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Abstract

In a recent paper, Žiga Virk defined a type of continuous map which preserves extension properties. We generalize this notion and call such maps extensional maps. In this paper we will establish many of the basic properties of extensional maps. We will then show that extensional maps are preserved by the limit of an inverse system. Finally, there is a generalization of inverse systems called approximate inverse systems, due to Mardešić and Rubin. We will prove several new results concerning these approximate systems, and then show that extensional maps are preserved by the limit of an approximate system as well.

Chapter 1

Introduction

1.1 Absolute Extensors and ANRs

In this paper, map will always refer to a continuous function. Spaces are topological spaces, with no other assumptions included. Generally, though, we will be concerned with either compact Hausdorff or compact metric spaces.

Definition 1.1.1. Given a space X, a subspace $A \subseteq X$, a space K, and a map $f : A \to K$, f is said to **extend over** X if there exists a map $F : X \to K$ such that $F|_A = f$.

The extension problem asks: Given a space X, a subspace $A \subseteq X$, a space K, and a map $f : A \to K$, under what conditions does the map f extend over X? The diagram below illustrates the situation.

$$
\begin{tikzcd}
X \arrow[dr, dashed, "F"] & \\
A \arrow[u, hook] \arrow[r, "f"'] & K
\end{tikzcd}
$$

The main notion in extension theory is that of an absolute extensor.

Definition 1.1.2. A space K is said to be an **absolute extensor** (AE) for a space X if for every closed subspace $A \subseteq X$ and every map $f : A \to K$, f extends over X. One writes either $K \in \mathrm{AE}(X)$ or $X\tau K$.

Using this notation, one can state the well-known Tietze Extension Theorem as: A space X is normal if and only if $X\tau\mathbb{R}$. That is, $\mathbb{R}$ is an absolute extensor for the class of normal spaces. Other examples of absolute extensors include the unit interval I, and, as the product of absolute extensors is again an absolute extensor, the cube I^n for any $n \in \mathbb{N}$, the space R^n for any $n \in \mathbb{N}$, and the Hilbert cube I^∞.

However, the sphere S^n, $n \in \mathbb{N}$ is not an absolute extensor for the class of normal spaces. For example, let $X = \mathbb{R}^2$, $A = S^1 \subseteq \mathbb{R}^2$, $K = S^1$ and $f = \mathrm{id}|_{S^1}$. Then by the no-retraction theorem, there cannot exist a map $F : \mathbb{R}^2 \to S^1$ such that $F|_{S^1} = \mathrm{id}|_{S^1}$. However, every map in a sphere *can* be extended over a neighborhood, which leads us to our next definition.

Definition 1.1.3. A space K is said to be an **absolute neighborhood extensor** (ANE) for a space X if for every closed subspace $A \subseteq X$ and every map $f : A \to K$, f extends over a neighborhood U of A in X. That is, there exists a map $F : U \to K$ such that $F|_A = f$.

The following theorem comes from [4].

Proposition 1.1.4. *If a contractible space Y is an* ANE *for the class $\mathcal{C}$, then Y is an* AE *for the class $\mathcal{C}$.*

For us, the most important example will be that every polyhedron is an ANE for the class of compact Hausdorff spaces.

Definition 1.1.5. By a **polyhedron** we will mean the geometric realization of a simplicial complex. If K is a simplicial complex, then $|K|$ will denote the geometric realization of K.

The following proposition concerning polyhedra will be useful.

Proposition 1.1.6. *Let X be a compact space. Suppose that $f : X \to |K|$ is a map where K is a simplicial complex. Then $f(X) \subseteq |L|$ for some finite subcomplex L of K.*

Closely related to AEs and ANEs are the notions of absolute retracts and absolute neighborhood retracts. We will first define what we mean by a retract and neighborhood retract.

Definition 1.1.7. Let $A \subseteq X$. Then A is said to be a **retract** of X if there exists a map $r : X \to A$ such that $r|_A = \mathrm{id}\,|_A$. The map r is called a **retraction**.

Definition 1.1.8. Let $A \subseteq X$. Then A is said to be a **neighborhood retract** of X if there exits a neighborhood U of A in X and a map $r : U \to A$ such that $r|_A = \mathrm{id}\,|_A$.

Definition 1.1.9. A metrizable space K is said to be an **absolute retract** (AR) if every embedding of K as a closed subspace of any metrizable space Z is a retract of Z.

Definition 1.1.10. A metrizable space K is said to be an **absolute neighborhood retract** (ANR) if every embedding of K as a closed subspace of any metrizable space Z is a neighborhood retract of Z.

As in the case of ANEs, every polyhedron is an ANR for the class of compact Hausdorff spaces.

The following two very useful facts about ANRs come from [4].

Theorem 1.1.11. *If K is an ANR then there exists an $\varepsilon > 0$ such that if $f, g : X \to K$ are any two maps defined on an arbitrary space X with the property that for all $x \in X$, $d(f(x), g(x)) < \varepsilon$, then f and g are homotopic.*

Proposition 1.1.12. *Every contractible ANR is an AR.*

1.2 Covering Dimension and Factorization Theorems

Definition 1.2.1. Given a collection of sets $\mathcal{U}$, the **order** of $\mathcal{U}$, written $\mathrm{ord}\,\mathcal{U}$, is the largest integer n such that there are $n+1$ members of $\mathcal{U}$ having a non-empty intersection.

Definition 1.2.2. A covering $\mathcal{V}$ is a **refinement** of a covering $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$.

Definition 1.2.3. A space X has **dimension** $\leq n$, $\dim X \leq n$, if and only if every finite open covering has a refinement of order $\leq n$.

The following classical result provides a link between dimension and extension theory:

Theorem 1.2.4. *For a normal space X, $\dim X \leq n$ if and only if $X \tau S^n$.*

Using 1.2.4, the Mardešić Factorization Theorem [9] can be restated as follows.

Theorem 1.2.5. *If X and Y are compact Hausdorff spaces, $X \tau S^n$, and $f : X \to Y$ is a map, then there exist a compact Hausdorff space Z and maps $g : X \to Z$ and $h : Z \to Y$, such that $f = hg$, $Z \tau S^n$, and $\mathrm{wt}(Z) \leq \mathrm{wt}(Y)$.*

Here, $\mathrm{wt}(Z)$ denotes the weight of Z, which is the minimum cardinality of a base for the topology of Z.

This result has been generalized many times. In [8], Levin, Rubin and Schapiro generalized this theorem so that one may replace S^n with any CW-complex. Their work was later generalized by Virk [13]. From [8] and [13] arose the notion of an *extensional map*.

The following definition comes from [13].

Definition 1.2.6. Let X and Z be spaces. A surjective map $f : X \to Z$ is said to be an **extensional map** if the following condition is satisfied: For every pair $A \subseteq B$ of closed subsets of Z and every map $\alpha : A \to K$, where K is any CW-complex, if $\alpha \circ f|_{f^{-1}(A)} : f^{-1}(A) \to K$ extends over $f^{-1}(B)$, then α extends over B.

$$\begin{tikzcd}
& f^{-1}(A) \arrow[r, hook] \arrow[d, "f"'] & f^{-1}(B) \arrow[d, "f"'] \arrow[dll, dashed, bend right, "\tilde{\alpha}"'] \\
K & A \arrow[l, "\alpha"'] \arrow[r, hook] & B \arrow[ll, dashed, bend left, "\beta"]
\end{tikzcd}$$

The question that we want to answer is: If I have a map $\alpha : A \to K$, can I extend that map over B? What an extensional map allows us to do is to "pull back" and look at the similar question for $f^{-1}(A)$, $f^{-1}(B)$ and $\alpha \circ f|_{f^{-1}(A)}$. Intuitively, this means that Z will have similar "extension properties" to X, or that f is a map that preserves extension properties. The situation is illustrated in the above diagram, where if the map $\tilde{\alpha}$ exists, then the map β will exist.

Using this definition we can state Virk's generalization of 1.2.5.

Theorem 1.2.7. *Suppose $g : X \to Y$ is a map defined on a compact Hausdorff space X. Then there exist a compact Hausdorff space Z with $\operatorname{wt} Z \leq \{\operatorname{wt} Y, \aleph_0\}$, a map $p : Z \to Y$, and a surjective map $f : X \to Z$ such that $g = pf$ and f is an extensional map.*

We are going to look at how extensional maps behave with respect to inverse limits, and the more general notion of approximate inverse limits. We will see that extensional maps are preserved by these limits. First, though, in the following section, we will generalize the idea of extensional maps and prove some basic properties concerning them.

1.3 Basic Results Concerning Extensional Maps

Definition 1.3.1. A surjective map $f : X \to Z$ is an **extensional map relative to a space** K if, given any pair $A \subset B$ of closed subsets of Z and any map $\alpha : A \to K$ for which $\alpha \circ f|_{f^{-1}(A)} : f^{-1}(A) \to K$ extends over $f^{-1}(B)$, the map α extends over B.

We can also consider extensional maps relative to a class of spaces.

Proposition 1.3.2. *Every homeomorphism is an extensional map relative to the class of all topological spaces.*

Proposition 1.3.3. *Every retraction is an extensional map relative to the class of all topological spaces.*

Proof. Let X be a space, Z a retract of X and $r : X \to Z$ a retraction. Let $A \subset B$ be closed subsets of Z and $\alpha : A \to K$ be a map, where K is a space.

Suppose $\alpha \circ r|_{r^{-1}(A)} : r^{-1}(A) \to K$ extends over $r^{-1}(B)$, say to $\tilde{\alpha}$. Then $\tilde{\alpha}|_B$ is an extension of α over B.

Indeed, since $B \subset r^{-1}(B)$, $\tilde{\alpha}$ is defined on all of B. Let $a \in A$. Then $a \in r^{-1}(A)$, and $(\tilde{\alpha}|_B)(a) = \alpha(r(a)) = \alpha(a)$. Thus we have the desired extension. □

Proposition 1.3.4. *If $f : X \to Y$ and $g : Y \to Z$ are extensional maps relative to a space K, then $g \circ f : X \to Z$ is an extensional map relative to K.*

Proof. Let $A \subset B$ be closed subsets of Z and $\alpha : A \to K$ a map. Suppose that $\alpha \circ (g \circ f)|_{(g\circ f)^{-1}(A)} : (g \circ f)^{-1}(A) \to K$ extends over $(g \circ f)^{-1}(B)$.

Since f is an extensional map relative to K, $g^{-1}(A) \subset g^{-1}(B)$ are closed subsets of Y, and $\alpha \circ g|_{g^{-1}(A)} : g^{-1}(A) \to K$ is a map to K, we can extend $\alpha \circ g$ over $g^{-1}(B)$.

Then since g is an extensional map relative to K, and $\alpha \circ g|_{g^{-1}(A)} : g^{-1}(A) \to K$ extends over $g^{-1}(B)$, we have that α extends over B. $\square$

The following proposition is obvious.

Proposition 1.3.5. *Let Z and K be spaces. If $Z\tau K$, then any continuous surjection $f : X \to Z$ defined on a space X is an extensional map relative to K.*

Using 1.3.5, and the fact that if Z is a compact Hausdorff space, then $Z\tau|K|$ for any contractible polyhedron $|K|$, we arrive at the following.

Proposition 1.3.6. *Let Z be a compact Hausdorff space and $|K|$ be a contractible polyhedron. If $f : X \to Z$ is a continuous surjection, then f is an extensional map relative to $|K|$.*

Proposition 1.3.7. *Let X and Z be spaces. If $f : X \to Z$ is an extensional map relative to a space K, and L is a retract of K, then f is an extensional map relative to L.*

Proof. Let $A \subseteq B$ be closed subspaces of Z, and $\alpha : A \to L$ be a map. Suppose that $\alpha \circ f|_{f^{-1}(A)} : f^{-1}(A) \to L$ extends over $f^{-1}(B)$, say to $\tilde{\alpha} : f^{-1}(B) \to L$. We wish to show that α extends over B.

Treat the maps $\alpha : A \to L \subseteq K$ and $\tilde{\alpha} : f^{-1}(B) \to L \subseteq K$ as maps to K. Then, since f is an extensional map relative to K, we have that α extends over B to a map $\beta : B \to K$ to K.

Let $r : K \to L$ be a retraction. Then $r \circ \beta : B \to L$ is the desired extension of α, and so f is an extensional map relative to L. $\square$

Lemma 1.3.8. *Let X and Z be spaces. If $f : X \to Z$ is an extensional map relative to a space K, L is a space that is homotopy dominated by K, and Z has the homotopy extension property relative to L, then f is an extensional map relative to L.*

Proof. Let $A \subset B$ be closed subsets of Z and $\alpha : A \to L$ a map. Let $g : L \to K$ and $h : K \to L$ be maps such that $h \circ g \simeq \mathrm{id}_L$.

