THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

AN INVESTIGATION INTO THE STRUCTURE OF ELECTRON FLUID-DYNAMICAL WAVES

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

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Norman, Oklahoma

1967

AN INVESTIGATION INTO THE STRUCTURE OF ELECTRON

FLUID-DYNAMICAL WAVES

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ACKNOWLEDGMENTS

The unfailing enthusiasm and personal interest shown by Professor Richard G. Fowler in directing this study are deeply appreciated. The patience, love, and understanding of my wife, Jo Ann, have been a source of strength throughout these years of graduate study. The financial support of the National Science Foundation has enabled me to carry this investigation to its present state. Acknowledgment is also due to all members of my committee for their interest and suggestions and to Mrs. Bennie Peterson for her skill in typing this thesis.

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LIST OF SYMBOLS

α	Nondimensionalized ionization potention.
β	Ionization frequency
$\Delta(y)$	Transfer operator for indicated quantity.
ε_o	MKS permittivity.
θ	Nondimensionalized electron temperature
κ	Constant determining relation of wave speed to applied field.
λ	Characteristic length used for nondimensionalizing equations.
μ	Nondimensionalized ionization frequency.
ν	Nondimensionalized electron density.
ξ	Nondimensionalized length coordinate.
ϕ_i	Ionization potential.
ψ	Nondimensionalized electron velocity.
e	Electron charge.
E	Electric field.
ε	Reduced electric field.
k	Boltzmann's constant.
m	Electron mass.
T	Temperature.

CHAPTER I

INTRODUCTION

Since the beginning of time, luminous pulses due to large potential differences between two points have been observed in the form of lightning strokes. To the best of the author's knowledge, Hauksbee in 1705 was the first man to pay serious attention to luminous pulses in evacuated chambers. As experimental evidence has accumulated, certain characteristics of such luminous pulses have begun to emerge. All observers report that the pulses travel with speeds approaching the speed of light. No Doppler shift of the emitted radiation has been detected so the excited atoms are not in motion and there is no mass motion. All attempts to establish these luminous pulses as solutions of Maxwell's equations have failed. On the basis of accumulated evidence, most observers have been led in recent years to the conclusion that these pulses are basically fluid phenomena. They are supersonic even with respect to electron acoustic speeds and, therefore, are shock waves. A fluid phenomenon involving no mass motion must be due to electron fluid action; hence, one derives the name electron fluid-dynamical waves.

In spite of the length of time these phenomena have been known, they are understood in only a rudimentary way. No work, experimental or theoretical, has yet resulted in a comprehensive classification, description, or understanding of the phenomena. Evaluation of the data recorded by experimenters has been complicated because data taken by one observer has little relation to the data taken by another observer. There are several reasons for this. First, each experimenter employs his own geometrical configuration for the discharge tube. Second, each observer reports his data in a form which seems meaningful under his operating conditions. Finally, there is no agreement as to what constitutes a significant variable.

The confusion which reigns is to be expected. Observation of electron fluid-dynamical waves involves resolution of events which occur in the period of about a nanosecond. Technology which permits such resolution is still in the embryonic stage; mastery of the art is itself an accomplishment independent of the value of the data obtained. More important, no theory has emerged that indicates the advantages of a particular geometry for the discharge tube - any configuration in which the desired phenomena can be produced has been acceptable. Lack of a good theory also makes determination of what constitutes a significant variable pure guess work.

The present work is an attempt to present a unified theory for electron fluid-dynamical waves. A one-dimensional, time independent, continuous theory is developed for proforce waves moving into neutral, non-ionized gas. It is shown that the wave may be divided into two regions: A thin sheath, located at the very front of the wave, in which the electric field falls rapidly to a negligible value and the electrons come to rest relative to the heavy particles, followed by an extended region of quasi-local neutrality in which ionization continues and the electron fluid cools down. The description of the thin sheath layer results in a dynamical theory of Debye layers which is of interest in its own right. Wave speed and degree of ionization resulting from a given wave are also determined as functions of applied electric field and initial pressure. It is anticipated that the approach employed may be extended to antiforce waves, different geometries, and to time dependent cases. It is also hoped that this work may serve as a tool in the planning of future experimental work.

CHAPTER II

BACKGROUND

Wheatstone's speculation[2] in 1835 that the luminous pulses observed from a low pressure discharge tube subjected to high potential differences were actually waves propagating down the tube was probably the first identification of ionizing potential waves. He was unable to verify his suspicions due to a lack of equipment with sufficient time resolution. In 1893 J. J. Thompson[3] reported observation of a fast moving luminous pulse generated in an evacuated, 15 meter long discharge tube which he studied by means of a rotating mirror arrangement. He concluded that the speed of the pulse was about one half the speed of light. For the next thirty years various attempts were made to obtain worthwhile data, but success was very limited. In 1930 Beams[4] confirmed Thompson's observations and offered a qualitative explanation of the phenomenon. Essentially his explanation was that the gas behind the pulse was electrically conducting so that the pulse carried the potential of the discharge electrode (the electrode to which the potential is applied). The high electric field at the front was considered to be responsible for breakdown of the gas in this region. The motion of the front was thought to be the result of the mass difference between positive ions and electrons. This mass difference creates a space charge which warps the field so as to cause the wave front to move away from the discharge electrode. It will be shown that this view is consistent with the present work. Snoddy, Beams, and Dietrich[5] published the first report of their experimental data in 1936. Beams and his associates[6,7] followed this report with a series of reports in which the wave velocity for various segments of the discharge tube was determined as a function of applied potential, initial pressure, and tube diameter. It was found that the waves accelerated as they traveled along the tube. Wave speed also increased with increasing potential difference and increasing tube diameter. Wave speed increased with initial pressure up to a few torr and then decreased. These reports also stated that the waves generally traveled from the discharge electrode to ground regardless of the polarity of the applied potentials. Potentials ranging from 19 to 117 KV were used to produce wave speeds between 5×10^8 cm/sec and 10^{10} cm/sec. Schonland[8] made extensive studies on the speed of lightning pilot streamers. Lack of knowledge of the conditions existing in lightning discharges limits the usefulness of this information for comparison with theory. Schonland[9] was able to make some predictions of the minimum propagation speed of pilot streamers from qualitative energy considerations. However, he did not give a theory for the propagation of these waves.

Loeb[10] and his associates have published numerous reports on their studies of corona discharges. The geometry involved in these discharges is quite different from that employed by the experimenters whose work has already been discussed. None the less, the same basic mechanisms are active in corona discharges as in breakdown in long discharge tubes. Loeb hypothesized a qualitative model for breakdown of a gas in a point-anode, plane-cathode geometry. In his view, photons from excited atoms propagate through the gas ionizing and exciting new atoms in front of the wave front. The newly excited atoms in turn emit photons which continue the process. The net result is a wave moving forward on photo ionization. The detailed analysis necessary to make this a complete model has not yet been carried out. Unexplained in this model is how a wave propagates in an atomic gas.

Interest in ionizing potential waves was stirred up anew when Fowler and Hood[11] and subsequently, Haberstitch[12] reported observation of precursors resembling breakdown waves in long discharge tubes in electrically driven shock tubes. Wave velocities observed were of the order of 10^9 cm/sec. These precursors differed from the shock waves usually observed in electrically driven shock tubes in that they have a much higher velocity and in that they involve electron fluid action rather than heavy particle fluid action. Haberstitch conducted a fairly comprehensive study of such waves in what will be referred to as a nosed-cone geometry. Measurements were made of wave speed as a function of front potential (potential across the wave front), pressure, and displacement from the discharge electrode. Electron density and wave thickness were also determined for some waves. The velocity data is quite similar to that reported by Snoddy et al. The density and thickness data are a truly significant step forward in the understanding of these waves. Haberstitch also attempted a theoretical analysis of his ionizing potential waves. Employing the one-dimensional, fluid dynamical production equations,

$$\frac{d}{dx}\,(\rho_e v) = -Ke$$

and

$$\frac{d}{dx}\,(\rho_i v) = Ke$$

with

$$K = \alpha\,\frac{\rho_e}{e}\,(V-v)$$

and Poisson's equation,

$$\frac{dE}{dx} = 4\pi\,(\rho_e + \rho_i),$$

Haberstitch assumed different forms of α and v-V as functions of E and examined the resulting wave profiles. Here x signifies the spatial coordinate, ρ_e and ρ_i the electron and ion charge densities respectively, and K the production coefficient. The results give some idea of the general nature of his waves but fail to qualify as a theory for propagation of the waves because the true forms of ρ_e, ρ_i, V, and v are never investigated.

The observation by Fowler and Hood of precursors in their electrically driven shock tube led Paxton and Fowler[13] to formulate a theory of breakdown wave propagation. Using a one-dimensional, time independent fluid model and assuming electron pressure to dominate the system, they wrote down the equations of conservation of mass, momentum, and energy:

$$MN_0V_0 = MNV + M_iN_iV_i + mnv$$

$$MN_0V_0^2 = MNV^2 + M_iN_iV_i^2 + mnv^2 + nkT_e + 1/2\ \varepsilon_0(E_0^2 - E^2),$$

$$MN_0V_0^3 = MNV^3 + M_iN_iV^3 + mnv^3 + 5nvkT_e.$$

It was then reasoned that, due to the large inertia of the ions and atoms, these particles would not be appreciably accelerated by the wave:

$$V_0 = V = V_i:$$

Furthermore, a zero current condition,

$$nv - N_iV_i = 0$$

was utilized. Defining the degree of ionization f by

$$N_i = N_0 - N = fN_0,$$

V_0 was determined as a function of the remaining variables. Employing a previously determined expression for electron temperature behind a shock wave, an approximate expression for V_0 as a function of applied field was obtained. The result showed fair agreement with experimental data obtained by different observers. However, this work ignored energy loss to ionization; Fowler[14] has since taken such losses into account with improved results.

Nelson[15], misunderstanding the nature of steady profile waves, has rejected the Paxton-Fowler approach. Instead, he proposed a photo-ionization model similar to the one proposed by Loeb; it also fails to explain wave propagation in an atomic gas.

Recently Winn[16] has published a report on the result of rapid application of a high potential pulse across a glow discharge. The resulting waves have characteristics similar to the waves previously studied by Snoddy et al and Haberstitch. Due to the presence of pre-ionization in his experiment, it can be expected to show some differences from waves propagating into a neutral gas.

All workers have recognized that breakdown waves, precursors, and corona discharges are very similar phenomena. Experimental work which shows the relationship between these phenomena has not been performed, and workers often use the above labels interchangeably. Paxton and Fowler and Haberstitch use the term "breakdown wave" for these phenomena. More recently, Loeb, seeking a label more descriptive of the common nature of these phenomena, has employed the label "ionizing waves of potential gradient." However, the author feels that the name "electron fluid-dynamical wave" represents a better description of the basic nature of the phenomena.

Following the approach employed by Paxton and Fowler, the author[17] attempted to describe the profile of electron fluid-dynamical waves. Under the assumption of constant heavy particle velocity, it was shown that the equations for a stable profile, one-dimensional fluid-dynamical model could be decoupled and solved. Physical interpretation of the obtained solutions proved difficult; attempts to understand these constant velocity solutions led to a deeper understanding of the model employed. The present work is the result of this new depth of understanding.

