Structural Theory for Laminated Anisotropic Elastic Shells

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A linear theory is formulated for analysis of small deflections of thin shells with arbitrary geometrical configuration and laminated of an arbitrary number of layers of different thicknesses, orientations, and anisotropic elastic coefficients. An accurate shell theory (Vlasov's) is used, and the composite-shell constitutive relation incorporates the anisotropic stretching-bending coupling effects considered by Stavsky. For shells of arbitrary geometry, it is found necessary to introduce a new parameter $F_{ij} \equiv \int_h z^3 Q_{ij} dz$ in the constitutive relation. This parameter is zero for homogeneous anisotropic materials and for anisotropic materials laminated symmetrically with respect to the middle surface. However, for a two-layer filament-wound shell, this parameter can increase the flexural rigidity by 3%, which is greater than a 2% effect considered in a previous layered-anisotropic cylindrical shell analysis.

INTRODUCTION

WITH THE DEVELOPMENT of structures laminated of anisotropic composite materials such as plywood and fiber-reinforced plastics, there has been considerable attention to the structural analysis of such shells. For example, in 1945, March et al. [1] considered cylindrical shells using constitutive equations of a homogeneous, anisotropic material.

The first analysis using a constitutive equation incorporating the coupling between stretching and bending effects is due to Ambartsumyan in 1953 [2]. However, in this analysis and his numerous subsequent analyses, summarized in Ref. 3, Ambartsumyan assumed that the individual layers were orthotropic (rather than generally anisotropic) and oriented so that the principal axes of material symmetry coincided with the principal coordinates of the shell reference

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surface. Thus, Ambartsumyan's work is applicable to what are more correctly termed laminated orthotropic shells rather than laminated anisotropic shells.

In 1959 Stavsky [4, 5] formulated a theory of plates of laminated anisotropic material. The Stavsky constitutive equations were applied by Dong et al. [6] in 1962 to the analysis of thin shells of arbitrary geometry. The shell theory used in Ref. 6 was essentially that now known as Love's first approximation theory [7]. There are a number of shell theories which are more accurate than the latter; these include Love's second approximation theory [7], Flügge's [8] (not derived for arbitrary shells), and Vlasov's [9]. The latter has the advantage of being both highly accurate and possessing symmetries not possessed by the Love's second and Flügge theories. Vlasov has found that in the case of statically loaded isotropic shells the difference between Love's second and Vlasov's never exceeded 5%. However, this question has not been investigated even for simple orthotropic shells much less for laminated anisotropic ones. Thus, it is of interest to investigate this factor for shells with lamination orientations and materials of engineering importance.

Cheng and Ho [10] presented an analysis of laminated anisotropic cylindrical shells using Flügge's shell theory. So far as is known, the present work is the first to combine one of the most accurate thin shell theories (Vlasov's) with the most general anisotropic constitutive equations (Stavsky's) for an arbitrary shell geometry.

HYPOTHESES

The following hypotheses of the classical linear theory of small deflections of thin elastic shells are retained:

H1. Displacements are assumed to be sufficiently small that linearized strain-displacement equations may be used.

H2. Transverse slopes are assumed to be sufficiently small that the linearized curvature expressions are adequate.

H3. The ratio of the thickness of the shell to the smallest radius of curvature is very small so that the following Bernoulli-Euler-Kirchhoff-Love hypotheses can be used:

(a) Plane, normal cross sections before deformation remain plane and normal to the reference surface during deformation. Thus, transverse-shear deformations are neglected.

(b) The shell is inextensible in the thickness direction, i.e., normal strains in the thickness direction are neglected.

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H4. The thickness is uniform and the elastic coefficients do not vary with position over the reference surface.

H5. Linearly elastic constitutive equations are used.

It is noted that H3(a) and (b) imply that each thin layer is perfectly bonded, i.e., by an adhesive of infinitesimal thickness but infinite shear and extensional stiffnesses. They also limit the analysis to shear-rigid laminated structures, excluding sandwich-type construction with shear flexible cores.

