HODGES, Billy Gene, 1929-
RECURRENT TENSOR FIELDS AND CONFORMAL RIEMANNIAN MANIFOLDS.

The University of Oklahoma, Ph.D., 1964
Mathematics

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

RECURRENT TENSOR FIELDS AND CONFORMAL RIEMANNIAN MANIFOLDS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

BILLY GENE HODGES

Norman, Oklahoma

1964

RECURRENT TENSOR FIELDS AND CONFORMAL RIEMANNIAN MANIFOLDS

APPROVED BY

T. K. Pan

Arthur Bernhart

John C. Brixey

William N. Huff

DISSERTATION COMMITTEE

ACKNOWLEDGMENT

I wish to acknowledge the consideration and encouragement which I received from Professor T. K. Pan during the preparation of this dissertation and during my entire graduate program at the University of Oklahoma.

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RECURRENT TENSOR FIELDS AND CONFORMAL RIEMANNIAN MANIFOLDS

CHAPTER I

INTRODUCTION

All of the results obtained in this dissertation are of a global nature, and the spaces to be considered in the following are restricted to be compact Riemannian manifolds with positive definite fundamental forms, such spaces being defined in Chapter II.

The classical results of the study of conformal correspondence in the small between Riemannian manifolds are well-known [1, pp. 89-95]. Riemannian manifolds of recurrent curvature, to be defined in Chapter II, were introduced by H. S. Ruse, whose work was extended by A. G. Walker [3, pp. 36-64]. Some of Walker's results are listed in Chapter II for reference in this dissertation.

One of the purposes of this dissertation is to determine if it is possible for a general Riemannian manifold, with certain restrictions on the curvature, to be globally conformal to a Riemannian manifold of recurrent curvature. Another purpose of this dissertation is to determine if it is possible for a Riemannian manifold of recurrent curvature to be globally conformal to a flat Riemannian manifold. These two problems are studied in Chapter V.

The method that is used to obtain the results of Chapter V gives rise to some results on global conformal correspondence between other types of Riemannian manifolds. These results, being more general than those obtained in Chapter V, are given in Chapter IV.

The property of recurrency that the curvature tensor field of a Riemannian manifold of recurrent curvature possesses is extended naturally to the definition of a general recurrent covariant tensor field. Chapter III contains results on the global existence of such tensor fields.

CHAPTER II

DEFINITIONS AND REFERENCE THEOREMS

Consider the real Euclidean space E_n of n dimensions. A continuous mapping F of an open subset S of E_n into E_n can be defined by expressing the coordinates $(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^n)$ of $\overline{P}$, the image of $P \in S$, as continuous functions of the coordinates $(x^1, x^2, \ldots, x^n)$ of P:

$$\overline{x}^i = f^i(x^1, x^2, \ldots, x^n). \tag{2.1}$$

(Throughout this dissertation all small Latin indices will have the range 1, 2, ..., n inclusive.) Such a mapping is said to be of class C^r, $r > 0$, if the functions f^i have continuous partial derivatives up to the order r inclusive and satisfy the condition that the Jacobian

$$\left|\frac{\partial \overline{x}^i}{\partial x^j}\right|$$

never vanishes in S.

An <u>n-dimensional manifold</u> is a connected Hausdorff space which has the property that each point of the space has a neighborhood homeomorphic to some open set of Euclidean n-space E_n.

A system D of differentiable coordinates in an n-dimensional manifold M_n consists of an indexed family $\{U_\alpha, \alpha \in A\}$ of open sets covering M_n, and, for each $\alpha \in A$, the index set, a homeomorphism

$$\theta_\alpha : S_\alpha \to U_\alpha ,$$

where S_α is an open set in E_n, such that the mapping

$$(2.2) \quad \theta_\alpha^{-1}\theta_\beta : \theta_\beta^{-1}(U_\alpha \cap U_\beta) \to \theta_\alpha^{-1}(U_\alpha \cap U_\beta) \quad , \alpha, \beta \in A$$

is differentiable of class C^r, $r > 0$.

Two systems of coordinates D, D' in M_n of class C^r are said to be <u>r-equivalent</u> if the composite families

$$\{U , U'\} , \quad \{\theta , \theta'\}$$

form a system of class C^r. This is a proper equivalence relation, so that the various systems of coordinates separate into disjoint equivalence classes.

An <u>n-dimensional differentiable manifold M_n of class C^r</u> is an n-dimensional manifold M_n, together with an r-equivalence class of systems of coordinates in M_n. The open sets U_α, $\alpha \in A$, are called <u>coordinate neighborhoods</u>, while the coordinate systems valid in U_α are called <u>local coordinate systems</u>. Let P be a point of M_n such that $P \in U_\alpha \cap U_{\bar{\alpha}}$. Then the homeomorphism θ_α gives P the coordinates

$x^1, x^2, \ldots, x^n$, while the homeomorphism $\theta_{\bar{\alpha}}$ gives P the coordinates $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$. These two sets of local coordinates are related by equations of the type

$$\bar{x}^i = f^i(x^1, x^2, \ldots, x^n),$$

where the functions f^i are of class C^r, and the Jacobian

$$\left|\frac{\partial \bar{x}^i}{\partial x^j}\right|$$

is non-zero in $U_\alpha \cap U_{\bar{\alpha}}$.

Let ϕ be a real-valued continuous function defined on M_n. Let P be a point of M_n and U_α a coordinate neighborhood con taining P, with local coordinates $x^1, x^2, \ldots, x^n$. Then, in U_α, ϕ can be considered as a function of the coordinates, $x^1, x^2, \ldots, x^n$, or briefly,

$$\phi = \phi(x^i) = \phi(x).$$

ϕ is said to be differentiable of class C^s at P if $\phi(x^i)$ has continuous partial derivatives through the s-th order at P. The function ϕ is called differentiable of class C^s on M_n if it is differentiable of class C^s at every point of M_n.