CHAPTER III

MODEL

Derivation of the Fluid Equations

It is the purpose of this work to develop a one-dimensional theoretical description of electron fluid-dynamical waves. As a first step, it seems appropriate to give attention to the derivation of the basic equations of production, momentum transfer, and energy transfer for a fluid system of charged particles subjected to an electric field. The basic concepts involved in the derivation of the fluid equations are well established so that one only has to apply these concepts to the system of current interest.
The production equation for a given species results from equating the time rate of change of the number of particles in a differential volume to the net gain or loss of particles per unit time due to creation and annihilation within the volume and due to fluxes of particles through the surfaces of the volume. Writing this mathematically for electrons gives the following

(Time rate of change of number in dV)	(Number created in dV per second)	(Particle flux into dV)	(Particle flux out of dV)
$\frac{\partial n}{\partial t}\,dxdydz$	$= \beta n\,dxdydz$	$+\, nv\,dydz$	$-\,(n + \frac{\partial n}{\partial x}dx)(v + \frac{\partial v}{\partial x}dx)dydz$

or

$$\frac{\partial n}{\partial t} + \frac{\partial (nv)}{\partial x} = \beta n.$$

The momentum transfer equation for electrons is derived similarly:

(Time rate of change of momentum in dv)	(Momentum flux into dV)	(Momentum flux out of dV)
$\frac{\partial}{\partial t}(mnv)\,dxdydz$	$= mnv^2 dydz$	$-\, m(n + \frac{\partial n}{\partial x}dx)(v + \frac{\partial v}{\partial x}dx)^2\,dydz$

(Volume force due to electric field)	(Elastic momentum loss to heavy particles)	(Inelastic momentum gain from heavy particles)	(Resultant force on dV due to pressure gradient)
$-\, enE\,dxdydz$	$-\,\Delta_c(mv)dxdydz$	$+\,\Delta_i(mv)dxdyz$	$+\,[p - (p + \frac{\partial p}{\partial x}dx)]\,dydz$

This reduces to

$$\frac{\partial}{\partial t}(mnv) + \frac{\partial}{\partial x}(mnv^2 + p) = -\,enE - \Delta_c(mv) + \Delta_i(mv)\ .$$

Doing the same thing for electron energy transport gives

$$\frac{\partial}{\partial t}(\tfrac{1}{2}mnv^2 + q)\,dxdydz = \tfrac{1}{2}mnv^3\,dydz - \tfrac{1}{2}m(n + \frac{\partial n}{\partial x}dx)(v + \frac{\partial v}{\partial x}dx)^3\,dydz$$

$$+\,[qv - (v + \frac{\partial v}{\partial x}dx)(q + \frac{\partial q}{\partial x}dx)]\,dydz - envE\,dxdyz$$

$$+\,[pv - (p + \frac{\partial p}{\partial x}dx)(v + \frac{\partial v}{\partial x}dx)]\,dydz$$

$$-\,\Delta_c(\tfrac{1}{2}mv^2)\,dxdydz + \Delta_i(\tfrac{1}{2}mv^2)\,dxdydz$$

which simplifies to

$$\frac{\partial}{\partial t}(\tfrac{1}{2}mnv^2 + q) + \frac{\partial}{\partial x}[\tfrac{1}{2}mnv^3 + (p + q)v] = -\,envE - \Delta_c(\tfrac{1}{2}mv^2) + \Delta_i(\tfrac{1}{2}mv^2)$$

In the energy equation the third term on the right hand side of the unsimilified equation is the flux of internal energy into the volume element, the fifth term is the work done against pressure as particles flow through the volume element.

The ionization frequency, that is, the number of ionizations per unit volume per second per electron is signified by β. The symbol Δ_c denotes a transfer operator for the quantity indicated in parenthesis from electrons to heavy particles due to elastic collisions; Δ_i is a similar operator for inelastic collisions (defined in terms of transfer from heavy particles to electrons). The number density and velocity are denoted by n and v, p is the electron pressure, and E is the electric field - applied field plus space charge field. Having demonstrated the derivation of the electron equations, we can easily write down the entire system of equations for all species. The production equations for electrons, ions, and neutral atoms are respectively:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (nv) = \beta n, \qquad \text{(1a}$$

$$\frac{\partial N_i}{\partial t} + \frac{\partial}{\partial x} (N_i V_i) = \beta n, \qquad \text{(1b}$$

and

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x} (NV) = -\beta n. \qquad \text{(1c}$$

The momentum equations for electrons and heavy particles are

$$\frac{\partial}{\partial t} (mnv) + \frac{\partial}{\partial x} (mnv^2 + p) = -enE - \Delta_c (mv) + \Delta_i (mv) \qquad \text{(2a}$$

and

$$\frac{\partial}{\partial t}(MNV + M_iN_iV_i) + \frac{\partial}{\partial x}(MNV^2 + M_iN_iV_i{}^2 + P + P_i) = eN_iE$$

$$+ \Delta_c\,(mv) - \Delta_i\,(mv). \qquad (2b$$

and finally, the energy equations for electrons and heavy particles are

$$\frac{\partial}{\partial t}(1/2\ mnv^2 + q) + \frac{\partial}{\partial x}[1/2\ mnv^3 + (q + p)\ v] = -envE - \Delta_c(1/2mv^2)$$

$$+ \Delta_i(1/2\ mv^2) \qquad (3a$$

and

$$\frac{\partial}{\partial t}(1/2MNV^2 + 1/2M_iN_iV^2 + Q_i + Q)$$

$$+ \frac{\partial}{\partial x}[1/2\ MNV^3 + 1/2M_iN_iV_i{}^3 + (Q + P)V + (Q_i + P_i)V_i]$$

$$= eN_iV_iE + \Delta_c\,(1/2mv^2) - \Delta_i\,(1/2mv^2). \qquad (3b$$

We note that only one momentum equation and only one energy equation is written for the heavy particles. This is the result of our concentration of interest on the electron fluid and of our disinterest in seeking all possible information about the interaction between ions and neutral atoms. As will be discussed, this suppression of heavy particle detail is possible because of the strong interaction between ions and neutral atoms. To make the above system of fluid-dynamical equations complete, one must write down Maxwell's equations. This step will be deferred until the problem of current interest has been more completely specified. It should also be mentioned that in the above equations thermal conduction has been assumed to be unimportant.

This assumption will be examined for validity after solutions to these equations have been found for a given case. Also neglected are relativistic correction terms; even so it is anticipated that this system of equations will prove adequate to describe all but the highest velocity waves ($V \geq 10^{10}$ cm/sec).

Determination of Operators

In order to use the above equations, one must know the ionization frequency, β, and the transfer operators Δ_c and Δ_i. The ionization frequency has been studied for the case of thermal ionization, that is, for the case where the electrons have no drift velocity relative to the neutral atoms and ionization is due to the thermal energy of the electrons. On the other hand, it is obvious that a directed beam of high energy electrons can ionize a gas just as effectively as a swarm of randomly directed electrons. Hence, it is necessary to take into account the directed motion of the electrons as well as their temperature when deriving the ionization frequency. Fowler[18] found that an ionization cross-section of the form

$$\sigma_i\ (v) = \sigma_0\ \left(\frac{v^2}{v_i^2} - 1\right)^{\alpha} \frac{v_i^2}{v^2} \quad \text{for } v \geq v_i \tag{4}$$

where

$$1/2mv_i^2 = e\phi_i$$

fitted the experimental data for thermal ionization taken by Smith[19] quite well. For most gases observed α was approximately unity.

Determination of an approximate β can be made by setting $\alpha = 1$ and integrating over the electron velocity distribution. For simplicity, assume that the electrons have a Boltzman velocity distribution in the electron drift frame; this will henceforth be referred to as a warped Boltzman distribution. Thus

$$\beta = \overline{\sigma N v} = \int N w \sigma_0 \left(\frac{w^2}{v_i^2} - 1\right) \frac{v_i^2}{w^2}\, df(\vec{w})$$

More explicitly,

$$\beta = \int \sigma_0 N \left(\frac{m}{2\pi k T_\varepsilon}\right)^{\frac{3}{2}} w \frac{v_i^2}{w^2} \left(\frac{w^2}{v_i^2} - 1\right) e^{-\frac{m}{2kT_e}\left[w_x^2 + w_y^2 + (w-u)^2\right]}\, dw_x\, dw_y\, dw$$

where u is the velocity of the electrons relative to the neutral atoms. Shifting to polar coordinates, this can easily be integrated to give

$$\beta = \left\{ \frac{2\sigma_0 N}{\pi^{1/2}} \left(\frac{2kT_e}{m}\right)^{1/2} e^{-\frac{mv_i^2}{2kT_e}} \right\}\left\{1/2 e^{\frac{mv_i^2}{2kT_e}}\right\}\left\{ e^{-\frac{m(v_i-u)^2}{2kT_e}} + e^{-\frac{m(v_i-u)^2}{2kT_e}} + \frac{1}{u}\left(\frac{2kT_e}{m}\right)^{1/2} \int_{x_1}^{x_2} x^2 e^{-x^2}\, dx + \frac{u^2 - v_i^2}{u}\left(\frac{m}{2kT_e}\right)^{1/2} \int_{x_1}^{x_2} e^{-x^2}\, dx \right\} \quad (5$$

x_1 and x_2 are defined as

$$x_1 = \left(\frac{m}{2kT_e}\right)^{1/2} (v_i - u)$$

and

$$x_2 = \left(\frac{m}{2kT_e}\right)^{1/2} (v_i + u).$$

In terms of the variables previously employed, $u = v-V$. The integrals appearing in equation 5 can be expressed as the sum or difference of imcomplete gamma functions and then found in tables[20]. The product of the last two brackets in equation 5 approaches unity as u approaches zero while the term in the first bracket is the ionization frequency for thermal ionization. It agrees quite well with the experimentally determined ionization frequencies. The expression for β is not simple and this will plague attempts to obtain simple analytic solutions to the fluid equations. Nevertheless, the above expression is essential for a proper understanding of the cases in which electrons have significant drift velocity relative to the neutral atoms.

Turning now to Δ_c (mv), we consider an electron with velocity $\vec{w} = [w_x, w_y, w_z]$ colliding with a neutral atom at rest (treated as a hard sphere) as shown in Figure 1

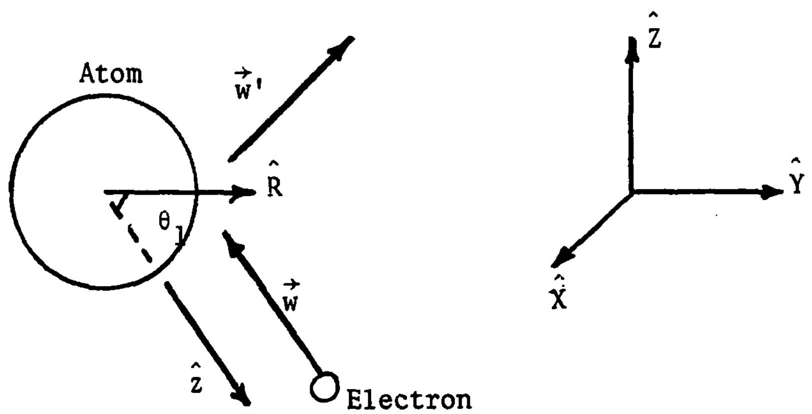

Fig. 1--Elastic collision between electron and atom.

where $\vec{w}$ is the electron's velocity before the collision and $\vec{w}'$ is its velocity after its collision with the atom and $\hat{z}$ is a unit vector parallel to the electron velocity. Since the mass of the atom is much larger than the mass of the electron, the magnitudes of $\vec{w}$ and $\vec{w}'$ will be essentially equal. Thus

$$m\vec{w}' - m\vec{w} = 2mw \cos\theta_1 \hat{R}. \tag{6}$$

An electron with velocity $\vec{w}$ may collide with an atom with any impact parameter, or equivalently, with any angle θ_1. This requires that the above expression (equation 6) be averaged over all angles:

$$\Delta(m\vec{w})_Z = N \int 2mw^2 \,|\hat{R}|_Z \, d\sigma.$$

Relating $\hat{R}$ to the XYZ frame in terms of θ_1, ϕ_1 and θ, ϕ and carrying out the necessary integration yields

$$\Delta(m\vec{w})_Z = - 4/3 \; mNw^2 \; (\pi R_1^{\;2}) \cos\theta. \tag{7}$$

Since R_1 is the radius of the atom, $\pi R_1^{\;2}$ is the atomic cross section for elastic collisions. The above expression represents the momentum loss per second for an electron with velocity $\vec{w}$ traveling through a gas containing N atoms per unit volume. In order to find the momentum loss for an electron swarm characterized by a drift velocity $\vec{u}$ relative to the atoms and a temperature T_e, the above expression must be averaged over the electron velocity distribution for the swarm. To carry out this averaging, the velocity dependence of the cross section must be known. It has been experimentally determined[21] that σ is inversely proportional to velocity:

$$\sigma = \sigma_0 \frac{V_0}{V}.$$

Inserting this expression into equation 7 and integrating over a warped Boltzman velocity distribution yields

$$\Delta(mu)_Z = -\frac{4N}{3}\sigma_0 v_0 mu.$$

This is the Z-component of the momentum lost by an average electron of a swarm characterized by a relative drift velocity $\vec{u} = [0,0,u]$ and a temperature T_e. From this one concludes that

$$\Delta^0{}_c(mv) = 4/3\sigma_0 N v_0 mn(v-V)$$

since, as before, $u = v-V$. This is the one-dimensional momentum transfer operator for the frame in which the atoms are at rest.