Suppose that $\alpha \circ f|_{f^{-1}(A)} : f^{-1}(A) \to L$ extends over $f^{-1}(B)$, say to $\tilde{\alpha} : f^{-1}(B) \to L$. Then $g \circ \tilde{\alpha} : f^{-1}(B) \to K$ is an extension of $g \circ \alpha \circ f|_{f^{-1}(A)} : f^{-1}(A) \to K$. Since f is an extensional map relative to K, $g \circ \alpha : A \to K$ extends over B, say to $\beta : B \to K$.

Consider $h \circ \beta : B \to L$. We have $h \circ \beta|_A = h \circ g \circ \alpha \simeq \mathrm{id}_L \circ \alpha = \alpha$. Thus, α is homotopic to a map that extends over B, and so α extends over B. $\square$

Proposition 1.3.9. *Let $\{K_\mu | \mu \in \Gamma\}$ be a collection of nonempty spaces, put K equal to the topological product of this collection, and let X and Z be spaces. If, for each $\mu \in \Gamma$, $f : X \to Z$ is an extensional map relative to K_μ, then f is an extensional map relative to K.*

Proof. Let $A \subset B$ be closed in Z, and $\alpha : A \to K$ a map. Suppose $\alpha \circ f|_{f^{-1}(A)} : f^{-1}(A) \to K$ extends over $f^{-1}(B)$, say to $\tilde{\alpha} : f^{-1}(B) \to K$. We wish to show that α extends over B.

Consider $\alpha_\mu = p_\mu \circ \alpha : A \to K_\mu$, where p_μ is the projection from K to K_μ. Then $\tilde{\alpha}_\mu = p_\mu \circ \tilde{\alpha} : f^{-1}(B) \to K_\mu$ is an extension of $\alpha_\mu \circ f|_{f^{-1}(A)} : f^{-1}(A) \to K$ over $f^{-1}(B)$.

Since for each $\mu \in \Gamma$, f is an extensional map relative to K_μ, α_μ extends over B to a map $\beta_\mu : B \to K_\mu$. Define a map $\beta : B \to K$ by taking $p_\mu(\beta(z)) = \beta_\mu(z)$ for every $z \in B$. Then β is an extension of α over B. $\square$

Some basic facts about extensional maps have now been established. We will now give a short outline of the rest of the paper. In chapter two we will show that extensional maps are preserved by limits of inverse systems. In chapter three we give the definition of the more general notion of the limit of an approximate inverse system, which was introduced by Mardešić and Rubin [10]. We establish several new results about closed subspaces of approximate inverse systems. We then use these results to show that extensional maps are preserved by limits of approximate inverse systems.

In chapter four we will look at a type of dimension introduced by Dranishnikov and Dydak [2] called extension dimension, and introduce a similar construction for extensional maps.

In chapter five we are concerned with the extension problem when the range is an inverse limit or an approximate inverse limit. In the case of inverse limits we show that under certain conditions maps to an inverse limit can be extended, but only to a neighborhood of the limit in the product space. In the case of approximate inverse limits, for a map to an approximate inverse limit there exists a map to a neighborhood of the limit in the product which is arbitrarily close to the original map.

In chapter six we look at a new definition of dimension, (m, n)-dim, introduced by Fedorchuk [3]. Using the tools that we established in chapter three, we show that this type of dimension is preserved by approximate inverse limits.

Chapter 2

Inverse Sequences and Systems

In this chapter we will show that extensional maps are preserved by inverse limits. In section one we will look at basic definitions and theorems concerning inverse sequences and systems. In section two we will see that extensional maps are preserved by the limit of an inverse sequence. In the third section we will see that extensional maps are preserved by the limit of an inverse system.

2.1 Definition

Definition 2.1.1. An ordered set $(\Gamma, \leq)$ is said to be a **directed set** if for for any two elements γ_1, $\gamma_2 \in \Gamma$ there exists a $\gamma_3 \in \Gamma$ such that $\gamma_1 \leq \gamma_3$ and $\gamma_2 \leq \gamma_3$. We call γ_3 a **successor** of γ_1 and γ_2, and we call γ_1 and γ_2 **predecessors** of γ_3.

Definition 2.1.2. A directed set in which each element has only a finite number of predecessors is said to be **cofinite**.

Definition 2.1.3. An inverse system $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, \Gamma\}$ of spaces consists of the following: a partially ordered set $(\Gamma, \leq)$ which is directed; for each $\gamma \in \Gamma$, a space X_γ; for each pair $\gamma \leq \gamma'$ from Γ, a map $p_{\gamma\gamma'} : X_{\gamma'} \to X_\gamma$. The maps must satisfy the following two conditions: (1) $p_{\gamma\gamma} = \mathrm{id}_{X_\gamma}$, and (2) if $\gamma \leq \gamma'$ and $\gamma' \leq \gamma''$, then $p_{\gamma\gamma''} = p_{\gamma\gamma'} \circ p_{\gamma'\gamma''}$.

Definition 2.1.4. An inverse system indexed by $\mathbb{N}$ is called an **inverse sequence**.

Definition 2.1.5. A point $x = (p_\gamma(x)) \in \prod_{\gamma\in\Gamma} X_\gamma$ belongs to $X = \lim \mathbf{X}$ provided the following condition is satisfied: For all $\gamma \in \Gamma$, if $\gamma \leq \gamma'$ then $p_\gamma(x) = p_{\gamma\gamma'} p_{\gamma'}(x)$.

The following lemmas and theorems concerning inverse systems and sequences will be useful.

Theorem 2.1.6. *If in an inverse system $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, \Gamma\}$, all X_γ are Hausdorff spaces, then $X = \lim \mathbf{X}$ is closed in $\prod_{\gamma\in\Gamma} X_\gamma$.*

Theorem 2.1.7. *If in an inverse system $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, \Gamma\}$, all X_γ are compact and nonempty, then $X = \lim \mathbf{X}$ is also compact and nonempty.*

Theorem 2.1.8. *For an inverse system $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, \Gamma\}$, the collection of $p_\gamma^{-1}(U)$, such that $\gamma \in \Gamma$ and $U \subseteq X_\gamma$ is open in X_γ, is a base for the topology of $X = \lim \mathbf{X}$.*

Theorem 2.1.9. *Let $\mathbf{X} = \{X_\gamma, p_{\gamma\gamma'}, \Gamma\}$ be an inverse system of compact Hausdorff spaces, $X = \lim \mathbf{X}$, A be a closed subspace of X, K be a CW-complex, and $f : A \to K$ be a map. Then for some $\gamma \in \Gamma$ there is a map $f_\gamma : p_\gamma(A) \to K$ such that $f \simeq f_\gamma \circ p_\gamma|_A$.*

2.2 Limit Theorem for Inverse Sequences

In this section, given an inverse sequence $\mathbf{Z} = (Z_i, p_{ij})$ and subset $A \subseteq Z$, let $A_k = p_k(A)$ for each $k \in \mathbb{N}$.

In Theorem 2.2.3 we will show that extensional maps are preserved by the limit of an inverse sequence. In the following section, Theorem 2.3.1 will show that extensional maps are preserved by the limit of an inverse system. While 2.3.1 is more general than 2.2.3, Theorem 2.2.3 and its proof are included due to the fact that they are somewhat simpler.

The following lemma will be needed in both the proof of 2.2.3 and 2.3.1.

Lemma 2.2.1. *Let K be an ANR, and X a normal space with the homotopy extension property with respect to K. Let A be a closed subset of X, $\alpha : A \to K$ a map and $\tilde{\alpha}$ an extension of α over X. Then, for any extension of α over a neighborhood U of A to a map $\beta : U \to K$, there exists a neighborhood $V \subset U$ of A and a map $\beta^* : V \to K$ such that (1) $\beta|_V = \beta^*$ and (2) β^* extends over X.*

Proof. Using 1.1.11, choose an open cover $\mathcal{W}$ of K such that for any space Z, any maps $f : Z \to K$ and $g : Z \to K$ that are star-$\mathcal{W}$ close are homotopic.

By continuity and the fact that U is a neighborhood of A in X, for each $x \in A$, there exists a neighborhood S_x of x in X such that $\tilde{\alpha}(S_x) \subseteq W$ for some $W \in \mathcal{W}$ and $S_x \subseteq U$. Also, there exists a neighborhood T_x of x in X such that $\beta(T_x) \subset W^*$ for some $W^* \in \mathcal{W}$.

Consider the open set $V^* = \bigcup\{S_x \cap T_x | x \in A\}$. Then V^* is a neighborhood of A contained in U. By normality, there exists a neighborhood V of A such that

$$A \subseteq V \subseteq \overline{V} \subseteq V^* \subseteq U. \tag{2.1}$$

The maps $\tilde{\alpha}$ and β are both defined on $\overline{V}$ (and so also on V). We wish to show that on $\overline{V}$, the maps $\tilde{\alpha}$ and β are star-$\mathcal{W}$ close. Let $z \in \overline{V}$. Then for some $x \in A$, $z \in S_x \cap T_x$. So since $z \in S_x$, for some $W \in \mathcal{W}$, $\tilde{\alpha}(z) \in W$ and $\tilde{\alpha}(x) = \alpha(x) \in W$. Furthermore, since $z \in T_x$, for some $W^* \in \mathcal{W}$, $\beta(z) \in W^*$ and $\beta(x) = \alpha(x) \in W^*$. Thus, $W \cap W^* \neq \varnothing$, and so $\tilde{\alpha}$ and β are star-$\mathcal{W}$ close.

So we have that on $\overline{V}$, the maps $\tilde{\alpha}$ and β are homotopic. The map $\tilde{\alpha}$ is defined on all of X, and so $\beta|_{\overline{V}}$ extends over all of X. Let $\beta^* = \beta|_V$. Then $\beta^* : V \to K$ extends over X. $\square$

Corollary 2.2.2. *In particular, Lemma 2.2.1 is true when K is a CW-complex and X is a compact Hausdorff space.*

Theorem 2.2.3. *Let $\mathbf{Z} = (Z_i, p_{ij})$ be an inverse sequence of compact Hausdorff spaces, and $Z = \lim \mathbf{Z}$. Let X be a compact Hausdorff space, $f : X \to Z$ a surjective map, and K a CW-complex. If $f_i = p_i \circ f : X \to Z_i$ is an extensional map relative to K for each $i \in \mathbb{N}$, then $f : X \to Z$ is an extensional map relative to K.*

$$\begin{tikzcd}
& & & & & X \arrow[dlllll, "f_1"'] \arrow[dllll, "f_2"] \arrow[dll, "f_i"] \arrow[d, "f"'] \\
Z_1 & Z_2 \arrow[l, "p_{1,2}"] & \cdots \arrow[l, "p_{2,3}"] & Z_i \arrow[l, "p_{i-1,i}"] & \cdots \arrow[l, "p_{i,i+1}"] & Z \arrow[ll, bend left, "p_i"] \arrow[llll, bend left, "p_2"] \arrow[lllll, bend left, "p_1"]
\end{tikzcd}$$

Proof. Let $A \subseteq B$ be closed subsets of Z, and $\alpha : A \to K$ a map. Suppose that $\alpha \circ f|_{f^{-1}(A)} : f^{-1}(A) \to K$ extends over $f^{-1}(B)$ to $\tilde{\alpha}$. Extend $\tilde{\alpha}$ to a neighborhood S of $f^{-1}(B)$. We will continue to call this new extension $\tilde{\alpha}$. Since f is a closed map, there exists a neighborhood U of B such that $f^{-1}(U) \subseteq S$.

By 2.1.9, for some $i \in \mathbb{N}$ there exists a map $\alpha_i : A_i \to K$ such that $\alpha_i \circ p_i|_A \simeq \alpha$.

Consider $p_i^{-1}(A_i)$. Then A is a closed subset of $p_i^{-1}(A_i)$. The map $\alpha_i \circ p_i|_A$ extends over $p_i^{-1}(A_i)$ to $\alpha_i \circ p_i|_{p_i^{-1}(A_i)}$. So by the homotopy extension property, α extends over $p_i^{-1}(A_i)$, to a map (1) α^* which is homotopic to $\alpha_i \circ p_i|_{p_i^{-1}(A_i)}$ on $p_i^{-1}(A_i)$. We can extend α^* to a neighborhood W of $p_i^{-1}(A_i)$, which is also then a neighborhood of A. Without loss of generality, we can assume that $W \subseteq U$, so that $f^{-1}(W) \subseteq S$.

We will now show that there exists a neighborhood $V \subseteq W$ of A such that $\alpha^* \circ f|_{f^{-1}(V)} : f^{-1}(V) \to K$ extends over S. First we note that $f^{-1}(W)$ is a neighborhood of $f^{-1}(A)$, and $\alpha^* \circ f|_{f^{-1}(W)} : f^{-1}(W) \to K$ is an extension of $\alpha \circ f|_{f^{-1}(A)}$ over $f^{-1}(W)$. So by Lemma 2.2.1, there exists a neighborhood T of $f^{-1}(A)$ such that (2) $\alpha^* \circ f|_T$ extends over S. Since f is a closed map, there exists a neighborhood $V \subseteq W$ of A such that $f^{-1}(V) \subseteq T$.