A differentiable manifold M_n of class C^r, $r > 0$, always admits a globally defined positive definite covariant tensor field of order 2 whose component functions

$$g_{jk}$$

are of class C^{r-1} on M_n, as was proved by N. E. Steenrod

[2, p. 120]. Thus, with each coordinate neighborhood U_α in M_n, we can associate a positive definite quadratic form in the differentials dx^i,

$$ds^2 = g_{jk}dx^j dx^k, \tag{2.3}$$

where $g_{jk}(x^i) = g_{kj}(x^i)$, and the repeated index, as it will throughout this dissertation, denotes the summation over its range. The quadratic form (2.3) does not depend on the coordinate system used and is called the <u>fundamental form</u> of the manifold. An n-dimensional differentiable manifold on which a fundamental form (2.3) is given is called an <u>n-dimensional Riemannian manifold</u>.

An n-dimensional Riemannian manifold is said to be <u>compact</u> if, for any system of coordinate neighborhoods $\{U_\alpha, \alpha \in A\}$ which covers the manifold, there exists a finite subfamily of the system which covers the manifold.

Because of the above-mentioned result of Steenrod, it will be assumed throughout this dissertation that, if M_n is a Riemannian manifold of class C^r, then the coefficients g_{jk} of the fundamental form of M_n are functions of class C^{r-1} on M_n. The symmetric second-order covariant tensor field on M_n whose components are the functions g_{jk} will be said to be of class C^{r-1} on M_n. This tensor field will also be denoted by g_{jk} and is called the <u>fundamental covariant tensor field</u>

of M_n.

Now let M_n be a Riemannian manifold of class C^r $(r \geqq 3)$. Then, since g_{jk} is of class C^{r-1} on M_n the coefficients of the Riemannian connection Υ of M_n, defined by

$$\Gamma^i_{jk}(x) = \tfrac{1}{2}\, g^{is}\left(\frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s}\right) , \tag{2.4}$$

are of class C^{r-2} on M_n, where the functions g^{is} are defined by

$$g^{is} = \frac{1}{g}\; G^{is}, \tag{2.5}$$

where

$$g = \left|g_{ij}\right| \neq 0, \tag{2.6}$$

and G^{is} is the cofactor of g_{si} in g. The curvature tensor field $R^h{}_{ijk}$ of M_n, whose components are defined by

$$R^h{}_{ijk}(x) = \frac{\partial \Gamma^h_{ik}}{\partial x^j} - \frac{\partial \Gamma^h_{ij}}{\partial x^k} + \Gamma^s_{ik}\,\Gamma^h_{sj} - \Gamma^s_{ij}\,\Gamma^h_{sk} , \tag{2.7}$$

is consequently of class C^{r-3} on M_n (Functions of class C^o on M_n are continuous on M_n). The Ricci tensor field R_{ij}, whose components are defined by

$$R_{ij} = g^{hk}R_{hijk} = R^h{}_{ijh}, \tag{2.8}$$

and the scalar curvature R of M_n, defined by

$$R = g^{ij}R_{ij} , \tag{2.9}$$

are of class C^{r-3} on M_n. Thus if $r \geqq 3$, R is continuous on M_n.

Let $B^{ij\ldots k}{}_{m\ldots p}$ be a tensor field of class C^1 on M_n. If comma (,) denotes the covariant derivative with respect to the Riemannian connection Υ, then

$$(2.10)\quad B^{ij\ldots k}{}_{\ell m\ldots p,t} = \frac{\partial B^{ij\ldots k}{}_{\ell m\ldots p}}{\partial x^t} + B^{sj\ldots k}{}_{\ell m\ldots p}\Gamma^{i}_{st}$$

$$+ \ldots + B^{ij\ldots s}{}_{\ell m\ldots p}\Gamma^{k}_{st} - B^{ij\ldots k}{}_{sm\ldots p}\Gamma^{s}_{\ell t}$$

$$- \ldots - B^{ij\ldots k}{}_{\ell m\ldots s}\Gamma^{s}_{pt}\ .$$

For a scalar field ϕ of class C^1 on M_n, we have its gradient

$$(2.11)\qquad \phi_{,i} = \frac{\partial\phi}{\partial x^i}\ .$$

The square of the length of the gradient is given by

$$(2.12)\qquad \Delta_1\phi = g^{ij}\phi_{,i}\phi_{,j}\ .$$

If ϕ is of class C^2 on M_n the Laplacian of ϕ is

$$(2.13)\qquad \Delta_2\phi = g^{ij}(\phi_{,i})_{,j} = (g^{ij}\phi_{,i})_{,j} = g^{ij}\phi_{,ij}\ ,$$

where $\phi_{,ij}$ denotes the second covariant derivative of ϕ with respect to the Riemannian connection Υ.

Now let M_n be an n-dimensional Riemannian manifold of class C^r, $r \geq 2$ (This restriction is placed on r so that the coefficients of the Riemannian connection will be continuous on M_n).

Definition 2.1. Let $T_{i_1\ldots i_p}$ be an arbitrary covariant tensor field of order p and of class C^s, $s \geq 1$, on

M_n. If $T_{i_1 \ldots i_p}$ is not a zero tensor field and if $T_{i_1 \ldots i_p}$ satisfies the condition

$$(2.14) \qquad T_{i_1 \ldots i_p, j} = T_{i_1 \ldots i_p} \lambda_j$$

for some continuous vector field λ_j on M_n, then $T_{i_1 \ldots i_p}$ is said to be a recurrent tensor field on M_n.

