71-12,579

KALBFLEISCH, John Howard, 1943-

THE ALGEBRAIC PARTITIONING OF FACTORIAL ARRANGEMENTS.

The University of Oklahoma, Ph.D., 1970
Biostatistics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

THIS DISSERTATION HAS BEEN MICROFILMED EXACTLY AS RECEIVED

Downloaded from SHAREOK. Persistent Handle URL: https://hdl.handle.net/11244/2932

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

# THE ALGEBRAIC PARTITIONING OF FACTORIAL ARRANGEMENTS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

JOHN H. KALBFLEISCH

Oklahoma City, Oklahoma

1970

THE ALGEBRAIC PARTITIONING OF

FACTORIAL ARRANGEMENTS

APPROVED BY

Roy B. Deal, Jr.

Paul S. Anderson, Jr.

John C. Driver

Harold Williams

Kathleen S. Sohle

DISSERTATION COMMITTEE

ACKNOWLEDGMENT

The author wishes to express his sincere appreciation and personal indebtness to Dr. D.L. Weeks, Department of Statistics, Oklahoma State University, for suggesting the dissertation topic and for his frequent suggestions and statistical guidance throughout the investigation.

Appreciation is also expressed to Dr. R.B. Deal, chairman of the dissertation committee, for his patience and mathematical and statistical guidance and to the other members of the dissertation committee, Dr. P. Anderson, Dr. J.C. Brixey, Dr. H. Williams and Dr. K. Sohler, for their suggestions and comments during the writing of the dissertation.

Thanks are also given to the Department of Statistics, Oklahoma State University, for the loan of several copies of department manuscripts.

Finally, the author wishes to express his appreciation to his wife, Nancy Lee, for her encouragement, understanding and moral support, and to his parents for help in many ways, throughout the entire period of graduate study.

TABLE OF CONTENTS


| | | Page |
|---|---|---|
| LIST OF TABLES | | v |
| LIST OF ILLUSTRATIONS | | vii |
| Chapter | | |
| I. | INTRODUCTION | 1 |
| II. | BASIC CONCEPTS AND NOTATION | 9 |
| III. | PLANS USING SOME OR ALL OF THE s-FAT's | 27 |
| | K-dimensional Rectangles | 28 |
| | Connected PLANs | 33 |
| IV. | BLOCKING AND MULTIPLE PARTITIONING | 37 |
| | Blocking Precedures for Equal Partitionings | 39 |
| | Blocking and Unequal Partitionings | 61 |
| | Multiple Partitionings | 71 |
| V. | STATISTICAL INFERENCE | 79 |
| | Brief Results for the General Case | 79 |
| | Analysis of Variance for Full Replicates | 85 |
| | Analysis of Variance for Multiple Full Replicates | 92 |
| | Analysis of Variance in the Presence of Blocks | 105 |
| VI. | DISCUSSION OF AN EXAMPLE | 117 |
| VII. | SUMMARY | 131 |
| LIST OF REFERENCES | | 134 |
| APPENDIXES | | |
| 1. | ELEMENTARY MATRIX CONCEPTS | 139 |
| 2. | A RESULT CONCERNING THE SUM OF SQUARES DUE TO CERTAIN EFFECTS | 140 |

LIST OF TABLES


| Table | | Page |
|---|---|---|
| 1. | Some Blocking PLANs for the Partitioning (9)..... | 43 |
| 2. | Some Blocking PLANs for example 9..... | 45 |
| 3. | Some Blocking PLANs for the partitioning (10)..... | 52 |
| 4. | Some Blocking PLANs for the Case $s_1 = s_2 = 2$ of the Partitioning (11)..... | 54 |
| 5. | Some Blocking PLANs for the Partitioning (12)..... | 56 |
| 6. | Some Blocking PLANs for example 10..... | 57 |
| 7. | Some Blocking PLANs for the Partitioning (13)..... | 60 |
| 8. | Some Blocking PLANs for the Partitioning (15)..... | 63 |
| 9. | Some Blocking PLANs for example 11..... | 65 |
| 10. | Confounding Schemes Used in Example 11..... | 66 |
| 11. | Some Blocking PLANs for the Case $n = 2$ and $s_1 \neq s_2$..... | 69 |
| 12. | Some Blocking PLANs for the 2-order Multiple Partitioning (22)..... | 73 |
| 13. | Some Blocking PLANs for the Partitioning (25)..... | 74 |
| 14. | Some Blocking PLANs for the Unequal Multiple Partitioning... | 76 |
| 15. | Abbreviated Analysis of Variance Table for a Full Replicate of a $P_1 \ldots P_n$-FAT..... | 85 |
| 16. | Abbreviated Analysis of Variance Table for a Full Replicate of a 2x2x2-FAT..... | 88 |
| 17. | Abbreviated Analysis of Variance Table for a Partitioning (28) (for $n = 2$)..... | 92 |
| 18. | Abbreviated Analysis of Variance Table for a Partitioned $P_1P_2$-FAT..... | 93 |
| 19. | Abbreviated Analysis of Variance Table for a Full Replicate of a $P_1P_2P_3$-FAT..... | 94 |
| 20. | Abbreviated Analysis of Varinace table for a Full Replicate of a $P_1 \ldots P_n$-FAT..... | 96 |

LIST OF TABLES (Continued)


| Table | | Page |
|---|---|---|
| 21. | Abbreviated Analysis of Variance Table for r Pooled Full Replicates of a $P_1 \ldots P_n$-FAT | 97 |
| 22. | Abbreviated Analysis of Variance Table for the $P_1P_2$-FAT Run in r Replicates | 100 |
| 23. | Abbreviated Analysis of Variance Table for a Partitioned $P_1 \ldots P_n$-FAT | 103 |
| 24. | Some Blocking PLANs for example 16 | 105 |
| 25. | Abbreviated AOV for Blocking PLAN (a) of Example 16 | 107 |
| 26. | Abbreviated AOV for Blocking PLAN (b) of Example 16 | 107 |
| 27. | Abbreviated AOV for Blocking PLAN ($g_1$) of Example 16 | 108 |
| 28. | Abbreviated AOV for Blocking PLAN ($h_7$) of Example 16 | 109 |
| 29. | Abbreviated AOV Table for Blocking PLAN (a) of Example 17 | 111 |
| 30. | Abbreviated AOV Table for Blocking PLAN (b) of Example 17 | 111 |
| 31. | Abbreviated AOV Table for Blocking PLAN (c) of Example 17 | 112 |
| 32. | Abbreviated AOV Table for Blocking PLAN (d) of Example 17 | 112 |
| 33. | Abbreviated AOV Table for Blocking PLAN (e) of Example 17 | 113 |
| 34. | Abbreviated AOV Table for the Initial Experiment | 121 |
| 35. | Analysis of Variance Table for Two Replicates of the 5x5x7-FAT | 124 |
| 36. | Abbreviated AOV for Two Replicates of a Partitioned $5_1 5_2 7_3$-FAT | 129 |

LIST OF ILLUSTRATIONS


| Figure | | Page |
|---|---|---|
| 1. | The Full Replicate and Four s-FAT's of Example 2..... | 13 |
| 2. | The Matrix Defining the Effects for a $\prod_{i=1}^{n} P_i$-FAT..... | 24 |
| 3. | The Matrix H Corresponding to the Matrix L of Figure 2..... | 25 |
| 4. | PLANs Involving FATs..... | 34 |
| 5. | Illustration of Blocking PLANs (a), (b), (c) and (d) for Example 9..... | 46 |
| 6. | Treatment Combination Correspondences..... | 48 |
| 7. | Methods to Obtain Blocking PLANs Consisting of 16 Blocks for the Partitioned 8x4-FAT of Example 9..... | 50 |
| 8. | Blocking PLANs ($g_1$) and ($h_7$) of Example 9..... | 51 |
| 9. | Illustration of Blocking PLANs (a), (b), (c) and (d) of Example 10..... | 58 |
| 10. | Illustrations of the Subsets of Treatment Combinations for Blocking PLANs (x) and (e)..... | 67 |
| 11. | The Matrix for example 14..... | 86 |
| 12. | The Matrix L for r Replicates of a $P_1P_2$-FAT..... | 99 |
| 13. | Allocation of Treatment Combinations to Blocks..... | 106 |
| 14. | A Matrix L for the PLAN Defined by Subset $S_1$ in Example 18.. | 114 |
| 15. | A Matrix L for the PLAN Defined by Subset $S_2$ in Example 18.. | 115 |
| 16. | A Representation of the 25 Food Doets..... | 118 |
| 17. | A Representation of the Four s-FAT's..... | 119 |
| 18. | Scheme Assigning Treatment Combinations to Blocks. A Rat in a Block is Randomly Assigned all Treatment Combinations in a Row..... | 127 |

# THE ALGEBRAIC PARTITIONING OF FACTORIAL ARRANGEMENTS

## CHAPTER I

## INTRODUCTION

The body of knowledge of medical and health-related phenomena is continually augmented by man's striving for an optimum state of health. Although new knowledge of such phenomena may arise from dreams as well as from scientific facts and logic, it proceeds further when guided by the rational framework of scientific investigation. The planned experiment is a common characteristic of scientific investigations in the health field. The design of the experiment plays a determining role in the success of each endeavor to obtain new knowledge or ascertain the validity of existing knowledge. A crucial aspect of medical and health-related experiments is the statistical design and analysis of the experiment and it is this phase of the scientific method to which the content of this dissertation addresses itself.

A common experimental situation might involve the application of a specified set of treatments to a group of experimental units with the objective of comparing the effects of the treatments on the units. Two main elements of the statistical design of an experiment are the physical design necessitated by the experiment and the treatment design. The

analysis procedure for an experiment is dependent upon the design of the experiment and a set of theoretical assumptions about the experimental units and the experimental process. For example, a clinical investigation might be designed to compare the effect of a new drug with the effect of a standard drug for reducing blood pressure in hypertensive patients. In this case the experimental unit would be the human being, or patient, and the treatment would be a predetermined dose of either the standard or test drug. Another comparative experiment might be designed to investigate several factors which are believed or known to have an effect on the experimental units. An example could occur in a setting similar to the above experiment except that the standard and test drugs, such as a diuretic or tranquilizer, are to be administered at various times of the day, say 9 A.M. and 7 P.M. The experiment now has two factors of interest. Factor one might be labeled medicine and it consists of two levels, where one level refers to the standard drug and the other level refers to the test drug. Factor two might be labeled time of day and it consists of two levels, represented by 9 A.M. and 7 P.M. A patient will randomly receive on of the four treatments,

(1) test drug dose at 9 A.M.,

(2) test drug dose at 7 P.M.,

(3) standard drug dose at 9 A.M. or

(4) standard drug dose at 7 P.M.

For an experiment designed this way it is possible to obtain information relating to differences between the standard drug and the test drug, to differences between the 9 A.M. administration and 7 P.M. administration and information concerning the relationship between the standard drug and

the test drug remain the same for both periods of administration. Thus, the scope of the experiment now includes the investigation of inter-factor and intra-factor relationships. The experimental unit is still the human being, or patient, but the treatment or treatment combination that each patient receives is a combination of levels, one level from each factor.

By the use of the treatment design known as the "factorial arrangement", effects corresponding to inter-factor and intra-factor relationships can be investigated.

Definition 1 : The treatment design of an experiment is said to be factorial if each treatment combination consists of a combination of levels, one level from each factor in the experiment.

Experiments that have a factorial treatment design are sometimes called factorial experiments. The design and analysis of factorial experiments was first described by Fisher (24) in 1926 and Yates (47). Since then most of the standard experimental design textbooks, such as Fisher (25), Cox (17), Davies (20), Cochran and Cox (13), Kempthorne (34) and Winer (46) have detailed accounts of the various statistical aspects of factorial experiments.

Definition 2 : If all factors in a factorial arrangement of treatments have the same number of levels, then it is referred to as being a symmetrical factorial arrangement of treatments, otherwise, if two or more factors have a different number of levels, then it is referred to as an asymmetrical or mixed factorial arrangement of treatments.

If one can apply all possible combinations of factor levels to the experimental material, the experiment is said to have a full replicate

of factorially arranged treatments. In the examples mentioned earlier, the treatment design which involved two drugs at two time periods is an example of a factorial arrangement while the treatment design of the example involving only two drugs is not factorial.

If, in the designing of an experiment, the situation arises where each experimental unit can receive only one treatment combination, then the problem may arise that a full replicate will require too many experimental units (where the number of units is restricted by size, obtainability or some other environmental or economic characteristic of the unit). For example, consider example 12.1 in Cox (17) where eleven essential amino acids are incorporated in a chemical medium in which the rate of growth of embryonic chick bones is measured. In this example each of the eleven amino acids is considered as a factor and each factor has two levels, those levels being the presence or absence of the amino acid. Consequently, a full replicate of the factorially arranged treatments would consist of $2^{11} = 2,048$ treatments, which, as is mentioned in example 12.1, is "quite impractible." One way to reduce the size of the experiment is to reduce the number of factors or the number of levels of some or all of the factors. However, this is not always possible. Another way to reduce the size of an experiment with a factorial arrangement of treatments is to consider only a subset of a full replicate of treatment combinations. The general idea is to obtain a subset of the treatment combinations that will yield a maximum amount of information about the effects of treatments that are considered important. When a subset of a full replicate of factorially arranged treatments is used, that subset of treatment combinations is usually referred to as a fract-

ional replicate (indicating that it is a fraction of the full treatment replicate).

Another common situation is where an experiment cannot be performed at one time, although it is possible to perform the entire experiment in parts, where each part might be performed at a different time or location. In this case a method is needed to separate the full treatment replicate into disjoint subsets so that one or more of the subsets can be chosen to represent each part of the experiment. So, if one is in a fractional replicate situation or a situation where the full treatment replicate is to be performed in parts, one must have a method to separate the full replicate into disjoint subsets. Present methods for obtaining disjoint subsets of the factorially arranged treatment combinations rely on the fact that certain comparisons among the treatments (most often the high order interactions) are of relatively little importance. Then one makes use of the well developed statistical theory (related to confounding schemes) to separate the full replicate of treatment combinations into subsets in such a manner that comparisons among the subsets are also comparisons among the treatments that are of little interest. The basis for the method of obtaining fractions of factorial arrangements was first introduced in 1945 by Finney (23) and an elementary account of confounding schemes for factorial experiments was described by Kempthorne (33) in 1947. Since then, descriptions of these methods and extensions of the methods are found in most experimental design textbooks.

There is also literature concerning fractional replicates of experiments with factorially arranged treatments that is not documented in the standard textbooks. A comprehensive account of fractional

replicate plans for the case where all the factors have two levels was published by the National Bureau of Standards (36) and Connor and Zelen (16) published fractional replicate plans for the case where all factors have three levels. Connor (14), Bose and Connor (8) and Connor and Young (15) published plans for the case where each factor has either two or three levels. Fractional replication was handled in general by Chakravarti (12) and Morrison (35). Recently, Daniel (19), Bose and Srivastava (9), Box and Hunter (10,11), John (30), Addelman (1), Banerjee (3), Dykstra (22), Banerjee and Federer (5) and Westlake (43) have published plans and methods concerning irregular fractions of factorial arrangements. Addelman (2) summarized many of the techniques for obtaining fractions of symmetrical and asymmetrical factorial experiments with orthogonal and non-orthogonal plans. Various properties of estimation procedures for factorial experiments were considered by Banerjee and Federer (4,6), Zacks (48,49), Addelman (2) and Shah (41). Sequential estimation problems in factorial experiments were considered by Huster (31) and sequential procedures are discussed for fractional replicates in the $2^{p-q}$ case by Daniel (19). Prairie and Zimmer (38,39) discussed plans and methods for the sequential treatment of factorial arrangements when the factors are applied sequentially. Confounding schemes for assigning a full replicate of factorially arranged treatment combinations to a set of blocks are discussed throughly in textbooks, such as Kempthorne (34), for $P^n$-factorial experiments, where P is a prime or prime power number. Confounding schemes for symmetrical factorial arrangements are mentioned in Addelman (2), Kempthorne (34,33), White and Hultquist (44) and Raktoe (40).

Most of the methods to generate fractional replicate plans and

confounding plans are indicated in the references in the preceding paragraph, however, these methods are not always satisfactory. For example, one may not want to sacrifice interaction information to arrive at a fraction of the treatment combinations. It might be that the investigator can place an interest priority in certain subsets of the levels of some or all of the factors in the experiment. If this is the case, the usual design procedures are not particularly adaptable to the investigation of intra-factor and inter-factor relationships and at the same time retain the priority desires. One might also be confronted with the situation of having the experimental units grouped in blocks of unequal size, which is not a very desirable situation since the usual confounding procedures generally require equal block sizes. Thus, there exists a need for other methods that will allow the partitioning of a full replicate of factorially arranged treatments into disjoint subsets so that some of these subsets can be run in the sense of a fractional replicate, or so that the entire experiment can be performed by assigning the subsets to blocks of experimental units.

Consider an experiment with factorially arranged treatments that is designed to investigate n factors, where each factor has P levels of interest. A full replicate of this experiment is referred to as a $P^n$-factorial arrangement of treatments. There are $P^n$ distinct treatment combinations in a full replicate of this experiment. Algebraically, one can express $P^n$ as

$$P^n = (P_1 + P_2)^n, \quad \text{where } P = P_1 + P_2 . \tag{1}$$

This expression gives a method to partition the full treatment replicate into subsets by consideration of the $2^n$ terms that appear on the right-

side of the equation (1). Each term in the algebraic expansion will define a subset of the treatment combinations. The subsets defined in this manner are disjoint and the union of all subsets will give us the set of all $P^n$ treatment combinations. Examples to illustrate this concept are given in the next chapter.

Now, consider the more general asymmetrical case where there are n factors and the i-th factor has $P_i$ levels, for i = 1, ...,n. Let $P_i \neq P_j$ for at least one i and j such that $i \neq j$. The set of $\prod_{i=1}^{n} P_i$ distinct treatment combinations can be partitioned into $\prod_{i=1}^{n} s_i$ subsets by the equation

$$\prod_{i=1}^{n} P_i = \prod_{i=1}^{n} (P_{i1} + \cdots + P_{is_i}), \quad \sum_{j=1}^{s_i} P_{ij} = P_i \cdot \qquad (2)$$

The first mention of this concept in the literature was made by Morrison (35). In 1961 Fry (27) used this method for the $3^2 = (2 + 1)^2$ factorial arrangement of treatments. In the unpublished doctoral theses of Williams (45) in 1963 and Thomas (42) in 1964, the cases $P^n = (P_1+P_2)^n$ and $P^n = (P_1 + \cdots + P_k)^n$ respectively, were considered in detail. This thesis will investigate the algebraic partitioning of experiments with symmetrical and asymmetrical factorial treatment arrangements. The following chapters will discuss notation schemes, methods for obtaining and combining subsets of treatment combinations, estimates of effects among the treatments, sequential methods for applying the subsets, analysis of variance methods for partitioned factorial treatment arrangements for the completely random and randomized block designs along with examples to illustrate relevant points and concepts.

## CHAPTER II

## BASIC CONCEPTS AND NOTATION

In this chapter the notation and basic concepts concerning the algebraic partitioning of a factorial arrangement of treatments is developed. Let the factorial arrangement of treatments consisting of $n$ factors, where the first factor has $P_1$ levels, ..., and the $n$-th factor has $P_n$ levels, be denoted by $(P_1 \cdots P_n)$-FAT or by $\prod_{i=1}^{n} P_i$-FAT. As mentioned in chapter I, if $P_1 = \cdots = P_n$ the $\prod_{i=1}^{n} P_i$-FAT is a symmetric factorial arrangement and if $P_i \neq P_j$ for some $i \neq j$, the $\prod_{i=1}^{n} P_i$-FAT is referred to as a asymmetrical or mixed factorial arrangement of treatments.

The $\prod_{i=1}^{n} P_i$-FAT is a collection of $\prod_{i=1}^{n} P_i$ different treatment combinations that represent the $n$ factors. The number $\prod_{i=1}^{n} P_i$ may be written

$$\prod_{i=1}^{n} P_i = \prod_{i=1}^{n} (P_{i1} + \cdots + P_{is_i}), \qquad \sum_{j=1}^{s_i} P_{ij} = P_i \tag{3}$$

The expression (3) can be used to define an algebraic partition on the set of $\prod_{i=1}^{n} P_i$ factorially arranged treatment combinations. The subsets of treatment combinations resulting from such a algebraic partitioning are denoted by the abbreviation "s-FAT." The algebraic partitioning of a $\prod_{i=1}^{n} P_i$-FAT is denoted by the expression

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1} + \cdots + P_{is_i})\text{-s-FAT's} . \tag{4}$$

The partitioning given by (4) indicates that the $P_i$ levels of the i-th factor, for $i = 1, \ldots, n$, are separated into $s_i$ groups or subsets, where the subsets are disjoint and are of size $P_{i1}, \ldots, P_{is_i}$. Consider the i-th factor in the partitioning (4). The actual assignment of the $P_i$ levels into the $s_i$ disjoint subsets is somewhat arbritrary and is mainly the choice of the investigator. For example, suppose the i-th factor in a medical experiment represented a certain amount of radiation the unit is exposed to, where there are seven levels of radiation exposure. The seven levels of radiation exposure can be grouped into two subsets of three and four levels the following ways. Let the seven levels of radiation exposure be represented by 0,1,2,3,4,5 and 6, where the higher numbers represent larger amounts of exposure. If the investigator knew very little about the effects of the different doses of radiation, then he might select the 0,2,4 and 6 levels for the levels in the subset of size four and the 1,3 and 5 levels for the other subset. It might also be the situation where the investigator knows that the very low dose levels will have slight effect and he is more concerned about the effects of the high dose levels. In this case the investigator might choose to group the four highest dose levels, 3,4,5 and 6, in one subset and group the three lowest dose levels, 0,1 and 2, in the other subset. Statistically, the important concept is that, say for the i-th factor in 2, there are $s_i$ disjoint subsets, where the first subset consists of $P_{i1}$ levels, ..., and the $s_i$-th subset consists of the remaining $P_{is_i}$ levels.

Definition 3: Denote the n factors of the partitioning (4) by $A_1, \ldots, A_n$. (Note: these letters will also be used to identify

sources of variation in an analysis of variance table).

Definition 4 : Consider the i-th factor, $A_i$ in a $\prod_{i=1}^{n} P_i$-FAT. It has $P_i$ levels and this set of levels is denoted by the symbol $T_i$ and $T_i = \{0,1, \ldots, P_i-1\}$ .

Definition 5 : For the $\prod_{i=1}^{n} P_i$-FAT define the set of design points corresponding to a full treatment combination replicate to be the set D, where $D = \{ (x_1, \ldots, x_n) : x_i \in T_i \text{ for } i = 1, \ldots, n \}$.

Example 1 : For the 4x5-FAT there are two factors, $A_1$ and $A_2$, where $A_1$ has 4 levels denoted by the elements of the set $T_1$, and $A_2$ has 5 levels denoted by the elements of $T_2$. $T_1$, $T_2$ and the set of design points, D, are given by

$$T_1 = \{0,1,2,3\}$$

$$T_2 = \{0,1,2,3,4\}$$

$$D = \{ (0,0),(0,1),(0,2),(0,3),(0,4),(1,0),(1,1),$$
$$(1,2),(1,3),(1,4),(2,0),(2,1),(2,2),(2,3),$$
$$(2,4),(3,1),(3,0),(3,2),(3,3),(3,4) \}.$$

For a partitioning given by (4) the i-th factor level set, $T_i$, is separated into $s_i$ subsets. Each of these subsets will be referred to as a pseudo-level, or more briefly, p-level. Thus, the i-th factor will have $s_i$ p-levels, where the first p-level represents a subset of size $P_{i1}$ of the original $P_i$ levels, ... , and the $s_i$-th subset represents a subset of size $P_{is_i}$ of the original $P_i$ levels.

Definition 6 : For the $s_i$ p-levels of the i-th factor in the partitioning (4) define the $s_i$ subsets $T_{i1}$, ..., and $T_{is_i}$ to be the sets of levels corresponding to each p-level.

Thus, $T_i = T_{i1} \cup T_{i2} \cup \ldots \cup T_{is_i}$ and $T_{ik} \cap T_{ik'} = \emptyset$ if $k \neq k'$, or else $= T_{ik}$.

The partitioning (4) separates the full replicate of $\prod_{i=1}^{n} P_i$ treatment combinations represented by the set of design points D, into $\prod_{i=1}^{n} s_i$ subsets (see remark 1).

Definition 7 : Consider the $\prod_{i=1}^{n} P_i$-FAT and the associated set of design points, D. Given the algebraic partitioning (4) and by considering only the p-levels for each factor, define the set of pseudo-design points, $S_D$, to be the set of n-tuples

$$S_D = \{(y_1, \ldots, y_n): y_i \in \{0,1, \ldots, s_i-1\} \text{ and for all } i = 1, \ldots, n\} .$$

Each element in $S_D$ represents an s-FAT and is described by the n-tuple $(y_1, \ldots, y_n)$, where $y_i$ indicates which of the $s_i$ p-levels of the i-th factor is being used to construct the particular s-FAT, for i = 1, ...,n.

Example 2 : Corresponding to example 1, consider the partitioning $4x5\text{-FAT} \longrightarrow (2 + 2)(2 + 3)\text{-s-FAT's}$ , or more explicitly, $4_1 5_2\text{-FAT} \longrightarrow (2_{11}+2_{12})(2_{21}+3_{22})\text{-s-FAT's}$. The four s-FAT's that result from this partitioning are obtained from the algebraic expansion of the right hand side of the partition expression, namely

$$(2_{11}+2_{12})(2_{21}+3_{22}) = 2_{11}2_{21}+2_{11}3_{22} +2_{12}2_{22} +2_{12}3_{22} .$$

Now, $S_D = \{(0,0),(0,1),(1,0),(1,1)\}$ and each element in $S_D$ indicates a s-FAT by the following correspondence scheme: for each $(y_1,y_2) \in S_D$ let $y_1 = 0$ refer to the p-level indicating $2_{11}$, $y_1 = 1$ refer to the p-level indicating $2_{12}$, $y_2 = 0$ refer to the p-level indicating $2_{21}$ and $y_2 = 1$ refer to the p-level indicating $3_{22}$. Thus, $(0,0) \in S_D$ indicates the $2_{11}2_{21}$ s-FAT, $(0,1) \in S_D$ indicates the $2_{11}3_{22}$ s-FAT, $(1,0) \in S_D$ indicates the $2_{12}2_{21}$ s-FAT and $(1,1) \in S_D$ indicates the $2_{12}3_{22}$ s-FAT.

Given a partitioning scheme for a $\prod_{i=1}^{n} P_i$-FAT and an element $(y_1, \ldots, y_n)$

in $S_D$, the set of design points of the corresponding s-FAT can be found by the cartesian product, $T_{1(y_1+1)} \times \cdots \times T_{n(y_n+1)}$, where $T_{jk} \subseteq T_j$ for $j = 1, \ldots, n$ and for $k = 1, \ldots, s_j$.

Example 2 (continued) : If $T_{11} = \{0,1\}$, $T_{12} = \{2,3\}$, $T_{21} = \{0,1\}$ and $T_{22} = \{2,3,4\}$ then

$(0,0) \in S_D$ is equivalent to $T_{11} \times T_{21}$ and the $2_{11}2_{21}$ s-FAT,

$(0,1) \in S_D$ is equivalent to $T_{11} \times T_{22}$ and the $2_{11}3_{22}$ s-FAT,

$(1,0) \in S_D$ is equivalent to $T_{12} \times T_{21}$ and the $2_{12}2_{21}$ s-FAT and

$(1,1) \in S_D$ is equivalent to $T_{12} \times T_{22}$ and the $2_{12}3_{22}$ s-FAT.

If a 4 by 5 square is used to represent the 20 treatment combinations of the 4x5-FAT, then the 4 s-FAT's are indicated in Figure 1. where rows represent levels of $A_2$ and columns represent rows of $A_1$.

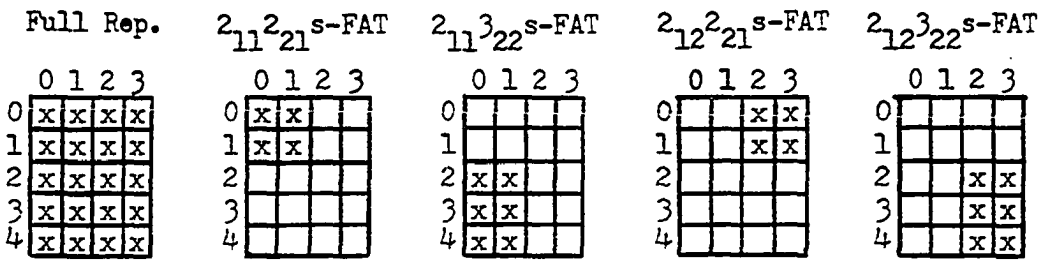


Figure 1. - The full replicate and four s-FAT's of example 2.

Remark 1 : Consider the partitioning

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1} + \ldots + P_{is_i})\text{-s-FAT's.}$$

If a $\prod_{i=1}^{n} P_i$-FAT is partitioned this way, then a total of $\prod_{i=1}^{n} s_i$ s-FAT's is obtained.

Proof: Consider the set of pseudo-design points, $S_D$.

$$S_D = \{(y_1, \ldots, y_n) : y_i \in \{0,1,\ldots,s_i-1\}, \text{ for } i = i,\ldots,n\}.$$

Since each element in $S_D$ describes exactly one s-FAT and no two elements in $S_D$ describe the same s-FAT, the number

of s-FAT's is equivalently the size or number of elements in $S_D$. Clearly, the size of $S_D$ is $\prod_{i=1}^{n} s_i$.

For a $\prod_{i=1}^{n} P_i$-FAT the set of design points D, is given by $D = T_1 \times \ldots \times T_n$, where $T_i$ represents the set of levels for the i-th factor. At times it may be desirable to express the set of levels of each factor in vector form.

Definition 8 : The vector of levels, $\underline{\theta}_i$, for the i-th factor, for $i = 1, \ldots, n$, in a $\prod_{i=1}^{n} P_i$-FAT is the $P_i$ by one vector whose (k,1) entry is k-1, for $k = 1, \ldots, P_i$. The components of $\underline{\theta}_i$ are elements of $T_i$.

Definition 9 : Let A be an n by m matrix and B be a p by q matrix. Define the matrix component composition, abbreviated "MCC," of A and B to be the np by mq matrix A*B, where

$$A*B = \begin{bmatrix} (a_{11},B) & \cdots & (a_{1m},B) \\ \vdots & & \vdots \\ (a_{n1},B) & \cdots & (a_{nm},B) \end{bmatrix} \quad \text{and}$$

$$(a_{ij},B) = \begin{bmatrix} (a_{ij},b_{11}) & \cdots & (a_{ij},b_{1q}) \\ \vdots & & \vdots \\ (a_{ij},b_{p1}) & \cdots & (a_{ij},b_{pq}) \end{bmatrix}.$$

Definition 10: If A is n by 1 and B is p by 1 (A and B are vectors) then $\underline{A}*\underline{B}$ shall be called the vector component composition, VCC, of $\underline{A}$ and $\underline{B}$ rather than the MCC of A and B.

Example 3 : For the $3x4$-FAT we have $\underline{\theta}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\underline{\theta}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$.

Now, $(\underline{\theta}_1 * \underline{\theta}_2)' = [(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),$
$(2,1),(2,2),(2,3)]'$

and it represents the full replicate of treatment combinations.

In general, for a $\prod_{i=1}^{n} P_i$-FAT, $\underline{\theta}_1 * \ldots * \underline{\theta}_n$ is the $\prod_{i=1}^{n} P_i$ by 1 vector whose

components comprise a full replicate of **factorially arranged treatment** combinations.