Now choose $k \geq i$ so that both $p_k^{-1}(A_k) \subseteq V$ and $p_k^{-1}(B_k) \subseteq U$. Note that $p_k^{-1}(A_k) \subseteq p_i^{-1}(A_i)$. By (1) α^* is homotopic to $\alpha_i \circ p_i|_{p_i^{-1}(A_i)}$ on $p_i^{-1}(A_i)$, so (3) $\alpha^*|_{p_k^{-1}(A_k)}$ is homotopic to $\alpha_i \circ p_i|_{p_k^{-1}(A_k)}$. Note that (4) $\alpha_i \circ p_i = \alpha_i \circ p_i^k \circ p_k$.

Since $p_k^{-1}(A_k) \subseteq V$, we have that $f^{-1}(p_k^{-1}(A_k)) \subseteq f^{-1}(V) \subseteq T$. By (2), we get that (5) $\alpha^* \circ f|_{f^{-1}(p_k^{-1}(A_k))} : f^{-1}(p_k^{-1}(A_k)) \to K$ extends over S, and so also extends over $f^{-1}(p_k^{-1}(B_k))$, since $f^{-1}(p_k^{-1}(B_k)) \subseteq f^{-1}(U) \subseteq S$.

Using (3) and (4) we get the following

$$\alpha^* \circ f|_{f^{-1}(p_k^{-1}(A_k))} \tag{2.2}$$

$$\simeq \alpha_i \circ p_i \circ f|_{f^{-1}(p_k^{-1}(A_k))} \tag{2.3}$$

$$= \alpha_i \circ p_i^k \circ p_k \circ f|_{f^{-1}(p_k^{-1}(A_k))}. \tag{2.4}$$

Applying (5) and the homotopy extension property, $\alpha_i \circ p_i^k \circ p_k \circ f|_{f^{-1}(p_k^{-1}(A_k))}$ extends over $f^{-1}(p_k^{-1}(B_k))$. By our assumption that $p_k \circ f$ is an extensional map relative to K, $\alpha_i \circ p_i^k|_{A_k}$ extends over B_k, say to $\beta_k : B_k \to K$. Then $\beta_k \circ p_k|_B : B \to K$ is an extension of $\alpha_i \circ p_i|_A : A \to K$. Since $\alpha_i \circ p_i|_A \simeq \alpha$, by the homotopy extension property, α extends over B. □

2.3 Limit Theorem for Inverse Systems

In this section, given an inverse system $\mathbf{Z} = (Z_\gamma, p_{\gamma\gamma'}, \Gamma)$ and subset $A \subseteq Z$, for each $\gamma \in \Gamma$, let $A_\gamma = p_\gamma(A)$.

Theorem 2.3.1. *Let* $\mathbf{Z} = (Z_\gamma, p_{\gamma\gamma'}, \Gamma)$ *be an inverse system of compact Hausdorff spaces, and* $Z = \lim \mathbf{Z}$*. Let* X *be a compact Hausdorff space,* $f : X \to Z$ *a surjective map, and* K *a CW-complex. If* $p_\gamma \circ f : X \to Z_\gamma$ *is an extensional map relative to* K *for each* $\gamma \in \Gamma$*, then* $f : X \to Z$ *is an extensional map relative to* K*.*

Proof. Let $A \subseteq B$ be closed subsets of Z, and $\alpha : A \to K$ a map. Suppose that $\alpha \circ f|_{f^{-1}(A)} : f^{-1}(A) \to K$ extends over $f^{-1}(B)$ to $\tilde{\alpha}$. Extend $\tilde{\alpha}$ to a neighborhood S of $f^{-1}(B)$. We will continue to call this new extension $\tilde{\alpha}$. Since f is a closed map, there exists a neighborhood U of B such that $f^{-1}(U) \subseteq S$.

By 2.1.9, for some $\gamma \in \Gamma$ there exists a map $\alpha_\gamma : A_\gamma \to K$ such that $\alpha_\gamma \circ p_\gamma|_A \simeq \alpha$.

Consider $p_\gamma^{-1}(A_\gamma)$. Then A is a closed subset of $p_\gamma^{-1}(A_\gamma)$. The map $\alpha_\gamma \circ p_\gamma|_A$ extends over $p_\gamma^{-1}(A_\gamma)$ to $\alpha_\gamma \circ p_\gamma|_{p_\gamma^{-1}(A_\gamma)}$. So by the homotopy extension property, α extends over $p_\gamma^{-1}(A_\gamma)$ to a map (1) α^* which is homotopic to $\alpha_\gamma \circ p_\gamma|_{p_\gamma^{-1}(A_\gamma)}$ on $p_\gamma^{-1}(A_\gamma)$. We can extend α^* to a neighborhood W of $p_\gamma^{-1}(A_\gamma)$, which is also then a neighborhood of A. Without loss of generality, we will assume that $W \subseteq U$, so that $f^{-1}(W) \subseteq S$.

We will now show that there exists a neighborhood $V \subseteq W$ of A such that $\alpha^* \circ f|_{f^{-1}(V)} : f^{-1}(V) \to K$ extends over S. We note that $f^{-1}(W)$ is a neighborhood of $f^{-1}(A)$. Since α^* is an extension of α, the map $\alpha^* \circ f|_{f^{-1}(W)} : f^{-1}(W) \to K$ is an extension of $\alpha \circ f|_{f^{-1}(A)}$ over $f^{-1}(W)$. So by Lemma 2.2.1, there exists a neighborhood $T \subseteq f^{-1}(W)$ of $f^{-1}(A)$ such that (2) $\alpha^* \circ f|_T$ extends over S. Since f is a closed map, there exists a neighborhood $V \subseteq W$ of A such that $f^{-1}(V) \subseteq T$.

Now choose $\gamma' \geq \gamma$ so that both $p_{\gamma'}^{-1}(A_{\gamma'}) \subseteq V$ and $p_{\gamma'}^{-1}(B_{\gamma'}) \subseteq U$. Note that $p_{\gamma'}^{-1}(A_{\gamma'}) \subseteq p_{\gamma}^{-1}(A_{\gamma})$. By (1), the map α^* is homotopic to $\alpha_\gamma \circ p_\gamma|_{p_\gamma^{-1}(A_\gamma)}$ on $p_\gamma^{-1}(A_\gamma)$, so (3) $\alpha^*|_{p_{\gamma'}^{-1}(A_{\gamma'})}$ is homotopic to $\alpha_\gamma \circ p_\gamma|_{p_{\gamma'}^{-1}(A_{\gamma'})}$. Note that (4) $\alpha_\gamma \circ p_\gamma = \alpha_\gamma \circ p_{\gamma\gamma'} \circ p_{\gamma'}$.

Since $p_{\gamma'}^{-1}(A_{\gamma'}) \subseteq V$, we have that $f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'})) \subseteq f^{-1}(V) \subseteq T$. Using (2), we get (5) $\alpha^* \circ f|_{f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'}))} : f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'})) \to K$ extends over S, and so also extends over $f^{-1}(p_{\gamma'}^{-1}(B_{\gamma'}))$, since $f^{-1}(p_{\gamma'}^{-1}(B_{\gamma'})) \subseteq f^{-1}(U) \subseteq S$.

Using (3) and (4),

$$\alpha^* \circ f|_{f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'}))} \tag{2.5}$$

$$\simeq \alpha_\gamma \circ p_\gamma \circ f|_{f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'}))} \tag{2.6}$$

$$= \alpha_\gamma \circ p_{\gamma\gamma'} \circ p_{\gamma'} \circ f|_{f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'}))}. \tag{2.7}$$

By the homotopy extension property and (5), $\alpha_\gamma \circ p_{\gamma\gamma'} \circ p_{\gamma'} \circ f|_{f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'}))}$ extends over $f^{-1}(p_{\gamma'}^{-1}(B_{\gamma'}))$. By our assumption that $p_{\gamma'} \circ f$ is an extensional map with respect to K, $\alpha_\gamma \circ p_{\gamma\gamma'}$ extends over $B_{\gamma'}$, say to $\beta_{\gamma'}$. Then $\beta_{\gamma'} \circ p_{\gamma'}|_B : B \to K$ is an extension of $\alpha_\gamma \circ p_\gamma|_A : A \to K$. Since $\alpha_\gamma \circ p_\gamma|_A \simeq \alpha$, the map α extends over B. $\square$

Chapter 3

Approximate Inverse Systems

3.1 Definition

We will now prove a theorem analogous to Theorem 2.2.3 in the more general setting of approximate inverse systems.

We will use the following from [10]:

Definition 3.1.1. An approximate inverse system $\mathbf{X} = \{X_\gamma, \varepsilon_\gamma, p_{\gamma\gamma'}, \Gamma\}$ of metric compacta consists of the following: a partially ordered set $(\Gamma, \leq)$ which is directed and has no maximal element; for each $\gamma \in \Gamma$, a compact metric space X_γ with metric d and a real number $\varepsilon_\gamma > 0$; for each pair $\gamma \leq \gamma'$ from Γ, a map $p_{\gamma\gamma'} : X_{\gamma'} \to X_\gamma$. Moreover, the following three conditions must be satisfied:

(A1) $d(p_{\gamma_1\gamma_2}p_{\gamma_2\gamma_3}, p_{\gamma_1\gamma_3}) \leq \varepsilon_{\gamma_1}, \gamma_1 \leq \gamma_2 \leq \gamma_3, p_{\gamma\gamma} = \text{id}$.

(A2) For all $\gamma \in \Gamma$ and $\eta > 0$ there exists a $\gamma' \geq \gamma$ such that for all $\gamma_2 \geq \gamma_1 \geq \gamma'$ we have that $d(p_{\gamma\gamma_1}p_{\gamma_1\gamma_2}, p_{\gamma\gamma_2}) \leq \eta$.

(A3) For all $\gamma \in \Gamma$ and $\eta > 0$ there exists a $\gamma' \geq \gamma$ such that for all $\gamma'' \geq \gamma'$

and $x, x' \in X_{\gamma''}$ we have that if $d(x, x') \leq \varepsilon_{\gamma''}$ then $d(p_{\gamma\gamma''}(x), p_{\gamma\gamma''}(x')) \leq \eta$.

Definition 3.1.2. A point $x = (p_\gamma(x)) \in \prod_{\gamma \in \Gamma} X_\gamma$ belongs to $X = \lim \mathbf{X}$ provided the following condition is satisfied.

(L) For all $\gamma \in \Gamma$ and $\eta > 0$ there exists $\gamma' \geq \gamma$ such that for all $\gamma'' \geq \gamma'$ we have that $d(p_\gamma(x), p_{\gamma\gamma''}p_{\gamma''}(x)) \leq \eta$.

The following theorems and lemmas will be useful.

Theorem 3.1.3. (Theorem 1 from [10]) *If in an approximate system* $\mathbf{X} = (X_\gamma, \varepsilon_\gamma, p_{\gamma\gamma'}, \Gamma)$, *all* $X_\gamma \neq \emptyset$, *then also* $X = \lim \mathbf{X} \neq \emptyset$.

Theorem 3.1.4. (Theorem 2 from [10]) *The limit* X *of an approximate system of compacta is a compact Hausdorff space.*

Lemma 3.1.5. (Lemma 3 from [10]) *The collection of all sets of the form* $p_\gamma^{-1}(V_\gamma)$, *where* $\gamma \in \Gamma$ *and* $V_\gamma \subseteq X_\gamma$ *is open, is a basis for the topology of* X.

Lemma 3.1.6. (Lemma 4 from [10]) *For every* $\gamma \in \Gamma$ *and* $\eta > 0$ *there is a* $\gamma' \geq \gamma$ *such that for every* $\gamma'' \geq \gamma'$ *one has* $d(p_{\gamma\gamma''}p_{\gamma''}, p_\gamma) \leq \eta$.

3.2 Closed Subspaces of Approximate Inverse Limits

We will now establish several propositions dealing with closed subspaces of approximate inverse limits that will be helpful in proving the main theorem.

In this section let $\mathbf{X} = \{X_\gamma, \varepsilon_\gamma, p_{\gamma\gamma'}, \Gamma\}$ be an approximate inverse system, $X = \lim \mathbf{X}$, and $A \subseteq X$ be closed. We will define $A_\gamma = p_\gamma(A)$. A_γ is clearly closed in X_γ.

Proposition 3.2.1. $A = \bigcap_{\gamma \in \Gamma} p_\gamma^{-1}(A_\gamma)$.

Proof. It is clear that $A \subseteq \bigcap_{\gamma \in \Gamma} p_\gamma^{-1}(A_\gamma)$.

We will now show that $\bigcap_{\gamma \in \Gamma} p_\gamma^{-1}(A_\gamma) \subseteq A$. To do this, let $x \in \bigcap_{\gamma \in \Gamma} p_\gamma^{-1}(A_\gamma)$. We will show that for every neighborhood U of x that $U \cap A \neq \emptyset$.