In Chapter III the existence of recurrent tensor fields on compact Riemannian manifolds is studied, the main result being that a tensor field which is itself the covariant derivative of another tensor field cannot be a recurrent tensor field.

In particular, if the curvature tensor field R_{hijk} of M_n satisfies (2.14) for some vector field λ_i on M_n, then M_n is said to be a Riemannian manifold of recurrent curvature. We classify Riemannian manifolds of recurrent curvature by the following definitions [3, p. 37]:

Definition 2.2. A K-manifold, denoted by K_n, is an n-dimensional Riemannian manifold which is not flat and whose curvature tensor field R_{hijk} satisfies the condition

$$(2.15) \qquad R_{hijk,\ell} = R_{hijk} \lambda_\ell$$

for some non-zero vector field λ_ℓ on K_n.

Definition 2.3. A K^*-manifold, denoted by K_n^*, is an n-dimensional Riemannian manifold which is either a K_n or satisfies both of the conditions

(2.16) $$R_{hijk,\ell} = 0$$

and

(2.17) $$R_{hijk}\lambda_\ell + R_{hik\ell}\lambda_j + R_{hi\ell j}\lambda_k = 0,$$

where λ_i is some non-zero vector field on K_n^*.

When (2.16) is satisfied by the curvature tensor field of a Riemannian manifold, the manifold is said to be a <u>symmetric Riemannian manifold</u>.

We now state a definition and two theorems of a local nature due to Walker [3, pp. 41-49]. A K^*-manifold which locally admits n-2 independent parallel vector fields is called a <u>locally simple K^*-manifold</u>.

<u>Theorem 2.1</u>. Every real K^*-manifold with a definite fundamental form is locally simple.

<u>Theorem 2.2</u>. A necessary and sufficient condition that a K^*-manifold be locally simple is that locally the components of the curvature tensor field have the form

(2.18) $$R_{hijk} = m_{hi}m_{jk} \quad ,$$

where

(2.19) $$m_{ij} = \rho_i \tau_j - \rho_j \tau_i \quad ,$$

ρ_i, τ_i being two vector fields such that the vector field λ_i appearing in (2.17) is locally of the form

(2.20) $$\lambda_i = \alpha \rho_i + \beta \tau_i \quad ,$$

α and β being scalars.

The proof of Theorem 2.2 depends on the fact that ρ_i and τ_i are linearly independent vector fields, which means that a locally simple K^*-manifold cannot be flat. For if the K^*-manifold were flat, we would have

(2.21) $$R_{hijk} = 0.$$

This would imply that m_{ij} is a zero tensor, or

(2.22) $$\rho_i\tau_j - \rho_j\tau_i = 0.$$

Multiplying (2.22) by ρ^i and summing on i give

(2.23) $$(\rho^i\rho_i)\,\tau_j - (\rho^i\tau_i)\,\rho_j = 0.$$

Thus, if the K^*-manifold is flat, ρ_i and τ_i are linearly dependent.

It should also be noted that the second-order covariant tensor field m_{ij} is anti-symmetric. That is,

(2.24) $$m_{ij} = -m_{ji} \,.$$

This dissertation is concerned only with the study of real K^*-manifolds with a positive definite fundamental form. By Theorem 2.1, such manifolds are locally simple. To obtain global results about K^*-manifolds, it is necessary to define simple (in the large) K^*-manifolds. By Theorem 2.2 and the discussion following it, we are led to the following definition:

<u>Definition 2.4</u>. A K^*-manifold, K_n^*, is said to be simple (in the large) if:

(i) K_n^* is not flat

(ii) The fundamental form of K_n^* is positive definite

(iii) The curvature tensor field of K_n^* satisfies

$$R_{hijk} = m_{hi}m_{jk} \tag{2.25}$$

throughout K_n^*, where

$$m_{ij} = \rho_i\tau_j - \rho_j\tau_i \quad . \tag{2.26}$$

In (2.26) we assume that ρ_i, τ_i are two vector fields defined on K_n^* such that the vector field λ_i appearing in (2.17) is, throughout K_n^*, of the form

$$\lambda_i = \alpha\rho_i + \beta\tau_i \quad , \tag{2.27}$$

α and β being scalar fields defined on K_n^*.

The conditions under which a Riemannian manifold is locally conformal to a flat Riemannian manifold (one on which $R_{hijk} = 0$) have been found [1, p. 92]. The present author, in an unpublished work, has extended this to a study of the conditions under which a Riemannian manifold is locally conformal to a K-manifold. In this dissertation, these same problems are studied globally.

Definition 2.5. Let M_n be a Riemannian manifold of class C^r, $r > 0$, with fundamental covariant tensor field g_{jk} of class C^{r-1} on M_n. A transformation of the fundamental covariant tensor field, defined by

$$\bar{g}_{jk} = e^{2\sigma}g_{jk} \ , \tag{2.28}$$

where σ is a scalar field of class C^s on M_n, is called a global conformal transformation of the fundamental covariant tensor field of M_n. If σ is a constant throughout M_n the transformation is a similarity transformation.

