Y_i}$	$\boldsymbol{I_i}$	$\boldsymbol{J_i}$	$\boldsymbol{K_i}$	**Early**	**Late**	$\boldsymbol{K_i}$	**Early**	**Late**
1	2	4	0	6	9		Yes (3)	12		Yes (6)
2	1	2	2	6	4	Yes (2)		4	Yes (2)	
3	2	1	4	10	6	Yes (4)		9	Yes (1)	
4	1	5	0	6	2	Yes (4)		3	Yes (3)	
5	2	2	2	6	7		Yes (1)	7		Yes (1)
Simulation 2: $N_P = 10$										
1	2	1	6	11	7	Yes (4)		10	Yes (1)	
2	2	1	4	10	7	Yes (3)		10	Yes (0)	
3	1	3	6	14	13	Yes (1)		16		Yes (2)
4	1	4	3	10	7	Yes (3)		7	Yes (3)	
5	5	1	9	11	10	Yes (1)		10	Yes (1)	
6	2	1	6	8	7	Yes (1)		10		Yes (2)
7	1	4	4	6	7		Yes (1)	10		Yes (4)
8	1	3	5	11	13		Yes (2)	16		Yes (5)
9	3	1	6	10	15		Yes (5)	10	Yes (0)	
10	4	2	3	10	13		Yes (3)	13		Yes (3)
Simulation 3: $N_P = 10$										
1	5	1	12	19	14	Yes (5)		14	Yes (5)	
2	3	1	12	14	14	Yes (0)		14	Yes (0)	
3	1	3	7	13	12	Yes (1)		12	Yes (1)	
4	5	4	3	11	11	Yes (0)		11	Yes (0)	
5	1	3	1	12	6	Yes (6)		12	Yes (0)	
6	2	5	7	17	18		Yes (1)	18		Yes (1)
7	1	2	7	10	9	Yes (1)		9	Yes (1)	
8	4	1	6	15	14	Yes (1)		14	Yes (1)	
9	2	1	3	8	8	Yes (0)		14		Yes (6)
10	2	3	5	12	12	Yes (0)		12	Yes (0)	

Table 4.8: Per-passenger comparison of arrival times for Simulations 1–3 under the proposed and fixed-route systems. For each passenger, the achieved arrival time is reported along with whether the passenger arrives early or late. Values in parentheses indicate the number of time steps the passenger is early or late relative to their desired arrival time.

Simulation 4: $N_P = 15$										
					Proposed System			**Fixed-Route System**		
Passenger i	X_i	Y_i	I_i	J_i	K_i	**Early**	**Late**	K_i	**Early**	**Late**
1	2	4	4	11	9	Yes (2)		14		Yes (3)
2	5	4	2	7	9		Yes (2)	8		Yes (1)
3	1	3	1	9	9	Yes (0)		9	Yes (0)	
4	3	1	7	13	11	Yes (2)		11	Yes (2)	
5	1	4	4	8	9		Yes (1)	8	Yes (0)	
6	1	4	6	9	9	Yes (0)		8	Yes (1)	
7	2	4	4	9	9	Yes (0)		14		Yes (5)
8	2	5	10	18	NOT DELIVERED			NOT DELIVERED		
9	5	3	4	9	18		Yes (9)	17		Yes (8)
10	3	2	4	9	15		Yes (6)	14		Yes (5)
11	2	1	4	8	6	Yes (2)		11		Yes (3)
12	2	5	6	11	14		Yes (3)	15		Yes (4)
13	1	3	3	9	9	Yes (0)		9	Yes (0)	
14	2	1	6	10	11		Yes (1)	11		Yes (1)
15	4	3	6	16	18		Yes (2)	17		Yes (1)
Simulation 5: $N_P = 15$										
1	1	5	4	12	9	Yes (3)		9	Yes (3)	
2	2	3	8	14	16		Yes (2)	17		Yes (3)
3	1	2	2	9	5	Yes (4)		5	Yes (4)	
4	2	1	10	13	13	Yes (0)		19		Yes (6)
5	2	1	7	13	13	Yes (0)		19		Yes (6)
6	2	1	4	7	6	Yes (1)		10		Yes (3)
7	2	1	3	11	6	Yes (5)		10	Yes (1)	
8	5	1	1	4	6		Yes (2)	5		Yes (1)
9	3	4	0	8	16		Yes (8)	14		Yes (6)
10	1	4	7	15	16		Yes (1)	14	Yes (1)	
11	1	5	9	17	17	Yes (0)		15	Yes (2)	
12	1	3	10	17	16	Yes (1)		17	Yes (0)	
13	5	1	8	13	13	Yes (0)		11	Yes (2)	
14	1	2	3	6	5	Yes (1)		5	Yes (1)	
15	5	2	9	14	11	Yes (3)		14	Yes (0)	