One can easily see how to generalize the above result to an arbitrary frame of reference by considering a collision between two particles; one can also gain insight into the nature of $\Delta_c(1/2mv^2)$ from such a consideration. If one views the collision of two particles M_1 and M_2 from some general frame of reference as shown in Figure 2 and if one applies conservation of momentum and energy, one

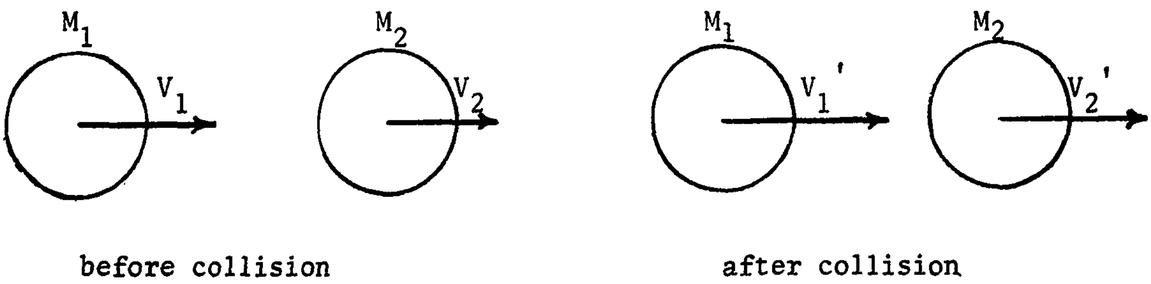

Fig. 2--Collision between two particles as seen from a general frame of reference.

can solve for V_2' :

$$V_2' = V_2 + \frac{2M_1}{M_1+M_2} (V_1-V_2).$$

Since $\Delta_c(mv)$ represents the momentum transferred from M_1 to M_2,

$$\Delta_c(mv) = M(V_2' - V_2) = \frac{2M_1M_2}{M_1+M_2} (V_1-V_2).$$

This expression is independent of the frame of reference from which the particles are viewed since it depends only on the relative velocity of the two particles. One concludes that the momentum transfer operator for a general system (composed of many particles of each species) is also frame invariant since it depends only on the drift velocity of one species relative to the other species:

$$\Delta_c(mv) = \Delta_c{}^0(mv).$$

Writing down the energy transferred from M_1 to M_2, one finds

$$\Delta_c(1/2mv^2) = 1/2M_2(V_2'{}^2 - V_2{}^2)$$

$$= \frac{2M_1M_2}{M_1+M_2} V_2(V_1-V_2)+1/2M_2 \left(\frac{2M_1}{M_1+M_2}\right)^2(V_1-V_2)^2.$$

The first term in this expression vanishes in the frame in which

M_2 is initially at rest so the second term is the energy transfer operator for this collision in the rest frame of M_2. Recalling the term above describing $\Delta_c(mv)$ for two particles, one can write

$$\Delta_c(1/2mv^2) = V_2\,\Delta_c(mv) + \Delta_c^{\,0}(1/2mv^2)$$

where $\Delta_c(1/2mv^2)$ is the elastic energy transfer operator in the frame of reference in which M_2 has velocity V_2 and $\Delta_c^{\,0}(1/2mv^2)$ is the same operator in the rest frame of M_2. Since in the present case $M_1 = m$ (electron mass) and $M_2 = M$ (atomic mass), $\Delta_c^{\,0}(1/2mv^2)$ is proportional to the square of the very small term $\frac{m}{m+M}$ and is negligible compared to $V\Delta_c(mv)$:

$$\Delta_c(1/2mv^2) = V\Delta_c(mv).$$

Hence, one does not need to proceed beyond the above consideration of the collision of two particles to gain an adequate knowledge of $\Delta_c(1/2mv^2)$ when electrons are transferring energy to heavy particles.

Turning now to the determination of the inelastic collision transfer operators, one finds that he is faced with quite a different situation. Inelastic collisions are very complex events often involving three bodies so that they are not yet fully understood and there are no simple arguments one can employ to derive the inelastic transfer terms. Rather than attempting to derive these terms, one can use another approach. One writes down the electron production and momentum transfer equations for a general frame assuming that Δ_i is unimportant. One now transforms these equations to a new frame of reference moving with velocity U_0 with respect to the first frame. The new coordinate x' is related to the old coordinate x by

$$x' = x - U_0 t$$

while obviously

$$t' = t$$

since one is dealing with a non-relativistic formulation of the fluid equations. By the chain rule of differentiation one has

$$\frac{\partial}{\partial x} = \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} = \frac{\partial}{\partial x'},$$

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} = \frac{\partial}{\partial t'} - U_0\frac{\partial}{\partial x'} .$$

Carrying out the algebraic manipulations involved in the transformation and setting

$$v' = v - U_0$$

and

$$V' = V - U_0,$$

one finds that the momentum transfer equation is not of the same form in the x' frame as it was in the original frame; there is an extra term $-\beta mnU_0$ appearing on the right hand side of the equation. In order to make the equation frame invariant, as it must be since it is derived from Newton's Second Law of Motion which is itself frame invariant, one must add a term βmnV to the original equation; one recognizes this as the inelastic collision momentum transfer term. So,

$$\Delta_i\ (mv) = \beta mnV.$$

The meaning of this term is obvious. From the viewpoint of the electron fluid, the main effect of inelastic collisions is to produce new electrons. These new electrons are created with no drift velocity relative to the neutral atoms, i.e., these new electrons have the same drift velocity as the heavy particles, and must be given momentum to join the moving frame.

Before one uses the same approach to derive the inelastic collision energy transfer operator, one must remember one additional fact. The ionization of an atom to produce a new electron requires the expenditure of an amount of energy $e\phi_i$, the ionization energy of the atom, by the electron swarm. Hence, one should put the term $-\beta n(e\phi_i)$ in equation 3a for $\Delta_i\ (1/2mv^2)$ and repeat the transformation procedure used to derive $\Delta_i\ (mv)$. The result is

$$\Delta_i(1/2mv^2) = \beta n(1/2mV^2 - e\phi_i).$$

Interpretation of this expression follows the same idea as interpretation of the momentum transfer term. One should note that the requirement of frame invariance is powerless to give information about scalar terms since scalars are frame invariant. Scalar terms must be obtained from other considerations; this is why the $e\phi_i$ term was inserted beforehand. It is possible that other scalar terms could appear in the fluid equations; for instance a term $\beta n(3/2nkT_i)$ could logically be included as the thermal energy a newly created electron possessed at creation due to the thermal motion of the heavy particles. However, $T_i << T_e$ in most cases so such a term is unimportant. Nonetheless, one should be aware that such scalar terms could exist in the fluid equations.

Summarizing the expressions derived in this section:

$$\beta = \{\frac{2}{\pi}(\frac{2kT_e}{M})^{1/2} e^{-\frac{mv_i^2}{2kTe}}\}\{1/2 e^{\frac{mv_i^2}{2kT_e}}\}\{e^{-\frac{m(v_i-u)^2}{2kT_e}} + e^{-\frac{m(v_i+u)^2}{2kT_e}}$$

$$+\frac{1}{u}(\frac{2kT_e}{m})^{1/2}\int_{x_1}^{x_2} x^2 e^{-x^2}\, dx + \frac{u^2-v_i^2}{u}(\frac{m}{2kT_e})^{1/2}\int_{x_1}^{x_2} e^{-x^2}\, dx\},$$

with

$$x_1 = (\frac{m}{2kT_e})^{1/2}(v_i - u),$$

$$x_2 = (\frac{m}{2kT_e})^{1/2}(v_i + u),$$

$$u = v - V.$$

$$\Delta_c(mv) = 4/3\sigma_0 v_0 Nmn(v - V).$$

$$\Delta_c(1/2mv^2) = V\Delta_c(mv) + \Delta^0{}_c(1/2mv^2) \simeq V\Delta_c(mv).$$

$$\Delta_i(mv) = \beta mnV.$$

$$\Delta_i(1/2mv^2) = \beta n(1/2mV^2 - e\phi_i).$$

Although many workers have employed basic fluid equations of the form presented in equations 1 through 3 and although some have even intuitatively employed expressions similar to some of the above derived transfer operators, this author knows of no one who has derived expressions for the ionization frequency and transfer operators as has been done above.

Having derived the one-dimensional fluid dynamic equations for a plasma and having developed expressions for the transfer operators and the ionization frequency, one is in a position to proceed with the detailed analysis for any of a broad class of problems. In particular, one can carry out the analysis of the profiles of electron fluid dynamical waves. The remainder of this paper is concerned with such an analysis.

CHAPTER IV

ANALYSIS OF GENERAL WAVE PROPERTIES

Solution of the one-dimensional fluid equations derived in the previous section plus Maxwell's equations is so difficult that no one has yet developed a method of attack. In order to establish an attack, the structure of a one-dimensional, constant velocity, steady-profile electron fluid-dynamical wave will be investigated. Such a wave might be realized if one were to release an infinitely extensive slab of electrons located between X and X + dX in an infinite volume of neutral gas subjected to an electric field E_o applied in the negative X-direction. If E_o were very small, the electrons would diffuse away from X and undergo mobility motion characterized by a drift velocity $\vec{u} = [u,o,o]$ and a temperature T_e. As E_o increases, u and T_e increases until the electrons attain sufficient energy to ionize neutral atoms of the gas. When sufficient ionization occurs in a region to replace the electrons flowing out of the region, one has a wave of the type under consideration. If an observer were to ride along with a steady-profile wave and look about him, he would see a wave structure which was not varying in time. Hence, a steady-profile wave has no time dependence in the wave frame, that is, in a frame moving along with the wave.

Certain other considerations concerning electron fluid-dynamical waves will also help simplify solution of the basic fluid equations. First though, let us specify a coordinate system. Since we have a one-dimensional system, this involves only the assignment of the positive x-direction. The positive x-direction will be defined as the direction in which the wave travels. It has been reported by most experimenters that electron fluid-dynamical waves travel from the electrode to which the potential is applied (regardless of its polarity) to the grounded electrode. This occurs because discharge tube design is usually such that ground is always nearby which results in a stronger field near the electrode to which the potential is applied. The wave initiates in the region of stronger field so that the wave travels away from the discharge electrode toward the grounded electrode. Therefore, the positive x-direction will generally point from the electrode to which the potential is applied toward the grounded electrode. Now consider the nature of collisions between atoms and ions. Such collisions are very effective in transferring energy and momentum from one species to the other because their masses are almost equal. Hence, equalization of velocity and temperature between ions and neutral atoms will occur within a very thin region; as far as the electron fluid is concerned, the heavy particles will all have the same velocity and temperature. As was previously mentioned, no Doppler shift has been detected in the waves in which we are interested so the heavy particles are nearly at rest in the lab frame. In the wave frame (moving with velocity V_o in the positive x-direction), these heavy particles will have velocity $V = V_o$ as they come into the wave. This velocity must remain approximately constant since no Doppler shift is detected. This will be verified in our analysis. Another factor which simplifies our problem is that each species behaves very much like an ideal gas.

