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INTRODUCTION

In [1], Auslander and Buchsbaum characterize separability for a Noetherian ring S over a base ring R in terms of ramification of prime ideals in S. They prove that, with rather general assumptions, S is R-separable if and only if each maximal ideal of S is unramified (Theorem 2.5). If more conditions are put on R and S, namely that R is an integrally closed Noetherian domain and S the integral closure of R in a separable field extension of the quotient field of R, with S R-projective, they achieve the following result: S is R-separable if and only if each prime ideal of height one in S is unramified (Corollary 3.7). We will give examples to show that this result can fail to hold if the Noetherian restriction on R is removed or if S is not R-projective.

We focus our attention here on the prime ideals of the base ring R, and call S <u>height 1 separable</u> over R if S is separable at each localization at a height 1 prime ideal of R. We establish some general properties of height 1 separable algebras, analogous to some of the basic properties of separable algebras, and we also give several examples of height 1 separable algebras which are not separable. In Chapter 4 we define and prove the existence of a height 1 separable closure of a given base ring. In the last chapter we restrict our attention to a certain class of domains, which we call semi-Krull domains, and, by generalizing somewhat the earlier notion of a height 1 closure, we are able to prove the existence of a unique height 1 closure. We then prove a Galois correspondence theorem which gives a one-to-one correspondence between integrally closed height 1 strongly separable subalgebras of the height 1 closure and certain subgroups of the automorphism group of the height 1 closure over the base ring.

All ring and algebras throughout this paper are assumed to be commutative with unit. The base ring will always be denoted by R and all unadorned tensors are taken over R .

CHAPTER 1

HEIGHT 1 SEPARABLE ALGEBRAS: BASIC PROPERTIES

The purpose of this chapter is to define height 1 separability and to establish some basic properties.

For a ring R, let $X'(R)$ denote the set of all prime ideals in R having rank (height) ≤ 1. If S is an R-algebra and Q is in $X'(S)$, then by $Q \cap R$ we mean the contraction of Q to R. We remark that if $R \subseteq S$, S is integral over R, and P is in $X'(R)$, then there exists a prime Q in $X'(S)$ with $Q \cap R = P$. We note also that if Going Down holds for the R-algebra S (e.g., if S is R-flat - see [15; p.33]), then $Q \cap R$ is in $X'(R)$ if Q is in $X'(S)$. We shall call this property NBU (for No Blowing Up, as in [7]).

<u>DEFINITION 1.1</u> An R-algebra S is <u>height 1 separable over</u> R if S_P is a separable R_P-algebra for all P in $X'(R)$. S is <u>strongly height 1 separable</u> if S is height 1 separable and finitely generated and projective as an R-module. S is <u>height 1 strongly separable</u> if S_P is separable as an R_P-algebra and finitely generated and projective as an R_P-module for each p in $X'(R)$. We remark

that strong height 1 separability implies height 1 strong separability.

Suppose that M is an $S \otimes S$ module for an R-algebra S. Then, for p in $X'(R)$, $M_p = M \otimes R_p$ is an $(S \otimes S)_p = S_p \otimes_{R_p} S_p$-module. Let $M^S = \{m \in M : (s \otimes 1)m = (1 \otimes s)m$, for all s in $S\}$. Define likewise $M_p^{S_p}$.

<u>PROPOSITION 1.2</u> If S is finitely generated as an R-algebra, and M is an $S \otimes S$ module, then $(M^S)_p \cong M_p^{S_p}$, for p in $X'(R)$.

<u>Proof:</u> Let s_i, $i = 1,\ldots,n$, generate S as an R-algebra, and define $f_i : M \to M$ by $f_i(m) = (1 \otimes s_i)m - (s_i \otimes 1)m$. Then we have the exact sequence $0 \longrightarrow M^S \longrightarrow M \xrightarrow{g} \Pi_1^n M$ (Π = product) where $g(m) = (f_1(m),\ldots,f_n(m))$, for m in M. For p in $X'(R)$ we have $(\Pi M_p) = \Pi(M_p)$, and so obtain the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (M^S)_p & \longrightarrow & M_p & \longrightarrow & (\Pi M)_p \\ & & & & \downarrow\wr & & \downarrow\wr \\ 0 & \longrightarrow & M_p^{S_p} & \longrightarrow & M_p & \longrightarrow & \Pi(M_p) \end{array} .$$

Thus we see that $(M^S)_p \cong M_p^{S_p}$.

Let S^e denote $S \otimes S$ and S_p^e denote $S_p \otimes S_p$.

<u>LEMMA 1.3</u> If M is an S^e-module for an R-algebra S, then $\mathrm{Hom}_{S_p^e}(S_p,M_p) \cong M_p^{S_p}$, by $f \to f(1)$, for all p in $X'(R)$.

<u>Proof:</u> Note that $f(1)$ is in $M_p^{S_p}$ if f is in $\mathrm{Hom}_{S_p^e}(S_p,M_p)$,

since $(s \otimes 1)f(1) = f((s \otimes 1)1) = f(s) = f((1 \otimes s)1) = (1 \otimes s)f(1)$, for all s in S_p . If m is in $M_p^{S_p}$ define $f : S_p \to M_p$ by $f(s) = sm$. Then f is in $\mathrm{Hom}_{S_p^e}(S_p, M_p)$. If $f(1) = 0$, then $f = 0$. The lemma follows.

If $h : M \to M'$ is a homomorphism of S^e-modules M and M' , and if p is in $X'(R)$, we have the induced map $h_p : M_p \to M'_p$, and the following diagram:

$$\begin{tikzcd} \mathrm{Hom}_{S_p^e}(S_p, M_p) \arrow[r, "h_p^*"', "f \to h_p \circ f"] \arrow[d, "\wr"] & \mathrm{Hom}_{S_p^e}(S_p, M'_p) \arrow[d, "\wr"] \\ M_p^{S_p} \arrow[r, "h_p^{S_p} = h_p|m_p"] & M'^{S_p}_p \end{tikzcd} .$$

We claim that $h_p^{S_p}$ is well-defined and the diagram commutes. If m is in $M_p^{S_p}$, then $h_p^{S_p}(m) = h_p(m)$, and $(s \otimes 1)h_p(m) = h_p((s \otimes 1)m) = h_p((1 \otimes s)m) = (1 \otimes s)h_p(m)$. Given $f : S_p \to M_p$ then $h_p^*(f) = h_p \circ f$, and we have $h_pf(1) = h_p(f(1)) = h_p^{S_p}(f(1))$, since $f(1)$ is in M_p . Thus we have

$$\begin{tikzcd} f \arrow[r] \arrow[d] & h_pf \arrow[d] \\ f(1) \arrow[r] & h_p^{S_p}(f(1)) = h_pf(1) \end{tikzcd}$$

, and the diagram commutes.

We can now state a characterization of height 1 separability, similar to that for separability found in [3; p.43].

<u>THEOREM 1.4</u> S is a height 1 separable R-algebra if and only if for all S^e-modules M and M' and all onto S^e homomorphisms

$h : M \to M'$, the map $h_p^{S_p} : M_p^{S_p} \to M_p'^{S_p}$ is onto, for all p in $X'(R)$. (Note that if h is onto then h_p is onto.)

Proof: If S is height 1 separable over R, then for p in $X'(R)$, S_p is S_p^e-projective. Thus, if $h_p : M_p \to M_p'$ is onto, we have h_p^* is onto, which implies that $h_p^{S_p}$ is onto.

Conversely, suppose the condition holds. To show S_p is R_p-separable, we show S_p is S_p^e-projective, for all p in $X'(R)$. Let $f : N \to N'$ be an onto homomorphism for S_p^e-modules N and N'. If we let $M = N_p$, $M' = N_p'$ and $h : M \to M'$ be the map f_p, then we have S^e-modules M and M' and an onto S^e-module homomorphism $h : M \to M'$ such that $M_p = N$, $M_p' = N'$ and $h_p = f$. Then, since h is onto, $h_p^{S_p}$ is onto and so $f^{S_p} : N^{S_p} \to N'^{S_p}$ is onto. Hence, $f^* : \mathrm{Hom}_{S_p^e}(S_p,N) \to \mathrm{Hom}_{S_p^e}(S_p,N')$ is onto, proving that S_p is S_p^e-projective.

We now prove some general facts about height 1 separability.

PROPOSITION 1.5 Suppose that R_1 and R_2 are R-algebras and that S_i is a height 1 separable R_i-algebra for $i = 1,2$. Let R_1 map to $R_1 \otimes R_2$ by $r \to r \otimes 1$ and let R_2 map to $R_1 \otimes R_2$ by $r \to 1 \otimes r$. Suppose that $R_1 \otimes R_2$ satisfies NBU over both R_1 and R_2. Then $S_1 \otimes S_2$ is a height 1 separable $R_1 \otimes R_2$-algebra.

Proof: Let p be in $X'(R_1 \otimes R_2)$ and let $p_i = p \cap R_i$, so that p_i is in $X'(R_i)$. Then, since $(S_i)_{p_i}$ is separable over $(R_i)_{p_i}$ we have that $(S_1)_{p_1} \otimes (S_2)_{p_2}$ is separable over $(R_1)_{p_1} \otimes (R_2)_{p_2}$. Localizing further we have $((S_1)_{p_1} \otimes (S_2)_{p_2})_p$ is separable over $((R_1)_{p_1} \otimes (R_2)_{p_2})_p$. Since $((S_1)_{p_1} \otimes (S_2)_{p_2})_p = (S_1 \otimes S_2)_p$, and likewise for $(R_1 \otimes R_2)_p$, the result follows.

<u>COROLLARY 1.6</u> If S_1 and S_2 are height 1 separable R-algebras, then so is $S_1 \otimes S_2$.

<u>Proof:</u> Let $R_1 = R_2 = R$ in (1.5), and note that if p is in $X'(R_1 \otimes R_2) = X'(R)$, then $p = p \cap R_1 = p \cap R_2$.

<u>COROLLARY 1.7</u> If S is a height 1 separable R-algebra and T is an R-algebra satisfying NBU over R , then $T \otimes S$ is height 1 separable over T .

<u>Proof:</u> In (1.5) let $S_2 = S$, $S_1 = T$, $R_1 = T$ and $R_2 = R$. If p is in $X'(R_1 \otimes R_2) = X'(T)$, then $p \cap T = p$, and $p \cap R$ is in $X'(R)$ by hypothesis. The conclusion follows by (1.5).

<u>COROLLARY 1.8</u> If I is an ideal of R such that R/I satisfies NBU over R and if S is a height 1 separable R-algebra, then $R/I \otimes S = S/IS$ is height 1 separable over R/I .

<u>Proof:</u> Follows directly from (1.7).

We remark that the hypothesis that R/I satisfies NBU over R is fairly restrictive.

If I contains a prime ideal of height ≥ 1 and R/I has a prime ideal P of height 1, then NBU is not satisfied; if P_0, P_1 are primes in R with $P_0 \subsetneq P_1 \subseteq I$, then $P_0 \subseteq P_1 \subsetneq P \cap R$, and so $P \cap R$ is not in $X'(R)$.

<u>PROPOSITION 1.9</u> If I is an ideal of S and S is a height 1 separable R-algebra, then S/I is a height 1 separable R-algebra.

<u>Proof:</u> If p is in $X'(R)$, then $(S/I)_p = S_p/I_p$. Since S_p is separable over R_p and I_p is an ideal in S_p , the result follows.

COROLLARY 1.10 If S is a height 1 separable R-algebra and $f : S \to S'$ is a surjective homomorphism, then S' is a height 1 separable R-algebra.

Proof: Since $S/\ker(f) = S'$, the result follows by (1.9).

Our next proposition deals with the R/I'-algebra S/I, where I is an ideal of S and I' is an ideal of R contained in $I \cap R$. But first we prove two lemmas.

LEMMA 1.11 Let I be an ideal of R and $p \supseteq I$ a prime ideal. Then $R_p/I_p = (R/I)_{p/I}$.

Proof: Define $f : R_p \to (R/I)_{p/I}$ by $r/s \to (r + I)/(s + I)$, for r in R and s in $R - p$. This is a well-defined surjection, and we show that its kernel is I_p. Suppose that $(r + I)/(s + I) = 0$ in $(R/I)_{p/I}$. Then there is an $s' + I$ in $R/I = p/I$ with $(s' + I)(r + I) = 0$; i.e., $s'r$ is in I. Note that s' is in $R - p$, since otherwise $s' + I$ would be in p/I. Then we have $s'r/s's = r/s$ is in I_p. On the other hand, if r/s is in I_p, then r is in I and clearly $f(r/s) = 0$. Thus the kernel of f is I_p and the proof is complete.

LEMMA 1.12 Let I be any ideal in R and let p be a prime in R/I. Denote $p \cap R$ by Q. Then, if M is any R-module with $IM = 0$, $M_Q = M_p$.

Proof: Follows by the following string of equalities: $M_Q = M \otimes R_Q = M/IM \otimes R_Q = (M \otimes R/I) \otimes R_Q = M \otimes R_Q/I_Q = M \otimes (R/I)_{Q/I}$ (by 1.11) $= M \otimes (R/I)_p = M \otimes_{R/I} (R/I)_p = M_p$.

PROPOSITION 1.13 Let S be a height 1 separable R-algebra and I an ideal in S. Let I' be an ideal of R contained in

$I \cap R$ such that R/I' satisfies NBU as an R-algebra. Then S/I is a height 1 separable R/I'-algebra.

Proof: Let p be in $X'(R/I')$; then $Q = p \cap R$ is in $X'(R)$. By (1.9), S/I is height 1 separable over R . Hence, $(S/I)_Q$ is a projective $(S/I \otimes S/I)_Q$-module. Since $I' \subseteq I \cap R$, $S/I \otimes S/I = S/I \otimes_{R/I'} S/I$. Also, $I'(S/I) = 0$ and $I'(S/I \otimes_{R/I'} S/I) = 0$. Thus, by (1.12), $(S/I)_p = (S/I)_Q$ and $(S/I \otimes_{R/I'} S/I)_p = (S/I \otimes_{R/I'} S/I)_Q$. Therefore $(S/I)_p$ is a projective $(S/I \otimes_{R/I'} S/I)_p$-module and a separable $(R/I')_p$-algebra.

Next, we have a version of transitivity.

PROPOSITION 1.14 Let T be a height 1 separable S-algebra, finitely generated as an S-module, where S is a height 1 separable extension of R and integral over R . Then T is a height 1 separable R-algebra.

Proof: Let p be in $X'(R)$. We show first that T_p is a height 1 separable S_p-algebra. For Q in $X'(S_p)$, let Q' be $Q \cap S$. Then, since $S_p = S \otimes R_p$ is a flat S-module, NBU holds and Q' is in $X'(S)$. Thus, $T_{Q'}$ is a separable $S_{Q'}$-algebra, and so $(T_{Q'})_p$ is separable over $(S_{Q'})_p$. (By $(T_{Q'})_p$ we mean $T_{Q'} \otimes R_p$.) Since $(T_{Q'})_p = (T_p)_Q$ and $(S_{Q'})_p = (S_p)_Q$, we see that T_p is height 1 separable over S_p , for each p in $X'(R)$.

Suppose M is a maximal ideal of S_p , for p in $X'(R)$. Since S_p is an integral extension of R_p , $M \cap R_p$ must be maximal in R_p ; i.e., $M \cap R_p = pR_p$. Since $ht(M) \leq ht(M \cap R_p)$ [11; p.30, 45] and $ht(M \cap R_p) = ht(pR_p) \leq 1$, we have $ht(M) \leq 1$; i.e., M is in $X'(S_p)$. Since T_p is a finitely generated S_p-module and $(T_p)_M$

is $(S_p)_M$-separable for all maximal ideals M is S_p , we have that T_p is separable over S_p [3; p.72]. Then, since S_p is separable over R_p , it follows that T_p is separable over R_p . We have shown that T is height 1 separable over R .

REMARK 1.15 We remark that in the proof of (1.14) we have shown the following: If S is an integral extension of R and T is a height 1 separable S-algebra, finitely generated as an S-module, then T_p is separable over S_p , for all p in $X'(R)$.

CHAPTER 2

POLYNOMIAL EXTENSIONS

In this chapter we examine R-algebras of the form $R[x]/(f(x))$, where $f(x)$ is a monic polynomial in $R[x]$. We also give an example of a height 1 separable R-algebra which is not separable.

DEFINITION 2.1 A monic polynomial $f(x)$ in $R[x]$ is a height 1 separable polynomial if $R[x]/(f(x))$ is a height 1 separable R-algebra.

We recall that for a field K, a polynomial $f(x)$ in $K[x]$ is separable if and only if there exist P and Q in $K[x]$ with $Pf + Qf' = 1$, where f' denotes the derivative of f. It is also known that this result holds for commutative rings (with 1). We note this result next.