Definition 11: The j-th factor, $A_j$, for $j = 1, \ldots, n$, in the algebraic partitioning $\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1}+\ldots+P_{is_i})\text{-s-FAT's}$, has level set $T_j$ which is separated into sub-level sets $T_{j1},\ldots,T_{js_j}$ by the partitioning. In a similar manner, define $\underline{\theta}_{jk}$ to be the vector of levels where the levels are elements of $T_{jk}$. If $T_{jk}$ has $P_{jk}$ elements, then $\underline{\theta}_{jk}$ is a $P_{jk}$ by 1 vector.

Definition 12 : Let $\underline{A}$ be an n by one vector and $\underline{B}$ be an n by one vector. By the Hadamard product (see Halmos (30)), abbreviated HP, of $\underline{A}$ and $\underline{B}$, and denoted by the symbol $\underline{A}\circledast\underline{B}$, we mean the n by one vector

$$\underline{A}\circledast\underline{B} = (a_1b_1, \ldots, a_nb_n)'$$

Remark 2 : If $\underline{A}$, $\underline{B}$ and $\underline{C}$ are n by 1 vectors, then

(i) $\underline{A}\circledast\underline{B} = \underline{B}\circledast\underline{A}$

(ii) $(\underline{A} + \underline{B})\circledast\underline{C} = (\underline{A}\circledast\underline{C}) + (\underline{B}\circledast\underline{C})$

(iii) $\underline{A}'(\underline{B}\circledast\underline{C}) = (\underline{A}\circledast\underline{B})'\underline{C} = (\underline{A}\circledast\underline{C})'\underline{B} = \underline{B}'(\underline{A}\circledast\underline{C}) = \underline{C}'(\underline{A}\circledast\underline{B})$

(iv) $\underline{A}'\underline{B} = J_n^1(\underline{A}\circledast\underline{B})$ , where $J_n^1$ is a 1 by n row vector of ones.

## Models

In order to talk about models for an experimental situation, the following quantities are given the appropriate meanings. **Capital** letters, except for those previously defined, will denote a matrix and **capital** letters underscored with a bar will denote column vectors. Certain vectors and matrices occur quite frequently and for this reason the following vectors and matrices are given special meaning. Let $\underline{\emptyset}_k$ be the k by one vector of zeros (each element is zero), let $\emptyset_b^a$ be the a by b matrix of zeros, let $I_n$ be the n by n identity matrix (diagonal elements are ones

and off-diagonal elements are zero) and $J_b^a$ be the a by b matrix where each entry is one. Furthermore, let $\underline{M}$ be a m by 1 vector of unknown constants and $\underline{e}$ an m by one vector of random error terms. Some of the usual assumptions concerning the distribution of $\underline{e}$ are that $E(\underline{e}) = \underline{\phi}_m$ and $E(\underline{e}\ \underline{e}') = \sigma_e^2 I_m$ ($\sigma_e^2$ is an unknown constant).

The initial process of describing an experimental situation in terms of a model involves the specification or defining of a set of (unknown) parameters that can be used to describe the basic experimental phenomena. This set of parameters will be components of the vector $\underline{\beta}$. Once the elements of $\underline{\beta}$ are specified, the next step in constructing a model is to assume the existence of a vector $\underline{M}$ that is some function of the vector of parameters, $\underline{\beta}$.

Definition 13 : If there exists a function f such that $f:\underline{\beta} \longrightarrow \underline{M}$, then the function f is said to define the population model $f(\underline{\beta}) = \underline{M}$. The population model should describe the basic or fundamental phenomenon that is under investigation in the experiment.

Definition 14 : If $f:\underline{\beta} \longrightarrow X\underline{\beta}$, where X is a known matrix of constants, the model $\underline{M} = X\underline{\beta} = f(\underline{\beta})$ is called a linear population model.

To further describe the model of an experimental situation, the vector of observations (or numerical results from the experiment), $\underline{Y}$, must be related to the population model.

Definition 16 : If there exists a function h such that $h:(\underline{M},\underline{e}) \longrightarrow \underline{Y}$, then h is said to define the observational model $\underline{Y} = h(\underline{M},\underline{e})$.

Definition 15 : If $\underline{Y} = h(\underline{M},\underline{e}) = \underline{M} + \underline{e}$, and $\underline{M} = X\underline{\beta}$, then $\underline{Y} = X\underline{\beta} + \underline{e}$ is said to be a linear observational model or simply linear model.

When defining a linear model $\underline{Y} = X\underline{\beta} + \underline{e}$ the matrix X, the vector $\underline{\beta}$ and

the distributional properties of $\underline{e}$ (and the joint distribution of $\underline{\beta}$ and $\underline{e}$ and the distribution of $\underline{\beta}$ if appropriate) must be specified.

Remark 3: Let $\underline{\beta}$ be a p by one vector of parameters and let $X^+$ be the generalized inverse of the matrix X in the linear model $\underline{Y} = X\underline{\beta} + \underline{e}$. If $\underline{M} = X\underline{\beta}$ is consistent then, (1) $XX^+\underline{M} = \underline{M}$ and

(2) $\underline{\beta} = X^+\underline{M} + (I_p - X^+X)\underline{\alpha}$, where $\underline{\alpha}$ is an arbritrary p by one vector.

Proof: The proof follows from theorems 6 and 7, appendix I, by letting $C = \underline{M}$, $A = X$, and $X = \underline{\beta}$. (See also Gateley (28)).

In the linear model $\underline{Y} = X\underline{\beta} + \underline{e}$ let $\underline{\beta}$ be a p by one vector of parameters, $\underline{Y}$ be an m by one vector of observations, $\underline{e}$ be an m by one vector of random error terms and X an n by p matrix of known constants. The distributional properties of $\underline{e}$ will be stated later. By remark 3, $\underline{\beta} = X^+\underline{M} + (I_p - X^+X)\underline{\alpha}$, for arbritrary $\underline{\alpha}$. Obviously $\underline{\beta}$ is not unique since it is a function of $\underline{\alpha}$, which can be arbritrarly chosen, unless the rank of the matrix X is p, and then by theorem 8 in appendix I, $X^+X = I_p$ and $\underline{\beta}$ is unique.

Definition 17 : The vector of parameters, $\underline{\beta}$, in the linear model $\underline{Y} = X\underline{\beta} + \underline{e}$, where $\underline{\beta}$ is p by one, $\underline{Y}$ and $\underline{e}$ are m by one and X is m by p, is said to be intrinsically defined if and only if the rank of X is p or if and only if $(I_p - X^+X) = \emptyset^p_p$. (See Gateley (28)).

In the linear model $\underline{Y} = X\underline{\beta} + \underline{e}$ suppose X is m by p and the rank of X is $q \leq p$.

Definition 18 : If $q = p$, then the model $\underline{Y} = X\underline{\beta} + \underline{e}$ is said to be a full rank linear model.

Definition 19 : If $q < p$, then the model $\underline{Y} = X\underline{\beta} + \underline{e}$ is said to be

less than full rank.

The observational model will always be a linear model, $\underline{Y} = \underline{M} + \underline{e}$, where $\underline{e}$ is a vector of random error terms. Since the only treatment design under consideration is the factorial arrangement, say a $\prod_{i=1}^{n} P_i$-FAT, the elements of $\underline{M}$ shall be called population means or cell means, indicating that they represent population means for the $m = \prod_{i=1}^{n} P_i$ treatment combinations. The vector of population means, $\underline{M}$, is intrinsically defined in the full rank model $\underline{Y} = \underline{M} + \underline{e}$ or $\underline{Y} = I_m\underline{M} + \underline{e}$ (see definition 17 and let $\underline{\beta} = \underline{M}$, $X = I_m$ and $p = m$). The vector $\underline{M}$ is estimable (see Graybill (29)), and therefore any linear function of the elements of $\underline{M}$ is estimable. Consequently, in the sequel the effects or comparisons of interest shall be defined as linear functions of the elements of $\underline{M}$ rather than as linear functions of the elements of $\underline{\beta}$ (if $\underline{M} = X\underline{\beta}$), thus avoiding some problems of estimability that occur in less than full rank design models, such as $\underline{Y} = X\underline{\beta} + \underline{e}$, where X is a design matrix.

## Effects

Attention is now focused on certain linear functions of the elements of $\underline{M}$ that are useful in the analysis of observations.

Definition 20 : An effect of the population model $\underline{M} = X\underline{\beta}$ is a linear combination of the elements of $\underline{M}$. Effects will be denoted by vector products $\underline{\lambda}'\underline{M}$, where $\underline{\lambda}$ is a m by 1 vector that is said to define the effect.

Definition 21 : Two effects $\underline{\lambda}_1'\underline{M}$ and $\underline{\lambda}_2'\underline{M}$ are orthogonal if $\underline{\lambda}_1$ and $\underline{\lambda}_2$ are orthogonal ($\underline{\lambda}_1'\underline{\lambda}_2 = 0$).

Definition 22 : A set of vectors is said to be orthogonal if every pair of distinct vectors in the set is an orthogonal pair. Two sets

of vectors are said to be orthogonal if every pair of vectors, taking one vector from each set, is an orthogonal pair.

Definition 23 : The overall mean effect of a $\prod_{i=1}^{n} P_i$-FAT is given by the effect $J\frac{1}{m}\underline{M}$, where $m = \prod_{i=1}^{n} P_i$.

Associated with the j-th factor in a $\prod_{i=1}^{n} P_i$-FAT are $P_j$ levels and $P_j$ level totals. The $k_j$-th level total of factor j, for $k_j = 1, \ldots, P_j$, is the sum of all elements in $\underline{M}$ that are designated by $k_j$-1 in the j-th position of the subscript. Thus, each level total is a sum of specified elements in $\underline{M}$. To define an effect on the level totals will be equivalent to defining an effect, $\underline{\lambda}'\underline{M}$, on the elements of $\underline{M}$ so that all elements composing a particular level total are assigned the same number in the appropriate positions of the vector $\underline{\lambda}$. Consequently, an effect defined on the level totals corresponds to an effect, $\underline{\lambda}'\underline{M}$, defined on $\underline{M}$.

Definition 24 : A main effect of the j-th factor in a $\prod_{i=1}^{n} P_i$-FAT is a set of $P_j$-1 orthogonal effects defined on the $P_j$ level totals (and therefore on the elements of $\underline{M}$) and such that each of these effects is orthogonal to the overall mean effect. The $P_j$-1 effects shall be referred to as components of the main effect.

For identification purposes in analysis of variance tables, let the symbol $A_j$ designate the source of variation due to the main effect of the j-th factor.

Definition 25 : A simple effect for the j-th factor in a $\prod_{i=1}^{n} P_i$-FAT is an effect orthogonal to the overall mean effect and defined on only two elements of $\underline{M}$ such that the subscripts of those two elements differ in only the j-th position. Thus, a vector $\underline{\lambda}$ that defines a simple effect $\underline{\lambda}'\underline{M}$ will contain zeros in all positions but two, and

those two positions will contain the numbers $+\theta$ and $-\theta$ (usually $\theta=1$). By the above definitions it is easy to show that some combination of simple effects will result in components of a main effect. Consider two simple effects

$$\underline{\lambda}_1'\underline{M} = m_{i_1,\ldots,i_{j_1},\ldots,i_{k_1},\ldots,i_n} - m_{i_1,\ldots,i_{j_2},\ldots,i_{k_1},\ldots,i_n} \quad \text{and}$$

$$\underline{\lambda}_2'\underline{M} = m_{i_1,\ldots,i_{j_1},\ldots,i_{k_2},\ldots,i_n} - m_{i_1,\ldots,i_{j_2},\ldots,i_{k_2},\ldots,i_n}$$

among the $j_1+1$ and $j_2+1$ levels of the j-th factor where one simple effect is at the $(k_1+1)$-st level of factor k and the other is at the $(k_2+1)$-st level of factor k.

Definition 26 : Given two simple effects $\underline{\lambda}_1'\underline{M}$ and $\underline{\lambda}_2'\underline{M}$, the effect that represents the difference between these two simple effects, $(\underline{\lambda}_1 - \underline{\lambda}_2)'\underline{M}$, is called the simple interaction effect among levels $j_1+1$ and $j_2+1$ of factor j and levels $k_1+1$ and $k_2+1$ of factor k.

Let the orthogonal sets of vectors $\{\underline{\lambda}(i)_1,\ldots,\underline{\lambda}(i)_{P_i-1}\}$ and $\{\underline{\lambda}(j)_1,\ldots,\underline{\lambda}(j)_{P_j-1}\}$ define main effects for factors i and j of a $\prod_{i=1}^{n} P_i$-FAT. These two orthogonal sets of vectors can be utilized to construct a third orthogonal set of $(P_i-1)(P_j-1)$ vectors by construction of all vectors of the form $\underline{\lambda}(ij)_{h_ih_j} = \underline{\lambda}(i)_{h_i} \otimes \underline{\lambda}(j)_{h_j}$ where $h_k \in \{1,\ldots,P_k-1 \text{ for } k = i,j\}$.

Definition 27 : The two factor interaction effect between factor i and factor j of a $\prod_{i=1}^{n} P_i$-FAT, given the main effects for factors i and j, is the orthogonal set of $(P_j-1)(P_i-1)$ effects

$$\left\{\underline{\lambda}'(ij)_{h_ih_j}\underline{M} : \underline{\lambda}(ij)_{h_ih_j} = \underline{\lambda}(i)_{h_i} \otimes \underline{\lambda}(j)_{h_j} \text{ for } h_i \in \{1,\ldots,P_i-1\} \text{ and } h_j \in \{1,\ldots,P_j-1\}\right\}.$$

For identification purposes in analysis of variance tables the symbol $A_i x A_j$ will designate the source of variation due to the two factor

interaction effect between factor i and factor j. The effect given by $\underline{\lambda}'(ij)_{h_i h_j}\underline{M}$ will be referred to as a component of the two factor interaction effect.

Definition 28 : Factor (main or interaction) effects of a $\prod_{i=1}^{n} P_i$-FAT are orthogonal if the orthogonal sets of vectors that represent them are orthogonal.

Remark 4 : Factor main effects for a $\prod_{i=1}^{n} P_i$-FAT are orthogonal.

Proof: Without loss of generality, the factor one and factor two main effects will be shown to be orthogonal. Let $\underline{\lambda}'(1)\underline{M}$ and $\underline{\lambda}'(2)\underline{M}$ be two arbritrary components of the factor one and factor two main effects, respectively, where

$$\underline{\lambda}(1) = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_1 \\ \vdots \\ \alpha_{P_1-1} \\ \vdots \\ \alpha_{P_1-1} \end{bmatrix}, \quad \underline{\lambda}(2) = \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_{P_2-1} \\ \beta_0 \\ \vdots \\ \beta_{P_2-1} \\ \vdots \\ \beta_0 \\ \vdots \\ \beta_{P_2-1} \end{bmatrix}, \quad \underline{M} = \begin{bmatrix} m(0, \ldots, 0) \\ \vdots \\ m(0, \ldots, P_n-1) \\ m(1, \ldots, 0) \\ \vdots \\ m(1, \ldots, P_n-1) \\ \vdots \\ m(P_1-1, \ldots, 0) \\ \vdots \\ m(P_1-1, \ldots, P_n-1) \end{bmatrix}$$

are the vectors. Since $\underline{\lambda}'(1)\underline{M}$ and $\underline{\lambda}'(2)\underline{M}$ are components of the main effects, $\underline{\lambda}'(1)J_1^m = \underline{\lambda}'(2)J_1^m = 0$ and, stated in other terms, $\sum_{i=1}^{P_1-1} \alpha_i = \sum_{j=0}^{P_2-1} \beta_j = 0$. Now,

$$\underline{\lambda}'(1)\underline{\lambda}(2) = (\prod_{k=3}^{n} P_k)(\sum_{i=0}^{P_1-1} \sum_{j=0}^{P_2-1} \alpha_i\beta_j) = \prod_{k=3}^{n} P_k(\sum_{i=0}^{P_1-1} \alpha_i)(\sum_{j=0}^{P_2-1} \beta_j)$$

$$= 0,$$

and the components $\underline{\lambda}'(1)\underline{M}$ and $\underline{\lambda}'(2)\underline{M}$ are orthogonal. This is obviously true for all choices of $\underline{\lambda}'(1)$ and $\underline{\lambda}'(2)$ in the factor one and factor two main effects, respectively. Thus, the factor one and factor two main effects are orthogonal and it follows that any two distinct factor main effects are orthogonal.

Remark 5 : The two factor interaction effect between factor i and factor j of a $\prod_{i=1}^{n} P_i$-FAT is orthogonal to the main effect of factor i and is orthogonal to the main effect of factor j.

Proof: It suffices to show orthogonality of the interaction effect and either main effect. Let $\underline{\lambda}'(ij)\underline{M} = (\underline{\lambda}(i) \otimes \underline{\lambda}(j))'\underline{M}$ be a component of the factor 1 - factor 2 interaction effect and let $\underline{\lambda}'(i')\underline{M}$ be a component of the factor one main effect. Designate components of $\underline{\lambda}_i$ by $\alpha$'s, $\underline{\lambda}_{i'}$ by $\delta$'s and elements of $\underline{\lambda}_j$ by $\beta$'s. It follows that ($k \neq i$ and $k \neq j$)

$$\underline{\lambda}'(i')\underline{\lambda}(ij) = \underline{\lambda}'(i')(\underline{\lambda}_i \otimes \underline{\lambda}_j)$$

$$= (\prod_{k=1}^{n} P_k)(\sum_{i=0}^{P_i-1} \sum_{j=0}^{P_j-1} \alpha_i \delta_{i'} \beta_j)$$

$$= (\prod_{k=1}^{n} P_k)(\sum_{i=0}^{P_i-1} \alpha_i \delta_{i'})(\sum_{j=0}^{P_j-1} \beta_j) = 0.$$

Thus, the components are orthogonal and the factor i main effect is orthogonal to the factor i - factor j interaction effect. Similarly, the factor j main effect is orthogonal to the factor i - factor j interaction effect.

By remark 5 all two factor interaction effects between factors i and j are orthogonal to the factor i main effect and factor j main effect, for all $i,j = 1, \ldots, n$. It can also be shown that an interaction effect

between factors i and j ($j \neq i$) is orthogonal to each factor k main effect, for k = 1, ...,n.

Definition 29 : For k = 2, ...,n, the k-factor interaction effect between factors $i_1$, ... and $i_k$ in a $\prod_{i=1}^{n} P_i$-FAT is the orthogonal set of $\prod_{h=1}^{k} (P_{i_h} - 1)$ effects, where the effects are determined by the vectors in the set

$$\{\underline{\lambda}(i_1 \cdots i_k)_{h_{i_1} \cdots h_{i_k}} : \underline{\lambda}(i_1 \cdots i_k) = \underline{\lambda}(i_1)_{h_{i_1}} \circledast \cdots \circledast \underline{\lambda}(i_k)_{h_{i_k}} \text{ for } h_{i_j} \in \{1, \ldots, P_{i_j} - 1\} \text{ for } j = 1, \ldots, k\}$$

Remark 6 : In a $\prod_{i=1}^{n} P_i$-FAT, all k-factor interaction effects and k'-factor interaction effects are orthogonal.

Proof: The method of proof is equivalent to the proof of remarks 4 and 5.

For the following definition two levels are chosen for each of k specified factors (k<n) in a $\prod_{i=1}^{n} P_i$-FAT and one level is chosen for each of the remaining n-k factors. Consider the $2^k$ design points that are composed of the chosen levels for each factor and call this subset of design points H.

Definition 30: A simple k-factor interaction effect among $2^k$ chosen levels of k specified factors (two levels per factor) is $\underline{\lambda}'\underline{M}$, where the elements of $\underline{\lambda}$ are zero if the design point corresponding to the element (in $\underline{\lambda}$) is not in H and plus or minus one if the design point corresponding to the element (in $\underline{\lambda}$) is in H and such that the sum of the elements in $\underline{\lambda}$ corresponding to each level of each of the k specified factors is zero ($\underline{\lambda}$ contains $m-2^k$ zeros, $2^{k-1}$ plus ones and $2^{k-1}$ minus ones).

To facilitate the analysis of observations (chapter V) of an experiment with a partitioned $\prod_{i=1}^{n} P_i$-FAT, the following quantities are defined. Let $L_1 = J_m^1$, the one by m row vector of ones, where $m = \prod_{i=1}^{n} P_i$. Let $L_{F_i}$ be the $(P_i-1)$ by m matrix that is determined by a set of $(P_i-1)$ row vectors that define a main effect for the i-th factor, for i = i, ...,n. Likewise, let $L_{F_{i_1}\ldots F_{i_k}}$ be the $\prod_{j=1}^{k}(P_{i_j}-1)$ by m matrix whose rows are the set of vectors defining the k-factor interaction effect between factors $i_1$, $i_2$, ... and $i_k$. Now, let L be the matrix, m by m, given in Figure 2.

$$L = \begin{bmatrix} L_1 \\ L_{F_1} \\ \vdots \\ L_{F_n} \\ L_{F_1F_2} \\ \vdots \\ L_{F_nF_{n-1}} \\ \vdots \\ L_{F_1\ldots F_{n-1}} \\ \vdots \\ L_{F_2\ldots F_n} \\ L_{F_1\ldots F_n} \end{bmatrix}$$

(overall mean effect)

(factor 1 main effect)

(factor n main effect)

(2-factor interaction effect between factors 1 and 2)

(2-factor interaction effect between factors n and n-1)

((n-1-factor interaction effect between factors 1, 2, ... and n-1)

((n-1)-factor interaction effect between factors 2, 3, ... and n)

(n-factor interaction effect between factors 1, 2, ... and n)

Figure 2. - The matrix defining effectsfor a $\prod_{i=1}^{n} P_i$-FAT.

Remark 7 : By construction, L'L = D, an m by m diagonal matrix. Let $\underline{h}_i$, the i-th row of the m by m matrix H, be the normalized i-th row $\lambda_i$, of the matrix L. The matrix H is similarly partitioned in Figure 3.

$$H = \begin{bmatrix} H_1 \\ H_{F_1} \\ \vdots \\ H_{F_n} \\ \vdots \\ H_{F_1F_2} \\ \vdots \\ H_{F_nF_{n-1}} \\ \vdots \\ H_{F_1\cdots F_{n-1}} \\ \vdots \\ H_{F_2\cdots F_n} \\ H_{F_1\cdots F_n} \end{bmatrix}$$

Figure 3. - The matrix H corresponding to the matrix L of figure 2.

Remark By construction, H is orthogonal ($H'H = HH' = I_m$).

Now, $H'H = H_1'H_1 + H_{F_1}'H_{F_1} + \cdots + H_{F_n}'H_{F_n} + \cdots + H_{F_1\cdots F_n}'H_{F_1\cdots F_n}$ and, letting $B_\theta = H_\theta'H_\theta$, term-wise substitution yields the following result,

$$H'H = B_1 + B_{F_1} + \cdots + B_{F_n} + B_{F_1F_2} + \cdots + B_{F_nF_{n-1}} + \cdots + B_{F_1\cdots F_n}.$$

Remark The matrices $B_\theta$, for $\theta = 1, F_1, \ldots, F_n, \ldots$ and $(F_1\cdots F_n)$, form an orthogonal idempotent decomposition for the identity matrix, $I_m$, or (1) each $B_\theta$ is an idempotent matrix,

(2) the rank of $B_\theta$ is the number of rows in $H_\theta$,

(3) $B_\theta B_{\theta'} = \emptyset_m^m$, for all $\theta \neq \theta'$ and

(4) $\sum_\theta B_\theta = I_m$.

Proof : Statement (4) is true by remark .

Since $B_\theta B_\theta = H_\theta'H_\theta H_\theta'H_\theta = H_\theta'I_rH_\theta = H_\theta'H_\theta = B_\theta$ (letting r

be the number of rows in $H_\theta$), statement (1) is proven. Now, $\text{rank}(B_\theta) = \text{rank}(H_\theta' H_\theta) = \text{rank}\ (H_\theta) = r_\theta$, where $r_\theta$ is the number of linearly independent rows or columns, whichever is fewer. For each $H_\theta$ there are fewer rows than columns, and since the rows are orthogonal the rank of $H_\theta$ is clearly the number of rows of $H_\theta$. Also, for $\theta \neq \theta'$, $B_\theta B_{\theta'} = H_\theta' H_\theta H_{\theta'}' H_{\theta'} = H_\theta' (\phi_c^a) H_{\theta'} = \phi_m^m$, where a is the rank of $H_\theta$ and c is the rank of $H_{\theta'}$. Thus statements (2) and (3) are proven.

# CHAPTER III

## PLANS USING SOME OR ALL OF THE s-FAT's

In the preceding chapter a total of $\prod_{i=1}^{n} s_i$ s-FAT's are obtained from the algebraic partitioning

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1} + \cdots + P_{is_i})\text{-s-FAT's.} \qquad (5)$$

Definition 31: For a $\prod_{i=1}^{n} P_i$-FAT the set of design points, D, represents one full replicate of the treatment combinations. Let F be a set composed of elements of D so that any element in D may not occur, may occur once or may occur more than once in F. The set F shall be called a PLAN.

Definition 32: For the algebraic partitioning (5), a subfactorial plan, denoted sPLAN, is the set of treatment combinations that is represented by any nonempty subset of $S_D$. In other words, an sPLAN is a set of s-FAT's generated by the algebraic partitioning (5).

An sPLAN can consist of one s-FAT or, if enough experimental units are available, an sPLAN can consist of two or more s-FAT's. Methods are developed in this chapter for sPLANs when the entire sPLAN can be performed at one time (with one block of units) or when the sPLAN consists of two or more s-FAT's that are performed in a sequence or in blocks. Methods that separate a set of treatment combinations into subsets, so the subsets can be assigned to blocks, are given in Chapter IV.

## K-dimensional Rectangles

Consider a full treatment replicate of a $\prod_{i=1}^{n} P_i$-FAT and the associated set of design points D, where D is given by

$$D = \left\{(x_1, \ldots, x_n): x_i \epsilon T_i = \{0,1, \ldots, P_i-1\} \text{ for } i = 1, \ldots, n \right\}.$$

Definition 33: The subset C of design points is said to be a k-dimensional rectangle, abbreviated "k-dim-rect.", for k = 1, ...,n, if C consists of $2^k$ distinct elements such that the n-tuples that represent them differ only in some k specified positions, where in each of the k positions only one of two numbers occurs.

For example, if k = 2, the design points $(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$, $(x_1, \ldots, x_i', \ldots, x_j, \ldots, x_n)$, $(x_1, \ldots, x_i, \ldots, x_j', \ldots, x_n)$ and $(x_1, \ldots, x_i', \ldots, x_j', \ldots, x_n)$ differ in the i-th and j-th positions and in the i-th position either $x_i$ or $x_i'$ occurs while either $x_j$ or $x_j'$ occurs in the j-th position. This set of four distinct design points forms a 2-dim-rect. and a specified linear combination of the observations corresponding to these design points will yield an estimate of a 2-factor simple interaction effect.

For a $\prod_{i=1}^{n} P_i$-FAT the vectors $\underline{\theta}_i$ are defined in definition 11 for i = 1, ..., n. Let $m = \prod_{i=1}^{n} P_i$ and let $\underline{D}$ be an m by one vector such that

$$\underline{D} = \underline{\theta}_1 * \ldots * \underline{\theta}_n . \tag{6}$$

Let $\underline{V}$ be an m by one vector of zeros, plus ones and minus ones and let $|\underline{V}|$ be the m by one vector where each entry in $|\underline{V}|$ is the absolute value of the corresponding entry in $\underline{V}$. If C is a subset of the set of design points, let a zero in the i-th position of $\underline{V}$ indicate the i-th component of $\underline{D}$ is not in C and a plus or minus one in the i-th position of $\underline{V}$ indicate the i-th component of $\underline{D}$ is in C. An element of $|\underline{V}| \circledast \underline{D}$ of the type

$0 \cdot (x_1, \ldots, x_n)$ will be written as 0, indicating that $(x_1, \ldots, x_n)$ is not in C and an element of the type $1 \cdot (x_1, \ldots, x_n)$ will be written as the symbol $(x_1, \ldots, x_n)$ indicating that the treatment combination $(x_1, \ldots, x_n)$ is in C. Thus, the non-zero entries of $\underline{V} \circledast \underline{D}$ are the elements of the subset C.

If the subset C represents a k-dim-rect. then a method is needed to select the elements of $\underline{V}$ so that $\underline{V}'\underline{M}$ is a k-factor simple interaction effect and so the non-zero components of $|\underline{V}| \circledast \underline{D}$ are the elements of C. Each of the m positions in $\underline{V}$ relates to an n-tuple or design point in D. Assign a 0 to those positions in $\underline{V}$ that correspond to design points that are not in C. Since C represents a k-dim-rect., C consists of $2^k$ n-tuples that differ in k of the n positions in such a way that in each of the k positions either one of two numbers occurs. Choose any four of the $2^k$ n-tuples that form a 2-dim-rect. Of these four n-tuples choose two that do not form a 1-dim-rect. and assign the value +1 to the corresponding positions in $\underline{V}$ and assign a -1 to the positions in $\underline{V}$ corresponding to the other two n-tuples. There remain $2^{k-2}$ positions in $\underline{V}$ to assign a +1 or -1. Choose a second set of four n-tuples such that they differ from the first set in only one position. For example, say the first four n-tuples chosen were $(x_1, x_2, x_3, \ldots, x_n)$

$$(x_1', x_2, x_3, \ldots, x_n)$$
$$(x_1, x_2', x_3, \ldots, x_n)$$
$$(x_1', x_2', x_3, \ldots, x_n).$$

Next, $(x_1, x_2', x_3, \ldots, x_n)$ and $(x_1', x_2, x_3, \ldots, x_n)$ are selected because they do not form a 1-dim-rect. and a +1 is assigned to the corresponding positions in $\underline{V}$. The number -1 is assigned to the positions in $\underline{V}$ corresponding to the n-tuples $(x_1, x_2, x_3, \ldots, x_n)$ and $(x_1', x_2', x_3', \ldots, x_n)$.

Now, a set of four n-tuples that differ from the first set in only one position could be

$$(x_1, x_2, x_3', \ldots, x_n)$$
$$(x_1, x_2', x_3', \ldots, x_n)$$
$$(x_1', x_2, x_3', \ldots, x_n)$$
$$(x_1', x_2', x_3', \ldots, x_n).$$

For each n-tuple in the second set there is exactly one n-tuple in the first set that is nearly identical. To each position in $\underline{V}$ corresponding to an n-tuple in the second set assign a -1 times the entry in $\underline{V}$ that corresponds to the nearly identical n-tuple in the first set selected. Thus, in the example mentioned, the numbers 1, -1, -1, and 1 would be assigned to the positions in $\underline{V}$ corresponding to the second set of four n-tuples in the order they were mentioned. The procedure of selecting a set of $2^h$ n-tuples nearly identical with the set of previously selected $2^h$ n-tuples (to which +1 or -1 are already assigned to positions in $\underline{V}$) is continued until all the elements of $\underline{V}$ are determined. The procedure will yield a vector $\underline{V}$ that has $2^k$ non-zero entries and $m-2^k$ zero entries. Since there are as many entires that are +1 as -1 and since +1 and -1 are the only non-zero entries, it is clear that $J_m^1 \underline{V} = 0$.

Example 4 : For the 3x4-FAT the vector of design points, $\underline{D}$, is obtained by $\underline{D} = \underline{\theta}_1 * \underline{\theta}_2$ , where $\underline{\theta}_1' = (0,1,2)$ and $\underline{\theta}_2' = (0,1,2,3)$.

$$\underline{D}' = ((00),(01),(02),(03),(10),(11),(12),(13),(20),(21),(22),(23))'$$

If $C = \{(00),(03),(10),(13)\}$ then $\underline{V}' = (1,0,0,-1,-1,0,0,1,0,0,0,0)'$

and $\underline{C}' = (|\underline{V}| \circledast \underline{D})' = ((00),0,0,(03),(10),0,0,(13),0,0,0,0)'$ .