Writing down the fluid equations under the above conditions, one has:

$$\frac{d(nv)}{dx} = \beta n, \tag{9a}$$

$$\frac{d(N_iV)}{dx} = \beta n, \tag{9b}$$

$$\frac{d(NV)}{dx} = -\beta n; \tag{9c}$$

$$\frac{d}{dx}(mnv^2 + nkT_e) = - enE - K_1\overline{mn}(v-V) + \beta mnV, \tag{10a}$$

$$\frac{d}{dx}[MNV^2 + M_iN_iV^2 + (N + N_i)kT_i] + eN_iE + K_1 mn(v-V) - \beta mnV; \tag{10b}$$

$$\frac{d}{dx}(mnv^3 + 5nvkT_e) = -2envE - 2VK_1mn(v-V) + \beta n(mV^2 - 2e\phi_i), \tag{11a}$$

$$\frac{d}{dx}[MNV^3 + M_iN_iV^3 + 5(N+N_i)VkT_i] = 2eN_iVE + 2VK_1 mn(v-V) - \beta mnV^2; \tag{11b}$$

where $K_1 = 4/3\ \sigma_o v_o N_o$. Before writing down Maxwell's equations, one should examine the fluid equations to see which of Maxwell's equations are important for this case. Subtracting 9a from 9b gives

$$\frac{d}{dx}\ (N_i V - nv) = 0.$$

Evaluating the constant of integration ahead of the wave where there are no ions or electrons, one has

$$N_i V - nv = 0.$$

This expression multiplied by the electron charge is the current; hence, there is no total current in this problem. With no time changes occurring, no current, and no magnetic field being applied, Maxwell's equations reduce to Poisson's equation:

$$\frac{dE}{dx} = \frac{e}{\varepsilon_o}\ (N_i - n). \tag{12}$$

This completes a system of eight equations containing eight unknowns which describes electron fluid-dynamical waves and which we hope to solve.

One can determine the nature of the variation of the heavy particle velocity through simple analysis. Adding equations 9b and 9c, integrating, and evaluating the constant of integration ahead of the waves, one finds that baryon flux is conserved:

$$(N + N_i)V = N_o V_o. \tag{13}$$

This, combined with the obvious relation $M_i = M\text{-}m$, can be used to express the heavy particle momentum and energy transfer equations in a different form:

$$\frac{d}{dx}[MN_oV_oV - mN_iV^2 + (N + N_i)\ kT_i] = eN_iE + K_1mn(v\text{-}V) - \beta mnV, \qquad (14$$

and

$$\frac{d}{dx}[MN_oV_oV^2 - mN_iV^3 + 5N_oV_okT_i] = 2eN_i\ VE + 2VK_1\ mn(v\text{-}V) - \beta mnV^2. \qquad (15$$

Carrying out the differentiation gives

$$(MN_oV_o\text{-}mN_iV)\ \frac{dV}{dx} + \frac{d}{dx}\ [(N\text{+}N_i)\ kT_i] = eN_iE + K_1mn(v\text{-}V) \qquad (16$$

and

$$2(MN_oV_oV - mN_iV^2)\frac{dV}{dx} + 5N_oV_o\ \frac{d}{dx}(kT_i) = 2eN_iVE + 2VK_1\ mn(v\text{-}V) \qquad (17$$

Multiplying equation 16 by 2V and subtracting from equation 17 allows one to express the derivative of the pressure as

$$\frac{d}{dx}[(N + N_i)\ kT_i] = -5/3\ mN_oV_o\ \frac{kT_i}{mv^2}\ \frac{dV}{dx.}$$

Substituting this expression into equation 16 gives

$$[MN_oV_o - mN_iV - 5/3\ mN_oV_o\ \frac{kT_i}{mV^2}]\ \frac{dV}{dx} = eN_iE + K_1mn(v\text{-}V).$$

Only the first term in the bracket is of importance since

$$mN_i,\; m \frac{kT_i}{mV^2} << MN_o.$$

So,

$$MN_oV_o \frac{dV}{dx} \simeq eN_iE + k_1\, mn(v-V).$$

Determing K_1 from experimental curves[22] gives $K_1 = 8.85 \times 10^8$ sec^{-1} for helium at unit density. Unit density will be defined as that density corresponding to a pressure of 1 mm Hg at 273 ^{0}K. Applied fields are usually of the order of 10^5 V/m; particle velocities are usually about 10^9 cm/sec; and the degree of ionization for electron fluid-dynamical waves is almost always less than 10^{-2}. These numbers indicate that $\frac{dV}{dx}$ is of the order of 10^2 sec^{-1}, or that

$$\frac{\Delta V}{V} \simeq 10^{-5}$$

since wave thickness are certainly no more than one meter. This seems adequate justification to conclude that V can be taken to be a constant. This author has carried out such an analysis[17] and has found that solution of the equations can indeed be effected. However, the constant velocity solutions require the electrons to always have a drift velocity relative to the heavy particles, and this in turn requires that the derivative of the electric field be non-zero. These conditions are simply not acceptable from a physical standpoint.

Further analysis of the basic equations shows that a constant velocity V is not really a good approximation. To see this, one should proceed as follows. One sees from equation 9a that

$$\frac{d(nv)}{dx} = \beta n$$

so that equation 11a can be written as

$$\frac{d}{dx}\left[mnv^3 + 5nv\, kT_e + nv(2e\phi_i)\right] = -2envE - 2VK_1\, mn(v-V) + \beta mnV^2.$$

The right hand side of this expression is of the same order of magnitude as the right hand side of equation 11b. Similarly, the right hand sides of equations 10a and 10b are of the same order of magnitude. This indicates that even though the heavy particle velocity does not change very much, the momentum and energy changes of the heavy particles in going through the wave are comparable with the momentum and energy changes of the electrons as they go through the wave. This is so because the very small change in heavy particle velocity is weighted by the large mass (relative to electron mass) of the heavy particles and by the overwhelming abundance of heavy particles relative to electrons. Therefore, even a small change in heavy particle velocity is significant, and it appears at first that one must keep V as a variable and seek sufficient accuracy to describe such small changes.

However, this is not necessary. If one examines only the equations describing electron behavior, one finds that nowhere in them is the heavy particle velocity weighted by either of the two factors discussed above which make the small velocity changes important. Hence, V can be set equal to a constant in the electron equations. With V constant and with zero current, one finds that the electron equations plus Poisson's equation form a system of four equations with four unknowns which completely describe electron fluid behavior.

Before restricting our attention to the electron equations, let us look at some rather general considerations. First, let us discuss the consequences of out choice of assignment of positive X - direction. Although the assignment of positive X is arbitrary, reversing the positive X - direction does make a difference. Inverting the positive X - direction reverses the sign of the electric field since E is the first derivative of the electric potential. On the other hand, the derivative of the field retains its sign under coordinate reversal. While this is not of any great consequence in reading this or any other particular work, it is of importance when trying to correlate different articles. For instance, Haberstitch chose his positive X - direction in the opposite sense from the choice made in the present work. Therefore, his positive field waves would be negative field waves in this work. Confusion thrives on such ambiguous terminology so a new terminology, fully defined, is in order. We will call waves proforce waves and antiforce waves. Proforce waves will be waves in which the electric force tends to accelerate electrons in the direction the wave is traveling. Antiforce waves will be the opposite case where the electric force tends to accelerate electrons in the direction opposite the direction the wave is traveling. This labeling is not ambiguous since it does not depend on the assignment of positive X. Haberstitch would have called proforce waves negative waves and antiforce waves positive waves. Paxton and Fowler had previously used the names negative and positive waves for proforce and antiforce waves respectively.

Boundary Conditions

Having arrived at unambiguous labels for electron fluid-dynamical waves, we now focus our attention on the conditions existing at the leading edge of the wave. To determine these conditions, we will examine the equations of conservation of momentum and conservation of energy for the total system. The conservation of momentum equation is obtained by adding equations 10a and 14, employing Poisson's equation, integrating the perfect differential obtained, and evaluating the constant of integration out in front of the wave. The result is

$$MN_oV_o(V-V_0) + mn\,v(v-V) + N_ok(T_i - T_o) + nkT_e = \frac{\varepsilon_o}{2}(E^2 - E_o^{\,2}) \qquad (18$$

where E_o and T_o are the applied field and initial temperature. Proceeding similarly with equations 9a, 11a, 15, and the zero current condition gives

$$MN_oV_o(V^2 - V_o^2) + mnv(v^2-V^2) + 5N_oV_ok(T_i - T_o) + 5nvkT_e + nv(2e\phi_i) = 0. \qquad (19$$

Now we must determine what it means for the wave front to reach us. First, we note that the wave front cannot be signified by an abrupt change in the electric field because a discontinuity in field results from a surface charge, or equivalently, an infinite volume charge density at the front. Therefore E must be equal to E_o at the wave front. Second, we would not expect that the heavy particles with their large inertia will experience a sharp change in either their velocity or temperature so we will take $V = V_o$, $T_i = T_o$ at the front. This leads to the conclusion that arrival of the wave must be signified by the existence of electrons with velocity v_1 and temperature $(T_e)_1$. Solving for these quantities by setting $E = E_o$, $V = V_o$, and $T_i = T_o$ in equations 18 and 19 gives

$$n_1 \left[v_1(v_1 - V_o) + \frac{k(T_e)_1}{m} \right] = 0$$

and

$$n_1v_1\left[v_1^2 - V_o^2 + 5\,\frac{k(T_e)_1}{m} + \frac{2e\phi_i}{m}\right] = 0.$$

These equations can be satisfied two ways. The first is $n_1 = 0$; this results in continuous solutions which, at present, are thought to describe antiforce waves. The detailed analysis of the case

$n_1 = 0$ has not yet been carried out. The second way is to require $n_1 \neq 0$; this results in discontinuous or shock solutions. These solutions describe proforce waves. Solving for v_1 and $(T_e)_1$, one finds

$$v_1 = \frac{5V_o}{8} \pm \frac{(9V_o^2 + 16 \frac{2e\phi_i}{m})^{1/2}}{8}$$

$$\frac{k(T_e)_1}{m} = v_1 \ (V_o - v_1). \qquad (20$$

Since $(T_e)_1 \geq 0$, $|v_1| < |V_o|$, and the negative sign must be taken:

$$v_1 = \frac{5V_o}{8} - \frac{(9V_o^2 + 16 \frac{2e\phi_i}{m})^{1/2}}{8} \ . \qquad (21$$

Also, since $(T_e)_1 \geq 0$, $v_1 \leq 0$ which requires

$$1/2mV_o^2 \geq e\phi_i$$

This imposes a lower limit on wave velocity. On the other hand, as V_o goes to infinity, $\frac{2e\phi_i}{m}$ becomes unimportant and v_1 approaches $\frac{V_o}{4}$. Thus,

$$0 \leq |v_1| \leq |\frac{V_o}{4}|.$$

Now we know v_1 and $(T_e)_1$ but n_1 is unspecified except that $n_1 \neq 0$.

Let us examine now what happens behind this wave front. There will be interplay between the electric field and the resistive forces due to the electrons moving past the heavy particles. The field will tend to accelerate the electrons in the positive direction (since we are restricting ourselves to proforce waves). The accumulation of electrons near the front of the wave will create a space charge field of opposing polarity to the applied field which will tend to cancel out the applied field. On the other hand, the resistive force terms described by the Δ operators will be opposing relative motion between the electrons and the heavy particles and trying to equalize their velocities. Equalization of the velocities (and densities) of electrons and ions will occur as the electric field (applied plus space charge) falls to zero permitting the existence of a neutral region. Ionization will continue in this neutral region as long as the electrons have sufficient thermal energy to produce ionization.

To summarize the structure of electron fluid-dynamical waves, electrons have a drift velocity v_1 such that $|v_1| < |V_0|$ at the front of the wave. Following this, the drift velocity of the electrons decreases (v, $V<0$) monotonically toward V. From Poisson's equation, which takes the form

$$\frac{dE}{dx} = \frac{e}{\varepsilon_0} n\left(\frac{v}{V} - 1\right)$$

by employing the zero current condition, the field increases from its negative value E_o at the wave front to its final limiting value of zero as $\frac{dE}{dx}$ goes to zero ($v \to V$). Quantitative solutions must have these general characteristics.