It follows from (2.28) that the components of the transformed fundamental covariant tensor field $\bar{g}_{jk}$ are of class C^t on M_n, where

(2.29) $$t = \mathrm{Min}[s, r-1].$$

The fundamental contravariant tensor field of M_n is transformed into

(2.30) $$\bar{g}^{ij} = e^{-2\sigma} g^{ij}$$

under the global conformal transformation defined by (2.28), and the components $\bar{g}^{ij}$ are seen to be of class C^t also, where t is defined by (2.29). The coefficients of the Riemannian connection Υ of M_n are transformed into

(2.31) $$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \sigma_{,k} + \delta^i_k \sigma_{,j} - g_{jk} g^{im} \sigma_{,m} \quad ,$$

where comma (,) denotes the covariant derivative with respect to Υ, and these functions are of class C^{t-1} on M_n. Let $\sigma_{,ij}$ denote the second covariant derivative of σ with respect to Υ and put

(2.32) $$\sigma_{ij} = \sigma_{,ij} - \sigma_{,i}\sigma_{,j} \; .$$

The curvature tensor field of M_n is transformed into

$$(2.34)\quad \bar{R}_{hijk} = e^{2\sigma}[R_{hijk} + g_{hk}\sigma_{ij} + g_{ij}\sigma_{hk} - g_{hj}\sigma_{ik} - g_{ik}\sigma_{hj} + (g_{hk}g_{ij} - g_{hj}g_{ik})\Delta_1\sigma] ,$$

and the components of the transformed curvature tensor field are of class C^{t-2} on M_n. Furthermore, the Ricci tensor field and the scalar curvature of M_n are transformed respectively into

$$(2.35)\qquad \bar{R}_{ij} = R_{ij} + (n-2)\sigma_{ij} + g_{ij}[\Delta_2\sigma + (n-2)\Delta_1\sigma]$$

and

$$(2.36)\qquad \bar{R} = e^{-2\sigma}[R + 2(n-1)\Delta_2\sigma + (n-1)(n-2)\Delta_1\sigma].$$

Therefore, the components of the transformed Ricci tensor field and the transformed scalar curvature are also of class C^{t-2} on M_n.

The following theorem, due to E. Hopf [5, p. 30], is instrumental in proofs of the major results of this dissertation.

Theorem 2.3. On a compact differentiable manifold M_n, if a scalar field $\phi(x)$ of class C^2 satisfies the condition

$$L(\phi) = g^{jk}(x)\,\frac{\partial^2\phi}{\partial x^j\partial x^k} + h^i(x)\,\frac{\partial\phi}{\partial x^i} \geqq 0 \ (\text{or} \leqq 0)$$

everywhere on M_n, where $g^{jk}(x)$ are coefficients of a positive definite quadratic form at every point of M_n, and $h^i(x)$ are continuous functions on M_n, then we have

$$\phi = \text{const.}$$

everywhere on M_n, and then automatically

$$L(\phi) = 0$$

throughout M_n.

CHAPTER III

RECURRENT TENSORS ON RIEMANNIAN MANIFOLDS

In this chapter we obtain some theorems which pertain to the existence (in the large) of recurrent tensor fields on compact Riemannian manifolds. Throughout this chapter M_n denotes an n-dimensional Riemannian manifold of class C^r, with r sufficiently large to insure that all theorems and their proofs will be valid and meaningful. It will also be assumed that the fundamental covariant tensor field of M_n is of class C^{r-1}, so that the coefficients of the Riemannian connection of M_n will be of class C^{r-2} on M_n. Let these coefficient functions be denoted by $\Gamma^i_{jk}(x)$. First we have

<u>Theorem 3.1</u>. Let M_n be a compact Riemannian manifold of class C^r ($r \geqq 2$) with a positive definite fundamental form $g_{ij}dx^i dx^j$, and let ϕ be a scalar field of class C^2 on M_n. Then the gradient vector field

$$\phi_{,i} = \frac{\partial\phi}{\partial x^i}$$

cannot be a recurrent vector field.

Proof: Let us assume that $\phi_{,i}$ is recurrent. Then, by Definition 2.1, $\phi_{,i}$ is not a zero vector field and satisfies

(3.1) $$\phi_{,ij} = \phi_{,i}\lambda_j \quad ,$$

where $\phi_{,ij} = (\phi_{,i})_{,j}$, and λ_j is a continuous vector field on M_n. In the expression $L(\phi)$ appearing in Theorem 2.3, let the functions $g^{jk}(x)$ be the components of the fundamental contravariant tensor field of M_n and choose the functions $h^k(x)$ to be

(3.2) $$h^k(x) = -g^{ij}\,\Gamma^k_{ij} - g^{kj}\,\lambda_j \;,$$

where Γ^k_{ij} are the coefficients of the Riemannian connection of M_n and are continuous on M_n, since M_n is of class C^r ($r \geqq 2$), and where λ_j are the components of the continuous vector field given in (3.1). The functions $h^k(x)$ are therefore continuous on M_n. Upon substituting from (3.2) into the expression for $L(\phi)$ and regrouping, we obtain

(3.3) $$L(\phi) = \Delta_2(\phi) - g^{kj}\lambda_j\phi_{,k} \;,$$

where $\Delta_2(\phi)$ is defined by (2.13). Substitution from (2.13) reduces (3.3) to

(3.4) $$L(\phi) = g^{kj}\phi_{,kj} - g^{kj}\lambda_{,j}\phi_{,k} \;.$$

It follows from (3.1) that

(3.5) $$L(\phi) = g^{kj}\phi_{,k}\lambda_j - g^{kj}\phi_{,k}\lambda_j = 0.$$

Thus, $L(\phi) = 0$ on M_n and, by Theorem 2.3,

$$\phi = \text{const.}$$

everywhere on M_n. Consequently, $\phi_{,i}$ is a zero vector field and cannot be a recurrent vector field.