Let U be a neighborhood of x. Then, by 3.1.5, there exists a basic open set of the form $p_\gamma^{-1}(U_\gamma)$, where U_γ is open in X_γ such that $x \in p_\gamma^{-1}(U_\gamma) \subseteq U$.

Since $p_\gamma(x) \in U_\gamma$, there exists an $\eta > 0$ such that $B(p_\gamma(x), \eta) \subseteq U_\gamma$. Let $\gamma' \geq \gamma$ satisfy 3.1.6 for γ and $\eta/3$.

Since $x \in \bigcap_{\gamma \in \Gamma} p_\gamma^{-1}(A_\gamma)$, we have $x \in p_{\gamma'}^{-1}(A_{\gamma'})$, and so there exists an $a \in A$ such that $p_{\gamma'}(x) = p_{\gamma'}(a)$. Then we have that

$$\begin{aligned}
&d(p_\gamma(x), p_\gamma(a)) \\
\leq &d(p_\gamma(x), p_{\gamma\gamma'}p_{\gamma'}(x)) + d(p_{\gamma\gamma'}p_{\gamma'}(x), p_\gamma(a)) \\
= &d(p_\gamma(x), p_{\gamma\gamma'}p_{\gamma'}(x)) + d(p_{\gamma\gamma'}p_{\gamma'}(a), p_\gamma(a)) \\
\leq &\eta/3 + \eta/3 < \eta.
\end{aligned}$$

So, $p_\gamma(a) \in B(p_\gamma(x), \eta) \subseteq U_\gamma$, and thus, $a \in p_\gamma^{-1}(U_\gamma) \subseteq U$. □

Proposition 3.2.2. *For any neighborhood U of A, there exists a $\gamma \in \Gamma$ such that for all $\gamma' \geq \gamma$, $p_{\gamma'}^{-1}(A_{\gamma'}) \subseteq U$.*

Proof. By 3.1.5, for each $a \in A$ there exists a basic open set of the form $p_{\gamma_a}^{-1}(V_{\gamma_a})$, where V_{γ_a} is open in X_{γ_a}, such that $a \in p_{\gamma_a}^{-1}(V_{\gamma_a}) \subseteq U$. Also, for each $a \in A$ there exists an $\eta_a > 0$ such that $B(p_{\gamma_a}(a), \eta_a) \subseteq V_{\gamma_a}$.

For each $a \in A$ let $W_a = p_{\gamma_a}^{-1}(B(p_{\gamma_a}(a), \eta_a/3))$. Then the collection $\{W_a | a \in A\}$ covers A. So, by the compactness of A, there exists a finite subset $A^* \subseteq A$ such that $\{W_a | a \in A^*\}$ covers A.

Using 3.1.6, choose a $\gamma \in \Gamma$ such that for all $a \in A^*$, $\gamma \geq \gamma_a$, and for all $\gamma' \geq \gamma$, we have that $d(p_{\gamma_a \gamma'} p_{\gamma'}, p_{\gamma_a}) \leq \eta_a/3$.

We will now show that for $\gamma' \geq \gamma$, $p_{\gamma'}^{-1}(A_{\gamma'}) \subseteq U$. To do this, let $x \in p_{\gamma'}^{-1}(A_{\gamma'})$. Then there exists an $a \in A$ such that $p_{\gamma'}(x) = p_{\gamma'}(a)$. Since $a \in A$, there exists $a^* \in A^*$ such that $a \in W_{a^*}$.

We now wish to show that $x \in p_{\gamma_{a^*}}^{-1}(V_{\gamma_{a^*}})$. The following inequalities hold by the choice of γ, and the fact that $a \in W_{a^*}$.

$$\begin{aligned} d(p_{\gamma_{a^*}\gamma'} p_{\gamma'}(x), p_{\gamma_{a^*}}(x)) &< \eta_{a^*}/3 \\ d(p_{\gamma_{a^*}\gamma'} p_{\gamma'}(a), p_{\gamma_{a^*}}(a)) &< \eta_{a^*}/3 \\ d(p_{\gamma_{a^*}}(a), p_{\gamma_{a^*}}(a^*)) &< \eta_{a^*}/3. \end{aligned}$$

And so, since $p_{\gamma_{a^*}\gamma'} p_{\gamma'}(x) = p_{\gamma_{a^*}\gamma'} p_{\gamma'}(a)$, the triangle inequality and the preceding three inequalities give us that $d(p_{\gamma_{a^*}}(x), p_{\gamma_{a^*}}(a^*)) < \eta_{a^*}$. Then, $p_{\gamma_{a^*}}(x) \in V_{\gamma_{a^*}}$. This implies that $x \in p_{\gamma_{a^*}}^{-1}(V_{\gamma_{a^*}}) \subseteq U$, and so $p_{\gamma'}^{-1}(A_{\gamma'}) \subseteq U$. $\square$

The following resolution property is well known for approximate inverse limits. Our goal now is to establish a similar property for a closed subspace A of an approximate inverse limit X.

Proposition 3.2.3. (Proposition 3.4 from [7]) *Let $\mathbf{X} = \{X_\gamma, \varepsilon_\gamma, p_{\gamma\gamma'}, \Gamma\}$ be an approximate system of compact metric spaces X_γ and $X = \lim \mathbf{X}$. Then for every map $h : X \to P$ into a compact polyhedron P, there exist a $\gamma \in \Gamma$ and a map $f : X_\gamma \to P$ such that $h \simeq f p_\gamma$.*

To accomplish this we will construct an approximate system having A as its approximate inverse limit.

Construction 3.2.4. Let $\mathbf{X} = \{X_\gamma, \varepsilon_\gamma, p_{\gamma\gamma'}, \Gamma\}$ be an approximate system with $(\Gamma, \leq)$ being an ordered set which is directed, has no maximal element and is cofinite. As before, $X = \lim \mathbf{X}$ and $A \subseteq X$ is closed.

For each $\gamma \in \Gamma$ our goal is to construct a closed neighborhood of A_γ with certain properties, and a number $\eta_\gamma > 0$. The neighborhood will be constructed inductively, and will be denoted $N(A_\gamma)$.

Let $\gamma' \in \Gamma$. If γ' has no predecessors, set $N(A_{\gamma'})$ to be the closed $\varepsilon_{\gamma'}$ neighborhood of $A_{\gamma'}$, and set $\eta_{\gamma'} = \varepsilon_{\gamma'}$.

Suppose that $N(A_{\gamma'})$ and $\eta_{\gamma'}$ have been constructed for all γ' with less than n predecessors.

Suppose now that γ' has n predecessors. Let $\gamma < \gamma'$ be a predecessor of γ'. Then $N(A_\gamma)$ has already been constructed. We will denote by $\gamma_1, \gamma_2, ...\gamma_k$ those predecessors of γ' for which $p_{\gamma\gamma'}(A_{\gamma'}) \subseteq \operatorname{int} N(A_\gamma)$. If no such predecessors exist, set $N(A_{\gamma'})$ equal to the closed $\varepsilon_{\gamma'}$ neighborhood of $A_{\gamma'}$, and $\eta_{\gamma'} = \varepsilon_{\gamma'}$. Otherwise, if $1 \leq k \leq n$, then there exists $\eta_{\gamma'} > 0$ such that the closed $\eta_{\gamma'}$ neighborhood of $A_{\gamma'}$ is contained in $\bigcap_{i=1}^{k} p_{\gamma_i\gamma'}^{-1}(N(A_{\gamma_i}))$ and $\eta_{\gamma'} \leq \varepsilon_{\gamma'}$. Set $N(A_{\gamma'})$ equal to the closed $\eta_{\gamma'}$ neighborhood of $A_{\gamma'}$. We note that $N(A_{\gamma'}) \subseteq \bigcap_{i=1}^{k} p_{\gamma_i\gamma'}^{-1}(N(A_{\gamma_i}))$.

Having completed the induction, we will construct a new relation $\leq'$ such that $\gamma <' \gamma'$ if and only if $\gamma < \gamma'$ and for every $\gamma'' \geq \gamma'$ we have that $d(p_{\gamma\gamma''}p_{\gamma''}, p_\gamma) < \eta_\gamma$. We will put $\gamma =' \gamma'$ if and only if $\gamma = \gamma'$.

Remark 3.2.5. Note that by our construction, if $\gamma \leq' \gamma'$ then $p_{\gamma\gamma'}(N(A_{\gamma'})) \subseteq N(A_\gamma)$. Indeed, suppose that $\gamma <' \gamma'$. Then $d(p_{\gamma\gamma'}p_{\gamma'}, p_\gamma) < \eta_\gamma$. So we have that $p_{\gamma\gamma'}(A_{\gamma'}) = p_{\gamma\gamma'}(p_{\gamma'}(A)) \subseteq N(p_\gamma(A)) = N(A_\gamma)$. Since $p_{\gamma\gamma'}(A_{\gamma'}) \subseteq N(A_\gamma)$, by the construction of $N(A_{\gamma'})$, we have $N(A_{\gamma'}) \subseteq p^{-1}_{\gamma\gamma'}(N(A_\gamma))$, and so $p_{\gamma\gamma'}(N(A_{\gamma'})) \subseteq N(A_\gamma)$.

Remark 3.2.6. Also note that if $\gamma <' \gamma'$ and $\gamma' \leq \gamma''$ then $\gamma <' \gamma''$.

Lemma 3.2.7. $(\Gamma, \leq')$ *as defined in 3.2.4 is a partially ordered set which is cofinite, has no maximal element and is directed.*

Proof. $(\Gamma, \leq')$ is obviously cofinite and partially ordered.

We will now show that $(\Gamma, \leq')$ has no maximal element. Let $\gamma \in \Gamma$. By Lemma 3.1.6 there exists a $\gamma' \geq \gamma$ such that for every $\gamma'' \geq \gamma'$ one has that $d(p_{\gamma\gamma''}p_{\gamma''}, p_\gamma) \leq \eta_\gamma$. Since $(\Gamma, \leq)$ has no maximal element, there exists a $\gamma'' > \gamma'$. Thus by our construction, $\gamma'' >' \gamma$.

Finally, we will show that $(\Gamma, \leq')$ is directed. Let γ_1, $\gamma_2 \in \Gamma$. Then since $(\Gamma, \leq')$ has no maximal element, there exists $\gamma_1' >' \gamma_1$ and $\gamma_2' >' \gamma_2$. Since $(\Gamma, \leq)$ is a directed set there exists $\gamma \in \Gamma$ such that $\gamma_1' \leq \gamma$ and $\gamma_2' \leq \gamma$. By Remark 3.2.6, $\gamma_1 <' \gamma$ and $\gamma_2 <' \gamma$. Thus, $(\Gamma, \leq')$ is a directed set. □

By Proposition 9 of [11] the following proposition is clear:

Proposition 3.2.8. *Let* $\mathbf{X} = \{X_\gamma, \varepsilon_\gamma, p_{\gamma\gamma'}, \Gamma\}$ *be an approximate system with* $(\Gamma, \leq)$ *being an ordered set which is directed, has no maximal element and is cofinite, and* $X = \lim \mathbf{X}$. *Let* $(\Gamma, \leq')$ *be as defined in 3.2.4,* $\mathbf{X}' = \{X_\gamma, \varepsilon_\gamma, p_{\gamma\gamma'}, \Gamma\}$ *be an approximate system using the ordered set* $(\Gamma, \leq')$*, and* $X' = \lim \mathbf{X}'$. *Then* $X = X'$.

Proposition 3.2.9. $\mathbf{A} = \{N(A_\gamma), \varepsilon_\gamma, p_{\gamma\gamma'}, (\Gamma\ \leq')\}$ *as defined in 3.2.4 is an approximate inverse system with* $\lim \mathbf{A} = A$.

Proof. One can easily check that $\mathbf{A}$ satisfies the axioms (A1), (A2), and (A3).

Let $A' = \lim \mathbf{A}$. We wish to show that $A' = A$.

Let $a \in A$. Of course, we wish to show that $a \in A'$. Clearly $a \in \prod_{\gamma\in\Gamma} N(A_\gamma)$. By condition (L), we need to show that for each $\gamma \in \Gamma$ and $\eta > 0$ there exists $\gamma_1 \geq' \gamma$ such that for all $\gamma'' \geq' \gamma_1$ we have that $d(p_\gamma(a), p_{\gamma\gamma''}p_{\gamma''}(a)) \leq \eta$.

Let $\gamma \in \Gamma$ and $\eta > 0$. Since $a \in A \subseteq X$, there exists $\gamma' \geq \gamma$ such that for all $\gamma'' \geq \gamma'$, $d(p_\gamma(a), p_{\gamma\gamma''}(p_{\gamma''}(a))) \leq \eta$. Choose γ_1 so that $\gamma_1 \geq' \gamma$ and $\gamma_1 \geq \gamma'$. So if $\gamma'' \geq' \gamma_1$, we have that $d(p_\gamma(a), p_{\gamma\gamma''}p_{\gamma''}(a)) \leq \eta$. And so, $a \in A'$.