Table 4.9: Per-passenger comparison of arrival times for Simulations 4 and 5 under the proposed and fixed-route systems. For each passenger, the achieved arrival time is reported along with whether the passenger arrives early or late. Values in parentheses indicate the number of time steps the passenger is early or late relative to their desired arrival time (continued).

4.3.2 Total Bus Operating Cost and Passenger Travel Time Cost

In this subsection, the performance of the proposed system is evaluated against the fixed-route system using aggregate system-level metrics across the five simulations. In particular, total bus operating cost and total passenger travel time cost are compared to assess the trade-off between operating efficiency and service quality.

The route assignments and corresponding travel times for each bus are shown in Table 4.11. These results are used to compute total bus operating cost using equation 2.33 from Section 2.5.1. Passenger travel time cost is computed using the per-passenger results presented in Table 4.8 and Table 4.9, based on equation 2.35 from Section 2.5.2. The resulting aggregate costs for each simulation are summarized in Table 4.10.

Simulation	**Proposed System**		**Fixed-Route System**	
	Bus Operating Cost	**Passenger Travel Time Cost**	**Bus Operating Cost**	**Passenger Travel Time Cost**
1	12.5	-2	12	8
2	18.5	9	16.5	27
3	20.5	-12	14.5	6
4	24.5	42	21.5	59
5	22.5	8	21.5	36
Total	98.5	45	86	136

Table 4.10: Comparison of total bus operating cost and passenger travel time cost across simulations for the proposed and fixed-route systems.

The results in Table 4.10 show that the proposed system achieves a substantial reduction in passenger travel time cost at the expense of a slight increase in bus operating cost compared to the fixed-route system. Across all simulations, total passenger travel time cost is reduced from 136, under the fixed-route system, to 45 under the proposed system. This corresponds to a reduction of approximately 67%. This signifi-

	Proposed System		**Fixed-Route System**	
Bus b	**Route Sequence**	**Travel Times**	**Route Sequence**	**Travel Times**
Simulation 1: $N_P = 5$				
1	1	7	1	7
2	3, 2	6, 5	2, 2	5, 5
Simulation 2: $N_P = 10$				
1	2, 1	5, 7	1, 1	7, 7
2	3, 1	6, 7	2	5
Simulation 3: $N_P = 10$				
1	1, 2	7, 5	1	7
2	2, 1, 2	5, 7, 5	2, 2, 2	5, 5, 5
Simulation 4: $N_P = 15$				
1	3, 2, 1	6, 5, 7	1, 1	7, 7
2	1, 3	7, 6	2, 2, 2	5, 5, 5
Simulation 5: $N_P = 15$				
1	1, 1	7, 7	1, 1	7, 7
2	3, 3, 2	6, 6, 5	2, 2, 2	5, 5, 5

Table 4.11: Bus route assignments and corresponding travel times for each bus under the proposed and fixed-route systems across simulations.

cant improvement in passenger service performance is achieved with a 14% increase in total bus operating cost.

The improvement in passenger travel time cost can be explained by the increased flexibility in route assignments enabled by the proposed model. As shown in Table 4.11, in Simulations 2 and 4, the proposed system performs one additional route assignment compared to the fixed-route system, allowing the system to better match service to passenger demand.

Furthermore, the ability to dynamically assign buses enables the proposed system to utilize route 3, which is inaccessible under the fixed-route configuration due to fleet limitations. This additional routing flexibility allows the system to reduce passenger travel time more effectively. For instance, in Simulation 5, the proposed system utilizes route 3 twice and achieves an approximate 78% reduction in passenger travel time cost while increasing bus operating cost by only approximately 5% relative to the fixed-route system.

