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COMPLETENESS IN TOPOLOGICAL SPACES

1. INTRODUCTION

One of the most significant generalizations of the concept of a metric space is that of a space satisfying the first three parts of R. L. Moore's Axiom 1 [16]. Such a space is called a Moore space while a space satisfying all of Axiom 1 [16] is called a complete Moore space. Every (complete) metric space is a (complete) Moore space, but there exist (complete) Moore spaces which are not metrizable. Since this paper is primarily concerned with various forms of completeness in topological spaces, the remaining comments will focus mainly on complete Moore spaces and further generalizations of complete metric spaces.

Roberts [20] showed that a metric space is metrically topologically complete if and only if it is a complete Moore space. Many theorems dealing with complete metric spaces are also valid for complete Moore spaces. The following examples were singled out by Armentrout in [3]:

Theorem A. [16; Chapter I, Theorem 162] The intersection of countably many open dense subsets of a complete Moore space S is dense in S.

Theorem B. [16; Chapter II, Theorem 1] A connected and locally connected complete Moore space is arcwise connected.

Theorem C. [16; Chapter I, Theorem 164] A complete Moore space is separable if and only if it does not contain uncountably many mutually exclusive open sets.

In 1950 Rudin [21] introduced a variation of completeness in Moore spaces and a space satisfying Axiom 1'' (equivalently Axiom 1''') of [21] is called a semicomplete Moore space. Every complete Moore space is semicomplete, but an example is given in [21; Theorem 9] of a semicomplete Moore space which is not complete. This apparently is the only known example of such a space, and it is natural to ask what additional properties are sufficient to guarantee that a semicomplete Moore space is complete. In [21] Roberts' result is extended to show that a metric space is complete if and only if it is a semicomplete Moore space. Although Rudin provides several interesting theorems and examples, results concerning semicomplete Moore spaces as such are relatively scarce. However, a number of theorems concerning complete Moore spaces which have appeared in the literature quite recently are valid for semicomplete Moore spaces and even spaces satisfying a more general completeness property described below. Armentrout observes in [3] that the three theorems previously mentioned in the discussion of complete Moore spaces are also valid for semicomplete Moore spaces.

In 1930 Aronszajn [4], in his study of arcwise connectedness, introduced the following completeness axiom for a T_3 (= regular Hausdorff) space: there is a sequence G of bases for the topology of the space such that if $g = g_1, g_2, \ldots$ is a sequence with $g_n \varepsilon G_n$ and $g_{n+1} \subset g_n$ for each n, then there is a point every neighborhood of which contains a term of g. Although not in chronological order, such a space is a generalization of a semicomplete Moore space. Aronszajn did not develop a theory of these spaces beyond showing that the previously mentioned arc theorem (Theorem B) holds in this setting. The arguments given by

Moore [16; loc. sit.] will show that Theorems A and C also have analogs for spaces satisfying Aronszajn's axiom. Zippin [35] extended Roberts' result by showing that a metric space is complete if and only if it satisfies Aronszajn's axiom and he also exhibited a space satisfying Aronszajn's axiom but which is not a Moore space.

In 1963 Arhangel'skii [2] introduced the concept of a base of countable order and proved that a space is metrizable if and only if it is a paracompact Hausdorff space having a base of countable order. In 1967 Wicke and Worrell appended a rather natural completeness property to the definition of a base of countable order and called the result a λ-base. For essentially T_1 spaces, the existence of a base of countable order is equivalent to the existence of a sequence of bases having a striking similarity to that in Aronszajn's axiom. In fact, a T_3 space satisfies Aronszajn's axiom if and only if it has a λ-base. Consequently, a T_3 space having a base of countable order (λ-base) is called an Aronszajn space (a complete Aronszajn space). These spaces are generalizations of Moore spaces and semicomplete Moore spaces respectively.

One major point on which Moore spaces and Aronszajn spaces differ from metric spaces is that, while every metric space is a dense subspace of a complete metric space, there is a Moore space which is not a subspace of any complete Aronszajn space. (See [21; Theorem 8] and [17].) Considerable work has been done by Alzoobaee [1], Green [12], Reed [19], and Whipple [23] in order to establish sufficient (and in some instances, necessary) conditions under which a Moore space has a completion or semicompletion. Apparently the completability of Aronszajn spaces has not been investigated.

Since 1965 Worrell and Wicke have engaged in a very thorough investigation of completeness in spaces having a base of countable order as well as spaces having a primitive base as described in Section 3 of this paper and non-first-countable analogs of spaces having a base of countable order such as the β_c and β_b spaces defined in [25]. The basis for the proofs of many of their results is essentially the technique employed by Rudin in [21; Theorem 2]. An extensive list of their works is found in the bibliography although many of their results have not been published. Many of the definitions used in this paper are due to Worrell and Wicke and can be found in [28] and [29].

The concept of a set of interior condensation, as introduced by Worrell and Wicke in [28] as a generalization of a G_δ set, has proved to be very useful in the investigation of topological completeness in spaces more general than metric spaces. In Section 4 of this paper some general properties of sets of interior condensation are presented to demonstrate the similarity to G_δ sets. These properties are also useful in Section 5 where several external characterizations of complete Aronszajn spaces are given. In Section 6 completeness and semicompleteness in Moore spaces are compared, in particular with regard to the similarity between the roles of G_δ sets and sets of interior condensation.

2. DEFINITIONS

Throughout this paper N denotes the set of positive integers and S denotes a topological space.

2.1 If M is a subset of a set X and F is a collection of subsets of X, then $\{f \cap M \mid f \in F\}$ is called the <u>trace of F on M</u> and is denoted by $Tr_M F$.

2.2 If G is a collection of sets, then G* denotes the union of all the members of G. If G is a well-ordered collection of sets and h is a set which is a subset of some member of G, then G(h) denotes the first member of G containing h. If h = {P} for some $P \in G^*$, then G(h) will be denoted by G(P).

2.3 A sequence G of sets is said to be <u>monotonic</u> if and only if $G_{n+1} \subset G_n$ for each n.

2.4 If $T \subset S$ and G is a sequence each term of which is a collection of sets open in T, then G is said to be <u>T-open</u>.

2.5 If $T \subset S$ and B is a collection of sets open in S such that if $P \in T$ and U is an open set in S containing P, there is a set in B containing P and contained in U, then B is said to be a <u>base for S at T</u>.

2.6 If G is a sequence of collections of sets and g is a monotonic sequence of nonempty sets such that $g_n \in G_n$ for each n, then g is called a <u>G-filterbase</u>. If each member of each term of G is a subset of S and g is a G-filterbase such that $\bar{g}_{n+1} \subset g_n$ for each n, then g is called a <u>G-nest</u>.

2.7 If each of f and g is a sequence (collection) of sets such that each term (member) of f contains a term (member) of g, then f is said to be <u>coarser</u> than g or equivalently g is <u>finer</u> than f.

2.8 If X is a set and $M \subset X$, then a sequence G is called a <u>monotonically contracting sequence of M in X</u> if for each n, G_n is a collection of subsets of X covering M such that if $g \in G_n$ and $x \in M \cap g$, there is a set in G_{n+1} containing x and contained in g.

2.9 If X is a set and $M \subset X$, then a sequence G is called a <u>primitive sequence of M in X</u> if for each n, G_n is a well-ordered collection of subsets of X covering M such that if $g \in G_n$, there is a point P in M such that $g = G_n(P)$ and if $Q \in M$, then $G_{n+1}(Q) \subset G_n(Q)$.

2.10 If X is a set and $M \subset X$ and G is a primitive sequence of M in X, then a G-filterbase g is called a <u>primitive representative of G</u> if for each n, there is a point P in S such that $g_n = G_n(P)$ and $g_{n+1} = G_{n+1}(P)$.

2.11 A collection of sets is said to be <u>perfectly decreasing</u> if and only if it contains a proper subset of each of its members.

2.12 A <u>base of countable order</u> for S is a base B for the topology of S such that if P is a point belonging to each member of a perfectly decreasing subcollection T of B, then T is a base for S at P.

2.13 A <u>λ-base</u> for S is a base of countable order B for S such that if T is a perfectly decreasing monotonic subcollection of B, there is a point every neighborhood of which contains a member of T.

2.14 S is <u>essentially T_1</u> if and only if for each two points x and y, either $\overline{\{x\}} = \overline{\{y\}}$ or $\overline{\{x\}} \cap \overline{\{y\}} = \phi$.

3. PRIMITIVE BASES

3.1 <u>DEFINITION</u>. A base B for the topology of S is called a <u>θ-base</u> for S if there is a sequence H of collections of open sets such that $B = \bigcup_{n=1}^{\infty} H_n$ and if D is an open set containing the point P, there is an n such that only finitely many sets in H_n contain P and one such set is a subset of D.

3.2 <u>DEFINITION</u>. A base B for the topology of S is called a <u>primitive base</u> for S if there is a sequence H of well-ordered collections of open sets such that $B = \bigcup_{n=1}^{\infty} H_n$ and if D is an open set containing the point P, then there exist integers n and k such that H_k has at least n elements containing P and the n'th such element is a subset of D.

3.3 REMARK. The concept of a θ-base as a generalization of a σ-locally finite (or σ-point finite) base is due to Worrell and Wicke [32] and the concept of a primitive base as a generalization of a θ-base is also due to Worrell and Wicke [33]. Apparently the actual term first appeared in [30] and [31]. Theorem 3.4 shows that, in essentially T_1 spaces, a primitive base is also a generalization of a base of countable order.

Example. The space X in [5; Example 2.3] is a paracompact quasi-developable space which is not a Moore space. By [6; Theorem 8] X has a θ-base, hence a primitive base. But X does not have a base of countable order since a paracompact space having a base of countable order is metrizable.

Example. The long line L [22] has a base of countable order, hence a primitive base. However, L does not have a θ-base for if it did, then L would be weakly θ-refinable, hence paracompact by [6; Theorem 11].

Example. Every space having a primitive base is first countable but the Sorgenfrey line S [14] is a first countable space which does not have a primitive base. For closed sets in S are G_δ sets [14] and if S has a primitive base, then by Theorem 4.9 S has a base of countable order and hence, being paracompact, is metrizable.

Remark. The first example given above shows that, unlike the case for spaces having a base of countable order, a paracompact space having a primitive base need not be metrizable. However, see Corollary 3.13.