GEOMETRICAL CONSIDERATIONS

A shell reference surface may be arbitrarily defined. Here, as is customary in shell theory, the reference surface used is the middle surface, defined as the surface located midway between the inner and outer surfaces of the composite shell. It is noted that in a multiply-laminated shell, the middle surface is not necessarily the neutral surface. In fact, there are usually numerous neutral surfaces (surfaces of zero strain) in contrast to a homogeneous shell which has only one. The quantities α, β are taken to be orthogonal curvilinear coordinates along the lines of principal curvature of the reference surface; z is the outward normal to the reference surface. Then the α, β, z coordinates constitute a three-dimensional, orthogonal, curvilinear coordinate system having a general differential line-element length ds given by [9]

$$(ds)^2 = H_1^2(d\alpha)^2 + H_2^2(d\beta)^2 + H_3^2(dz)^2 \tag{1}$$

where, here

$$H_1 \equiv \overline{A}(1 + R_1^{-1}z),\ H_2 \equiv \overline{B}(1 + R_2^{-1}z),\ H_3 \equiv 1 \tag{2}$$

and $\overline{A}$, $\overline{B}$ are the reference-surface metric coefficients (i.e., coefficients of the first fundamental quadratic form of the reference surface) and R_1, R_2 are the principal radii of curvature of the reference surface. Then the pertinent strain-displacement relations exact within the limitation of hypothesis H1 are [9]:

$$\begin{aligned}
\epsilon_{\alpha\alpha} &= H_1^{-1}U_{\alpha,\alpha} + (H_1H_2)^{-1}H_{1,\beta}U_\beta + (H_1H_3)^{-1}H_{1,z}U_z \\
\epsilon_{\beta\beta} &= H_2^{-1}U_{\beta,\beta} + (H_2H_3)^{-1}H_{2,z}U_z + (H_1H_2)^{-1}H_{2,\alpha}U_\alpha \\
\epsilon_{\alpha\beta} &= (H_1/H_2)(U_\alpha/H_1)_{,\beta} + (H_2/H_1)(U_\beta/H_2)_{,\alpha}
\end{aligned} \tag{3}$$

where $\epsilon_{\alpha\alpha}$, $\epsilon_{\beta\beta}$, $\epsilon_{\alpha\beta}$ are the in-surface strains at an arbitrary point (α,β,z); U_α, U_β, U_z are the displacements in the α, β, z directions; a

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subscript comma denotes partial differentiation with respect to the subscript quantity following the comma.

When H_1 and H_2 as given by Eqs. (2) are expanded in powers of z and substituted into Eqs. (3), the results can be expressed as follows:

$$\epsilon_{\alpha\alpha} = e_1 + X_1 z,\ \epsilon_{\beta\beta} = e_2 + X_2 z,\ \epsilon_{\alpha\beta} = e_6 + X_6 z \tag{4}$$

where e_1, e_2, e_6 are the reference-surface strains (normal in directions α and β, and shear along α or β directions, respectively) which are given by:

$$\begin{aligned} e_1 &= \bar{A}^{-1} u_{,\alpha} + (\overline{AB})^{-1} \bar{A}_{,\beta} v + R_1^{-1} w \\ e_2 &= (\overline{AB})^{-1} \bar{B}_{,\alpha} u + \bar{B}^{-1} v_{,\beta} + R_2^{-1} w \\ e_6 &= (\bar{A}/\bar{B})(u/\bar{A})_{,\beta} + (\bar{B}/\bar{A})(v/\bar{B})_{,\alpha} \end{aligned} \tag{5}$$

and the curvature changes X_1, X_2 and twist change X_6 of the reference surface are given by:

$$\begin{aligned} X_1 &= (1/R_1)_{,\alpha}(u/\bar{A}) + (1/R_1)_{,\beta}(v/\bar{B}) - R_1^{-2} w \\ &\quad - \bar{A}^{-1}(\bar{A}^{-1} w_{,\alpha})_{\alpha\alpha} - (\overline{AB^2})^{-1} \bar{A}_{,\beta} w_{,\beta} \\ X_2 &= (1/R_2)_{,\alpha}(u/\bar{A}) + (1/R_2)_{,\beta}(v/\bar{B}) - R_2^{-2} w \\ &\quad - \bar{B}^{-1}(\bar{B}^{-1} w_{,\beta})_{,\beta} - (\overline{A^2 B})^{-1} \bar{B}_{,\alpha} w_{,\alpha} \end{aligned} \tag{6}$$