Taken together, the results demonstrate that allowing dynamic assignment of buses across routes fundamentally improves system performance. The model consistently adapts to varying demand conditions, synchronizes transfers, responds more effectively to passenger schedules, and allocates heterogeneous buses in a manner that reflects both route structure and passenger requirements. At the passenger level, this leads to fewer late arrivals, reduced delay, and significantly earlier arrivals, with late passengers reduced by approximately 21%, total delay reduced by approximately 38%, and total early arrival time increasing by approximately 69%.

At the system level, these improvements are achieved with only a slight increase in operating cost, with bus operating cost increasing by approximately 14% while passenger travel time cost is reduced by approximately 67%. Overall, the results show that dynamic assignment enables a limited fleet to be utilized more effectively, improving both reliability and responsiveness.

Chapter 5

Conclusion

5.1 Summary

Current research in public transit optimization has primarily evolved along two parallel directions, fixed-route system optimization and demand-responsive transportation. In fixed-route systems, the dominant focus has been on timetable coordination, headway control, and fleet scheduling, with the objective of improving service reliability and reducing passenger waiting times. These approaches have demonstrated that coordinating departures across routes and adjusting headways can significantly enhance transfer synchronization and overall passenger experience. However, such models typically assume static route assignments and homogeneous fleets, limiting their ability to respond to fluctuations in passenger demand or to operate efficiently in resource-constrained environments with vehicles with varying capacity.

In contrast, demand-responsive transportation systems introduce a high degree of flexibility by adapting routes and schedules to individual passenger requests. These systems incorporate passenger-specific information such as origin, destination, and time constraints, enabling more efficient routing and reduced waiting times. Prior work has shown that such approaches can significantly decrease travel distance, number of stops, and passenger delays. However, these benefits are typically achieved using small homogeneous vehicles, and often rely on near curb-to-curb service or dynamically generated routes. As a result, many DRT models are not directly applicable to large-scale urban transit systems, where predefined routes, structured transfer networks, and high-capacity fleets remain essential for operational feasibility.

Taken together, the existing literature reveals a clear gap between the rigidity of fixed-route systems and the flexibility of demand-responsive approaches. Fixed-route models provide structure and scalability but lack adaptability, while DRT models offer responsiveness but often sacrifice operational simplicity and scalability. Despite significant advances in both areas, there remains a need for optimization frameworks that bridge this gap by introducing controlled flexibility into fixed-route systems without abandoning their underlying structure.

To address this gap, this thesis develops a flexible fixed-route transit framework that introduces dynamic bus reassignment within a predefined route structure. The proposed approach retains the operational simplicity and scalability of fixed-route systems while incorporating elements of demand-awareness typically associated with demand-responsive transportation. Rather than allowing vehicles to deviate from routes or generate new routes dynamically, buses are reassigned across a set of predefined routes at discrete time steps based on evolving passenger demand. This enables the system to adapt to temporal and spatial demand variations without sacrificing the structural constraints required for large-scale transit operations.

The problem is formulated as a multi-objective mixed-integer linear program that jointly determines bus-route assignments, departure timing, and passenger-to-bus allocation over a finite planning horizon. The system consists of a heterogeneous fleet of buses operating on predefined unidirectional routes that originate from and terminate at a central depot. These buses are not permanently tied to specific routes but are dynamically reassigned over time. Passenger demand is modeled at the individual level, with each passenger characterized by a specific origin, destination, and desired arrival time. The objective function balances passenger service quality and operational efficiency by minimizing a weighted combination of passenger travel time cost and bus operating cost. Passenger travel time is modeled asymmetrically, penalizing late arrivals more heavily than early arrivals, while bus operating cost reflects the duration for which buses are actively assigned to routes.

This formulation introduces several key contributions relative to existing work. First, it enables dynamic reassignment of a heterogeneous fleet across fixed routes, allowing capacity to be allocated where and when it is needed. Second, it models passenger demand at the individual level, enabling precise scheduling decisions that account for passenger-specific time constraints. Additionally, the use of an asymmetric passenger travel time cost incentivizes punctuality without allowing early arrivals to offset delays. Third, the use of a discrete time representation allows the system to explicitly capture boarding, riding, transferring, and disembarking events, ensuring physical consistency. This enables detailed simulation of system behavior, providing practical insights for transit planners. Finally, the model provides a transparent framework for analyzing trade-offs between service quality and operating cost, allowing planners to evaluate how increases in operating cost improve passenger service and how cost reductions impact service quality.