3.4 THEOREM. If S is essentially T_1, then S has a base of countable order if and only if there is an S-open primitive sequence G of S such that if P is a point belonging to each term of a primitive representative g of G, then g is a base for S at P.

Proof. Suppose S has a base of countable order. By [32; Theorem 2] there is a sequence B of bases for S such that if P is a point belonging to each term of a B-filterbase b, then b is a base for S at P and by [29; Lemma 2.1] there is a primitive sequence G of S such that $G_n \subset B_n$ for each n. G has the desired property.

Conversely, suppose G is an S-open primitive sequence of S having the indicated property. The arguments for Lemmas 2.2 and 2.3 of [29] show that there is a monotonic sequence B of open covers of S such that every B-filterbase is finer than some primitive representative of G. Since each term of B is clearly a base for S, S has a base of countable order by [32; Theorem 2].

3.5 LEMMA. Suppose $X \subseteq Y \subset S$ and $X \subset T \subset S$ and B is a sequence of well-ordered bases for S at Y and for each n, $H_n = (H_{nm})_{m=1}^{\infty}$ is a Y-open primitive sequence of X. Then there is a T-open primitive sequence W of X such that for each $n \in N$ and $P \in X$, (i) $W_n \subset Tr_T B_n$; (ii) $W_{n+1}(P) \subset B_n(P)$; and (iii) $W_n(P) \cap Y \subset \bigcap\{H_{ij}(P) \mid i+j = n+1\}$.

Proof. For each n, let $Tr_T B_n$ be well-ordered. $V_1 = \{U \in Tr_T B_1 \mid U \cap Y$ is contained in some set in $H_{11}\}$ is a base for T at X. Let $x \leq y$ in V_1 if and only if (1) $H_{11}(x \cap Y) < H_{11}(y \cap Y)$ or (2) $H_{11}(x \cap Y) = H_{11}(y \cap Y)$ and $x \leq y$ in $Tr_T B_1$. V_1 is well-ordered by this ordering. Thus $W_1 = \{V_1(P) \mid P \in X\}$ is a well-ordered subcollection of $Tr_T B_1$ covering X such that if $w \in W_1$, there is a point Q in X such that $w = W_1(Q)$ and if $P \in X$, then $W_1(P) = V_1(P)$.

Let $P \varepsilon X$. Then $H_{11}(P) \leq H_{11}(W_1(P) \cap Y)$. Suppose $H_{11}(P) < H_{11}(W_1(P) \cap Y)$. There is a set v in V_1 such that $P \varepsilon v$ and $v \cap Y \subset H_{11}(P)$. So $H_{11}(v \cap Y) = H_{11}(P) < H_{11}(W_1(P) \cap Y)$ implies $v < W_1(P)$ in V_1. Hence, $V_1(P) \leq v < W_1(P)$ which is a contradiction. So $H_{11}(P) = H_{11}(W_1(P) \cap Y)$, whence $W_1(P) \cap Y \subset H_{11}(P)$.

Suppose k is a positive integer and $W_1,\ldots,W_k$ are well-ordered T-open covers of X such that if $n \leq k$, then

(I_1) if $w \varepsilon W_n$, there is a point P in X such that $w = W_n(P)$;

(I_2) $W_n \subset Tr_T B_n$;

(I_3) $W_n(P) \cap Y \subset \bigcap\{H_{ij}(P) \mid i+j = n+1\}$ for each $P \varepsilon X$;

and if $k > 1$ and $n \leq k-1$, then

(I_4) $W_{n+1}(P) \subset W_n(P) \cap B_n(P)$ for each $P \varepsilon X$.

If $i \leq k+1$, let $H_{i,k+2-i}$ be denoted by G_i. Let $V_{k+1} = \{U \varepsilon Tr_T B_{k+1} \mid$ some set in W_k and some set in B_k contain U and if $i \leq k+1$, some set in G_i contains $U \cap Y\}$. V_{k+1} is a base for T at X. V_{k+1} is well-ordered so that $x \leq y$ if and only if

(1') there is a positive integer $j \leq k+1$ such that $G_j(x \cap Y) \neq G_j(y \cap Y)$ and if i is the least such integer, then $G_i(x \cap Y) < G_i(y \cap Y)$; or

(2') $G_i(x \cap Y) = G_i(y \cap Y)$ for $i = 1,\ldots,k+1$ and

(a) $W_k(x) < W_k(y)$; or

(b) $W_k(x) = W_k(y)$ and $B_k(x) < B_k(y)$; or

(c) $W_k(x) = W_k(y)$ and $B_k(x) = B_k(y)$ and $x \leq y$ in $Tr_T B_{k+1}$.

Then $W_{k+1} = \{V_{k+1}(P) \mid P \varepsilon X\}$ is a well-ordered subcollection of $Tr_T B_{k+1}$ covering X such that if $w \varepsilon W_{k+1}$, there is a point Q in X such that $w = W_{k+1}(Q)$ and if $P \varepsilon X$, then $W_{k+1}(P) = V_{k+1}(P)$. Thus, properties (I_1) and (I_2) hold for $n = k+1$.

Let $P \varepsilon X$. There is a set v in V_{k+1} such that $P \varepsilon v$, $v \cap Y \subset \bigcap_{i=1}^{k+1} G_i(P)$ and $v \subset W_k(P) \cap B_k(P)$. Suppose there is a positive integer j such that $G_j(P) \neq G_j(W_{k+1}(P) \cap Y)$ and let i be the least such integer. Clearly $G_i(P) < G_i(W_{k+1}(P) \cap Y)$ and for each $n \leq k+1$, $G_n(P) = G_n(v \cap Y)$. If there is a positive integer $n < i$, then $G_n(v \cap Y) = G_n(P) = G_n(W_{k+1}(P) \cap Y)$. Since $G_i(v \cap Y) = G_i(P) < G_i(W_{k+1}(P) \cap Y)$, it follows from property (1') of the order on V_{k+1} that $v < W_{k+1}(P)$. This is a contradiction since $W_{k+1}(P) = V_{k+1}(P) \leq v$. So $G_j(P) = G_j(W_{k+1}(P) \cap Y)$ for each $j \leq k+1$. Hence, $W_{k+1}(P) \cap Y \subset \bigcap_{j=1}^{k+1} G_j(p) = \bigcap\{H_{ij}(P) | i+j = k+2\}$ and property (I_3) holds for $n = k+1$. If $W_k(P) < W_k(W_{k+1}(P))$, then since $G_j(v \cap Y) = G_j(P) = G_j(W_{k+1}(P) \cap Y)$ for $j = 1,\ldots,k+1$ and $W_k(v) = W_k(P)$, it follows that $v < W_{k+1}(P)$ in V_{k+1} which is again a contradiction. Clearly $W_k(P) \leq W_k(W_{k+1}(P))$ so $W_k(P) = W_k(W_{k+1}(P))$. Similarly $B_k(P) = B_k(W_{k+1}(P))$ hence $W_{k+1}(P) \subset W_k(P) \cap B_k(P)$ and property (I_4) holds for $n = k$.

By induction there exists a sequence W as described.

3.6 <u>THEOREM</u>. S has a primitive base if and only if there is an S-open primitive sequence W of S such that if D is an open set containing the point P, there is an n such that $W_n(P) \subset D$.

Proof. Suppose S has a primitive base and let H be a sequence as in Definition 3.2. For each pair of positive integers n and k, let $V_{nk} = \{(v_1,\ldots,v_n) |$ there is a point P such that for each $i \leq n$, v_i is the i'th set in H_k containing P$\}$. If $v = (v_1,\ldots,v_n) \varepsilon V_{nk}$, let $v_* = \bigcap_{i=1}^{n} v_i$. Let V_{nk} be well-ordered by the lexicographic order and let $F_{nk} = \{v_* | v \varepsilon V_{nk}\}$ be well-ordered so that $x < y$ in F_{nk} if and only if the first element v of V_{nk} such that $x = v_*$ strictly precedes the first element w of V_{nk} such that $y = w_*$. Let D be an open set containing the point P. There exist integers n and k such that H_k has at least n elements containing P and the n'th such element is a subset of D. For $1 \leq i \leq n$, let v_i be the i'th element of H_k containing P and let $v = (v_1,\ldots,v_n)$. v_* is the first element of F_{nk} containing x.

Let $G_1, G_2,\ldots$ be an enumeration of $\{F_{nk}|n, k \in N\}$. If D is an open set containing the point P, there is an n such that $P \in G_n{}^*$ and $G_n(P) \subset D$. For each n, let $T_n = \{U|U$ is open in S, $U \notin G_n$ and there is a point P in U such that no proper open subset of U contains P$\}$. Let $K_n = G_n \cup T_n$ be well-ordered so that each element of G_n precedes each element of T_n and the order of G_n is preserved. $B_n = \bigcup_{i=n}^{\infty} K_i$ is well-ordered so that $x \leq y$ in B_n if and only if it is true that if i (resp. j) is the least integer no less than n such that $x \in K_i$ (resp. $y \in K_j$), then (1) $i < j$ or (2) $i = j$ and $x \leq y$ in K_i.

Let V be an open set containing the point P and let $n \in N$. Let U be an open subset of V containing P such that if i is a positive integer less than n and $P \in K_i{}^*$, then $U \subset K_i(P)$. If no proper open subset of U contains P, then $U \in K_n \subset B_n$ and $P \in U \subset V$. Otherwise, let D be a proper open subset of U containing P. There is an i such that $G_i(P) \subset D$ and clearly $i \geq n$. So $G_i(P) \in K_i \subset B_n$ and $P \in G_i(P) \subset V$. In either case there is a set in B_n containing P and contained in V. Thus, B_n is a base for S.

So $B = B_1, B_2,\ldots$ is a monotonic sequence of well-ordered bases for S such that if D is an open set containing the point P, there is an n such that $B_n(P) \subset D$. By Lemma 3.5 (with $X = Y = T = S$ and $H_{ij} = \{S\}$) there is an S-open primitive sequence W of S such that if $n \in N$ and $P \in S$, then $W_{n+1}(P) \subset B_n(P)$. It follows that W has the desired property. The converse follows directly from the definitions.