Now let us focus our attention on the electron fluid equations. The boundary conditions on the back side of the wave are obtained by taking limits on these equations. Remembering that V can be treated as a constant in the electron equations, one can then write them in the non-dimensionalized form

$$\frac{d}{d\xi}(\nu\psi) = \mu\nu,$$

$$\frac{d}{d\xi}[\nu\psi\ (\psi - 1) + \nu\theta] = -\nu\varepsilon - \kappa\nu(\psi-1),$$

$$\frac{d}{d\xi}\ [\nu\psi(\psi^2-1) + 5\nu\psi\theta + \nu\psi\alpha] = -2\nu\psi\varepsilon - 2\kappa\nu(\psi-1),$$

and

$$\frac{d\varepsilon}{d\xi} = \frac{\nu}{\alpha}\ (\psi - 1).$$

$$\nu = \frac{2e\phi_i}{\varepsilon_o E_o^2}\, n$$

$$\psi = \frac{v}{V},$$

$$\theta = \frac{kT_e}{mV^2},$$

$$\varepsilon = \frac{E}{E_o},$$

and

$$\mu = \frac{\lambda\beta}{V} = \frac{\kappa\beta}{K_1}$$

and as new constants

$$\alpha = \frac{2e\phi_i}{mV^2}$$

$$\kappa = \frac{\lambda K_1}{V} = \frac{mV}{eE_o} K_1$$

and as the characteristic length

$$\lambda = \frac{mV^2}{eE_o} .$$

The characteristic length λ is intrinsically negative, so in going through the wave, one travels from some negative ξ toward $\xi = +\infty$. The initial or boundary values expressed in terms of the new variables are

$$\nu_1 \neq 0,$$

$$\psi_1 = \frac{5}{8} - \frac{(9 + 16\alpha)1/2}{8}$$

$$\theta_1 = \psi_1(1-\psi_1) \qquad (27$$

and

$$\varepsilon_1 = 1.$$

Subtracting twice the momentum equation (equation 23) from the energy equation (equation 24) gives a perfect differential which, when integrated and the constant of integration evaluated ahead of the wave, results in the expression

$$\nu\psi\ (\psi - 1)^2 + \nu(5\psi - 2)\ \theta + \nu\psi\alpha + \alpha(\varepsilon^2 - 1) = 0. \qquad (28$$

Applying the conditions known to exist far behind the wave front

$(\psi \to 1,\ \varepsilon \to 0$ as $\xi \to \infty)$, one finds

$$\nu_f\ (\alpha + 3\theta_f) = \alpha \qquad (29$$

where the subscript f denotes the final value of the variable. Since the electrons eventually come into thermal equilibrium with the heavy particles, $\alpha \gg \theta_f$, and

$$\nu_f \simeq 1.$$

From the definition of ν, one realizes that this says that the energy density $1/2\varepsilon_o E_o^2$ goes into ionizing the atoms.

From our qualitative discussion of the wave profile we recall that $\psi > \psi_1$. Poisson's equation shows that

$$\kappa\nu(\psi - 1) = \alpha\ \kappa\ \frac{d(\varepsilon - 1)}{d\xi}$$

Employing these expressions in the energy equation (equation 24), one has

$$\frac{d}{d\xi}\left[\nu\psi(\psi^2 - 1) + 5\nu\psi\theta + \nu\psi\alpha + 2\alpha\kappa(\varepsilon - 1)\right] \leq -2\psi_1\ \nu\varepsilon.$$

Substituting for $\nu\varepsilon$ from the momentum equation (equation 23) and integrating yields

$$\nu\psi(\psi^2-1) + 5\nu\psi\theta + \nu\psi\alpha + 2\alpha\kappa(\varepsilon - 1) \leq 2\psi_1\ [\nu\psi(\psi-1) + \nu\theta + \alpha\kappa(\varepsilon-1)].$$

Evaluating this expression as $\xi \to \infty$ gives

$$2\nu_f\ \theta_f\ (1 - \psi_1) + \nu_f(\alpha + 3\theta_f) \geq 2\alpha\kappa(1-\psi_1).$$

Employing equation 29 and recalling that $\theta_f << \alpha$, one finally obtains

$$\kappa \geq \frac{1}{2(1 - \psi_1)}. \tag{30}$$

The constant κ determines wave speed as a function of applied electric field and initial pressure. Hence, it is important to determine κ more precisely. Exact determination of κ can be carried out only when the detailed structure of the wave is known. Even so, some approximate calculations give a good indication of the nature of the dependence of κ on wave speed (or applied field). The momentum equation (equation 23) can be written as

$$\frac{d}{d\xi}\ [\nu\psi(\psi - 1) + \nu\theta + \alpha\kappa(\varepsilon - 1)] = -\nu\varepsilon.$$

Substituting for ν on the right hand side of this expression by means of Poisson's equation gives

$$-\nu\varepsilon = -\frac{\alpha}{\psi-1}\frac{d\varepsilon}{d\xi}\varepsilon = -\frac{\alpha}{\psi-1}\,1/2\,\frac{d\varepsilon^2}{d\xi}\ .$$

Integration by parts leads to

$$\nu\psi\,(\psi-1) + \nu\theta + \alpha\kappa(\varepsilon-1) = \frac{\alpha}{2}\left[\frac{\varepsilon^2}{\psi-1} - \frac{1}{\psi_1-1}\right] - (\alpha/2)\int_{\psi_1}^{\psi}\frac{\varepsilon^2}{(\psi-1)^2}\,d\psi.$$

Taking the limit of this expression as $\psi \to 1$ ($\xi \to \infty$) results in

$$\nu_f\,\theta_f - \alpha\kappa = \frac{\alpha}{2(\psi_1-1)} - \frac{\alpha}{2}\int_{\psi_1}^{1}\frac{\varepsilon^2}{(\psi-1)^2}\,d\psi,$$

Since $\nu_f\,\theta_f \ll \alpha$,

$$\kappa = \frac{1}{2(1-\psi_1)} + 1/2\int_{\psi_1}^{1}\frac{\varepsilon^2}{(\psi-1)^2}d\psi. \tag{31}$$

The integrand $\frac{\varepsilon^2}{(\psi-1)^2}$ has an initial value of $\frac{1}{(\psi_1-1)^2}$ and falls to zero as $\psi \to 1$. For all cases which have been investigated by more exact calculations, $\frac{\varepsilon^2}{(\psi-1)^2}$ has been a monotone, decreasing function of ψ. Taking the integrand to be a linear function of ψ yields

$$\kappa = \frac{3}{4(\psi_1-1)}\ . \tag{32}$$

The exactness of this expression is something which only detailed calculations can reveal, but it will prove accurate enough to serve as a general guide for calculations. Since ψ_r varies over only a narrow range, κ is approximately constant, or wave speed is approximately proportional to applied field, a conclusion in full agreement with experimental results.

CHAPTER V

ANALYSIS OF WAVE PROFILES

Having considered the gross character of electron fluid-dynamical waves, i.e., the degree of ionization resulting from a given applied field, the wave speed for a given field, and the boundary conditions corresponding to a specified wave speed, we shall now proceed with a detailed analysis of the wave profile. One would like to be able to manipulate the four electron equations and to decouple them in order to obtain a differential equation containing only one of the four unknowns. The equations could then be integrated one by one. However, as often happens, this ideal cannot be realized and we are forced to seek solutions by some less elementary approach. Since we have only a vague idea of the nature of the desired solutions, we are forced to resort to a very powerful, laborious method which requires a minimum of foreknowledge. The method of successive approximations is just such a method. To employ this method, one must put the system of equations in the form

$$\frac{d\vec{y}}{d\xi} = \vec{F}(\xi,\vec{y})$$

the solution then takes the form

$$\vec{y}_\eta = \vec{y}_o + \int_o^\xi \vec{F}(\xi,\vec{y}_{n-1})d\xi$$

where $\vec{y}_\eta$ is the Nth approximation to $\vec{y}$ and $\vec{y}_o$ denotes the known boundary values at ξ=0. In order to put the system of electron equations in this form, one manipulates the production, momentum, and energy equations (equations 22, 23, and 24) to obtain new equations to replace the momentum and energy equations. This results in

$$\frac{d\psi}{d\xi} = \frac{3\psi\varepsilon+\kappa(5\psi-2)(\psi-1)+\mu(4\psi^2-5\psi+1-\alpha)}{5\theta-3\psi^2} \tag{35}$$

and

$$\frac{d}{d\xi}(\psi^{2/3}\theta) = \frac{2\kappa}{3}\ \psi^{-1/3}(\psi-1)^2-\mu\psi^{-1/3}[\theta-1/3[(\theta-1)^2-\alpha]\} \tag{36}$$

Therefore,

$$\vec{y} = \begin{pmatrix} \nu\psi \\ \varepsilon \\ \psi^{2/3}\theta \\ \psi \end{pmatrix}$$

and

$$\vec{F}(\xi,\vec{y}) = \begin{pmatrix} \mu\nu \\ \frac{\nu}{\alpha}(\psi-1) \\ 2/3\kappa\psi^{-1/3}(\psi-1)^2-1/3\mu\psi^{-1/3}[3\theta-(\psi-1)^2+\alpha] \\ \dfrac{3\psi\varepsilon+\kappa(5\psi-2)(\psi-1)+\mu(4\psi^2-5\psi+1-\alpha)}{5\theta-3\psi^2} \end{pmatrix}$$

The boundary (initial) values of ψ, θ, and ε are known. However, the boundary value of ν is not known; ν_1 (boundary value) is to be determined from the requirement that $\nu \to 1$ as $\xi \to \infty$. Any given case, that is, any specified wave speed can now be carried out step by step. For purpose of illustration, we take a wave speed of $V = 10^9$ cm/sec in Helium at unit density. For this wave speed

$$\alpha = .086,$$

$$\psi_1 = .222, \qquad (37$$

$$\theta_1 = .173.$$

Calculating κ from the approximate relation given in equation 32 gives

$$\kappa = .96 \qquad (38$$

Using the above values to calculate $\mu_1 = \frac{\kappa}{K_1} \beta_1$ from equation 5 results in $\mu_1 = 1.62$. The value of $(\frac{d\psi}{d\xi})_1$ is therefore found to be

$$(\frac{d\psi}{d\xi})_1 = .93 + .97\kappa = 1.86 \qquad (39$$

By successive approximations,

$$\nu_2\psi_2 = \nu_1\psi_1 + 1.62\nu_1\xi,$$

and

$$\psi_2 = \psi_1 + 1.86\xi.$$

Taking the ratio of these two expressions yields ν_2 which must approach $\nu_f = 1$ as ξ approaches ∞:

$$\frac{1.62\nu_1}{1.86} = 1;$$

$$\nu_1 = 1.15.$$

Using this value for ν_1 and inserting the above expressions into Poisson's equation and integrating to find ε_3 gives

$$\varepsilon_3 = 1-8.66\xi+10.82\xi^2-.212\ln(1+8.38\xi).$$

This expression falls rapidly from the value $\varepsilon_1 = 1$ to negative values. Recalling the nature of the solutions obtained by successive approximations, we know that successive solutions are descriptive of the true solution in successively broader regions. From this we conclude that our results indicate that the field falls rapidly to a negligible value as one goes through the wave. Actually, there is little more that we can tell from our results thus far because many more iterations would have to be carried out before representative solutions would emerge for all variables, and this is not the most profitable way to proceed since we now know that the field falls rapidly.

The rapid drop in the electric field described above can be easily understood. If a contained volume of plasma is subjected to an electric field, a Debye sheath layer will form. Excess charges of one polarity create a space charge field in the layer which cancels out the applied field so that the interior region of the plasma is essentially field free and neutral. The above results simply indicate that this is happening in the dynamical situation under consideration. Therefore, an electron fluid-dynamical wave is composed of two parts: A thin Debye sheath region in which the field falls to a negligible value and the electrons come to rest relative to the heavy particles, and a rather broad region in which the field is negligible and the electrons and heavy particles have the same velocity (and ion and electron densities are equal).