Theorem 3.2. Let M_n be a compact Riemannian manifold of class C^r ($r \geq 2$) with a positive definite fundamental form $g_{ij}dx^i dx^j$, and let ξ_i be a covariant vector field of class C^2 on M_n. Then the second-order covariant tensor field

$$\xi_{i,j}$$

cannot be a recurrent tensor field.

Proof: If $\xi_{i,j}$ were a recurrent tensor field, then, by Definition 2.1, $\xi_{i,j}$ would not be a zero tensor field and would satisfy

(3.6) $$\xi_{i,jk} = \xi_{i,j}\lambda_k,$$

where $\xi_{i,jk} = (\xi_{i,j})_{,k}$ and λ_k is a continuous vector field on M_n. In the expression $L(\phi)$ in Theorem 2.3, let

(3.7) $$\phi = \xi^s\xi_s$$

and $h^k(x)$ be given by (3.2).

Then we have

(3.8) $$L(\phi) = L(\xi^s\xi_s) = \Delta_2(\xi^s\xi_s) - g^{jk}\lambda_j(\xi^s\xi_s)_{,k}.$$

Substituting

$$(3.9) \qquad \Delta_2(\xi^s\xi_s) = 2\xi^{i,j}\xi_{i,j} + 2\xi^s g^{jk}\xi_{s,jk}$$

and

$$(3.10) \qquad (\xi^s\xi_s)_{,k} = 2\xi^s\xi_{s,k}$$

into (3.8), we obtain

$$(3.11) \qquad L(\xi^s\xi_s) = 2\xi^{i,j}\xi_{i,j} + 2\xi^s g^{jk}\xi_{s,jk} - 2\xi^s g^{jk}\xi_{s,k}\lambda_j ,$$

which reduces, by (3.6), to

$$(3.12) \qquad L(\xi^s\xi_s) = 2\xi^{i,j}\xi_{i,j} .$$

Since

$$\xi^{i,j}\xi_{i,j} = g^{ab}g^{cd}\xi_{a,c}\xi_{b,d}$$

is a positive definite form in $\xi_{a,c}$, it follows that

$$(3.13) \qquad L(\xi^s\xi_s) = 2\xi^{i,j}\xi_{i,j} \geqq 0.$$

Therefore, by Theorem 2.3,

$$(3.14) \qquad L(\xi^s\xi_s) = 2\xi^{i,j}\xi_{i,j} = 0$$

everywhere on M_n. Since $\xi^{i,j}\xi_{i,j}$ is positive definite, it follows that

$$\xi_{i,j} = 0.$$

Thus, $\xi_{i,j}$ is a zero tensor field and, consequently, cannot be a recurrent tensor field.

Theorem 3.1 is valid for a scalar field, a tensor field of order zero, and Theorem 3.2 is valid for a covariant vector field, a covariant tensor field of order one. We now generalize these results in the following theorem, valid for a covariant tensor field of arbitrary order.

Theorem 3.3. Let M_n be a compact Riemannian manifold of class C^r ($r \geqq 2$) with a positive definite fundamental form $g_{ij}dx^i dx^j$, and let $\xi_{i_1 \dots i_p}$ be a covariant tensor field of order p and of class C^2 on M_n. Then the covariant tensor field of order p + 1,

$$\xi_{i_1 \dots i_p, j} ,$$

cannot be a recurrent tensor field.

Proof: If $\xi_{i_1 \dots i_p, j}$ were a recurrent tensor field, then, by Definition 2.1, $\xi_{i_1 \dots i_p, j}$ would not be a zero tensor field and would satisfy the condition

$$\xi_{i_1 \dots i_p, jk} = \xi_{i_1 \dots i_p, j} \lambda_k , \tag{3.15}$$

where $\xi_{i_1 \dots i_p, jk} = (\xi_{i_1 \dots i_p, j})_{,k}$, and λ_k is a continuous vector field on M_n. Now consider the scalar field

$$\phi = \xi^{i_1 \dots i_p} \xi_{i_1 \dots i_p} .$$

If, in the expression $L(\phi)$, we choose the functions $h^k(x)$ as defined by (3.2), we have

$$L(\xi^{i_1 \dots i_p} \xi_{i_1 \dots i_p}) = \Delta_2 (\xi^{i_1 \dots i_p} \xi_{i_1 \dots i_p}) - g^{jk} \lambda_j (\xi^{i_1 \dots i_p} \xi_{i_1 \dots i_p})_{,k} . \tag{3.16}$$

Upon substituting from

$$(\xi^{i_1 \dots i_p} \xi_{i_1 \dots i_p})_{,k} = 2 \xi^{i_1 \dots i_p} \xi_{i_1 \dots i_p, k} \tag{3.17}$$

and

$$(3.18) \qquad \Delta_2(\xi^{i_1\cdots i_p}\xi_{i_1\cdots i_p}) = 2\xi^{i_1\cdots i_p,j}\xi_{i_1\cdots i_p,j} + 2\xi^{i_1\cdots i_p}\, g^{jk}\xi_{i_1\cdots i_p,jk} \, ,$$

(3.16) becomes

$$(3.19) \quad L(\xi^{i_1\cdots i_p}\xi_{i_1\cdots i_p}) = 2\xi^{i_1\cdots i_p,j}\xi_{i_1\cdots i_p,j} + 2\xi^{i_1\cdots i_p}\, g^{jk}\xi_{i_1\cdots i_p,jk} - g^{jk}\lambda_j(2\xi^{i_1\cdots i_p}\xi_{i_1\cdots i_p,k}) \, ,$$

which reduces, by (3.15), to

$$(3.20) \qquad L(\xi^{i_1\cdots i_p}\xi_{i_1\cdots i_p}) = 2\xi^{i_1\cdots i_p,j}\xi_{i_1\cdots i_p,j} \, .$$