3.7 THEOREM. If S has a primitive base and A is a base for S, then A contains a primitive base for S.

Proof. Let G be an S-open primitive sequence of S such that if D is an open set containing the point P, there is an n such that $G_n(P) \subset D$. Let W be an S-open primitive sequence of S as in Lemma 3.5 with $X = Y = T = S$ and for each n, $H_n = G$ and $B_n = A$. Then $\bigcup_{n=1}^{\infty} W_n$ is a subcollection of A which is a primitive base for S.

3.8 THEOREM. If S has a primitive base and X is a subspace of S, then X has a primitive base.

Proof. Let G be an S-open primitive sequence of S such that if D is an open set containing the point P, there is an n such that $G_n(P) \subset D$. For each n, let $F_n = \{G_n(P) | P \in X\}$. Then $F = F_1, F_2, \ldots$ is an S-open primitive sequence of X. With $Y = S$ and $T = X$ and for each n, $H_n = F$ and B_n the topology of S, by Lemma 3.5 there is an X-open primitive sequence W of X such that $W_n(P) \subset F_n(P)$ for each $P \in X$ and $n \in N$. Let $P \in X$ and let D be an open set in X containing P. If U is an open set in S such that $U \cap X = D$, there is an n such that $F_n(P) = G_n(P) \subset U$ hence $W_n(P) \subset D$. So X has a primitive base.

3.9 THEOREM. If every point of S has a neighborhood having a primitive base, then S has a primitive base.

Proof. By Theorem 3.8 each point Q of S has an open neighborhood $U(Q)$ having a primitive base. Let $H^Q = H_1^Q, H_2^Q, \ldots$ be a $U(Q)$-open (hence S-open) primitive sequence of $U(Q)$ such that if $P \in U(Q)$ and D is an open set containing P, there is an n such that $H_n^Q(P) \subset D$. Let S be well-ordered and if $n \in N$ and x is a set contained in some set in $\bigcup\{H_n^Q | Q \in S\}$, let $Q_n(x)$ denote the first point Q of S such that some set in $H_n{}^Q$ contains x. If $Q \varepsilon S$ and $n \varepsilon N$ and x is either a point of S or a set contained in some set in $H_n{}^Q$, let $H_n{}^Q(x)$ be denoted by $H_n(Q,x)$. Let τ (the topology of S) be well-ordered.

Suppose $k \varepsilon N$ and G_k is a well-ordered open cover of S each member of which is contained in a set in $\{H_k{}^Q | Q \varepsilon S\}^*$ and if $g \varepsilon G_k$, there is a point P such that $g = G_k(P)$. $V_k = \{U \varepsilon \tau | U$ is contained in some set in G_k and some set in $\{H_{k+1}^Q | Q \varepsilon S\}^*\}$ is well-ordered so that $x \leq y$ in V_k if and only if

(1) $Q_k(x) < Q_k(y)$; or

(2) $Q_k(x) = Q_k(y)$ and $H_k(Q_k(x),x) < H_k(Q_k(y),y)$; or

(3) $Q_k(x) = Q_k(y)$ and $H_k(Q_k(x),x) = H_k(Q_k(y),y)$ and $G_k(x) < G_k(y)$; or

(4) $Q_k(x) = Q_k(y)$ and $H_k(Q_k(x),x) = H_k(Q_k(y),y)$ and $G_k(x) = G_k(y)$ and $x \leq y$ in τ.

Let $G_{k+1} = \{V_k(P) | P \varepsilon S\}$. If $P \varepsilon S$, then $G_{k+1}(P) = V_k(P)$.

Let $P \varepsilon S$ and let $Q = Q_k(G_{k+1}(P))$. Then $H_k(Q,P) \leq H_k(Q,G_{k+1}(P))$. Suppose $H_k(Q,P) < H_k(Q,G_{k+1}(P))$. V_k is a base for S so there is a set v in V_k such that $P \varepsilon v \subset H_k(Q,P) \cap G_k(P)$. $Q_k(v) \leq Q$ since $v \subset H_k(Q,P)$. If $Q_k(v) < Q$, then by the definition of the order on V_k, $v < G_{k+1}(P)$, a contradiction since $G_{k+1}(P) = V_k(P) \leq v$. Thus $Q_k(v) = Q$ and $H_k(Q_k(v),v) = H_k(Q,v) = H_k(Q,P) < H_k(Q,G_{k+1}(P))$. Hence $v < G_{k+1}(P)$, which is again a contradiction. So $H_k(Q,P) = H_k(Q,G_{k+1}(P))$, that is, $H_k(Q_k(G_{k+1}(P)),P) = H_k(Q_k(G_{k+1}(P)),G_{k+1}(P))$.

Clearly $G_k(P) \leq G_k(G_{k+1}(P))$. Suppose $G_k(P) < G_k(G_{k+1}(P))$ and let v be as above. The preceding argument shows that $Q_k(v) = Q_k(G_{k+1}(P))$ and $H_k(Q_k(v),v) = H_k(Q_k(G_{k+1}(P)),P) = H_k(Q_k(G_{k+1}(P)),G_{k+1}(P))$. Since

$G_k(v) = G_k(P) < G_k(G_{k+1}(P))$, it follows from the definition of the order on V_k that $v < G_{k+1}(P)$, again a contradiction. So $G_k(P) = G_k(G_{k+1}(P))$.

Thus there is an S-open primitive sequence G of S such that if $P \in S$ and $k \in N$, then $G_{k+1}(P) \subset H_k(Q_k(G_{k+1}(P)),P)$. Let $P \in S$ and $k \in N$ and let v be an open set such that $P \in v \subset H_{k+2}(Q_k(G_{k+1}(P)),P) \cap G_{k+1}(P)$. Since $H_{k+2}(Q_k(G_{k+1}(P)),P) \subset H_{k+1}(Q_k(G_{k+1}(P)),P)$, it follows that $Q_{k+1}(v) \leq Q_k(G_{k+1}(P))$. v is contained in some set in G_{k+1} and some set in $\{H^Q_{k+2} | Q \in S\}^*$, hence $v \in V_{k+1}$ and $G_{k+2}(P) = V_{k+1}(P) \leq v$. Thus $Q_{k+1}(G_{k+2}(P)) \leq Q_{k+1}(v) \leq Q_k(G_{k+1}(P))$.

Let D be an open set containing the point P. $(Q_k(G_{k+1}(P)))_{k=1}^{\infty}$ is a non-increasing sequence in S so there are an integer m and a point Q such that $Q_k(G_{k+1}(P)) = Q$ for all $k \geq m$. There is an integer n such that $H_n(Q,P) \subset D$. If $k = n+m$, then $G_{k+1}(P) \subset H_k(Q_k(G_{k+1}(P)),P) = H_k(Q,P) \subset D$. Thus S has a primitive base.

3.10 DEFINITION . S is a λ_c (λ_b) space if and only if there is an S-open monotonically contracting sequence G of S such that if g is a G-filterbase, then $\bigcap \bar{g}_n$ is a nonempty countably compact (compact) set such that if U is an open set containing $\bigcap \bar{g}_n$, then U contains a term of g. It is an easy consequence of [29; Lemma 2.1], [25; Lemma 5.1], and the proofs of [29; Lemmas 2.2 and 2.3] that a T_1 space S is a λ_c (λ_b) space if and only if there is an S-open primitive sequence G of S such that if g is a primitive representative of G, then $\bigcap \bar{g}_n$ is a nonempty countably compact (compact) set such that if U is an open set containing $\bigcap \bar{g}_n$, then U contains a term of g. Also [25; Lemma 2.1] shows that a T_3 space is a λ_b-space if and only if it satisfies Condition K of [28].

3.11 REMARK. For Hausdorff spaces, conditions λ_c and λ_b are generalizations of the concept of a λ-base (see [25; Proposition 2.1]) to spaces which need not be first countable. However, a first countable Hausdorff λ_b space does not necessarily have a λ-base, e.g., a first countable compact non-metrizable Hausdorff space (see [22; Examples 48, 97, and 107]).

3.12 THEOREM. A T_3 λ_c-space has a λ-base if and only if it has a primitive base.

Proof. Suppose S is a T_3 λ_c-space having a primitive base. As in the proof of Theorem 3.6, there is a sequence B of well-ordered bases for S such that if $P \varepsilon S$, then $\{B_n(P) | n \varepsilon N\}$ is a base for S at P. Let G be an S-open monotonically contracting sequence of S as in Definition 3.10. By [29; Lemma 2.1] there is a primitive sequence F of S such that $F_n \subset G_n$ for each n. By Lemma 3.5 (with $X = Y = T = S$ and $H_n = F$ for each n), there is an S-open primitive sequence W of S such that if $P \varepsilon S$ and $n \varepsilon N$, then $W_n(P) \subset F_n(P)$ and $W_{n+1}(P) \subset B_n(P)$. For each $P \varepsilon S$, $\{W_n(P) | n \varepsilon N\}$ is a base for S at P. By [29; Lemma 2.4] there is an S-open primitive sequence Z of S such that if $P \varepsilon S$, $\overline{Z}_{n+1}(P) \subset Z_n(P)$ and $Z_n(P) \subset W_n(P)$ for each n. Hence $\{Z_n(P) | n \varepsilon N\}$ is a base for S at P. It follows from the proofs of [29; Lemmas 2.2 and 2.3] that there is a monotonic sequence V of bases for S such that every V-filterbase is finer than some primitive representative of Z. Let v be a V-nest and let z be a primitive representative of Z coarser than v. For each n, there is a point P_n such that $z_n = Z_n(P_n)$ and $z_{n+1} = Z_{n+1}(P_n)$. Thus $\overline{z}_{n+1} \subset z_n$. By [29; Lemma 2.2] there is a W-filterbase w coarser than z and an F-filterbase f coarser than w. Since f is a G-filterbase, $\bigcap \overline{f}_n$ is a nonempty countably compact subset of S such that if U is an open set containing $\bigcap \overline{f}_n$, then U contains a term of f. For each n, there is a j such that $z_j \subset f_n$. By [25; Lemma 5.1] $M = \bigcap \overline{z}_n$ is a nonempty countably compact set such that if U is an open set containing M, then U contains a term of z. By [25; Remark 5.1], $\{P_n | n \in N\}$ either is finite or has a limit point in M.