Solution of the Quasi-Neutral Region

Solutions for the broad quasi-neutral region following the sheath layer can be obtained easily. In this region ψ is approximately unity since electron and ion densities are approximately equal. Whenever ψ appears alone, it will be set equal to unity; on the other hand, ψ-1 will be retained. Equations 22, 23, and 24 become

$$\frac{d\nu}{d\xi} = \mu\nu, \tag{40}$$

$$\frac{d}{d\xi}\left[\nu(\psi-1)+\nu\theta\right] = -\nu\varepsilon-\kappa\nu(\psi-1), \tag{41}$$

$$\frac{d}{d\xi}\left[2\nu(\psi-1)+5\nu\theta+\nu\alpha\right] = -2\nu\varepsilon-2\kappa\nu(\psi-1). \tag{42}$$

Equations 41 and 42 can be combined to obtain an integrable expression which, when integrated, gives

$$\nu(\alpha + 3\theta) = \text{constant}.$$

The value of this constant was determined in equation 29; thus

$$\nu(\alpha+3\theta) = \alpha \qquad (43$$

In equation 40, μ is a function of θ only since the velocity of the electrons relative to the neutral atoms is negligible. From equation 5 and $\mu = \frac{\kappa\beta}{K_1}$, one has

$$\mu = \left(\frac{8}{\pi}\right)^{1/2}\theta^{1/2}\; V\sigma_o N_o\; e^{-\frac{\alpha}{2\theta}}$$

Using equation 43 and the above expression for μ, equation 40 becomes

$$\int_{\left(\frac{\alpha}{2\theta_2}\right)^{1/2}}^{\left(\frac{\alpha}{2\theta}\right)^{1/2}} \frac{e^{\rho^2}}{\rho^2 + 3/2}\, d\rho = \frac{2}{3\pi^{1/2}}(\alpha V^2)^{1/2}\; \sigma_o N_o\; \frac{\kappa}{K_1}\; \xi. \qquad (44$$

One notes that $\alpha V^2 = \frac{2e\phi_i}{m}$ which is independent of wave speed (or applied field) while θ_2 is the value of θ at the back side of the sheath layer and will be determined from a detailed analysis of the sheath layer. The integral in equation 44 is divergent as the upper limit approaches infinity; however, this only says that as $\xi \to \infty$ θ decreases toward its final value. The value of the integral can be found by computer or numerical methods for each θ. The resulting curves for ν and θ (ν determined from equation 43) are shown in figure 3. Note that the solutions are expressed in terms of the variables ν and $\frac{\theta}{\alpha}$ and the reduced coordinate $\kappa\xi$ so that only scale factors change as the applied field, and therefore wave speed, changes. These curves completely describe the broad quasi-neutral region once θ_2 is known.

Solution of the Thin Sheath Layer

All that remains to be done in order to complete the description of the profile of a proforce electron fluid-dynamical wave is to obtain solutions for the electron variables in the sheath layer. In this layer many things are happening. The field is changing rapidly due to the presence of a space charge; the electrons are accelerating due to the presence of the field and due to friction against the heavy particles; ionization is occurring; and the electron temperature is changing. All these significant variations make meaningful approximation difficult. To gain insight into the variations of the variables in the sheath layer, one can examine the values of the derivatives of the variables at the wave front for a typical wave; as before, take a typical wave in He to have $V = 10^9$ cm/sec. Referring back to the values quoted in equations 37, 38, and 39 and substituting these values into equations 22, 34, and 36, one can calculate

$$\left[\frac{d(\ln\psi)}{d\xi} \right]_1 = 8.38,$$

$$\left[\frac{d(\ln\nu)}{d\xi} \right]_1 = -1.08,$$

$$\left[\frac{d(\ln\theta)}{d\xi} \right]_1 = 4.52.$$

Now knowing ν_1 one cannot calculate $(\frac{d\varepsilon}{d\xi})_1$, but one would anticipate that ε would be a rapidly varying function. In summary, ε and ψ vary rapidly, θ varies somewhat more slowly, and ν varies much more slowly. Even though this knowledge will prove helpful later, it does not give any real indication of a method of attack on the problem. However, there is one factor which is in our favor; the sheath layer should be relatively thin.

The thickness of the sheath layer will be determined by how long it takes ψ to attain a value close to unity. To obtain a limit on this thickness, eliminate ν between equations 22 and 25 to obtain

$$\frac{1}{\psi(1-\psi)} \frac{d\psi}{d\xi} = \frac{\mu}{\psi} \left(\frac{d\varepsilon}{d\xi}\right)^{-1} \frac{d^2\varepsilon}{d\xi^2} . \qquad (45$$

At the front of the wave $\frac{d\varepsilon}{d\xi}$ starts off with a negative value. As ψ approaches unity, $\frac{d\varepsilon}{d\xi}$ approaches zero so $\frac{d\varepsilon}{d\xi}$ is a negative, increasing function; $\frac{d^2\varepsilon}{d\xi^2}$ is positive. Therefore, neglecting the last term in equation 45 will give an underestimate of $\frac{d\psi}{d\xi}$ everywhere and will give an upper limit on the thickness of the sheath layer. Under the assumption $\left(\frac{d\varepsilon}{d\xi}\right)^{-1} \frac{d^2\varepsilon}{d\xi^2} << \frac{\mu}{\psi}$, equation 45 becomes

$$\frac{1}{1-\psi} \frac{d\varepsilon}{d\xi} = \mu$$

Since μ is a function of ψ and θ, the above expression cannot be integrated directly. However, one can use this equation and equation 36 to obtain numerical solutions by the method of numerical iterations for and specific case. For $V = 10^9$ cm/sec in He one finds the thickness of the sheath ξ_2 to be bounded by

$$\kappa\xi_2 = 1,$$

or since $\kappa \simeq .96$

$$\xi_2 = 1.04.$$

The above approach in which the curvature of the field is taken to be small is equivalent to assuming ε to be linear:

$$\varepsilon = 1-a\xi.$$

As a next step, take the field as having a constant curvature which is determined at the front of the wave. This should be the maximum curvature of the field, and this approximation inserted in equation 45 will over approximate $\frac{d\psi}{d\xi}$ and give a lower limit on ξ_2. Setting

$$\varepsilon = 1 - a\xi(1 - b\xi),$$

equation 45 becomes

$$\frac{1}{\psi(1-\psi)} \frac{d\psi}{d\xi} = \frac{\mu}{\psi} + \frac{2b}{1-2b\xi} .$$

Since the value of $\frac{d\psi}{d\xi}$ at the front is known to be

$$\left(\frac{d\psi}{d\xi}\right)_1 = \frac{3\psi_1 + \kappa(\psi_1-1)(5\psi_1-2)}{5\theta_1-3\psi_1^{\,2}} , \tag{46}$$

b is determined. The constant a is determined by requiring that the ν and θ curves have the same values at ξ_2 when determined from the sheath solutions as when determined from the quasi-neutral solutions. In the quasi-neutral region

$$\nu(\alpha + 3\theta) = \alpha$$

Setting $\psi = 1$ in equation 28 and comparing with the above expression shows that

$$\varepsilon_2 = 0. \tag{47}$$

Rewriting equation 25 as

$$\psi = 1 + \frac{\alpha}{\nu} \frac{d\varepsilon}{d\xi} \tag{48}$$

it is obvious that ψ approaching unity is the same as $\frac{d\varepsilon}{d\xi}$ approaching zero. Hence, at ξ_2

$$\varepsilon_2 = 0,$$

$$\left(\frac{d\varepsilon}{d\xi}\right)_2 = 0$$

while

$$b = \frac{1}{2\psi_1(1-\psi_1)} \left(\frac{d\psi}{d\xi}\right)_1 - \frac{\mu_1}{2\psi_1} \tag{49}$$

This gives

$$\xi_2 = \frac{1}{2b}$$

and

$$a = 4b.$$

Again putting in the values for $V = 10^9$ cm/sec in He at unit density, one finds $\xi_2 = .30$. This is a lower limit on ξ_2 so

$$.30 \leq \xi_2 \leq 1.04 \tag{50}$$

for $V = 10^9$ cm/sec. From these considerations one concludes that the sheath layer is thin but is not a discontinuity for this case.

A second order series for ε can satisfy the boundary conditions at the leading edge of the wave and also permits solutions for ν and θ which satisfy equation 43 at ξ_2. However, the curves for the sheath layer and for the quasi-neutral region do not go together smoothly; the curves have different slopes as they approach ξ_2 from opposite sides. Eliminating ψ between equations 22 and 25 gives

$$\mu\nu - \frac{d\nu}{d\xi} = \alpha\frac{d^2\varepsilon}{d\xi^2} \tag{51}$$

This expression is valid throughout the wave. On the other hand, for ψ close to unity, we previously found

$$\frac{d\nu}{d\xi} = \mu\nu$$

For this to be true, $\frac{d^2\varepsilon}{d\xi^2}$ must approach zero as ψ approaches unity. This condition cannot be satisfied by a field with constant curvature. A third order power series for ε,

$$\varepsilon = 1 - a\xi\ (1-b\xi+c\xi^2), \tag{52}$$

can satisfy this condition as well as the other conditions on ε which were previously derived. Since b will be determined exactly as it was for the the second order series, it is still given by equation 49. The parameters ξ_2, a, and c are determined in terms of b from combination of the above discussed conditions

$$\varepsilon_2 = 0,$$

$$\left(\frac{d\varepsilon}{d\xi}\right)_2 = 0,$$

$$\left(\frac{d^2\varepsilon}{d\xi^2}\right)_2 = 0$$

with the results

$$\xi_2 = \frac{1}{b},$$

$$a = 3b,$$

$$c = \frac{b^2}{3}.$$

Equation 45 then gives

$$\psi \doteq 1 + \frac{3\alpha b}{\nu}\,(1 - 2b\xi + b^2\xi^2). \tag{53}$$

Recalling that the velocity ψ varies much more rapidly than the density ν, one realizes that the above expression for ψ must vary mainly due to the series term in the numerator resulting from differentiation of ε with only secondary dependence on ν. Therefore, good values for ψ can be obtained from an approximate form for ν. To approximate ν, one can substitute the third order series for ε (equation 52) into the differential equation for ν given in equation 51 and integrate holding μ constant. The resulting expression for ν,

$$\nu = \nu_1 e^{\mu\xi} + 2a\left(\frac{6c}{\mu^2} - \frac{2b}{\mu}\right)(e^{\mu\xi} - 1) - 2a\frac{6c\xi}{\mu}, \tag{54}$$

will prove adequate to determine ψ to a high degree of accuracy because the numerator of the ψ expression will have fallen to a low value by the time μ has changed significantly from μ_1. ν_1 is determined from Poisson's equation to be

$$\nu_1 = \frac{\alpha a}{1-\psi_1}.$$

It is now a simple matter to calculate values of ψ and ε for any wave speed; the curves for ψ and ε enable one to, in turn calculate a more accurate κ from equation 31. Calculating ε and ψ for $V = 10^9$ cm/sec in He at unit density and then recalculating κ gives $\kappa = .90$, a value which agrees well with the approximate value of .96 previously obtained. Henceforth, $\kappa = .90$ will be taken for $V = 10^9$ cm/sec. The corresponding curves for ψ and ε are shown in figures 4 and 5.

Before attempting to solve equation 36 for $\psi^{2/3}\,\theta$ (and thus for θ), it will prove profitable to examine the nature of μ over the range of θ's and ψ's which will be encountered. Carrying out the calculations for various ψ's and θ's and examining the results, one finds that μ can be expressed as

$$\mu = \mu_c(\theta)\; f(\psi) \tag{55}$$

without a significant loss of accuracy. Attempts to simplify equation 5 to the above form by making suitable approximations have not succeeded. None the less, the form of μ given in equation 55 fits the calculated values of μ for all cases thus far carried out and we anticipate that it is generally valid. The expressions

$$\mu_c = \kappa B\theta$$

and

$$f(\psi) = \psi^{-P}$$

fit the above numerical data.

One may now proceed to solve for $\psi^{2/3}\theta$. Employing the above form for μ, equation 36 becomes

$$\frac{d}{d\xi}(\psi^{2/3}\theta) = \frac{2\kappa}{3}\psi^{-1/3}(\psi-1)^2 - 1/3\kappa B\psi^{-(P+1/3)}\theta[3\theta+\alpha-(\psi-1)^2] \quad (56$$

Since ψ is a known function of ξ, this is a first order, non-linear differential equation in one variable. The non-linearity of the equation prevents one from integrating it to obtain a standard form. The best that we can do from this approach is to numerically integrate this equation by numerical interations; the results for $V = 10^9$ cm/sec in He at unit density is shown in figure 6. Having only numerical solutions for θ, one can only obtain numerical curves for ν and μ. Figures 7 and 8 show the numerical curves for ν and μ when $V = 10^9$ cm/sec in He at unit density.