$$\begin{aligned} X_6 &= (R_1^{-1} - R_2^{-1})[(\bar{A}/\bar{B})(u/\bar{A})_{,\beta} - (\bar{B}/\bar{A})(v/\bar{B})_{,\alpha}] \\ &\quad - (2/\overline{AB})[w_{,\alpha\beta} - \bar{A}^{-1}\bar{A}_{,\beta} w_{,\alpha} - \bar{B}^{-1}\bar{B}_{,\alpha} w_{,\beta}] \end{aligned}$$

where u, v, w are the reference-surface displacements.

When it is desired to determine the exact static stress distribution in the shell, it is necessary to derive a compatibility equation from Eqs. (5) and (6) which ensures compatibility of the reference-surface strains and curvature changes (see Ref. 9, p. 327, Eqs. 16.2). However, in many problems of interest, namely buckling and vibration, it is usually sufficiently accurate to assume a simple form (usually harmonic in the reference-surface coordinates) and then check to make certain that the compatibility equation is satisfied.

EQUILIBRIUM CONSIDERATIONS

The equilibrium equations are as follows:

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$$
\begin{aligned}
&(\overline{B}N_1)_{,\alpha} - \overline{B}_{,\alpha}N_2 + (\overline{A}N_{21})_{,\beta} + \overline{A}_{,\beta}N_{12} + (\overline{AB}/R_1)Q_1 + \overline{AB}q_1 = 0\\
&(\overline{A}N_2)_{,\beta} - \overline{A}_{,\beta}N_1 + (\overline{B}N_{12})_{,\alpha} + \overline{B}_{,\alpha}N_{21} + (\overline{AB}/R_2)Q_2 + \overline{AB}q_2 = 0\\
&-R_1^{-1}N_1 - R_2^{-1}N_2 + (\overline{AB})^{-1}[(\overline{B}Q_1)_{,\alpha} + (\overline{A}Q_2)_{,\beta}] + q_3 = 0 \quad \text{(7a-e)}\\
&(\overline{B}M_{12})_{,\alpha} + \overline{B}_{,\alpha}M_{21} - (\overline{A}M_2)_{,\beta} + \overline{A}_{,\beta}M_1 - \overline{AB}Q_2 = 0\\
&(\overline{A}M_{21})_{,\beta} + \overline{A}_{,\beta}M_{12} - (\overline{B}M_1)_{,\alpha} + \overline{B}_{,\alpha}M_2 - \overline{AB}Q_1 = 0
\end{aligned}
$$

where the stress resultants are defined as follows:

$$
\begin{aligned}
N_1 &= \overline{B}^{-1}\int_h H_2\sigma_\alpha dz = \int_h (1+CR_2^{-1}z)\,\sigma_\alpha dz\\
N_2 &= \overline{A}^{-1}\int_h H_1\sigma_\beta dz = \int_h (1+CR_1^{-1}z)\,\sigma_\beta dz\\
N_{12} &= \overline{B}^{-1}\int_h H_2\sigma_{\alpha\beta} dz = \int_h (1+CR_2^{-1}z)\,\sigma_{\alpha\beta} dz\\
N_{21} &= \overline{A}^{-1}\int_h H_2\sigma_{\beta\alpha} dz = \int_h (1+CR_1^{-1}z)\,\sigma_{\beta\alpha} dz\\
M_1 &= \overline{B}^{-1}\int_h H_2\sigma_\alpha z dz = \int_h (1+CR_2^{-1}z)\,z\sigma_\alpha dz\\
M_2 &= \overline{A}^{-1}\int_h H_1\sigma_\beta z dz = \int_h (1+CR_1^{-1}z)\,z\sigma_\beta dz\\
M_{12} &= \overline{B}^{-1}\int_h H_2\sigma_{\alpha\beta} z dz = \int_h (1+CR_2^{-1}z)\,z\sigma_{\alpha\beta} dz\\
M_{21} &= \overline{A}^{-1}\int_h H_1\sigma_{\beta\alpha} z dz = \int_h (1+CR_1^{-1}z)\,z\sigma_{\beta\alpha} dz\\
Q_1 &= \overline{B}^{-1}\int_h H_2\sigma_{\alpha z} dz\ ;\ Q_2 = \overline{A}^{-1}\int_h H_1\sigma_{\beta z} dz
\end{aligned}
\qquad (8)
$$