Now let $a \in A'$. We will show that $a \in \bigcap_{\gamma\in\Gamma} p_\gamma^{-1}(A_\gamma)$, and so by Proposition 3.2.1 we will have $a \in A$.

Suppose on the contrary that $a \notin \bigcap_{\gamma\in\Gamma} p_\gamma^{-1}(A_\gamma)$. There there exists $\gamma \in \Gamma$ such that $p_\gamma(a) \notin A_\gamma$. So there exists $\eta > 0$ such that $B(p_\gamma(a), 2\eta) \cap A_\gamma = \emptyset$.

Choose γ' large enough so that it simultaneously satisfies condition (L) for the point a, γ and $\eta/3$, condition (A3) for the system $\mathbf{A}$, γ and $\eta/3$, and Lemma 3.1.6 for γ and $\eta/3$.

By condition (L) we have

$$d(p_\gamma(a), p_{\gamma\gamma'}p_{\gamma'}(a)) \leq \eta/3. \tag{3.1}$$

Since $N(A_{\gamma'})$ is an $\eta_{\gamma'}$-neighborhood of $A_{\gamma'}$, $p_{\gamma'}(a)$ is a distance of less than or equal to $\varepsilon_{\gamma'}$ from $A_{\gamma'}$. So there exists a point $b \in A$ such that $p_{\gamma'}(b) \in A_{\gamma'}$ and $d(p_{\gamma'}(a), p_{\gamma'}(b)) \leq \varepsilon_{\gamma'}$. Then by our application of condition (A3) we have that

$$d(p_{\gamma\gamma'}(p_{\gamma'}(a)), p_{\gamma\gamma'}(p_{\gamma'}(b))) \leq \eta/3. \tag{3.2}$$

By our application of Lemma 3.1.6 we have

$$d(p_{\gamma\gamma'}(p_{\gamma'}(b)), p_{\gamma}(b)) \leq \eta/3. \tag{3.3}$$

Finally, (3.1), (3.2), (3.3) and the triangle inequality give us

$$d(p_{\gamma}(a), p_{\gamma}(b)) \leq \eta. \tag{3.4}$$

Since $b \in A$, $p_{\gamma}(b) \in A_{\gamma}$. This contradicts the fact that $B(p_{\gamma}(a), 2\eta) \cap A_{\gamma} = \emptyset$, and so we conclude that $a \in \bigcap_{\gamma \in \Gamma} p_{\gamma}^{-1}(A_{\gamma})$, and so $a \in A$. $\square$

Applying Proposition 3.2.3 to the approximate inverse system $\mathbf{A}$ that we have constructed we arrive at the following:

Proposition 3.2.10. *Let $\mathbf{X} = \{X_{\gamma}, \varepsilon_{\gamma}, p_{\gamma\gamma'}, \Gamma\}$ be an approximate system with $(\Gamma, \leq)$ being an ordered set which is directed, has no maximal element and is cofinite. Let $X = \lim \mathbf{X}$ and $A \subseteq X$ be a closed subspace. Then for every map $h : A \to K$ into a compact polyhedron K, there exist a $\gamma \in \Gamma$ and a map $f : N(A_{\gamma}) \to K$ such that $h \simeq f p_{\gamma}|_{A}$.*

We will need the following lemma in the proof of the main theorem in the next section.

Lemma 3.2.11. *Let $\mathbf{X} = \{X_{\gamma}, \varepsilon_{\gamma}, p_{\gamma\gamma'}, \Gamma\}$ be an approximate system with $(\Gamma, \leq)$ being an ordered set which is directed, has no maximal element and is cofinite. Let $X = \lim \mathbf{X}$ and $A \subseteq X$ be a closed subspace. If $\alpha_{\gamma} : N(A_{\gamma}) \to K$*

is a map into a compact polyhedron K, then there exists $\gamma' \geq' \gamma$ such that for all $\gamma'' \geq' \gamma'$, we have $\alpha_\gamma \circ p_\gamma|_{p_{\gamma''}^{-1}(N(A_{\gamma''}))} \simeq \alpha_\gamma \circ p_{\gamma\gamma''} \circ p_{\gamma''}|_{p_{\gamma''}^{-1}(N(A_{\gamma''}))}$.

Proof. Using Theorem 1.1 of section IV of [4], it is easy to show that there exists a finite open cover $\mathcal{W} = \{W_i | i = 1, 2, 3, ...n\}$ of K such that if two maps to K defined on an arbitrary space are $\mathcal{W}$-close, they are homotopic.

Consider the collection $\{\alpha_\gamma^{-1}(W_i) | i = 1, 2, 3, ...n\}$. This is a collection of open sets which covers $N(A_\gamma)$. Let $\eta > 0$ be a Lebesgue number for this cover.

Choose $\gamma' \geq' \gamma$ so that γ' satisfies Lemma 3.1.6 for γ and $\eta/2$. Then for every $\gamma'' \geq' \gamma'$ one has $d(p_{\gamma\gamma''}p_{\gamma''}, p_\gamma) \leq \eta/2$.

Now let $\gamma'' \geq' \gamma'$. We will show that $\alpha_\gamma \circ p_\gamma|_{p_{\gamma''}^{-1}(N(A_{\gamma''}))}$ and $\alpha_\gamma \circ p_{\gamma\gamma''} \circ p_{\gamma''}|_{p_{\gamma''}^{-1}(N(A_{\gamma''}))}$ are $\mathcal{W}$-close.

Let $x \in p_{\gamma''}^{-1}(N(A_{\gamma''}))$. Since η is a Lebesgue number for $\{\alpha_\gamma^{-1}(W_i) | i = 1, 2, 3, ...n\}$, $B(p_\gamma(x), \eta) \subseteq \alpha_\gamma^{-1}(W_i)$ for some $1 \leq i \leq n$. Then since we know that $d(p_{\gamma\gamma''}p_{\gamma''}(x), p_\gamma(x)) \leq \eta/2$, we have that $p_{\gamma\gamma''}p_{\gamma''}(x), p_\gamma(x) \in \alpha_\gamma^{-1}(W_i)$. And so $\alpha_\gamma p_{\gamma\gamma''}p_{\gamma''}(x), \alpha_\gamma p_\gamma(x) \in W_i$. So $\alpha_\gamma \circ p_\gamma|_{p_{\gamma''}^{-1}(N(A_{\gamma''}))}$ and $\alpha_\gamma \circ p_{\gamma\gamma''} \circ p_{\gamma''}|_{p_{\gamma''}^{-1}(N(A_{\gamma''}))}$ are $\mathcal{W}$-close, and thus $\alpha_\gamma \circ p_\gamma|_{p_{\gamma''}^{-1}(N(A_{\gamma''}))} \simeq \alpha_\gamma \circ p_{\gamma\gamma''} \circ p_{\gamma''}|_{p_{\gamma''}^{-1}(N(A_{\gamma''}))}$. □

3.3 Limit Theorem for Approximate Inverse Systems with Cofinite Indexing Set

Theorem 3.3.1. *Let $\mathbf{Z} = \{Z_\gamma, \varepsilon_\gamma, p_{\gamma\gamma'}, \Gamma\}$ be an approximate system with $(\Gamma, \leq)$ being an ordered set which is directed, has no maximal element and is cofinite. Let X be a compact metric space, $Z = \lim \mathbf{Z}$ and $f : X \to Z$ a* *surjective map. If, for each $\gamma \in \Gamma$, $p_\gamma \circ f : X \to p_\gamma(Z)$ is an extensional map relative to a polyhedron K, then $f : X \to Z$ is an extensional map relative to K.*

Proof. Let $A \subseteq B$ be closed subsets of Z. Let $\alpha : A \to K$ be a map and suppose that $\alpha \circ f|_A : A \to K$ extends over $f^{-1}(B)$, say to $\tilde{\alpha} : B \to K$. Extend $\tilde{\alpha}$ over a neighborhood S of $f^{-1}(B)$. Continue to call the new extension $\tilde{\alpha}$. Since f is a closed map, there exists a neighborhood T of B such that $f^{-1}(T) \subseteq S$.

By Proposition 3.2.10, there exists $\gamma \in \Gamma$ such that there is a map $\alpha_\gamma : N(A_\gamma) \to K$ such that $\alpha_\gamma \circ p_\gamma|_A \simeq \alpha$. Since $p_\gamma^{-1}(N(A_\gamma))$ is a closed neighborhood of A, using the homotopy extension property, α extends over $p_\gamma^{-1}(N(A_\gamma)$ to a map $\alpha^* : p_\gamma^{-1}(N(A_\gamma) \to K$ such that (1) $\alpha_\gamma \circ p_\gamma|_{p_\gamma^{-1}(N(A_\gamma))} \simeq \alpha^*$. Note that we can choose γ large enough so that $p_\gamma^{-1}(N(A_\gamma) \subseteq T$.

(2) By Lemma 3.2.11 there exists $\gamma' \geq' \gamma$ such that for all $\gamma'' \geq' \gamma'$, we have $\alpha_\gamma \circ p_\gamma|_{p_{\gamma''}^{-1}(N(A_{\gamma''}))} \simeq \alpha_\gamma \circ p_{\gamma\gamma''} \circ p_{\gamma''}|_{p_{\gamma''}^{-1}(N(A_{\gamma''}))}$.

As in the proof of Theorem 2.2.3, we will show that there exists a neighborhood $V \subset p_\gamma^{-1}(N(A_\gamma))$ of A such that $\alpha^* \circ f|_{f^{-1}(V)} : f^{-1}(V) \to K$ extends over S.

First we note that $f^{-1}(p_\gamma^{-1}(N(A_\gamma)))$ is a neighborhood of $f^{-1}(A)$, and $\alpha^* \circ f|_{f^{-1}(p_\gamma^{-1}(N(A_\gamma)))} : f^{-1}(p_\gamma^{-1}(N(A_\gamma))) \to K$ is an extension of $\alpha \circ f|_{f^{-1}(A)}$ over $f^{-1}(p_\gamma^{-1}(N(A_\gamma)))$. So by Proposition 2.2.1, there exists a neighborhood W of $f^{-1}(A)$ such that $\alpha^* \circ f|_W$ extends over S. Since f is a closed map, there exists a neighborhood $V \subset p_\gamma^{-1}(N(A_\gamma))$ of A such that $f^{-1}(V) \subset W$. And so since $\alpha^* \circ f|_W$ extends over S, we have that $\alpha^* \circ f|_{f^{-1}(V)}$ extends over S.

Now choose $\gamma' \geq' \gamma$ such that $p_{\gamma'}^{-1}(A_{\gamma'}) \subseteq V$, $p_{\gamma'}^{-1}(B_{\gamma'}) \subseteq T$, and γ' satisfies (2).

Since $f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'})) \subseteq f^{-1}(V)$, and $\alpha^* \circ f|_{f^{-1}(V)}$ extends over S, we have that $\alpha^* \circ f|_{f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'}))}$ extends over S. By (1) we have, $\alpha^* \circ f|_{f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'}))} \simeq \alpha_\gamma \circ p_\gamma \circ f|_{f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'}))}$. Using this and (2) we have $\alpha^* \circ f|_{f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'}))} \simeq \alpha_\gamma \circ p_{\gamma\gamma'} \circ p_{\gamma'} \circ f|_{f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'}))}$. So, by the homotopy extension property, $\alpha_\gamma \circ p_{\gamma\gamma'} \circ p_{\gamma'} \circ f|_{f^{-1}(p_{\gamma'}^{-1}(A_{\gamma'}))}$ extends over S, and since $f^{-1}(p_{\gamma'}^{-1}(B_{\gamma'})) \subseteq f^{-1}(T) \subseteq S$, we have that it also extends over $f^{-1}(p_{\gamma'}^{-1}(B_{\gamma'}))$. By our assumption that $p_{\gamma'} \circ f$ is an extensional map, $\alpha_\gamma \circ p_{\gamma\gamma'}|_{A_{\gamma'}}$ extends over $B_{\gamma'}$, say to $\beta_{\gamma'} : B_{\gamma'} \to K$.

Then, $\beta_{\gamma'} \circ p_{\gamma'}$ is defined on all of B. So since $\beta_{\gamma'} \circ p_{\gamma'}|_A = \alpha_\gamma \circ p_{\gamma\gamma'} \circ p_{\gamma'}|_A \simeq \alpha_\gamma \circ p_\gamma|_A \simeq \alpha$, by the homotopy extension property, α extends over B. $\square$

3.4 Mardešić Trick for Approximate Inverse Systems

The well-known Mardešić trick provides a cofinite indexing set in the case of commutative inverse systems. We wish to extend this method to the case of approximate inverse systems. Construction 3.4.1 and the proof of 3.4.2 are due to [12] and correspondence with Vlasta Matijevic.

Construction 3.4.1. We will begin our construction with an approximate system $\mathbf{Z} = \{Z_a, \varepsilon_a, p_{aa'}, A\}$ with $(A, \leq)$ being an ordered set which is directed and has no maximal element.