Case 1. Suppose $\{P_n | n \in N\}$ is finite. There is a point P such that $P_n = P$ for infinitely many n. Since $\{Z_k(P) | k \in N\}$ is a base for S at P, it follows that z is a base for S at P. So v converges to P and $P \in \bigcap \bar{v}_n = \bigcap v_n$.

Case 2. Suppose $\{P_n | n \in N\}$ has a limit point P in M. Let $k \in N$. There is an $n \geq k$ such that $P_n \in Z_k(P)$. So $Z_k(P_n) \leq Z_k(P)$. Since P is in $M = \bigcap \bar{z}_j = \bigcap z_j$, P is in $z_n = Z_n(P_n) \subset Z_k(P_n)$. Thus, $Z_k(P) \leq Z_k(P_n)$ and $z_n \subset Z_k(P_n) = Z_k(P)$. So $\{Z_k(P) | k \in N\}$ is coarser than z, hence coarser than v. Since $\{Z_k(P) | k \in N\}$ is a base for S at P, v converges to P and $P \in \bigcap \bar{v}_n = \bigcap v_n$.

In both cases S has a λ-base by [24; Theorems 1 and 4]. The converse follows from Theorems 3.4 and 3.6.

3.13 <u>COROLLARY</u>. A paracompact locally countably compact space is metrizable if and only if it has a primitive base.

Proof. This follows from Theorem 3.12 and the fact that a locally countably compact T_3 space is a λ_c space (see Theorem 3.14) and a paracompact space having a base of countable order is metrizable.

3.14 <u>THEOREM</u>. If S is a T_3 space every point of which has a λ_c (λ_b) neighborhood, then S is a λ_c (λ_b) space.

Proof. By Theorem 5.7 every point Q of S has an <u>open</u> neighborhood U(Q) which is a λ_c (λ_b) space. Let $H^Q = H_1^Q, H_2^Q, \ldots$ be a U(Q)-open primitive sequence of U(Q) such that if h is a primitive representative of H^Q, then $M = \bigcap \bar{h}_n^{U(Q)}$ is a nonempty countably compact (compact) subset of U(Q) such that if V is an open set in U(Q) containing M, then V contains a term of h. By Lemma 3.5 there is a U(Q)-open primitive sequence W^Q

of $U(Q)$ such that $W_n^Q(P) \subset H_n^Q(P)$ for each $P \in U(Q)$ and if $w \in W_n^Q$, then $\bar{w}^S \subset U(Q)$. As in the proof of Theorem 3.9, there is an S-open primitive sequence G of S such that if $P \in S$ and $k \in N$, then $G_{k+1}(P) \subset W_k(Q_k(G_{k+1}(P)),P)$ and $Q_{k+1}(G_{k+2}(P)) \leq Q_k(G_{k+1}(P))$. (This notation is defined in the proof of Theorem 3.9 for an arbitrary well-order on S.) Let g be a primitive representative of G. For each n, there is a point P_n such that $g_n = G_n(P_n)$ and $g_{n+1} = G_{n+1}(P_n)$. $(Q_k(g_{k+1}))_{k=1}^{\infty}$ is a non-increasing sequence with respect to the well-order on S so there is an integer m and a point Q such that $Q_k(g_{k+1}) = Q$ for $k \geq m$. Thus $(g_{k+m})_{k=1}^{\infty}$ is a decreasing sequence of sets such that for each k, $g_{k+m} = G_{k+m}(P_{k+m}) \subset W_{k+m-1}(Q,P_{k+m}) \subset W_k(Q,P_{k+m})$. By [29; Lemma 2.2] there is a primitive representative w of W^Q coarser than g and a primitive representative h of H^Q coarser than w. The proof of [25; Lemma 5.1] shows that $M = \bigcap \bar{g}_n^{U(Q)}$ is a nonempty countably compact (compact) subset of $U(Q)$ such that if V is an open set in $U(Q)$ containing M, then V contains a term of g. Since there is an integer k such that $\bar{g}_k^{U(Q)} = \bar{g}_k^S$, it follows that S is a λ_c (λ_b) space.

4. SETS OF INTERIOR CONDENSATION

4.1 DEFINITION. A subset M of S is said to be a set of interior condensation in S if and only if there is an S-open monotonically contracting sequence G of M such that if P is a point belonging to each term of a G-filterbase, then P is in M. Such a sequence G will be called an S-sequence for M. Obviously a G_δ set is a set of interior condensation. However, it follows from the results in Section 6 that if S is a semi-complete Moore space which is not complete (such as the example given in [21; Theorem 9]), then S is a set of interior condensation in the Wallman compactification ωS of S but S is not a G_δ in ωS.

4.2 THEOREM. M is a set of interior condensation in S if and only if there is an S-open primitive sequence H of M such that if P is a point which belongs to each term of a primitive representative of H, then P is in M.

Proof. If M is a set of interior condensation in S and G is an S-sequence for M, then by [29; Lemma 2.1] there is a primitive sequence H of M such that for each n, $H_n \subset G_n$. H has the desired property. Conversely, let H be an S-open primitive sequence of M having the indicated property. By [29; Lemma 2.3] there is an S-open monotonically contracting sequence G of M such that every G-filterbase is finer than some primitive representative of H. It follows that G is an S-sequence for M and M is a set of interior condensation in S.

4.3 THEOREM. The intersection of countably many sets of interior condensation is a set of interior condensation.

Proof. Let $M_1, M_2, \ldots$ be sets of interior condensation in S and for each n, let $H_n = (H_{nm})_{m=1}^{\infty}$ be an S-sequence for M_n. Let $M = \bigcap_{n=1}^{\infty} M_n$ and for each n, let $G_n = (G_{nm})_{m=1}^{\infty}$ be a primitive sequence of M_n such that $G_{nm} \subset H_{nm}$ [29; Lemma 2.1]. By Lemma 3.5 there is an S-open primitive sequence W of M such that if $P \in M$ and $n \in N$, then $W_n(P) \subset \bigcap\{G_{ij}(P) \mid i+j = n+1\}$. Suppose P is a point belonging to each term of a primitive representative w of W. For each n, G_n is a primitive sequence of M in S and $(w_{n+m-1})_{m=1}^{\infty}$ is a decreasing sequence of sets such that for each m, there is a point P_{nm} in $w_{n+m-1} \cap M$ such that $w_{n+m-1} = W_{n+m-1}(P_{nm})$, hence, $w_{n+m-1} \subset G_{nm}(P_{nm})$. By [29; Lemma 2.2] there is a G_n-filterbase g_n coarser than $(w_{n+m-1})_{m=1}^{\infty}$, hence coarser than w. P belongs to each term of g_n so P is in M_n for each n. By Theorem 4.2 M is a set of interior condensation in S.

4.4 LEMMA. If S has a primitive base and M is a set of interior condensation in S, there is an S-sequence for M each term of which is a base for S at M.

Proof. Let W be a primitive sequence of S as in Theorem 3.6. For each n, let $B_n = \bigcup_{i=n}^{\infty} W_i$ be well-ordered so that $x \leq y$ in B_n if and only if it is true that if i (resp. j) is the first integer no less than n such that $x \in W_i$ (resp. $y \in W_j$), then (1) $i < j$ or (2) $i = j$ and $x \leq y$ in W_i. B_n is a base for S and $B_n(P) = W_n(P)$ for each $P \in S$. Let H be an S-open primitive sequence of M as in Theorem 4.2. By Lemma 3.5 there is an S-open primitive sequence G of M such that if $P \in M$ and $n \in N$, then $G_{n+1}(P) \subset B_n(P)$ and $G_n(P) \subset H_n(P)$. For each n, let $F_n = \bigcup_{i=n}^{\infty} G_i$. If $P \in M$, $n \in N$, and D is an open set in S containing P, then there is a $k \geq n$ such that $G_{k+1}(P) \subset B_k(P) = W_k(P) \subset D$. So F_n is a base for S at M for each n. Suppose P is a point belonging to each term of an F-filterbase f. For each $i \in N$, there is an integer $n_i \geq i$ such that $f_i \in G_{n_i}$ and there is a point P_i in M such that $f_i = G_{n_i}(P_i) \subset H_{n_i}(P_i) \subset H_i(P_i)$. By [29; Lemma 2.2] there is a primitive representative h of H coarser than f. P belongs to each term of h so P is in M. Thus F is an S-sequence for M.

4.5 THEOREM. If X is a set of interior condensation in Y and either (1) Y is a G_δ in S or (2) S has a primitive base and Y is a set of interior condensation in S, then X is a set of interior condensation in S.

Proof. In either case, there is an S-sequence B for Y such that B_n is a base for S at Y for each n. By Theorem 4.2 there is a Y-open primitive sequence H of X such that if P is a point of Y belonging to each term of a primitive representative of H, then P is in X. By Lemma 3.5 there is an S-open primitive sequence W of X such that for each $n \in N$, $W_n \subset B_n$ and if $P \in X$, then $W_n(P) \cap Y \subset H_n(P)$.

Let P be a point belonging to each term of a primitive representative w of W. For each n, there is a point P_n in X such that $w_n = W_n(P_n)$. Thus $(w_n \cap Y)_{n=1}^{\infty}$ is a decreasing sequence of sets such that for each n, P_n is a point of $w_n \cap X$ such that $w_n \cap Y = W_n(P_n) \cap Y \subset H_n(P_n)$. By [29; Lemma 2.2] there is a primitive representative h of H coarser than $(w_n \cap Y)_{n=1}^{\infty}$. w is a B-filterbase and P belongs to each term of w so P is in Y. Thus P is in each term of h, hence P is in X. So X is a set of interior condensation in S by Theorem 4.2.

4.6 THEOREM. If S is an essentially T_1 space having a base of countable order and M is a closed set in S, then M is a set of interior condensation in S.

Proof. Let B be a sequence of bases for S such that if P is a point belonging to each term of a B-filterbase b, then b is a base for S at P [32; Theorem 2]. For each n, let $G_n = \{b \in B_n | b \cap M \neq \phi\}$. Clearly, G is an S-open monotonically contracting sequence of M. Let P be a point which belongs to each term of a G-filterbase g. If P is not in M, then S − M contains a term of g, a contradiction. So $P \in M$ and G is an S-sequence for M.