where C is a dummy parameter or tracer to show the difference between subsequent results for the Love first approximation theory ($C = 0$) and the Vlasov theory ($C = 1$). It is noted that the variables Q_1 and Q_2 can be eliminated by substitution from Eqs. (7d) and (7e) into Eq. (7c).

CONSTITUTIVE RELATION

In view of hypothesis H3(b), the following constitutive equation is the most general (anisotropic) possible for each individual layer:

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$$\begin{bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{\alpha\alpha} \\ \epsilon_{\beta\beta} \\ \epsilon_{\alpha\beta} \end{bmatrix} \tag{9}$$

where Q_{ij} are the reduced stiffness coefficients that is $Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}}$ and the general anisotropic symmetry requirement that $Q_{ji} = Q_{ij}$ has already been incorporated.

In Stavsky's coupled layered anisotropic theory of plates [5], the following symbols have become customary to denote composite-plate stiffness coefficients:

$$(A_{ij}, B_{ij}, D_{ij}) = \int_h (1, z, z^2) Q_{ij} dz \tag{10}$$

where $i,j = 1,2,6$, and the A_{ij}, B_{ij}, D_{ij} are symmetric because the Q_{ij} are symmetric. These same coefficients have been used also in coupled layered anisotropic shell theories [6,10].

Here it is found necessary to define a new composite-shell stiffness coefficient not found previously:

$$F_{ij} = \int_h z^3 Q_{ij} \, dz \tag{11}$$

Inserting Eqs. (4) and (9) into Eqs. (8), performing the indicated integrations and expressing the results in terms of Eqs. (10) and (11) yields the following composite-shell constitutive relation:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_{12} \\ N_{21} \\ M_1 \\ M_2 \\ M_{12} \\ M_{21} \end{bmatrix} = \left[\begin{array}{c|c} & \\ a_{ij} & b_{ij} \\ & \\ \hline & \\ b_{ij} & d_{ij} \\ & \end{array}\right] \begin{bmatrix} e_1 \\ e_2 \\ e_6 \\ X_1 \\ X_2 \\ X_6 \end{bmatrix} \tag{12}$$

where the elements of the submatrices are:

$$a_{ij} = A_{ij} + K_j B_{ij}, \quad b_{ij} = B_{ij} + K_j D_{ij},$$
$$d_{ij} = D_{ij} + K_j F_{ij}; \quad i = 1,2,3,4; j = 1,2,6; \text{ no sum} \tag{13a, b, c}$$

where $K_j = C/R_j$.

It is noted that for arbitrary layer orientation schemes for anisotropic layers, in general

$$B_{ij}/A_{ij} \neq B_{kl}/A_{kl}, \quad D_{ij}/B_{ij} \neq D_{kl}/B_{kl}, \quad F_{ij}/D_{ij} \neq F_{kl}/D_{kl} \tag{14}$$

due to the different z integrations entering into Eqs. (10) and (11).

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However, for the case of a single-layer (macroscopically homogeneous) general anisotropic shell, inequalities (14) would become equalities. This is the basic difference between an arbitrarily layered anisotropic shell (or plate) and an ordinary (macroscopically homogeneous) anisotropic shell (or plate).

Another important point to note is that, in general ($K_1 \neq K_2$), the a_{ij}, b_{ij}, d_{ij} submatrices are not symmetric. This is in contrast to the more approximate layered anisotropic theory formulated in Ref. 6; the latter theory can be obtained merely by setting $C = 0$ ($K_j = 0$) in the present theory. This is the fundamental difference between first-order and second-order theories.