PROPOSITION 2.2 For a commutative ring R, a monic polynomial $f(x)$ in $R[x]$ is separable if and only if there exist P and Q in $R[x]$ with $Pf + Qf' = 1$.

Proof: Let M be a maximal ideal in R, and let I be the ideal in $R[x]$ generated by f and f'. Then

$R[x]/I \otimes R/M = (R/M[x])/\bar{I}$, where $\bar{I}$ is the ideal generated by the canonical images $\bar{f},\bar{f}'$ of f,f' , respectively, in $R[x]/M$.

Let $S = R[x]/(f)$; then S is a finitely generated R-module. Thus S is R-separable if and only if S/MS is R/M-separable for every maximal ideal M in R . Note that $S/MS = (R/M[x])/(\bar{f})$. Since R/M is a field, $\bar{f}$ is separable if and only if $\bar{I} = (\bar{f},\bar{f}') = R/M[x]$; i.e., if and only if $(R/M[x])/\bar{I} = 0$. By the above, we see that S is R-separable if and only if $R[x]/I \otimes R/M = 0$ for all maximal ideals M in R . By the Nakayama Lemma, $R[x]/I \otimes R/M = 0$ for all maximal ideals M if and only if $R[x]/I = 0$. This proves the proposition.

2.3 From (2.2) we see that if $f(x)$ is a separable polynomial and $f(a) = 0$, then $f'(a)$ is a unit.

COROLLARY 2.4 A monic polynomial f in $R[x]$ is height 1 separable if and only if $I_p = R_p[x]$ for each p in $X'(R)$, where I is the ideal in R generated by f and f' .

Proof: Follows directly from (2.2), since $(R[x]/f)_p = R_p[x]/f_p$ and I_p is generated by f_p and f'_p , where f_p denotes the polynomial f with its coefficients considered as elements of R_p .

2.5 As in (2.3), we note that if $f(x)$ is height 1 separable and $S = R[x]/f$, then whenever a is a root of f in S , $f'(a)$ is a unit in S_p , for all p in $X'(R)$.

Let $f(x)$ be a height 1 separable polynomial in $R[x]$, and let $S = R[x]/f$. Let $a = x + (f)$ in S . Then a is a root of f in S , and, since the leading coefficient of $x - a$ is a unit, the

Euclidean algorithm is valid here [12; p.120]. Hence, $x - a$ divides f in $S[x]$. Let

$$\begin{aligned} f(x) &= (x - a)(b_0 + \cdots + b_{n-2}x^{n-2} + x^{n-1}) \\ &= (b_0x + \cdots + b_{n-2}x^{n-1} + x^n) - (ab_0 + \cdots + ab_{n-2}x^{n-2} + ax^{n-1}) \\ &= x^n + (b_{n-2} - a)x^{n-1} + (b_{n-3} - ab_{n-2})x^{n-2} \\ &\quad + \cdots + (b_0 - ab_1)x - ab_0 \end{aligned}$$

Since $f(a) = 0$, we have the following:

$$(*) \quad a^n = \sum_{i=0}^{n-1} (ab_i - b_{i-1})a^i \text{ , where } b_{n-1} = 1 \text{ and } b_{-1} = 0 .$$

Recall that if T is a separable R-algebra then the separability idempotent for T is the unique idempotent e in $T \otimes T$ such that e maps to 1 under the multiplication map $T \otimes T \to T$ and $(1 \otimes t)e = (t \otimes 1)e$, for all t in T. We are going to describe the separability idempotent for S_p in terms of a, b_i, and $f'(a)$, for p in $X'(R)$.

<u>PROPOSITION 2.6</u> In the setting described above, the element $e = \sum_{i=0}^{n-1} a^i \otimes \frac{b_i}{f'(a)}$ in $S_p \otimes S_p$, for p in $X'(R)$, is the separability idempotent for S_p.

<u>Proof</u>: If $m : S_p \otimes S_p \to S_p$ is the multiplication map, then we have $m(e) = \sum_{i=0}^{n-1} \frac{a^i b_i}{f'(a)} = \frac{f'(a)}{f'(a)} = 1$. To show that $(1 \otimes s)e = (s \otimes 1)e$ for all s in S_p, we need only show $(1 \otimes a)e = (a \otimes 1)e$, since S_p is generated over R_p by a. We have the following:

$$(1 \otimes a)e = \sum_{i=0}^{n-1} a^i \otimes \frac{ab_i}{f'(a)} \quad \text{and} \quad (a \otimes 1)e = \sum_{i=0}^{n-1} a^{i+1} \otimes \frac{b_i}{f'(a)} .$$

From (*) we obtain $a^n \otimes \frac{1}{f'(a)} = \left[\sum_{i=0}^{n-1} (ab_i - b_{i-1})a^i\right] \otimes \frac{1}{f'(a)} =$

$\sum_{i=0}^{n-1} a^i \otimes \frac{ab_i - b_{i-1}}{f'(a)}$, since the terms $ab_i - b_{i-1}$ are in R. Then,

$$(a \otimes 1)e = \sum_{i=0}^{n-1} \left[a^i \otimes \frac{b_{i-1}}{f'(a)} + a^i \otimes \frac{ab_i - b_{i-1}}{f'(a)}\right] = \sum_{i=0}^{n-1} \left[a^i \otimes \frac{ab_i}{f'(a)}\right] =$$

$(1 \otimes a)e$. Thus e is the separability idempotent.

The key to the above is the fact that if S_p is R_p-separable, then $f'(a)$ is a unit in S_p . If $f'(a)$ is not a unit in S , then S itself cannot be R-separable. If we let

$e' = \sum_{i=0}^{n-1} a^i \otimes b_i$, an element of $S \otimes S$, then the above shows that it is still true that $(1 \otimes s)e' = (s \otimes 1)e'$ for all s in S , but it is no longer true that $\sum a^i b_i = 1$, since $\sum a^i b_i = f'(a)$. However, the element e' acts like a "pre-image" for the separability idempotent of each localization S_p , p in $X'(R)$, in the sense that the ideal generated by $m(e')$ in S_p , namely S_p itself, is the same ideal generated by $m(e)$, where m is the multiplication map, since $m(e') = f'(a)$ is a unit in S_p .

Next we give an example which is based on the following exercise in [11; #8, p.114]. Let k be a field, and let R' be the ring of all polynomials in $k[x,y]$ having no term in a power of x alone. Let M be the prime ideal in R' consisting of all polynomials having constant term zero. Note that the ideal (y) is not prime in R' , since $(xy)^2$

is in (y) but xy is not in (y). Suppose there is a prime ideal P with $(y) \subsetneq P \subsetneq M$. If f belongs to $M - P$, then at least one term of f is not in P; this term cannot be in (y), so it must be of the form rx^iy, $i \geq 1$. But then $(rx^iy)^2 = r^2x^{2i}y^2$ is in $(y) \subset P$, implying rx^iy is in P, a contradiction. Thus, M is minimal over (y). Further, $\operatorname{rank}(M) \geq 2$ since M contains the zero ideal and the prime ideal of all polynomials with constant term equal to zero and no term in a power of y alone.

<u>EXAMPLE 2.7</u> (A height 1 separable R-algebra which is not separable.) Using the notation in the above exercise, let $R = R'_M$. Then, MR_M is minimal over (y), and by localizing we have ensured that the element y cannot be contained in any prime in $X'(R)$, for the existence of such a prime would contradict the minimality of MR_M over (y). Let $f(t) = t^n - y$, where $1/n$ is in R, and let $S = R[t]/(f)$. Then S is a finitely generated projective R-module. Since y is not a unit in R, S is not R-separable [10; 2.4]. But, since y is not in any prime p in $X'(R)$, y is a unit in R_p, and so $S_p = R_p[t]/(f)$ is R_p-separable. (We note that (2.4) also shows that S is height 1 separable over R, since $(-1/y)f(t) + (t/ny)f'(t) = (-1/y)(t^n - y) + (t/ny)(nt^{n-1}) = 1$.)

<u>2.8</u> We also show now that the domain R in (2.7) is integrally closed. This will follow if we show that R' is integrally closed. Suppose g is in the quotient field of R' and integral over R'. Then, since g is also in the quotient field of $k[x,y]$ and integral over $k[x,y]$, g is in $k[x,y]$. Write $g = g(x,y)$ as a polynomial in y:

$$g(x,y) = g_0(x) + g_1(x)y + \cdots + g_n(x)y^n .$$

Since $g_1(x)y + \cdots + g_n(x)y^n$ is in R' we see that $g_0(x)$ must be integral over R'. Thus, g_0 satisfies a monic polynomial in $R'[t]$:

$$g_0(x)^n + a_1(x,y)g_0(x)^{n-1} + \cdots + a_n(x,y) = 0 ,$$

where each a_i is in R'.

Since each $a_i(x,y)$ contains no terms in powers of x alone, we see that $c_i = a_i(x,0)$ must be an element of k, $i = 1,\ldots,n$. Thus

$$g_0(x)^n + c_1g_0(x)^{n-1} + \cdots + c_n = 0 .$$

This says that $g_0(x)$ is integral over the field k, but this is impossible unless $g_0(x)$ is a constant. Thus we have

$$g(x,y) = \text{constant} + g_1(x)y + \cdots + g_n(x)y^n$$

and we have that $g(x,y)$ is in R'. This shows that R' is integrally closed, and, hence, that $R = R'_M$ is also integrally closed.

CHAPTER 3

HEIGHT 1 GALOIS EXTENSIONS

In this chapter we define height 1 Galois and give some examples of height 1 Galois extensions, which will also be examples of height 1 separable extensions that are not separable.

Throughout this chapter let S be an extension of the base ring R, and let G be a finite group of R-algebra automorphisms of S. If g is in G and p is in $X'(R)$, let $g_p : S_p \to S_p$ be the induced map given by $g_p(s/t) = g(s)/t$, for s in S and t in $R - p$. Then it is easy to check that g_p is an R_p-automorphism of S_p. Denote the set of all g_p, for g in G, by G_p.

3.1 Let S^G be the subring of G-invariant elements of S, and let $S_p^{G_p}$ be the G_p-invariants of S_p. Under the identification of $S \otimes R_p$ with S_p, given by the map $s \otimes r/t \to rs/t$, we may identify the group G_p with $G \otimes 1$, as shown in the following commutative diagram:

$$\begin{array}{ccc}
S \otimes R_p & \xrightarrow{\quad g \otimes 1 \quad} & S \otimes R_p \\
\big\downarrow & \begin{array}{ccc} s \otimes r/t & \longrightarrow & g(s) \otimes r/t \\ \big\downarrow & & \big\downarrow \\ rs/t & \longrightarrow & rg(s)/t \\ & & = g(rs)/t \end{array} & \big\downarrow \\
S_p & \xrightarrow[\quad g_p \quad]{} & S_p
\end{array}$$

Since G is a finite group, and so "almost finite" as defined in [13], Theorem 1.13 in [13] shows that $(S \otimes R_p)^{G \otimes 1} = S^G \otimes R_p$; i.e., $S_p^{G_p} = (S^G)_p$.

We establish some notation in order to present our next theorem. The constructions we will make are basically the same as presented in [3; Chap. III].

3.2 Let $S\{G\}$ be the free S-module with basis consisting of elements u_g , for each g in G . We make $S\{G\}$ an S-algebra by defining multiplication as follows: $u_g \cdot a = g(a)u_g$, for a in S , and $(\sum a_g u_g)(\sum b_f u_f) = \sum a_g g(b_f) u_{fg}$. If p is in $X'(R)$, then $(S\{G\})_p = S_p\{G\}$ (here $u_g \cdot a = g_p(a)u_g$ for a in S_p) by the map $au_g/t \to (a/t)u_g$, for a in S and t in $R - p$.

Define $j : S\{G\} \to \mathrm{Hom}_R(S,S)$ by $j(\sum a_g u_g)(s) = \sum a_g g(s)$, for s and a_g in S . Similarly we define $j_p : S_p\{G\} \to \mathrm{Hom}_{R_p}(S_p,S_p)$ by $j_p(\sum a_g u_g)(s) = \sum a_g g_p(s)$, for a_g and s in S_p . Note that if S is finitely presented as an R-module then $\mathrm{Hom}_{R_p}(S_p,S_p) = \mathrm{Hom}_R(S,S)_p$, and the map j_p is induced by j .

3.3 Let $C(G,S)$ be the ring of S-valued functions on G. Then $C(G,S)$ is an R-algebra, and we claim that $C(G,S)_p = C(G,S_p)$, for p in $X'(R)$, by the map $f/t \to (1/t)f$, for f in $C(G,S)$, t in $R - p$; i.e., $(f/t)(g) = f(g)/t$, where g is in G. To show this map is injective, suppose that $f(g)/t = 0$, for each g in G. Then, for each g, there is a t_g in $R - p$ with $t_g f(g) = 0$. Set $t' = \Pi\, t_g$. Then t' is in $R - p$ and $t'f(g) = 0$, for each g, and so $f/t = 0$. To show the map is surjective, let h be in $C(G,S_p)$, and suppose $G = \{g_1,\ldots,g_n\}$. If $h(g_i) = s_i/t_i$, for s_i in S and t_i in $R - p$, let $t = \Pi\, t_i$. Define $f : G \to S$ by $f(g_i) = (\prod_{j\neq i} t_j)s_i$. Then, since $(f/t)(g_i) = f(g_i)/t = s_i/t_i = h(g_i)$, we have $f/t \to h$.

Now define $L : S \otimes S \to C(G,S)$ by $L(\sum a_i \otimes b_i)(g) = \sum a_i g(b_i)$, for a_i,b_i in S and g in G. Then, since $(S \otimes S)_p = S_p \otimes_{R_p} S_p$, we have the induced map $L_p : S_p \otimes_{R_p} S_p \to C(G,S_p)$ given by $L_p(\sum a_i \otimes b_i)(g) = \sum a_i g_p(b_i)$, for a_i,b_i in S_p and g in G. We will define a height 1 Galois extension of R to be an extension which is a Galois extension as defined in [3] at each localization at a prime in $X'(R)$. Thus the following theorem is basically Proposition 1.2 of [3; Chap. III] applied to each localization S_p.

THEOREM 3.4 Using the above notation, the following are equivalent:

(1) a. $(S^G)_p = R_p$ for each p in $X'(R)$.

b. For each nonzero idempotent e in S_p and f_p, g_p

in G_p with $f_p \neq g_p$, there is an x in S_p such that $f_p(x)e \neq g_p(x)e$.

c. S is a height 1 separable R-algebra.

(2) a. $(S^G)_p = R_p$ for each p in $X'(R)$.

b. For each p in $X'(R)$ there exist elements $x_1,\ldots,x_n$, $y_1,\ldots,y_n$ in S_p with $\sum x_j f_p(y_j) = \delta_{f_p,1}$, where f_p is in G_p .

(3) a. S_p is a finitely generated projective R_p-module for each p in $X'(R)$.

b. For each p in $X'(R)$ the map j_p is an isomorphism.

(4) a. $(S^G)_p = R_p$ for each p in $X'(R)$.

b. For each p in $X'(R)$ the map L_p is an isomorphism.

(5) a. $(S^G)_p = R_p$ for each p in $X'(R)$.

b. For p in $X'(R)$ and each $f_p \neq 1$ in G_p , and for each maximal ideal M in S_p , there is an x in S_p with $f_p(x) - x$ not in M .

Proof: (1) implies (2). Localize at p in $X'(R)$; the proof is exactly as in [3].

(2) implies (3). Localize first; proof as in [3].

(3) implies (4). First we show $S_p^{G_p} = R_p$, for p in $X'(R)$. Clearly $R_p \subseteq S_p^{G_p}$. Now, since j_p is an isomorphism, we see that j_p(Center of $S_p\{G\}$) = Center($\mathrm{Hom}_{R_p}(S_p,S_p)$) = R_p . Thus, we need only show $S_p^{G_p} \subseteq$ Center($S_p\{G\}$) . Let x be in $S_p^{G_p}$. Then xu_1 is in $S_p\{G\}$, and $(xu_1)(\sum s_f u_f) = \sum xs_f u_f$, for each $\sum s_f u_f$ in $S_p\{G\}$. Also, $(\sum s_f u_f)(xu_1) = \sum s_f f_p(x) u_f$. Since x is in $S_p^{G_p}$, $f_p(x) = x$,

and so x (identified with xu_1) is in the center of $S_p\{G\}$. Hence, $S_p^{G_p} \subseteq R_p$, and we have equality. The rest of the proof is as in [3].

(4) implies (2). Let the element h of $C(G,S_p)$ be given by $h(f) = \delta_{f_p,1}$. Since L_p is an isomorphism, there are elements $x_1,\ldots,x_n,y_1,\ldots,y_n$ in S_p such that $L_p(\sum x_i \otimes y_i) = h$. Then, $h(f) = L_p(\sum x_i \otimes y_i)(f) = \sum x_i f_p(y_i) = \delta_{f_p,1}$.