Since

$$\xi^{i_1\cdots i_p,j}\xi_{i_1\cdots i_p,j}$$

is a positive definite form in $\xi_{i_1\cdots i_p,j}$, we have

$$(3.21) \quad L(\xi^{i_1\cdots i_p}\xi_{i_1\cdots i_p}) = 2\xi^{i_1\cdots i_p,j}\xi_{i_1\cdots i_p,j} \geqq 0$$

on M_n. Hence, by Theorem 2.3,

$$(3.22) \qquad L(\xi^{i_1\cdots i_p}\xi_{i_1\cdots i_p}) = 2\xi^{i_1\cdots i_p,j}\xi_{i_1\cdots i_p,j} = 0$$

on M_n. Since

$$\xi^{i_1\cdots i_p,j}\xi_{i_1\cdots i_p,j}$$

is positive definite, we conclude from (3.22) that

$$\xi_{i_1\cdots i_p,j} = 0 \, .$$

Therefore, $\xi_{i_1\cdots i_p,j}$ is a zero tensor field and, consequently, cannot be a recurrent tensor field.

When the tensor field $\xi_{i_1\ldots i_p}$ is itself a recurrent tensor field, we have the following special case of Theorem 3.3:

Corollary 3.3.1. Let M_n be a compact Riemannian manifold of class C^r $(r \geqq 2)$ with a positive definite fundamental form. If $\xi_{i_1\ldots i_p}$ is a recurrent tensor field of class C^2 on M_n, it cannot be recurrent twice.

Another consequence of Theorem 3.3 is obtained when M_n is a K-manifold of class C^r $(r \geqq 3)$. Since $r \geqq 3$, the curvature tensor field of M_n is a continuous tensor field and, by Definition 2.2, is a recurrent tensor field. We have

Corollary 3.3.2. Let K_n be a compact Riemannian K-manifold of class C^r $(r \geqq 3)$ with a positive definite fundamental form. Then there exists no covariant tensor field

$$T_{hij}$$

of order 3 and of class C^1 on K_n for which the relation

$$R_{hijk} = T_{hij,k} \tag{3.23}$$

holds.

Proof: From Theorem 3.3, no recurrent tensor field can be the covariant derivative of another covariant tensor field. Thus, in a K-manifold, (3.23) is impossible.

As a conclusion to this chapter, we shall apply

Theorem 3.3 to obtain a generalization of the following theorem due to S. Bochner [5, p. 170]:

Theorem 3.4. On a compact Riemannian manifold of class C^r $(r \geqq 2)$ with a positive definite fundamental form, if, for an arbitrary covariant tensor (or scalar) field, the r-th successive covariant derivative vanishes,

(3.24) $$\xi_{i_1 \ldots i_p, k_1 \ldots k_r} = 0 ,$$

then the first derivative already vanishes. That is,

(3.25) $$\xi_{i_1 \ldots i_p, k} = 0 .$$

Theorem 3.5. On a compact Riemannian manifold M_n of class C^r $(r \geqq 2)$, if, for an arbitrary covariant tensor (or scalar) field, the r-th successive covariant derivative can be expressed as

(3.26) $$\xi_{i_1 \ldots i_p, k_1 \ldots k_{(r-1)} k_r} = \xi_{i_1 \ldots i_p, k_1 \ldots k_{(r-1)}} \lambda_{k_r}$$

for some continuous vector field λ_i on M_n, then the first derivative already vanishes. That is,

(3.27) $$\xi_{i_1 \ldots i_p, k} = 0.$$

Proof: By the method of proof of Theorem 3.3, (3.26) implies

$$\xi_{i_1 \ldots i_p, k_1 \ldots k_{(r-1)}} = 0 .$$

Consequently, by Theorem 3.4, (3.27) follows.

CHAPTER IV

CONFORMAL RIEMANNIAN MANIFOLDS

Let M_n be a Riemannian manifold with fundamental covariant tensor field g_{jk} and curvature tensor field R_{hijk}. Let σ be a scalar field on M_n defining a global conformal transformation of the fundamental covariant tensor field of M_n, as in Definition 2.5. If the transformed curvature tensor field $\bar{R}_{hijk}$ satisfies the condition

$$(4.1) \qquad \bar{R}_{hijk} = 0$$

throughout M_n, then M_n is said to be reduced to a flat Riemannian manifold by the global conformal transformation of the fundamental covariant tensor field. It is shown in this chapter that for certain types of Riemannian manifolds such a global conformal transformation of the fundamental covariant tensor field is impossible. The method of this chapter also leads to the main results of Chapter V, in which K^*-manifolds are studied.

If $n \neq 1$, (2.36) can be written as

(4.2)
$$\Delta_2\sigma + \frac{n-2}{2}\Delta_1\sigma = \frac{e^{2\sigma}\bar{R} - R}{2\ (n-1)}$$

This equation will be used in the proof of

<u>Theorem 4.1.</u> Let $M_n (n \geqq 2)$ be a compact Riemannian manifold of class C^r $(r \geqq 3)$ with a positive definite fundamental form $g_{ij}dx^i dx^j$, such that the scalar curvature R of M_n satisfies the condition

(4.3)
$$R \geqq 0$$

throughout M_n. There exists no global conformal transformation of the fundamental covariant tensor field which is determined by a scalar field σ of class C^s $(s \geqq 2)$ on M_n, such that the transformed scalar curvature $\bar{R}$ satisfies the condition

(4.4)
$$\bar{R} \leqq 0$$

throughout M_n, unless

(4.5)
$$R = 0$$

throughout M_n. Then automatically

(4.6)
$$\bar{R} = 0$$

throughout, and the global conformal transformation is just a similarity transformation.