Remark 10 : If a $\prod_{i=1}^{n} P_i$ by 1 vector $\underline{V}$ consisting of zeros and +1 or -1 entries, then $|\underline{V}| \circledast \underline{D}$ does not necessarily represent a k-dim-rect. for

$k = 1, \ldots, n$. For example, if $\underline{V}' = (1,0,0,-1,0,-1,0,0,0,0,+1,0)'$ in example 4, then $(|\underline{V}| \otimes \underline{D})' = ((00),0,0,(03),0,(11),0,0,0,0,(22),0)'$ and the design points (00),(03), (11) and (22) do not represent a 2-dim-rect.

If the non-zero elements of $|\underline{V}| \otimes \underline{D}$ represent a k-dim-rect., then $\underline{V}'\underline{Y}$ is an estimate of some linear combination of population means (elements of $\underline{M}$) that corresponds to a simple interaction effect among k of the factors at two specified levels of each of the k factors and one fixed level of the other n-k factors. The existence of a k-dim-rect. is a necessary condition for the existence of an estimate of a k-factor simple interaction effect among the population means. The existence of a k-dim-rect. is not in general a sufficient condition for existence of an estimate of a k-factor interaction effect among the population means.

Example 5 : Consider a $2^3$-FAT. The set of design points, D, is $D = \{(000),(001),(010),(011),(100),(101),(110),(111)\}$. Now, let

$$\underline{V}' = (\ 1,\ -1,\ -1,\ \ 1,\ \ 0,\ \ 0,\ \ 0,\ \ 0)' \text{ and}$$

$$\underline{W}' = (\ \tfrac{1}{2},\ \ \tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{2},\ \ \tfrac{1}{2},\ \ \tfrac{1}{2})' .$$

The non-zero elements of $|\underline{V}| \otimes \underline{D}$ correspond to the subset of design points $\{(000),(001),(010),(011)\}$ and these design points form a 2-dim-rect. Also, assuming the observational model is $\underline{Y} = \underline{M} + \underline{e}$ and $E(\underline{e}) = \underline{0}$, we obtain $E(\underline{V}'\underline{Y}) = \underline{V}'\underline{M} = m_{111} - m_{112} - m_{121} + m_{122}$ which represents a 2-factor simple interaction effect between factors one and two at level one of factor one. The set of design points indicated by the non-zero elements of $|\underline{W}| \otimes \underline{D}$ does not represent a 2-dim-rect. but

$$E(\underline{W}'\underline{Y}) = \bar{m}_{11.} - \bar{m}_{21.} - \bar{m}_{12.} + \bar{m}_{22.}$$

which represents the 2-factor simple interaction effect between factors

one and two averaged over levels of factor three.

Definition 34: The set of vectors $\underline{V}_1, \ldots, \underline{V}_h$ is said to be a linearly dependent set if there exists scalars (real numbers) $a_1, \ldots, a_h$, not all zero, such that

$$a_1\underline{V}_1 + \ldots + a_h\underline{V}_h = \underline{\emptyset} \quad (\underline{\emptyset} \text{ is the vector of zeros}).$$

In the contrary case, the vectors $\underline{V}_1, \ldots, \underline{V}_h$ are said to be linearly independent, in other words, if $a_1\underline{V}_1 + \ldots + a_h\underline{V}_h = \underline{\emptyset}$, then the scalars $a_1, \ldots, a_h$ must all be zero.

Definition 35: For a $\prod_{i=1}^{n} P_i$-FAT let the vectors $\underline{V}_1, \ldots, \underline{V}_h$ be such that the non-zero elements of $|\underline{V}_1| @ \underline{D}$, ... and $|\underline{V}_h| @ \underline{D}$ all represent k-dim-rect.'s for some k, k = 1, ...,n. The set of k-dimirect.'s is said to be a set of h linearly independent k-dim-rect.'s if and only if the vectors $\underline{V}_1$, ...and $\underline{V}_h$ are linearly independent.

Definition 36: The set of vectors $\underline{V}_1$, ... and $\underline{V}_h$ is said to be an orthonormal set of vectors if it is an orthogonal set and if $(\underline{V}_i'\underline{V}_i)^{\frac{1}{2}}=1$ for all i = 1, ...,h.

Remark 11 : It can be shown that an orthonormal set of vectors is a linearly independent set of vectors, an orthogonal set of vectors not containing the zero vector is a linearly independent set of vectors and a linearly independent set of vectors may or may not be an orthogonal set of vectors.

Definition 37: For a $\prod_{i=1}^{n} P_i$-FAT let the vectors $\underline{V}_1$, ...and $\underline{V}_h$ be such that the non-zero elements of $|\underline{V}_1| @ \underline{D}$, ... and $|\underline{V}_h| @ \underline{D}$ all represent k-dim-rect.'s for some k = 1, ...,n. This set of k-dim-rect.'s is said to be a set of h orthogonal (orthonormal) k-dim-rect.'s if and only if $\underline{V}_1$, ... and $\underline{V}_h$ is a set of orthogonal (orthonormal) vectors.

## Connected PLANs

For the following definitions let F be a nonempty subset of D, the set of design points of a $\prod_{i=1}^{n} P_i$-FAT. Also, let $d_i$ be the number of distinct elements (representing distinct levels) in $D_i$, where $D_i$ is a subset of $T_i$, for i = 1, ..., n ($T_i$ is the set of levels for the i-th factor in a $\prod_{i=1}^{n} P_i$-FAT).

Definition 38: A PLAN F is said to be connected, denoted cPLAN, if for every pair of design points, $f_i$ and $f_j$, in the PLAN F there exists a sequence of design points $f_i = h_1, h_2, \ldots, h_{r-1}, h_r = f_j$ in F such that every two adjacent design points in the sequence differ in exact ly one position.

Definition 39: A PLAN F is said to be a complete PLAN if

$$F = D_1 X \cdots X D_n .$$

Remark 12 : Every complete PLAN is also a cPLAN, however a cPLAN is not necessarily a complete PLAN.

Definition 40 : The PLAN F is a weak-k-cPLAN, denoted w-k-cPLAN, if F is connected and if F contains at least one k-dim-rect. and if F is also a w-k'-cPLAN for k' = 1, ..., k-1.

Definition 41 : The PLAN F is a strong-k-cPLAN, denoted s-k-cPLAN, if F is connected, if every point in F belongs to a k-dim-rect. and if F is also a s-k'-cPLAN for k' = 1, ..., k-1.

Definition 42 : A PLAN F is said to be a complete-k-PLAN at a given k factors ($k \leq n$) if $d_i > 1$ for those k factors, and if PLAN F is complete.

Definition 43 : A PLAN F is said to be a completely connected PLAN, denoted ccPLAN, if $d_i > 1$ for i = 1, ..., n, and if F is complete.

Remark 13 : If k = 1, then a w-1-cPLAN and a s-1-cPLAN are referred to as a cPLAN, since in reality they are the same.

Example 6 : In Figure 4 let PLANs (a), (b) and (c) be taken from a $3^2$-FAT, PLAN (d) from a $3^3$-FAT and PLAN (e) from a $4^2$-FAT.

| PLAN (a) | PLAN (b) | PLAN (c) | PLAN (d) | PLAN (e) |
|---|---|---|---|---|
| (00) | (00) | (00) | (012) (200) | (00) |
| (01) | (01) | (01) | (011) (202) | (02) |
| (10) | (10) | (02) | (020) (210) | (11) |
| (12) | (11) | (10) | (022) (211) | (13) |
| (21) | (12) | (11) | (100) (220) | (20) |
| (22) | (21) | (12) | (102) (221) | (22) |
| | | (20) | (120) (222) | (31) |
| | | (21) | (122) | (33) |

Figure 4. - PLANs involving FATs.

PLAN (a) is a cPLAN; PLAN (b) is a cPLAN that is also a w-2-cPLAN but not a s-2-cPLAN; PLAN (c) is a cPLAN that is also a s-2-cPLAN but not a ccPLAN; PLAN (d) is a s-3-cPLAN but not a ccPLAN and PLAN (e); although at first it may appear to be a s-2-cPLAN, it is not even a cPLAN.

The set of design points for a complete PLAN is given by $D_1 X \ldots X D_n$, and if $d_i$ is the number of distinct elements in $D_i$, this PLAN can be thought of as a full replicate of a $\prod_{i=1}^{n} d_i$-FAT. A matrix similar to the matrix L of chapter II can be constructed, where the rows define the overall mean effect, factor main effects (if $d_i \geq 2$) and factor interaction effects. If $d_i = 1$ for the i-th factor, then, obviously, no main effect can be defined for factor i and there will be no interaction effects involving the i-th factor.

Example 7 : For a $4^2$-FAT let a partitioning be given by

$$4_1 4_2\text{-FAT} \longrightarrow (2_{11} + 2_{12})(2_{21} + 2_{22})\text{-s-FAT's} \quad .$$

Let $2_{11}$ and $2_{21}$ refer to the lowest two levels of factor one and factor two and let $2_{12}$ and $2_{22}$ refer to the two highest levels of factor one

and factor two. If the treatment combinations from the $2_{11}2_{21}$-s-FAT and the $2_{12}2_{22}$-s-FAT are considered as a PLAN, then the PLAN is not connected.

Definition 44 : A set of s-FAT's from a partitioning of the type

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1} + \ldots + P_{is_i})\text{-s-FAT's}$$

is said to form a csPLAN (complete sPLAN, w-k-csPLAN, s-k-csPLAN, complete-k-sPLAN or ccsPLAN), if the set of pseudo-design points in $S_D$ that represent them form a cPLAN (complete PLAN, w-k-cPLAN, s-k-cPLAN, complete-k-PLAN or ccPLAN).

If the pseudo-design points that represent a set of s-FAT's form a ccsPLAN (or complete-k-sPLAN, s-k-csPLAN, w-k-csPLAN, complete sPLAN or csPLAN), then the set of design points that the s-FAT's represent also form a ccPLAN (complete-k-PLAN, s-k-cPLAN, w-k-cPLAN, complete PLAN or cPLAN), since each s-FAT is a ccPLAN.

A full treatment replicate that can not be run at one time might be run in parts, where each part is a s-FAT or group of s-FAT's that result from a partitioning of the original FAT. The sequence of s-FAT's is important. The sequence of s-FAT's should be chosen so that if the experiment is terminated prematurely, then the s-FAT's that have been run form at least a cPLAN of some type. For example, complete preferable to not complete and completely connected preferable to strong connected preferable to weak connected and the degree of connectedness (k) as high as possible. The concepts of connectedness and completeness can also be applied to sequences of pseudo-design points.

Example 8 : For the algebraic partitioning

$$8_1 8_2\text{-FAT} \longrightarrow (2_{11}+2_{12}+2_{13}+2_{14})(2_{21}+2_{22}+2_{23}+2_{24})\text{-s-FAT's}$$

the set of pseudo-design points is

$$S_D = \{(00),(01),(02),(03),(10),(11),(12),(13),(20),(21),(22),(23),(30),(31),(32),(33)\} .$$

The sequence (00),(03),(33),(30) is preferable to the sequence (00),(11),(22),(33) because, if the experiment is ended after step 4 (or 3 or 2) in the sequence, then the first sequence is a ccsPLAN (or a csPLAN (representing a s-2-cPLAN) for termination after either 3 or 2) while the second sequence is not a connected PLAN and not a complete PLAN.

# CHAPTER IV

## BLOCKING AND MULTIPLE PARTITIONING

A usual blocking procedure consists of assigning a set (or sets) of treatment combinations to a group (or groups) of the same number of experimental units. The entire set of treatment combinations is separated into subsets in such a manner that the number of treatment combinations in each set will also be the number of units in the blocks. It is desirable, if possible, to randomly assign the sets of treatment combinations to the blocks. The method by which the full set of treatment combinations is separated into subsets is now of extreme importance. If there is no reason to consider the blocks of units as an additional source of variation which must be accounted for in the analysis of the experiment, then a random assignment of subsets of treatment combinations to the experimental units is adequate. However, if there is reason to consider the blocks as a source of variation, then some of the comparisons among the observations that estimate certain treatment effects will also estimate certain block effects, In this case those treatment effects are said to be confounded with block effects. The manner in which the set of treatment combinations is separated into subsets can dictate the treatment effects that are confounded with block effects. Methods that allow one to separate the full set of treatment combinations into subsets so the subsets may be assigned to blocks of units are henceforth called

blocking procedures. Most of the current blocking procedures require blocks of equal size and this is a desirable condition simply from an analysis point of view.

When a $\prod_{i=1}^{n} P_i$-FAT is algebraically partitioned via,

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1} + \ldots + P_{is_i})\text{-s-FAT's}, \tag{7}$$

into $\prod_{i=1}^{n} s_i$ s-FAT's, then parts of individual s-FAT's or one or more of the s-FAT's can be assigned to a block of units. Methods developed in this chapter will allow the assigning of groups of treatment combinations to a set of blocks, where the blocks may or may not be of the same size. The main method of separating the full set of treatment combinations into appropriate subsets, given the available blocks and block sizes, is to algebraically partition the $\prod_{i=1}^{n} P_i$-FAT via (7) and arrive at a method of assigning the s-FAT's to the available blocks by consideration of confounding schemes involving the set of pseudo-design points, $S_D$. There are $\prod_{i=1}^{n} s_i$ elements in $S_D$ and these pseudo-design points designate a pseudo-factorial arrangement of treatments, hereafter denoted as a $\prod_{i=1}^{n} s_i$-p-FAT. The treatment combinations of a $\prod_{i=1}^{n} s_i$-FAT can be assigned to blocks and similarly, the pseudo-design points of a $\prod_{i=1}^{n} s_i$-p-FAT can be assigned to blocks, or more properly labeled, pseudo-blocks. Since the set of all pseudo-design points is a complete PLAN, main effects and interaction effects can be defined for the factors in the $\prod_{i=1}^{n} s_i$-p-FAT. When components of these main effects and interactions are confounded with pseudo-block effects, blocking procedures for partitioned FAT's result. Blocking procedures are considered for partitioned FAT's for the case when $s_1 = \ldots = s_n$ and when $s_i \neq s_j$ for at least one pair $i \neq j$.

**Definition 45** : If $s_1 = \ldots = s_n$ in the partitioning (7), the partitioning is called an equal partitioning. Otherwise, if $s_i \neq s_j$ for some $i \neq j$, the partitioning is referred to as an unequal partitioning.

In the case where all the partition numbers are equal, for the i-th factor, let $P_{i.1} = P_{i1} = \ldots = P_{is_i}$, and denote the sum $(P_{i1} + \ldots + P_{is_i})$ by $s_i(P_{i.1})$. Thus, if $P_{i1} = \ldots = P_{is_i} = P_{i.1}$, for $i = 1, \ldots, n$, the algebraic partitioning (7) is written as follows

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (s_i(P_{i.1}))\text{-s-FAT's} \; . \tag{8}$$

## Blocking Precedures for Equal Partitionings

For an algebraic partitioning of the type (7) the set of pseudo-design points representing a $\prod_{i=1}^{n} s_i$-p-FAT can be used to formulate confounding schemes and blocking procedures. If the algebraic partitioning (7) is an equal partitioning, then $s = s_1 = \ldots = s_n$ and the set of pseudo-design points represents a $s^n$-p-FAT. Since it is usually desirable to confound high order interaction effects with blocks, we shall try to confound interaction effects in the $s^n$-p-FAT with pseudo-blocks as a means of arriving at a blocking procedure. If interaction effects in the $s^n$-p-FAT are confounded with pseudo-blocks, then some of the interaction effects in the $\prod_{i=1}^{n} P_i$-FAT are confounded with blocks. In some cases the confounding of some of the components of factor main effects with block effects is unavoidable, as is the case where each s-FAT is assigned to a block of units. Blocking procedures are now discussed for equal partitions where s is an arbritrary integer greater than one.

Confounding schemes for $s^n$-FAT's where s is a prime or prime power number are given in Kempthorne (34). If blocks of size $s^m$ ($m<n$)

are available for confounding effects of a full replicate of a $s^n$-FAT with block effects, then a total of $s^{n-m}$ blocks are required. To separate the set of $s^n$ treatment combinations into sets of size $s^m$, one must choose n-m linearly independent effects to confound with blocks. Counting these n-m linearly independent effects and their generalized interactions, a total of $(s^{n-m})/(s-1)$ effects are confounded with blocks. For blocks of size $s^m$ there are a total of $\prod_{i=0}^{n-m-1} ((s^n-s^i)/(s^{n-m}-s^i))$ systems of confounding to choose from (see Kempthorne (34)). Usually a system where the high order interactions are confounded with blocks is preferable to a system where the main effects or components of main effects are confounded with blocks. Examples of equal partitionings when s is a prime and prime power number are given in the sequel.

Once the set of pseudo-design points is separated into subsets these subsets can be randomly assigned to blocks of the appropriate size. Assigning subsets of pseudo-design points to pseudo-blocks is essentially the same as assigning subsets of the $\prod_{i=1}^{n} P_i$ treatment combinations to blocks of appropriate size. The confounding schemes are used to separate the set of treatment combinations into subsets and the subsets are randomly assigned to the blocks of experimental units.

If s in the $s^n$-p-FAT is a prime power number then additional methods labeled pseudo-factorials in Kempthorne (34) can be used to obtain conofunding schemes for a $s^n$-FAT. If s is a product of prime numbers, then the theory recently developed by White and Hultquist (44) and Raktoe (40) can be used to construct confounding schemes.

Whether or not blocks of equal size can be accommodated depends entirely upon the algebraic partitions $P_i = P_{i1} + \ldots + P_{is_i}$, for all

$i = 1, \ldots, n$. The size of a s-FAT is determined by the numbers of the type $\prod_{i=1}^{n} P_{ij_i}$, where $j_i \in \{1, \ldots, s_i\}$. If $P_{i1} = \ldots = P_{is_i}$, for all $i = 1, \ldots, n$, it is obvious that blocks admitting the same number of s-FAT's are of equal size. However, it is not necessary that $P_{i1} = \ldots = P_{is_i}$ for $i = 1, \ldots, n$, in order to arrive at a PLAN involving equal block sizes (an example is given in the sequel). As will be seen later, it is also possible to have $s_i \neq s_j$ for $i \neq j$ and arrive at equal block sizes when the s-FAT's of an algebraic partition are applied to blocks of experimental units.

It is also possible that blocking procedures can be obtained by constructing confounding schemes in each s-FAT. This would normally be the case when there are a lot of blocks available and the blocks have a relatively small number of experimental units. If confounding schemes within each s-FAT are used and if a combined analysis of the observations of all the s-FAT's is to be performed, then the confounding schemes in each s-FAT must be chosen so an overall analysis is possible. For example, one might confound interaction effects in all of the s-FAT's or one might confound main effects or components of main effects in all of the s-FAT's, but confounding components of main effects in some of the s-FAT's and interactions in some of the s-FAT's will probably lead to complicated analysis procedures, if any analysis procedure exists for all the s-FAT's as a whole.

Definition 46 : To denote a partitioned factorial arrangement of treatments that is applied to a set of blocks, the symbol A : B will be used, where the symbol A denotes the algebraic partitioning and the symbol B is a set of numbers indicating block sizes.

Thus, for n = 2, the expression $(2(P_{1.1}))(2(P_{2.1})):2P_{1.1}P_{2.1}$ indicates that a $P_1P_2$-FAT is partitioned according to

$$P_1P_2\text{-FAT} \longrightarrow (P_{1.1} + P_{1.1})(P_{2.1} + P_{2.1})\text{-s-FAT's},$$

and the whole set of treatment combinations is assigned to two blocks, where both blocks are of size $2P_{1.1}P_{2.1}$. For the algebraic partitioning

$$P_1P_2\text{-FAT} \longrightarrow (s(P_{1.1}))(s(P_{2.1}))\text{-s-FAT's} \qquad (9)$$

the possibilities for blocking PLANs are enumerated in Table 1. The first column in table 1 indicates pseudo-block size in the $s^2$-p-FAT: the second column indicates the block size in terms of the original full replicate of the $P_1P_2$-FAT: the third column indicates the number of blocks required for a full treatment replicate and the fourth column gives the blocking PLAN notation. The four PLANs in column four are henceforth referred to as blocking PLANs (a), (b), (c) and (d). The number of pseudo-blocks is equal to $s^2$ divided by the pseudo-block size. The degrees of freedom available for confounding is the number of blocks minus one. In blocking PLAN (a) there are s pseudo-blocks and a suitable confounding scheme can be obtained by confounding components of either fator main effects with pseudo-blocks, or if s is a prime or prime power number, by confounding components of interaction effects with pseudo-block effects. If s is not prime or prime power or equal to one, effects corresponding to interaction might be confounded with block effects. In blocking PLAN (a) there are s s-FAT's that are assigned to one block of units. In blocking PLAN (b) all effects in the p-FAT are confounded with pseudo-block effects. Blocking PLANs (c) and (d) are obtained by confounding all effects of the p-FAT with pseudo-block effects (as in blocking PLAN (b)) and then confounding effects within each s-FAT with

TABLE 1

SOME BLOCKING PLANS FOR THE PARTITIONING (9)

| Block size $s^2$-p-FAT | Block size $P_1P_2$-FAT | number of blocks | Blocking PLAN | |
|---|---|---|---|---|
| s | $sP_{1.1}P_{2.1}$ | s | $(s(P_{1.1}))(s(P_{2.1})):sP_{1.1}P_{2.1}$ | (a) |
| 1 | $P_{1.1}P_{2.1}$ | $s^2$ | ( " )( " ):$P_{1.1}P_{2.1}$ | (b) |
| 1 | $P_{1.1}$ | $s^2P_{2.1}$ | ( " )( " ):$P_{1.1}$ | (c) |
| 1 | $P_{2.1}$ | $s^2P_{1.1}$ | ( " )( " ):$P_{2.1}$ | (d) |
| OTHER | | | | |

block effects. The confounding schemes available within each s-FAT depend upon the numbers $P_{i.j}$ and thus, the word "other" appears in the table to allow for schemes confounding other effects (whenever possible) with block effects. No PLANs are developed that would result in assigning parts of different s-FAT's to the same block of units.

Example 9 : Consider the algebraic partitioning

$$8_1 4_2\text{-FAT} \longrightarrow (4_{11} + 4_{12})(2_{21} + 2_{22})\text{-s-FAT's.}$$

Let $4_{11}$ and $2_{12}$ refer to the lowest four and lowest two levels of factors one and two respectively. Likewise, let $4_{12}$ and $2_{22}$ refer to the highest four and highest two levels of factor one and factor two respectively. For this example, the p-FAT is equivalent to a $2^2$-FAT. The blocking PLANs are given in table 2, which follows the form of Table 1. Blocking PLANs (a), (b), (c) and (d) are obtained by confounding the effects of the $2^2$-p-FAT with pseudo-block effects. Blocking PLAN (a) is obtained by confounding the pseudo-factor one main effect with pseudo-block effects; blocking PLAN (b) by confounding the pseudo-factor two main effect with pseudo-block effects and blocking PLAN (c) by confounding the pseudo-interaction effect with pseudo-block effects. Blocking PLAN (d) is obtained by confounding all effets of the $2^2$-p-FAT with pseudo-block effects. Once the p-FAT confounding procedure separates the elements of $S_D$ into subsets, each element, or pseudo-design point, is replaced by the design points it represents. These sets of design points are then randomly assigned to the blocks. In Figure 5 the blocking PLANs (a), (b), (c) and (d) are represented. The numbers in Figure 5 indicate which subset that particular treatment combination is assigned to. Rows represent levels of factor one

TABLE 2

SOME BLOCKING PLANS FOR EXAMPLE 9

| Block size $2^2$-p-FAT | Block size 8x4-FAT | Number of Blocks | Blocking PLANs |
|---|---|---|---|
| 2 | 16 | 2 | (a)<br>(b)<br>(c) |
| 1 | 8 | 4 | (d) |
| 1 | 4 | 8 | (e), $(f_1)$, $(f_2)$, $(f_3)$, $(g_1)$, $(g_2)$, $(g_3)$ |
| 1 | 2 | 16 | $(h_1)$, $(h_2)$, $(h_3)$, $(h_4)$, $(h_5)$, $(h_6)$, $(h_7)$ |

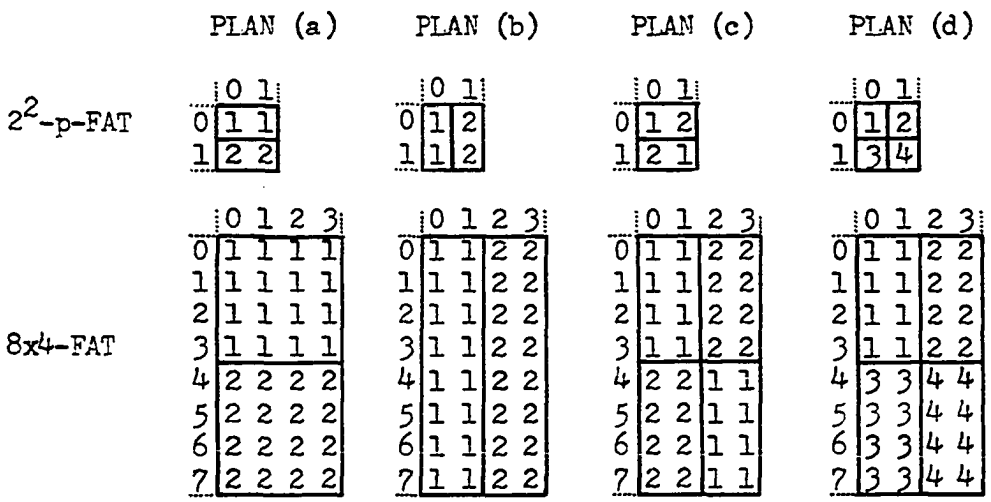


Figure 5. - Illustration of blocking PLANs (a), (b), (c) and (d) for example 9.

and columns represent levels of factor two. The blocking PLANs for block sizes four and two are obtained by first separating the full replicate of 32 treatment combinations into four sets by blocking PLAN (d) and then separating each of the four sets into either two or four smaller sets, depending upon the block sizes. Each of the four sets is an s-FAT obtained from the partitioning. Each s-FAT is equivalent to a 4x2-FAT. Confounding schemes for a 4x2-FAT, labeling the factors as $B_1$ and $B_2$, can be obtained from confounding schemes in a $2^3$-FAT, labeling the factors as A, B and C. The correspondence between the $2^3$-FAT and 4x2-FAT factors is

(1) A represents $B_2$

(2) B, C and BC represent $B_1$

(3) AB, AC and BC represent $B_1B_2$.

This correspondence procedure is the procedure labeled "pseudo-factors" in chapter seventeen of Kempthorne (34). From the correspondences (1), (2) and (3) above, the treatment combination relationships in Figure 6 result. From these correspondences in Figure 6, it is easy to conclude that a main effect defined for factor A is equivalent to a main effect defined for factor $B_2$; a main effect for factors B and C and a 2-factor interaction effect for factors B and C is equivalent to a main effect defined for factor $B_1$ and the interaction effects defined for AB, AC and ABC are equivalent to a two factor interaction effect defined for factors $B_1$ and $B_2$. To obtain blocking PLANs for eight and sixteen blocks the blocking PLAN (d) is first performed. Now, to obtain blocking PLAN (e), the main effect of factor A in the $2^3$-FAT is confounded with pseudo-blocks of size four, and consequently, $B_2$ or a component of the factor two main effect in the $P_1P_2$-FAT, is confounded with block effects (the

| Design Point in 2x2x2-FAT | Design Point in 4x2-FAT |
|---|---|
| (000) | (00) |
| (001) | (10) |
| (010) | (20) |
| (011) | (30) |
| (100) | (01) |
| (101) | (11) |
| (110) | (21) |
| (111) | (31) |

Figure 6. - Treatment combination correspondences.

confounding of factor A is carried out in each of the $2^3$-FAT's representing the 4x2-FAT's or s-FAT's). Similarly, components of the factor one main effect will be confounded with block effects if either of B, C or BC is confounded with pseudo-blocks in each of the s-FAT's. These three schemes represent blocking PLANs $(f_1)$, $(f_2)$ and $(f_3)$. Blocking PLANs $(g_1)$, $(g_2)$ and $(g_3)$ are those obtained by confounding components of the two factor interaction with block effects, and thus, can be obtained by confounding either of AC, AB or ABC with pseudo-blocks of size four in each s-FAT. For those blocking PLANs incorporating sixteen blocks, the methods to obtain the PLANs are given in Figure 7. Blocking PLANs $(g_1)$ and $(h_7)$ are chosen as representatives of the blocking PLANs for block sizes four and two. For blocking PLAN $(g_1)$ there are eight sets of four treatment combinations each (two sets per s-FAT) and for blocking PLAN $(h_7)$ there are sixteen sets of two treatment combinations each (four sets per s-FAT). These blocking PLANs are represented in Figure 8.

Table 3 represents a summary of the blocking PLANs for the equal partitioning

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (s(P_{i.1}))\text{-s-FAT's.} \tag{10}$$

In Table 3 the first n blocking PLANs are obtained by confounding the effects of the $s^n$-p-FAT with pseudo-block effects. The remaining blocking PLANs are arrived at by first invoking the blocking PLAN with pseudo-block size one and then separating each set by further confounding effects within each s-FAT with block effects. The word "other" appears in the table to indicate that there might be other confounding schemes available that would lead to other blocking PLANs, but the availability depends largely upon the numbers $P_{1.1}$, ... and $P_{n.1}$. The greater the number of

| Pseudo-block size | Effects confounded in the | | | PLAN |
|---|---|---|---|---|
| | $2^3$-FAT | 4x2-p-FAT | 8x4-FAT | |
| 4 | A,B,AB | $B_2$,<br>part of $B_1$<br>part of $B_1B_2$ | a component of $F_2$<br>a component of $F_1$<br>a component of $F_1xF_2$ | $(g_1)$ |
| 4 | B,C,BC | $B_1$ | a component of $F_1$ | $(g_2)$ |
| 4 | A,C,AC | $B_2$<br>part of $B_1$<br>part of $B_1B_2$ | a component of $F_2$<br>a component of $F_1$<br>a component of $F_1F_2$ | $(g_7)$ |
| 4 | A,BC,ABC | $B_2$<br>part of $B_1$<br>part of $B_1B_2$ | a component of $F_2$<br>a component of $F_1$<br>a component of $F_1F_2$ | $(g_3)$ |
| 4 | B,AC,ABC | part of $B_1$<br>part of $B_1B_2$ | a component of $F_1$<br>a component of $F_1F_2$ | $(g_4)$ |
| 4 | C,AB,ABC | part of $B_1$<br>part of $B_1B_2$ | a component of $F_1$<br>a component of $F_1F_2$ | $(g_5)$ |
| 4 | AC,BC,AB | part of $B_1$<br>part of $B_1B_2$ | a component of $F_1$<br>a component of $F_1F_2$ | $(g_6)$ |

Figure 7. - Methods to obtain blocking PLANs consisting of 16 blocks for the partitioned 8x4-FAT of example 9.

Blocking PLAN ($g_1$)

| Factor one \ Factor two | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |
| 1 | 2 | 1 | 4 | 3 |
| 2 | 1 | 2 | 3 | 4 |
| 3 | 2 | 1 | 4 | 3 |
| 4 | 5 | 6 | 7 | 8 |
| 5 | 6 | 5 | 8 | 7 |
| 6 | 5 | 6 | 7 | 6 |
| 7 | 6 | 5 | 8 | 7 |

Blocking PLAN ($h_7$)

| Factor one \ Factor two | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | 4 | 5 | 8 |
| 1 | 3 | 2 | 7 | 6 |
| 2 | 1 | 4 | 5 | 8 |
| 3 | 3 | 2 | 7 | 6 |
| 4 | 9 | 12 | 13 | 16 |
| 5 | 11 | 10 | 15 | 14 |
| 6 | 9 | 12 | 13 | 16 |
| 7 | 11 | 10 | 15 | 14 |

Figure 8. - Blocking PLANs $(g_1)$ and $(h_7)$ of example 9.