(2) implies (1) and (5) equivalent to (2) are as in [3].

DEFINITION 3.5 Let S be an extension of R and let G be a finite group of automorphisms of S . S is a height 1 Galois extension of R with group G if any of the five equivalent conditions in (3.4) are satisfied.

We will now characterize height 1 Galois extensions in a special setting.

Recall that a prime ideal Q in an R-algebra S is said to be unramified (see [1]) if $p = Q \cap R$ satisfies the following:

(1) $pS_Q = QS_Q$;

(2) S_Q/pS_Q is a separable field extension of R_p/pR_p .

Let R and S be integrally closed domains and G a finite group of automorphisms of S such that $S^G = R$, and suppose that, for each p in $X'(R)$, R_p is Noetherian. (This will be the case, for example, if S is a Krull domain — see [7; p.82].) Since G is finite, each element s in S satisfies the relation $\Pi\,(s - g_i(s)) = 0$, product taken over all g_i in G . The coefficients are symmetric functions of $g_i(s)$ and hence belong to $S^G = R$. Thus, S is integral over R . Since R is integrally closed, Going Down holds for S over R [2; pp.343,344]; thus, if Q is in $X'(S)$, then $p = Q \cap R$ is in $X'(R)$. If L is the quotient field of S then any element in L can be written as a quotient with numerator in S and denominator in R (see [18; p.78]). Letting G act on L , we see that if K is the quotient field of R , then $K = L^G$, and so L is a Galois extension of K [3; 1.4, p.86].

PROPOSITION 3.6 With S and R as described above, S is a height 1 Galois extension of R if and only if S is unramified at all the primes Q in $X'(S)$ with $ht(Q) = 1$.

Proof: Suppose that S is height 1 Galois over R and let Q be a height 1 prime in S ; let $p = Q \cap R$. Note that since $pS_Q \subseteq QS_Q$ and $pS_Q \neq 0$, then pS_Q will equal QS_Q , since QS_Q is of height 1 , provided that pS_Q is a prime ideal in S_Q . This follows from the separability of S_p over R_p , since then we have that S_p/pS_p is separable over the field R_p/pR_p and is therefore a finite product of fields. Then, S_Q/pS_Q is also a product of fields and hence a field itself, being a local ring, since it is just a further localization of S_p/pS_p . Hence, pS_Q is a prime ideal of S_Q and $pS_Q = QS_Q$. By the above we have also shown that S_Q/pS_Q is separable over R_p/pR_p . Thus we have that S is unramified over R at all the height 1 primes in $X'(S)$.

Conversely, suppose that S is unramified at the height 1 primes in $X'(S)$. Let p be in $X'(R)$. Then, by (3.1), $(S_p)^G = (S^G)_p = R_p$, and, as in the discussion preceding the proposition, the quotient field of S_p is a Galois field extension of the quotient field of R_p, hence a finite separable field extension. Then, since R_p is integrally closed and Noetherian, and S_p is the integral closure of R_p in the quotient field of S_p, we have that S_p is a finitely generated R_p-module, [6; 5.48, p.185]. Thus, if we can show that S_p/pS_p is separable over R_p/pR_p, it follows that S_p is R_p-separable [3; 7.1, p.72].

Since R is integrally closed and S is integral over R such that the quotient field of S is a finite separable field extension of the quotient field of R, there are only are only a finite number of primes in S lying over p [8; 12.5, p.121]; denote these by $Q_1,\ldots,Q_n$. Each Q_i is in $X'(S)$, and we are given that $S_{Q_i}/Q_iS_{Q_i}$ is a separable field extension of R_p/pR_p, for each i. We will show that S_p/pS_p is R_p/pR_p-separable by showing that $S_p/pS_p = \Pi\, S_{Q_i}/Q_iS_{Q_i}$. For the moment replace R_p by R, pR_p by p, S_p by S, and the ideals Q_iS_p by Q_i. Then we still have $Q_i \cap R = p$, and, since p is maximal in R, Q_i is maximal in S. Further, if Q is any maximal ideal of S, then $Q \cap R = p$, and so Q must be one of the Q_i's. Thus, $Q_1,\ldots,Q_n$ are all the maximal ideals in S. Let $I = Q_1 \cap \cdots \cap Q_n$. Then $pS \subseteq I$. Now, for each i, we have $(pS)_{Q_i} = (Q_i)_{Q_i} = (Q_1 \cap \cdots \cap Q_n)_{Q_i} = I_{Q_i}$. Hence, $pS = I$. By the Chinese Remainder Theorem we have $S/pS = \Pi\, S/Q_i$. Since $S/Q_i = S/Q_i \otimes_{S_{Q_i}} S_{Q_i} = S/Q_i \otimes_S S_{Q_i} = S_{Q_i}/Q_iS_{Q_i}$, we see that

$$S/pS = \Pi\, S_{Q_i}/Q_iS_{Q_i} \ .$$

Going back to our original notation, the above says that $S_p/pS_p = \Pi\, S_{Q_i}/Q_iS_{Q_i}$ (note that $(S_p)_{Q_i} = S_{Q_i}$). As we noted above, this shows that S_p is R_p-separable. Finally, since S_p is connected for each p in $X'(R)$, we see that S is height 1 Galois over R by (3.4-1). This completes the proof of the proposition.

We remark that if S and R are as in Example 2.7, then the first part of the proof of (3.6) shows that S is unramified at each height 1 prime ideal of S .

Recall that if R and S are domains with $R \subseteq S$, and such that the quotient field of S , say L , is a finite extension of the quotient of R , then the complementary module $C(S/R)$ is defined to be the set of all elements x in L such that $tr(xS) \subseteq R$, where tr is the trace map, and the Dedekind different $D(S/R)$ is then defined to be the set of all x in L such that $xC(S/R) \subseteq S$. As in [7; Chap. IV], we now consider the situation where S is a Krull domain and G is a finite group of automorphisms of S . Then, letting $R = S^G$, R is also Krull, and so both R and S are integrally closed domains. If L and K are the quotient fields of S and R , respectively, then $K = L^G$, and so L is a Galois extension of K . The following appears as Proposition 1.6.3 in [7; p.84]: In the setting just described, S is unramified over R at p in $X'(S)$ if and only if p does not contain $D(S/R)$. Fossum uses this result in the following test for ramification. Let u be a primitive element for L over K such that u is in S , and let $f(t)$ be its minimal polynomial. Then, one can check that $C(S/R) \subseteq f'(u)^{-1}S$, and so $f'(u)$ is in $D(S/R)$. Thus, if there are primitive elements $u_1,\ldots,u_n$ in S such the elements $f_1'(u_1),\ldots,f_n'(u_n)$ are not in any height 1 prime ideal of S, then S is unramified at all height 1 primes in $X'(S)$. This result is used in the following two examples of height 1 Galois extensions (which are also examples of height 1 separable extensions that are not separable).

EXAMPLE 3.7 (See [7; p.85] or [17; p.58].) Let k be a field of characteristic p and let n be an integer relatively prime to p, and suppose k contains a primitive n-th root of unity, w. The map on $S = k[x_1,\ldots,x_r]$, $r \geq 2$, defined by $x_i \to wx_i$ and extending linearly, is a k-automorphism. The cyclic group G generated by this automorphism is the finite group Z/nZ. The fixed subring $R = S^G$ is the subalgebra of S generated by all monomials of degree n. Each x_i is a primitive element with minimal polynomial $f_i(t) = t^n - x_i^n$. It is easy to see that no height 1 prime ideal in S can contain all the elements $f_i'(x_i) = nx_i^{n-1}$, $i = 1,\ldots,r$. Thus, S is unramified at each height 1 prime in $X'(S)$, and by (3.6) is a height 1 Galois extension of R.

In particular, S is height 1 separable over R. We show that S is not R-separable. Let I be the ideal in R generated by all monomials of degree n. Then $R/I = k$, a field. Now, S/IS cannot be separable over the field R/I because S/IS contains the nilpotent elements $x_i + IS$. Thus, S is not separable over R.

We further note that, since R/I is a field, S/IS is not height 1 separable over R/I. However, this does not violate (1.8) because R/I does not satisfy NBU over R. We see that $X'(R/I) = \bar{0}$, and $\bar{0} \cap R = I$. Since $(0) \subsetneq (x_1^n) \subsetneq I$, I is not in $X'(R)$.

The next example is similar to (3.7) and is also found in [17; p.58].

EXAMPLE 3.8 Let k, w, and n be as in (3.7). Let $S = k[x,y]$. The map $x \to wx$, $y \to w^{-1}y$ is a k-automorphism of S. If we let $R = S^G$, where G is the group generated by this automorphism, then we see that $R = k[x^n,y^n,xy]$. As in (3.7), the elements x and y are primitive elements for the quotient field of S over that of R, with minimal polynomials $f_1(t) = t^n - x^n$ and $f_2(t) = t^n - y^n$, respectively. Since no height 1 prime ideal in S can contain both nx^{n-1} and ny^{n-1}, we see that S is unramified at the height 1 primes in $X'(S)$ and is therefore a height 1 Galois extension of R.

As in (3.7), we see that S is not separable over R. Let I be the ideal (x^n,y^n,xy) in R. Then $R/I = k$, and S/IS is not separable over R/I since S/IS contains nilpotent elements.

We remark that in both (3.7) and (3.8) we have the following situation: R is integrally closed and Noetherian and S is the integral closure of R is a separable field extension of the quotient field of R (see (5.7)). Since S is unramified at each minimal prime ideal in S but S is not R-separable, S is not R-projective [1; 3.7, p.759]. Thus the hypothesis that S be R-projective cannot be removed in [1; 3.7].

CHAPTER 4

HEIGHT 1 SEPARABLE CLOSURES

In this chapter we define what a height 1 separable closure is and show that under certain conditions a height 1 separable closure does exist.

There are two basic approaches to showing the existence of a separable closure of a connected base ring R . One way, as in [3; Chap. III] or [10], is to start with the base ring and construct larger and larger extensions having the desired properties, until the process finally stops, which it will, by a cardinality argument. The other approach, as used in [14] is to start at the top and tensor together representatives of all possible extensions with the required properties, and then "discard" just enough in order to arrive at a connected extension. The "discarding" is done by modding out an ideal generated by idempotents, leaving a connected quotient ring. We will show in Example 4.18 that certain difficulties arise if this second approach is used in the height 1 separable case. We will use the first method described above to construct a height 1 separable closure.

DEFINITION 4.1 An R-algebra S is called height 1 connected over R if S_p is connected for each p in $X'(R)$. If R is height 1 connected over itself we simply say that R is height 1 connected.

EXAMPLE 4.2 (A connected extension which is not height 1 connected.) Let R be the domain in Example 2.7. Recall that the element y in R is not a unit and is not in any height 1 prime of R. Let $S = R[t]/(t^2 - y^2)$. Then, as in (2.7), S is a strongly height 1 separable extension of R. S is not a domain, but S is connected, as we show next. Suppose $r + st$ is an element of S such that $(r + st)^2 = r + st$. Then $r^2 + 2rst + s^2t^2 = r^2 + s^2y^2 + 2rst = r + st$. Hence, $2rs = s$ and $r^2 + s^2y^2 = r$. If $s \neq 0$, then $r = \frac{1}{2}$, and we get $s^2y^2 = \frac{1}{4}$, or, $4s^2y^2 = 1$. But this cannot happen since y is not a unit in R, and so $s = 0$ and $r = 0$ or 1. S is not, however, height 1 connected. If p is in $X'(R)$, then y is a unit in R_p, and we see from the above that the element $\frac{1}{2} + \frac{1}{2}y^{-1}t$ is idempotent in S_p.

DEFINITION 4.3 S is height 1 separably closed if, whenever T is a strongly height 1 separable extension of S which is height 1 connected over S, then $T = S$.

If R is a height 1 connected ring an extension S is a height 1 separable closure of R if:

(i) S is a locally strongly height 1 separable extension of R; i.e., S is a direct limit of strongly height 1 separable subextensions of R; and

(ii) S is height 1 separably closed and height 1 connected over R.

In addition to the height 1 connectedness of the base ring R, we will need the following property in order to show the existence of a height 1 separable closure.

DEFINITION 4.4 An R-module M is height 1 divided over R (or simply height 1 divided, if there is no confusion) if, whenever a and b are in M with $a \neq b$ there is a prime ideal p in $X'(R)$ with $a_p \neq b_p$, where a_p and b_p denote the elements $a/1$ and $b/1$ in M_p; in other words, M is height 1 divided if, whenever m is in M and $m_p = 0$ for all p in $X'(R)$, then $m = 0$.

If M is height 1 divided, then the map $M \to \prod_{p \text{ in } X'(R)} M_p$ given by $m \to (m_p)_{p \text{ in } X'(R)}$ is an injection.

Notice that M will be height 1 divided if, for each nonzero m in M there is a p in $X'(R)$ with $\mathrm{Ann}_R(m) \subseteq p$. For then, if $m_p = 0$, there is an s in $R - p$ with $sm = 0$. But then s is in $\mathrm{Ann}_R(m) \subseteq p$, and so we must have $m = 0$. In particular, if R is a domain, R is height 1 divided.

EXAMPLE 4.5 (A ring which is not height 1 divided.) Let R' be a commutative ring, and M a maximal ideal in R' with rank greater than 2. Let $R = R' \oplus R'/M$, with multiplication defined by $(r,\bar{s})(r',\bar{s}') = (rr', r\bar{s}' + r'\bar{s})$, where r and s are in R' and $\bar{s}$ denotes the element $s + M$.

Let s be in $R' - M$, so that $(0,\bar{s}) \neq 0$ in R. Then we claim that $(0,\bar{s})_p = 0$ in R_p for each p in $X'(R)$. If p is in

$X'(R)$, then there exists an element $(r,\bar{t})$ in $R - p$, with r in M and t in R' ; for, if $(r,\bar{t})$ is in p for each r in M and t in R' , then $(P_0,R'/M) \subsetneq (P_1,R'/M) \subsetneq p$, where $P_0 \subsetneq P_1 \subsetneq M$ are primes in R' and $(P_i,R'/M)$ are primes in R . This contradicts the fact that p is in $X'(R)$. Then, we have $(r,\bar{t})(0,\bar{s}) = (r \cdot 0, r\bar{s} + 0\bar{t}) = (0,r\bar{s}) = (0,\bar{0})$, and so $(0,\bar{s})_p = 0$ in R_p , for each p in $X'(R)$. Note that for this element $(0,\bar{s})$, $\mathrm{Ann}_R((0,\bar{s})) = (M,R'/M)$, which is not contained in any prime in $X'(R)$.

LEMMA 4.6 Suppose R is height 1 divided and M is a finitely generated projective R-module. Then M is height 1 divided.

Proof: First, we show the lemma is true if M is a finite free R-module; i.e., $M = R^n$. This follows immediately from the observation that $(R^n)_p = (R_p)^n$, for p in $X'(R)$. Hence, if $(x_1,\ldots,x_n)$ is in R^n and $(x_1,\ldots,x_n)_p = 0$ for each p in $X'(R)$, then $(x_i)_p = 0$ for each p and so by hypothesis, $x_i = 0$, for $i = 1,\ldots,n$.

Now, if M is a finite projective R-module, then M is a direct summand of R^n , for some n . Since R^n is height 1 divided by the above, the result follows.

PROPOSITION 4.7 Suppose R is height 1 divided, T is a strongly height 1 separable R-algebra, S is a locally strongly height 1 separable R-algebra, and S is height 1 connected over R . Then, $|\mathrm{Alg}_R(T,S)|$ is bounded by cardinal depending only on R .

Proof: Let $S = \mathrm{dir\,lim}\, S_i$, where each S_i is a strongly height 1 separable R-algebra. Note that $\mathrm{Alg}_R(T,S) \subseteq \mathrm{Hom}_R(T,S)$. Also, since T is finitely presented over R , $\mathrm{Hom}_R(T,S)_p = \mathrm{Hom}_{R_p}(T_p,S_p)$, for each p in $X'(R)$. Thus we identify an element f_p in $\mathrm{Hom}_R(T,S)_p$ with the map $f_p : T_p \to S_p$ in $\mathrm{Hom}_{R_p}(T_p,S_p)$. We show first that $\mathrm{Hom}_R(T,S)$ is height 1 divided as an R-module. Suppose f is in $\mathrm{Hom}_R(T,S)$ and f_p is the zero map, for each p in $X'(R)$. Since T is finitely generated over R , there is an index i such that f belongs to $\mathrm{Hom}_R(T,S_i)$. Since T and S_i are finitely generated projective R-modules, $\mathrm{Hom}_R(T,S_i)$ is finitely generated and projective over R also. Thus, by (4.6), since f_p is in $\mathrm{Hom}_R(T,S_i)_p$ and is identically zero for each p in $X'(R)$, f must be identically zero also, showing that $\mathrm{Hom}_R(T,S)$ is height 1 divided.