4.7 DEFINITION. S is said to be developable if and only if there is a sequence G of open covers of S such that if $P \in S$ and U is an open set containing P, there is an integer n such that $st(P,G_n) \subset U$. Such a sequence G is called a development for S. It is well-known that if S is developable, then S has a monotonic development.

4.8 <u>THEOREM</u>. If M is a set of interior condensation in a developable space S, then M is a G_δ in S.

Proof. Let B be a development for S and let H be a primitive sequence of M in S as in Theorem 4.2. By Lemma 3.5 there is a primitive sequence W of M in S such that for each n, $W_n \subset B_n$ and if $P \in M$, then $W_n(P) \subset H_n(P)$. Let $P \in \bigcap_{n=1}^{\infty} W_n^*$ and for each n, let w_n be a set in W_n containing P. It follows from the nature of a development that $w = (w_n)_{n=1}^{\infty}$ has a monotonic subsequence $w' = (w_n')_{n=1}^{\infty}$. For each n, there is an integer $j_n \geq n$ and a point P_n in M such that $w_n' = W_{j_n}(P_n) \subset W_n(P_n) \subset H_n(P_n)$. By [29; Lemma 2.2] there is a primitive representative h of H coarser than w'. P is in each term of h, hence P is in M. Thus $\bigcap_{n=1}^{\infty} W_n^* \subset M$ and clearly $M \subset \bigcap_{n=1}^{\infty} W_n^*$. So M is a G_δ in S.

4.9 <u>THEOREM</u>. If S is essentially T_1, then S has a base of countable order if and only if S has a primitive base and closed sets in S are sets of interior condensation.

Proof. By Theorem 4.3 closed sets in any space X are sets of interior condensation if and only if for each open set D in X, closed sets in D are sets of interior condensation (in X). Thus this theorem is equivalent to [34; Theorem 1].

4.10 <u>COROLLARY</u>. [32; Theorem 4] If S has a θ-base and closed sets in S are G_δ sets, then S is developable.

Proof. A space in which closed sets are G_δ sets is essentially T_1 and a space having a θ-base has a primitive base. So S has a base of countable order by Theorem 4.9. By [32; Theorem 3] it suffices to show that S is θ-refinable. Let $B = \bigcup_{n=1}^{\infty} B_n$ be a θ-base for S and let G be an open cover of S. For each n, let $H_n = \{b \in B_n | b$ is contained in some set in G} and let $(U_{nm})_{m=1}^{\infty}$ be a non-increasing sequence of open sets in S such that $\bigcap_{m=1}^{\infty} U_{nm} = S - H_n^*$. If for each n and k, $H_{nk} = H_n \cup \{g \cap U_{nk} | g \in G\}$, then $\bigcup\{H_{nk} | n,k \in N\}$ is a θ-refinement of G. Thus S is θ-refinable.

5. ARONSZAJN SPACES

5.1 DEFINITION. A T_3 space having a base of countable order (λ-base) is called an Aronszajn space (a complete Aronszajn space). The following characterizations follow from [32; Theorem 2] and [29; Theorems 3.2 and 3.3]: A T_3 space S is an Aronszajn space (a complete Aronszajn space) if and only if there is a sequence G of bases for the topology of S such that if g is a G-filterbase, then g converges to (some point of S and to) each point of $\bigcap g_n$. Thus a T_3 space is a complete Aronszajn space if and only if it satisfies Aronszajn's axiom (see Introduction). A sequence G as above is called an Aronszajn sequence (a complete Aronszajn sequence) for S and by the proofs of [32; Theorem 2] and [29; Theorem 3.2], G may be assumed to be monotonic.

5.2 DEFINITION. A Tychonoff (= completely regular Hausdorff) space S is Čech complete if and only if S is a G_δ in its Stone-Čech compactification βS.

5.3 DEFINITION. S is Hausdorff embedded in a space T provided S is a subspace of T and for each x ε S and y ε T, there exist mutually exclusive open sets in T containing x and y respectively.

5.4 THEOREM. Let S be a T_3 space. Then the following statements are equivalent:

(1) S is a complete Aronszajn space;

(2) S is an open continuous image of a complete metric space;

(3) S is an open continuous image of a space having a λ-base.

Also, the following are equivalent and (1) implies (4). Furthermore, if S has a primitive base, then (4) implies (1).

(4) S is a λ_b space;

(5) S is an open continuous image of a Čech complete space;

(6) S is a set of interior condensation in every space in which it is dense and Hausdorff embedded;

(7) S is a set of interior condensation in its Wallman compactification ωS.

Proof. (1) and (2) are equivalent by [24; Theorem 4]. Since a complete metric space has a λ-base, (2) implies (3). (3) implies (1) by [27; Theorem 1]. (4) implies (5) by [28; Theorem 1] and [25; Lemma 2.1].

Suppose S is an open continuous image of a Čech complete space X. X is a set of interior condensation in βX. Theorem 2 of [28] remains valid if the phrase "T_2 space of which it is a dense subspace" is replaced by the phrase "space in which it is dense and Hausdorff embedded" from which it follows that (5) implies (6).

S is Hausdorff embedded in ωS by [8; Lemma 6], hence (6) implies (7).

The proof of [28; Theorem 3] will establish the following: If S is a set of interior condensation in a T_1 compact space E in which it is dense and regularly embedded, then S satisfies condition K (equivalently S is a λ_b-space [25; Lemma 2.1]). (Condition (S2) as used in the proof of [28; Theorem 3] is incorrect and should read as follows: If $P \in S$ and $n < k$, then $\overline{H_k(P)}^E \subset H_n(P)$. The existence of a sequence H having this stronger property is a consequence of [29; Lemma 2.4] assuming only that the subspace is regularly embedded in the containing space.) Thus (7) implies (4).

(1) implies (4) by [25; Proposition 2.1] and if S has a primitive base, then (4) implies (1) by Theorem 3.12.

5.5 REMARK. A mapping f of S onto a space R is said to be *inductively open* if and only if there is a subspace S' of S such that the restriction of f to S' is open and f(S') = R. Since both [28; Theorem 2] and [27; Theorem 1] are valid for inductively open mappings (for the latter, see the comment on page 261 of [27]), Theorem 5.4 remains valid if the word "open" is replaced by "inductively open."

5.6 REMARK. It is well known that a subspace of a complete metric space is complete (with respect to a compatible metric) if and only if it is a G_δ set. An analogous result holds for complete and semicomplete Moore spaces. A G_δ subset of a complete Aronszajn space is a complete Aronszajn space but the converse is not true. For example, the long line L [22] is a complete Aronszajn space and the set M of ordinal points of L is closed in L and therefore is a complete Aronszajn space. However, M is not a G_δ in L.

5.7 THEOREM. If S is a T_3 λ_c (λ_b) space and M is a set of interior condensation in S, then M is a λ_c (λ_b) space. Thus a subspace of a complete Aronszajn space is complete if and only if it is a set of interior condensation.

Proof. Theorem 4.5 and [28; Theorem 3] show that a set of interior condensation in a Čech complete space is a λ_b space.

Let X be a T_3 complete M-space (see [25; Definition 3.2]) and let Y be a set of interior condensation in X. By [25; Proposition 3.1] there is a sequence U of open covers of X such that if K is a decreasing sequence of nonempty closed sets in X such that for each n, some set in U_n contains a term of K, then $\bigcap K_n \neq \phi$. It follows from [29; Lemmas 2.1, 2.4 and 2.2] that there is an X-open primitive sequence G of Y such that if P is a point of X belonging to each term of a G-filterbase, then P is in Y and if $Q \in Y$ and $n \in N$, then $\overline{G_{n+1}(Q)} \subset G_n(Q)$. For each n, let $B_n = \{V | V$ is open in X and $\overline{V}$ is contained in some set in $U_n\}$. By Lemma 3.5 there is a Y-open primitive sequence W of Y such that $W_n \subset Tr_Y B_n$ and $W_n(P) \subset G_n(P)$ for each $n \in N$ and $P \in Y$. Let w be a W-filterbase. For each n, $\bar{w}_n$ (the closure of w_n in X) is contained in some set in U_n, so $A = \bigcap \bar{w}_n \neq \phi$. There is a G-nest g coarser than w, hence $A \subset \bigcap g_n \subset Y$. Suppose there is a sequence $x_1, x_2, \ldots$ of distinct points of A such that $\{x_k | k \in N\}$ has no limit point. For each n, let $K_n = \{x_k | k \geq n\}$. Then $K_1, K_2, \ldots$ is a decreasing sequence of closed sets such that for each n, K_n is contained in some set in U_n. So $\bigcap K_n \neq \phi$, which yields a contradiction. Thus A is countably compact. Suppose there is an open set V in X containing A but containing no term of w. There is a sequence $x_1, x_2, \ldots$ of distinct points of Y such that if $k \in N$, there is a $j \geq k$ such that $x_k \in w_j - V$. By the argument given above, $\{x_k | k \in N\}$ has a limit point P which clearly is in A. Thus P is in V and V contains some x_k. This is a contradiction. So Y is a λ_c space.

Now suppose S is a T_3 $\lambda_c(\lambda_b)$ space and M is a set of interior condensation in S. By [25; Theorem 4.1] there exist a T_3 complete M-space (paracompact Čech complete space) X and a continuous open surjection f of X onto S. Let H be an S-sequence for M and for each n, let $f^{-1}(H_n) = \{f^{-1}(h) | h \in H_n\}$. Then $f^{-1}(H) = (f^{-1}(H_n))_{n=1}^{\infty}$ is an X-sequence for $f^{-1}(M)$. For clearly $f^{-1}(H)$ is an X-open monotonically contracting sequence of $f^{-1}(M)$. If x is a point of X belonging to each term of an $f^{-1}(H)$-filterbase b, then f(x) belongs to each term of f(b) which is an H-filterbase. Thus f(x) is in M and x is in $f^{-1}(M)$. So $f^{-1}(M)$ is a set of interior condensation in X, hence $f^{-1}(M)$ is a λ_c (λ_b) space. By [25; Theorem 4.3] M is a λ_c (λ_b) space.

Suppose S is a complete Aronszajn space and M is a complete subspace of S. By part (6) of Theorem 5.4, M is a set of interior condensation in $\overline{M}$ and by Theorem 4.6, $\overline{M}$ is a set of interior condensation in S. So M is a set of interior condensation in S by Theorem 4.5. The converse follows from Theorem 3.12.