TABLE 3

SOME BLOCKING PLANS FOR THE PARTITIONING (10)

| Block size $s^n$-p-FAT | Block size $P_1..P_n$-FAT | Number of blocks |
|---|---|---|
| $s^{n-1}$ | $s^{n-1}P_{1.1}\cdots P_{n.1}$ | $s$ |
| $s^{n-2}$ | $s^{n-2}P_{1.1}\cdots P_{n.1}$ | $s^2$ |
| ⋮ | ⋮ | ⋮ |
| $s$ | $sP_{1.1}\cdots P_{n.1}$ | $s^{n-1}$ |
| 1 | $P_{1.1}\cdots P_{n.1}$ | $s^n$ |
| 1 | $P_{1.1}$ | $s^nP_{2.1}\cdots P_{n.1}$ |
| 1 | $P_{2.1}$ | $s^nP_{1.1}P_{3.1}\cdots P_{n.1}$ |
| ⋮ | ⋮ | ⋮ |
| 1 | $P_{n.1}$ | $s^nP_{1.1}\cdots P_{n-1.1}$ |
| CTHEP | | |

blocks the more difficult it is to find a blocking PLAN in which components of main effects remain unconfounded with block effects. For this reason, blocking PLANs with a few large sized blocks are often preferable to blocking PLANs with relatively small sized blocks.

So far, only blocking procedures for partitions of the type (10) have been mentioned. In the general partitioning expression (7) it might be the case that $P_{ij_i} \neq P_{ij_{i'}}$, for the i-th factor and $j_i \neq j_{i'}$. Consequently, the s-FAT's do not necessarily have to be the same size. For $n = 2$ and $s_1 = s_2 = s$, assume that the numbers $P_{11}$, ... and $P_{1s_1}$ are not all equal in the partitioning

$$P_1P_2\text{-FAT} \longrightarrow (P_{11} + \ldots + P_{1s_1})(P_{21} + \ldots + P_{2s_2})\text{-s-FAT's.} \qquad (11)$$

For the partitioning (11) the $s^2$ s-FAT's are of sizes $P_{11}P_{21}$, ... and $P_{1s_1}P_{2s_2}$. For a matter of simplicity, let $s = 2$ and $h_{ij} = P_{1i}P_{2j}$, for $i = 1,2$ and $j = 1,2$. Blocking PLANs derivalbe from confounding schemes in the $2^2$-p-FAT are given in Table 4. PLAN (a) of Table 4 and PLAN (b) of table 4 are obtained by confounding factor main effects with pseudo-block effects in the $2^2$-p-FAT. PLAN (c) is obtained by confounding the two-factor interaction effect of the $2^2$-p-FAT (which is a component of the two-factor interaction effect in the $P_1P_2$-FAT) with pseudo-block effects. PLAN (d) results from confounding all effects of the $2^2$-p-FAT with pseudo-block effects. Thus, for PLAN (d) one component of each of the factor main effects will be confounded with block effects and one component of the two-factor interaction effect in the $P_1P_2$-FAT will be confounded with block effects.

For the case when $s = 3$, the partitioning

$$P_1P_2\text{-FAT} \longrightarrow (P_{11}+P_{12}+P_{13})(P_{21}+P_{22}+P_{23})\text{-s-FAT's} \qquad (12)$$

TABLE 4

SOME BLOCKING PLANS FOR THE CASE $s_1=s_2=2$ OF THE PARTITIONING (11)

| Block size $2^2$-p-FAT | Block size in the $P_1P_2$-FAT | Number of blocks | PLANs |
|---|---|---|---|
| 2 | $(h_{11}+h_{12}),(h_{21}+h_{22})$ | 2 | (a) |
| | $(h_{11}+h_{21}),(h_{12}+h_{22})$ | 2 | (b) |
| | $(h_{11}+h_{22}),(h_{12}+h_{21})$ | 2 | (c) |
| 1 | $h_{11}$, $h_{12}$, $h_{21}$, $h_{22}$ | 4 | (d) |

results in nine s-FAT's of sizes $h_{ij} = P_{1i}P_{2j}$, for i = 1,2,3 and j = 1, 2,3. Table 5 gives the blocking PLANs derived from the $3^2$-p-FAT. In Table 5 there are four PLANs available for a partitioned factorial arrangement of treatments of the type (12), if the partitioned factorial is to be run in three blocks. Two of these PLANs ((c) and (d)) result from confounding a factor main effect in the $3^2$-p-FAT with pseudo-block effects and the other two PLANs ((a) and (b)) are obtained by confounding components of the two-factor interaction effect in the $3^2$-p-FAT with pseudo-block effects. PLAN (e) results from confounding all the effects of the $3^2$-p-FAT with pseudo-block effects.

Example 10 : Consider the algebraic partitioning

$10_1 9_2$-FAT $\longrightarrow (2_{11} + 3_{12} + 5_{13})(2_{21} + 3_{22} + 4_{23})$-s-FAT's.

For simplicity, let $2_{11}$ and $2_{21}$ refer to the lowest two levels of factors one and two; let $5_{13}$ and $4_{23}$ refer to the highest five and four levels of factor one and factor two, respectively, and let $3_{12}$ and $3_{22}$ correspond to the three middle levels of factors one and two. In this example $S_D$ corresponds to a $3^2$-p-FAT, so confounding schemes for a $3^2$-FAT will be used to arrive at some of the blocking PLANs. The nine s-FAT's resulting from the partition are of sizes $h_{11} = 4$, $h_{12} = 6$, $h_{13} = 8$, $h_{21} = 6$, $h_{22} = 9$, $h_{23} = 12$, $h_{31} = 10$, $h_{32} = 15$ and $h_{33} = 20$. Some of the blocking PLANs are given in Table 6. Blocking PLANs within the s-FAT's are omitted at this point because the s-FAT's are of different sizes. For blocking PLANs (a), (b), (c) and (d), Figure 9 illustrates how the confounding schemes in the $3^2$-p-FAT designate blocking PLANs for the 10x9-FAT. As before, the rows of the squares in Figure 9 represent levels of factor one and the columns of

TABLE 5

SOME BLOCKING PLANS FOR THE PARTITIONING (12)

| Block size $3^2$-p-FAT | Block size in $P_1P_2$-FAT | Number of blocks | PLAN |
|---|---|---|---|
| 3 | $(h_{11}+h_{23}+h_{32}),(h_{12}+h_{21}+h_{33}),(h_{13}+h_{22}+h_{31})$ | 3 | PLAN (a) |
| | $(h_{11}+h_{22}+h_{33}),(h_{13}+h_{21}+h_{32}),(h_{12}+h_{23}+h_{31})$ | 3 | PLAN (b) |
| | $(h_{11}+h_{21}+h_{31}),(h_{12}+h_{22}+h_{32}),(h_{31}+h_{32}+h_{33})$ | 3 | PLAN (c) |
| | $(h_{11}+h_{12}+h_{13}),(h_{21}+h_{22}+h_{23}),(h_{31}+h_{32}+h_{33})$ | 3 | PLAN (d) |
| 1 | $h_{11},h_{12},h_{13},h_{21},h_{22},h_{23},h_{31},h_{32},h_{33}$ | 9 | PLAN (e) |

TABLE 6

SOME BLOCKING PLANS FOR EXAMPLE 10

| Block size $3^2$-p-FAT | Block size in 10x9-FAT | Number of blocks | Blocking PLAN |
|---|---|---|---|
| 3 | 31, 32, 27 | 3 | PLAN (a) |
| | 33, 28, 29 | 3 | PLAN (b) |
| | 20, 30, 40 | 3 | PLAN (c) |
| | 18, 27, 45 | 3 | PLAN (d) |
| 1 | 4,6,6,8,9,10,12,15,20 | 9 | PLAN (e) |

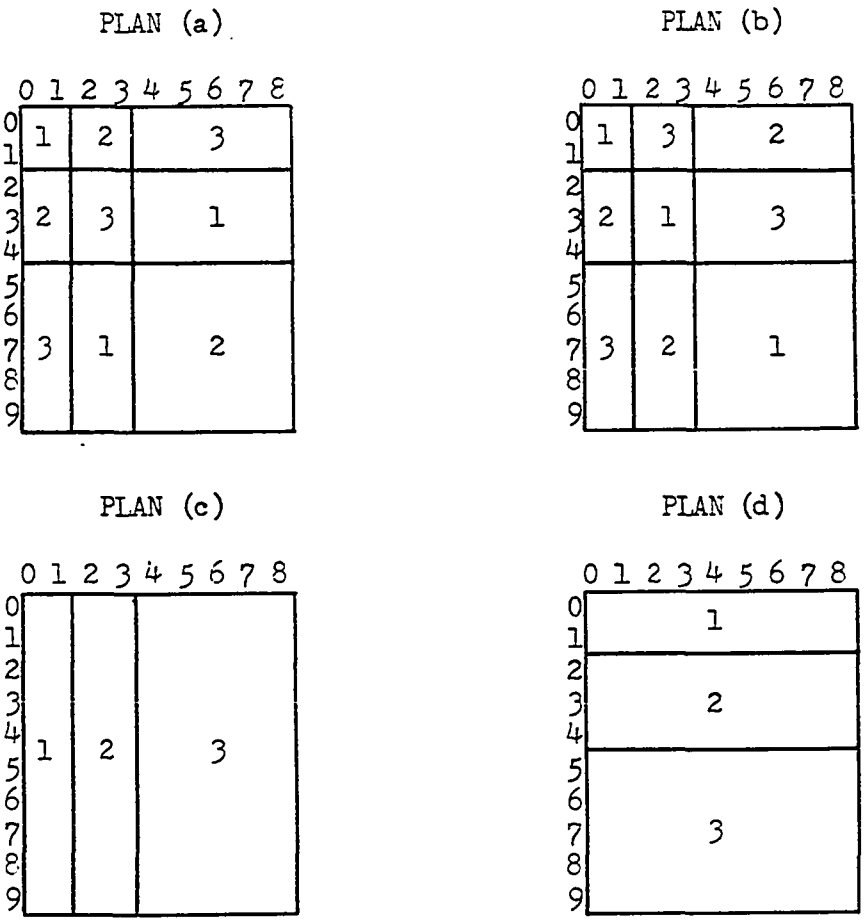


Figure 9. - Illustration of blocking PLANs (a), (b), (c) and (d) example 10.

the square represent levels of factor two. The numbers 1, 2 and 3 indicate the three sets of treatment combinations that result from the various blocking PLANs. Blocking PLAN (e) is obtained by confounding all effects in the $3^2$-p-FAT with pseudo-blocks. A diagrammatic representation of blocking PLAN (e) is omitted.

For the partitioning of a $\prod_{i=1}^{n} P_i$-FAT when $n = 3$ and $s_1 = s_2 = s_3 = 2$, or more explicitly,

$$P_1P_2P_3\text{-FAT} \longrightarrow (P_{11}+P_{12})(P_{21}+P_{22})(P_{31}+P_{32})\text{-s-FAT's}, \tag{13}$$

and, letting $h_{ijk} = P_{1i}P_{2j}P_{3k}$, for $i = 1,2$, $j = 1,2$ and $k = 1,2$, the blocking PLANs listed in Table 7 can be obtained from confounding schemes in the $2^3$-p-FAT. PLANs (a), (b) and (c) result from confounding factor main effects with pseudo-block effects; PLANs (d), (e), (f) and (g) result from confounding interaction effects of the $2^3$-p-FAT with pseudo-block effects; PLANs (h), (i) and (j) result from confounding two distinct factor main effects and their generalized interaction effect with pseudo-block effects; PLANs (k), (l) and (m) result from confounding one factor main effect, the three-factor interaction effect and their generalized interaction effect with pseudo-block effects and PLAN (n) is obtained by confounding the two-factor interaction effects of the $2^3$-p-FAT with pseudo-block effects. PLANs (d), (e), (f), (g) and (n) are the only PLANs in which interaction effects are confounded with block effects.

For the general partitioning

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1} + \ldots + P_{is_i})\text{-s-FAT's} \tag{14}$$

where, for at least one $i \in \{1, \ldots, n\}$, the numbers $P_{i1}$, ... and $P_{is_i}$ are not all equal, blocking PLANs can be obtained from confounding schemes in $\prod_{i=1}^{n} s_i$-p-FAT. Naturally, it is easier to obtain PLANs if all the $s_i$ are

TABLE 7

SOME BLOCKING PLANS FOR THE PARTITIONING (13)

| Block size $2^3$-p-FAT | Block size in $P_1P_2P_3$-FAT | No. of Blocks | PLAN |
|---|---|---|---|
| 4 | $(h_{111}+h_{112}+h_{121}+h_{122}),(h_{211}+h_{212}+h_{221}+h_{222})$ | 2 | (a) |
| | $(h_{111}+h_{112}+h_{211}+h_{212}),(h_{121}+h_{122}+h_{221}+h_{222})$ | 2 | (b) |
| | $(h_{111}+h_{121}+h_{211}+h_{221}),(h_{112}+h_{122}+h_{212}+h_{222})$ | 2 | (c) |
| | $(h_{111}+h_{112}+h_{221}+h_{222}),(h_{121}+h_{122}+h_{211}+h_{212})$ | 2 | (d) |
| | $(h_{111}+h_{222}+h_{121}+h_{212}),(h_{112}+h_{122}+h_{211}+h_{221})$ | 2 | (e) |
| | $(h_{111}+h_{122}+h_{211}+h_{222}),(h_{112}+h_{121}+h_{212}+h_{221})$ | 2 | (f) |
| | $(h_{111}+h_{122}+h_{212}+h_{221}),(h_{112}+h_{121}+h_{211}+h_{222})$ | 2 | (g) |
| 2 | $(h_{111}+h_{112}),(h_{121}+h_{122}),(h_{221}+h_{222}),(h_{211}+h_{212})$ | 4 | (h) |
| | $(h_{111}+h_{121}),(h_{112}+h_{122}),(h_{212}+h_{222}),(h_{211}+h_{221})$ | 4 | (i) |
| | $(h_{111}+h_{211}),(h_{112}+h_{212}),(h_{122}+h_{222}),(h_{121}+h_{221})$ | 4 | (j) |
| | $(h_{111}+h_{122}),(h_{112}+h_{121}),(h_{211}+h_{222}),(h_{212}+h_{221})$ | 4 | (k) |
| | $(h_{212}+h_{111}),(h_{112}+h_{211}),(h_{121}+h_{222}),(h_{122}+h_{221})$ | 4 | (l) |
| | $(h_{111}+h_{221}),(h_{121}+h_{211}),(h_{112}+h_{222}),(h_{122}+h_{212})$ | 4 | (m) |
| | $(h_{111}+h_{222}),(h_{112}+h_{221}),(h_{122}+h_{211}),(h_{212}+h_{121})$ | 4 | (n) |
| 1 | $h_{111}, h_{112}, h_{121}, h_{122}, h_{211}, h_{212}, h_{221}, h_{222}$ | 8 | (r) |

some power of a specific prime number. If the $s_i$ are powers of different prime numbers or products of different prime numbers, then the theory of White and Hultquist (44) and Raktoe (40) might be used to develop a confounding scheme that confounds interactions of the p-FAT with pseudo-blocks. Perhaps it might be easier and quicker to partition the original s-FAT so that the set of pseudo-design points representing the s-FAT's is easier to separate into subsets that can be assigned to blocks of units. Whether or not more than one partitioning of a FAT is possible depends upon the way each set of levels for each factor is separated into subsets and, in a blocking situation, upon available block sizes.

## Blocking and Unequal Partitionings

An algebraic partition of a $\prod_{i=1}^{n} P_i$-FAT of the type (7) is called an unequal partition if $s_i \neq s_j$ for some $i \neq j$, $i = 1,\ldots,n$ and $j = 1,\ldots,n$. The set of pseudo-design points, $S_D$, now corresponds to a mixed or asymmetrical factorial arrangement of treatments. The statistical theory that leads to confounding schemes in mixed factorial treatment arrangements has been developed in various ways. Geometrical methods have been used to obtain a mathematical basis for the development of confounding schemes. The use of the mathematical properties of finite fields, or Galois fields, also leads to confounding schemes for prime symmetrical factorial treatment arrangements. Recently, White and Hultquist (44) and Raktoe (40) present methods to combine Galois fields with a different number of prime elements in such a manner as to retain the properties of a finite field, thus, providing a mathematical basis for mixed factorial condounding schemes. A method of blocking can also be obtained by confounding procedures that take into account only a subset of the factors

that make up a treatment combination. For example, in a $\prod_{i=1}^{n} P_i$-FAT representing factors $A_1$, ... and $A_n$, confounding methods can be used on the set of treatment combinations forming a $P_1P_2$-FAT (factors $A_1$ and $A_2$) to arrive at a confounding scheme in the $\prod_{i=1}^{n} P_i$-FAT. Each treatment combination $(x_1,x_2)$ in the $P_1P_2$-FAT is replaced by a set of treatment combinations designated by $x_1$ and $x_2$ in the first and second positions of the n-tuples representing the treatment combinations in the $\prod_{i=1}^{n} P_i$-FAT.

The confounding schemes for a $\prod_{i=1}^{n} P_i$-FAT depend largely upon the set of pseudo-design points, $S_D$. Attention is now directed to the situation where $S_D$ represents a mixed p-FAT. If a $P_1P_2$-FAT is partitioned

$$P_1P_2\text{-FAT} \longrightarrow (s_1(P_{1.1}))(s_2(P_{2.1}))\text{-s-FAT's}, \tag{15}$$

then the possibilities for blocking PLANs for the partitioned $P_1P_2$-FAT are given in Table 8. The word "other" appears in Table 8 to allow for other confounding schemes concerning the s-FAT's that might lead to blocking PLANs. More can be said about blocking precedures for the partition (15) if $s_1 \neq s_2$ and if $s_1$ and $s_2$ are powers of the same prime number. For this case, the methods mentioned in Kempthorne (34) concerning pseudo-factors are appropriate for confounding schemes in the p-FAT of the partition. Also, if each s-FAT represents a $q^n$-FAT, where q is a prime power number, then confounding schemes within each s-FAT are easily constructed by the pseudo-factor method mentioned in Kempthorne (34). An example of these concepts is now provided.

Example 11 : Consider the unequal algebraic partitioning

$$6x6x4\text{-FAT} \longrightarrow (3_1(2_{1.1}))(3_2(2_{2.1}))(2_3(2_{3.1}))\text{-s-FAT's}.$$

Let the sets of six levels for the first and second factors be partitioned into low two, intermediate two and high two level subsets and

TABLE 8

SOME BLOCKING PLANS FOR THE PARTITIONING (15)

| Block size $s_1 s_2$-p-FAT | Block size $P_1 P_2$-FAT | Number of blocks | Blocking PLANs |
|---|---|---|---|
| $s_1$ | $s_1 P_{1.1} P_{2.1}$ | $s_2$ | $(s_1(P_{1.1}))(s_2(P_{2.1})):s_1 P_{1.1} P_{2.1}$ |
| $s_2$ | $s_2 P_{1.1} P_{2.1}$ | $s_1$ | ( " )( " ):$s_2 P_{1.1} P_{2.1}$ |
| 1 | $P_{1.1} P_{2.1}$ | $s_1 s_2$ | ( " )( " ):$P_{1.1} P_{2.1}$ |
| 1 | $P_{1.1}$ | $s_1 s_2 P_{2.1}$ | ( " )( " ):$P_{1.1}$ |
| 1 | $P_{2.1}$ | $s_1 s_2 P_{1.1}$ | ( " )( " ):$P_{2.1}$ |
| OTHER | | | |

let the third factor level set be partitioned into two subsets, one subset representing the low two levels and the other representing the high levels. The set of pseudo-design points is equivalent to a 3x3x2-FAT. Each s-FAT represents a $2^3$-FAT. For this example, confounding schemes in a 3x3x2-FAT and $2^3$-FAT lead to the blocking PLANs listed in the fourth column of Table 9. To obtain the blocking PLANs mentioned in Table 9, some of the components of the main effects and some of the components of the interaction effects of the 6x6x4-FAT must be confounded with block effects. Let those components of factor main effects and components of interaction effects attributed to between s-FAT effects be represented by A for factor one; B for factor two; C for the third factor; BC for the factor 2 - factor 3 interaction and $AB^1$ and $AB^2$ for the usual components of the factor 1 - factor 2 interaction effects. Table 10 indicates the confounding schemes that are needed to obtain the blocking PLANs mentioned in Table 9. Single letters in Table 10 indicate that components of factor main effects are confounded with block effects. Two or more letters indicate that a component of an interaction effect is confounded with block effects. For example, FDE in blocking PLAN (q) indicates that the 3-factor interaction effect among factors one, two and three in each s-FAT is confounded with block effects. The subsets of treatment combinations are given in Figure 10 for blocking PLANs (e) and (x), where the numbers in the boxes indicate which subset the treatment combination defined by the row and column indices is placed. For blocking PLAN (x) in Figure 10, the confounding of all effects in the 3x3x2-p-FAT with pseudo-block effects results in eighteen different

TABLE 9

BLOCKING PLANS FOR EXAMPLE 11

| Block size 3x3x2-p-FAT | Block size 6x6x4-FAT | Number of blocks | Blocking PLAN |
|---|---|---|---|
| 9 | 72 | 2 | (a) |
| 6 | 48 | 3 | (b),(c),(d),(e) |
| 3 | 24 | 6 | (f),(g) |
| 2 | 16 | 9 | (h) |
| 1 | 8 | 18 | (i) |
| 1 | 4 | 36 | (j),(k),(l),(m), (n),(p),(q) |
| 1 | 2 | 72 | (r),(s),(t),(u), (v),(w),(x) |

TABLE 10

CONFOUNDING SCHEMES USED IN EXAMPLE 11

| Blocking PLAN | Effects to confound with block effects in | |
|---|---|---|
| | 3x3x2-p-FAT | 2x2x2-s-FAT |
| (a) | C | none |
| (b) | A | none |
| (c) | B | none |
| (d) | $AB^1$ | none |
| (e) | $AB^2$ | none |
| (f) | A, C, AC | none |
| (g) | B, C, BC | none |
| (h) | A, B, $AB^1$, $AB^2$ | none |
| (i) | all | none |
| (j) | all | F |
| (k) | all | D |
| (l) | all | E |
| (m) | all | FD |
| (n) | all | FE |
| (p) | all | DE |
| (q) | all | FDE |
| (r) | all | F, D, FD |
| (s) | all | F, E, FE |
| (t) | all | D, E, DE |
| (u) | all | F, DE, FDE |
| (v) | all | D, FE, FDE |
| (w) | all | E, FD, FDE |
| (x) | all | FD, DE, FE |

Blocking PLAN (x)

3x3x2-p-FAT

| factor 2 | factor 3 | factor 1 = 0 | factor 1 = 1 | factor 1 = 2 |
|---|---|---|---|---|
| 0 | 0 | 1 | 5 | 9 |
| | 1 | 13 | 17 | 21 |
| 1 | 0 | 25 | 29 | 33 |
| | 1 | 37 | 41 | 45 |
| 2 | 0 | 49 | 53 | 57 |
| | 1 | 61 | 65 | 69 |

6x6x4-FAT

| factor 3 | factor 1 = 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 3 | 5 | 7 | 9 | 11 |
| | 2 | 4 | 6 | 8 | 10 | 12 |
| | 4 | 2 | 8 | 6 | 12 | 10 |
| | 3 | 1 | 7 | 5 | 11 | 9 |
| 1 | 13 | 15 | 17 | 19 | 21 | 23 |
| | 14 | 16 | 18 | 20 | 22 | 24 |
| | 16 | 14 | 20 | 18 | 20 | 22 |
| | 15 | 13 | 19 | 17 | 23 | 21 |
| 2 | 25 | 27 | 29 | 31 | 33 | 35 |
| | 26 | 28 | 30 | 32 | 34 | 36 |
| | 28 | 26 | 32 | 30 | 36 | 34 |
| | 27 | 25 | 31 | 29 | 35 | 33 |
| 3 | 37 | 39 | 41 | 43 | 45 | 47 |
| | 38 | 40 | 42 | 44 | 46 | 48 |
| | 40 | 38 | 44 | 42 | 48 | 46 |
| | 39 | 37 | 43 | 41 | 47 | 45 |
| 4 | 49 | 51 | 53 | 55 | 57 | 59 |
| | 50 | 52 | 54 | 56 | 58 | 60 |
| | 52 | 50 | 56 | 54 | 60 | 58 |
| | 51 | 49 | 55 | 53 | 59 | 57 |
| 5 | 61 | 63 | 65 | 67 | 69 | 71 |
| | 62 | 64 | 66 | 68 | 70 | 72 |
| | 64 | 62 | 68 | 66 | 72 | 70 |
| | 63 | 61 | 67 | 65 | 71 | 69 |

Blocking PLAN (e)

3x3x2-p-FAT

| factor 2 | factor 3 | factor 1 = 0 | factor 1 = 1 | factor 1 = 2 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| | 1 | 1 | 2 | 3 |
| 1 | 0 | 3 | 1 | 2 |
| | 1 | 3 | 1 | 2 |
| 2 | 0 | 2 | 3 | 1 |
| | 1 | 2 | 3 | 1 |

6x6x4-FAT

| factor 3 | factor 1 = 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 2 | 3 | 3 |
| | 1 | 1 | 2 | 2 | 3 | 3 |
| | 1 | 1 | 2 | 2 | 3 | 3 |
| | 1 | 1 | 2 | 2 | 3 | 3 |
| 1 | 1 | 1 | 2 | 2 | 3 | 3 |
| | 1 | 1 | 2 | 2 | 3 | 3 |
| | 1 | 1 | 2 | 2 | 3 | 3 |
| | 1 | 1 | 2 | 2 | 3 | 3 |
| 2 | 3 | 3 | 1 | 1 | 2 | 2 |
| | 3 | 3 | 1 | 1 | 2 | 2 |
| | 3 | 3 | 1 | 1 | 2 | 2 |
| | 3 | 3 | 1 | 1 | 2 | 2 |
| 3 | 3 | 3 | 1 | 1 | 2 | 2 |
| | 3 | 3 | 1 | 1 | 2 | 2 |
| | 3 | 3 | 1 | 1 | 2 | 2 |
| | 3 | 3 | 1 | 1 | 2 | 2 |
| 4 | 2 | 2 | 3 | 3 | 1 | 1 |
| | 2 | 2 | 3 | 3 | 1 | 1 |
| | 2 | 2 | 3 | 3 | 1 | 1 |
| | 2 | 2 | 3 | 3 | 1 | 1 |
| 5 | 2 | 2 | 3 | 3 | 1 | 1 |
| | 2 | 2 | 3 | 3 | 1 | 1 |
| | 2 | 2 | 3 | 3 | 1 | 1 |
| | 2 | 2 | 3 | 3 | 1 | 1 |

Figure 10. - Illustrations of the subsets of treatment combinations for blocking PLANs (x) and (e).

sets (one pseudo-design point per set), represented by the numbers 1, 5, 9, ... and 69. Next, within each of the eighteen s-FAT's the FD, DE and FE interaction effects are confounded with block effects. Thus, subset 1 in the 3x3x2-p-FAT is replaced by the four subsets 1, 2, 3 and 4 in the 6x6x4-FAT.

Let a $P_1P_2$-FAT be partitioned according to (where $s_1 \neq s_2$)

$$P_1P_2\text{-FAT} \longrightarrow (P_{11}+ \cdots +P_{is_i})(P_{21}+ \cdots +P_{2s_2})\text{-s-FAT's.}$$

For simplicity, let $h_{ij} = P_{1i}P_{2j}$, for $i\epsilon\{1, \ldots,s_1\}$ and $j\epsilon\{1,\ldots,s_2\}$ . The blocking PLANs that are obtained by confounding methods in the $s_1s_2$-p-FAT are given in table 11. Also, let

$$h_{\cdot j} = \sum_i h_{ij} \quad \text{and} \quad h_{i\cdot} = \sum_i h_{ij} \ .$$

In Table 8 there can be more than one PLAN that will give the indicated number of blocks for the partitioned factorial arrangement. The number of PLANs depends upon the numbers $s_1$ and $s_2$. For example, if $s_2 = (s_1)^k$, where k is some positive integer, then there are

$$\prod_{i=0}^{k-m} (( s_1^{k+1} - s_1)/(s_1^{k-m+1} - s_1^i))$$

confounding schemes for pseudo-block size $s_1^m$ that will lead to blocking PLANs requiring $s_1^{k-m+1}$ blocks. For the more general algebraic partition (7), the set of pseudo-design points represents a $\prod_{i=1}^{n} s_i$-p-FAT. The blocking procedures depend mainly upon the availability of confounding schemes for the p-FAT, which depends upon the numbers $s_1$, ... and $s_n$ and the sizes of the $\prod_{i=1}^{n} s_i$ s-FAT's. Consequently, there is no general statement that is made concerning blocking procedures for unequal partitioned FAT's. The concept of multiple partitioning, first mentioned by Thomas (42), is now briefly discussed.

TABLE 11

SOME BLOCKING PLANS FOR THE CASE $n=2$ and $s_1 \neq s_2$

| Block size $s_1 s_2$-p-FAT | Block size in $P_1 P_2$-FAT | Number of blocks in $P_1 P_2$-FAT |
|---|---|---|
| $s_1$ | $h_{.1}$, $h_{.2}$, ... and $h_{.s_2}$ | $s_2$ |
| $s_2$ | $h_{1.}$, $h_{2.}$, ... and $h_{s_1.}$ | $s_1$ |
| 1 | $h_{11}$, $h_{12}$, ... and $h_{s_1 s_2}$ | $s_1 s_2$ |
| OTHER | | |

## Multiple Partitioning

Consider the special case of the algebraic partitioning

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1} + \ldots + P_{is_i})\text{-s-FAT's}, \tag{16}$$

where $P_{i1} = \ldots = P_{is_i}$, for $i = 1, \ldots, n$. As mentioned earlier, this partitioning is denoted

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (s_i(P_{i.1}))\text{-s-FAT's}. \tag{17}$$

It might be the case that the s-FAT's furnished by the partitioning (17) are still prohibitive for some reason. The size of each s-FAT might be reduced by partitioning it algebraically and thus, it is replaced by two or more smaller s-FAT's. In the partitioning (17), if $P_{i.1} = s_{i.2}(P_{i.2})$, for $i = 1, \ldots, n$, then each s-FAT can be partitioned according to

$$\prod_{i=1}^{n} P_{i.1} \longrightarrow \prod_{i=1}^{n} (s_{i.2}(P_{i.2}))\text{-s-FAT's}, \tag{18}$$

since each s-FAT represents a factorial arrangement of size $\prod_{i=1}^{n} P_{i.1}$.

Definition 46 : The expression $P_i = s_{i.1}s_{i.2}\ldots s_{i.k}(P_{i.k})$ indicates that the i-th factor in a $\prod_{i=1}^{n} P_i$-FAT is equally partitioned k times.

Definition 47 : If each factor in a $\prod_{i=1}^{n} P_i$-FAT is equally partitioned k times, then the partitioning

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (s_{i.1}\ldots s_{i.k}(P_{i.k}))\text{-s-FAT's}, \tag{19}$$

is said to be an equal k-order multiple partitioning of the $\prod_{i=1}^{n} P_i$-FAT.

If $k = 1$, the partitioning is of the type previously mentioned and $s_{1.1}$ is written as $s_1$. For an equal k-order multiple partitioning (19) there are a total of $\prod_{i=1}^{n} (s_{i.1}\ldots s_{i.k})$ s-FAT's and each s-FAT consists of $\prod_{i=1}^{n} P_{i.k}$ treatment combinations. In the general situation, a $\prod_{i=1}^{n} P_i$-FAT may be multiply partitioned according to

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1} + \ldots + P_{is_i})\text{-s-FAT's}, \tag{20}$$

where one, some or all of the numbers $P_{ih}$ may be expressed as

$$P_{ih} = P_{ih.1} + \cdots + P_{ih.t_{ih}}.$$

If this is dome for each i and h, i = 1, ..., n and h = 1, ..., $s_i$, the partitioning (20) is a 2-order multiple partitioning and is denoted by

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} \Big(\sum_{k=1}^{s_i} (P_{ik.1} + \cdots + P_{ik.t_{ik}})\Big)\text{-s-FAT's.} \quad (21)$$

If $P_{ih.j} \neq P_{ih.k}$ for some h = 1, ..., $s_i$ and for some j, k 1,...,$t_{ih}$, $j \neq k$, then the expression

$$P_{ih} = P_{ih.1} + \cdots + P_{ih.t_{ih}}$$

indicates that the i-th factor is unequally partitioned two times, or just partitioned two times. The concept of an unequally k-partitioned factor is a direct generalization of the 2-order partitioning of a factor. A k-order multiple partitioning is a partitioning of a $\prod_{i=1}^{n} P_i$-FAT, where each factor level set is partitioned k times and at least one factor level set is unequally partitioned k times.