Now, since T_p is a strongly separable R_p-algebra and S_p is connected, for p in $X'(R)$, we have $|\mathrm{Hom}_{R_p}(T_p,S_p)| \leq \mathrm{rank}_{R_p}(T_p)$ by [3; 1.6, p.88]. Thus, we have $\mathrm{Alg}_R(T,S) \subseteq \mathrm{Hom}_R(T,S) \subseteq \prod_{p \text{ in } X'(R)} \mathrm{Hom}_{R_p}(T_p,S_p)$ (this last inclusion by the height 1 dividedness of $\mathrm{Hom}_R(T,S)$), and we see that $|\mathrm{Alg}_R(T,S)| \leq \sum_{p \text{ in } X'(R)} \mathrm{rank}_{R_p}(T_p)$.

<u>COROLLARY 4.8</u> If R is height 1 divided and S is a locally strongly height 1 separable R-algebra, height 1 connected over R , then the cardinality of S is bounded by some cardinal M depending only on R .

<u>Proof:</u> Let $\mathcal{S}(R)$ be a set of representatives of isomorphism classes of strongly height 1 separable R-algebras. Since S is locally strongly height 1 separable over R , S is the union of strongly height 1 separable R-subalgebras S_i of S . Each S_i in this union is isomorphic to some S_i' in $\mathcal{S}(R)$. By (4.7),

$|\mathrm{Alg}_R(S'_i,S)|$ is bounded by $c_{S'_i} = \sum_{p \text{ in } X'(R)} \mathrm{rank}_{R_p}(S'_i)_p$. Thus, $|S| \le \sum c_{S'_i}|S'_i|$. If we let $M = \sum_{T \text{ in } S(R)} c_T|T|$, then we have $|S| \le M$.

Corollary 4.8 is the first step in showing the existence of a height 1 separable closure, as it provides a bound on the cardinality of the type of extension we need. The next major step is provided in Proposition 4.10. First, we prove a lemma.

<u>LEMMA 4.9</u> Suppose T is a direct limit of rings T_i , for i in some index set I , where each T_i is a strongly height 1 separable extension of R . If T_0 is any one of the T_i's , then T is a locally height 1 strongly separable T_0-algebra.

<u>Proof</u>: Let T_0 be one of the T_i's ; then $T = \operatorname{dir\,lim}_{i \ge 0} T_i$. We need to show that each T_i , for $i \ge 0$, is a height 1 strongly separable T_0-algebra.

Let p be in $X'(T_0)$, and let $p \cap R = Q$. Since T_0 is a projective R-module, Q is in $X'(R)$ [15; p.33]. Thus, $(T_i)_Q$ is strongly separable over R_Q , $(T_0)_Q \subseteq (T_i)_Q$, and $(T_0)_Q$ is also strongly separable over R_Q . Hence, $(T_i)_Q$ is strongly separable over $(T_0)_Q$ [3; 2.4, p.94]. Localizing further at p in $X'(T_0)$ shows that $((T_i)_Q)_p$ is strongly separable over $((T_0)_Q)_p$. The lemma will follow once we show that $((T_i)_Q)_p = (T_i)_p$. This isomorphism is given by the map $1/t(a/b) \to a/bt$, for a in T_i , b in $R - Q$, and t in $T_0 - p$. This map is well-defined, for if $1/t(a/b) = 1/s(c/d)$, then there is an r in $T_0 - p$ with $r(s \cdot a/b - t \cdot c/d) = 0$; thus, $bdr(s \cdot a/b - t \cdot c/d) = 0$, and so $r(dsa - btc) = 0$, showing $a/bt = c/ds$ in $(T_i)_p$. The map is clearly onto, and is injective: if $a/bt = c/ds$, with a,c in T_i , b,d in $R - Q$, and t,s in $T_0 - p$, then there is an r in $T_0 - p$ with $r(dsa - btc) = 0$; then, $rbd(s \cdot a/b - t \cdot c/d) = 0$, with rbd in $T_0 - p$, showing that $1/t(a/b) = 1/s(c/d)$ in $((T_i)_Q)_p$. This completes the proof.

PROPOSITION 4.10 If S is a strongly height 1 separable extension of T and T is a locally strongly height 1 separable extension of R , then S is a locally strongly height 1 separable extension of R .

Proof: Let $T = \operatorname{dir\,lim} T_i$, where each T_i is a strongly height 1 separable R-algebra. Since S is finitely presented over T , there is an index i , say $i = 0$, and a finitely presented T_0-module, say S_0 , such that $S = S_0 \otimes_{T_0} T$ [9; 6.3.3, p.134]. We claim that S_0 can actually be chosen to be a ring which is finitely generated and projectvie as a T_0-module. Let $x_1,\ldots,x_n$ generate S as a T-module. We may assume that S_0 contains each x_i and all the products $x_i x_j$, and that S_0 is generated as a T_0-module by $x_1,\ldots,x_n$. Since S is a projective T-module, there are elements $f_1,\ldots,f_n$ in $\operatorname{Hom}_T(S,T)$ such that $x = \sum f_i(x)x_i$, for each x in S . We may assume that T_0 has been picked large enough to contain all the $f_i(x_j)$, $i,j = 1,\ldots,n$. Then, for each x in S_0 , $f_i(x)$ is in T_0 . Thus, each f_i is an element of $\operatorname{Hom}_{T_0}(S_0,T_0)$, and the elements x_i,f_i , $i = 1,\ldots,n$, form a finite dual basis, showing that S_0 is a finite projective T_0-module. S_0 is a ring because it contains all the products $x_i x_j$ of the generators x_i .

Next, we show that S_0 is a height 1 separable extension of T_0. Note that T is integral over T_0 since it is integral over R. Now, since S is strongly height 1 separable over T, then S_p is separable over T_p for each p in $X'(T_0)$, by (1.15). Thus, $S_p = (S_0)_p \otimes_{(T_0)_p} T_p$ is T_p-separable. By (4.9), T_p is a locally strongly separable $(T_0)_p$-algebra. Then, T_p is faithfully flat over $(T_0)_p$, which implies that $(S_0)_p$ is separable over $(T_0)_p$ [20; 2.9, p.93]. Thus, S_0 is height 1 separable over T_0.

If we can show that $S_0 \otimes_{T_0} T_i$, $i \geq 0$, is a strongly height 1 separable R-algebra, then the proof will be complete since $S = S_0 \otimes_{T_0} T = \operatorname{dir\,lim}_{i \geq 0}(S_0 \otimes_{T_0} T_i)$. Now, if NBU holds for T_i over T_0, then by (1.7), $S_0 \otimes_{T_0} T_i$ is height 1 separable over T_i; we show next that NBU does hold for T_i over T_0. Let p be in $X'(T_i)$. Since T_i is flat as an R-module, Going Down holds, and $ht(p \cap R) = ht(p) \leq 1$. Since T_i is integral over T_0, $ht(p) \leq ht(p \cap T_0)$ [11; p.30,45]. Hence $ht(p) \leq ht(p \cap T_0) \leq ht((p \cap T_0) \cap R) = ht(p \cap R) \leq 1$, and so $p \cap T_0$ is in $X'(T_0)$.

Since $S_0 \otimes_{T_0} T_i$ is height 1 separable and finitely generated over T_i, and T_i is height 1 separable and integral over R, by transitivity (1.14) we have that $S_0 \otimes_{T_0} T_i$ is a height 1 separable R-algebra. Finally, since $S_0 \otimes_{T_0} T_i$ is a finitely generated projective T_i-module and T_i is a finitely generated projective R-module, we see that $S_0 \otimes_{T_0} T_i$ is a finitely generated projective R-module.

As remarked earlier, this finishes the proof.

LEMMA 4.11 If T is a height 1 connected extension of R and S is a height 1 connected extension of T, finitely generated and projective as a T-module, then S is a height 1 connected extension of R.

Proof: Let p be in $X'(R)$; then T_p is connected. Suppose there is an idempotent e in S_p. Then, $S_p = S_p e \times S_p(1 - e)$, where $S_p e$ and $S_p(1 - e)$ are finitely generated and projective over T_p. If Q is in $X'(T_p)$, then $Q' = Q \cap T$ is in $X'(T)$, and so $S_{Q'}$ is connected. Now, $S_{Q'} = (S_p)_Q$, and so $S_{Q'} = (S_p e)_Q \times (S_p(1 - e))_Q$. Thus, either $(S_p e)_Q$ or $(S_p(1 - e))_Q$ must be zero. Since T_p is connected, this implies that either $S_p e$ or $S_p(1 - e)$ is zero; i.e., either $e = 0$ or $1 - e = 0$, proving the lemma.

THEOREM 4.12 If R is height 1 connected and height 1 divided then a height 1 separable closure of R exists.

Proof: Suppose there is no height 1 separable closure of R. Then, whenever T is a locally strongly height 1 separable extension of R, height 1 connected over R, T is not height 1 separably closed; i.e., there exists an extension S which is strongly height 1 separable and height 1 connected over T, with T properly contained in S. By (4.10), S is a locally strongly height 1 separable extension of R; by (4.11), S is height 1 connected over R. Thus, if the theorem is false, there must exist a transfinite collection of extensions of R, $\{S_i\}$, indexed by a class of ordinals $\{i\}$, with the following properties:

(1) S_i is a locally strongly height 1 separable extension of R which is height 1 connected over R ; and,

(2) if $i \leq j$, then $S_i \subsetneq S_j$.

For (1), if i is a nonlimit ordinal the extension S_i of S_{i-1} exists by the discussion above; if i is a limit ordinal, then S_i is found by taking the direct limit of all S_j with $j < i$.

For each S_i in this collection we have $|S_i| \leq M$, where M is a fixed cardinal depending only on R , by (4.8). Thus, if we take k to be any ordinal with cardinality greater than M , then $i < k$ for all i in the collection $\{i\}$. Forming the direct limit of S_i , $i < k$, we have $|\text{dir lim } S_i| > M$, a contradiction. Thus, a height 1 separable closure for R exists.

It is at this point, if we continue to parallel the development of the usual separable closure of a connected ring, that we would like to be able to prove that a height 1 separable closure is unique up to isomorphism. This proves to be a stubborn result here. The key to proving uniqueness in the separable case is the fact that, if S is the separable closure of R and T is a strongly separable extension of R , then the number of R-algebra homomorphisms from T to S is precisely $\text{rank}_R(T)$ (see [3; Chap. III]). This allows one to prove that, if S and S' are both separable closures of R , then there exist R-algebra injections $S \to S'$ and $S' \to S$ and that the composition is an isomorphism. In the height 1 case, unfortunately, there seems to be no convenient way to determine the number of R-algebra homomorphisms from a strongly height 1 separable extension to a height 1 separable closure.

When R is a domain there is another technique for attempting to show uniqueness. If S is a height 1 separable closure of R which is a domain also, then S is contained in a separable closure of the quotient field of R, since the quotient field of S is a separable (algebraic) field extension of the quotient of R (see Chapter 5, (5.8)). Hence, if S and T are two height 1 separable closures of R, each a domain, we might as well assume they are both contained in the same separable closure of the quotient field of R. Suppose that $T = \text{dir lim } T_i$. If it is true that ST_i is a strongly height 1 separable extension of S, then $S = ST_i$, and so $S = \text{dir lim}(ST_i) = ST$. By symmetry we would have $T = ST$, and so $S = T$. (Note that the height 1 connectedness is no problem here since these are all domains.) Since the map $S \otimes T_i \to ST_i$ is a surjection, ST_i will be a finitely generated height 1 separable extension of S provided that NBU is satisfied for the R-algebra S. This we will show in (4.13). The problem is to show that ST_i is a projective S-module. In the next chapter we will look at a certain class of domains and weaken the definition of height 1 separable closure somewhat by replacing "strongly height 1 separable" with "height 1 strongly separable." There we will be able to prove uniqueness as outlined above. Thus, it appears that, although the height 1 separable closure found here is a maximal extension satisfying the given conditions, it is not quite large enough to be unique.

We finish this chapter with a few more general results related to the above discussion and one more example.

PROPOSITION 4.13 Suppose $S = \text{dir lim } S_i$, where each S_i is an extension of R finitely generated and projective as an R-module. Then

Going Down holds for S over R .

Proof: Suppose $P' \subsetneq P$ are primes in R and Q is a prime in S with $P = Q \cap R$. We need to find a prime Q' in S , $Q' \subsetneq Q$, with $Q' \cap R = P'$.

Let T_i be the set of all primes J in S_i with $J \subseteq Q \cap S_i$ and $J \cap R = P'$. Since Going Down is satisfied for S_i over R (S_i is a flat R-module), T_i is not empty. Further, since S_i is a finitely generated R-module, T_i is finite [2; p.327].

If $S_i \subseteq S_j$, the map $T_j \to T_i$ given by $J \to J \cap S_i$, is well-defined. Thus, the sets T_i form an inverse system of nonempty, compact Hausdorff spaces (in the discrete topology), and so the inverse limit is nonempty. If $(J_i)_{i \in I}$ is in this inverse limit, we have $J_i \cap R = P'$ for each i , and, if $S_i \subseteq S_j$, $J_i = J_j \cap S_i$. Let $Q' = \text{dir lim } J_i$, and we see that Q' is prime in S , $Q' \subseteq Q$, and $Q' \cap R = P'$.

LEMMA 4.14 If R is an integrally closed domain and S is a strongly separable connected extension of R , then S is a domain.

Proof: Let $R = \text{dir lim } R_i'$, where each R_i' is a finitely generated algebra over the integers, hence, Noetherian. Then, since the ring of integers is a Nagata ring (see [15; p.231]), each R_i' is Nagata [15; p.240]. This implies the integral closure of each R_i' , say R_i , is a finite R_i'-module (a property of Nagata rings). So each R_i is Noetherian and integrally closed, and we have $R = \text{dir lim } R_i$. Since S is a strongly separable extension of R , there is an index i , say $i = 0$, and a strongly separable extension of R_0 , say S_0 , such that $S = S_0 \otimes_{R_0} R$ [14; III.6, p.49]. Then $S = \text{dir lim}(S_0 \otimes_{R_0} R_i)$, limit taken over $i \geq 0$.

Since S_0 is strongly separable over R_0, $S_0 \otimes_{R_0} R_i$ is strongly separable over R_i. Hence, $R_i \subseteq S_0 \otimes_{R_0} R_i$ [3; 2.3, p.94]. Also, $S_0 \otimes_{R_0} R_i \subseteq S_0 \otimes_{R_0} R = S$ because S_0 is R_0-flat, and so we have $S_0 \otimes_{R_0} R_i$ is connected. Then, by [10; 4.2, p.473], $S_0 \otimes_{R_0} R_i$ is a domain, and so S itself is a domain.

PROPOSITION 4.15 Let R be an integrally closed domain and let S be a strongly height 1 separable extension of R with S_p connected for all p in $X'(R)$ with $ht(p) = 1$. Then S is a domain.

Proof: Let K be the quotient field of R. Since K is flat over R, $K \otimes S$ is height 1 separable over K by (1.7). Since K is a field, this means that $K \otimes S$ is a separable K-algebra. Thus we have $K \otimes S = (K \otimes S)e_1 \times \cdots \times (K \otimes S)e_n$, where each e_i is an idempotent in $K \otimes S$, and each $(K \otimes S)e_i$ is a field.

Consider the map $S \to (1 \otimes S)e_i$ given by $s \to (1 \otimes s)e_i$. Let P_i be the kernel of this map. Each P_i is a prime ideal in S since $(1 \otimes S)e_i$ is a domain. Suppose r is in R and $(1 \otimes r)e_i = 0$. Then $(1 \otimes r)e_i = r(1 \otimes 1)e_i = 0$, which implies that $r = 0$. Hence, $P_i \cap R = (0)$, and P_i is a minimal prime in S, as S is integral over R.

Suppose x belongs to $\cap P_i$. Then, $(1 \otimes x)e_i = 0$ for each i. Since S is flat over R, S may be identified with $1 \otimes S \subseteq K \otimes S$. Identifying x with $1 \otimes x$, we have $x = (1 \otimes x)1 = (1 \otimes x)\sum e_i = \sum (1 \otimes x)e_i = 0$. Thus $\cap P_i = (0)$. Notice also that if $xy = 0$, for x and y in S and x not in any P_i, then y

belongs to every P_i , and so $y = 0$. Hence, the set of zero divisors in S is contained in $\cup P_i$. Now we can show that S is domain. Suppose x and y are nonzero elements of S with $xy = 0$. If Q is a height 1 prime ideal in R , then S_Q is a domain. ([10; p.473].) Then, in S_Q , either x or y is zero; say $x = 0$. Then there is an element q in $R - Q$ with $qx = 0$, and so q must be in some P_i . But then, q belongs to $P_i \cap R = (0)$, a contradiction. So $xy \neq 0$, and S is a domain.