Let $B = \{b \subseteq A | b \text{ is finite and } \max(b) \text{ exists}\}$. We will order B by inclusion. Define a function $t : B \to A$ by $t(b) = \max(b)$ for $b \in B$.

We claim that t is surjective and nondecreasing. To show surjectivity, let

$a \in A$. Then $t(\{a\}) = a$. To show that t is nondecreasing, let $b_1, b_2 \in B$ with $b_1 \leq b_2$. Then $b_1 \subseteq b_2$, and so $\max(b_1) \leq \max(b_2)$. So we have $t(b_1) \leq t(b_2)$.

We will now define a new approximate system $\mathbf{Y} = \{Y_b, \varepsilon_b, q_{bb'}, B\}$. $(B, \leq)$ is defined as above. We will set $Y_b = Z_{t(b)}$, $\varepsilon_b = \varepsilon_{t(b)}$, and $q_{bb'} = p_{t(b)t(b')}$.

Proposition 3.4.2. $\mathbf{Y}$ *as defined in 3.4.1 is an approximate inverse system.*

Proof. By Definition 3.1.1 we have to show that $\mathbf{Y}$ satisfies conditions (A1), (A2) and (A3).

(A1) Let $b_1 \leq b_2 \leq b_3$. We must show that $d(q_{b_1b_2}q_{b_2b_3}, q_{b_1b_3}) \leq \varepsilon_{b_1}$. Indeed, $\max(b_1) \leq \max(b_2) \leq \max(b_3)$, and so $t(b_1) \leq t(b_2) \leq t(b_3)$. By condition (A1) for the approximate system $\mathbf{Z}$, we have

$$d(p_{t(b_1)t(b_2)}p_{t(b_2)t(b_3)}, p_{t(b_1)t(b_3)}) \leq e_{t(b_1)}.$$

So clearly by our construction, $d(q_{b_1b_2}q_{b_2b_3}, q_{b_1b_3}) \leq \varepsilon_{b_1}$.

(A2) Let $b \in B$ and $\eta > 0$. By (A2) for $\mathbf{Z}$, we have that there exists $a' \geq t(b)$ such that for all $a_2 \geq a_1 \geq a'$ we have that $d(p_{t(b)a_1}p_{a_1a_2}, p_{t(b)a_2}) \leq \eta$. Put $b' = b \cup \{a'\}$. Note that $t(b') = a'$, and $b' \geq b$. Let $b_2 \geq b_1 \geq b'$. Then we have $t(b_2) \geq t(b_1) \geq t(b') = a'$. And so, $d(p_{t(b)t(b_1)}p_{t(b_1)t(b_2)}, p_{t(b)t(b_2)}) \leq \eta$. This gives us that $d(q_{bb_1}q_{b_1b_2}, q_{bb_2}) \leq \eta$, and proves property (A2) for $\mathbf{Z}$.

(A3) Let $b \in B$ and $\eta > 0$. By (A3) for $\mathbf{Z}$, there exists an $a' \geq t(b)$ such that for every $a'' \geq a'$, and for all $z, z' \in Z_{a''}$, we have that if $d(z, z') \leq \varepsilon_{a''}$ then $d(p_{t(b)a''}(z), p_{t(b)a''}(z')) \leq \eta$. Let $b' = b \cup \{a'\}$. Then $t(b') = a'$ and $b' \geq b$. Let $b'' \geq b'$. Then $t(b'') \geq t(b') = a'$. Let $y, y' \in Y_{b''} = Z_{t(b'')}$, and $d(y, y') \leq \varepsilon_{b''} = \varepsilon_{t(b'')}$. Then we have that $d(p_{t(b)t(b'')}(y), p_{t(b)t(b'')}(y')) \leq \eta$, and so $d(q_{bb''}(y), q_{bb''}(y')) \leq \eta$. $\square$

Proposition 3.4.3. *Let* $\mathbf{Z}$ *and* $\mathbf{Y}$ *be as in 3.4.1, and* $Z = \lim \mathbf{Z}$ *and* $Y = \lim \mathbf{Y}$*. Then the function* $h : Z \to \prod_{b \in B} Y_b$ *such that* $q_b(h(z)) = p_{t(b)}(z)$*, is continuous, injective, and* $h(Z) = Y$*, showing that* $Z \simeq Y$*.*

Proof. Clearly h is continuous.

To show h is injective, let $z_1, z_2 \in Z$, with $z_1 \neq z_2$. Then there exists $a \in A$ such that $p_a(z_1) \neq p_a(z_2)$. So, since $t(\{a\}) = a$, we have $q_{\{a\}}(h(z_1)) = p_a(z_1) \neq p_a(z_2) = q_{\{a\}}(h(z_2))$, and thus $h(z_1) \neq h(z_2)$.

We will now show that $h(Z) = Y$.

Let $z \in Z$. We need to show that $h(z) \in Y$. That is, we need to show that for each $b \in B$ and $\eta > 0$ there exists $b' \geq b$ such that for all $b'' \geq b'$ we have $d(q_b(h(z)), q_{bb''}q_{b''}(h(z))) \leq \eta$. To do this, let $b \in B$ and $\eta > 0$. We first note that $t(b) \in A$, and so since $z \in Z$, there exists $a' \geq t(b)$ such that for all $a'' \geq a'$ we have $d(p_{t(b)}(z), p_{t(b)a''}p_{a''}(z)) \leq \eta$. We will set $b' = b \cup \{a'\}$. Then $b' \in B$ and $b' \geq b$. Note that $t(b') = a'$.

Let $b'' \geq b'$. Then $t(b'') \geq t(b') = a'$. So then, $d(q_b(h(z)), q_{bb''}q_{b''}(h(z))) = d(p_{t(b)}(z), p_{t(b)t(b'')}p_{t(b'')}(z)) \leq \eta$. Thus, $h(z) \in Y$.

Let $y \in Y$. We are going to construct a point $z \in \prod_{a \in A} Z_a$ such that $h(z) = y$. For each $a \in A$, we will define z by $p_a(z) = q_{\{a\}}(y)$. We need to show two things. First, we need to show that $z \in Z$. That is, that z is actually in the limit. Second, we need to show that $h(z) = y$.

To accomplish these two things, we must first establish that if (1) $b_1, b_2 \in B$ with $t(b_1) = t(b_2)$, then $q_{b_1}(y) = q_{b_2}(y)$. Suppose on the contrary that $b_1, b_2 \in B$, $t(b_1) = t(b_2)$ but that $q_{b_1}(y) \neq q_{b_2}(y)$.

Note that $Y_{b_1} = Y_{b_2} = Z_{t(b_1)}$. Let $\eta = d(q_{b_1}(y), q_{b_2}(y))$. Since $y \in Y$, by condition (L), there exists $b \geq b_1, b_2$ such that for all $b' \geq b$ we have

$d(q_{b_1}(y), q_{b_1 b'} q_{b'}(y)) \leq \eta/3$ and $d(q_{b_2}(y), q_{b_2 b'} q_{b'}(y)) \leq \eta/3$. By our construction $q_{b_1 b'} = p_{t(b_1)t(b')} = p_{t(b_2)t(b')} = q_{b_2 b'}$.

So by the triangle inequality we have,

$$\begin{aligned}
& d(q_{b_1}(y), q_{b_2}(y)) \\
\leq & d(q_{b_1}(y), q_{b_1 b'} q_{b'}(y)) + d(q_{b_1 b'} q_{b'}(y), q_{b_2}(y)) \\
= & d(q_{b_1}(y), q_{b_1 b'} q_{b'}(y)) + d(q_{b_2 b'} q_{b'}(y), q_{b_2}(y)) \\
\leq & \eta/3 + \eta/3 \\
< & \eta.
\end{aligned}$$

This is a contradiction. So if $b_1, b_2 \in B$ with $t(b_1) = t(b_2)$, then $q_{b_1}(y) = q_{b_2}(y)$.

Now we wish to show that $z \in Z$. To do this we must show that condition (L) is satisfied. Let $a \in A$ and $\eta > 0$. We need to find an $a' \geq a$ such that for all $a'' \geq a'$ we have $d(p_{aa'} p_{a''}(z), p_a(z)) \leq \eta$. Since $y \in Y$ there exists $b' \in B$ such that $b' \geq \{a\}$ (that is, $b' \supseteq \{a\}$ and so $a \in b'$) such that for all $b'' \geq b'$ we have $d(q_{\{a\}b''} q_{b''}(y), q_{\{a\}}(y)) \leq \eta$.

We will set $a' = \max(b')$. Then $a' \geq a$. Let $a'' \geq a'$. Consider $b'' = b' \cup \{a''\}$. Then $b'' \geq b'$ and $t(b'') = a''$. So we have $p_{a''}(z) = q_{\{a''\}}(y) = q_{b''}(y)$ and $q_{\{a\}b''} = p_{aa''}$. Thus, $d(p_{aa''} p_{a''}(z), p_a(z)) = d(q_{\{a\}b''} q_{b''}(y), q_{\{a\}}(y)) \leq \eta$. And so $z \in Z$.

Now we will show that $h(z) = y$. Let $b \in B$. Then, $q_b(h(z)) = p_{t(b)}(z) = q_{\{t(b)\}}(y) = q_b(y)$. The last equality follows from the fact that $t(\{t(b)\}) = t(b)$ and the above argument. And so $h(z) = y$. Thus, h is surjective, and so is a homeomorphism. $\square$

3.5 Limit Theorem for Approximate Inverse Systems

Using the Mardešić trick we established in the previous section, we can drop the assumption of cofiniteness from the main result.

Theorem 3.5.1. *Let* $\mathbf{Z} = \{Z_a, \varepsilon_a, p_{aa'}, A\}$ *be an approximate system with* $(A, \leq)$ *being an ordered set which is directed and has no maximal element. Let* X *be a compact metric space,* $Z = \lim \mathbf{Z}$ *and* $f : X \to Z$ *a surjective map. If, for each* $a \in A$, $p_a \circ f : X \to p_a(Z)$ *is an extensional map relative to a polyhedron* K*, then* $f : X \to Z$ *is an extensional map relative to* K*.*

Proof. Using 3.4.1 construct a new system $\mathbf{Y} = \{Y_b, \varepsilon_b, q_{bb'}, B\}$ such that B is cofinite and $Z \cong Y = \lim \mathbf{Y}$. Let $h : Z \to Y$ be the homeomorphism described in the previous section.

For $b \in B$, denote the identity map from Y_b to $X_{t(b)}$ as i_b. Then $p_{t(b)} = i_b \circ p_b \circ h$. The map $p_{t(b)} \circ f : X \to Z_{t(b)}$ is an extensional map relative to K, and so $i_b \circ p_b \circ h \circ f : X \to Z_{t(b)}$ is an extensional map relative to K. By 1.3.2 and 1.3.4, we have that $p_b \circ h \circ f : X \to Y_b$ is an extensional map relative to K. By 3.3.1, we have that $h \circ f : X \to Y$ is an extensional map relative to K. Using 1.3.2 and 1.3.4 once again, we get that $f : X \to Z$ is an extensional map relative to K. $\square$

Chapter 4

Extensional Map Dimension

In this chapter our goal is to construct a notion of dimension for extensional maps similar to the extension dimension. First, we will define extension dimension. We will then establish two theorems that will be necessary to prove the existence of extensional map dimension. Finally, we will define extensional map dimension, and prove its existence.

4.1 Extension Dimension

We have seen, 1.2.4, that covering dimension can be defined in terms of extensions of maps in spheres. Also, while this paper will not discuss cohomologial dimension in any detail, a similar characterization exists for this type of dimension. One has $\dim_G X \leq n$ if and only if $X \tau K(G, n)$ where $K(G, n)$ is an Eilenberg MacLane complex. This led to the development by Dranishnikov and Dydak [2] of the idea of extension dimension.

The following construction comes from [6].

Construction 4.1.1. Let $\mathcal{C}$ be a class of spaces, $\mathcal{T}$ a class of CW-complexes, and K, $K' \in \mathcal{T}$. If it is true that for all $X \in \mathcal{C}$, $X\tau K$ implies $X\tau K'$, then we write $K \leq_{(\mathcal{C},\mathcal{T})} K'$. This defines a preorder on $\mathcal{T}$. We define an equivalence relation on $\mathcal{T}$ by $K \sim_{(\mathcal{C},\mathcal{T})} K'$ if and only if $K \leq_{(\mathcal{C},\mathcal{T})} K'$ and $K' \leq_{(\mathcal{C},\mathcal{T})} K$. The equivalence class of K under this relation is called the extension type of K relative to $(\mathcal{C}, \mathcal{T})$. By $\mathrm{ET}_{(\mathcal{C},\mathcal{T})}$ we mean the class of extension types relative to $(\mathcal{C}, \mathcal{T})$. The relation $\leq_{(\mathcal{C},\mathcal{T})}$ induces a partial order on the extension types $\mathrm{ET}_{(\mathcal{C},\mathcal{T})}$, which we will also denote $\leq_{(\mathcal{C},\mathcal{T})}$.