Remark 14 : For the 2-order multiple partitioning of a $\prod_{i=1}^{n} P_i$-FAT given by (21), there are a total of $\prod_{i=1}^{n} \big(\sum_{k=1}^{s_i} t_{ik}\big)$ s-FAT's.

Proof : The $P_i$ levels of the i-th factor are separated into $t_{i1} + \cdots + t_{is_i}$ sets of levels. Proceeding to treat the partitioning as if $s_i = t_{i1} + \cdots + t_{is_i}$ and following the proof of remark 1, the result is obtained.

A 2-order multiple partitioning of the type (21) for $2 = t_{1s_1} = t_{2s_2}$ and n = 2 is given by

$$P_1P_2\text{-FAT} \longrightarrow (s_1s_{1.2}(P_{1.2}))(s_2s_{2.2}(P_{2.2}))\text{-s-FAT's.} \quad (22)$$

This indicates that a $P_1P_2$-FAT was first partitioned according to

$$P_1P_2\text{-FAT} \longrightarrow (s_1(P_{1.1}))(s_2(P_{2.1}))\text{-s-FAT's.} \tag{23}$$

Then, since $P_{1.1} = s_{1.2}(P_{1.2})$ and $P_{2.1} = (s_{2.2}(P_{2.2}))$, and since each s-FAT in (23) represents a $P_{1.1}P_{2.1}$-FAT, each s-FAT is subjected to a partitioning of the type

$$P_{1.1}P_{2.1}\text{-s-FAT} \longrightarrow (s_{1.2}(P_{1.2}))(s_{2.2}(P_{2.2}))\text{-s-FAT's.} \tag{24}$$

Now, combining (23) and (24), (22) is obtained. For (22) the possible blocking PLANs are listed in Table 12. The first fourteen blocking PLANs in Table 12 are obtained by confounding main effects or interaction effects in the $s_1s_{1.2}s_2s_{2.2}$-p-FAT with pseudo-block effects. The fifteenth PLAN is obtained by confounding all effects in the p-FAT with pseudo-block effects and the remaining PLANs are obtained by confounding all effects in the p-FAT with pseudo-block effects and by confounding effects within each s-FAT with block effects.

It is not necessary that all factors be equally k-ordered, multiply partitioned. For example, consider a $P_1P_2$-FAT where the first factor of $P_1$ levels is 2-order partitioned and the second factor of $P_2$ levels is 1-order partitioned. This partitioning is given by

$$P_1P_2\text{-FAT} \longrightarrow (s_1s_{1.2}(P_{1.2}))(s_2(P_{2.1}))\text{-s-FAT's.} \tag{25}$$

The blocking PLANs for the partitioned FAT in (25) are given in Table 13. The first seven blocking PLANs of Table 13 are obtained by confounding schemes applied to the $s_1s_{1.2}s_2$-p-FAT and the remaining blocking PLANs are obtained by confounding all effects of the p-FAT with pseudo-block effects and by confounding effects within each s-FAT (the same effects for all s-FAT's) with block effects. A brief example of some of these concepts is now given.

TABLE 12

SOME BLOCKING PLANS FOR THE 2-ORDER MULTIPLE PARTITIONING (22)

| Block size $s_1 s_{1.2} s_2 s_{2.2}$-p-FAT | Block size $P_1 P_2$-FAT | Number of blocks in $P_1 P_2$-FAT |
|---|---|---|
| $s_1$ | $s_1 P_{1.2} P_{2.2}$ | $s_2 s_{1.2} s_{2.2}$ |
| $s_{1.2}$ | $s_{1.2} P_{1.2} P_{2.2}$ | $s_1 s_2 s_{2.2}$ |
| $s_2$ | $s_2 P_{1.2} P_{2.2}$ | $s_1 s_{1.2} s_{2.2}$ |
| $s_{2.2}$ | $s_{2.2} P_{1.2} P_{2.2}$ | $s_1 s_{1.2} s_2$ |
| $s_1 s_{1.2}$ | $s_1 s_{1.2} P_{1.2} P_{2.2}$ | $s_2 s_{2.2}$ |
| $s_1 s_2$ | $s_1 s_2 P_{1.2} P_{2.2}$ | $s_{1.2} s_{2.2}$ |
| $s_1 s_{2.2}$ | $s_1 s_{2.2} P_{1.2} P_{2.2}$ | $s_{1.2} s_2$ |
| $s_2 s_{1.2}$ | $s_2 s_{1.2} P_{1.2} P_{2.2}$ | $s_1 s_{2.2}$ |
| $s_2 s_{2.2}$ | $s_2 s_{2.2} P_{1.2} P_{2.2}$ | $s_1 s_{1.2}$ |
| $s_{1.2} s_{2.2}$ | $s_{1.2} s_{2.2} P_{1.2} P_{2.2}$ | $s_1 s_2$ |
| $s_1 s_{1.2} s_2$ | $s_1 s_{1.2} s_2 P_{1.2} P_{2.2}$ | $s_{2.2}$ |
| $s_1 s_{1.2} s_{2.2}$ | $s_1 s_{1.2} s_{2.2} P_{1.2} P_{2.2}$ | $s_2$ |
| $s_{1.2} s_2 s_{2.2}$ | $s_{1.2} s_2 s_{2.2} P_{1.2} P_{2.2}$ | $s_1$ |
| $s_1 s_2 s_{2.2}$ | $s_1 s_2 s_{2.2} P_{1.2} P_{2.2}$ | $s_{1.2}$ |
| 1 | $P_{1.2} P_{2.2}$ | $s_1 s_{1.2} s_2 s_{2.2}$ |
| 1 | $P_{1.2}$ | $s_1 s_{1.2} s_2 s_{2.2} P_{1.2}$ |
| 1 | $P_{2.2}$ | $s_1 s_{1.2} s_2 s_{2.2} P_{1.2}$ |
| OTHER | | |

TABLE 13

SOME BLOCKING PLANS FOR THE PARTITIONING (25)

| Block size $s_1 s_{1.2} s_2$-p-FAT | Block size $P_1 P_2$-FAT | No. of Blocks $P_1 P_2$-FAT |
|---|---|---|
| $s_1$ | $s_1 P_{1.2} P_{2.1}$ | $s_{1.2} s_2$ |
| $s_{1.2}$ | $s_{1.2} P_{1.2} P_{2.1}$ | $s_1 s_2$ |
| $s_2$ | $s_2 P_{1.2} P_{2.1}$ | $s_1 s_{1.2}$ |
| $s_1 s_{1.2}$ | $s_1 s_{1.2} P_{1.2} P_{2.1}$ | $s_2$ |
| $s_1 s_2$ | $s_1 s_2 P_{1.2} P_{2.1}$ | $s_{1.2}$ |
| $s_{1.2} s_2$ | $s_{1.2} s_2 P_{1.2} P_{2.1}$ | $s_1$ |
| 1 | $P_{1.2} P_{2.1}$ | $s_1 s_{1.2} s_2$ |
| 1 | $P_{1.2}$ | $s_1 s_{1.2} s_2 P_{2.1}$ |
| 1 | $P_{2.1}$ | $s_1 s_{1.2} s_2 P_{1.2}$ |

Example 12 : Consider the algebraic partitioning

$$8_1 6_2\text{-FAT} \longrightarrow (2_1(4_{1.1}))(2_2(3_{2.1}))\text{-s-FAT's.}$$

This partitioning yields four s-FAT's of size twelve and will admit blocking PLANs with block sizes 3, 4, 12, 24 and 48. Since $4 = 2x2$, the number $4_{1.1}$ can be represented by $4_{1.1} = 2_{1.2}(_{1.2})$. Thus, the partitioning mentioned above becomes the partitioning

$$8_1 6_2\text{-FAT} \longrightarrow (2_1 2_{1.2}(2_{1.2}))(2_2(3_{2.1}))\text{-s-FAT's.}$$

This partitioning now admits blocking PLANs with block sizes 2, 3, 6, 12 and 24.

It is not necessary that all factors in a $\prod_{i=1}^{n} P_i$-FAT be multiply partitioned. If the level sets of some (not necessarily all) factors are multiply partitioned, then the entire partitioning is referred to simply as a multiple partition. For example, an unequal 2-order partition on factor one and a 1-order partition on factor two is indicated by the multiple partition

$$P_1 P_2\text{-FAT} \longrightarrow ((P_{11.1}+P_{11.2})+(P_{12.1}+P_{12.2}))(P_{21}+P_{22})\text{-s-FAT's.}$$

Letting $h_{ijk} = P_{1i.j}P_{2k}$ for $i,j,k = 1,2$, it is seen that $S_D$ corresponds to a 4x2 FAT. The blocking PLANs obtained by confounding procedures in the 4x2-p-FAT are given in Table 14. In this case $s_1 = s_2 = s_{1.2} = 2$ and $s_1 s_{1.2} = s_1 s_2 = s_{1.2} s_2 = 4$. The methods to obtain the blocking PLANs in Table 14 are indicated by the arrangements of x's is the last four columns of Table 14. The x's indicate that effects in the 4x2-p-FAT are confounded with pseudo-blocks. An x in column one indicates that the entire factor one main effect of the 4x2-p-FAT is confounded with pseudo-block effects; an x in column two indicates that a component of the factor one main effect of the 4x2-p-FAT is confounded with pseudo-block

TABLE 14

SOME BLOCKING PLANS FOR AN UNEQUAL MULTIPLE PARTITION

| Block size 4x2-p-FAT | Block size in $P_1P_2$-FAT | Number blocks | Methods 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| 4 | $(h_{111}+h_{121}+h_{112}+h_{122}),(h_{211}+h_{221}+h_{212}+h_{222})$ | 2 | | | x | |
| 4 | $(h_{111}+h_{112}+h_{211}+h_{212}),(h_{121}+h_{122}+h_{221}+h_{222})$ | 2 | | | x | |
| 4 | $(h_{111}+h_{121}+h_{221}+h_{211}),(h_{222}+h_{212}+h_{122}+h_{112})$ | 2 | | x | | |
| 4 | $(h_{111}+h_{112}+h_{221}+h_{222}),(h_{121}+h_{122}+h_{211}+h_{212})$ | 2 | | | x | |
| 4 | $(h_{111}+h_{121}+h_{212}+h_{222}),(h_{122}+h_{112}+h_{211}+h_{221})$ | 2 | | | | x |
| 4 | $(h_{111}+h_{122}+h_{211}+h_{222}),(h_{112}+h_{121}+h_{212}+h_{221})$ | 2 | | | | x |
| 4 | $(h_{111}+h_{122}+h_{212}+h_{221}),(h_{112}+h_{211}+h_{121}+h_{222})$ | 2 | | | | x |
| 2 | $(h_{111}+h_{112}),(h_{121}+h_{122}),(h_{211}+h_{212})$ and $(h_{221}+h_{222})$ | 2 | x | | | |
| 2 | $(h_{111}+h_{121}),(h_{112}+h_{122}),(h_{212}+h_{222})$ and $(h_{211}+h_{221})$ | 4 | | x | x | x |
| 2 | $(h_{111}+h_{211}),(h_{121}+h_{221}),(h_{112}+h_{212})$ and $(h_{122}+h_{222})$ | 4 | | x | x | x |
| 2 | $(h_{111}+h_{122}),(h_{112}+h_{121}),(h_{221}+h_{222})$ and $(h_{212}+h_{221})$ | 4 | | | x | x |
| 2 | $(h_{111}+h_{212}),(h_{112}+h_{211}),(h_{121}+h_{222})$ and $(h_{122}+h_{211})$ | 4 | | | x | x |
| 2 | $(h_{111}+h_{221}),(h_{112}+h_{222}),(h_{121}+h_{221})$ and $(h_{212}+h_{122})$ | 4 | | x | x | x |
| 2 | $(h_{111}+h_{222}),(h_{112}+h_{221}),(h_{121}+h_{212})$ and $(h_{122}+h_{211})$ | 4 | | | x | x |
| 1 | $h_{111}, h_{112}, h_{121}, h_{122}, h_{211}, h_{212}, h_{221}$ and $h_{222}$ | 8 | | x | x | x |

effects; an x in column three indicates that a component of the factor two main effect is confounded with pseudo-block effects and an x in column four indicates that a component of the factor one - factor two interaction effect is confounded with pseudo-block effects. An example is now given of an unequal multiple partition of a FAT so that the blocking PLANs the partitioning admits will be analogous to those presented in Table 14.

Example 13 : First consider the partitioning of a 12x5-FAT given by $12_1 5_2$-FAT $\longrightarrow (7_{11}+5_{12})(2_{21}+3_{22})$-s-FAT's. This partitioning results in four s-FAT's of sizes 10, 14 15 and 21. The blocking PLANs admitted by this partitioning by confounding effects in the $2^2$-p-FAT will have block sizes (i) 25 and 35, (ii) 24 and 36, (iii) 31 and 29 and (iv) 10, 14, 15 and 21. A second order partition can be performed on the first factor by the following

$$(7_{11} + 5_{12}) \longrightarrow ((4_{11.1} + 3_{11.2}) + (2_{12.1} + 3_{12.2})).$$

The partitioning is now represented by

$$12_1 5_2\text{-FAT} \longrightarrow ((4_{11.1}+3_{11.2})+(2_{12.2}+3_{12.2}))(2_{21}+3_{22})\text{-s-FAT's.}$$

From this partitioning there result eight s-FAT's of sizes 4, 6, 6, 6, 8, 9, 9 and 12. This partitioning admits blocking PLANs with block sizes

| | | | |
|---|---|---|---|
| (i) | 24 and 36 | (vi) | 14, 10, 21 and 15 |
| (ii) | 35 and 25 | (vii) | 12, 12, 18 and 18 |
| (iii) | 30 and 30 | (viii) | 20, 10, 15 and 15 |
| (iv) | 29 and 31 | (ix) | 17, 18, 13 and 18 |
| (v) | 14, 15, 16 and 15 | (x) | 4, 6, 6, 6, 8, 9, 9 and 12. |

# CHAPTER V

## STATISTICAL INFERENCE

It is possible to obtain observations from one or more of the s-FAT's of an algebraic partitioning

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1} + \ldots + P_{is_i})\text{-s-FAT's.}$$

Given a set of observations the problems of statistical inference, namely estimation and significance testing, are now discussed. In this chapter methods are given for unbiased estimation of certain functions of the unknown parameters and methods are developed for constructing tables appropriate for the analysis of variance as a means for significance testing under normal theory and approximate significance testing under randomization theory.

### Brief Results for the General Case

In the following discussion let $\underline{Y}$ be an m by one vector of observations, $\underline{M}$ an m by one vector of population means and let $\underline{e}$ be an m by one vector of error terms such that $E(\underline{e}) = \underline{\phi}_1^m$ and $\text{var}(\underline{e}\ \underline{e}') = \sigma^2 I_m$. It shall also be assumed that the observational model, $\underline{Y} = h(\underline{M},\underline{e})$, is a linear model, in other words, $\underline{Y} = \underline{M} + \underline{e}$. Let $\underline{b}_k$ be an m by one vector of constants, for k = 1, ...,m, such that, for $k \neq k'$

$$\underline{b}_{k'}'\underline{b}_k = 0 \tag{26}$$

The choice of $\underline{b}_k$, for k = 1, ...,m, can be made so they define a set of

effects among the elements of $\underline{M}$ that correspond to components of factor main effects and factor interaction effects.

Theorem 1 : If $\underline{b}_k'\underline{M}$ is the effect of interest, then $\underline{b}_k'\underline{Y}$ is an unbiased estimate of that effect and $\text{var}(\underline{b}_k'\underline{Y}) = \sigma^2\underline{b}_k'\underline{b}_k$.

Proof: (1) $E(\underline{b}_k'\underline{Y}) = \underline{b}_k'E(\underline{Y}) = \underline{b}_k'\underline{M}$ and

(2) $\text{var}(\underline{b}_k'\underline{Y}) = \underline{b}_k'\text{var}(\underline{Y})\underline{b} = \underline{b}_k'\text{var}(\underline{e})\underline{b} = \underline{b}_k'(\sigma^2 I_m)\underline{b} =$

$= \sigma^2\underline{b}_k'\underline{b}$.

In general, $\text{cov}(\underline{b}_k'\underline{Y}, \underline{b}_{k'}'\underline{Y}) = \underline{b}_k'\text{cov}(\underline{Y},\underline{Y})\underline{b}_{k'} = \sigma^2\underline{b}_k'\underline{b}_{k'}$. If $\underline{b}_k'\underline{b}_{k'} = 0$, then the estimates $\underline{b}_k'\underline{Y}$ and $\underline{b}_{k'}'\underline{Y}$ are uncorrelated and if $\underline{e}$ is normally distributed, the estimates are independently (and normally) distributed. Now assume that $\underline{e}$ is normally distributed, or $\underline{e} \approx N(\underline{\emptyset}_1^m, \sigma^2 I_m)$. It is easily shown that $\underline{Y} \approx N(\underline{M}, \sigma^2 I_m)$ and $\underline{b}_k'\underline{Y} \approx N(\underline{b}_k'\underline{M}, \sigma^2\underline{b}_k'\underline{b}_k)$.

Theorem 2 : If $\underline{b}_k'\underline{Y} \approx N(\underline{b}_k'\underline{M}, \sigma^2\underline{b}_k'\underline{b}_k)$ then

$$\frac{(\underline{b}_k'\underline{Y})^2}{(\sigma^2\underline{b}_k'\underline{b}_k)} \approx \chi'^2\left(1,\ q = \tfrac{1}{2}\frac{(\underline{b}_k'\underline{M})'(\underline{b}_k'\underline{M})}{(\sigma^2\underline{b}_k'\underline{b}_k)}\right)$$

Proof: Since $\underline{b}_k'\underline{Y} \approx N(\underline{b}_k'\underline{M}, \sigma^2\underline{b}_k'\underline{b}_k)$ it is also known that

$$\frac{1}{\sqrt{\sigma^2\underline{b}_k'\underline{b}_k}}\underline{b}_k'\underline{Y} \approx N\left((1/(\sqrt{\sigma^2\underline{b}_k'\underline{b}_k}))\underline{b}_k'\underline{M},\ I_1 = 1\right).$$

Using theorem 4.1 in Graybill (29) and noting that $\underline{b}_k'\underline{M}$ and $\underline{b}_k'\underline{Y}$ are scalars, it follows that

$$\frac{(\underline{b}_k'\underline{Y})^2}{(\sigma^2\underline{b}_k'\underline{b}_k)} \approx \chi'^2\left(1,\ q = \tfrac{1}{2}\frac{(\underline{b}_k'\underline{M})^2}{(\sigma^2\underline{b}_k'\underline{b}_k)}\right), \qquad (27)$$

where q is the noncentrality parameter.

The statistic of theorem 2 can be expressed the following way:

$$\frac{(\underline{b}_k'\underline{Y})^2}{(\sigma^2\underline{b}_k'\underline{b}_k)} = \frac{(\underline{b}_k'\underline{Y})'(\underline{b}_k'\underline{Y})}{(\sigma^2\underline{b}_k'\underline{b}_k)} = \frac{\underline{Y}'\underline{b}_k\underline{b}_k'\underline{Y}}{(\sigma^2\underline{b}_k'\underline{b}_k)} = \underline{Y}'B_k\underline{Y}\left(\frac{1}{\sigma^2}\right) \text{ where}$$

$B_k = (1/(\underline{b}_k'\underline{b}_k))(\underline{b}_k\underline{b}_k')$. Thus, $(1/(\sigma^2))\underline{Y}'B_k\underline{Y} \approx \chi'^2(1, q = \frac{1}{2\sigma^2}(\underline{b}_k'\underline{M})'(\underline{b}_k'\underline{M}))$ and the term $\underline{Y}'B_k\underline{Y}$ shall be referred to as the sum of squares corresponding to the effect $\underline{b}_k'\underline{M}$.

Theorem 3 : If $\underline{b}_k'\underline{b}_{k'} = 0$ for all $k \neq k'$ and $k,k' = 1, \ldots, m$, then $\underline{Y}'\underline{Y} = \underline{Y}'(B_1 + \ldots + B_m)\underline{Y}$, where $B_k = (1/(\underline{b}_k'\underline{b}_k))(\underline{b}_k\underline{b}_k')$.

Proof: $\underline{Y}'\underline{Y} = \underline{Y}'I_m\underline{Y}$, so it remains to be shown that $\sum_{i=1}^{m} B_i = I_m$.

Since $\underline{b}_k'\underline{b}_k = 0$ if $k \neq k'$, let R be a m by m matrix where the i-th row of R is given by $\underline{R}' = (1/\sqrt{\underline{b}_i'\underline{b}_i})\underline{b}_i'$. Now, for $i \neq j$, it follows that $\underline{R}_i'\underline{R}_j = 0$, $\underline{R}_i'\underline{R}_i = 1$ and that R is an orthogonal matrix. Thus,

$$R'R = I_m = (\underline{R}_1, \ldots, \underline{R}_m)\begin{bmatrix}\underline{R}_1' \\ \vdots \\ \underline{R}_m'\end{bmatrix}$$

$$= \sum_{i=1}^{m} \underline{R}_i\underline{R}_i' = \sum_{i=1}^{m} B_i.$$

In the analysis of variance tables the total sum of squares can always be represented by $\underline{Y}'\underline{Y}$, and the usual correction factor can be represented by $\underline{Y}'(\frac{1}{m}J_m^m)\underline{Y}$. Thus, the total corrected sum of squares is $\underline{Y}'(I_m - \frac{1}{m}J_m^m)\underline{Y}$. Let $\underline{Y}$ be a vector of $m = \prod_{i=1}^{n} P_i$ observations from a completely random design where the treatment design is a $\prod_{i=1}^{n} P_i$-FAT. In chapter III the matrices $L_1$, $L_{F_1}$, ..., $L_{F_n}$, ..., and $L_{F_1\ldots F_n}$ and the matrices $B_1$, $B_{F_1}$, ... and $B_{F_1\ldots F_n}$ were defined for factor main effects and factor interaction effects. In this case $B_1 = \frac{1}{m}J_m^m$ and $I_m = B_1 + B_{F_1} + \ldots + B_{F_1\ldots F_n}$. Let $d_\theta$ be the rank of $B_\theta$.

Theorem 4 : If $\underline{Y} \approx N(\underline{M}, \sigma^2 I_m)$ then, for $\theta \neq \theta'$ and $\theta,\theta' = 1, F_1, \ldots, F_n, \ldots, (F_1\ldots F_n)$

(i) $\frac{1}{\sigma^2}\underline{Y}'B_\theta\underline{Y} \approx \chi'^2(d_\theta, \frac{1}{2\sigma^2}(\underline{M})'B_\theta(\underline{M}))$ and

(ii) $\underline{Y}'B_\theta\underline{Y}$ and $\underline{Y}'B_{\theta'}\underline{Y}$ are independent.

Proof: $B_\theta B_\theta = H'_\theta H_\theta H'_\theta H_\theta = H'_\theta (H_\theta H'_\theta) H_\theta = H'_\theta (I_{d_\theta}) H_\theta = B_\theta$ , so $B_\theta$ is an idempotent matrix, and by using corollary 4.7.1 in Graybill (29), the result (i) is easily verified. Also, since $B_\theta B_{\theta'} = \phi_c^a$, where $a = d_\theta$ and $c = d_{\theta'}$, and by using theorem 4.15 in Graybill (29) the result (ii) is easily verified.

The term $\underline{Y}'B_\theta\underline{Y}$ is referred to as the sum of squares corresponding to the effect $L'_\theta\underline{M}$.

Remark 15 : Plackett (37), (see also Addelman (2)), has shown that for a k-way classification the main effects and interaction effects are orthogonal if and only if the following condition holds:

$$n_{i_1 \cdots i_k} = \frac{1}{N^{k-1}} \prod_{j=i_1}^{i_k} n(j) \text{ , where}$$

$$n(j) = \sum_{i_1=1}^{P_1} \cdots \sum_{i_k=1}^{P_k} n_{i_1 \cdots i_k} ,$$

$$N = \sum_{i_1} \cdots \sum_{i_k} n_{i_1} \cdots n_{i_k} ,$$

$P_{i_j}$ is the number of levels for the $i_j$-th factor and $n_{i_1 \cdots i_k}$ is the number of observations for the $(i_1,\ldots,i_k)$-cell in the k-way classification.

If the experiment consists of one or more full replicates of a $\prod_{i=1}^{n} P_i$-FAT (run in a completely random design) it is easy to see that, in view of remark 15, there exist orthogonal main effects and interaction effects.

If the experimental design is a block design then the existence of orthogonality of main and interaction effects must be investigated. For a partitioned $\prod_{i=1}^{n} P_i$-FAT where the blocking PLAN indicates that the s-FAT's are assigned to blocks, the question immediately presents itself

as to whether the between block comparisons can be attributed to orthogonal components of factor main and interaction effects. Let the s-FAT's of a partitioning of a $\prod_{i=1}^{n} P_i$-FAT (or groups of s-FAT's) be assigned to blocks by an appropriate blocking procedure. Consider each pseudo-design point in $S_D$ as being replicated by the number of treatment combinations it represents in the $\prod_{i=1}^{n} P_i$-FAT. Thus,

$$n_{i_1 \ldots i_j \ldots i_n} = P_{1i_1} \ldots P_{ji_j} \ldots P_{ni_n} \quad \text{for } i_j \in \{0, \ldots, s_j - 1\}$$

$$n_{\bullet, \ldots, i_j, \ldots, \bullet} = n(i_j) = \Big(\prod_{\substack{i=1 \\ i \neq j}}^{n} P_i\Big) P_{ji_j} \quad \text{and}$$

$$N = n_{\bullet, \ldots, \bullet} = \prod_{i=1}^{n} P_i \ .$$

Now, $\frac{1}{N^{n-1}} \Big( \prod_{k=i_1}^{i_n} n(k) \Big) = \frac{1}{(P_1 \ldots P_n)^{n-1}} (\theta)$ where

$$\theta = (P_2 \ldots P_n) P_{1i_1} \ \ldots \ (P_1 \ldots P_{n-1}) P_{ni_n}, \text{ and}$$

after some manipulation,

$$\frac{1}{N^{n-1}} \Big( \prod_{k=1}^{n} n(k) \Big) = n_{i_1 \ldots i_n} \ .$$

This indicates that main effects and interaction effects can be defined in the $\prod_{i=1}^{n} s_i$-p-FAT that retain the orthogonality properties, regardless of the size of the s-FAT's, consequently, regardless of the equality of block sizes. If the blocks are of equal size it is possible to confound the s-FAT's with blocks in such a way that only interaction effects are confounded with block effects. If the blocks are of unequal size, then in most cases main effects and interaction effects will have to be confounded with block effects and the number of confounding schemes is limited (as the number of confounding schemes for asymmetrical factorials is limited) as was observed in chapter IV.

For significance testing the highest order interaction can be assumed to be zero (i.e., assume $L_{F_1 \ldots F_n}\underline{M} = \underline{\emptyset}_1^d$, where $d = (P_1-1)\ldots(P_n-1)$) to obtain an error term in the AOV table. Now, letting $\theta = F_1 \ldots F_n$,

$$E(\underline{Y}'B_\theta\underline{Y}) = 2\underline{M}'B_\theta\underline{M} + tr(B_\theta(\sigma^2 I_m)) = 0 + \sigma^2 \text{rank}(B_\theta) = d_\theta\sigma^2, \text{ since}$$

$B_\theta$ is idempotent. Thus,

$$E(\frac{1}{d_\theta}\underline{Y}'B_\theta\underline{Y}) = \sigma^2.$$

Now, consider some other factor main effect or interaction effect $L_\lambda\underline{M}$. The null hypothesis that $L_\lambda\underline{M} = \underline{\emptyset}_1^{d_\lambda} = \underline{\emptyset}$ is equivalent to the hypothesis that $\underline{M}'B_\lambda\underline{M} = \underline{\emptyset}$ since,

$$\begin{aligned} \text{if } L_\lambda\underline{M} = \underline{\emptyset} \text{ then } D_\lambda L_\lambda\underline{M} &= \underline{\emptyset} \\ H_\lambda\underline{M} &= \underline{\emptyset} \\ (\underline{M}'H_\lambda)(H_\lambda\underline{M}) &= \underline{\emptyset} \\ \underline{M}'B_\lambda\underline{M} &= \underline{\emptyset}\ , \end{aligned}$$

and, consequently, the statistic

$$\frac{d_\theta\underline{Y}'B_\lambda\underline{Y}}{d_\lambda\underline{Y}'B_\theta\underline{Y}} \approx F(\,d_\lambda, d_\theta\,)$$

provides a means for siginficance testing, for $\lambda = F_1,\ldots,F_n,\ldots,(F_2\ldots F_n)$. By the remark in appendix II, if $L_w$ defines an effect $L_w\underline{M}$ and if $L_a$ can be expressed as $L_a = GL_w$, then $\underline{Y}'B_w\underline{Y} = \underline{Y}'B_a\underline{Y}$ and the hypothesis that $\underline{M}'B_w\underline{M} = \underline{\emptyset}$ is equivalent to the hypothesis that $\underline{M}'B_a\underline{M} = \underline{\emptyset}$. A statistic that provides a significance test for $H_o$: $\underline{M}'B_w\underline{M} = \underline{\emptyset}$ also provides a signficance test for the hypothesis $H_o$: $\underline{M}'B_a\underline{M} = \underline{\emptyset}$. Also, if the conditions in the remark of appendix II are satisfied, it makes no difference if either $L_w$ or $L_a$ was chosen to define the effect because the sum of squares corresponding to $L_w\underline{M}$ and $L_a\underline{M}$ are equal. In the sequel, sums of squares will be computed by the easiest method.

## Analysis of Variance for Full Replicates

Methods are now developed that will allow the construction of analysis of variance tables for full treatment combination replicates of partitioned FAT's in the absence of block effects. This case reduces to the analysis of unpartitioned factorial arrangements, since there is no importance associated with the fact that the full treatment combination replicate was performed in pieces, or in s-FAT's.

The set of $\prod_{i=1}^{n} s_i$-s-FAT's resulting from the partitioning

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1} + \ldots + P_{is_i})\text{-s-FAT's}$$

is considered as a full replicate of the $\prod_{i=1}^{n} P_i$-FAT. In this case,

$$n_{i_1 \ldots i_n} = 1$$

$$n(k) = \prod_{i \neq k} P_i$$

$$n = \prod_{i=1}^{n} P_i \quad \text{and}$$

$$(1/N^{n-1})\left(\prod_{k=i_1}^{i_n} n(k)\right) = 1 = n_{i_1 \ldots i_n},$$

so, when n is any positive integer, by remark 15, there exists orthogonality of main effects and interaction effects. The matrix L of chapter III defines factor main effects and factor interaction effects and the overall mean effect for a full replicate of a $\prod_{i=1}^{n} P_i$-FAT. Since $E(L\underline{Y}) = L\underline{M}$, $L\underline{Y}$ is an unbiased estimate of the effect $L\underline{M}$, and $\text{var}(L\underline{Y}) = \sigma^2 LL'$. Given a vector $\underline{Y}$ of observations of the $\prod_{i=1}^{n} P_i$-FAT (run in a completely random design) and since

$$I_m = B_1 + B_{F_1} + \ldots + B_{F_n} + B_{F_1F_2} + \ldots + B_{F_nF_{n-1}} + \ldots + B_{F_1 \ldots F_n}$$

the following abbreviated analysis of variance (first three columns) table is easily constructed (see Table 15).