We remark that Example 4.2 shows that if S is not height 1 connected over R , then S may not be a domain even if S itself is connected and strongly height 1 separable over R .

COROLLARY 4.16 If R is an integrally closed domain, then a height 1 separable closure of R is also a domain.

Proof: First note that if R is a domain, then R is height 1 divided and height 1 connected, so R has a height 1 separable closure S . Then, S is a direct limit of strongly height 1 separable, height 1 connected extensions of R , each of which is a domain by (4.15). Thus, S is a domain.

PROPOSITION 4.17 Let R be an integrally closed domain and let S and T be strongly height 1 separable, height 1 connected extensions of R . Assume that S and T are contained in the same separable closure of the quotient field of R . Then, ST is a height 1 strongly separable extension of R .

Proof: Notice that S and T are domains whose quotient fields are separable algebraic extensions of the quotient field of R (see

(5.7)), so it makes sense to assume they are both contained in the same separable closure. Then, ST is also a domain contained in this separable closure. Let p be in $X'(R)$. Then $S_p \otimes_{R_p} T_p$ is a strongly separable extension of R_p. The multiplication map $m : S_p \otimes T_p \to (ST)_p$ is a surjection, and so $(ST)_p$ is a finitely generated separable extension of R_p.

Since $S_p \otimes T_p$ is finitely generated, projective, and separable over R_p, we have that $S_p \otimes T_p$ is a product of domains $D_1 \times \cdots \times D_n$ (this follows from (4.14)). Each D_i is a strongly separable extension of R_p. Let Q be the kernel of the map m. Then Q is a prime ideal in $S_p \otimes T_p = D_1 \times \cdots \times D_n$, and so Q must be of the form $D_1 \times \cdots \times Q_i \times \cdots \times D_n$ (all factors are D_j except the i-th) for some prime ideal Q_i in D_i. Since $R_p \subseteq S_p \otimes T_p$, $Q \cap R_p = (0)$. Thus, $Q_i \cap R_p = (0)$, and, since D_i is integral over R_p, $Q_i = (0)$. Hence, $(ST)_p = (S_p \otimes T_p)/Q = D_i$. This makes $(ST)_p$ projective over $S_p \otimes T_p$ and therefore projective over R_p. We have shown that ST is a height 1 strongly separable extension of R.

We noted at the beginning of this chapter an alternative way of constructing a separable closure of a connected ring [14; Chap. III]. An important part of this construction is the following: If R is connected, T a locally strongly separable extension of R and I an ideal of T generated by idempotents, then T/I is also a locally strongly separable extension of R [14; Thm. III.5, p.48]. The proof of this depends on the fact that, if $T = \text{dir lim } T_i$, where each T_i is a strongly separable subextension of T, then $I \cap T_i$ is an ideal in T_i generated by idempotents. Our next example will show that this can fail to hold for locally strongly height 1 separable extensions.

For reference we recall the definition of the Boolean spectrum $X(R)$ of a commutative ring (see, e.g., [19] or [14; Chap. II]). $X(R)$ is the set of all maximal Boolean ideals of R, where a maximal Boolean ideal is a nonempty set E of idempotents in R satisfying the following:

(i) for every idempotent e of R, either e or $1 - e$ is in E, but not both; and,

(ii) if e and f are idempotents of R then ef is in E if and only if e is in E or f is in E.

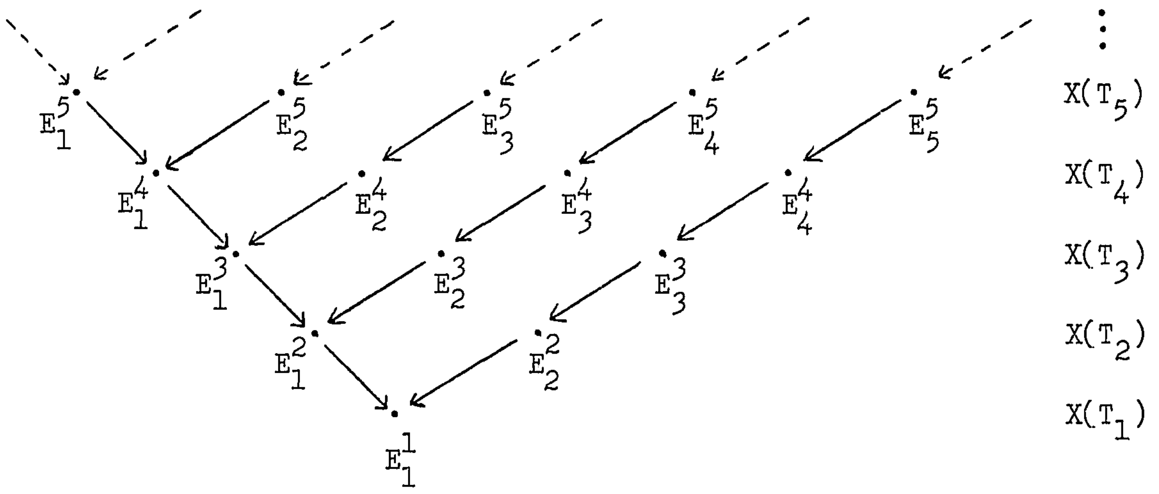
If $f : R \to S$ is a ring homomorphism, the induced continuous map $\mathrm{Spec}(f) : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ in turn induces a continuous map $X(f) : X(S) \to X(R)$. This map takes a maximal Boolean ideal of idempotents in $X(S)$ to the set of idempotents that map into it under f.

<u>EXAMPLE 4.18</u> Let R and S be as in Example 4.2. Thus, $S = R[t]/(t^2 - y^2)$ is a connected strongly height 1 separable extension of R. Define the map $h' : S \to R \times R$ by $h'(r + st) = (r + sy, r - sy)$. Then h' is an injection. Let $h : S \to S \times R$ be the composition of h' with the inclusion map $R \times R \hookrightarrow S \times R$. We have the following directed system: $S \xrightarrow{h} S \times R \xrightarrow{h \times 1} S \times R \times R \xrightarrow{h \times 1 \times 1} S \times R \times R \times R \to \cdots$, where 1 is the identity on R. To simplify the notation, rewrite this sequence as $T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} T_3 \xrightarrow{f_3} \cdots$, where $T_1 = S$, $T_2 = S \times R$, $f_1 = h$, $f_2 = h \times 1$, etc. Each T_i is a strongly height 1 separable extension of R, and so $T = \mathrm{dir\,lim}\, T_i$ is a

locally strongly height 1 separable extension of R. We will show that $X(T) = \text{proj lim } X(T_i)$ is homeomorphic to the subspace $\{1,1/2,1/3,\ldots,0\}$ of the real numbers.

Each T_n has the n minimal idempotents $e_1^n = (1,0,\ldots,0),\ldots,$ $e_n^n = (0,\ldots,0,1)$. Thus each $X(T_n)$ consists of n maximal Boolean ideals E_j^n, $j = 1,\ldots,n$, where E_j^n is the set of all idempotents in T_n with a 0 in the j-th component.

Consider the map $f_i : T_i \to T_{i+1}$. For any idempotent e in T_i, $f_i(e)$ is the idempotent in T_{i+1} obtained by writing down the first component of e and then writing down e itself as the next i components; i.e., $f_i(e)$ is just e with the first component repeated. Thus, f maps E_1^i into both E_1^{i+1} and E_2^{i+1}, but for $j > 2$, f maps E_j^i into E_{j+1}^{i+1} only. So the map $X(f_i) : X(T_{i+1}) \to X(T_i)$ is given by $X(f_i)(E_j^{i+1}) = \begin{cases} E_1^i, & \text{for } j = 1,2 \\ E_{j-1}^i, & \text{for } j \geq 3 \end{cases}$. We may picture the inverse system $\{X(T_i),X(f_i)\}$ as in the following diagram:



We see that $X(T) = \text{proj lim } X(T_i)$ consists of the following coherent sequences:

$$S_1 = (E_1^1, E_2^2, E_3^3, \ldots)$$

$$S_2 = (E_1^1, E_1^2, E_2^3, E_3^4, \ldots)$$

$$S_3 = (E_1^1, E_1^2, E_1^3, E_2^4, E_3^5, \ldots)$$

$$\vdots$$

$$S_0 = (E_1^1, E_1^2, \ldots, E_1^i, \ldots) \quad .$$

The bijection between $X(T)$ and $\{1, \frac{1}{2}, \ldots, 0\}$ is given by $S_i \leftrightarrow 1/i$, $S_0 \leftrightarrow 0$. Moreover, the topologies of the two spaces are equivalent. As a subspace of the reals, $\{1, \frac{1}{2}, \ldots, 0\}$ has as a basis of open sets the sets $\{1/n\}$, $\{1/n, 1/n+1, 1/n+2, \ldots, 0\}$, for $n \geq 1$. As a subspace of $\Pi\, X(T_i)$, where each $X(T_i)$ has the discrete topology, $X(T)$ has as a basis of open sets all sets of the form $(A_1 \times A_2 \times \cdots) \cap X(T)$, where $A_i = X(T_i)$ for all but finitely many indices. From the above diagram it is easy to see that these sets are precisely $\{S_n\}$, $\{S_n, S_{n+1}, \ldots, S_0\}$, for $n \geq 1$.

We claim now that if I is the ideal in T generated by the idempotents in the element S_0 of $X(T)$, then $T/I = R$. Let $I_i = I \cap T_i$. Then $T/I = \text{dir lim } T_i/I_i$.

We show first that $R \subseteq T/I$. Since $R \subseteq S$, R is embedded in T by the map which identifies an element r in R with the equivalence class of elements $\{r, (r,r), (r,r,r), \ldots\}$ in $\text{dir lim } T_i = T$. Suppose this equivalence class is the zero element in T/I. Then, for some index i, $(r, r, \ldots, r)$ is in I_i. Since the generators of I

all have a 0 in the first conponent, the first component of each element in I_i is 0 ; i.e., $r = 0$, and so $R \subseteq T/I$.

Next we show that R actually maps onto T/I . For any index i , $T_i = S \times R \times \cdots \times R$. Since $T_i/I_i \subseteq T/I$ and T/I is connected, T_i/I_i is also connected. We see that $S \times R \times \cdots \times R$ maps onto T_i/I_i , and so all but one of the minimal idempotents in $S \times R \times \cdots \times R$ maps onto 0 in T_i/I_i , and the remaining idempotent maps onto the identity. Thus, either S or R maps onto T_i/I_i . Now, S maps to T_i/I_i by $r + st \to (r + sy, r - sy, \ldots, r - sy) + I$. The element $r + sy$ in R maps to $(r + sy, r + sy, \ldots, r + sy) + I$. Note that $(r + sy, r + sy, \ldots, r + sy) - (r + sy, r - sy, \ldots, r - sy) = (0, 2sy, \ldots, 2sy)$. Since sy is not a unit in R , the equivalence class containing $(0, 2sy, \ldots, 2sy)$, namely $\{(0,2sy,\ldots,2sy),(0,2sy,\ldots,2sy,2sy),\ldots\}$, is in I . Thus, $(0,2sy,\ldots,2sy)$ is in I_i , and we see that $r + st$ and $r + sy$ map onto the same coset in T_i/I_i . So, if S maps onto T_i/I_i , this mapping factors through R , and R maps onto T_i/I_i . Therefore, R maps onto T/I , and we have that $T/I = R$.

Consider now the ideal I_2 in $T_2 = S \times R$. Since $R \subseteq T_2/I_2 \subseteq T/I = R$, we have $T_2/I_2 = R$. If I_2 is generated by idempotents in $S \times R$, then I_2 must be generated by the idempotent $(0,1)$. But then $T_2/I_2 = S$, a contradiction. Hence, I_2 is not generated by idempotents.

We remark that the above proof fails if $S = R[t]/t^2 - y^2$ is strongly separable over R (as it should, by the proof of III.5, p.48 in [14]), because then y is a unit in R and R does not map onto T/I , as in the proof above.

CHAPTER 5

SEMI-KRULL DOMAINS

In this chapter we define a class of domains called semi-Krull domains, and, by generalizing the notion of a height 1 separable closure, we are able to show that a unique (up to isomorphism) height 1 separable closure exists for a semi-Krull domain. We then prove a Galois correspondence theorem, showing that there is a one-to-one correspondence between integrally closed height 1 strongly separable subextensions of the height 1 separable closure S of the semi-Krull domain R and closed groups of finite index of the full automorphism group of S over R, in the usual sense of Galois theory.

Throughout this chapter we will make a slight change in the meaning of $X'(R)$: $X'(R)$ will denote the set of all prime ideals in R which are of height one. In this chapter R will always be a domain; if S_p is R_p-separable for some height 1 prime p in R, then it follows automatically that $S_{(0)}$ is $R_{(0)}$-separable (by a further localization). Also in this chapter, by a finite height 1 strongly separable extension of R we will mean a height 1 strongly separable extension which is finitely generated as an R-module.

We recall that a domain R is a Krull domain if R_p is a discrete valuation ring (DVR) for each p in $X'(R)$, $R = \cap R_p$, the intersection taken over all p in $X'(R)$, and if any nonzero element of R lies in at most a finite number of primes in $X'(R)$. A Krull domain is integrally closed. It turns out that Krull domains provide a convenient setting in which to work with height 1 separability. However, the condition that $R = \cap R_p$ will not be needed in what we do, so we make the following definition.

DEFINITION 5.1 A domain R is a semi-Krull domain if:

(i) R is integrally closed,

(ii) R_p is a DVR for each p in $X'(R)$, and

(iii) each nonzero element of R is contained in at most a finite number of primes in $X'(R)$.

EXAMPLE 5.2 (A semi-Krull domain which is not Krull.) Let R be the domain of Example 2.7. In (2.8) we showed that R is integrally closed. We show next that if p is in $X'(R)$, then R_p is a DVR. Recall that $R = R'_M$, where $R' \subseteq k[x,y]$ contains all polynomials with no term in a power of x alone and M is the maximal ideal of all polynomials with zero constant term. Since a height 1 prime in R corresponds to a height 1 prime of R' contained in M, we may as well just show that R'_p is a DVR, where p is in $X'(R')$ and p is contained in M. Since y is not in p (recall that y is not an element of any height 1 prime contained in M), we have $x = xy/y$ is in R'_p. Thus, $k[x,y] \subseteq R'_p$. Let $P_0 = pR'_p \cap k[x,y]$. Then $k[x,y]_{P_0} \subseteq R'_p$.

Suppose that f/g is in R'_p, where f is in R' and g is in $R' - p$. If g were in P_0, then we would have g in pR'_p, say $g = a/b$, for a in p and b in $R' - p$. But then $gb = a$ is in p, a contradiction. Thus, f/g is in $k[x,y]_{P_0}$, and we have $k[x,y]_{P_0} = R'_p$. This shows that R'_p is Noetherian. Since R'_p is a one-dimensional integrally closed Noetherian domain, R'_p is a PID (see [11; #14, p.73]). Then, because R'_p is a PID and a local ring distinct from its quotient field, R'_p is a DVR [4; 7.7, p.50].

We show now that a nonzero element of R cannot lie in infinitely many height 1 prime ideals. If $f/g \neq 0$ is in $R = R'_M$ and f/g lies in a height 1 prime, say pR'_M, where p is a height 1 prime of R' contained in M, then f is in p. So if f/g lies in infinitely many height 1 primes of R, then f is in infinitely many height 1 primes of R' contained in M. We show this cannot happen in R'.

Suppose $f \neq 0$ is in R' and f lies in p, a height 1 prime of R' contained in M. As above, let $P_0 = k[x,y] \cap pR'_p$, and we have that $k[x,y]_{P_0} = R'_p$. Hence, P_0 is a height 1 prime of $k[x,y]$, and f is in P_0. If P and Q are height 1 primes of R' with $P \neq Q$, then $R'_P \neq R'_Q$. To see this, pick an element b in $Q - P$ and an a in $R' - Q$. Then a/b is in R'_P. If $a/b = c/d$, where c is in R' and d is in $R' - Q$, then $ad = bc$. But this cannot happen, since ad is in $R' - Q$ and bc is in Q. Thus, $R'_P \neq R'_Q$. Now, since $k[x,y]_{P_0} = R'_P$ and $k[x,y]_{Q_0} = R'_Q$, where P_0 and Q_0 are as defined above, then $P_0 \neq Q_0$ if $P \neq Q$. Thus, if f lies in infinitely many height 1 primes of R' , then f lies in infinitely many height 1 primes of $k[x,y]$. But this cannot happen since $k[x,y]$ is Krull. Hence, we have shown that R is a semi-Krull domain.