5.8 LEMMA. Let S be a T_3 space and T a dense subspace of a space X and f a continuous mapping of T into S. If Y is a subspace of X containing T such that for each $x \in Y$, $f(\mathrm{Tr}_T \mathrm{Nbd}_X x)$ is a convergent filterbase in S, then the function g from Y to S defined by $g(x) = \lim f(\mathrm{Tr}_T \mathrm{Nbd}_X x)$ is a continuous extension of f to Y.

Proof. If $x \in T$, then clearly $f(x) = \lim f(\mathrm{Tr}_T \mathrm{Nbd}_X x)$ so g is an extension of f. Let U be an open set in X and let y be a point of $U \cap Y$ and let V be an open set in S containing g(y). There is a neighborhood W of y in X such that $f(W \cap T) \subset V$. T is dense in X so $W \cap U \cap T \neq \phi$ and $V \cap f(U \cap T) \neq \phi$. So $g(y) \in \overline{f(U \cap T)}$ and $g(U \cap Y) \subset \overline{f(U \cap T)}$. Thus, if $x \in Y$ and V is an open set in S containing g(x), there is a neighborhood U of x in X such that $g(U \cap T) \subset \overline{f(U \cap T)} \subset \overline{V}$. Since S is T_3, it follows that g is continuous.

5.9 LEMMA. If S is a complete Aronszajn space and $M \subset S$, there is an Aronszajn sequence G for M such that every G-filterbase converges in S.

Proof. Let B be a complete Aronszajn sequence for S and let H be a primitive sequence of M such that $H_n \subset B_n$ [29; Lemma 2.1]. Let $K_1 = \{h \cap M | h \in H_1\}$ and for $n > 1$, let $K_n = \{k \cap H_n(P) | k \in K_{n-1}$ and $P \in k\}$. Let $G_n = \bigcup_{i=n}^{\infty} K_i$ and let g be a G-filterbase. For each i, there is an integer $n_i \geq i$ such that $g_i \in K_{n_i}$ and there is a point $P_i \in g_i$ such that $g_i \subset H_{n_i}(P_i) \subset H_i(P_i)$. By [29; Lemma 2.2] there is an H-filterbase h coarser than g. Since h converges in S, g converges in S and if there is a point x in $\bigcap g_n$, then h converges to x, hence g converges to x. If $P \in M$ and $k_i = H_1(P) \cap \ldots \cap H_i(P) \cap M$, then $k_i \in K_i$ and $\{k_i | i \in N\}$ is a base for M at P since $\{H_i(P) | i \in N\}$ is a base for S at P. So each G_n is a base for M and G is an Aronszajn sequence for M.

5.10 <u>THEOREM</u>. If S is a T_3 space, then the following statements are equivalent:

(1) S is a complete Aronszajn space;

(2) S is first countable and for each developable space X and subspace T of X and continuous function f from T into S, there exist a G_δ set Y in X containing T and a continuous extension of f to Y;

(3) S is first countable and for each space X and dense subspace T of X and continuous function f from T into S, there exist a set of interior condensation Y in X containing T and a continuous extension of f to Y.

Proof. Suppose S is a complete Aronszajn space, X is a space, T is a dense subspace of X, and f is a continuous function from T into S. Let G be an Aronszajn sequence for f(T) such that every G-filterbase converges in S. For each n, let $f^{-1}(G_n) = \{f^{-1}(g) | g \in G_n\}$. Then $f^{-1}(G) = (f^{-1}(G_n))_{n=1}^{\infty}$ is a T-open monotonically contracting sequence of T. Let H be a primitive sequence of T such that $H_n \subset f^{-1}(G_n)$ for each n [29; Lemma 2.1]. By Lemma 3.5 there is an X-open primitive sequence W of T such that if $P \varepsilon T$ and $n \varepsilon N$, then $W_n(P) \cap T \subset H_n(P)$. If $w \varepsilon W_{n+1}$, there is a point P of T such that $w = W_{n+1}(P) \subset W_n(P)$ and it follows that $W^*_{n+1} \subset W^*_n$. Let $V_1 = W_1$ and for $n > 1$, let $V_n = \{v \cap W_n(P) | v \varepsilon V_{n-1}$ and $P \varepsilon v \cap W^*_n\}$. Let n be a positive integer such that $V^*_{n-1} = W^*_{n-1}$. If $P \varepsilon W^*_n \subset W^*_{n-1} = V^*_{n-1}$, there is a set v in V_{n-1} such that $P \varepsilon v$, hence $P \varepsilon v \cap W_n(P) \varepsilon V_n$. Thus $W_n{}^* \subset V_n{}^*$ and clearly $V_n{}^* \subset W_n{}^*$. Since $V_1{}^* = W_1{}^*$ it follows that $V_n{}^* = W_n{}^*$ for all n. Let $Y = \{x \varepsilon X | x$ belongs to each term of a V-filterbase$\}$. If $x \varepsilon T$ and $v_n(x) = W_1(x) \cap \ldots \cap W_n(x)$, then $v = v_1, v_2, \ldots$ is a V-filterbase each term of which contains x, so $x \varepsilon Y$ and $T \subset Y$. Let $x \varepsilon Y$ and $n \varepsilon N$ and let v be set in V_n containing x. Since x is in V^*_{n+1} and $V^*_{n+1} = W^*_{n+1}$, it follows that $v \cap W_{n+1}(x)$ is a set in V_{n+1} containing x and contained in v. Thus V is a monotonically contracting sequence of Y which is clearly an X-sequence for Y. So Y is a set of interior condensation in X.

Let $x \varepsilon Y$ and let v be a V-filterbase each term of which contains x. By [29; Lemma 2.2] there is a W-filterbase w coarser than v and there is an H-filterbase h coarser than $Tr_T w$. Since f(h) is a G-filterbase, f(h) converges in S, hence $f(Tr_T Nbd_X x)$ converges in S. Thus f has a continuous extension to Y by Lemma 5.8. Also any space having a base of countable order is first countable so (1) implies (3).

That (3) implies (2) follows from Theorem 4.8 by first extending the given function to a G_δ in $\overline{T}^X$. Since closed sets in X are G_δ sets in X, the implication follows.

Suppose (2) holds. A theorem of Ponomarev [18] asserts that every first countable T_1 space is an open continuous image of a metrizable space. Let f be a continuous open surjection of a metric space X onto S and let $\hat{X}$ be a completion of X. There exist a G_δ subset Y of $\hat{X}$ containing X and a continuous extension g of f to Y. Y is a complete metric space and g is an inductively open continuous mapping of Y onto S. So S is a complete Aronszajn space by Remark 5.5. Thus (2) implies (1).

5.11 DEFINITION. A (complete) centered base for S is a base of countable order (λ-base) B such that every perfectly decreasing subcollection T of B is regular (i.e. each set in T contains the closure of a set in T).

5.12 REMARK. The concept of a (complete) centered base was introduced in [11] using somewhat different terminology from that above but the two are easily shown to be equivalent. An essentially T_1 space having a centered base is regular so if S is a T_1 space having a (complete) centered base, then S is an Aronszajn space (a complete Aronszajn space). However, no example of an Aronszajn space (a complete Aronszajn space) which does not have a (complete) centered base is given in [11] and to the author's knowledge, no such example has been found. In an attempt to find a relatively large class of T_3 spaces having a (complete) centered base, it is proved in [11] that (1) if S is a T_3 space having (complete) centered bases locally, then S has a (complete) centered base and (2) every metacompact (complete) Moore space has a (complete) centered base. This latter result is extended to a larger class of spaces in Theorem 6.15 but the technique used there is essentially that used in [11]. However, the purpose for introducing these concepts here is to obtain an extension property for open continuous mappings similar to those in Theorem 5.10. The following lemma is needed first.

5.13 LEMMA. An essentially T_1 space S has a (complete) centered base if and only if there is a monotonic sequence G of bases for S such that if g is a G-filterbase, then g is regular and g converges to (some point of S and to) each point of $\bigcap g_n$.

Proof. Suppose S is an essentially T_1 space having a (complete) centered base B. By [32; Theorem 2'] ([29; proof of Theorem 3.2]) there is a monotonic sequence G of bases for S such that for each n, $G_n \subset B$ and if g if a G-filterbase, then g converges to (some point of S and to) each point of $\bigcap g_n$. If g is perfectly decreasing, then g is regular. Suppose g is not perfectly decreasing. There is point x in $\bigcap g_n$ and g is a base for S at x. Thus g is regular since S is regular.

Conversely, let G be a monotonic sequence as described and let H be a primitive sequence of S such that $H_n \subset G_n$ [29; Lemma 2.1]. For each n, H_n contains no perfectly decreasing subcollection. For suppose K is a perfectly decreasing subcollection of H_n for some n and let k_1, k_2,... be a sequence such that for each i, k_i properly contains k_{i+1}. For each j, there is a point P_j of S such that $k_j = H_n(P_j)$. But since $H_n(P_{j+1}) = k_{j+1} \subset k_j$, it follows that $k_{j+1} < k_j$. So k_1, k_2,... is a strictly decreasing sequence with respect to the well order on H_n. This yields a contradiction.

Let $B = \bigcup_{n=1}^{\infty} H_n$ and let T be a perfectly decreasing subcollection of B. Let t_1 be a set in T. There is an integer n_1 such that $t_1 \in H_{n_1} \subset G_{n_1} \subset G_1$. Since no H_n contains a perfectly decreasing subcollection, there exist an integer $n_2 > n_1$ and a set $t_2 \in H_{n_2} \subset G_{n_2} \subset G_2$ such that $t_2 \subset t_1$. This process may be continued to construct a G-filterbase t which, by the supposition on G, is regular. Thus t_1 contains the closure of some set in T and T is regular. The proof given in [32; Theorem 2] ([29; Theorem 3.3]) shows that B is a base of countable order (λ-base) for S, hence B is a (complete) centered base for S.

5.14 THEOREM. If S is a T_3 space having a complete centered base, X is an essentially T_1 space having a base of countable order and f is an open continuous mapping of a subspace T of X into S, then there exist a set Y of interior condensation in X containing T and an open continuous extension of f to Y.