The numerical curves obtained above cannot be considered the general solution for the profile of an electron fluid-dynamical wave. However, these curves which were obtained for a typical wave speed can serve as a guide as to how to obtain more acceptable solutions. Looking at the curve for ψ in figure 4, one sees that in the sheath ψ is almost linear in ξ:

$$\psi = \psi_1 + \psi_2 \xi .$$

Therefore,

$$\frac{d}{d\xi} = \psi_2 \frac{d}{d\psi}$$

and one can effect a change of independent variable throughout our system of equations.

Carrying out the differentiation in equation 56, the equation for θ becomes

$$\frac{d\theta}{d\psi} + \frac{\kappa B}{\psi_2} \psi^{-(P+1)} \theta^2 - 1/3 \frac{\kappa B}{\psi_2} [(\psi-1)^2 - \alpha] \psi^{-(P+1)} \theta - \frac{2\kappa}{3\psi_2} \psi^{-1} (\psi-1)^2 = 0. \qquad (57$$

This is in the form of Riccati's equation so it can be transformed to a homogenous linear second order differential equation by the charge of variable

$$\theta = \frac{\psi_2}{\kappa B} \psi^{P+1} \frac{d}{d\psi} [\ln u].$$

This gives

$$\frac{d^2u}{d\psi^2} + \psi^{-1}\{5/3+p-1/3\ \frac{\kappa B}{\psi_2}[(\psi-1)^2-\alpha]\psi^{-p}\}\ \frac{du}{d\psi}$$

$$-\psi^{-2}2/3(\frac{\kappa}{\psi_2})^2 B(\psi-1)^2\psi^{-p}u = 0. \qquad (58$$

Due to the nature of the coefficients of u and $\frac{du}{d\psi}$, one experiences difficulty in determining u. Therefore, we will numerically analyze these coefficients and obtain mathematically simpler expressions through curve fitting the numerical values. One then obtains the differential equation

$$\frac{d^2u}{d\psi^2} + \psi^{-1}(2.39\ -27\psi^{-1})\ \frac{du}{d\psi}\ -\ \psi^{-2}\ .31(\psi^{-1}-1)\ u = 0$$

for the case of $V = 10^9$ cm/sec in He at unit density. The solution to this equation is

$$u = \psi^{-1.11}[C_1 - C_2(.27)^{.83}\int_{(\frac{.27}{\psi_1})}^{(\frac{.27}{\psi})} x^{-1.83}\ e^{-x}\ dx].$$

Therefore,

$$\theta = \frac{\psi_2}{\kappa B}\ \psi^p\{\psi^{.83}e^{-\frac{.27}{\psi}}[k_o-(.27)^{.83}\int_{(\frac{.27}{\psi_1})}^{(\frac{.27}{\psi}} x^{-1.83}\ e^{-x}dx]^{-1}-1.11\} \qquad (59$$

where k_o is the ratio of C_1 to C_2 and is determined by requiring $\theta=\theta_1$ when $\psi=\psi_1$. The integral can be expressed as an incomplete Γ- function and looked up in tables[20]. The resulting curve for θ when $V = 10^9$ cm/sec in He at unit density is also shown in figure 6.

Knowing θ as a function of ψ, one can calculate μ as a function of ψ; the result for the above θ curve is shown in figure 9. Except in the neighborhood of ψ_1, μ is approximately linear in ψ:

$$\mu = \mu_o - \mu_2 \psi .$$

Changing independent variables in the production equation gives

$$\frac{d}{d\psi}[\ln (\nu\psi)] = \frac{\mu}{\psi_2 \psi} . \tag{60}$$

Using the linear μ approximation, this can be integrated to obtain

$$\nu = \nu_1 \psi_1^{-\left(\frac{\mu_o}{\psi_2} - 1\right)} e^{\frac{\mu_2}{\psi_2}\psi_1} \psi^{\left(\frac{\mu_o}{\psi_2} - 1\right)} e^{-\frac{\mu_2}{\psi_2}\psi} \tag{61}$$

a graph for this expression when $V = 10^9$ cm/sec in He is also shown in figure 7. Inserting this expression for ν into Poisson's equation and integrating with respect to ψ yields

$$\varepsilon = 1 + \left[\frac{\nu_1}{\alpha\psi_2} \psi_1^{\left(1 - \frac{\mu_o}{\psi_2}\right)} e^{\frac{\mu_2}{\psi_2}\psi_1}\right]\Big\{\frac{\psi_2}{\mu_0}\left(\psi_1^{\frac{\mu_o}{\psi_2}} e^{-\frac{\mu_2}{\psi_2}\psi_1} - \psi^{\frac{\mu_o}{\psi_2}} e^{-\frac{\mu_2}{\psi_2}\psi}\right)$$

$$+ \left(1 - \frac{\mu_2}{\mu_o}\right)\left(\frac{\psi_2}{\mu_2}\right)^{\left(\frac{\mu_o}{\psi_2} + 1\right)} \Gamma\left(\frac{\mu_o}{\psi_2} + 1\right)\left[I\left(k_1\psi, \frac{\mu_o}{\psi_2}\right) - I\left(k_1\psi_1, \frac{\mu_o}{\psi_2}\right)\right]\Big\} \tag{62}$$

where I (x,p) is the incomplete gamma function and

$$k_1 = \frac{\mu_2}{\psi_2}\left(\frac{\mu_o}{\psi_2} + 1\right)^{-1/2} .$$

One can determine ν_1 by requiring that ε go to zero as ψ approaches unity. The resulting curve when $V = 10^9$ cm/sec in He at unit density is shown along with the series solution in figure 5. Calculations reveal that the dependence of ε on the incomplete gamma function in equation 62 is small ~~so~~ that the basic dependence of ε on ψ is

$$\varepsilon \simeq C_1 + C_2\, \psi^{\frac{\mu_0}{\psi_2}}\, e^{-\frac{\mu_2}{\psi_2}\psi} . \tag{63}$$

Thus, under a linear ψ approximation, one is able to effect a working description of the variables in the sheath layer.

The above solutions for the sheath layer are only known to be valid for wave speeds of around 10^9 cm/sec in He at unit density. Let us now examine what happens as wave speed varies. Utilizing our knowledge of ψ_1 in equation 32, we conclude that κ is almost independent of wave speed. As V increases from 10^9 cm/sec toward infinity, α decreases from .086 toward zero, ψ_1 increases from .222 toward .250, and θ_1 increases from .173 toward .187. Sheath thickness, ξ_2, increases monotonically from .56; the existence of an upper limit on the thickness of the sheath layer cannot be established for two reasons. First, the expression used for σ_i in deriving equation 5 is quite probably not valid for extremely high electron velocities so we do not know μ for high wave speeds. Second, particle velocities would need to increase by no more than a factor of ten from the values for $V = 10^9$ cm/sec for relativistic corrections to become important. Hence, one concludes that sheath thickness increases with increasing wave speed as long as our formulation of the problem is valid. Since θ_1 and ψ_1 do not change appreciably as V increases, one would not expect b to change appreciably with increasing V, and would expect the sheath layer to remain thin but finite. On the other hand, as V decreases toward its limiting value of $(\frac{2e\phi_i}{m})^{1/2}$, α increases toward unity while ψ_1 and θ_1 approach zero. From equation 35, $(\frac{d\psi}{d\xi})_1$ becomes infinite so b becomes infinite (equation 49) and ξ_2 goes to zero, i.e., the sheath layer becomes a discontinuity. Hence, the above solutions for the sheath layer are valid for all wave speeds for which our formulation of the fluid equation are valid. Coupling the sheath solution to the quasi-neutral solutions gives a complete description of proforce electron fluid-dynamical waves. Figures 10 and 11 show the curves for ν and θ for the entire wave when $V = 10^9$ cm/sec in He at unit density. Since ψ and ε do not vary in the quasi-neutral region, their profiles are completely shown in figures 4 and 5. In order to indicate the meaning of the scale factors for the different reduced variables, we will give the value of each variable in terms of common units corresponding to a typical value of the reduced variable. Thus, $\nu = 1$ corresponds to an electron density of $N = 2.91 \times 10^9$ cm^{-3}, $\psi = 1$ corresponds to an electron velocity of $V = 10^9$ cm/sec, $\theta = .10$ corresponds to an electron temperature of 6.6×10^5 °K, and $\varepsilon = 1$ corresponds to an electric field of $E = 5.52 \times 10^3$ V/m for a wave speed of $V = 10^9$ cm/sec in He at unit density. We now have a complete understanding of the profile of proforce waves.

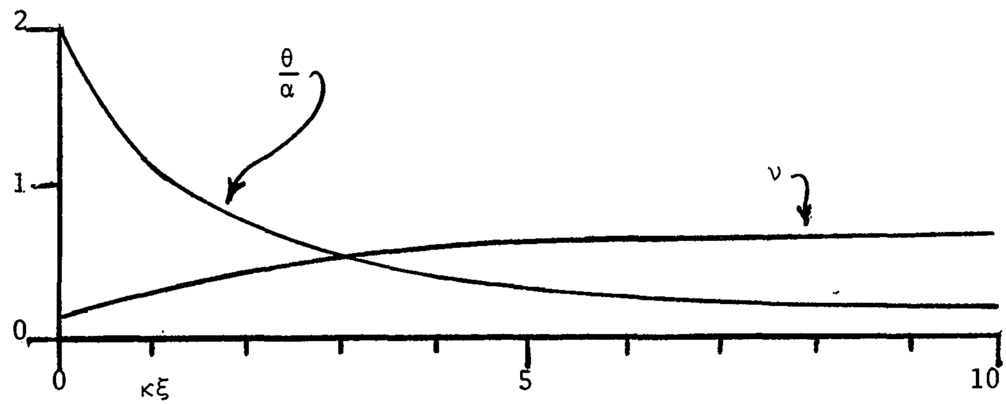

Figure 3 Electron temperature and density versus position for quasi-neutral region graph.

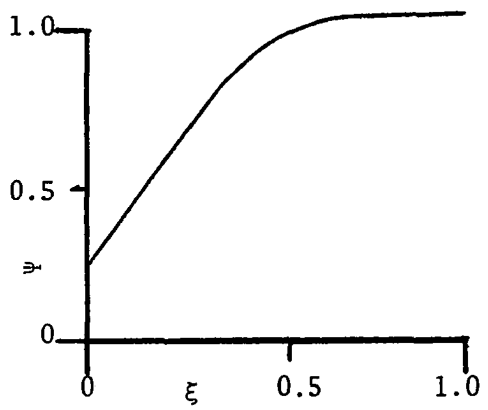

Figure 4 Electron velocity versus position graph.

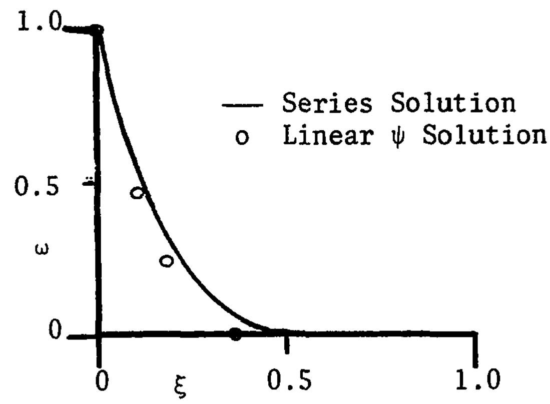

Figure 5 Electric field versus position graph.

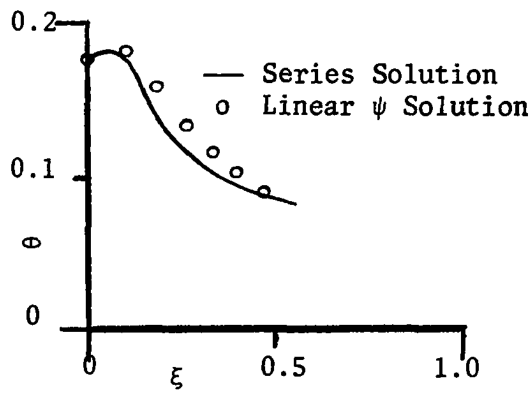

Figure 6 Electron temperature versus position for sheath layer graph.