Proof: Let M_n be a compact Riemannian manifold on which the given hypotheses are satisfied. Since M_n is of class C^r ($r \geqq 3$) and σ is of class C^s ($s \geqq 2$), conditions (2.28) through (2.36) are satisfied throughout M_n. Since $n \geqq 2$, $n - 1 \neq 0$ and (4.2) is satisfied throughout M_n. Furthermore, R and $\overline{R}$ are continuous on M_n. By virtue of (2.12) and (2.13), (4.2) can be written as

$$g^{ij}\sigma_{,ij} + \frac{n-2}{2} g^{ij}\sigma_{,i}\sigma_{,j} = \frac{e^{2\sigma}\overline{R}-R}{2(n-1)} \tag{4.7}$$

Now consider the expression

$$L(\sigma) = g^{jk}(x)\frac{\partial^2\sigma}{\partial x^j \partial x^k} + h^i(x)\frac{\partial\sigma}{\partial x^i}\ . \tag{4.8}$$

If we choose $h^i(x)$ as

$$h^i(x) = -g^{jk}\Gamma^i_{jk} + \frac{n-2}{2} g^{ij}\sigma_{,j} \quad , \tag{4.9}$$

we have, by virtue of (4.7),

$$L(\sigma) = \frac{e^{2\sigma}\overline{R}-R}{2(n-1)} \tag{4.10}$$

It follows from (4.3) and (4.4) that

$$L(\sigma) \leqq 0$$

throughout M_n. Thus, by Theorem 2.3,

$$\sigma = \text{const.}$$

throughout M_n, and

$$L(\sigma) = 0$$

throughout M_n. Therefore, by (4.10), it follows that

$$R = \overline{R} = 0$$

and the proof is complete.

Theorem 4.2. Let M_n $(n \geqq 2)$ be a compact Riemannian manifold of class C^r $(r \geqq 3)$ with a positive definite fundamental form $g_{ij}dx^i dx^j$, such that the scalar curvature R of M_n satisfies the condition

(4.11) $$R \leqq 0$$

throughout M_n. There exists no global conformal transformation of the fundamental covariant tensor field which is determined by a scalar field σ of class C^s $(s \geqq 2)$ on M_n, such that the transformed scalar curvature $\bar{R}$ satisfies the condition

(4.12) $$\bar{R} \geqq 0$$

throughout M_n, unless

(4.13) $$R = 0$$

throughout M_n. Then automatically

(4.14) $$\bar{R} = 0$$

throughout, and the global conformal transformation is just a similarity transformation.

Proof: The method of proof is the same as the proof of Theorem 4.1, except in the present case we find

$$L(\sigma) \geqq 0$$

throughout M_n. Theorem 2.3 also applies here, and the conclusion follows.

A useful consequence of Theorems 4.1 and 4.2 is

Theorem 4.3. Let M_n $(n \geqq 2)$ be a compact Riemannian manifold of class C^r $(r \geqq 3)$ with a positive definite fundamental form $g_{ij}dx^i dx^j$, such that the scalar curvature R of M_n satisfies either of the conditions

(4.15) $$R \geqq 0$$

or

(4.16) $$R \leqq 0$$

throughout M_n. There exists no global conformal transformation of the fundamental covariant tensor field which is determined by a scalar field σ of class C^s $(s \geqq 2)$ on M_n, such that the transformed scalar curvature $\bar{R}$ satisfies the condition

(4.17) $$\bar{R} = 0$$

throughout M_n, unless

(4.18) $$R = 0$$

throughout M_n, and then the global conformal transformation is just a similarity transformation.

Proof: If (4.15) is satisfied, the hypotheses of Theorem 4.1 are satisfied, and the conclusion follows. If (4.16) is satisfied, the hypotheses of Theorem 4.2 are satisfied, and the conclusion follows.

As a consequence of Theorem 4.3, we have

Theorem 4.4. Let M_n $(n \geqq 2)$ be a compact Riemannian manifold of class C^r ($r \geqq 3$) with a positive definite fundamental form $g_{ij}dx^i dx^j$, such that the scalar curvature R of M_n satisfies either

$$R \geqq 0$$

or

$$R \leqq 0$$

throughout M_n. There exists no global conformal transformation of the fundamental covariant tensor field which is determined by a scalar field σ of class C^s ($s \geqq 2$) on M_n and which reduces M_n to a flat Riemannian manifold, unless $R = 0$ throughout M_n, and then the global conformal transformation is a similarity transformation.

Proof: If M_n were reduced to a flat Riemannian manifold by a global conformal transformation, the transformed curvature tensor field would have to satisfy

$$\bar{R}_{hijk} = 0$$

throughout M_n. It would follow immediately that

$$\bar{R} = 0$$

throughout M_n. The hypotheses of Theorem 4.3 would be satisfied, and, therefore, the conclusion follows.

When M_n is of constant curvature, we have the following

<u>Corollary 4.4.1</u>. Let M_n ($n \geqq 2$) be a compact

Riemannian manifold of class C^r ($r \geqq 3$) with a positive definite fundamental form $g_{ij}dx^i dx^j$, such that the scalar curvature R of M_n is a constant throughout M_n. There exists no global conformal transformation of the fundamental covariant tensor field which reduces M_n to a flat Riemannian manifold unless R is identically zero on M_n, and then the global conformal transformation is a similarity.