Definition 4.1.2. The **extension dimension** of X relative to $(\mathcal{C}, \mathcal{T})$, denoted $\operatorname{extdim}_{(\mathcal{C},\mathcal{T})} X$, is the initial element, if it exists, of the class of extension types $\left\{D \in \mathrm{ET}_{(\mathcal{C},\mathcal{T})} \,|X\tau L \text{ for all } L \in D\right\}$.

See [6] for several examples of classes $\mathcal{C}$ and $\mathcal{T}$ for which $\operatorname{extdim}_{(\mathcal{C},\mathcal{T})}$ exists.

4.2 The Ψ^∞ Operator

The following construction comes from [5].

Construction 4.2.1. For each simplicial complex K, let $\mathcal{F}_K$ be the collection of nonempty, finite subcomplexes of K. Fix a simplicial complex K, and let $M \in \mathcal{F}_K$. Let $\mathcal{D}_{(M,K)}$ be the set of $D \in \mathcal{F}_K$ such that $M \subseteq D$. We will define a relation $\sim_{(M,K)}$ on $\mathcal{D}_{(M,K)}$. For $D, C \in \mathcal{D}_{(M,K)}$, $D \sim_{(M,K)} C$ if there exists a simplicial isomorphism from D to C which is the identity on M. Clearly $\sim_{(M,K)}$ is an equivalence relation on $\mathcal{D}_{(M,K)}$. We denote the equivalence class of $D \in \mathcal{D}_{(M,K)}$ as $[D]_{(M,K)}$.

Let $\mathcal{E}_{(M,K)}$ be the set of equivalence classes of $\mathcal{D}_{(M,K)}$ under the relation $\sim_{(M,K)}$. The set $\mathcal{E}_{(M,K)}$ is countable. Using the Axiom of Choice, we will fix one representative from each equivalence class. That is, we will fix a function $\theta_{(M,K)} : \mathcal{E}_{(M,K)} \to \mathcal{D}_{(M,K)}$ so that $\theta_{(M,K)}([D]_{(M,K)}) \in [D]_{(M,K)}$ for each $D \in \mathcal{D}_{(M,K)}$.

Assume that the preceding construction has been applied to each $M \in \mathcal{F}_K$.

For $M \in \mathcal{F}_K$ and $E \subseteq \mathcal{E}_{(M,K)}$, $\theta_{(M,K)}(E)$ is a subcomplex of K containing M. Thus, (i) for all $M \in \mathcal{F}_K$, $\bigcup \theta_{(M,K)}(\mathcal{E}_{(M,K)})$ is a subcomplex of K containing the subcomplex M.

We now define a function Ψ from the set of subcomplexes L of K to the set of subcomplexes of K by

$$\Psi(L) = \bigcup \left\{ \bigcup \theta_{(M,K)}(\mathcal{E}_{(M,K)}) | M \in \mathcal{F}_L \right\}.$$

By (i), for each pair of subcomplexes $L \subseteq L'$ of K,

(ii) $\Psi(L)$ is a subcomplex of K, and $L \subseteq \Psi(L)$, and

(iii) $\Psi(L) \subseteq \Psi(L')$.

We will define Ψ^∞ inductively. Let $\Psi^0(L) = L$. For each $k \in \mathbb{N}$, if $\Psi^{k-1}(L)$ has been defined, then by $\Psi^k(L)$ we mean $\Psi(\Psi^{k-1}(L))$. Put,

$$\Psi^\infty(L) = \bigcup \left\{ \Psi^k(L) | k \in \mathbb{N} \right\}.$$

Then, $\Psi^\infty(L)$ is a subcomplex of K.

Lemma 4.2.2. (Lemma 3.1 from [5]) *Let K be a simplicial complex and L a subcomplex of K. Then $\Psi^1(\Psi^\infty(L)) = \Psi^\infty(L)$, and hence $\Psi^\infty(\Psi^\infty(L)) = \Psi^\infty(L)$.*

Lemma 4.2.3. (Lemma 3.2 from [5]) *Let K be a simplicial complex and L a subcomplex of K. If*

1. $L = \emptyset$, *then* $\Psi^1(L) = \emptyset$,

2. L *is finite, then* $\mathrm{card}(\Psi^1(L)) \leq \aleph_0$,

3. L *is infinite and* $k \in \mathbb{N}$, *then we may conclude that*

   $card(\Psi^k(L)) = \mathrm{card}(\Psi^\infty(L)) = \mathrm{card}(L)$.

Proposition 4.2.4. (Corollary 4.5 from [5]) *Let K be a simplicial complex and X a Hausdorff σ-compactum with $X\tau|K|$. Then for every subcomplex L of K, $X\tau|\Psi^\infty(L)|$.*

Proposition 4.2.5. *Let X and Z be compact Hausdorff spaces, K a simplicial complex, and $f : X \to Z$ an extensional map relative to $|K|$. Then for any nonempty subcomplex $L \subseteq K$, $f : X \to Z$ is an extensional map relative to $|\Psi^\infty(L)|$.*

Proof. Fix a subcomplex L of K. Let $A \subseteq B$ be closed in Z, and $\alpha : A \to |\Psi^\infty(L)|$ a map. Suppose that $\alpha \circ f|_{f^{-1}(A)} : f^{-1}(A) \to |\Psi^\infty(L)|$ extends over $f^{-1}(B)$, say to $\tilde{\alpha} : f^{-1}(B) \to |\Psi^\infty(L)|$. We wish to extend α over B.

Treat α and $\tilde{\alpha}$ as maps to $|K|$. Then, since f is an extensional map relative to $|K|$, there exists a map $\beta : B \to |K|$ such that $\beta|_A = \alpha$.

There exists $M \in \mathcal{F}_{\Psi^\infty(L)}$ such that $\beta(A) \subseteq |M|$. Now $\beta(B) \subseteq |M'|$ for some finite subcomplex M' of K, where $M \subseteq M'$. By the definition of $\Psi^1(\Psi^\infty(L))$, we may as well assume that $M' \subseteq \Psi^1(\Psi^\infty(L)) = \Psi^\infty(L)$. Hence, $\beta(B) \subseteq |\Psi^\infty(L)|$, and so we may treat β as a map to $|\Psi^\infty(L)|$. $\square$

4.3 Wedge Theorem

To achieve a similar theorem for wedges, we will have to make use of a strengthened version of extensional maps.

Definition 4.3.1. A surjective map $f : X \to Y$ is a *strong extensional map* relative to K if for each pair $A \subset B$ of closed subsets of Z and any map $\alpha : A \to K$ for which $\alpha \circ f : f^{-1}(A) \to K$ extends to a map $\tilde{\alpha} : f^{-1}(B) \to K$, there exists a map $\beta : B \to K$ such that $\beta \circ f = \tilde{\alpha} : f^{-1}(B) \to K$.

Theorem 4.3.2. *Let X and Z be compact Hausdorff spaces. Let $f : X \to Z$ be a strong extensional map relative to an arbitrary wedge of intervals. Let $\{K_\mu | \mu \in \Gamma\}$ be a collection of nonempty simplicial complexes. Put $K = \bigvee_v \{K_\mu | \mu \in \Gamma\}$, where say v is a vertex common to K_μ for all $\mu \in \Gamma$. Suppose that for each $\mu \in \Gamma$, f is an extensional map relative to $|K_\mu|$. Then f is an extensional map relative to $|K|$.*

Proof. Select a regular neighborhood vC of v which is the cone from v to a closed subpolyhedron C of $|K|$. Choose C' in such a way that $C' \cong C$, vC' is also a cone neighborhood of v and $vC \setminus \operatorname{int} vC' \cong C \times I$ with say $C \times \{0\}$ corresponding to C' and $C \times \{1\}$ corresponding to C.

For each $\mu \in \Gamma$, let $C_\mu = C \cap |K_\mu|$ and $C'_\mu = C' \cap |K_\mu|$. The preceding construction is to be done so that $vC'_\mu \subset vC_\mu$ are cone neighborhoods of v in $|K_\mu|$, and that $vC_\mu \setminus \operatorname{int} vC'_\mu \cong C_\mu \times I$ via restriction of the preceding identification.

For each $\mu \in \Gamma$, let $I_\mu = [0, 1]$ and put $L = \bigvee_0 \{I_\mu | \mu \in \Gamma\}$. We fix a map $g : |K| \to |L|$ so that for each $\mu \in \Gamma$,

1. $g(|K_\mu|) \subset I_\mu$;

2. $g(C'_\mu) = \{\frac{1}{2}\}$, $g(vC'_\mu) \subset [0, \frac{1}{2}]$;

3. $g(|K_\mu| \setminus vC_\mu) = 1$; and

4. $g(vC_\mu \setminus \operatorname{int} vC'_\mu) \subset [\frac{1}{2}, 1]$.

The existence of such a map can be seen readily from the preceding description of the cone neighborhood of v. Let $A \subset B$ be closed subsets of Z and $\alpha : A \to |K|$ a map. Suppose that $\alpha \circ f|_{f^{-1}(A)} : f^{-1}(A) \to |K|$ extends over $f^{-1}(B)$, say to $\tilde{\alpha} : f^{-1}(B) \to |K|$. We wish to show that α extends over B. Note that since A is compact, we can assume $|K|$ to be finite.

Consider $g \circ \alpha : A \to |L|$. Then $g \circ \alpha \circ f|_{f^{-1}(A)} : f^{-1}(A) \to |L|$ extends over $f^{-1}(B)$ to $g \circ \tilde{\alpha} : f^{-1}(B) \to |L|$. Since f is a strong extensional map relative to any wedge of intervals, $g \circ \alpha$ extends over B, say to $\beta : B \to |L|$, in such a way that $\beta \circ f = g \circ \tilde{\alpha} : f^{-1}(B) \to |L|$.

Let $D \subset |L|$ be the cone from the vertex 0 having the property that $D \cap I_\mu = [0, \frac{1}{2}]$ for each μ. Note that $g^{-1}(D) = vC'$. Put $B_0 = \beta^{-1}(D)$ and $A_0 = A \cap \beta^{-1}(D)$.

Surely $\alpha(A_0) \subset g^{-1}(D) = vC'$. As a contractible polyhedron, $vC' \subset |K|$ is a retract of $|K|$. Let $r_1 : |K| \to vC'$ be a retraction. Then it is easily seen that $r_1 \circ \tilde{\alpha}|_{f^{-1}(B_0)} : f^{-1}(B_0) \to vC'$ is an extension of $\alpha \circ f|_{f^{-1}(A_0)} : f^{-1}(A_0) \to vC'$. Thus, since f is an extensional map relative to contractible polyhedra, $\alpha|_{A_0}$ extends over B_0 to a map $\alpha_0 : B_0 \to vC'$.

For each $\mu \in \Gamma$, let $B_\mu = \beta^{-1}([\frac{1}{2}, 1])$, $[\frac{1}{2}, 1] \subset I_\mu$. Then $\{B_\mu | \mu \in \Gamma\}$ is a discrete collection of closed subspaces of B. Write $A_\mu = B_\mu \cap A$. Then $A = A_0 \cup \bigcup\{A_\mu | \mu \in \Gamma\}$. Also, $B = B_0 \cup \bigcup\{B_\mu | \mu \in \Gamma\}$.

One checks that $\alpha(A_\mu) \subset |K_\mu| \cup vC'$, and the latter is homotopy equivalent to $|K_\mu|$. Put $\alpha_\mu : B_0 \cup A_\mu \to |K_\mu| \cup vC'$ equal to α_0 on B_0 and α on A_μ.

Now consider $f^{-1}(B_0 \cup A_\mu)$. We wish to extend $\alpha_\mu \circ f|_{f^{-1}(B_0 \cup A_\mu)}$ over $f^{-1}(B_0 \cup B_\mu)$. The two maps $\tilde{\alpha}|_{f^{-1}(B_0 \cup A_\mu)}$ and $\alpha_\mu \circ f|_{f^{-1}(B_0 \cup A_\mu)}$ are defined on $f^{-1}(B_0 \cup A_\mu)$. We will show that they are homotopic on $f^{-1}(B_0 \cup A_\mu)$.