There is an interesting cyclic nature in the form of Eqs. (13): B_{ij} appears as a secondary term in the expression for a_{ij}, but B_{ij} plays the primary role in b_{ij}; likewise D_{ij} is secondary in b_{ij} and primary in d_{ij}. Thus, in general,

$$s_{ij}^{(m)} = S_{ij}^{(m)} + K_j S_{ij}^{(m+1)} \tag{15}$$

where $s_{ij}^{(m)}$ are the composite-shell stiffness coefficients ($s_{ij}^{(1)} \equiv a_{ij}$, $s_{ij}^{(2)} \equiv b_{ij}$, etc.) and $S_{ij}^{(m)}$ are Stavsky's composite-plate stiffness coefficients ($S_{ij}^{(1)} \equiv A_{ij}$, etc.). Apparently this cyclic characteristic is due to the nature of the asymptotic expansion in powers of z; see Eq. (4).

When the present composite-shell constitutive relations, Eqs. (12) and (13), are reduced to the special case of a cylindrical shell ($K_1 = 0$), they differ from those of Ref. 10 because of slight differences in the form of the strain-displacement relations they used (Flügge instead of Vlasov) and because they neglected F_{ij}. Ref. 9 presents some reasons supporting the use of the Vlasov relations.

DETERMINATION OF RELATIVE CONTRIBUTIONS OF SECONDARY TERMS IN SHELL CONSTITUTIVE COEFFICIENTS

In order to assess the relative importance of the $K_j F_{ij}$ term appearing in Eq. (13c), it is of interest to determine the effect of $K_j B_{ij}$ in relation to A_{ij}, $K_j D_{ij}$ compared to B_{ij}, as well as $K_j F_{ij}$ as a fraction of D_{ij}.

First of all, it is noted that all of these secondary effects are not present in plates (because then $K_1 = K_2 = 0$). Furthermore, only half of them are present in a cylindrical shell ($K_1 = 0$).

Inspection of Eqs. (13) shows that B_{ij} and F_{ij} are both zero for single-layer (homogeneous) shells regardless of whether the material is anisotropic and for layered shells which are laminated symmetri-

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cally about the middle surface. The most common example of the latter is one having a total number of layers (n) which is odd, assuming that each layer is identical in thickness and elastic behavior and that the lamination orientations are symmetric about the middle surface.

To consider the relative importance of the secondary effects, a series of calculations are made for a shell having an even number (n) of identical layers oriented in such a way that successive alternating layers have elastic coefficients Q_{ij} and rQ_{ij}. The shell is assumed to be spherical (so that $K_j = K_i$) with a thickness/radius ratio $h/R = 0.1$ (considered to be the upper limit of thin-shell theory). Sample calculations for $n = 2$ are

$$A_{ij}/Q_{ij}h = (1/2)\ (1) + (1/2)\ (r) = (1/2)\ (1 + r)$$

$$B_{ij}/Q_{ij}h^2 = (1/2)\ [(1/2)^2(1) - (1/2)^2(r)] = (1/8)\ (1 - r)$$

$$D_{ij}/Q_{ij}h^3 = (1/3)\ [(1/2)^3(1) + (1/2)^3(r)] = (1/24)\ (1 + r)$$

$$F_{ij}/Q_{ij}h^4 = (1/4)\ [(1/2)^4(1) - (1/2)^4(r)] = (1/64)\ (1 - r)$$

Then

$$K_iB_{ij}/A_{ij} = [0.1(1/8)\ (1 - r)]/[(1/2)\ (1 + r)] = 0.0250\ (1 - r)/(1 + r)$$

$$K_jD_{ij}/B_{ij} = [0.1(1/24)\ (1 + r)]/[(1/8)\ (1 - r)] = 0.0333\ (1 + r)/(1 - r)$$

$$K_jF_{ij}/D_{ij} = [0.1(1/64)\ (1 - r)]/[(1/24)\ (1 + r)] = 0.0375\ (1 - r)/(1 + r)$$