Proof. By Theorems 4.5 and 4.6 it may be assumed that T is dense in X. Let G be a sequence of bases for S as in Lemma 5.13. By [32; Theorem 2] there is a sequence B of bases for X such that if P is a point of X belonging to each term of a B-filterbase b, then b is a base for X at P. For each n, let $K_n = \{U | U$ is open in X and $f(U \cap T) \in G_n\}$. Suppose $n \in N$, $x \in T$, and V is an open set in X containing x. Since f is open, there is a set g in G_n such that $f(x) \in g$ and $g \subset f(V \cap T)$. If U is an open set in X such that $U \cap T = f^{-1}(g)$, then $f(U \cap V \cap T) = g$ and $U \cap V$ is a set in K_n containing x and contained in V. So K_n is a base for X at T. Let H be a primitive sequence of T such that $H_n \subset B_n$ [29; Lemma 2.1]. By a simple modification of [29; Lemma 2.4], there is a primitive sequence W of T such that for each n, $W_n \subset K_n$ and $W_n(P) \subset H_n(P)$ for each $P \in T$. Let V and Y be as in the proof of (1) implies (3) in Theorem 5.10. Suppose $x \in Y$ and let v be a V-filterbase each term of which contains x. As in the proof of Theorem 5.10, there is a W-filterbase w coarser than v and by [29; Lemma 2.2] there is an H-filterbase h coarser than w. $f(\mathrm{Tr}_T w)$ is a G-filterbase, hence is regular and converges to some point in S. So $f(\mathrm{Tr}_T \mathrm{Nbd}_X x)$ converges in S and by Lemma 5.8 the mapping g from Y into S defined by $g(x) = \lim f(\mathrm{Tr}_T \mathrm{Nbd}_X x)$ is a continuous extension of f to Y. If U is an open set in X, $x \in U \cap Y$ and v, w, and h are as above, then there is an integer n such that $h_n \subset U$ and there is an integer m such that $w_m \subset h_n$. $f(Tr_T w)$ is regular so there is an integer j such that $\overline{f(w_j \cap T)} \subset f(w_m \cap T)$. Now $g(x) \in \overline{f(w_j \cap T)} \subset f(w_m \cap T) \subset f(U \cap T)$. Thus $g(U \cap Y) \subset f(U \cap T)$ and clearly $f(U \cap T) \subset g(U \cap Y)$. So $g(U \cap Y) = f(U \cap T)$ and $g(U \cap Y)$ is open. Thus g is an open mapping.

6. MOORE SPACES

6.1 DEFINITIONS. A T_3 developable space is called a Moore space. Since every developable space has a base of countable order, every Moore space is an Aronszajn space. A development G for S is called complete if and only if G is monotonic and if $M_1, M_2, \ldots$ is a decreasing sequence of nonempty closed sets in S such that for each n, M_n is a subset of the closure of some set in G_n, then $\bigcap M_n \neq \phi$. A development G for S is called semicomplete if and only if G is monotonic and if g is a G-filterbase, then $\bigcap \bar{g}_n \neq \phi$. A Moore space is said to be complete (semicomplete) if and only if it has a complete (semicomplete) development. Every complete Moore space is semicomplete but there is a semicomplete Moore space that is not complete [21].

6.2 THEOREM. A Moore space is semicomplete if and only if it is a complete Aronszajn space.

Proof. Suppose S is semicomplete and let G be a semicomplete development for S. Let $H_1 = G_1$ and for $n > 1$, let $H_n = \{g \in G_n | \bar{g}$ is contained in some set in $H_{n-1}\}$. It is easy to show that $H = H_1, H_2, \ldots$ is also a semicomplete development for S. Let h be an H-filterbase. There is a point x in $\bigcap \bar{h}_n$. Let U be an open set containing x and let n be an integer such that $st(x,G_n) \subset U$. There is a set g in G_n which contains $\bar{h}_{n+1}$ and since $x \in \bar{h}_{n+1}$, it follows that $\bar{h}_{n+1} \subset g \subset U$. Thus h converges to x and H is a complete Aronszajn sequence for S.

Conversely, suppose S is a complete Aronszajn space and let B be a monotonic complete Aronszajn sequence for S. Let H be a development for S and for each n, let $G_n = \{b \in B_n | b$ is contained in some set in $H_n\}$. $G = G_1, G_2, \ldots$ is a monotonic development for S. If g is a G-filterbase, then g is a B-filterbase and g converges to some point x of S and clearly $x \in \bigcap \bar{g}$. So G is a semicomplete development for S and S is semicomplete.

6.3 <u>COROLLARY</u>. (cf. Corollary 6.13) If S is a Moore space, then each of the statements in Theorems 5.4 and 5.10 is equivalent to S being semicomplete. In particular, the following are equivalent:

(1) S is semicomplete;

(2) S is a set of interior condensation in every space in which it is dense and Hausdorff embedded;

(3) S is a set of interior condensation in ωS;

(4) If T is a dense subspace of a space X and f is a continuous mapping of T into S, then f has a continuous extension to a set of interior condensation in X.

6.4 <u>DEFINITION</u>. If H is a sequence of open covers of S such that if $P \in S$, some term of H is finite at P (i.e., contains only finitely many sets which contain P), then $\bigcup_{n=1}^{\infty} H_n$ is called a <u>θ-cover of S</u>. If K is a θ-cover of S which is a refinement of a cover G of S, then K is called a <u>θ-refinement of G</u>. If every open cover of S has a θ-refinement, then S is said to be <u>θ-refinable</u>.

6.5 LEMMA. If G is an open cover of a developable space S, then G has a θ-refinement $H = \bigcup_{n=1}^{\infty} H_n$ such that if $P \in S$, there is an integer n such that H_i is finite at P for all $i \geq n$.

Proof. By [7; Theorem 9] there is a σ-discrete closed refinement $K = \bigcup_{n=1}^{\infty} K_n$ of G covering S such that for each n, every set in K_n is a subset of a set in K_{n+1}. For each $n \in N$ and $k \in K_n$, let $U_n(k)$ be an open set in S contained in some set in G and containing k such that $U_n(k) \cap (K_n - \{k\})^* = \phi$. For each $P \in S - K_n^*$, let $V_n(P)$ be an open set containing P and contained in some set in G such that $V_n(P) \cap K_n^* = \phi$. Then $H_n = \{U_n(k) | k \in K_n\} \cup \{V_n(P) | P \in S - K_n^*\}$ is a refinement of G covering S which is finite at each point of K_n^* and $H = \bigcup_{n=1}^{\infty} H_n$ is a θ-refinement of G. Since $K_n^* \subset K_{n+1}^*$, it follows that if $P \in S$ and n is an integer such that $P \in K_n^*$, then H_i is finite at P for all $i \geq n$.

6.6 LEMMA. If S has a θ-cover each element of which is developable, then S is developable.

Proof. Let $G = \bigcup_{n=1}^{\infty} G_n$ be a θ-cover of S such that each set in G is developable. Let F be an open cover of S and for each $n \in N$ and $g \in G_n$, let $H_n(g) = \bigcup_{k=1}^{\infty} H_{nk}(g)$ be a θ-refinement (in g) of the open cover $F_n(g) = \{f \cap g | f \in F\}$ of g having the property in Lemma 6.5. For each pair of positive integers n and k, let $H_{nk} = \bigcup_{g \in G_n} H_{nk}(g)$ and let $H = \bigcup_{n,k=1}^{\infty} H_{nk}$. Each H_{nk} is a refinement of F covering S. Let $P \in S$ and let n be an integer such that G_n is finite at P. Let $g_1, \ldots, g_j$ be the sets in G_n which contain P. For each $i \leq j$, there is an integer n_i such that if $m \geq n_i$, then $H_{nm}(g_i)$ is finite at P. If $k = n_1 + \ldots + n_j$, then H_{nk} is finite at P. So H is a θ-refinement of F and S is θ-refinable.

Each set in G is developable, hence essentially T_1. Let x and y be points of S such that $x \in \overline{\{y\}}^S$ and let g be a set in G containing

x. Then $y \in g$, hence $x \in g \cap \overline{\{y\}}^S = \overline{\{y\}}^g$. By [13; Theorem 1] $y \in \overline{\{x\}}^g \subset \overline{\{x\}}^S$ and S is essentially T_1. Each set in G has a base of countable order so by [32; Theorem 1] S has a base of countable order. Thus S is developable by [32; Theorem 3].

6.7 THEOREM. If S is a T_3 space having a θ-cover each element of which is a semicomplete Moore space, then S is a semicomplete Moore space.

Proof. By Lemma 6.6 S is a Moore space and the semicompleteness of S follows from Theorem 6.2 and [29; Theorem 3.1].

6.8 EXAMPLE. The space Σ in [21; Theorem 9] is a non-complete Moore space which is the union of a countable point-finite collection of complete open subspaces.

Proof. Using the notation of [21], $S_X = b_X \cup \{F_t | t \in \bigcup_{i=1}^{\infty} I_{iX}\}^*$ is an open subspace of Σ for each $X \in M$. If K is the semicomplete development for Σ defined in [21], then $(Tr_{S_X} K_n)_{n=1}^{\infty}$ is a complete development for S_X and S_X is complete. By [15; Theorems 2 and 3] every Moore space is countably metacompact so the countable open cover $\{S_X | X \in M\}$ of Σ has a precise, hence countable, point-finite open refinement H. Each set in H is an open subspace of some S_X hence is a complete Moore space and the result follows.

6.9 THEOREM. If S is a T_3 space having a locally finite open cover each set of which is a complete Moore space, then S is a complete Moore space.