TABLE 15

ABBREVIATED ANALYSIS OF VARIANCE TABLE FOR A FULL REPLICATE OF A $P_1 \ldots P_n$-FAT

| Source | DF | Sum of Squares |
|---|---|---|
| Total | $m = \prod P_i$ | $\underline{Y}'\underline{Y}$ |
| (overall)mean | 1 | $\underline{Y}'B_1\underline{Y}$ |
| $A_1$ (factor 1 main effect) | $(P_1-1)$ | $\underline{Y}'B_{F_1}\underline{Y}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $A_n$ (factor n main effect) | $(P_n-1)$ | $\underline{Y}'B_{F_n}\underline{Y}$ |
| $A_1xA_2$ | $(P_1-1)(P_2-1)$ | $\underline{Y}'B_{F_1F_2}\underline{Y}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $A_nxA_{n-1}$ | $(P_n-1)(P_{n-1}-1)$ | $\underline{Y}'B_{F_nF_{n-1}}\underline{Y}$ |
| $A_1xA_2xA_3$ | $(P_1-1)(P_2-1)(P_3-1)$ | $\underline{Y}'B_{F_1F_2F_3}\underline{Y}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $A_{n-2}xA_{n-1}xA_n$ | $(P_n-1)(P_{n-1}-1)(P_{n-2}-1)$ | $\underline{Y}'B_{F_nF_{n-1}F_{n-2}}\underline{Y}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $A_1x \ldots xA_{n-1}$ | $(P_1-1)\ldots(P_{n-1}-1)$ | $\underline{Y}'B_{F_1\ldots F_{n-1}}\underline{Y}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $A_2x \ldots xA_n$ | $(P_2-1)\ldots(P_n-1)$ | $\underline{Y}'B_{F_2\ldots F_n}\underline{Y}$ |
| $A_1x \ldots xA_n$ | $(P_1-1)\ldots(P_n-1)$ | $\underline{Y}'B_{F_1\ldots F_n}\underline{Y}$ |

Example 14 : For $n = 3$ and $P_1 = P_2 = P_3 = 2$ the sets of levels for each of the three factors are $T_1 = T_2 = T_3 = \{0,1\}$ and it follows that $T = T_1 X\, T_2 X\, T_3 = \{(000),(001),(010),(011),(100),(101),(110),(111)\}$ . Each element of T designates a treatment combination and specifically, a 0 denotes the low level of the first, second or third factor and a 1 denotes the high level of the first, second or third factor. To build the matrix L the first row is chosen to be $J_8^1$ and the next three rows are chosen so they represent main effects for the first, second and third factors. The last four rows are obtained by taking the appropriate HD of rows 2 and 3, 2 and 4, 3 and 4 and 2, 3 and 4. The matrix L is given in Figure 11, where the columns correspond to the design points (000), (001), ... and (111).

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_{F_1} \\ L_{F_2} \\ L_{F_3} \\ L_{F_1F_2} \\ L_{F_1F_3} \\ L_{F_2F_3} \\ L_{F_1F_2F_3} \end{bmatrix} = L$$

Figure 11. - The matrix for example 14.

To obtain H from L the rows of L must be normalized. In this case the problem of normalization is easy because each row contains a plus one or minus one in each position, so, if $l_k$ is a row of L, then $(1/\sqrt{8})l_k$ is the normalized row $(=h_k)$. Thus, $H = (1/\sqrt{8})L$. The matrices $B_1 = H_1' H_1$

$B_{F_1} = H'_{F_1} H_{F_1}$, ..., $B_{F_3} = H'_{F_3} H_{F_3}$, ... and $B_{F_1F_2F_3} = H'_{F_1F_2F_3} H_{F_1F_2F_3}$ are constricted. If this is done, the following matrices are obtained.

$$B_1 = (1/8)J_8^8 \qquad B_{F_1} = (1/8)\begin{bmatrix} J_4^4 & -J_4^4 \\ -J_4^4 & J_4^4 \end{bmatrix}$$

$$B_{F_2} = \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix} \qquad B_{F_3} = \frac{1}{8}\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

$$B_{F_1F_2} = \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix} \qquad B_{F_1F_3} = \frac{1}{8}\begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

$$B_{F_2F_3} = \frac{1}{8}\begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix} \qquad B_{F_1F_2F_3} = \frac{1}{8}\begin{bmatrix} 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

The first three columns of an analysis of variance table are presented in Table 16.

For the partitioning

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1} + \ldots + P_{is_i})\text{-s-FAT's}$$

the i-th factor main effect consists of $(P_i - 1)$ components, for $i = 1, \ldots, n$.

To facilitate analysis procedures a set of orthogonal components for each

TABLE 16

ABBREVIATED ANALYSIS OF VARINACE TABLE FCR A
FULL REPLICATE OF A 2x2x2-FAT

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | 7 | $\underline{Y}'(I_8-\frac{1}{8}J_8^8)\underline{Y} = Y_{ijk}^2 - (1/8)(Y...)^2$ |
| $A_1$ | 1 | $\underline{Y}'B_{F_1}\underline{Y} = (1/8)(Y_{1..}-Y_{0..})^2$ |
| $A_2$ | 1 | $\underline{Y}'B_{F_2}\underline{Y} = (1/8)(Y_{.1.}-Y_{.0.})^2$ |
| $A_3$ | 1 | $\underline{Y}'B_{F_3}\underline{Y} = (1/8)(Y_{..1}-Y_{..0})^2$ |
| $A_1xA_2$ | 1 | $\underline{Y}'B_{F_1F_2}\underline{Y} = (1/8)(Y_{00.}-Y_{01.}-Y_{10.}+Y_{11.})^2$ |
| $A_2xA_3$ | 1 | $\underline{Y}'B_{F_2F_3}\underline{Y} = (1/8)(Y_{.00}-Y_{.01}-Y_{.10}+Y_{.11})^2$ |
| $A_1xA_3$ | 1 | $\underline{Y}'B_{F\ F}\underline{Y} = (1/8)(Y_{0.0}-Y_{0.1}-Y_{1.0}+Y_{1.1})^2$ |
| $A_1xA_2xA_3$ | 1 | $\underline{Y}'B_{F_1F_2F_3}\underline{Y} = (1/8)(Y_{000}-Y_{001}-Y_{010}+Y_{011}-Y_{100}+Y_{110}+Y_{101}-Y_{111})^2$ |

factor main effect are chosen the following way. First, choose a set of orthogonal components of the factor main effect in the $\prod_{i=1}^{n} P_i$-FAT that represents a main effect in the $\prod_{i=1}^{n} s_i$-p-FAT. For the i-th factor, for i = 1, ...,n, there will be $(s_i-1)$ components of the i-th factor main effect that represent effects defined between the p-levels of the i-th factor (ie: the main effect in the $\prod_{i=1}^{n} s_i$-p-FAT). Now, within the k-th p-level of the i-th factor, there are $P_{ik}$ levels and consequently, $(P_{ik}-1)$ components of the factor i main effect can be defined, for i = 1, ...,n and k = 1, ...,$s_i$. Thus, a total of $(s_i-1) + \sum_{j=1}^{s_i} (P_{ij}-1) = (P_i-1)$ components of the factor i main effect have been defined, and this set, if it is an orthogonal set, is a main effect for factor i.

The source of variation in an analysis of variance table due to the factor i main effect has been denoted by $A_i$, for i = 1, ...,n. Now, in view of the partitioning (28), $A_i$ can be separated into a between p-level source of variation, denoted by $A_{i.1}$, and a within p-level source of variation for each p-level, denoted by $A_{i.(1,k)}$ for the k-th p-level of factor i. Since there are $s_i$ p-levels for factor i, the sources of variation $A_{i.(1,1)}$, ... and $A_{i.(1,s_i)}$ will be combined into one quantity representing those components of the factor i main effect attributable to the within p-level effects and it will be denoted by $A_{i.(1,.)}$. The matrices $L_{F_i}$, $H_{F_i}$ and $B_{F_i}$ can be expressed as follows

$$L_{F_i} = \begin{bmatrix} L_{F_i.1} \\ L_{F_i.(1,.)} \end{bmatrix} \quad H_{F_i} = \begin{bmatrix} H_{F_i.1} \\ H_{F_i.(1,.)} \end{bmatrix}$$

$$\text{and} \quad B_{F_i} = B_{F_i.1} + B_{F_i.(1,.)} . \tag{29}$$

The sum of squares corresponding to the i-th factor main effect, $\underline{Y}' B_{F_i} \underline{Y}$,

can be expressed as follows

$$\underline{Y}'B_{F_i}\underline{Y} = \underline{Y}'B_{F_{i.1}}\underline{Y} + \underline{Y}'B_{F_{i.(1,.)}}\underline{Y}. \tag{30}$$

As a matter of computational convenience it is suggested that $\underline{Y}'B_{F_i}\underline{Y}$ is first computed and then $\underline{Y}'B_{F_{i.1}}\underline{Y}$ is computed (using totals corresponding to p-levels) so that $\underline{Y}'B_{F_{i.(1,.)}}\underline{Y}$ is then obtained by subtraction,

$$\underline{Y}'B_{F_{i.(1,.)}}\underline{Y} = \underline{Y}'B_{F_i}\underline{Y} - \underline{Y}'B_{F_{i.1}}\underline{Y}.$$

Since the factor main effects are expressed as the sum of between and within p-level effects, a k-factor interaction effect, $L_{F_{i_1}\cdots F_{i_k}}\underline{M}$ is expressed as the sum of $2^k$ sets of effects defined by the $2^k$ matrices $L_{F_{i_1}j_1} \otimes \cdots \otimes L_{F_{i_k}j_k}$, where $j_h = .1$ or $.(1,.)$ for all $h = 1, \ldots, k$.

For example, if $k = 2$, then

$$L_{F_1F_2} = \begin{bmatrix} L_{F_{1.1}} \otimes L_{F_{2.1}} \\ L_{F_{1.1}} \otimes L_{F_{2.(1,.)}} \\ L_{F_{1.(1,.)}} \otimes L_{F_{2.1}} \\ L_{F_{1.(1,.)}} \otimes L_{F_{2.(1,.)}} \end{bmatrix}$$

Of the $2^k$ sets of components of the k-factor interaction effect only the set of components defined by $L_{F_{i_1}.1} \otimes \cdots \otimes L_{F_{i_k}.1}$ completely represents a k-factor interaction effect among k factors in the $\prod_{i=1}^{n} s_i$-p-FAT. The sum of squares corresponding to a k-factor interaction effect is typically $\underline{Y}'B_{F_{i_1}\cdots F_{i_k}}\underline{Y}$ and it can be separated into a quantity corresponding to a k-factor interaction effect in the $\prod_{i=1}^{n} s_i$-p-FAT, denoted by

$\underline{Y}'B_{F_{i_1}.1\cdots F_{i_k}.1}\underline{Y}$, and a quantity corresponding to the other components of the interaction effect, which is computed

$$\underline{Y}'B_{F_{i_1}\cdots F_{i_k}}\underline{Y} - \underline{Y}'B_{F_{i_1}.1\cdots F_{i_k}.1}\underline{Y}.$$

For the partitioning (28) when n = 2, an abbreviated analysis of variance table (first three columns) can be constructed according to Table 17, letting $m = P_1P_2$. Considering the $s_1s_2$ s-FAT's of the partitioning, the first three columns of an analysis of variance table can be written in the form of Table 18 (for the partitioning (28) and n = 2). The symbol "$A_1xA_2$" in Tables 17 and 18 denotes the source of variation for all components of the factor one factor two interaction effect except that set of components that is also a between s-FAT effect ($A_{1.1}xA_{2.1}$). Table 19 is the abbreviated analysis of variance table for the partitioning

$$\prod_{i=1}^{3} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{3} (P_{i1} + \cdots + P_{is_i})\text{-s-FAT's},$$

letting $m = P_1P_2P_3$.

Abbreviated analysis of variance tables for full replicates of partitionings of the type (28) involving more than three factors is a direct generalization of Tables 18 and 19.

## Analysis of Varinace for Multiple Full Replicates

The construction of analysis of variance tables is briefly examined for the situation where the full treatment replicate of a $\prod_{i=1}^{n} P_i$-FAT is performed r times. The general method is to construct the abbreviated analysis of variance table for each of the r repititions of the experiment and then to add corresponding degrees of freedom and sums of squares in the r tables. This addition of the sum of squares for factor main effects or factor interaction effects gives a sum of squares corresponding to a

TABLE 17

ABBREVIATED ANALYSIS OF VARIANCE TABLE FOR THE PARTITIONING (28) (for n=2)

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | $P_1P_2-1$ | $\underline{Y}'( I_m - \frac{1}{m}J^m_m )\underline{Y}$ |
| $A_1$ | $(P_1-1)$ | $\underline{Y}'B_{F_1}\underline{Y}$ |
| $A_{1.1}$ | $(s_1-1)$ | $\underline{Y}'B_{F_{1.1}}\underline{Y}$ |
| $A_{1.(1,.)}$ | $(P_1-s_1)$ | $\underline{Y}'B_{F_1}\underline{Y} - \underline{Y}'B_{F_{1.1}}\underline{Y}$ |
| $A_2$ | $(P_2-1)$ | $\underline{Y}'B_{F_2}\underline{Y}$ |
| $A_{2.1}$ | $(s_2-1)$ | $\underline{Y}'B_{F_{2.1}}\underline{Y}$ |
| $A_{2.(1,.)}$ | $(P_2-s_2)$ | $\underline{Y}'B_{F_2}\underline{Y} - \underline{Y}'B_{F_{2.1}}\underline{Y}$ |
| $A_1xA_2$ | $(P_1-1)(P_2-1)$ | $\underline{Y}'B_{F_1F_2}\underline{Y}$ |
| $A_{1.1}xA_{2.1}$ | $(s_1-1)(s_2-1)$ | $\underline{Y}'B_{F_{1.1}F_{2.1}}\underline{Y}$ |
| $A_{1.1}xA_{2.(1,.)}$ | $(s_1-1)(P_2-s_2)$ | $\underline{Y}'B_{F_{1.1}F_{2.(1,.)}}\underline{Y}$ |
| $A_{1.(1,.)}xA_{2.1}$ | $(P_1-s_1)(s_2-1)$ | $\underline{Y}'B_{F_{1.(1,.)}F_{2.1}}\underline{Y}$ |
| $A_{1.(1,.)}xA_{2.(1,.)}$ | $(P_1-s_1)(P_2-s_2)$ | $\underline{Y}'B_{F_{1.(1,.)}F_{2.(1,.)}}\underline{Y}$ |
| $A_{1.1}xA_{2.1}$ | $(s_1-1)(s_2-1)$ | $\underline{Y}'B_{F_{1.1}F_{2.1}}\underline{Y}$ |
| "$A_1xA_2$" | $(P_1-1)(P_2-1)-$ $(s_1-1)(s_2-1)$ | $\underline{Y}'B_{F_1F_2}\underline{Y}$ - $\underline{Y}'B_{F_{1.1}F_{2.1}}\underline{Y}$ |

TABLE 18

ABBREVIATED ANALYSIS OF VARIANCE TABLE
FOR A PARTITIONED $P_1P_2$-FAT

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | $P_1P_2-1$ | $\underline{Y}'(I_m - \frac{1}{m}J_m^m)\underline{Y}$ |
| Between all s-FAT's | $s_1s_2-1$ | $\underline{Y}'(B_{F_{1.1}} + B_{F_{2.1}} + B_{F_{1.1}F_{2.1}})\underline{Y}$ |
| $A_{1.1}$ | $s_1-1$ | $\underline{Y}'B_{F_{1.1}}\underline{Y}$ |
| $A_{2.1}$ | $s_2-1$ | $\underline{Y}'B_{F_{2.1}}\underline{Y}$ |
| $A_{1.1}xA_{2.1}$ | $(s_1-1)(s_2-1)$ | $\underline{Y}'B_{F_{1.1}F_{2.1}}\underline{Y}$ |
| Within all s-FAT's | $P_1P_2 - s_1s_2$ | $\underline{Y}'(I_m-B_1-B_{F_{1.1}}-B_{F_{2.1}}-B_{F_{1.1}F_{2.1}})\underline{Y}$ |
| $A_{1.(1,.)}$ | $P_1-s_1$ | $\underline{Y}'B_{F_1}\underline{Y} - \underline{Y}'B_{F_{1.1}}\underline{Y}$ |
| $A_{2.(1,.)}$ | $P_2-s_2$ | $\underline{Y}'B_{F_2}\underline{Y} - \underline{Y}'B_{F_{2.1}}\underline{Y}$ |
| "$A_1xA_2$" | $(P_1-1)(P_2-1)-(s_1-1)(s_2-1)$ | $\underline{Y}'B_{F_1F_2}\underline{Y} - \underline{Y}'B_{F_{1.1}F_{2.1}}\underline{Y}$ |

TABLE 19

ABBREVIATED ANALYSIS OF VARIANCE TABLE FOR A FULL REPLICATE OF A $P_1P_2P_3$-FAT

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | $P_1P_2P_3-1$ | $\underline{Y}'(I_m - \frac{1}{m} J_m^m)\underline{Y}$ |
| Between all s-FAT's | $s_1s_2s_3-1$ | $\underline{Y}'(a_1+a_2+a_3+a_4+a_5+a_6+a_7)\underline{Y}$ |
| $A_{1.1}$ | $s_1-1$ | $\underline{Y}'B_{F_{1.1}}\underline{Y} = a_1$ |
| $A_{2.1}$ | $s_2-1$ | $\underline{Y}'B_{F_{2.1}}\underline{Y} = a_2$ |
| $A_{3.1}$ | $s_3-1$ | $\underline{Y}'B_{F_{3.1}}\underline{Y} = a_3$ |
| $A_{1.1}xA_{2.1}$ | $(s_1-1)(s_2-1)$ | $\underline{Y}'B_{F_{1.1}F_{2.1}}\underline{Y} = a_4$ |
| $A_{1.1}xA_{3.1}$ | $(s_1-1)(s_3-1)$ | $\underline{Y}'B_{F_{1.1}F_{3.1}}\underline{Y} = a_5$ |
| $A_{2.1}xA_{3.1}$ | $(s_2-1)(s_3-1)$ | $\underline{Y}'B_{F_{2.1}F_{2.1}}\underline{Y} = a_6$ |
| $A_{1.1}xA_{2.1}xA_{3.1}$ | $(s_1-1)(s_2-1)(s_3-1)$ | $\underline{Y}'B_{F_{1.1}F_{2.1}F_{3.1}}\underline{Y} = a_7$ |
| Within all s-FAT's | $P_1P_2P_3-s_1s_2s_3$ | |
| $A_{1.(1,.)}$ | $P_1-s_1$ | $\underline{Y}'B_{F_1}\underline{Y} - a_1$ |
| $A_{2.(1,.)}$ | $P_2-s_2$ | $\underline{Y}'B_{F_2}\underline{Y} - a_2$ |
| $A_{3.(1,.)}$ | $P_3-s_3$ | $\underline{Y}'B_{F_3}\underline{Y} - a_3$ |
| "$A_1xA_2$" | $(P_1-1)(P_2-1)-(s_1-1)(s_2-1)$ | $\underline{Y}'B_{F_1F_2}\underline{Y} - a_4$ |
| "$A_2xA_3$" | $(P_2-1)(P_3-1)-(s_2-1)(s_3-1)$ | $\underline{Y}'B_{F_2F_3}\underline{Y} - a_6$ |
| "$A_1xA_3$" | $(P_1-1)(P_3-1)-(s_1-1)(s_3-1)$ | $\underline{Y}'B_{F_1F_3}\underline{Y} - a_5$ |
| "$A_1xA_2xA_3$" | $(P_1-1)(P_2-1)(P_3-1) - (s_1-1)(s_2-1)(s_3-1)$ | $\underline{Y}'B_{F_1F_2F_3}\underline{Y} - a_7$ |

factor main effect in replicates (hereafter abbreviated "Rep.s"), or factor interaction effect in Rep.s. Next, assuming each of the n factors is a fixed effect factor and since the same set of factorially arranged treatment combinations appears in each of the r repitions, the factor effect in Rep.s sum of squares can be separated into a sum of squares term corresponding to the factor effect and a sum of squares term corresponding to a factor by Rep.s interaction effect. If there is no reason for treating the r repititions as a source of variation that must be accounted for in the analysis, then all of the sum of squares corresponding to factor by Rep.s interaction effects may be pooled to obtain a residual or error sum of squares, providing the Rep.s are assumed to be of random effects. If there is reason to consider the r repititions of the experiment as a source of variation to be accounted for in the analysis, say as r randomized blocks, then the usual advice is to leave the factor by Rep.s interaction terms unpooled. In this case, if the blocks or Rep.s are random, then the factor by Rep.s interaction terms can be used as error terms for significance testing purposes. The following discussion will serve to illustrate the above mentioned concepts. Abbreviated analysis of variance tables will contain only the first two columns, however, sum of squares will be exhibited for a case when n = 2.

Consider a $\prod_{i=1}^{n} P_i$-FAT that is performed r times, or in r Rep.s. The abbreviated analysis of variance table for each of the r Rep.s of the experiment is given in Table 20. Pooling the r analysis of variance tables yields an analysis of variance table of the form given in Table 21. Although the means to obtain the sum of squares is probably obvious, the special case where n = 2 is examined to illustrate the analysis procedure. obtaining the sum of square

TABLE 20

ABBREVIATED ANALYSIS OF VARIANCE TABLE FOR
A FULL REPLICATE OF A $P_1 \ldots P_n$-FAT

| Source | DF |
|---|---|
| Total (corrected) | $(P_1 \ldots P_n)-1$ |
| $A_1$<br>.<br>.<br>$A_n$ | $P_1-1$<br>.<br>.<br>$P_n-1$ |
| $A_1 x A_2$<br>.<br>.<br>$A_n x A_{n-1}$ | $(P_1-1)(P_2-1)$<br>.<br>.<br>$(P_n-1)(P_{n-1}-1)$ |
| .<br>.<br>$A_1 x \ldots x A_n$ | .<br>.<br>$(P_1-1)\ldots(P_n-1)$ |

TABLE 21

ABBREVIATED ANALYSIS OF VARIANCE TABLE FOR r POOLED FULL REPLICATES OF A $P_1 \ldots P_n$-FAT

| Source | DF |
|---|---|
| Total (corrected) | $(P_1 \ldots P_n)-1$ |
| Between all Replicates | $r - 1$ |
| Within all Replicates | $r(P_1 \ldots P_n-1)$ |
| $A_1$ in Rep.s | $r(P_1-1)$ |
| $A_1$ | $(P_1-1)$ |
| $A_1$xRep.s | $(P_1-1)(r-1)$ |
| . | . |
| . | . |
| $A_n$ in Rep.s | $r(P_n-1)$ |
| $A_n$ | $(P_n-1)$ |
| $A_n$xRep.s | $(P_n-1)(r-1)$ |
| $A_1$x$A_2$ in Rep.s | $r(P_1-1)(P_2-1)$ |
| $A_1$x$A_2$ | $(P_1-1)(P_2-1)$ |
| $A_1$x$A_2$xRep.s | $(P_1-1)(P_2-1)(r-1)$ |
| . | . |
| . | . |
| $A_n$x$A_{n-1}$ in Rep.s | $r(P_n-1)(P_{n-1}-1)$ |
| $A_n$x$A_{n-1}$ | $(P_n-1)(P_{n-1}-1)$ |
| $A_n$x$A_{n-1}$xRep.s | $(P_n-1)(P_{n-1}-1)(r-1)$ |
| . | . |
| . | . |
| $A_1$x...x$A_n$ in Rep.s | $r(P_1-1)\ldots(P_n-1)$ |
| $A_1$x...x$A_n$ | $(P_1-1)\ldots(P_n-1)$ |
| $A_1$x...x$A_n$xRep.s | $(P_1-1)\ldots(P_n-1)(r-1)$ |

Suppose there are r repititions of a $P_1P_2$-FAT. For each repitition the matrix I can be constructed, or say I(i) is the matrix L for the i-th replicate, for i = 1, ...,n. Given the matrices L(i), the following procedure is used to build a matrix L for the r replicates of the $P_1P_2$-FAT.

(1) Let the first row of L be $J^1_{rP_1P_2} = L_1 = (L(1)_1,\ldots,L(r)_1)$.

(2) Choose the next r-1 rows of L, call them $L_R$ to be, for $m = P_1P_2$

$$L_R = \begin{bmatrix} J^1_m & -J^1_m & \phi^1_m & \cdots & \phi^1_m & \phi^1_m \\ J^1_m & J^1_m - 2J^1_m & & \cdots & \phi^1_m & \phi^1_m \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ J^1_m & J^1_m & J^1_m & & J^1_m - (r-1)J^1_m & \end{bmatrix}$$

(3) Inspect the matrices $L(i)_\theta$ to make sure that $L(i)_\theta = L(j)_\theta$ for $i \neq j$ and $\theta = 1, F_1, F_2$ and $F_1F_2$.

(4) From the matrices L(1), ...,and L(r) form the following,

$$L_{F_1} = (L(1)_{F_1}, \ldots, L(r)_{F_1})$$

$$L_{F_2} = (L(1)_{F_2}, \ldots, L(r)_{F_2})$$

$$L_{F_1F_2} = (L(1)_{F_1F_2}, \ldots, L(r)_{F_1F_2}) .$$

(5) Let $L_{\theta R} = L_\theta \circledast L_R$ for $\theta = F_1$, $F_2$ and $F_1F_2$, thus constructing $L_{F_1R}$, $L_{F_2R}$ and $L_{F_1F_2R}$.

(6) For notation purposes, let $L_{\theta \text{ in } R} = \begin{bmatrix} L_\theta \\ L_{\theta R} \end{bmatrix}$ for $\theta = F_1, F_2$ and $F_1F_2$

(7) To form the matrix L for all r replicates of the $P_1P_2$-FAT, the results of steps (2), (4) and (5) are combined with the $rP_1P_2$ row vector of ones obtained in step one. See Figure 12.

$$I = \begin{bmatrix} L_1 \\ L_R \\ L_{F_1 \text{ in } R} \\ L_{F_2 \text{ in } R} \\ L_{F_1F_2 \text{ in } R} \end{bmatrix} = \begin{bmatrix} L_1 \\ L_R \\ L_{F_1} \\ L_{F_1R} \\ L_{F_2} \\ L_{F_2R} \\ L_{F_1F_2} \\ L_{F_1F_2R} \end{bmatrix}$$

Figure 12. - The matrix I for r replicates of a $P_1P_2$-FAT.

Given I the matrix H is obtained by normalizing the rows of I. By partitioning H into ($H_1'$, $H_{F_1 \text{ in } R}'$, $H_{F_2 \text{ in } R}'$, $H_{F_1F_2 \text{ in } R}'$) the matrices $B_\theta$ are constructed by

$$H'H = H_1'H_1 + H_R'H_R + H_{F_1 \text{in} R}'H_{F_1 \text{in} R} + H_{F_2 \text{in} R}'H_{F_2 \text{in} R} + H_{F_1F_2 \text{in} R}'H_{F_1F_2 \text{in} R}$$

$$= B_1 + B_R + B_{F_1 \text{ in } R} + B_{F_2 \text{ in } R} + B_{F_1F_2 \text{ in } R}$$

$$= B_1 + B_R + B_{F_1} + B_{F_1R} + B_{F_2} + B_{F_2R} + B_{F_1F_2} + B_{F_1F_2R} .$$

The first three columns of an analysis of variance Table are given in Table 22. The sum of squares may be written in terms of the observations in such a way that computation is straightforward. If an element of the vector of observations, $\underline{Y}$, is represented by $y_{ijk}$ for the observation of the ij-th treatment combination of the k-th replicate, for $i = 0,\ldots,P_1-1$, for $j = 0,\ldots,P_2-1$ and for $k = 1,\ldots,r$, then the sums of squares in Table 22 can be expressed as follows:

$$\underline{Y}'(I_{P_1P_2} - H_1'H_1)\underline{Y} = \sum_{ijk}\sum\sum y_{ijk}^2 - (1/rP_1P_2)(y_{...})^2$$

$$\underline{Y}'B_R\underline{Y} = \sum_k (y_{..k})^2 - (1/rP_1P_2)(y_{...})^2$$

TABLE 22

ABBREVIATED ANALYSIS OF VARIANCE TABLE FOR THE $P_1P_2$-FAT RUN IN r REPLICATES

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | $rP_1P_2-1$ | $\underline{Y}'(I_{rP_1P_2} - H_1'H_1)\underline{Y}$ |
| $A_1$ in Rep.s | $r(P_1-1)$ | $\underline{Y}'B_{F_1 \text{ in } R}\underline{Y}$ |
| $A_1$ | $(P_1-1)$ | $\underline{Y}'B_{F_1}\underline{Y}$ |
| $A_1$xRep.s | $(P_1-1)(r-1)$ | $\underline{Y}'B_{F_1R}\underline{Y}$ |
| $A_2$ in Rep.s | $r(P_2-1)$ | $\underline{Y}'B_{F_2 \text{ in } R}\underline{Y}$ |
| $A_2$ | $(P_2-1)$ | $\underline{Y}'B_{F_2}\underline{Y}$ |
| $A_2$xRep.s | $(P_2-1)(r-1)$ | $\underline{Y}'B_{F_2R}\underline{Y}$ |
| $A_1$x$A_2$ in Rep.s | $r(P_1-1)(P_2-1)$ | $\underline{Y}'B_{F_1F_2 \text{in } R}\underline{Y}$ |
| $A_1$x$A_2$ | $(P_1-1)(P_2-1)$ | $\underline{Y}'B_{F_1F_2}\underline{Y}$ |
| $A_1$x$A_2$xRep.s | $(P_1-1)(P_2-1)(r-1)$ | $\underline{Y}'B_{F_1F_2R}\underline{Y}$ |

$$\underline{Y}'B_F \text{ in } R\underline{Y} = (1/P_2)\sum\sum_{ik}(y_{i.k})^2 - (1/P_1P_2)\sum_k(y_{..k})^2$$

$$\underline{Y}'B_{F_1}\underline{Y} = (1/rP_2)\sum_i(y_{i..})^2 - (1/rP_1P_2)(y_{...})^2$$

$$\underline{Y}'B_{F_1R}\underline{Y} = (1/P_2)\sum\sum_{ik}(y_{i.k})^2 - (1/rP_2)\sum_i(y_{i..})^2 - (1/P_1P_2)\sum_k(y_{..k})^2 + (1/rP_1P_2)(y_{...})^2$$

$$\underline{Y}'B_{F_2} \text{ in } R\underline{Y} = (1/P_1)\sum\sum_{jk}(y_{.jk})^2 - (1/P_1P_2)\sum_k(y_{..k})^2$$

$$\underline{Y}'B_{F_2}\underline{Y} = (1/rP_1)\sum_j(y_{.j.})^2 - (1/rP_1P_2)(y_{...})^2$$

$$\underline{Y}'B_{F_2R}\underline{Y} = (1/P_1)\sum\sum_{jk}(y_{.jk})^2 - (1/rP_1)\sum_j(y_{.j.})^2 - (1/P_1P_2)\sum_k(y_{..k})^2 + (1/rP_1P_2)(y_{...})^2$$

$$\underline{Y}'B_{F_1F_2} \text{ in } R\underline{Y} = \sum\sum\sum_{ijk}y_{ijk}^2 - (1/P_2)\sum\sum_{ik}(y_{i.k})^2 - (1/P_1)\sum\sum_{jk}(y_{.jk})^2 + (1/P_1P_2)\sum_k(y_{..k})^2$$

$$\underline{Y}'B_{F_1F_2}\underline{Y} = (1/r)\sum\sum_{ij}(y_{ij.})^2 - (1/rP_2)\sum_i(y_{i..})^2 - (1/rP_1)\sum_j(y_{.j.})^2 + (1/rP_1P_2)(y_{...})^2$$

$$\underline{Y}'B_{F_1F_2R}\underline{Y} = \sum\sum\sum_{ijk}y_{ijk}^2 - (1/r)\sum\sum_{ij}(y_{ij.})^2 - (1/P_2)\sum\sum_{ik}(y_{i.k})^2 - (1/P_1)\sum\sum_{jk}(y_{.jk})^2 + (1/P_1P_2)\sum_k(y_{..k})^2 + (1/rP_1)\sum_j(y_{.j.})^2 + (1/rP_2)\sum_i(y_{i..})^2 - (1/rP_1P_2)(y_{...})^2.$$

Example 15 : Suppose a $2^2$-FAT is run in a completely random design and that it was repeated three times. For each repitition the matrix L(i) that defines the effects is, for i = 1,2 and 3,

$$L(1) = L(2) = L(3) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

To construct an analysis of variance table for all three replicates the matrix I must be formed. Following the aforementioned procedure

the matrix L is given by

$$L = \begin{bmatrix} L_1 \\ L_R \\ L_{F_1} \\ L_{F_1R} \\ L_{F_2} \\ L_{F_2R} \\ L_{F_1F_2} \\ L_{F_1F_2R} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -2 & -2 & 2 & 2 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -2 & 2 & -2 & 2 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -2 & 2 & 2 & -2 \end{bmatrix}$$

If there are r replicates of a $\prod_{i=1}^{n} P_i$-FAT, where in each replicate the $\prod_{i=1}^{n} P_i$-FAT is partitioned via:

$$\prod_{i=1}^{n} P_i\text{-FAT} \longrightarrow \prod_{i=1}^{n} (P_{i1} + \ldots + P_{is_i})\text{-s-FAT's},$$

then an appropriate abbreviated analysis of variance table is given in Table 23, letting $m = \prod_{i=1}^{n} P_i$. In Table 23 the symbol $\underline{Y}(j)'B_{F_k}\underline{Y}(j)$ denotes the sum of squares due to the factor k main effect in the j-th replicate, for $j = i, \ldots, r$. Thus, the term $\sum_{j=1}^{r} \underline{Y}(j)'B_{F_k}\underline{Y}(j)$ is the sum of the sums of squares due to the main effect of factor k for all r replicates of the experiment and this source of variation was previously denoted $A_k$ in Rep.s. In a similar manner, $\sum_{j=1}^{r} \underline{Y}(j)'B_{F_{i_1}\ldots F_{i_k}}\underline{Y}(j)$ is the sum of the sums of squares due to the k-factor interaction effect among factors $i_1$, ... and $i_k$ for all r replicates of the experiment.