Now, for a Riemannian manifold M_n, consider the Ricci quadratic form

$$R_{ij}\xi^i\xi^j,$$

where ξ^i is an arbitrary contravariant vector field on M_n. If

$$R_{ij}$$

are the coefficients of a positive definite quadratic form on M_n, then the scalar curvature

$$R = g^{ij}R_{ij}$$

satisfies

$$R > 0$$

on M_n, since g^{ij} are the coefficients of the positive definote fundamental form on M_n. Similarly, if

$$R_{ij}$$

are the coefficients of a negative definite form on M_n, then

$$R < 0$$

on M_n. By Theorem 4.4, neither of these situations is possible if M_n is compact and of class C^r $(r \geqq 3)$, such that there exists a global conformal transformation of the fundamental covariant tensor field which reduces M_n to a flat Riemannian manifold. The above consideration leads to

<u>Theorem 4.5</u>. Let M_n $(n \geqq 2)$ be a compact Riemannian manifold of class C^r $(r \geqq 3)$ with a positive definite fundamental form $g_{ij}dx^i dx^j$. Let there be a global conformal transformation of the fundamental covariant tensor field which is determined by a scalar field σ of class C^s $(s \geqq 2)$ on M_n and which reduces M_n to a flat Riemannian manifold. Then the Ricci quadratic form

$$R_{ij}\xi^i\xi^j$$

is neither positive definite nor negative definite throughout M_n.

CHAPTER V

RIEMANNIAN MANIFOLD AND K^*-MANIFOLD

This chapter is concerned with the study of two problems, the first of which is to determine if a compact Riemannian manifold can be reduced to a K^*-manifold by a suitable global conformal transformation of the fundamental covariant tensor field. A necessary condition is found. The scalar curvature of the Riemannian manifold cannot be non-negative throughout the manifold.

The second problem is to determine if a compact K^*-manifold can be reduced to a flat Riemannian manifold by a suitable global conformal transformation of the fundamental covariant tensor field. This is found to be impossible for a compact simple (in the large) K^*-manifold.

We are concerned here only with Riemannian manifolds with positive definite fundamental forms. By Theorem 2.1, a K^*-manifold with a positive definite fundamental form is at least locally simple. Only global results are of interest here, and, therefore, the K^*-manifolds to be considered are those K^*-manifolds which are simple (in the large) in the sense of Definition 2.4.

For the purpose of obtaining the results mentioned above, we first prove

<u>Theorem 5.1</u>. Let K_n^* be a simple (in the large) K^*-manifold. Then the scalar curvature R of K_n^* satisfies the condition

$$R \leqq 0$$

throughout K_n^*. Furthermore, R cannot be identically zero on K_n^*.

Proof: By Definition 2.4, there exists an anti-symmetric second-order covariant tensor field m_{ij} on K_n^* such that the curvature tensor field of K_n^* satisfies

$$R_{hijk} = m_{hi}m_{jk} \tag{5.1}$$

throughout K_n^*. From (5.1), it follows that

$$R_{ij} = g^{hk}R_{hijk} = g^{hk}m_{hi}m_{jk} \quad . \tag{5.2}$$

This in turn gives

$$R = g^{ij}R_{ij} = g^{ij}g^{hk}m_{hi}m_{jk} \quad , \tag{5.3}$$

which reduces to

$$R = -m^{ij}m_{ij} \quad . \tag{5.4}$$

Since

$$m^{ij}m_{ij}$$

is a positive definite form in m_{ij}, it follows from (5.4) that

$$R \leqq 0$$

throughout K_n^*. Furthermore, if R were identically zero on K_n^*, then m_{ij} would be identically zero on K_n^*. Thus, R_{hijk} would be a zero tensor field throughout K_n^*. K_n^* would be flat, contradicting the assumption that it is simple (in the large).

We are now able to prove

Theorem 5.2. Let M_n $(n \geqq 2)$ be a compact Riemannian manifold of class C^r $(r \geqq 3)$ with a positive definite fundamental form $g_{ij}dx^i dx^j$, such that the scalar curvature R of M_n satisfies

$$R \geqq 0$$

throughout M_n. There exists no global conformal transformation of the fundamental covariant tensor field which is determined by a scalar field σ of class C^s $(s \geqq 2)$ on M_n and which reduces M_n to a simple (in the large) K^*-manifold.

Proof: Suppose there is a global conformal transformation of the fundamental covariant tensor field which reduces M_n to a simple (in the large) K^*-manifold. By Theorem 5.1, the transformed scalar curvature $\bar{R}$ satisfies

$$\bar{R} \leqq 0$$

throughout M_n. The hypotheses and, hence, the conclusion of Theorem 4.1 are satisfied. That is,

$$R = \overline{R} = 0$$

throughout M_n. By Theorem 5.1, $\overline{R}$ cannot be zero throughout, since M_n was assumed to be reduced to a simple (in the large) K^*-manifold by the global conformal transformation. This contradiction establishes the theorem.

We now apply Theorem 4.4 to obtain the final result of this chapter.

Theorem 5.3. Let K_n^* ($n \geqq 2$) be a compact simple (in the large) K^*-manifold of class C^r ($r \geqq 3$). There exists no global conformal transformation of the fundamental covariant tensor field which is determined by a scalar field σ of class C^s ($s \geqq 2$) on K_n^* and which reduces K_n^* to a flat Riemannian manifold.

Proof: By Theorem 5.1, the hypotheses of Theorem 4.4 are satisfied in the present case. Therefore, if there were a global conformal transformation which reduces K_n^* to a flat Riemannian manifold, the scalar curvature of K_n^* would satisfy

$$R = 0$$

throughout K_n^*, by Theorem 4.4. This is impossible, by Theorem 5.1. This contradiction establishes the theorem.

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