Finally, we show that R is not Krull; i.e., that $R \neq \cap R_p$, the intersection taken over all height 1 primes p . As we have noted, the element $x = xy/y$ lies in each R_p , for p in $X'(R)$. But x is not in $R = R'_M$, for if $x = f/g$, for f in R' and g in $R' - M$, then $gx = f$. But this is a contradiction since the left side contains an x-term (g has a nonzero constant) and f contains no terms in x alone.

Before we examine the height 1 separable closure of a semi-Krull domain, we prove two very useful results.

DEFINITION 5.3 An R-algebra S is height 1 separable almost everywhere in $X'(R)$ if S_p is a separable R_p-algebra for all but possibly a finite number of primes p in $X'(R)$.

PROPOSITION 5.4 Let R be semi-Krull and let S , a domain, be an R-algebra finitely generated as an R-module such that the quotient field of S is a finite separable field extension of the quotient field of R . Then S is height 1 separable almost everywhere in $X'(R)$.

Proof: Let L,K be the quotient fields of S,R , respectively. Since L is a finite separable extension of K , $L = K(u)$, for some primitive element u in S . Let $f(x)$ be the minimal polynomial of u . Since u is integral over R and R is integrally closed, $f(x)$ is in $R[x]$ [2; pp.310,311]. Let r be the resultant of f and f' , $r = \mathrm{Res}(f,f')$. If b is the leading coefficient of f' , then, since r and b are in R , there are at most a finite number of primes in

$X'(R)$ containing r or b.

Suppose p is in $X'(R)$ and neither r nor b is in p. Let $T_p = R_p[x]/(f)$. We show that T_p is a strongly separable R_p-algebra. Clearly T_p is a finitely generated projective R_p-module. Let k be the field R_p/pR_p. Then, $T_p/(pR_p)T_p = (R_p[x]/(f))/(pR_p)T_p = k[x]/(f)$. Since $\mathrm{Res}(f,f')$ is not in p (and b is not in P), $\mathrm{Res}(f,f') \neq 0$ in k. Hence, f is a separable polynomial. Since pR_p is the only maximal ideal in R_p and T_p is a finite R_p-module, it follows that T_p is a separable R_p-algebra.

Next we show that $S_p = T_p$. Note that $T_p \subseteq S_p$ since T_p and S_p are both contained in L. Since R_p is a PID and S_p is a torsion-free R_p-module, we see that S_p is R_p-flat [2; 3.2, p.15]. Since S_p is also finitely generated as an R_p-module, we have that S_p is R_p-projective [5; 2, p.287]. Now, with T_p separable over R_p and S_p projective over R_p, S_p is projective over T_p [3; 2.3, p.48]. Thus, $S_p = T_p \oplus M$, for some projective T_p-module M. We claim that $M = (0)$. We have $S_p \otimes_{R_p} K = L$. Thus,

$(T_p \oplus M) \otimes_{R_p} K = (T_p \otimes K) \oplus (M \otimes K) = K[x]/(f) \oplus (M \otimes K) = L \oplus (M \otimes K)$.

Hence, $L \oplus (M \otimes K) = L$, and so $M \otimes K = (0)$. Since M is a free R_p-module we have $M = (0)$. Thus, $S_p = T_p$, and the proof is complete.

<u>PROPOSITION 5.5</u> If R is a PID and S is a domain which is strongly separable over R, then S is integrally closed.

<u>Proof</u>: Let L be the quotient field of S and let T be the integral closure of S in L. Let x be in T. Then $S[x]$ is a torsion-free R-module, and is thus flat over R. $S[x]$ is also

finitely generated as an R-module, and so by [5; 2, p.287], $S[x]$ is a projective R-module. Furthermore, since S is R-separable, we have that $S[x]$ is projective as an S-module [3; 2.3, p.48]. Thus, $S[x] = S \oplus M$, for some projective S-module M. We show that $M = (0)$. We have $S[x] \otimes_S L = L[x] = L$. Also, $(S \oplus M) \otimes_S L = (S \otimes_S L) \oplus (M \otimes_S L) = L \oplus (M \otimes_S L)$. Hence, $M \otimes_S L = (0)$, and, since M is S-projective, we have $M = (0)$. Therefore $S = S[x]$, proving that x is in S, and so S is integrally closed.

We now define the height 1 separable closure of a semi-Krull domain.

<u>DEFINITION 5.6</u> Let R be a semi-Krull domain. A domain S which is an extension of R is said to be a <u>height 1 separable closure of</u> R if the following hold:

(i) S is a locally finite height 1 strongly separable extension of R; i.e., S is a direct limit of finite height 1 strongly separable subextensions of R.

(ii) Whenever T is a domain which is a finite height 1 strongly separable extension of S, then $T = S$; i.e., S is <u>height 1 separably closed</u>.

We will show that a height 1 separable closure exists in these circumstances and is unique up to isomorphism.

<u>LEMMA 5.7</u> Suppose S and T are domains with quotient fields K and L, respectively. If T is a height 1 separable extension of S then L is a finite separable (algebraic) field extension of K.

<u>Proof</u>: Let p be in $X'(S)$. Then, since T_p is

S_p-separable, we see that $(S-0)^{-1}T_p = (S-0)^{-1}T$ is a separable K-algebra. The field L is just a further localization of $(S-0)^{-1}T$, and so, by transitivity we have that L is a separable K-algebra. Since K is a field, then L must be a finite separable (and hence, algebraic) field extension of K.

<u>COROLLARY 5.8</u> If S and T are domains such that T is a locally height 1 separable extension of S, then the quotient field of T is a separable algebraic field extension of the quotient field of S.

<u>Proof</u>: Let L,K be the quotient fields of T,S, respectively. Any finitely generated sub-K-algebra of L is contained in the quotient field of some height 1 separable extension of S. This quotient field is a separable algebraic extension of K by (5.7). Thus T itself is a separable algebraic field extension of K.

A useful characterization of strong separability which we will use in the next theorem is given in [3; 2.1, p.92]. We note it here for reference.

<u>5.9</u> S is a strongly separable extension of R if and only if there is an element t in $\mathrm{Hom}_R(S,R)$ and elements $x_1,\ldots,x_n,y_1,\ldots,y_n$ in S such that $\sum x_j y_j = 1$ and $\sum x_j t(y_j x) = x$ for all x in S.

<u>THEOREM 5.10</u> If R is a semi-Krull domain, then a height 1 separable closure of R exists.

<u>Proof</u>: Suppose S is a domain which is a locally finite height 1 strongly separable extension of R. If p is any prime in $X'(R)$, then S_p is a locally strongly separable extension of R_p, and so the

cardinality of S_p is bounded by some cardinal number depending only on R_p [3; 3.3, p.103]. Since $S \subseteq S_p$, we see that the cardinality of S is also bounded by a cardinal number depending only on R.

Now let T be a domain containing S and a finite height 1 strongly separable extension of S. We will show that T is a locally finite height 1 strongly separable extension of R.

Let $x_1, \ldots, x_n$ generate T as an S-module. Let $x_i x_j = \sum a_{ijk} x_k$, for a_{ijk} in S. Pick a finite subset F of T. Each element f of F is of the form $f = \sum s_i x_i$, s_i in S. If $S = \text{dir lim } S_i$, where each S_i is a finite height 1 strongly separable R-subalgebra of S, then there is an index i, say $i = 0$, such that S_0 contains all the elements a_{ijk} and s_i described above. Let $T_0 = \sum S_0 x_i$. Then T_0 is an S_0-algebra, finitely generated as an S_0-module.

Without loss of generality we may assume that the quotient field of T_0 is a finite separable field extension of the quotient field of R. (See Remark 5.11 following the theorem.) Thus, by (5.4), T_0 is height 1 separable over R almost everywhere in $X'(R)$.

Suppose that p is in $X'(R)$ and $(T_0)_p$ is not R_p-separable. We will show that we can replace S_0 by some S_i, $i \geq 0$, so that the "new" $(T_0)_p$ is R_p-separable, and that raising S_0 like this will not affect the separability of T_0 at primes in $X'(R)$ where it was already separable. We show this last point first.

If $S_0 \subseteq S_i$, let $T_i = \sum S_i x_i$. Suppose that $(T_0)_Q$ is R_Q-separable for Q in $X'(R)$. By the construction of T_i, we see that the map $S_i \otimes_R T_0 \to T_i$ is onto, and hence, the map

$(S_i)_Q \otimes_{R_Q} (T_0)_Q \to (T_i)_Q$ is onto. Since $(S_i)_Q$ and $(T_0)_Q$ are each R_Q-separable, it follows that $(S_i)_Q \otimes (T_0)_Q$ is R_Q-separable, and therefore $(T_i)_Q$ is separable over R_Q. Now $(T_i)_Q \subseteq T_Q$, a domain, and so $(T_i)_Q$ is a torsion-free R_Q-module. Since $(T_i)_Q$ is finitely generated as an R_Q-module, we have $(T_i)_Q$ is projective over R_Q. Thus we see that $(T_i)_Q$ is strongly separable over R_Q. We have shown that if we replace S_0 by a larger S_i and therefore T_0 by T_i, T_i is strongly separable at any prime in $X'(R)$ where T_0 is strongly separable. (Notice that if p is in $X'(R)$ and $(T_0)_p$ is R_p-separable, then $(T_0)_p$ is automatically strongly separable over R_p just as in the argument above for $(T_i)_Q$.)

We now return to the first part of the claim. Suppose that p is in $X'(R)$ and T_{0_p} is not separable over R_p. We know that T_p is separable over S_p by (1.15). Further, we show that T_p is S_p-projective. As in the proof of (1.14), we have that any maximal ideal in S_p is actually in $X'(S_p)$. If Q is in $X'(S_p)$, then $Q' = Q \cap S$ is in $X'(S)$, and so $T_{Q'}$ is strongly separable over $S_{Q'}$. Therefore, $(T_{Q'})_p = T_{Q'} \otimes R_p$ is a strongly separable extension of $(S_{Q'})_p$. Since $(T_p)_Q = (T_{Q'})_p$ and $(S_p)_Q = (S_{Q'})_p$, we see that each localization of T_p at a maximal ideal of S_p yields a finitely generated flat module. Thus, T_p is a flat S_p-module. Since T_p is finitely generated and flat over S_p, T_p is projective over S_p by [5; 2, p.287]. Hence, T_p is strongly separable over S_p.

To simplify the notation we will drop the subscript p for the moment. We have the following situation: T is a strongly separable extension of S , S is a locally strongly separable extension of R , $S_0 \subseteq S$ is a strongly separable extension of R , and $T_0 \subseteq T$ is an S_0-algebra, finitely generated as a module. There are elements z_i, y_i $i = 1, \ldots, n$, in T and t in $\mathrm{Hom}_S(T,S)$ such that $\sum z_i y_i = 1$ and $\sum z_j t(y_j x) = x$, for each x in T . Let $z_i = \sum a_{ij} x_j$ and $y_i = \sum b_{ij} x_j$, for a_{ij} and b_{ij} in S . If necessary, replace S_0 by a larger S_i so that S_0 contains all a_{ij} , b_{ij} , and $t(x_i)$. Then $t(\sum s_i x_i) = \sum s_i t(x_i)$ is in S_0 , and so $t|_{T_0}$ is in $\mathrm{Hom}_{S_0}(T_0, S_0)$. Also, T_0 contains the elements z_i, y_i . Hence, T_0 is strongly separable over S_0 , and thus strongly separable over R . In our original notation, we have shown that T_{0_p} is strongly separable over R_p .

We may repeat this process finitely many times, if necessary, in order to obtain $T_0 \subseteq T$ such that T_0 is a finite height 1 strongly separable extension of R . Since we can find such a T_0 containing any finite subset of T , we have shown that T is a locally finite height 1 strongly separable extension of R .

A cardinality argument as in (4.12) shows that a height 1 separable closure of R must exist.

REMARK 5.11 In the proof of (5.10) it was noted that the quotient field of $T_0 = \sum S_0 x_i$ could be assumed to be a finite separable field extension of the quotient field of R . This can be assured by picking a large enough S_0 , as we show now. Let L, K, L_0, K_0 be the quotient fields of T, S, T_0, S_0 , respectively. We know that L is a finite separable field extension of K by (5.7). Hence, L is strongly separable as a K-algebra. Let $L = K(u)$, for a primitive element u in S. Each element x_i in $T \subseteq L$ (recall that $x_1, \ldots, x_n$ generate T over S) can be expressed as a polynomial in u, say $f_i(u)$, with coefficients in K. Now, if necessary, raise S_0 so that S_0 contains the element u and K_0 contains all the coefficients of the $f_i(u)$. Then $L_0 = K_0(u)$.

Since L is strongly separable over K, there are the elements z_i, y_i, $i = 1, \ldots, m$, in L and t in $\mathrm{Hom}_K(L,K)$ as in (5.9). If necessary, raise S_0 again so that all the z_i, y_i are in L_0 and $t(u)$ is in K_0. Then, $t|_{L_0}$ is in $\mathrm{Hom}_{K_0}(L_0, K_0)$, and it follows that L_0 is a strongly separable extension of K_0. K_0 is a finite separable extension of the quotient field of R by (5.7), and so we have that L_0 is also a finite separable field extension of the quotient field of R.

Before we prove uniqueness, we note the following.

<u>LEMMA 5.12</u> If R is semi-Krull and $S = \operatorname{dir\,lim} S_i$, where each S_i is a domain extension of R, finitely generated as an R-module, then NBU holds for S over R.

<u>Proof:</u> Let Q be in $X'(S)$ and let $p = Q \cap R$. Suppose there is a prime P' in $\mathrm{Spec}(R)$ with $(0) \subsetneq P \subsetneq p$. Note that S is a domain, and since each S_i is finitely generated over R, S is integral over R. Then, since R is integrally closed, there is a prime Q' in $\mathrm{Spec}(S)$ with $Q' \subseteq Q$ and $Q' \cap R = P'$ [2; 4, p.343], a contradiction. Thus p is in $X'(R)$.

THEOREM 5.13 The height 1 separable closure of a semi-Krull domain R is unique, up to isomorphism.

Proof: If S is a height 1 separable closure of R, then the quotient field of S is a separable algebraic extension of the quotient field of R by (5.8). Hence, S is contained in a separable closure of the quotient field of R. We will show that, inside this separable closure, S is unique. Since any two separable closures are isomorphic, the result will follow.

Suppose that both S and T are height 1 separable closures of R contained in the same separable closure of the quotient field of R. Let $T = \text{dir lim } T_i$, where each T_i is a finite height 1 strongly separable extension of R. We claim that ST_i is a finite height 1 strongly separable extension of S, for each i. Since NBU holds for S over R (5.12), we have that $S \otimes T_i$ is height 1 separable over S by (1.7). Since $S \otimes T_i$ maps onto ST_i, ST_i is also height 1 separable over S by (1.10). Hence, we need only show that $(ST_i)_Q$ is projective over S_Q for each Q in $X'(S)$. Let $p = Q \cap R$; then p is in $X'(R)$. To simplify the notation, we replace R_p, S_p, and T_{i_p} with R, S, and T, respectively. We have the following situation: R is a DVR (and so Noetherian, integrally closed), S is a locally strongly separable extension of R, and T is a strongly separable extension of R. Let $S = \text{dir lim } S_i$, where each S_i is a strongly separable extension of R.

Since each $S_i \otimes T$ is a strongly separable R-algebra, $S_i \otimes T$ is a finite product of domains [10; 4.2, p.473]. We claim that $S \otimes T$ is also a finite product of domains. Since $S \otimes T$ is finitely generated and projective over S, $S \otimes T$ contains only a finite number of idempotents, $e_1, \ldots, e_n$, and we have $S \otimes T = (S \otimes T)e_1 \times \cdots \times (S \otimes T)e_n$. Let $(S \otimes T)e$ be any one of the summands. Suppose that a and b are in $S \otimes T = \text{dir lim}(S_i \otimes T)$ such that $(ae)(be) = 0$. There is an index i such that a, b, and e are in $(S_i \otimes T) \subseteq S \otimes T$ (T is R-flat). Thus, e is an idempotent in $S_i \otimes T$ and in the domain $(S_i \otimes T)e$, $(ae)(be) = 0$. Thus, either $ae = 0$ or $be = 0$ in $(S \otimes T)e$, and so $(S \otimes T)e$ is a domain.