TABLE 23

ABBREVIATED ANALYSIS OF VARIANCE TABLE FOR A r REPLICATES OF A PARTITIONED $P_1 \ldots P_n$-FAT

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | $rm - 1$ | $\underline{Y}'(I_m - (1/rm)J^{rm}_{rm})\underline{Y}$ |
| Replicates | $r - 1$ | $\underline{Y}'B_R\underline{Y}$ |
| Between s-FAT's in Rep.s | $r(s_1 \ldots s_n - 1)$ | |
| $A_{1.1}$ in Rep.s | $r(s_1 - 1)$ | $\underline{Y}'B_{F_{1.1}\text{in}R}\underline{Y}$ |
| $A_{1.1}$ | $(s_1 - 1)$ | $\underline{Y}'B_{F_{1.1}}\underline{Y} = a_1$ |
| $A_{1.1}$xRep.s | $(r-1)(s_1 - 1)$ | $\underline{Y}'B_{F_{1.1}R}\underline{Y} = a_2$ |
| . | . | . |
| . | . | . |
| $A_{n.1}$ in Rep.s | $r(s_n - 1)$ | $\underline{Y}'B_{F_{n.1}\text{in}R}\underline{Y}$ |
| $A_{n.1}$ | $(s_n - 1)$ | $\underline{Y}'B_{F_{n.1}}\underline{Y} = a_3$ |
| $A_{n.1}$xRep.s | $(r-1)(s_n - 1)$ | $\underline{Y}'B_{F_{n.1}R}\underline{Y} = a_4$ |
| . | . | . |
| . | . | . |
| $A_{1.1}x \ldots xA_{n.1}$in Rep.s | $r(s_1 - 1)\ldots(s_n - 1)$ | $\underline{Y}'B_{F_{1.1}\ldots F_{n.1}\text{in}R}\underline{Y}$ |
| $A_{1.1}x \ldots xA_{n.1}$ | $(s_1 - 1)\ldots(s_n - 1)$ | $\underline{Y}'B_{F_{1.1}\ldots F_{n.1}}\underline{Y} = a_5$ |
| $A_{1.1}x \ldots xA_{n.1}$xRep.s | $(r-1)(s_1 - 1)\ldots(s_n - 1)$ | $\underline{Y}'B_{F_{1.1}\ldots F_{n.1}R}\underline{Y} = a_6$ |
| Withis all s-FAT's in Rep.s | $r(P_1 \ldots P_n - s_1 \ldots s_n)$ | |
| $A_{1.(1,.)}$ | $P_1 - s_1$ | $\underline{Y}'B_{F_1}\underline{Y} - a_1$ |
| $A_{1.(1,.)}$xRep.s | $(r-1)(P_1 - s_1)$ | $\sum_j \underline{Y}(j)'B_{F_1}\underline{Y}(j) - (a_2+a_1) - \underline{Y}'B_{F_1}\underline{Y}$ |
| $A_{n.(1,.)}$ | $P_n - s_n$ | $\underline{Y}'B_{F_n}\underline{Y} - a_3$ |
| $A_{n.(1,.)}$xRep.s | $(r-1)(P_n - s_n)$ | $\sum_j \underline{Y}(j)'B_{F_n}\underline{Y}(j) - (a_3+a_4) - \underline{Y}'B_{F_n}\underline{Y}$ |
| "$A_1x \ldots xA_n$" | $(P_1-1)\ldots(P_n-1) - (s_1-1)\ldots(s_n-1)$ | $\underline{Y}'B_{F_1\ldots F_n}\underline{Y} - a_5$ |
| "$A_1x \ldots xA_n$"xRep.s | $(r-1)((P_1-1)..(P_n-1) - (s_1-1)\ldots(s_n-1))$ | $\sum_j \underline{Y}(j)'B_{F_1\ldots F_n}\underline{Y}(j) - a_5 - a_6 - \underline{Y}'B_{F_1\ldots F_n}\underline{Y}$ |

## Analysis of Variance in the Presense of Blocks

Analysis of variance tables can be constructed for the blocking PLANs mentioned in chapter IV. In general, the total (corrected) sum of squares is expressed as the sum of (1) the sum of squares due to between block effects (this source of variation will be denoted B.A.B. (between all blocks)) and (2) the sum of squares due to within block effects (this source of variation will be denoted W.A.B. (within all blocks)). The B.A.B. sum of squares is obtained from the block totals. This B.A.B. sum of squares may be expressed (if desired) as the total of the sums of squares representing all effects that are confounded with block effects. The W.A.B. sum of squares can be expressed in terms of sums of squares corresponding to factor main effects and factor interaction effects (or unconfounded components of the main or interaction effects). The following examples serve to illustrate relevant concepts.

Example 16 : For the partitioning of example 4.1,

$s_1 4_2$-FAT $\longrightarrow$ $(4_{11} + 4_{12})(2_{21} + 2_{22})$-s-FAT's

the blocking PLANs (a), (b), (c), ... and ($h_7$) are obtained by confounding the components of main and interaction effects indicated in Table 24. Each s-FAT is equivalent to a 4x2-FAT. The source of variation due to the effects that are components of the factor one main effect are denoted by B, C and BC, the component of the factor two main effect is denoted by A and the components of the factor one-factor two interaction effect by AB, AC and ABC. The sum of squares for these components are given by

$$\underline{Y}'B_{F_{1.(1,.)}}\underline{Y} = \underline{Y}'B_B\underline{Y} + \underline{Y}'B_C\underline{Y} + \underline{Y}'B_{BC}\underline{Y}$$

$$\underline{Y}'B_{F_{2.(1,.)}}\underline{Y} = \underline{Y}'B_A\underline{Y} \text{ and}$$

TABLE 24

SOME BLOCKING PLANS FOR EXAMPLE 16

| Blocking PLAN | Components confounded with block effects | |
|---|---|---|
| | between s-FAT effects | within s-FAT effects |
| (a) | $A_{1.1}$ | none |
| (b) | $A_{2.1}$ | none |
| (c) | $A_{1.1}{}^{x}A_{2.1}$ | none |
| (d) | $A_{1.1}$, $A_{2.1}$, $A_{1.1}{}^{x}A_{2.1}$ | none |
| (e) | all | A |
| ($f_1$) | all | B |
| ($f_2$) | all | C |
| ($f_3$) | all | BC |
| ($g_1$) | all | AB |
| ($g_2$) | all | AC |
| ($g_3$) | all | ABC |
| ($h_1$) | all | A, B and AB |
| ($h_2$) | all | B, C and BC |
| ($h_3$) | all | A, C and AC |
| ($h_4$) | all | A, BC and ABC |
| ($h_5$) | all | B, AC and ABC |
| ($h_6$) | all | C, AB and ABC |
| ($h_7$) | all | AC, AB and BC |

$$\underline{Y}'B_{\text{"}F_1F_2\text{"}}\underline{Y} = \underline{Y}'B_{AB}\underline{Y} + \underline{Y}'B_{AC}\underline{Y} + \underline{Y}'B_{ABC}\underline{Y} .$$

The abbreviated analysis of variance tables are given for blocking PLANs (a), (d), ($g_1$) and ($h_7$) in Tables 25, 26, 27 and 28, respectively, The letters $b_i$ in tables 25, 26, 27 and 28 represent block totals of observations, where the blocks received the treatment combinations in Figure 13.

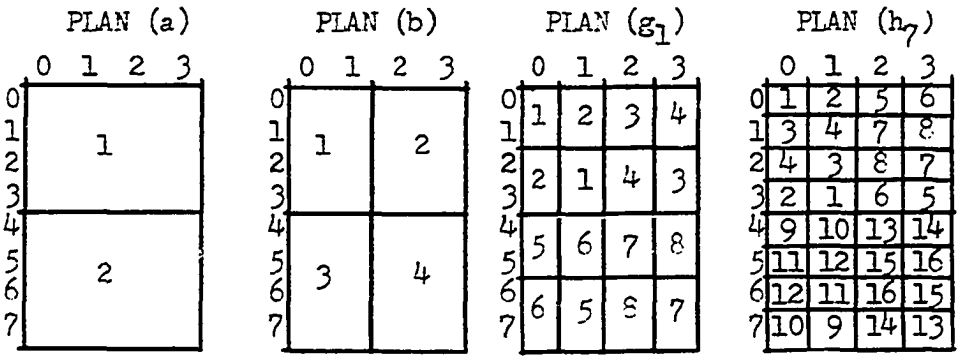


Figure 13. - Allocation of treatment combinations to blocks.

Also, the sums of squares in Table 27 are

$$a_1 = (1/16)((b_1+b_2+b_3+b_4)^2 + (b_5+b_6+b_7+b_8)^2) - (1/32)(y_{...})^2$$

$$a_2 = (1/16)((b_1+b_2+b_5+b_6)^2 + (b_3+b_4+b_7+b_8)^2) - (1/32)(y_{...})^2$$

$$a_3 = (1/16)((b_1+b_2+b_7+b_8)^2 + (b_3+b_4+b_5+b_6)^2) - (1/32)(y_{...})^2$$

$$a_4 = (1/16)((b_1+b_3+b_5+b_7)^2 + (b_2+b_4+b_6+b_8)^2) - (1/32)(y_{...})^2$$

$$a_5 = (1/16)((b_1+b_3+b_6+b_8)^2 + (b_2+b_4+b_5+b_7)^2) - (1/32)(y_{...})^2$$

$$a_6 = (1/16)((b_1+b_4+b_5+b_8)^2 + (b_2+b_3+b_6+b_7)^2) - (1/16)(y_{...})^2$$

$$a_7 = (1/16)((b_1+b_4+b_6+b_7)^2 + (b_2+b_3+b_5+b_8)^2) - (1/16)(y_{...})^2$$

and the sums of squares for table 28 are $a_1$, $a_2$, ... and $a_{16}$, where

$$x_1=(\tfrac{1}{16})((b_1+b_2+b_3+b_4+b_5+b_6+b_7+b_8)^2+(b_9+b_{10}+b_{11}+b_{12}+b_{13}+b_{14}+b_{15}+b_{16})^2)$$

$$x_2=(\tfrac{1}{16})((b_1+b_2+b_3+b_4+b_9+b_{10}+b_{11}+b_{12})^2+(b_5+b_6+b_7+b_8+b_{13}+b_{14}+b_{15}+b_{16})^2)$$

TABLE 25

ABBREVIATED AOV FOR BLOCKING PLAN (a) OF EXAMPLE 16

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | 31 | $\underline{Y}'(I_{32} - (1/32)\, J^{32}_{32})\underline{Y} = \sum\sum_{ij} y_{ij}^2 - \frac{1}{32}y_{..}^2$ |
| B.A.B. | 1 | |
| $A_{1.1}$ | 1 | $\underline{Y}'B_{F_{1.1}}\underline{Y} = \frac{1}{16}b_1^2 + \frac{1}{16}b_2^2 - \frac{1}{32}(y_{..})^2 = a_1$ |
| W.A.B. | 30 | |
| $A_{1.(1,.)}$ | 6 | $\underline{Y}'B_{F_1}\underline{Y} - a_1 = \frac{1}{4}\sum_i (y_{i.})^2 - \frac{1}{32}(y_{..})^2 - a_1$ |
| $A_2$ | 3 | $\underline{Y}'B_{F_2}\underline{Y} = (1/8)\sum_j (y_{.j})^2 - (1/32)y_{..}^2$ |
| $A_1 x A_2$ | 21 | $\underline{Y}'B_{F_1F_2}\underline{Y} = \sum\sum_{ij} y_{ij}^2 - (1/4)\sum_j y_{.j}^2 - (1/8)\sum_i y_{i.}^2 + (1/32)y_{..}^2$ |

TABLE 26

ABBREVIATED AOV FOR BLOCKING PLAN (b) OF EXAMPLE 16

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | 31 | $\underline{Y}'(I_{32} - (1/32)\, J^{32}_{32})\underline{Y}$ |
| B.A.B. | 3 | |
| $A_{1.1}$ | 1 | $a_1 = \underline{Y}'B_{F_{1.1}}\underline{Y} = \frac{1}{16}((b_1+b_2)^2 + (b_3+b_4)^2) - \frac{1}{32}y_{..}^2$ |
| $A_{2.1}$ | 1 | $a_2 = \underline{Y}'B_{F_{2.1}}\underline{Y} = \frac{1}{16}((b_1+b_3)^2 + (b_2+b_4)^2) - \frac{1}{32}y_{..}^2$ |
| $A_{1.1} x A_{2.1}$ | 1 | $\underline{Y}'B_{F_1F_2}\underline{Y} = (1/8)(b_1^2+b_2^2+b_3^2+b_4^2) - a_1 - a_2 - \frac{1}{32}y_{..}^2$ |
| W.A.B. | 28 | |
| $A_{1.(1,.)}$ | 6 | $\underline{Y}'B_{F_1}\underline{Y} - a_1$ |
| $A_{2.(1,.)}$ | 2 | $\underline{Y}'B_{F_2}\underline{Y} - a_2$ |
| "$A_1 x A_2$" | 20 | $\underline{Y}'B_{F_1F_2}\underline{Y} - \underline{Y}'B_{F_{1.1}F_{2.1}}\underline{Y}$ |

TABLE 27

ABBREVIATED AOV FOR BLOCKING PLAN ($g_1$) OF EXAMPLE 16

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | 31 | $\Sigma\Sigma y_{ij}^2 - (1/32)(y_{..})^2$ |
| B.A.B. | 7 | $(1/4)\Sigma_k b_k^2 - (1/32)(y_{..})^2$ |
| $A_{1.1}$ | 1 | $\underline{Y}'B_{F_{1.1}}\underline{Y} = a_1$ |
| $A_{2.1}$ | 1 | $\underline{Y}'B_{F_{2.1}}\underline{Y} = a_2$ |
| $A_{1.1}xA_{2.1}$ | 1 | $\underline{Y}'B_{F_{1.1}F_{2.1}}\underline{Y} = a_3$ |
| AC | 1 | $\underline{Y}'B_{AC}\underline{Y} = a_4$ |
| $A_{1.1}xAC$ | 1 | $\underline{Y}'B_{F_{1.1}AC}\underline{Y} = a_5$ |
| $A_{2.1}xAC$ | 1 | $\underline{Y}'B_{F_{2.1}AC}\underline{Y} = a_6$ |
| $A_{1.1}xA_{2.1}xAC$ | 1 | $\underline{Y}'B_{F_{1.1}F_{2.1}AC}\underline{Y} = a_7$ |
| W.A.B. | 24 | |
| $A_{1.(1,.)}$ | 6 | $\underline{Y}'B_{F_1}\underline{Y} - a_1$ |
| $A_{2.(1,.)}$ | 2 | $\underline{Y}'B_{F_2}\underline{Y} - a_2$ |
| "$A_1xA_2$" | 16 | $\underline{Y}'B_{F_1F_2}\underline{Y} - (a_3 + a_4 + a_5 + a_6 + a_7)$ |

TABLE 28

ABBREVIATED AOV FOR BLOCKING PLAN ($h_7$) IF EXAMPLE 16

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | 31 | $\underline{Y}'(I_{32}-(1/32)J^{32}_{32})\underline{Y}$ |
| B.A.B. | 15 | $\frac{1}{2}\sum_k b_k^2 - (1/32)(y_{..})^2$ |
| $A_{1.1}$ | 1 | $a_1$ in table 27 |
| $A_{2.1}$ | 1 | $a_2$ in table 27 |
| $A_{1.1}xA_{2.1}$ | 1 | $a_3$ in table 27 |
| BC | 1 | $\frac{1}{4}\sum_{k=1}^{8}(b_{2k-1}+b_{2k})^2 - (1/32)(y_{..})^2 = a_4$ |
| AC | 1 | |
| AB | 1 | |
| $A_{1.1}xAC$ | 1 | |
| $A_{2.1}xAC$ | 1 | |
| $A_{1.1}xA_{2.1}xAC$ | 1 | |
| $A_{1.1}xBC$ | 1 | $a_5 = \frac{1}{2}\sum_{k=1}^{16} b_k^2 - a_1 - a_2 - a_3 - a_4 - \frac{1}{32}(y_{..})^2$ |
| $A_{2.1}xBC$ | 1 | |
| $A_{1.1}xA_{2.1}xBC$ | 1 | |
| $A_{1.1}xAB$ | 1 | |
| $A_{2.1}xAB$ | 1 | |
| $A_{1.1}xA_{2.1}xAB$ | 1 | |
| W.A.B. | 16 | |
| $A_{1.(1,.)}$ | 2 | $\underline{Y}'B_{F_1}\underline{Y} - a_1$ |
| $A_{2.(1,.)}$ | 5 | $\underline{Y}'B_{F_2}\underline{Y} - a_2 - a_4$ |
| "$A_1xA_2$" | 9 | $\underline{Y}'B_{F_1F_2}\underline{Y} - a_3 - a_5$ |

$$x_3=(\tfrac{1}{16})((b_1+b_2+b_3+b_4+b_{13}+b_{14}+b_{15}+b_{16})^2+(b_5+b_6+b_7+b_8+b_9+b_{10}+b_{11}+b_{12})^2)$$

$$x_4=(\tfrac{1}{16})((b_1+b_4+b_5+b_8+b_9+b_{12}+b_{13}+b_{16})^2+(b_2+b_3+b_6+b_7+b_{10}+b_{11}+b_{14}+b_{15})^2)$$

$$x_5=\sum_{k=1}^{16} b_k^2 - (1/32)(y_{..})^2 - a_1 - a_2 - a_3 - a_4$$

and $a_1 = x_1 - (1/32)(y_{..})^2$

$a_2 = x_2 - (1/32)(y_{..})^2$

$a_3 = x_3 - (1/32)(y_{..})^2$

$a_4 = x_4 - (1/32)(y_{..})^2$

$a_5 = x_5$ .

Example 17 : Consider the partitioning of example 10,

$10x9\text{-FAT} \longrightarrow (2_{11} + 3_{12} + 5_{13})(2_{21} + 3_{22} + 4_{23})\text{-s-FAT's.}$

The abbreviated analysis of variance table for blocking PLANs (a), (b), (c), (d) and (e) of Table 9 are given in Tables 29, 30, 31, 32 and 33.

The letters $a_1$, $a_2$, $a_3$ and $a_4$ will have the same meaning in all tables. An analysis of variance table can be constructed from a matrix L. In the following example the matrix L is given for a less than full replicate of a partitioned FAT.

Example 18 : Consider the partitioning

$7_1 6_2\text{-FAT} \longrightarrow (2_{11} + 3_{12} + 2_{13})(2_{21} + 2_{22} + 2_{23})\text{-s-FAT's.}$

Let $2_{11}$ and $2_{21}$ refer to the two lowest levels of factors one and two, let $2_{13}$ and $2_{23}$ refer to the two highest levels of factors one and two and let $3_{12}$ and $2_{22}$ refer to the middle levels of factors one and two. A matrix L is given for two PLANs, where the PLANs are defined by the subsets $S_1$ and $S_2$ of $S_D$, in Figure 14 and 15, respectively,

$S_1 = \{(00), (01), (10), (11), (12), (21)\}$

$S_1 = \{(00), (01), (10), (11), (12), (20), (21)\}$ .

TABLE 29

ABBREVIATED AOV TABLE FOR BLOCKING PLAN (a) OF EXAMPLE 17

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | 89 | $\underline{Y}'(I_{90} - (1/90)J_{90}^{90})\underline{Y} = \sum\sum_{ij} (y_{ij})^2 - (1/90)y_{..}^2$ |
| B.A.B. | 2 | $a_1 =$ |
| $A_{1.1}$ | 2 | $(\frac{1}{18})((b_1+b_2+b_3)^2+(b_4+b_5+b_6)^2+(b_7+b_8+b_9)^2)-\frac{1}{90}y_{..}^2$ |
| W.A.B. | 87 | |
| $A_{1.(1,.)}$ | 7 | $\underline{Y}'B_{F_1}\underline{Y} - a_1 = (1/9)\Sigma(y_{i.})^2 - (1/32)(y_{..})^2$ |
| $A_2$ | 8 | $\underline{Y}'B_{F_2}\underline{Y} = (1/10)\Sigma(y_{.j})^2 - (1/32)(y_{..})^2$ |
| $A_1 x A_2$ | 72 | $\underline{Y}'B_{F_{12}}\underline{Y} = \sum\sum_{ij} y_{ij}^2 - (1/10)\Sigma(y_{.j})^2 - (1/9)\Sigma(y_{i.})^2 + (1/90)(y_{..})^2$ |

TABLE 30

ABBREVIATED AOV TABLE FOR BLOCKING PLAN (b) OF EXAMPLE 17

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | 89 | $\underline{Y}'(I_{90} - (1/90)J_{90}^{90})\underline{Y}$ |
| B.A.B. | 2 | |
| $A_{2.1}$ | 2 | $a_2 = \frac{1}{20}((b_1+b_4+b_7)^2+(b_2+b_5+b_8)^2+(b_3+b_6+b_9)) - (1/90)(y_{..})^2$ |
| W.A.B. | 87 | |
| $A_1$ | 9 | $(1/9)\Sigma(y_{i.})^2 - (1/32)(y_{..})^2 = \underline{Y}'B_{F_1}\underline{Y}$ |
| $A_{2.(1,.)}$ | 6 | $(1/10)\Sigma(y_{.j})^2 - (1/32)(y_{..})^2 - a_2 = \underline{Y}'B_{F_2}\underline{Y} - a_2$ |
| $A_1 x A_2$ | 72 | $\underline{Y}'B_{F_1F_2}\underline{Y}$ |

TABLE 31

ABBREVIATED AOV TABLE FOR BLOCKING PLAN (c) OF EXAMPLE 17

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | 89 | $\underline{Y}'(I_{90} - (1/90)J_{90}^{90})\underline{Y}$ |
| B.A.B.<br>part of<br>$A_{1.1}xA_{2.1}$ | 2<br><br>2 | $a_3 = \frac{(b_1+b_6+b_8)^2}{31} + \frac{(b_2+b_4+b_9)^2}{32} + \frac{(b_3+b_5+b_7)^2}{27} - \frac{y_{..}^2}{90}$ |
| W.A.B. | 87 | |
| $A_1$ | 9 | $\underline{Y}'B_{F_1}\underline{Y}$ |
| $A_2$ | 8 | $\underline{Y}'B_{F_2}\underline{Y}$ |
| "$A_1xA_2$" | 70 | $\underline{Y}'B_{F_1F_2}\underline{Y} - a_3$ |

TABLE 32

ABBREVIATED AOV TABLE FOR BLOCKING PLAN (d) OF EXAMPLE 17

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | 89 | $\underline{Y}'(I_{90} - (1/90)J_{90}^{90})\underline{Y}$ |
| B.A.B.<br>part of<br>$A_{1.1}xA_{2.1}$ | 2<br><br>2 | $a_4 = \frac{(b_1+b_5+b_9)^2}{33} + \frac{(b_3+b_4+b_8)^2}{29} + \frac{(b_2+b_6+b_7)^2}{28} - \frac{y_{..}^2}{90}$ |
| W.A.B. | 87 | |
| $A_1$ | 9 | $\underline{Y}'B_{F_1}\underline{Y}$ |
| $A_2$ | 8 | $\underline{Y}'B_{F_2}\underline{Y}$ |
| "$A_1xA_2$" | 70 | $\underline{Y}'B_{F_1F_2}\underline{Y} - a_4$ |

TABLE 33

ABBREVIATED AOV TABLE FOR BLOCKING PLAN (e) OF EXAMPLE 17

| Source | DF | Sum of Squares |
|---|---|---|
| Total (corrected) | 89 | $\underline{Y}'(I_{90} - (1/90)J_{90}^{90})\underline{Y}$ |
| B.A.B. | 8 | |
| $A_{1.1}$ | 2 | $a_1$ |
| $A_{2.1}$ | 2 | $a_2$ |
| $A_{1.1}xA_{2.1}$ | 4 | $a_3 + a_4$ |
| W.A.B. | 81 | |
| $A_{1.(1,.)}$ | 7 | $\underline{Y}'B_{F_1}\underline{Y} - a_1$ |
| $A_{2.(1,.)}$ | 6 | $\underline{Y}'B_{F_2}\underline{Y} - a_2$ |
| "$A_1xA_2$" | 68 | $\underline{Y}'B_{F_1F_2}\underline{Y} - a_3 - a_4$ |

| 00 | 01 | 02 | 03 | 10 | 11 | 12 | 13 | 20 | 21 | 22 | 23 | 24 | 25 | 30 | 31 | 32 | 33 | 34 | 35 | 40 | 41 | 42 | 43 | 44 | 45 | 52 | 53 | 62 | 63 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $L_1$ |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | 0 | 0 | 0 | 0 | |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | -13 | -13 | -13 | -13 | $L_{F_{1.1}}$ |
| 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | -2 | -2 | -2 | -2 | -2 | -2 | 0 | 0 | 0 | 0 | $L_{F_{1.(1,.)}}$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | -1 | -1 | |
| 7 | 7 | -5 | -5 | 7 | 7 | -5 | -5 | 7 | 7 | -5 | -5 | 0 | 0 | 7 | 7 | -5 | -5 | 0 | 0 | 7 | 7 | -5 | -5 | 0 | 0 | -5 | -5 | -5 | -5 | |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | -4 | -4 | 1 | 1 | 1 | 1 | -4 | -4 | 1 | 1 | 1 | 1 | -4 | -4 | 1 | 1 | 1 | 1 | $L_{F_{2.1}}$ |
| 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 1 | -1 | $L_{F_{2.(1,.)}}$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | |
| 1 | -1 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | -2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | -3 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | -4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 1 | 1 | -2 | 0 | -1 | -1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | -2 | -2 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | 0 | 0 | -3 | -3 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | 0 | 0 | 1 | 1 | -2 | 0 | 0 | 0 | -4 | -4 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 1 | 1 | 1 | -3 | -1 | -1 | -1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $L_{F_1 x F_2}$ |
| 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | -2 | -2 | -2 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | 0 | 0 | -3 | -3 | -3 | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | 0 | 0 | 1 | 1 | 1 | -3 | 0 | 0 | -4 | -4 | -4 | 12 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 | -1 | -1 | 2 | 2 | 1 | 1 | 1 | 1 | -2 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 | -1 | -1 | 2 | 2 | -1 | -1 | -1 | -1 | 2 | 2 | 2 | 2 | 2 | 2 | -4 | -4 | 0 | 0 | 0 | 0 | |
| 0 | 0 | -2 | 2 | 0 | 0 | -2 | 2 | 0 | 0 | -2 | 2 | 0 | 0 | 0 | 0 | -2 | 2 | 0 | 0 | 0 | 0 | -2 | 2 | 0 | 0 | 5 | -5 | 5 | -5 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | -1 | 1 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | -2 | 2 | 0 | 0 | 0 | 0 | |

Figure 14. - A matrix L for the PLAN defined by subset $S_1$ in example 18.

| | 00 | 01 | 02 | 03 | 10 | 11 | 12 | 13 | 20 | 21 | 22 | 23 | 24 | 25 | 30 | 31 | 32 | 33 | 34 | 35 | 40 | 41 | 42 | 43 | 44 | 45 | 50 | 51 | 52 | 53 | 60 | 61 | 62 | 63 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| L = | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | -4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | -13 | -13 | -13 | -13 | -13 | -13 | -13 | -13 |
| | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | -2 | -2 | -2 | -2 | -2 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 |
| | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 |
| | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 1 | -1 | -1 | 1 | 1 | -1 | -1 | 1 | 1 | -1 | -1 | 0 | 0 | 1 | 1 | -1 | -1 | 0 | 0 | 1 | 1 | -1 | -1 | 0 | 0 | 1 | 1 | -1 | -1 | 1 | 1 | -1 | -1 |
| | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | -14 | -14 | 3 | 3 | 3 | 3 | -14 | -14 | 3 | 3 | 3 | 3 | -14 | -14 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| | 1 | -1 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | -2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | -3 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | -4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | -5 | 5 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | -6 | 6 | 0 | 0 |
| | 1 | 1 | -2 | 0 | -1 | -1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | -2 | -2 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | 0 | 0 | -3 | -3 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | 0 | 0 | 1 | 1 | -2 | 0 | 0 | 0 | -4 | -4 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | 0 | 0 | 1 | 1 | -2 | 0 | 0 | 0 | 1 | 1 | -2 | 0 | 0 | 0 | -5 | -5 | 10 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | 1 | 1 | -2 | 0 | 0 | 0 | 1 | 1 | -2 | 0 | 0 | 0 | 1 | 1 | -2 | 0 | 0 | 0 | 1 | 1 | -2 | 0 | -6 | -6 | 12 | 0 |
| | 1 | 1 | 1 | -3 | -1 | -1 | -1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | -2 | -2 | -2 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | 0 | 0 | -3 | -3 | -3 | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | 0 | 0 | 1 | 1 | 1 | -3 | 0 | 0 | -4 | -4 | -4 | 12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | 0 | 0 | 1 | 1 | 1 | -3 | 0 | 0 | 1 | 1 | 1 | -3 | 0 | 0 | -5 | -5 | -5 | 15 | 0 | 0 | 0 | 0 |
| | 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | 1 | 1 | 1 | -3 | 0 | 0 | 1 | 1 | 1 | -3 | 0 | 0 | 1 | 1 | 1 | -3 | 0 | 0 | 1 | 1 | 1 | -3 | -6 | -6 | -6 | 18 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | -4 | -4 | -1 | -1 | -1 | -1 | 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | -4 | -4 | 1 | 1 | 1 | 1 | -4 | -4 | -2 | -2 | -2 | -2 | 8 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Figure 15. - A matrix L for the PLAN defined by subset $S_2$ in example 18.