Proof. By Lemma 6.6 S is a Moore space. Let G be a monotonic development for S and let U be a locally finite cover of S by open complete subspaces of S. For each set u in U, let G(u) be a monotonic development for u. Let V be an open cover of S each member of which intersects only finitely many sets in U. For each n, let $H_n = \{h \mid$ for some $v \in V$ and $u \in U$, h is a set in $G_n(u)$, $\bar{h} \subset v \cap u$ and h is contained in some set in $G_n\}$. Then $H = H_1, H_2, \ldots$ is a monotonic development for S. Let $M_1, M_2, \ldots$ be a decreasing sequence of closed nonempty sets in S such that for each n, M_n is contained in the closure of some set h_n in H_n. Let v be a set in V containing $\bar{h}_1$. v intersects only finitely many sets $u_1, \ldots, u_k$ in U. For each n, $\bar{h}_n$ is contained in some set u in U and since $M_n \subset M_1 \subset \bar{h}_1 \subset v$ and $M_n \subset \bar{h}_n \subset u$, it follows that v intersects u and $u = u_j$ for some $j \leq k$. So there exist an integer $j \leq k$ and an increasing sequence $n_1, n_2, \ldots$ of positive integers such that $\bar{h}_{n_i} \subset u_j$ and $h_{n_i} \in G_{n_i}(u_j) \subset G_i(u_j)$. M_{n_i} is closed in u_j so by the completeness of $G(u_j)$, $\bigcap_{i=1}^{\infty} M_i = \bigcap_{i=1}^{\infty} M_{n_i} \neq \phi$. Thus H is a complete development for S and S is complete.

6.10 DEFINITION. An essentially T_1 space S is called strongly Čech complete if and only if there is a sequence G of open covers of S such that if U is an open filterbase such that for each n, some set in G_n contains a set in U, then U converges. Such a sequence G is called a complete sequence of open covers of S.

6.11 THEOREM. If S is a T_3 strongly Čech complete space and T is a dense subspace of a space X and f is a continuous mapping of T into S, then there exist a G_δ set Y in X containing T and a continuous extension of f to Y.

Proof. Let H be a complete sequence of open covers of S and for each n, let $G_n = \{U \mid U$ is open in S and U is contained in some member of each of $H_1, H_2, \ldots, H_n\}$. Then $G = G_1, G_2, \ldots$ is a monotonic complete sequence of bases for S. For each n, let $K_n = \{U \mid U$ is open in X and $f(U \cap T)$

is contained in some set in $G_n\}$. $Y = \bigcap_{n=1}^{\infty} K_n^*$ is a G_δ in X containing T. Let $x \in Y$ and let $F = \{V \in G_1 | \text{for some open set } U \text{ in } X \text{ containing } x, f(U \cap T) \subset V\}$. If V_1 and V_2 are sets in F and U_1 and U_2 are open sets in X containing x such that $f(U_i \cap T) \subset V_i$ for $i = 1, 2$, then $V = V_1 \cap V_2$ is a set in G_1 and $U = U_1 \cap U_2$ is an open set in X containing x such that $f(U \cap T) \subset V$. Since T is dense in X, no set in F is empty. Thus F is an open filterbase in S. If n is a positive integer and U is a set in K_n containing x, then $f(U \cap T)$ is contained in some set V in $G_n \subset G_1$. So V is in F and for each n, $F \cap G_n \neq \phi$. Thus F converges in S and the result follows from Lemma 5.8.

6.12 THEOREM. If S is a Tychonoff space having the extension property in Theorem 6.11, then S is Čech complete.

Proof. Let X be a G_δ in βS containing S and let g be a continuous extension to X of the identity on S. Since $\beta X = \beta S$, X is Čech complete and by [10; Theorem 2.8] there is a sequence H of open covers of X such that if U is an open filtersubbase in X such that for each n, H_n contains a member of U, then $\bigcap \bar{U} \neq \phi$. For each n, let $G_n = \{h \cap S | h \in H_n\}$ and suppose F is an open filtersubbase in S such that for each n, G_n contains a member of F. For each $f \in F$, let U(f) be an open set in X such that $U(f) \cap S = f$ and U(f) is in H_n if f is in G_n. Then $U = \{U(f) | f \in F\}$ is an open filtersubbase in X such that for each n, H_n contains a member of U. So there is a point x in $\bigcap \bar{U}^X$. If $f \in F$, then $x \in \overline{U(f)}^X$ hence $g(x) \in g(\overline{U(f)}^X) \subset \overline{g(U(f))}^S = \overline{g(U(f) \cap S)}^S = \overline{g(f)}^S = \bar{f}^S$. Thus $g(x) \in \bigcap \bar{F}^S$ and G is a complete sequence of open covers of S in the sense of [10]. By [10; Theorem 2.8] S is Čech complete.

6.13 COROLLARY. (cf. Corollary 6.3) If S is a Moore space, then the following statements are equivalent:

(1) S is complete;

(2) S is a G_δ in every space in which it is dense and Hausdorff embedded;

(3) S is a G_δ in ωS;

If S is completely regular, then each of the above is equivalent to:

(4) if T is a dense subspace of a space X and f is a continuous mapping of T into S, then f has a continuous extension to a G_δ in X.

Proof. That (1) implies (2) is observed in [9; Theorem 2.13] (although the word "dense" was omitted). That (2) implies (3) follows from [8; Lemma 6] and (3) implies (1) by [8; Theorems 7 and 3]. (1) implies (4) in general by Theorem 6.11 and [9; Theorem 2.13]. If S is completely regular, then (4) implies (1) by Theorem 6.12 and [8; Theorem 3].

6.14 DEFINITION. S is said to be orthocompact if and only if every open cover G of S has an open refinement H covering S such that if x is a point of S, then the intersection of all the sets in H which contain x is open. H is called a Q-refinement of G.

6.15 THEOREM. (1) If S is an orthocompact Moore space, then S has a centered base. (2) If S is a complete Aronszajn space which has a centered base, then S has a complete centered base.

Proof. (1) The argument for this part of the theorem is essentially that in [11; Theorem 9]. Let G be a monotonic development for S and for each n, let Q_n be a Q-refinement of G_n. Let $C_1 = Q_1$ and for each n, let $C_{n+1} = \{U \in Q_{n+1} | \text{if } V \text{ is in } \bigcup_{i \leq n} C_i, \text{ then either } \overline{U} \subset V \text{ or } U \cap (S - V) \neq \phi\}$. Let $B_m = \bigcup_{i \geq m} C_i$. Suppose there is an integer $m > 1$ such that B_m does not cover S and let $P \in S - B_m^*$. For each $i < m$, let U_i be the intersection of all the sets in C_i containing P if $P \in C_i^*$ and otherwise let $U_i = S$. There is an integer $k \geq m$ such that $\overline{st(P,G_k)} \subset \bigcap_{i<m} U_i$. Let V be a set in Q_k containing P. Then $\bar{V}$ is contained in every set in $\bigcup_{i<m} C_i$ containing P so either $V \in C_k$ or there is a set in $\bigcup_{m \leq i<k} C_i$ containing V (hence P) but not containing $\bar{V}$. In either case some set in B_m contains P.

Since B_m is a refinement of G_m, $B = B_1, B_2, \ldots$ is a monotonic sequence of bases for S. Let b be a B-filterbase. For each i, there is an integer $n_i \geq i$ such that $b_i \in C_{n_i}$. Let j and k be positive integers with $k > n_j$. Then $b_k \in C_{n_k}$ and $b_j \in C_{n_j} \subset \bigcup_{i \leq n_k - 1} C_i$ and $b_k \cap (S - b_j) = \phi$. Thus $\bar{b}_k \subset b_j$ and b is regular. Suppose there is a point x in $\bigcap b_n$ and U is an open set containing x. There is an i such that $st(x,G_i) \subset U$. b_i is contained in some set in $G_{n_i} \subset G_i$ so $b_i \subset U$ and b converges to x. Thus S has a centered base by Lemma 5.13.

(2) Let A be a complete Aronszajn sequence for S. By Lemma 5.13 there is a monotonic sequence B of bases for S such that if b is a B-filterbase, then b is regular and b converges to each point of $\bigcap b_n$. Let H be a primitive sequence of S such that $H_n \subset A_n$ [29; Lemma 2.1]. By Lemma 3.5 there is a primitive sequence W of S such that if $n \in N$, $W_n \subset B_n$ and $W_n(P) \subset H_n(P)$ for each $P \in S$. Let $G_n = \bigcup_{i \geq n} W_i$. Since $\{W_i(x) | i \in N\}$ is a monotonic base for S at x for each $x \in S$, $G = G_1, G_2, \ldots$ is a monotonic sequence of bases for S. Suppose g is a G-filterbase. For each i, there is an $n_i \geq i$ such that $g_i \in W_{n_i} \subset B_{n_i} \subset B_i$. So g is a B-filterbase hence g is regular and if there is a point x in $\bigcap g_n$, then g converges to x. For each i, there is a point P_i of S such that $g_i = W_{n_i}(P_i) \subset W_i(P_i) \subset H_i(P_i)$ and by [29; Lemma 2.2] there is an H-filterbase h coarser than g. h is an A-filterbase and h converges to some

point x of S. Therefore, g converges to x and S has a complete centered base by Lemma 5.13.

6.16 EXAMPLE. The tangent disc space X [22] is an orthocompact Moore space which is not metacompact.

Proof. Let R be the set of real numbers. For each $n \geq 1$ and $k \geq 0$, there is a positive number $\delta(n,k)$ such that if I is an open interval whose length does not exceed $\delta(n,k)$ and $S = \{(x,y) \mid x \in I \text{ and } \frac{1}{2(2n+k)} < y < \frac{1}{2n+k}\}$ intersects an open ball in the plane having center $(r, \frac{1}{2n})$ and radius $\frac{1}{2n}$, then S is contained in the open ball having center $(r, \frac{1}{n})$ and radius $\frac{1}{n}$. Let G be an open cover of X and for each $r \in R$, let n_r be a positive integer such that the open ball having center $(r, \frac{1}{n_r})$ and radius $\frac{1}{n_r}$ is contained in some set in G. Let $U(n,k,j) = \{(x,y) \mid \frac{j}{2}\,\delta(n,k) < x < \frac{j+2}{2}\,\delta(n,k)$ and $\frac{1}{2(2n+k)} < y < \frac{1}{2n+k}\}$ for $j = 0, \pm 1, \pm 2, \ldots$ and $k = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$. For each $r \in R$, let $U(r) = \{(r,0)\} \cup \bigcup \{U(n,k,j) \mid n = n_r$ and $U(n,k,j)$ intersects the open ball having center $(r, \frac{1}{2n})$ and radius $\frac{1}{2n}\}$. Then $U(r)$ is an open set in X containing $(r,0)$ and contained in some set in G. If H is a point finite open cover of $\{(x,y) \mid y > 0\}$ each set of which is contained in some set in G, then $H \cup \{U(r) \mid r \in R\}$ is a Q-refinement of G. Thus X is orthocompact.

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