Write $S \otimes T = D_1 \times \cdots \times D_n$, a product of domains. Since each D_i is strongly separable over S (because $S \otimes T$ is), we have $S \subseteq D_i$. Consider the surjection $D_1 \times \cdots \times D_n \to ST$ which is the multiplication map $S \otimes T \to ST$. Since $S \subseteq S \otimes T$ and $S \subseteq ST$, the kernel of this map intersects S in (0). This kernel is a prime ideal of the form $B = D_1 \times \cdots \times p_i \times \cdots \times D_n$, for some prime p_i in one of the D_i's. Then, $B \cap S = (0)$ implies that $p_i \cap S = (0)$, and so $p_i = (0)$, since D_i is an integral extension of S. Thus, D_i is isomorphic to ST as S-modules, and we have ST is a projective $S \otimes T$-module. Since $S \otimes T$ is projective over S, so is ST. In our original notation we have shown that $S_p T_{i_p} = (ST_i)_p$ is a projective S_p-module, for $p = Q \cap R$, Q in $X'(S)$. Localizing at p in $\mathrm{Spec}(R)$ and then localizing again at Q in $\mathrm{Spec}(S)$ is just the same as localizing at Q to begin with, since $R \subseteq S$ and $p \subseteq Q$. Hence, $(ST_i)_Q$ is a projective S_Q-module. Clearly, $(ST_i)_Q$ is finitely generated over S_Q . We have shown that ST_i is a finite height 1 strongly separable extension of S . Since S is height 1 separably closed, $S = ST_i$, for each i . Thus, $S = \text{dir lim } ST_i = ST$. The symmetry of the argument shows that $T = ST$, and so $S = T$, as desired.

We remark that if R is a semi-Krull domain, then R is certainly height 1 connected and height 1 divided. If S is a height 1 separable closure as defined in Chapter 4, then S is contained in the height 1 separable closure we have just exhibited, for if S is a locally strongly height 1 separable extension of R , then S is a locally finite height 1 strongly separable extension of R . Thus, as we remarked following (4.12), the closure of Chapter 4 is "large enough" to be maximal, but not "large enough" to be unique.

<u>PROPOSITION 5.14</u> If R is semi-Krull with height 1 separable closure S , then S is integrally closed.

<u>Proof</u>: We show first that S_p is integrally closed if p is in $X'(S)$. For such a p , let $Q = p \cap R$. Then Q is in $X'(R)$ by (5.12). Since S_Q is locally strongly separable

over R_Q , a PID, we seee that S_Q is a direct limit of strongly separable extensions of R_Q , each of which is integrally closed by (5.5). Hence, S_Q is integrally closed. Then it follows that S_p , being a further localization on S_Q , is integrally closed also. Now suppose that x is an element of the quotient field of S and x is integral over S . Then x is in the quotient field of S_p , for any p in $X'(S)$, and x is integral over S_p . Thus, x is in S_p . Consider now the finitely generated S-algebra $S[x]$. Since x is integral over S , we see that $S[x]$ is actually finitely generated as an S-module. If p is in $X'(S)$, we have shown above that $S[x]_p = S_p$. Hence, $S[x]$ is a finite height 1 strongly separable extension of S . Since S is height 1 separably closed, we have $S[x] = S$. Thus, x is in S , and S is integrally closed.

We make a few observations now.

<u>5.15</u> With R and S as above and T a locally finite height 1 strongly separable domain extension of R , then there is an R-algebra injection $T \to S$, since T is contained in a height 1 separable closure of R which is isomorphic to S .

<u>5.16</u> With R and S as above and T a height 1 strongly separable domain extension of R , then there are only finitely many morphisms in $\mathrm{Alg}_R(T,S)$.

Proof: Let p be in $X'(R)$. Since T_p is a finitely generated separable R_p-algebra and S_p is connected, there are only finitely many R_p-homomorphisms from T_p to S_p by [3; 1.6, p.88]. If $f : T \to S$ and $g : T \to S$ are R-algebra maps with $f \neq g$, then the induced R_p-algebra maps f_p and g_p are not equal. Hence there cannot be infinitely many R-algebra maps from T to S .

5.17 Suppose that S is contained in W , a separable closure of K , the quotient field of R . If f is a K-automorphism of W , then $f(S) = S$, since $f(S)$ is also a height 1 separable closure of R contained in W . Also, if p is in $X'(R)$, then $f(S_p) = S_p$.

PROPOSITION 5.18 Let T be an integrally closed domain extension of the semi-Krull domain R , with T finitely generated as an R-module. Then T is a semi-Krull domain.

Proof: Let p be in $X'(T)$. Then, since R is integrally closed and T is integral over R , $Q = p \cap R$ is in $X'(R)$. R_Q is Noetherian, and, since T_Q is a finite R_Q-module, T_Q is Noetherian. Hence, T_p , a further localization of T_Q , is also Noetherian. Then, since T_p is Noetherian, integrally closed, and has only one nonzero prime, pT_p , T_p is a DVR [4; 10.7, p.69].

Next, let t be a nonzero element of T , and suppose t belongs to infinitely many primes in $X'(T)$. Since t is integral over R, there are elements a_i in R with $t^n + a_1t^{n-1} + \cdots + a_n = 0$. Then, we have

$$(*) \qquad a_n = -t^n - a_1t^{n-1} - \cdots - a_{n-1}t .$$

The element a_n, then, belongs to infinitely many primes in $X'(T)$. Each prime in $X'(T)$ contracts to a prime in $X'(R)$, and, further, there are only a finite number of primes in $X'(T)$ lying over any prime in $X'(R)$, since T is a finite R-module [2; p.327]. Thus, a_n belongs to infinitely many primes in $X'(R)$, and so $a_n = 0$. From $(*)$ we then obtain $a_{n-1} = -t^{n-1} - a_1t^{n-2} - \cdots - a_{n-2}t$. The above argument shows that $a_{n-1} = 0$. Repeating this argument, we eventually see that all $a_i = 0$, a contradiction since $t \neq 0$. Hence, t lies in at most finitely many primes in $X'(T)$. Since T is integrally closed by hypothesis, the proof is complete.

The above proof can be modified slightly to prove the same conclusion with different assumptions on T.

PROPOSITION 5.19 Let R be semi-Krull, and let T be an integrally closed domain extension of R, integral and height 1 strongly separable over R. Then T is semi-Krull.

Proof: If p is in $X'(T)$, the proof that T_p is a DVR is exactly as in (5.18). The second part of the proof also follows as in (5.18), except that the reason that there are only a finite number of primes in T lying over any prime in R follows from the fact that T is a height 1 separable extension of R. For then, the quotient field of T is a finite separable extension of the quotient field of R by (5.7). By [8; 12.5, p.121], there can be only finitely many primes in T lying over any prime in R .

PROPOSITION 5.20 Let R be semi-Krull with height 1 separable closure S . If T is an integrally closed domain contained in S which is a finite height 1 strongly separable extension of R , then S is a height 1 separable closure of T also.

Proof: By (5.18), T is semi-Krull. If $S = \text{dir lim } S_i$, where each S_i is a finite height 1 strongly separable extension of R , there is an index i , say $i = 0$, such that $T \subseteq S_i$. Thus, $S = \text{dir lim } S_i$, limit taken over $i \geq 0$. We show that each S_i in this direct limit is also height 1 strongly separable over T . If p is in $X'(T)$, let $Q = p \cap R$. Then Q is in $X'(R)$. Since S_{i_Q} is a strongly separable extension of R_Q and T_Q is a strongly separable R_Q-extension contained in S_{i_Q} , S_{i_Q} is a strongly separable extension of T_Q [3; 2.4, p.94]. Then, since $R \subseteq T$ and $Q \subseteq p$, S_{i_p} and T_p are just further localizations of S_{i_Q} and T_Q . Hence, S_{i_Q} is a strongly separable extension of T_p . This finishes the proof.

COROLLARY 5.21 With R , T , and S as in (5.20), any R-algebra morphism $f : T \to S$ is induced by an automorphism of S .

Proof: Let S' denote S considered as a T-algebra by the action of f ; i.e., $t \cdot s = f(t)s$, for t in T and s in S . S is a height 1 separable closure of T by (5.20), and hence, so is S' . Thus, there is an isomorphism $g : S \to S'$. For all s in S and t in T we have $g(ts) = t \cdot g(s) = f(t)g(s)$. If $s = 1$, we get $g(t) = f(t)$, for all t in T .

We will show in (5.23) that if f is any R-endomorphism of S, then f is actually an automorphism. First, a lemma.

LEMMA 5.22 Let T' be a height 1 strongly separable domain extension of R and let T be a height 1 separable domain extension of R. If f is an R-algebra homomorphism from T to T', then f is an injection.

Proof: If x is in the kernel of f, then $f_p(x) = 0$ for each p in $X'(R)$, where f_p is the induced map from T_p to T'_p. But the kernel of f_p is (0) by [3; 2.6, p.96]. Thus, $tx = 0$, for some t in $R - p$, and so $x = 0$.

PROPOSITION 5.23 Let R be semi-Krull with height 1 separable closure S. If f is any R-algebra homomorphism from S to S, then f is an automorphism of S.

Proof: Suppose $f(x) = 0$ for some x in S. Let T be a finite height 1 strongly separable extension of R contained in S such that x is in T. If $t_1,\dots,t_n$ generate T as an R-module, let T' be a finite height 1 strongly separable extension of R in S containing $f(t_i)$, $i = 1,\dots,n$. By (5.22), the restriction of f to T, mapping T to T', is an injection. Hence, $x = 0$, and f is an injection.

We show now that f is onto. Let y be in S and let T be a finite height 1 strongly separable extension of R in S containing y. By (5.16) there are only finitely many embeddings of T into S, say $g_1,\dots,g_n$. Let g_1 be the inclusion map, $g_1(x) = x$ for x in T. Since $f|_T : T \to S$ is an injection, $\{f \circ g_1,\dots,f \circ g_n\} = \{g_1,\dots,g_n\}$. Thus, $g_1 = f \circ g_i$, for some i. Then we have

$y = g_1(y) = f(g_i(y))$, showing f is onto.

Our objective in the rest of this chapter is to examine the Galois theory connected with the height 1 separable closure of a semi-Krull domain. This means we will be interested in the groups $\mathrm{Aut}_T(S)$ and $\mathrm{Aut}_{T_p}(S_p)$, for certain subextensions T of R in S and p in $X'(R)$. We begin to look at these groups in the next proposition.

PROPOSITION 5.24 Let R be semi-Krull with height 1 separable closure S . Let T be any extension of R contained in S , and let L,F be the quotient fields of S,T , respectively. Then, for any prime p in $X'(R)$, the groups $\mathrm{Aut}_F(L)$, $\mathrm{Aut}_T(S)$, and $\mathrm{Aut}_{T_p}(S_p)$ are isomorphic.

Proof: We show first that $\mathrm{Aut}_T(S)$ and $\mathrm{Aut}_{T_p}(S_p)$ are isomorphic. Let f be in $\mathrm{Aut}_T(S)$. Then $f_p : S_p \to S_p$, given by $f_p(s/r) = f(s)/r$ for s in S , r in $R - p$, is in $\mathrm{Aut}_{T_p}(S_p)$.

Suppose that $\bar{f}$ is in $\mathrm{Aut}_{T_p}(S_p)$. We will show that $f = \bar{f}|_S$ is in $\mathrm{Aut}_T(S)$. The map $\bar{f}$ extends to a map $g : L \to L$ given by $g(s/t) = \bar{f}(s)/\bar{f}(t)$, for s,t in S and $t \neq 0$. It is straightforward to check that g is in $\mathrm{Aut}_F(L)$. Now, by (5.8), L is a separable algebraic field extension of K , the quotient field of R . Hence, L is contained in a separable closure W of K . Since W is unique (inside a fixed algebraic closure of K), g extends to an automorphism $\bar{g}$ of W [12; 2, p.171]. By (5.17), $\bar{g}|_S$ is in $\mathrm{Aut}_T(S)$. By the construction of g we see that $\bar{g}|_S = \bar{f}|_S = f$. The composition $\mathrm{Aut}_T(S) \to \mathrm{Aut}_{T_p}(S_p) \to \mathrm{Aut}_T(S)$, given by $f \to f_p \to f_{p|S} = f$ is the identity on $\mathrm{Aut}_T(S)$, and the composition $\mathrm{Aut}_{T_p}(S_p) \to \mathrm{Aut}_T(S) \to \mathrm{Aut}_{T_p}(S_p)$, given by $\bar{f} \to \bar{f}_{|S} = f \to f_p = \bar{f}$ is the identity on $\mathrm{Aut}_{T_p}(S_p)$ ($f_p = \bar{f}$ because $f_p(s/r) = f(s)/r = \bar{f}(s/r)$, for s in S and r in $R - p$).

Now we show that $\mathrm{Aut}_F(L)$ and $\mathrm{Aut}_T(S)$ are isomorphic. As we noted above, if f is in $\mathrm{Aut}_T(S)$, then $\bar{f} : L \to L$ given by $\bar{f}(a/b) = f(a)/f(b)$ for $a,b \neq 0$ in S , is in $\mathrm{Aut}_F(L)$. If we start with some $\bar{f}$ in $\mathrm{Aut}_F(L)$, as above, $\bar{f}$ extends to an automorphism g of the separable closure of the quotient field of R , and $f = g|_S$ is in $\mathrm{Aut}_T(S)$. It is easy to see by the construction of these mappings that they describe the desired isomorphism between $\mathrm{Aut}_T(S)$ and $\mathrm{Aut}_F(L)$.

The next proposition is the first step towards the Galois correspondence we will establish.

PROPOSITION 5.25 Let R be semi-Krull with height 1 separable closure S . Let T be an integrally closed height 1 strongly separable extension of R contained in S . Then, if $H = \mathrm{Aut}_T(S)$, $T = S^H$.

Proof: Let x be an element of $S - T$. Let F be the quotient field of T and K be the quotient field of R . Then F is a finite separable extension of K by (5.7). Suppose that W is a separable closure of K containing both S and F . Since S is a direct limit of finitely generated R-modules, S is integral over R , and thus integral over T . Since T is integrally closed, we see that x is in $W - F$. Thus, by standard Galois theory, there is a g in $\mathrm{Aut}_F(W)$ with $g(x) \neq x$. Hence, $f = g|_S$ is in $\mathrm{Aut}_T(S)$ with $f(x) \neq x$, and so $S^H = T$.

5.26 We note in particular that (5.25) implies that $S^G = R$, where $G = \mathrm{Aut}_R(S)$. If H is any subgroup of G , then H is almost finite, as defined in [13], since there are only finitely many restrictions of H to any finitely generated subalgebra of S by (5.16). Hence, by [13; Thm. 1.13], $(S^H)_p = (S_p)^H$, where we have identified H_p with H . If $H = G$, then $R_p = S_p^H$, and we see that S_p is an infinite Galois extension of R_p . Further, we show that $L^G = K$, where L and K are the quotient fields of S and R , respectively, and G acts on L by $f(s/t) = f(s)/f(t)$, for s,t in S and f in G . Let W be a separable closure of K containing L . If x is any element of $L - K$, then x is in $W - K$. Hence, there is an f in $\mathrm{Aut}_K(W)$ such that $f(x) \neq x$. Let $g = f|_S$, and suppose $x = a/b$, for a,b in S . We see that g is in $\mathrm{Aut}_R(S)$ and g extends to $\bar{g}$ in $\mathrm{Aut}_K(L)$ as described above. Thus, $\bar{g}(a/b) = g(a)/g(b) = f(a)/f(b) = f(a/b) \neq a/b$. Therefore, $K = L^G$, and L is an infinite Galois extension of K .

PROPOSITION 5.27 Let R be semi-Krull with height 1 separable closure S , and let H be a subgroup of $\mathrm{Aut}_R(S)$. Then S^H is integrally closed.

Proof: If x is in the quotient field of S^H and integral over S^H , then x is in S , since S is integrally closed (5.14). Then, since x is left fixed by all the elements of H , x is in S^H .