# CHAPTER VI

## DISCUSSION OF AN EXAMPLE

In this chapter an example is presented to illustrate how the use of methods developed in the preceding chapters can aid in the design and analysis of a real experimental situation. The main prerequisite for the use of partitioned factorial arrangement schemes is that the objective of the experiment be to investigate inter-factor and intra-factor relationships among two or more factors.

Consider an experiment designed to investigate the metabolism of protein in rats with induced pseudo-phenylketonuria, which is a condition assumed to be equivalent to phenylketonuria. Various amounts of the amino acids tyrosine and phenylalanine are added or deleted from the diets of the rats for a two week period. The two amino acids are to be studied at three levels: almost absent, normal and large amounts. After the two week feeding period, amounts of homogentisic acid (a measure of protein metabolism) are measured in daily urine samples for a seven day period. Analysis of these measurements can determine whether the response is affected by different levels of each amino acid, and if the response pattern for levels of one amino acid is the same at each level of the other amino acid. If the results of this study indicate that the response is not different for the various levels of the amino acids, then the study

can be terminated. On the other hand, if the results of this study indicate that the response is significantly affected by the various amounts of amino acids in the diet, then the investigator might desire to enlarge upon the experiment, utilizing additional levels (say five) to obtain more definitive information. In the enlarged experiment, the five levels could be, almost absent, below normal, normal, above normal and extremely large.

Since various amounts of the amino acids can be added or deleted from the diets, the factorial arrangement is an obvious choice for the treatment design. The factorial treatment design will allow the investigation of inter-amino acid and intra-amino acid relationships. The experimental unit is the rat and, since groups of homogeneous rats are readily available, a completely random assignment of the treatment combinations to units is sufficient. With the aim of studying all five levels of each amino acid, the total number of different treatment combinations (or diets) in a factorial arrangement is 25, where each treatment combination is a combination of levels, one level (amount) of each amino acid. In the context of this dissertation, the 25 diets are analogous to the 25 treatment combinations of a $5_1 5_2$-FAT. In the $5_1 5_2$-FAT, one factor is the phenylalanine and the other factor is the tyrosine. The levels of the two amino acids are represented by the numbers 0, 1, 2, 3 and 4, where

0 represents almost absent,

1 represents below normal,

2 represents normal,

3 represents above normal, and

4 represents extremely large.

The 25 diets composed of the varying amounts of the two amino acids, are represented by the 25 individual cells in Figure 16.

| Factor 1 (Phenylalanine) | Factor 2 (Tyrosine) 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 6 | 7 | 8 | 9 | 10 |
| 2 | 11 | 12 | 13 | 14 | 15 |
| 3 | 16 | 17 | 18 | 19 | 20 |
| 4 | 21 | 22 | 23 | 24 | 25 |

Figure 16. - A representation of the 25 food diets.

In view of the investigator's desire to run an initial experiment to determine the first objective, namely whether there are significant effects with the three different levels of each amino acid, the investigator could partition the experiment utilizing only the lowest and highest levels along with the middle level, as one set (this set would correspond to levels 0, 2 and 4). The second set would include the other two levels, 1 and 3. Thus, the five levels for each factor have been separated into two subsets. These sets of levels are represented by $3_{11}$ and $2_{12}$ for factor one and $3_{21}$ and $2_{22}$ for factor two. The algebraic partitioning

$$5_1 5_2\text{-FAT} \longrightarrow (\, 3_{11} + 2_{12} \,)(\, 3_{21} + 2_{22} \,)\text{-s-FAT's} \qquad (30)$$

results in the four s-FAT's $3_{11}3_{21}$-s-FAT, $3_{11}2_{22}$-s-FAT. $2_{12}3_{21}$-s-FAT and $2_{12}2_{22}$-s-FAT. These s-FAT's are represented in Figure 17 by the letters "a", "b", "c" and "d". The nine a's represent the $3_{11}3_{21}$-s-FAT, which is a combination of the lowest, middle and highest levels of each factor; while the four d's represent the $2_{12}2_{22}$-s-FAT, which corresponds to the combinations of the remaining two levels of each factor. The letters b and c correspond to the $3_{11}2_{22}$-s-FAT and $2_{12}3_{21}$-s-FAT. These s-FAT's are

Factor 2 (Tyrosine)

Factor 1 (Phenylalanine)

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | a | b | a | b | a |
| 1 | c | d | c | d | c |
| 2 | a | b | a | b | a |
| 3 | c | d | c | d | c |
| 4 | a | b | a | b | a |

Figure 17. - A representation of the four s-FAT's.

composed of treatment combinations of the three levels of one amino acid and the two levels of the other amino acid. Thus, the $3_{11}2_{22}$-s-FAT, designated by the letter "b", represents combinations of the low, middle and high levels of phenylalanine with the one and three (below normal and above normal) levels of tyrosine; while the $2_{12}3_{21}$-s-FAT, designated by the letter "c", represents combinations of the two levels (1 and 3) of phenylalanine with the low, middle and high (0, 2 and 4) levels of tyrosine.

The initial experiment is equivalent to running the $3_{11}3_{21}$-s-FAT. Since rats are likely to be readily available, the statistician can suggest that two rats receive each treatment combination. The urine of each rat is measured each day for seven consecutive days. The seven days can be considered as seven levels of a third factor and, in view of this third factor, the partitioning (30) can be expressed as

$$5_1 5_2 7_3\text{-FAT} \longrightarrow (3_{11} + 2_{12})(2_{21} + 2_{22})7 \text{ -s-FAT's}, \qquad (31)$$

so the experimental situation is more adequately described. The partitioning (31) results in the four s-FAT's $3_{11}3_{21}7_3$-s-FAT, $3_{11}2_{22}7_3$-s-FAT, $2_{12}3_{21}7_3$-s-FAT and $2_{12}2_{22}7_3$-s-FAT. The initial study is now equivalent to running two replicates of a 3x3x7-FAT, where the factors one and two

represent the amino acids (at levels 0, 2 and 4) and factor three is number of days after the initial two week feeding period (letting 0 represent day one, 1 represent day two, ..., and 6 represent day 7). The analysis of the observations of the $3_{11}3_{21}7_3$-s-FAT are summarized in Table 34. In the experiment, reasonable statements to investigate are that the effects of the three levels of phenylalanine are not different with respect to the response measured (amounts of homogentisic acid) and that the effects of the three levels of tyrosine are not different with respect to the response measured. In statistical terminology, these two statements are equivalent to hypotheses of zero main effects for factors one and two. Another aim of the initial study is to determine whether or not the pattern of response for one factor is the same at each level of the other factor. This aim can be statistically investigsted by obtaining evidence for or against a hypothesis of zero interaction between factor one and factor two.

If, in fact, the three levels of factor one (phenylalanine) do affect the response measured, then, hopefully, the results of the initial experiment will produce evidence for rejecting the hypothesis of a zero factor one main effect. A similar statement can be made for factor two (tyrosine); factor three (days); and for the factor interactions. The fact that each rat is measured on seven consecutive days puts the experimental design in a repeated measures situation. Since each treatment combination is applied to two units, the MS(e) of Table 34 is an appropriate term for significance testing purposes (because it is a measure of the failure of units (rats) treated alike to respond alike, which is experimental error). In Table 34, for i = 1, ..., 7, the significance

TABLE 34

ANALYSIS OF VARIANCE TABLE FOR THE INITIAL EXPERIMENT

| Source | DF | SS | MS | MSR | SL |
|---|---|---|---|---|---|
| Total (corrected) | 125 | | | | |
| Phenylalanine (0,2,4) | 2 | SS(P) | MS(P) | MS(P)/MS(e) | $a_1$ |
| Tyrosine (0,2,4) | 2 | SS(T) | MS(T) | MS(T)/MS(e) | $a_2$ |
| Days | 6 | SS(D) | MS(D) | MS(D)/MS(e) | $a_3$ |
| Phenylalanine x Tyrosine | 4 | SS(PxT) | MS(PxT) | MS(PxT)/MS(e) | $a_4$ |
| Phenylalanine x Days | 12 | SS(PxD) | MS(PxD) | MS(PxD)/MS(e) | $a_5$ |
| Tyrosine x Days | 12 | SS(TxD) | MS(TxD) | MS(TxD)?MS(e) | $a_6$ |
| Phenyl. x Tyrosine x Days | 24 | SS(PxTxD) | MS(PxTxD) | MS(PxTxD)?MS(e) | $a_7$ |
| Residual | 63 | SS(e) | MS(e) | | |

levels $a_i$ indicate the strength of the evidence against the hypothesis that main effects or interaction effects, whichever the case may be, are zero. If either $a_1$ or $a_2$ is judged significant (about .05 or smaller), while $a_4$, $a_5$, $a_6$ and $a_7$ are judged not significant, then there is evidence for a difference in response due to difference in effects of levels of factor one or two. Of course, the experiment can be continued for other reasons (to study the phenylalanine by tyrosine interaction, if $a_4$ is judged significant) and the experiment can be terminated for other reasons (although there may be statistical evidence for differences, the differences exhibited by the data are so small that they are of no practical importance).

Suppose the decision is made to continue the experiment by running the three remaining s-FAT's. The sequence or order in which the three s-FAT's are run might or might not be important. The three s-FAT's might be run at one time in a completely random design. Perhaps the investigator can run only one s-FAT at a time. If this is the case, then the following sequenced PLANs exist:

PLAN (1): $3_{11}2_{22}7_3$-s-FAT $\longrightarrow$ $2_{12}3_{21}7_3$-s-FAT $\longrightarrow$ $2_{12}2_{22}7_3$-s-FAT,

PLAN (2): $3_{11}2_{22}7_3$-s-FAT $\longrightarrow$ $2_{12}2_{22}7_3$-s-FAT $\longrightarrow$ $2_{12}3_{21}7_3$-s-FAT,

PLAN (3): $2_{12}3_{21}7_3$-s-FAT $\longrightarrow$ $3_{11}2_{22}7_3$-s-FAT $\longrightarrow$ $2_{12}2_{22}7_3$-s-FAT,

PLAN (4): $2_{12}3_{21}7_3$-s-FAT $\longrightarrow$ $2_{12}2_{22}7_3$-s-FAT $\longrightarrow$ $3_{11}2_{22}7_3$-s-FAT,

PLAN (5): $2_{12}2_{22}7_3$-s-FAT $\longrightarrow$ $3_{11}2_{22}7_3$-s-FAT $\longrightarrow$ $2_{12}3_{21}7_3$-s-FAT and

PLAN (6): $2_{12}2_{22}7_3$-s-FAT $\longrightarrow$ $2_{12}3_{21}7_3$-s-FAT $\longrightarrow$ $3_{11}2_{22}7_3$-s-FAT.

Writing these PLANs, including the initial experiment ((00)), in terms of pseudo-design points, one obtains

PLAN (1): (00) $\longrightarrow$ (01) $\longrightarrow$ (10) $\longrightarrow$ (11),

PLAN (2): (00)⟶(01)⟶(11)⟶(10),

PLAN (3): (00)⟶(10)⟶(01)⟶(11),

PLAN (4): (00)⟶(10)⟶(11)⟶(01),

PLAN (5): (00)⟶(11)⟶(01)⟶(10) and

PLAN (6): (00)⟶(11)⟶(10)⟶(01).

In the context of this dissertation, it is easy to see that all six PLANs, if they are performed as an entire experiment, are complete PLANs, and consequently, are connected PLANs. If the sequence of application is taken into account, then only PLANs (2) and (4) are connected. If foresight indicates the experiment run in sequence might be prematurely ended, then either PLAN (2) or PLAN (4) is a suitable choice, since they are step-wise connected. Moreover, if the analysis of the initial experiment indicates that the factor one main effect is highly significant while the factor two main effect is not significant, then it seem reasonable that additional levels of factor one should be next in order of investigation. Thus, PLAN (4) is preferable to PLAN (2) since the application of the second s-FAT involves different levels of factor one, while application of the second s-FAT in PLAN (2) involves different levels of factor two.

Now, suppose all four s-FAT's had been run. The entire experiment is now equivalent to a $5_1 5_2 7_3$-FAT run in two replicates. The results of the experiment can be summarized in the analysis of variance table given in Table 35. In a manner similar to the analysis of the initial experiment, the significance levels $a_1$ through $a_7$ in Table 35 can be used to assess the strength of the evidence against hypotheses of zero main effects and zero interaction effects. Of course, it must be realized that

TABLE 35

ANALYSIS OF VARIANCE FOR TWO REPLICATES OF THE 5x5x7-FAT

| Source | DF |
|---|---|
| Total (corrected) | 349 |
| Phenylalanine (0,1,2,3,4) | 4 |
| Tyrosine (0,1,2,3,4) | 4 |
| Days (0,1,2,3,4,5,6) | 6 |
| Phenylalanine x Tyrosine | 16 |
| Phenylalanine x Days | 24 |
| Tyrosine x Days | 24 |
| Phenylalanine x Tyrosine x Days | 96 |
| Residual | 175 |

inferences concerning the factor one and two main effects and interaction effects are made with respect to the five levels for each factor.

Next, suppose that some distinguishing characteristic of the units (rats), such as type or strain, can be used to separate the group of fifty rats into two smaller groups. A rat is either of strain A or strain B, and thus, the experimental units can be divided into subgroups according to this characteristic. For this illustrative example, these groups are labeled $G_A$ and $G_B$. In the context of this thesis, the groups $G_A$ and $G_B$ are referred to as blocks (of rats). If the strain of rat is known or suspected to have an effect on the ellicited measurement, then the rats of the two strains will respond differently to the treatments. The difference in response due to strain is automatically a part of the experiment and must be dealt with in the designing and analysis of the experiment.

Precaution must be taken in the assignment of treatment combinations to rats so that the between strain (or between groups) effect will not bias any of the between level comparisons for either factor one or factor two. In other words, the experiment must be designed so factor effects can be investigated irrespective of the strain effect. To illustrate why this precaution must be taken, suppose the rats of strain A receive all the treatments involving the 0 and 1 levels of factor one and the rats of strain B receive all the treatments involving the 2, 3 and 4 levels of factor one. Now, the difference between, (1) the average of the responses for the rats receiving treatments involving the 0 and 1 levels of factor one and, (2) the average of the responses of the rats receiving treatments involving the 2, 3 and 4 levels of factor one, is

a measure of the strain effect and also, a measure of the effect of levels 0 and 1 versus levels 2, 3 and 4 of factor one. A difference between the averages (1) and (2) (say (1) minus (2)) is hard to intrepret because one cannot be sure whether this difference is due to strain, levels of factor one or a combination of strain and levels of factor one. In this situation and in the context of this thesis, the strain effect (block effect) is said to be confounded with a component of the factor one main effect (average of levels 0 and 1 versus average of levels 2, 3 and 4). Since the purpose of the experiment is to investigate the effects of different levels of factors one and two, it is imperative not to confound the strain effects with the two factor (main) effects.

Suppose there are 24 rats in $G_A$ (strain A) and 26 rats in $G_B$ (strain B). Previously, the experiment was described as two full replicates of a $5_1 5_2 7_3$-FAT. By the partitioning mentioned earlier, namely

$$5_1 5_2 7_3\text{-FAT} \longrightarrow (3_{11}+2_{12})(3_{21}+2_{22})7_3\text{-s-FAT's},$$

four s-FAT's resulted. By methods developed in chapter four, one can obtain a scheme that assigns the treatments of the $3_{11}3_{21}$-s-FAT and the $2_{12}2_{22}$-s-FAT to the group of 26 rats and the treatments of the $3_{11}2_{22}$-s-FAT and $2_{12}3_{21}$-s-FAT to the group of 24 rats. Figure 18 gives a more detailed of the assignment of treatments to rats. The result of this assignment scheme is that the strain effect is not confounded with any part of a main effect for factor one or two. However, to obtain this clarity on the information relating to the effects of levels of the factors, one must sacrifice clarity in some other aspect of the experiment. In this case, the strain effect has been confounded with a component of the interaction between factors one and two. Analytical

Block 1 (group of 26 rats)

| | | | | | | |
|---|---|---|---|---|---|---|
| (000) | (001) | (002) | (003) | (004) | (005) | (006) |
| (010) | (011) | (012) | (013) | (014) | (015) | (016) |
| (020) | (021) | (022) | (023) | (024) | (025) | (026) |
| (100) | (101) | (102) | (103) | (104) | (105) | (106) |
| (110) | (111) | (112) | (113) | (114) | (115) | (116) |
| (120) | (121) | (122) | (123) | (124) | (125) | (126) |
| (200) | (201) | (202) | (203) | (204) | (205) | (206) |
| (210) | (211) | (212) | (213) | (214) | (215) | (216) |
| (220) | (221) | (222) | (223) | (224) | (225) | (226) |
| (330) | (331) | (332) | (333) | (334) | (335) | (336) |
| (340) | (341) | (342) | (343) | (344) | (345) | (346) |
| (430) | (431) | (432) | (433) | (434) | (435) | (436) |
| (440) | (441) | (442) | (443) | (444) | (445) | (446) |

Block 2 (group of 24 rats)

| | | | | | | |
|---|---|---|---|---|---|---|
| (030) | (031) | (032) | (033) | (034) | (035) | (036) |
| (040) | (041) | (042) | (043) | (044) | (045) | (046) |
| (130) | (131) | (132) | (133) | (134) | (135) | (136) |
| (140) | (141) | (142) | (143) | (144) | (145) | (146) |
| (230) | (231) | (232) | (233) | (234) | (235) | (236) |
| (240) | (241) | (242) | (243) | (244) | (245) | (246) |
| (300) | (301) | (302) | (303) | (304) | (305) | (306) |
| (310) | (311) | (312) | (313) | (314) | (315) | (316) |
| (320) | (321) | (322) | (323) | (324) | (325) | (326) |
| (400) | (401) | (402) | (403) | (404) | (405) | (406) |
| (410) | (411) | (412) | (413) | (414) | (415) | (416) |
| (420) | (421) | (422) | (423) | (424) | (425) | (426) |

Figure 18. - Scheme assigning treatment combinations to blocks. A rat in a block is randomly assigned all treatment combinations in a row.

procedures for this kind of situation are developed in Chapter five of this thesis. For this particular situation, an analysis of variance table will be identical to Table 36. In Table 36 attention is directed to the fact that one degree of freedom of the phenylalanine by tyrosine interaction is lost (compare with Table 35). This one degree of freedom is now attributed to the between groups source of variation. Information on the other sources of variation (factor main effects and interaction effects) is the same as in Table 35.

TABLE 36

ABBREVIATED AOV TABLE FOR TWO REPLICATES OF A PARTITIONED $5_1 5_2 7_3$-FAT RUN IN TWO BLOCKS

| Source | DF |
|---|---|
| Total | 349 |
| Between Groups (blocks) | 1 |
| part of (Phenylalanine x Tyrosine) | 1 |
| Within Groups (blocks) | 348 |
| Phenylalanine | 4 |
| Tyrosine | 4 |
| Days | 6 |
| Phenylalanine x Tyrosine | 15 |
| Phenylalanine x Days | 24 |
| Tyrosine x Days | 24 |
| Phenylalanine x Tyrosine x Days | 96 |
| Residual | 175 |

# CHAPTER VII

## SUMMARY

This dissertation investigated some of the statistical design and analysis problems occurring in comparative experiments that are formulated to study inter-factor and intra-factor relationships among several factors of interest. More specifically, experiments having a factorial treatment design and a completely random (unit) design or block design were considered in detail. Methods were developed that allow the partitioning of a full replicate of factorially arranged treatment combinations (referred to in this study as a FAT) into disjoint subsets of factorially arranged treatment combinations (referred to in this study as s-FAT's). These procedures can be used for experiments that cannot be performed at one time or in one place and must therefore be performed in parts. The generating of partitioned factorial arrangements is especially suited for experimental situations in which a priority of interest can be placed on the levels of some or all of the factors under investigation. These methods can also be used to combine experiments that investigate the same factors, but not necessarily the same levels. This is accomplished by treating the seperate experiments as parts or pieces of a larger experiment in a manner such that the seperate experiments can be obtained by some partitioning of the larger experiment. Schemes incorporating various combinations of the s-FAT's were developed for completely random

designs (no blocks) and for experimental situations where blocks were present (these schemes were referred to as PLANs or blocking PLANs, whichever the case may be).

If the order in which the groups of s-FAT's are performed is important, then the concepts of complete and connected designs were found to be useful in selecting a sequence of s-FAT's that assures the attainment of statistical information about inter-factor and intra-factor relationships. The methods and analysis procedures required the assumption of a linear observational model. The statistical concepts of effects, factor main effects and interaction effects among factors were given meaning with respect to population means and unbiased estimates of these effects were given. Methods were developed that led to the construction of analysis of variance tables for full replicates, multiple full replicates and full replicates of partitioned factorial arrangements performed in the presence of blocks (with confounding of various treatment effects with block effects). A specific example was given for a situation having observations of a less than full replicate of a partitioned factorial arrangement. In Chapter VI an example was presented to illustrate the use of methods that were developed in preceding chapters.

There are several problems concerning partitioned factorials that remain uninvestigated. The running of s-FAT's in sequence and the sequential analysis of this sequence needs statistical inquiry. Associated with this sequential aroblem are problems of response surface methodology. The use of partitioned factorial arrangements for combining experiments needs to be expanded as does further investigation of analysis procedures for the case where some, but not all, of the s-FAT's have observations.

In particular, experiments in which a large number of factors and levels make full replicates of the treatment combinations virtually impossible or impracticable, need more thorough investigation. Finally, it is suggested that the use of graph theory in a more thorough study of connectedness and tensor products in investigating the structure of design matrices may prove profitable.

LIST OF REFERENCES


1. Addelman, S. 1962 Symmetrical and asymmetrical fractional factorial plans. Technometrics 4: 47-58.

2. Addelman, S. 1963 Techniques for constructing fractional replicate plans. J. Amer. Stat. Assn. 58: 45-71.

3. Banerjee, K.S. 1963 Index numbers for factorial effects and their connection with a special kind of irregular fractional plan of a factorial experiment. J. Amer. Stat. Assn. 58: 497-512.

4. Banerjee, K.S. and Federer, W.T. 1964 Estimates of effects for fractional replicates. Ann. Math. Stat. 35: 711-715.

5. Banerjee, K.S. and Federer, W.T. 1965 On a special subset giving an irregular fractional replicate of a $2^n$ factirial experiment. J. Royal Stat. Soc., Series B 29: 292-303.

6. Banerjee, K.S. and Federer, W.T. 1966 On estimation and construction in fractional replications. Ann. Math. Stat. 37: 1033-1039.

7. Bose, R.C. 1947 Mathematical theory of the symmetrical factorial design. Sankhya 8: 107-166.

8. Bose, R.C. and Connor, W.S. 1960 Analysis of fractionally replicated $2^n3^m$ designs. Bulletin de l'institut International de Statistique 37: 141-160.

9. Bose, R.C. and Srivastava, J.N. 1964 Analysis of irregular factorial fractions. Sankhya A 26: 117-144.

10. Box, G.E.P. and Hunter, J.S. 1961 The $2^{k-p}$ fractional factorial designs I. Technometrics 3: 311-352.

11. Box, G.E.P. and Hunter, J.S. 1961 The $2^{k-p}$ fractional factorial designs II. Technometrics 3: 449-458.

12. Chakravarti, I.M. 1956 Fractional replication in asymmetrical factorial designs and partially balanced arrays. Sankhya 17: 143-164.

13. Cochran, W.G. and Cox, G.M. 1957 Experimental Designs. John Wiley and Sons, Inc., New York.

14. Connor, W.S. 1960 Fractional factorial experiment designs of mixed $2^n3^m$ series. Ind. Eng. Chem. 52: 69A-71A.

15. Connor, W.S. and Young, S. 1961 Fractional factorial designs for experiments with factors at 2 and 3 levels. Nat. Bur. of Standards Applied Mathematics Series 58.

16. Connor, W.S. and Zelen, M. 1959 Fractional factorial experiment design for factors at three levels. Nat. Bur. of Standards Applied Mathematics Series 54.

17. Cox, D.R. 1958 Planning of Experiments. John Wiley and Sons, Inc., New York.

18. Daniel, C. 1956 Fractional replication in industrial research. Berkeley Symposium on Mathematical Statistics and Probability volumne V, 87-98.

19. Daniel, C. 1962 Sequences of fractional replicates in the $2^{p-q}$ series. J. Amer. Stat. Assn. 57: 403-429.

20. Davies, O.L. 1954 Design and Analysis of Industrial Experiments. Oliver and Boyd, Ltd., London.

21. Davies, O.L. and Hay, W.A. 1950 The construction and uses of fractional factorial designs in industrial research. Biometrics 6: 233-249.

22. Dykstra, S. 1959 Partial duplication of factorial experiments. Technometrics 1: 63-75.

23. Finney, D.J. 1945 The fractional replication of factorial arrangements. Ann. of Eugenics 12: 291-301.

24. Fisher, R.A. 1925 Statistical Methods for Research Workers. Oliver and Boyd, Ltd., London.

25. Fisher, R.A. 1935 The Design of Experiments. Oliver and Boyd, Ltd., London.

26. Fisher, R.A. 1945 A system of confounding for factors with more than two alternatives giving completely orthogonal cubes and higher powers. Ann. of Eugenics 12: 283-290.

27. Fry, R.E. 1961 Finding new fractions of factorial experimental designs. Technometrics 3: 359-370.

28. Gateley, W.Y. 1962 Application of the Generalized Inverse Concept to the Theory of Linear Statistical Models. Unpublished doctoral dissertation, Oklahoma State University.

29. Graybill, F.A. 1961 An Introduction to Linear Statistical Models. McGraw-Hill Book Company, New York.

30. Halmos, P.R. 1958 Finite-Dimensional Vector Spaces. D.VanNorstrand Co., Inc., Princeton, New Jersey.

31. Huster, J.S. 1964 Sequential factorial estimation. Technometrics 6: 41-49.

32. John, P.M.W. 1961 Three quarter replicates of $2^4$ and $2^5$ designs. Biometrics 17: 319-321.

33. Kempthorne, O. 1947 A simple approach to confounding and fractional replication in factorial experiments. Biometrika 34: 255-274.

34. Kempthorne, O. 1952 The Design and Analysis of Experiments. John Wiley and Sons, Ltd., New York.

35. Morrison, M. 1956 Fractional replication for mixed series. Biometrics 12: 1-19.

36. National Bureau of Standards 1957 Fractional factorial experiment designs for factors at two levels. Nat. Bur. of Standards Applied Mathematics Series 48.

37. Plackett, R.L. 1946 Some generalizations in the multifactorial design. Biometrika 33: 328-332.

38. Prairie, R.R. and Zimmer, W.J. 1964 $2^P$ factorial experiments with factors applied sequentially. J. Amer. Stat. Assn. 59: 1205-1216.

39. Prairie, R.R. and Zimmer, W.J. 1968 Fractional replicates of $2^P$ factorial experiments with factors applied sequentially. J. Amer. Stat. Assn. 63: 644-652.

40. Raktoe, B.L. 1969 Combining elements from distinct finite fields in mixed factorials. Ann. Math. Stat. 40: 498-504.

41. Shah, K.R. 1969 Uniformly better combined estimators in factorial arrangements with confounding. J. Amer. Stat. Assn. 62: 638-642.

42. Thomas, H.L. 1964 Partitioned Factorials. Unpublished doctoral dissertation, Oklahoma State University.

43. Westlake, W.J. 1965 Composite designs based on irregular fractions of factorials. Biometrics 21: 324-336.

44. White, D. and Hultquist, R.A. 1965 Construction of confounding plans for mixed factorial designs. Ann. Math. Stat. 36: 1256-1271.

45. Williams, D.R. 1963 A New Approach of Factorial Experimentation. Unpublished doctoral dissertation, Oklahoma State University.

46. Winer, B.J. 1962 Statistical Principles in Experimental Design. McGraw-Hill Book Company, New York.

47. Yates, F. 1933 Complex experiments. Supplement to J. Royal Stat. Society 2: 181-247.

48. Zacks, S. 1963 On a complete class of linear unbiased estimators for randomized factorial experiments. Ann. Math. Stat. 34: 769-779.

49. Zacks, S. 1964 Generalized least squares estimators for randomizes fractional replication designs. Ann. Math. Stat. 35: 696-704.

# APPENDIXES

## APPENDIX 1

## ELEMENTARY MATRIX CONCEPTS

Let A be an n by m nontrivial matrix and let A' be the transpose of A.

Theorem 1: If the rank of A is r, then A = BC, where B is an n by r matrix and C is an r by m matrix. (This factorization is not necesarily unique).

Definition 1: The generalized inverse of A, denoted by $A^+$, is $A^+ = C'(CC')^{-1}(B'B)^{-1}B'$ if A = BC. (See Gateley(28)).

Theorem 2: $A^+$ is unique.

Theorem 3: Given A, if there exists X such that AXA = A, XAX = X, AX = (AX)' and XA = (xa)', then $X = A^+$.

Theorem 4: $(A^+)' = (A')^+$.

Theorem 5: Let the rank of A be denoted by r(A). Then $r(A) = r(A^+) = r(A^+A) = r(AA^+) = tr(A^+A) = tr(AA^+)$, where tr(A) denotes the trace of the matrix A.

Theorem 6: AX = C is consistent if and only if $AA^+C = C$.

Theorem 7: If AX = C is consistent, then the general solution is $X = A^+C + (I - A^+A)Y$, where I is the identity matrix and Y is arbritrary.

Theorem 8: If r(A) = m, then $A^+ = (A'A)^{-1}A'$ and $A^+A = I_m$.

If r(A) = n, then $A^+ + A'(A'A)^{-1}$ and $AA^+ = I_n$.

## APPENDIX 2

### A RESULT CONCERNING THE SUM OF SQUARES DUE TO CERTAIN EFFECTS

Let $\underline{Y}$ be an m by one vector of observations from the linear observational model $\underline{Y} = \underline{M} + \underline{e}$ and assume that $E(\underline{Y}) = \underline{M}$. Also, let $L_w$ be a d by m matrix defining the effect $L_w\underline{M}$ and such that the rows of $L_w$ form an orthogonal set of one by m vectors. The matrix $H_w$ is the row-wise normalized matrix $L_w$, therefore $H_w = D_w L_w$ ($D$ is diagonal). Let $B_w = H_w'H_w$ and suppose $H_a$ is a d by m matrix and let $B_a = H_a'H_a$.

Theorem 1: For all m by one vectors $\underline{Y}$, $\underline{Y}'B_w\underline{Y} = \underline{Y}'B_a\underline{Y}$ if and only if there exists an orthogonal matrix G such that $H_a = GH_w$.

Proof: $\underline{Y}'B_a\underline{Y} = \underline{Y}'B_w\underline{Y}$ if and only if $B_a = B_w$ (for all $\underline{Y}$).

Now, $H_a'H_a = H_w'H_w$ and $H_a = (H_wH_a')^{-1}H_w = GH_w$, since $H_wH_a'$ is d by d of rank d. So far, a matrix G exists, namely $G = (H_wH_a')^{-1}$. G is orthogonal since

$H_a'H_a = H_w'G'GH_w = H_w'H_w$, $H_wH_w'G'GH_wH_w' = H_wH_w'H_wH_w' = I_d$ and $G'G=I$.

Now assume there exists a G such that $H_a = GH_w$.

$B_a = H_a'H_a = H_w'G'GH_w = H_w'H_w = B_w$ and $\underline{Y}'B_a\underline{Y} = \underline{Y}'B_w\underline{Y}$ for all $\underline{Y}$.

Remark 1: In the context of the theorem, note that $H_aH_a' = I_d$.

Remark 2: If $H_a$ is written as $H_a = D_aL_a$, where $D_a$ is diagonal and $L_a$ is row-wise orthogonal, then it can be shown that $L_a = CL_w$, where $C = D_a^{-1}(H_wH_a')^{-1}D_w$, C is nonsingular and $(CD_w^{-1})(CD_w^{-1})' = D_a^{-2}$.