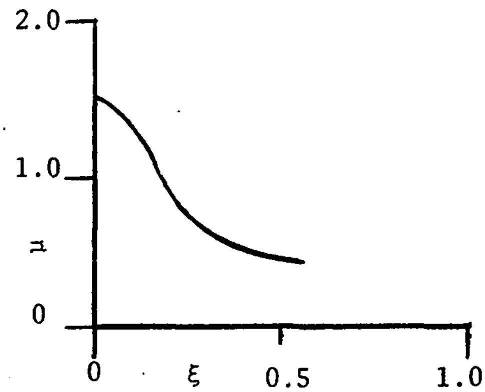

Figure 7 Ionization frequency versus position for sheath layer graph.

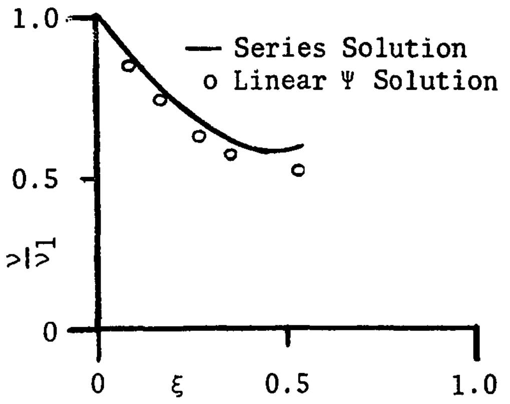

Figure 8 Electron density versus position for sheath layer graph.

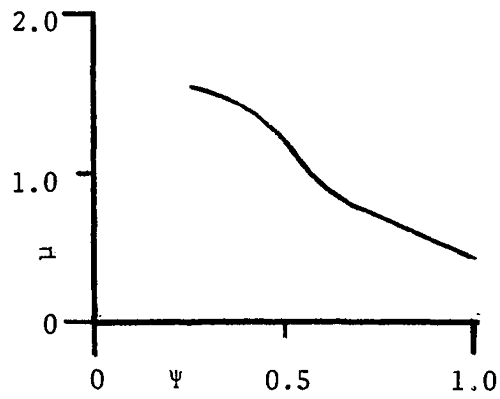

Figure 9 Ionization frequency versus electron velocity graph.

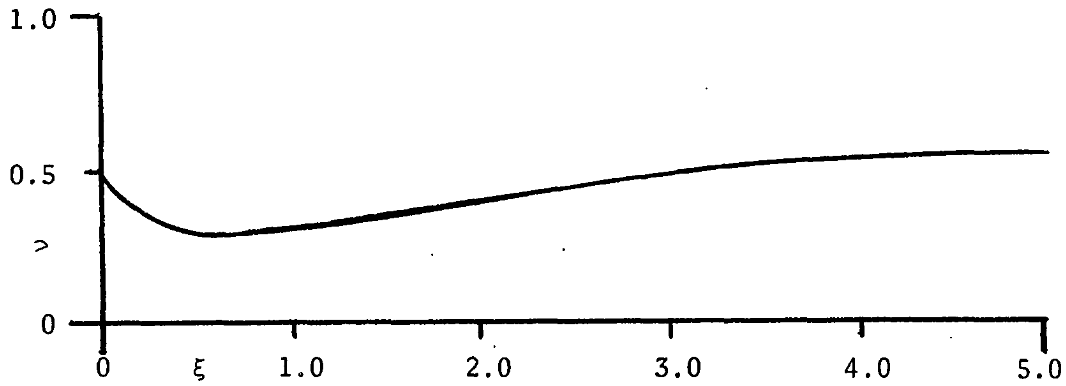

Figure 10 Electron density versus position graph.

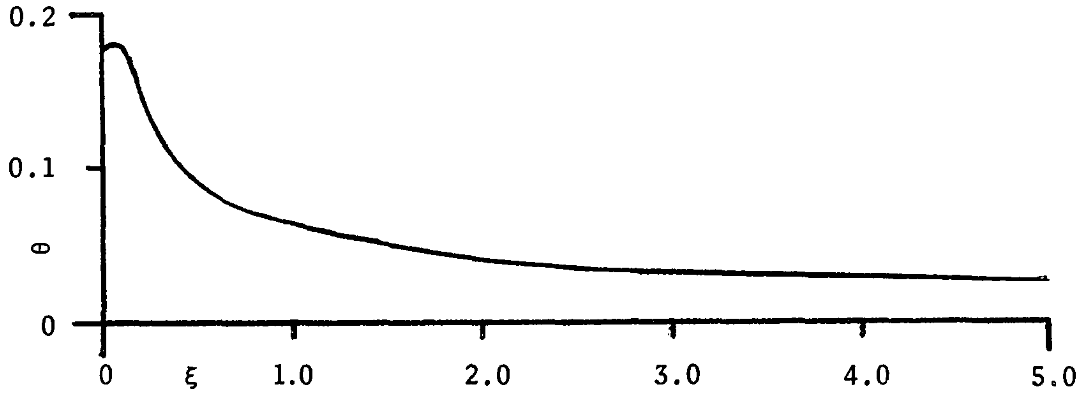

Figure 11 Electron temperature versus position graph.

CHAPTER VI

ANALYSIS OF MODEL

Having obtained solutions from the fluid dynamical model, one is in a position to analyze the validity of the model. First, one must answer the question of whether the fluid dynamical equations can constitute a valid model for electron fluid-dynamical waves. This is equivalent to asking if the individual particles of a particular species experience a sufficient number of collisions within the wave thickness so that the particle velocities of the species are correlated. The mean free path for Heluim at unit density is about 1.7×10^{-2} cm[23]. Since it was found above that the wave thickness is of the order of 1 cm, the heavy particles will experience many collisions as they go through the wave and may definitely be treated as a fluid. On the other hand, electrons do not experience definite events which can be labeled as collisions in going through the wave. However, this is misleading. An electron does interact strongly with the electric field so it experiences a "collision" with the space charge field due to all the other electrons and ions about it. These "collisions" bring about correlation of the electron velocities within lengths short compared with the wave thickness, and the electrons do behave as a fluid, so that the fluid equations can form a valid model for electron fluid-dynamical waves.

Let us now examine the assumption of constant wave velocity for a steady profile wave under a constant applied field. Most observers have reported that they observed accelerating waves. This may or may not indicate that assuming a constant wave velocity is incorrect. Let us consider a discharge tube of length L with a constant potential V_o applied across it. At time t = 0, the wave has not propagated away from the discharge electrode, and the applied field E_o is $E_o = -\frac{V_o - \Delta V}{L}$ where ΔV is in the front potential which is constant for a steady profile wave. At a later time when the wave has propagated a distance x from the discharge electrode, $E_o = -\frac{V_o - \Delta V}{L - x}$ since the wave carries the potential of the discharge electrode. Hence, E_o increases in magnitude; and since wave speed is approximately proportional to applied field, the wave accelerates. This acceleration results from maintaining the applied potential constant rather than the applied field. Since experimental work is usually done with a constant potential applied across the electrodes, it seems likely that the observed acceleration of electron fluid-dynamical waves is due to increasing field. Care must be exercised to perform experiments conforming to the conditions of the theoretical model.

Finally, one should analyze the importance of thermal conduction which has been ignored in the present model. One can derive a rough estimate of the thermal conduction term from elementary kinetic theory considerations. The flux of thermal energy through a surface can be shown to be

$$dQ = K\frac{dT}{dX}$$

where

$$K = n\lambda k(\frac{8kT}{\pi m})^{1/2}.$$

Since n, T, and $\frac{dT}{dX}$ can easily be found from the solutions derived in the last chapter, one can find the flux of thermal energy through unit area if he knows λ. In the present case electrons experience events which can be labeled collisions only infrequently so λ is quite large and thermal conductivity is apparently significant. However, this is the same problem previously encountered when discussing whether the electrons behave as a fluid. There it was pointed out that even though an electron does not undergo specific particle collisions, it does "collide" with the space charge field due to the other electrons and ions about it. If one considers λ to be the length over which an electron interacts significantly with other particles, then one concludes from the sheath solutions that $\lambda \simeq .5$ cm, $n \simeq 3 \times 10^9$ cm^{-3}, and $T \simeq 10^6$ °K; $K \simeq 1.3 \times 10^2$ ergs per cm per sec per °K. Since $\frac{\Delta T}{\Delta X} \simeq 5 \times 10^5$ °K (again for $V = 10^9$ cm/sec in He at unit density),

$$K\frac{dT}{dX} \simeq 6.5 \times 10^7 \text{ ergs sec}^{-1} \text{ cm}^{-2}.$$

This heat flux must be compared with the flux of kinetic energy through unit area, $1/2mv^3$. Putting in the appropriate numbers, one finds

$$1/2mnv^3 \simeq 1.4 \times 10^9 \text{ ergs sec}^{-1} \text{ cm}^{-2}.$$

The thermal conduction term is only about 5% of the kinetic energy flux term so thermal conduction is indeed unimportant.

Since the results obtained from a fluid model are consistent with the restrictions placed upon the model, the fluid-dynamical model is self consistent. This does not insure that the solutions obtained will present a correct picture of the problem under consideration but it makes it highly probable; when agreement is obtained between such a self consistent model and experiment, one can be almost certain that the model is correct. Since the results obtained from the fluid model show general agreement with experimental evidence, we feel justified in asserting that the fluid-dynamical equations are a proper model for electron fluid-dynamical waves.

CHAPTER VII

CONCLUSION

It has been shown that the structure of a one-dimensional, steady profile, electron fluid-dynamical wave can be derived from the fluid-dynamical equations. The proforce wave is composed of two distinct regions. The first is a thin Debye sheath type of layer in which the electric field falls to a negligible value and the electrons and heavy particles come to rest relative to one another. Following this is a region of local neutrality in which the electrons still have sufficient thermal energy for ionization to continue. The number density of electrons (and ions) far behind the wave was found to be

$$n = \frac{\varepsilon_o E_o^2}{2e\phi_i}$$

when E_o is the applied field. Essentially all the energy of the electric field goes into ionizing the neutral atoms. It has also been shown that the wave speed must satisfy

$$1/2mV^2 \geq e\phi_i$$

so there is a lower limit on wave speed. The wave speed corresponding to an applied field E_o is given approximately by

$$V = \frac{3}{4(1-\psi_1)} \frac{e}{mK_1} E_o .$$

Since V_o has a lower limit, so does E_o. The term in brackets varies by less than 25% as E_o and V vary over their entire ranges so V is approximately linear in E_o. This is in agreement with experimental evidence. Calculating numerical values for a wave speed of 10^9 cm/sec in He at unit density as has been done throughout this thesis, one finds that electron densities are of the order of 3 x 10^9 cm^{-3}, electron velocities are of the order of 10^9 cm/sec, and electron temperatures are of the order of 10^6 °K. The applied field necessary to obtain a wave speed of 10^9 cm/sec is 5.52 X 10^4 V/M. This field is within order of magnitude agreement with the fields present in the discharge tubes of Snoddy et al and Haberstitch. A quantitative determination of the fields present in their discharge tubes is almost impossible so a quantitative correlation of experimental and theoretical data cannot be made. Haberstitch measured electron densities of 10^{10} to 10^{11} cm^{-3}, but this was for antiforce waves. An electron density of 3 X 10^9 cm^{-3} may be a proper value for proforce waves of V = 10^9 cm/sec in He. Another factor influencing the electron density in a discharge tube is losses to the walls; these losses do not enter into a one-dimensional approach. These losses also influence the wave thickness; even so, the thickness of about 2 cm which we found for the above case agrees well with the thicknesses reported by Haberstitch.

Even though it is not possible to match theory to experiment due to lack of sufficient data, the experimental data do not indicate any reason to doubt the theory. On the contrary, all data seem to confirm the theory.

This thesis has demonstrated the analysis of only one class of electron-fluid waves -- the proforce waves in one-dimensional, time independent situations. However, it has shown that the fluid-dynamical equations provide a good model for electron fluid-dynamical waves; this model will certainly serve as the basis for analyzing proforce waves in more complex geometries, antiforce waves, and time dependent waves. It is hoped that the present work might be a useful tool in future experimental theoretical work.

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