5.28 As we have shown above, the groups $\mathrm{Aut}_K(L)$ and $\mathrm{Aut}_R(S)$ are isomorphic, where L and K are the respective quotient fields. $\mathrm{Aut}_K(L)$ has the usual Krull topology: a set $H \subseteq \mathrm{Aut}_K(L)$ is open if and only if $H = \mathrm{Aut}_E(L)$, where $K \subseteq E \subseteq L$ and E is a finite extension of K ; a subset of $\mathrm{Aut}_K(L)$ is closed if and only if it is an intersection of open subgroups. We use the topology on $\mathrm{Aut}_K(L)$ to define a topology on $\mathrm{Aut}_R(S)$ as follows: A set $H \subseteq \mathrm{Aut}_R(S)$ is open if and only if $H = \mathrm{Aut}_T(S)$, where T is a height 1 strongly separable extension of R contained in S . For such an extension T , the quotient field of T , say F , is a finite (separable) extension of K , and so $\mathrm{Aut}_F(L)$ is an open subgroup of $\mathrm{Aut}_K(L)$. By (5.24), $\mathrm{Aut}_F(L) = \mathrm{Aut}_T(S)$. Conversely, let $H = \mathrm{Aut}_F(L)$ be an open subgroup of $\mathrm{Aut}_K(L)$, for some finite extension F of K . If $T = S^H$, then F is the quotient field of T (actually we know that any element of F is of the form t/r , for t in T and r in R by [18; p.78]). We claim that T is height 1 strongly separable over R . Let p be in $X'(R)$. Since T_p is the integral closure of R_p in F and R_p is Noetherian and integrally closed, T_p is finitely generated as an R_p-module [6; 5.48, p.185]. Furthermore, since R_p is a PID, T_p is a flat R_p-module. Thus, by [5; 2, p.287], T_p is a projective R_p-module. Since S_p is locally strongly separable over R_p , there exists a strongly separable extension of R_p , say S_0 , with $R_p \subseteq T_p \subseteq S_0$. By [3; 1.12, p.46], S_0 is separable over T_p . Now, T_p is Noetherian, since R_p is, and T_p is integrally closed; thus, by [10; 4.2], S_0 is a projective T_p-module. Hence, T_p is separable over R_p . Thus, T is a height 1 strongly separable extension of R , and $\mathrm{Aut}_T(S) = \mathrm{Aut}_F(L)$ by (5.24).

We note also that since S_p is an infinite Galois extension of R_p , for p in $X'(R)$, $\mathrm{Aut}_{R_p}(S_p)$ has the usual Krull topology (a subset is open if and only if it is of the form $\mathrm{Aut}_{T'}(S_p)$ for some strongly separable extension T' of R_p), and this topology coincides with that of $\mathrm{Aut}_K(L)$. We have seen that if F is a finite extension of K , $T' = (S^H)p$ is a strongly separable extension of R_p and $\mathrm{Aut}_{T'}(S_p) = \mathrm{Aut}_F(L)$. If T' is a strongly separable extension of R_p , then F , the quotient field of T' , is a finite extension of K and $\mathrm{Aut}_{T'}(S_p) = \mathrm{Aut}_F(L)$.

An immediate result of the preceding discussion is the next proposition.

PROPOSITION 5.29 Let R be semi-Krull with height 1 separable closure S , and let T be a height 1 strongly separable extension of R contained in S . Then the integral closure of T is also height 1 strongly separable over R .

Proof: Let $H = \mathrm{Aut}_T(S)$. Then, if L and F are the quotient fields of S and T , respectively, we have $\mathrm{Aut}_T(S) = \mathrm{Aut}_F(L)$, and F is a finite (separable) extension of the quotient field of R . As in (5.28), S^H is an integrally closed height 1 strongly separable extension of R , with quotient field F . Since $T \subseteq S^H$ we see that S^H must be the integral closure of T .

For reference, we quote here the following results found in [16]:

5.30 Let S be a connected (commutative) ring which is a locally finitely generated separable R-algebra and suppose $R = S^G$ for some automorphism group G of S . (Then S is automatically locally strongly separable over R by Theorem 1 in this same paper.)

Then, if R' is a finitely generated separable R-subalgebra of S, $R' = S^H$, where H is the set of all automorphisms in G which are the identity on R', and $(G : H) < \infty$. Conversely, if H is a subgroup of G with $(G : H) < \infty$, then S^H is a finitely generated separable R-subalgebra of S (Theorem 2(ii)).

5.31 Let S be a connected (commutative) ring which is a locally finitely generated separable R-algebra and let $R = S^G$ for some automorphism group G of S. Then there exists the usual one-to-one Galois correspondence between locally finitely generated separable R-subalgebras of S and closed subgroups of $\mathrm{Aut}_R(S)$. (Theorem 3(i).)

PROPOSITION 5.32 Let R be semi-Krull with height 1 separable closure S, and let H be a closed subgroup of finite index in $G = \mathrm{Aut}_R(S)$. Then, $T = S^H$ is an integrally closed height 1 strongly separable extension of R and $\mathrm{Aut}_T(S) = H$.

Proof: S^H is integrally closed by (5.27). If p is in $X'(R)$, then S_p is locally strongly separable over R_p. Since H is a subgroup of $\mathrm{Aut}_R(S) = \mathrm{Aut}_{R_p}(S_p)$ of finite index, $T_p = (S^H)_p = S_p^H$ is a finitely generated separable extension of R_p, as in (5.30). T_p is R_p-projective since it is contained in some strongly separable extension of R_p. Thus T is a height 1 strongly separable extension of R.

To finish the proof, we need to show that $\mathrm{Aut}_T(S) = H$. This follows easily from (5.31). Since H_p ($= H$) is a closed subgroup of G_p ($= G$), $R_p = S_p^{G_p}$, and $T_p = S_p^{H_p}$, we have $\mathrm{Aut}_{T_p}(S_p) = H_p$.

Therefore, $\mathrm{Aut}_T(S) = H$.

We now state the Galois theorem we have been leading up to.

THEOREM 5.33 Suppose R is a semi-Krull domain with height 1 separable closure S , and let $G = \mathrm{Aut}_R(S)$. Then there exists a one-to-one correspondence between integrally closed height 1 strongly separable extensions T of R contained in S and closed subgroups H of finite index in G , given by $T \to \mathrm{Aut}_T(S)$, $H \to S^H$.

Proof: If T is an integrally closed height 1 strongly separable extension of R and $H = \mathrm{Aut}_T(S)$, then $T = S^H$ by (5.25). We show now that H is closed and has finite index in G . If p is in $X'(R)$, we have $G = \mathrm{Aut}_{R_p}(S_p)$ and $H = \mathrm{Aut}_{T_p}(S_p)$. Also, S_p is an infinite Galois extension of R_p and T_p is a strongly separable extension of R_p . Thus, H has finite index in G by (5.30). By (5.31), we have that H is a closed subgroup of G . The rest of the proof follows by (5.32).

We do not expect that if T is an integrally closed height 1 strongly separable extension of R then T must be finitely generated over R . Such an extension T is the integral closure of R in a finite separable extension of the quotient field of R , and in general for T to be finitely generated over R we would need to assume that R is Noetherian (e.g., see [6; p.185]). We shall show in the next example that in our setting such a T may not be finitely generated. The example also shows that a finitely generated height 1 strongly separable extension need not be integrally closed.

EXAMPLE 5.34 We return once more to the ring in Example 2.7: k is a field, $R' \subseteq k[x,y]$ is the ring of all polynomials with no term in a power of x alone, M is the prime ideal in R' of all polynomials with zero constant term, and $R = R'_M$. R is semi-Krull by (5.2). Also, the element y belongs to no prime in $X'(R)$.

Let $S = R[t]/t^2 - y = R[\sqrt{y}]$. Then, as in (2.7), S is a finite height 1 strongly separable extension of R . We will show that (1) the integral closure of S , say T , is not finitely generated as an R-module, and, (2) T is height 1 strongly separable over R .

Proof of (1): Let T' be the integral closure of $R'[\sqrt{y}]$. An element of T' is of the form $a + b\sqrt{y}$, where a and b are in the quotient field of R' . Now, for each $i \geq 0$, the element $t_i = x^i\sqrt{y}$ is in T' , because $x^i = x^iy/y$ is in the quotient field of R' and $x^i\sqrt{y}$ is integral over R' . Let N' be the R'-module generated by the elements $1, t_i$ for $i \geq 0$. We claim that $N' = T'$. By the above, we see that $N' \subseteq T'$. Now suppose that $a + b\sqrt{y}$ is in T' , for elements a,b in the quotient field of R' . Since $a + b\sqrt{y}$ is integral over $R'[\sqrt{y}]$, it is integral over R' . If K is the quotient field of R' , then $a + b\sqrt{y}$ belongs to the finite extension of K with basis $\{1,\sqrt{y}\}$ and the characteristic polynomial of $a + b\sqrt{y}$ is $(t - (a + b\sqrt{y}))(t - (a - b\sqrt{y})) = t^2 - 2at + (a^2 - b^2y)$. The coefficients of this polynomial are in R' since $a + b\sqrt{y}$ is integral over R' [2; p.310]. Thus, a is in R' (since $\frac{1}{2}$ is in R') and so b^2y is in R' . Let $b = f/g$, where f and g are in $R' \subseteq k[x,y]$, and, as elements of $k[x,y]$, f and g are relatively prime. Then, we have f^2y/g^2 is in R' , which implies that g^2 must divide y . Thus, g is actually a constant; and we see that the element b lies in $k[x,y]$. Write $b = \sum c_i x^i + h(x,y)$, where c_i is in k , and the polynomial $h(x,y)$ has no terms in a power of x alone. Then $a + b\sqrt{y} = a + h(x,y)\sqrt{y} + \sum c_i t^i$, an element of N' . Therefore, $T' \subseteq N'$, and we have equality. We have shown that, as an R'-algebra, $T' = R'[t_0,t_1,\ldots]$. Now, since T' is integrally closed, so is T'_M and we have $T'_M = (R'[t_0,t_1,\ldots])_M = R'_M[t_0,t_1,\ldots] = R[t_0,t_1,\ldots]$. Thus $R[t_0,t_1,\ldots]$ is integrally closed. Since $S \subseteq R[t_0,t_1,\ldots] \subseteq T$, the integral closure of S , we have $T = R[t_0,t_1,\ldots]$. Further, we have actually shown above that T is generated as an R-module by the elements $1,t_1,t_2,\ldots$.

We show that T cannot be finitely generated as an R-module by showing that t_n cannot be written as an R-linear combination of $1,t_1,\ldots,t_{n-1}$, for $n \geq 1$. Suppose that $x^n\sqrt{y} = r_0 + r_1 x\sqrt{y} + \cdots + r_{n-1}x^{n-1}\sqrt{y}$, where each r_i is in R . Then, $r_0 = 0$, and we have $x^n = r_1 x + \cdots + r_{n-1}x^{n-1}$. Let $r_i = f_i/g_i$, where f_i,g_i are in R' (and so have no terms in a power of x alone), and g is in $R' - M$ (and so g_i has a constant term). Clearing denominators we obtain $(g_1 g_2 \cdots g_{n-1})x^n = (g_2 g_3 \cdots g_{n-1})f_1 x + \cdots + (g_1 g_2 \cdots g_{n-2})f_{n-1}x^{n-1}$. Setting $y = 0$ in the above equation, we see that the left hand side is a nonzero polynomial in x of degree n , while the right hand side is a polynomial in x of degree at most $n - 1$, a contradiction. Thus T cannot be finitely generated as an R-module.

Proof of (2): This follows easily now. We know that T is generated as an R-module by the elements $1,\sqrt{y},x\sqrt{y},x^2\sqrt{y},\ldots$. If p is any prime in $X'(R)$, $x^i = x^i y/y$ is in R_p , since y is in $R - p$.

Hence, T_p is generated over R_p by the elements $1, \sqrt{y}$, and we have $T_p = S_p$, which is strongly separable over R_p .

While an integrally closed height 1 strongly separable extension T of R may not be finitely generated, it is true that T is the integral closure of a finitely generated height 1 strongly separable extension, as in the above example. We prove this next.

<u>PROPOSITION 5.35</u> If $T \subseteq S$ is an integrally closed height 1 strongly separable extension of R , then there is an R-subalgebra $F \subseteq T$, finitely generated and height 1 strongly separable over R , such that the integral closure of F is T . Furthermore, $F_p = T_p$, for each p in $X'(R)$.

<u>Proof</u>: Let L and K be the quotient fields of T and R , respectively. Then L is a finite separable extension of K . Let a_i/b_i , $i = 1,\ldots,n$, generate L over K , where a_i and $b_i \neq 0$ are in T . Let B be the R-subalgebra of T generated by the elements a_i, b_i . Since B is integral over R , we see that B is actually finitely generated as a module over R . It is straightforward to check that the quotient field of B is L . By (5.4), B is height 1 separable almost everywhere in $X'(R)$.

We show next that if B_p is R_p-separable for p in $X'(R)$, then $B_p = T_p$. Since $R_p \subseteq B_p \subseteq T_p$ and T_p is strongly separable over R_p , T_p is strongly separable over B_p , and thus $T_p = B_p \oplus M$, for some projective B_p-module M . Tensoring both sides of this equation with L over B_p we obtain $L = L \oplus (M \otimes_{B_p} L)$. Hence,

$M \otimes_{B_p} L = (0)$, and, since M is projective over B_p , we have $M = (0)$.

Suppose now that p is in $X'(R)$ and B_p is not separable over R_p . By the strong separability of T_p over R_p there are elements z_i, y_i , $i = 1,\ldots,n$, in T_p and t in $\mathrm{Hom}_{R_p}(T_p, R_p)$ with $\sum z_i y_i = 1$ and $\sum z_j t(y_j x) = x$, for each x in T_p . Let T' be an R-subalgebra of T , finitely generated as an R-module, with $B \subseteq T' \subseteq T$, and such that T'_p contains the elements z_i, y_i . Then $t|_{T'_p}$ is in $\mathrm{Hom}_{R_p}(T'_p, R_p)$, and we see that T'_p is strongly separable over R_p .

Repeating this process finitely many times (if necessary) we finally arrive at a finitely generated height 1 strongly separable R-algebra F with $B \subseteq F \subseteq T$ and such that the quotient field of F is L . Since T is integrally closed, we see that T must be the integral closure of F . Finally, we note that we have shown above that for the F we have constructed, $F_p = T_p$ for each p in $X'(R)$.

We finish this chapter with the following Galois correspondence for integrally closed extensions which are direct limits of height 1 strongly separable extensions.

PROPOSITION 5.36 Let R be semi-Krull with height 1 separable closure S , and let $G = \mathrm{Aut}_R(S)$. Then there exists a one-to-one correspondence between integrally closed locally height 1 strongly separable extensions T of R contained in S and closed subgroups H of G , given by $T \to \mathrm{Aut}_T(S)$, $H \to S^H$.

Proof: Suppose T is integrally closed and $T = \text{dir lim } T_i$, where each T_i is height 1 strongly separable over R and $T_i \subseteq T$. Then we have $H = \mathrm{Aut}_T(S) = \mathrm{Aut}_{\cup T_i}(S) = \cap \mathrm{Aut}_{T_i}(S)$, and so H is closed in G. Since the quotient field of T is a separable algebraic extension of the quotient field of R, we have $T = S^H$ just as in the proof of (5.25).

Conversely, suppose H is closed in G. Then $H = \cap \mathrm{Aut}_{T_i}(S)$, for some set of height 1 strongly separable extensions T_i of R. We may assume that each T_i is integrally closed, for if $\bar{T}_i$ is the integral closure of T_i, and L, F_i are the quotient fields of S, T_i, we have $\mathrm{Aut}_{F_i}(L) = \mathrm{Aut}_{T_i}(S) = \mathrm{Aut}_{\bar{T}_i}(S)$. If $T_{i_1}, \ldots, T_{i_n}$ is any finite set of the T_i's, then $\mathrm{Aut}_{T_{i_1} \cdot T_{i_2} \cdots T_{i_n}}(S) = \mathrm{Aut}_{T_{i_1}}(S) \cap \cdots \cap \mathrm{Aut}_{T_{i_n}}(S)$. Furthermore, since $T_{i_1} \otimes \cdots \otimes T_{i_n}$ is height 1 separable over R (1.6), the product $T_{i_1} \cdots T_{i_n}$ is height 1 separable over R. If p is in $X'(R)$, then we see also that $(T_{i_1} \cdots T_{i_n})_p$ is actually strongly separable over R_p, since it is finitely generated over R_p and contained in some strongly separable subalgebra of S_p. Thus, $T' = \cup T_{i_1} \cdots T_{i_n}$, where the union is taken over all finite products of the T_i's, is a locally height 1 strongly separable extension of R, and $H = \cap \mathrm{Aut}_{T_{i_1} \cdots T_{i_n}}(S)$, where the intersection is taken over all finite products of the T_i's. As we noted above, we may as well assume that each of the finite products $T_{i_1} \cdots T_{i_n}$ is integrally closed, and hence that T' is integrally closed. Thus, we have

$S^H = S^{\mathrm{Aut}_{T'}(S)} = T'$ by the first part of the proof, and so S^H is integrally closed and height 1 strongly separable over R. To complete the proof, we need to show that $\mathrm{Aut}_{S^H}(S) \subseteq H$, since it is always true that $H \subseteq \mathrm{Aut}_{S^H}(S)$. Let $H_i = \mathrm{Aut}_{T_i}(S)$. Noting that $S^H = S^{\cap H_i} \supseteq \cup S^{H_i}$ and $S^{H_i} = T_i$ by (5.35), we have $\mathrm{Aut}_{S^H}(S) \subseteq \mathrm{Aut}_{\cup S^{H_i}}(S) \subseteq \cap \mathrm{Aut}_{S^{H_i}}(S) = \cap \mathrm{Aut}_{T_i}(S) = H$. This completes the proof.

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