5.2 Results

The effectiveness of the proposed framework is evaluated through a series of structured simulation scenarios designed to highlight both qualitative system behavior and quantitative performance improvements. The results demonstrate that the model produces physically consistent and operationally meaningful solutions while capturing key behaviors expected in real-world transit systems. In particular, the simulations show that the system is capable of dynamically reassigning buses across routes, coordinating transfers between vehicles, and allocating a heterogeneous fleet in a manner that reflects both operating cost and passenger demand. This flexibility allows the system to serve more routes than buses, as buses are reassigned over time to match demand. The model also adapts to passenger-specific arrival constraints, prioritizing those with tighter deadlines, and responds predictably to changes in objective function weights, shifting between service-oriented and cost-efficient solutions.

Beyond these qualitative behaviors, a comparative analysis against a fixed-route system demonstrates the practical benefits of introducing controlled flexibility into route assignments. At the passenger level, the proposed model reduces the number of late arrivals by 21% and decreases total delay by 38%, while also increasing both the frequency and magnitude of early arrivals. These improvements indicate that the system not only serves more passengers on time, but also aligns more effectively with passenger schedules overall. At the system level, these gains in service quality are achieved with only a modest increase in operating cost. Specifically, passenger travel time cost is reduced by 67%, while bus operating cost increases by 14%, indicating a favorable trade-off between efficiency and service quality.

Taken together, these findings confirm that dynamic reassignment of buses across predefined routes enables a limited and heterogeneous fleet to be utilized more effectively. The model consistently adapts to varying demand conditions, improves transfer coordination, and enhances responsiveness to passenger needs, while maintaining feasibility and respecting operational constraints. Overall, the results support the conclusion that introducing flexibility within a fixed-route structure can significantly improve both service reliability and system performance.

5.3 Limitations and Future Work

While the proposed framework demonstrates strong performance across a range of scenarios, several limitations should be acknowledged. First, the model assumes that all passenger demand is known at the beginning of the planning horizon, which may not fully reflect real-world transit systems where demand evolves dynamically over time. Incorporating real-time or stochastic demand would introduce additional complexity but would improve the practical applicability of the model. Second, the formulation assumes deterministic travel times and does not account for traffic variability or disruptions, which can significantly impact transit operations in practice. Third, the computational complexity of the mixed-integer linear programming formulation may limit scalability as the size of the network, fleet, and passenger set increases, particularly for real-time deployment. Finally, while the model captures detailed operational dynamics, it does not consider strategic decisions such as route design, frequency setting, or long-term fleet planning, which are often integrated into broader transit planning frameworks.

These limitations present several opportunities for future work. One natural extension is the incorporation of stochastic or real-time passenger demand, enabling the model to operate in a rolling-horizon or online optimization framework. This would allow the system to continuously update decisions as new information becomes available. Another important direction is the integration of uncertainty in travel times, potentially through robust or stochastic optimization techniques. In addition, integrating predictive models for demand or traffic conditions could allow the system to anticipate future states and make more informed decisions. From a computational perspective, decomposition methods or learning-based approaches could be explored to improve scalability and enable faster solution times for larger problem instances.

In conclusion, this thesis presents a flexible fixed-route transit framework that bridges the gap between traditional fixed-route systems and demand-responsive transportation. By enabling dynamic reassignment of buses across predefined routes, the proposed model introduces a level of adaptability that significantly improves both passenger service quality and system efficiency. The results demonstrate that even limited flexibility in fleet deployment can lead to substantial reductions in passenger delay, improved alignment with passenger schedules, and more effective utilization of a heterogeneous fleet. These findings suggest that incorporating controlled flexibility into existing transit systems represents a promising direction for improving urban mobility, particularly in resource-constrained environments where efficient use of limited infrastructure is critical.

Ultimately, this work shows that transit systems do not need to abandon fixed routes to become adaptive, but instead can achieve significant gains by rethinking how limited resources are deployed within them.

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