In carrying out similar calculations for other even values of n, it turns out that $A_{ij}/Q_{ij}h$ is always *equal* to $(1/2)(1 + r)$; $D_{ij}/Q_{ij}h^3$ is always proportional to $(1 + r)$; and $B_{ij}/Q_{ij}h^2$ and $F_{ij}/Q_{ij}h^3$ are always proportional to $(1 - r)$. Results of a series of such calculations are given in *Table I*. It is noted that as the number of layers is increased hKB_{ij}/A_{ij} and hKF_{ij}/D_{ij} decrease rapidly but that hKF_{ij}/D_{ij} is always greater than hKB_{ij}/A_{ij}, thus justifying the inclusion of the new quantity F_{ij}.

Table I. Secondary Contributions to the Composite-Shell Stiffness Coefficients on a Relative Basis for a Spherical Shell Having R/h = 10 and r = 0.1.

Ratio	Value of Ratio for n =		
	2	4	6
hKB_{ij}/A_{ij}	0.020	0.010	0.007
hKD_{ij}/B_{ij}	0.041	0.081	0.131
hKF_{ij}/D_{ij}	0.031	0.026	0.018

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GOVERNING EQUATIONS IN TERMS OF DISPLACEMENTS

To obtain a set of three coupled, linear, partial differential equations in terms of the reference-surface displacements (u, v, w), it is necessary to substitute Eqs. (5), (6), and (12) into Eqs. (7a-c) after eliminating Q_1 and Q_2 by means of Eqs. (7d,e). However, the results for the general case are too lengthy to reproduce here.

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NOMENCLATURE

a_{ij}, b_{ij}, d_{ij} = composite-shell stiffness coefficients, Eq. (13)
A_{ij}, B_{ij}, D_{ij} = Stavsky composite-plate stiffness coefficients, Eq. (10)
$\overline{A}, \overline{B}$ = metric coefficients of the reference surface
C = tracer: zero for Love first approximation theory, unity for Vlasov theory
C_{ij} = Cauchy stiffness coefficients for generalized Hooke's law in 3-dimensional space
e_i = reference-surface strains
F_{ij} = new composite-shell stiffness coefficient defined by Eq. (11)
h = total thickness of composite shell
H_1, H_2, H_3 = line-element parameters, Eq. (2)
K_j = C/R_j
M_1, M_2, M_{12}, M_{21} = bending and twisting moment resultants
n = number of layers
N_1, N_2, N_{12}, N_{21} = in-surface normal and shear force resultants
q_1, q_2, q_3 = components of distributed load intensity (per unit of reference-surface area)
Q_1, Q_2 = transverse shear force resultants
Q_{ij} = reduced stiffness matrix = $C_{ij} - \dfrac{C_{i3}C_{j3}}{C_{33}}$
r = ratio of $Q_{ij}^{(k+1)}/Q_{ij}^{(k)}$
R_1, R_2 = principal radii of curvature of the reference surface
$s_{ij}^{(m)}$ = a_{ij}, b_{ij}, d_{ij}
$S_{ij}^{(m)}$ = A_{ij}, B_{ij}, D_{ij}, F_{ij}

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U_α, U_β, U_z = displacements of any point in the shell

u, v, w = displacements of an arbitrary point on the reference surface

X_i = changes in curvature ($i = 1,2$) or in twist ($i = 6$) of the reference surface

z = outer normal coordinate, measured from the reference surface

α, β = orthogonal curvilinear coordinates on the reference surface

ϵ_{ij} = strain components at any point in the shell

σ_{ij} = stress components, acting in a plane parallel to the local tangent to the reference plane, at any point in the shell

sub i = 1, 2, 6

sub ij $\begin{cases} = i = 1,2,3,4 \text{ and } j = 1,2,6 \text{ for } s_{ij}^{(m)} = (a_{ij}, b_{ij}, d_{ij}) \\ = ij = \alpha\alpha, \beta\beta, \alpha\beta \text{ for } \epsilon_{ij} \text{ and } \sigma_{ij} \end{cases}$

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