One can check that $\tilde{\alpha}(f^{-1}(B_0)) \subset vC'$ and $(\alpha_\mu \circ f)(f^{-1}(B_0)) \subset vC'$. For $x \in f^{-1}(A_\mu)$ we have $\tilde{\alpha}(x) = \alpha_\mu \circ f(x)$. We wish to construct a map $\hat{H} : (B_0 \cup A_\mu) \times I \to |K_\mu| \cup vC'$, such that $\hat{H}(x,0) = \alpha_\mu \circ f(x)$ and $\hat{H}(x,1) = \tilde{\alpha}(x)$. Consider the closed subpace

$$E = (B_0 \times \{0\}) \cup ((B_0 \cap A_\mu) \times I) \cup (B_0 \times \{1\}) \subseteq B_0 \times I.$$

Define a map $h : E \to vC'$ by,

$$\begin{aligned} h(x,0) &= (\alpha_\mu \circ f)(x) && x \in B_0 \\ h(x,t) &= (\alpha_\mu \circ f)(x) = \tilde{\alpha}(x) && x \in B_0 \cap A_\mu; \quad 0 \le t \le 1 \\ h(x,1) &= \tilde{\alpha}(x) && x \in B_0. \end{aligned}$$

Since vC' is an absolute extensor for $B_0 \times I$ we have that h extends over $B_0 \times I$, say to a map $H : B_0 \times I \to vC'$. We will now define a map $\hat{H} : (B_0 \cup A_\mu) \times I \to |K_\mu| \cup vC'$ by,

$$\begin{aligned} \hat{H}(x,t) &= H(x,t) && x \in B_0 \quad 0 \le t \le 1 \\ \hat{H}(x,t) &= (\alpha_\mu \circ f)(x) = \tilde{\alpha}(x) && x \in A_\mu \quad 0 \le t \le 1. \end{aligned}$$

So, $\tilde{\alpha}|_{f^{-1}(B_0 \cup A_\mu)}$ and $\alpha_\mu \circ f|_{f^{-1}(B_0 \cup A_\mu)}$ are homotopic on $f^{-1}(B_0 \cup A_\mu)$.

Since $\tilde{\alpha} : f^{-1}(B_0 \cup B_\mu) \to |K|$ is defined on all of $f^{-1}(B_0 \cup B_\mu)$, by the homotopy extension property we can extend $\alpha_\mu \circ f|_{f^{-1}(B_0 \cup A_\mu)}$ over $f^{-1}(B_0 \cup B_\mu)$, say to $\tilde{\alpha}_\mu : f^{-1}(B_0 \cup B_\mu) \to |K|$.

Since f is an extensional map relative to $|K_\mu|$, and $|K_\mu|$ is homotopy equivalent to $|K_\mu| \cup vC'$, by Lemma 1.3.8, f is an extensional map relative to $|K_\mu| \cup vC'$. Let $r_2 : |K| \to |K_\mu| \cup vC'$ be a retraction. Then $r_2 \tilde{\alpha_\mu}|_{f^{-1}(B_0 \cup B_\mu)} : f^{-1}(B_0 \cup B_\mu) \to |K_\mu| \cup vC'$ is an extension of $\alpha_\mu f|_{f^{-1}(B_0 \cup A_\mu)}$. Thus, there is an extension of α_μ over $B_0 \cup B_\mu$, call it $\alpha^*_\mu : B_0 \cup B_\mu \to |K_\mu| \cup vC'$. For $a \in B_0 \cup A_\mu$, $\alpha^*_\mu(a) = r_2(\alpha_\mu(a)) = \alpha_\mu(a)$.

Finally, we define $\alpha^* : B \to |K|$ to be $\bigcup\{\alpha^*_\mu | \mu \in \Gamma\}$. The map α^* is the desired extension of α. $\square$

4.4 Extensional Map Dimension: Definition and Existence

Now that we have established the Ψ^∞ operator and a wedge theorem, we will define the extensional map dimension, and prove its existence. The construction parallels the construction of the extension dimension.

Construction 4.4.1. Let $\mathcal{C}$ be a class of spaces, $\mathcal{T}$ a class of CW-complexes, and K, $K' \in \mathcal{T}$. If it is true that for all $X, Z \in \mathcal{C}$ and all maps $f : X \to Z$, f is an extensional map relative to K implies f is an extensional map relative to K', then we write $K \leq_{(\mathcal{C},\mathcal{T})} K'$. This defines a preorder on $\mathcal{T}$. We define an equivalence relation on $\mathcal{T}$ by $K \sim_{(\mathcal{C},\mathcal{T})} K'$ if and only if $K \leq_{(\mathcal{C},\mathcal{T})} K'$ and $K' \leq_{(\mathcal{C},\mathcal{T})} K$. The equivalence class of K under this relation is called the extensional map type of K relative to $(\mathcal{C}, \mathcal{T})$. By $\mathrm{EMT}_{(\mathcal{C},\mathcal{T})}$ we mean the class of extension types relative to $(\mathcal{C},\mathcal{T})$. The relation $\leq_{(\mathcal{C},\mathcal{T})}$ induces a partial order on the extensional map types $\mathrm{EMT}_{(\mathcal{C},\mathcal{T})}$, which we will also denote $\leq_{(\mathcal{C},\mathcal{T})}$.

Definition 4.4.2. The **extensional map dimension** of $f : X \to Z$ relative to $(\mathcal{C},\mathcal{T})$, denoted $\text{extmap-dim}_{(\mathcal{C},\mathcal{T})} X$, is the initial element, if it exists, of the class of extensional map types $\{D \in \mathrm{EMT}_{(\mathcal{C},\mathcal{T})} \,|\, f : X \to Z$ is an extensional map relative to L for all $L \in D\}$.

Theorem 4.4.3. *Let $\mathcal{T}$ be the class of polyhedra, and $\mathcal{C}$ be the class of compact Hausdorff spaces. If $X, Z \in \mathcal{C}$ and $f : X \to Z$ is a surjective map, which is a strong extensional map relative to an arbitrary wedge of intervals, then $\text{extmap-dim}_{(\mathcal{C},\mathcal{T})} f$ exists.*

Proof. Choose a collection $\mathcal{U}$ of polyhedra $|M|$, each M having cardinality $\leq \aleph_0$, so that $\mathcal{U}$ has the property that if L is a simplicial complex with $\operatorname{card} L \leq \aleph_0$ then for some $|M| \in \mathcal{U}$, L is simplicially isomorphic to M, and if $|M|, |N| \in \mathcal{U}$ with M simplicially isomorphic to N, then $M = N$. Then $\operatorname{card} \mathcal{U} \leq 2^{\aleph_0}$. We may assume that there is a fixed 0-simplex v such that for each $|M| \in \mathcal{U}$, $v \in M$.

Let $K = \bigvee_v \{M \,|\, |M| \in \mathcal{U}$ and f is an extensional map relative to $|M|\}$. Since $\operatorname{card} \mathcal{U} \leq 2^{\aleph_0}$, the number of summands in K is at most $2^{\aleph_0}$. By 4.3.2, f is an extensional map relative to $|K|$. We claim that $|K|$ is a representative of $\text{extmap-dim}_{(\mathcal{C},\mathcal{T})} f$.

Let $|L| \in \mathcal{T}$ and f be an extensional map relative to $|L|$. We must show that $|K| \leq |L|$. That is, we must show that if $g : W \to Y$ is an extensional map relative to $|K|$, where $W, Y \in \mathcal{C}$, then g is an extensional map relative to $|L|$.

Let $A \subseteq B$ be closed subsets of Y and $\alpha : A \to |L|$ be a map. Assume that $\alpha \circ g|_{g^{-1}(A)} : g^{-1}(A) \to |L|$ extends over $g^{-1}(B)$, say to $\tilde{\alpha} : g^{-1}(B) \to |L|$. We wish to extend α over B. We note that $\alpha(A) \subseteq \tilde{\alpha}(g^{-1}(B)) \subseteq |L_0|$, where L_0 is a finite subcomplex of L.

By 4.2.5, since f is an extensional map relative to $|L|$, it is also an extensional map relative to $|\Psi^\infty(L_0)|$. By 4.2.3 (2), $\operatorname{card} \Psi^\infty(L_0) \leq \aleph_0$. By our construction of $\mathcal{U}$ we can assume that $|\Psi^\infty(L_0)| \in \mathcal{U}$, and thus is a summand of $|K|$. Thus, by 1.3.7, g is an extensional map relative to $|\Psi^\infty(L_0)|$. So there exists an extension of α over B, say $\beta : B \to |\Psi^\infty(L_0)| \subseteq |L|$, proving that g is an extensional map relative to $|L|$. $\square$

Chapter 5

Extensions into Neighborhoods of the Limit

5.1 Inverse Sequences

Theorem 5.1.1. *Let* $\mathbf{Z} = \{Z_i, p_i^{i+1}\}$ *be an inverse sequence of compact metric spaces with surjective bonding maps and limit* Z*. For a space* X*, if* $X\tau Z_i$ *for each* $i \in \mathbb{N}$*, then for any neighborhood* U *of* Z *in the topological product* $\prod_1^\infty Z_i$*, for any closed subspace* A *of* X*, and for any map* $f : A \to Z$*, there exists a map* $F : X \to U$ *such that* $F|_A = f$*.*

Proof. Fix a neighborhood U of Z in the product $\prod_1^\infty Z_i$, a closed subspace A of X, and a map $f : A \to Z$. For each $i \in \mathbb{N}$ let p_i denote the projection map in Z_i.

Then $p_i f : A \to Z_i$ is a map, and so by our assumption that $X\tau Z_i$, the map $p_i f$ extends over X, say to $g_i : X \to Z_i$.

Set

$$G_1 = (g_1, g_2, g_3, ...).$$

Then $G_1 : X \to \prod_1^\infty Z_i$ is a map. We will now construct a sequence of maps from X to $\prod_1^\infty Z_i$. Set

$$G_2 = (p_1^2 g_2, g_2, g_3, g_4, ...)$$

$$G_3 = (p_1^3 g_3, p_2^3 g_3, g_3, g_4, ...)$$

$$...$$

$$G_i = (p_1^i g_i, p_2^i g_i, ..., p_{i-1}^i g_i, g_i, g_{i+1}, ...).$$

Clearly for each $i \in \mathbb{N}$, $G_i : X \to \prod_1^\infty Z_i$ is a map. Moreover, $G_i|_A = f$. Indeed, on A,

$$\begin{aligned} G_i &= (p_1^i g_i, p_2^i g_i, ..., p_{i-1}^i g_i, g_i, g_{i+1}, ...) \\ &= (p_1^i p_i g, p_2^i p_i g, ..., p_{i-1}^i p_i g, p_i g, p_{i+1} g, ...) \\ &= (p_1 g, p_2 g, p_3 g, ..., p_i g, p_{i+1} g, ...) = f. \end{aligned}$$

We claim that there exists an $n \in \mathbb{N}$ such that $G_n(X) \subseteq U$. Since Z is compact, there exists a finite collection $\{U_i | 1 \leq i \leq m\}$ of basic open sets in the product topology that covers Z, and whose union is contained in U. Since U_i is a basic open set, it is of the form $\prod_1^\infty V_j$ where $V_j \neq Z_j$ only finitely many times. Let $n_i \in \mathbb{N}$ be a number so that $V_j = Z_j$ for all $j \geq n_i$. Let $n = \max\{n_i | 1 \leq i \leq m\}$.

We now wish to show that $G_n(X) \subseteq U$. Let $x \in X$. Since each p_i^{i+1} is surjective, $Z_i = p_i(Z)$. Since $g_n(X) \subseteq Z_n$, there exists some $z \in Z$ such that

$p_n(z) = g_n(x)$. The collection $\{U_i | 1 \le i \le m\}$ covers Z, and so there exists a k such that $z \in U_k$. We wish to show that $G_n(x) \in U_k \subseteq U$. Because of our choice of n, $U_k = \prod_1^\infty V_j$ where $V_j = Z_j$ for all $j \ge n$. Since $z \in U_k$, we have that $p_j(z) \in V_j$ for all $j \in \mathbb{N}$. We will now show that $G_n(x) \in U_k$. For $j \le n$, $p_j(G_n(x)) = (p_j^n g_n)(x) = p_j^n(p_n(z)) = p_j(z)$. And so, for $j \le n$, $p_j(G_n(x)) \in V_j$. For $j \ge n$, $V_j = Z_j$, and so of course $p_j(G_n(x)) \in V_j$. We have shown that $G_n(x) \in U_k \subseteq U$. We set $F = G_n$ to conclude the proof. $\square$

5.2 Inverse Systems

Theorem 5.2.1. *Let* $\mathbf{Z} = \{Z_\gamma, p_{\gamma\gamma'}, \Gamma\}$ *be an inverse system of spaces with surjective bonding maps and compact limit* Z*. For a space* X*, if* $X \tau Z_\gamma$ *for each* $\gamma \in \Gamma$*, then for any neighborhood* U *of* Z *in the topological product* $\prod_{\gamma \in \Gamma} Z_\gamma$*, for any closed subspace* A *of* X*, and for any map* $f : A \to Z$*, there exists a map* $F : X \to U$ *such that* $F|_A = f$*.*

Proof. Fix a neighborhood U of Z in the product $\prod_{\gamma \in \Gamma} Z_i$, a closed subspace A of X, and a map $f : A \to Z$. For each $\gamma \in \Gamma$, let p_γ denote the projection map to Z_γ.

Then, $p_\gamma f : A \to Z_\gamma$ is a map, and so by our assumption that $X \tau Z_\gamma$, the map $p_\gamma f$ extends over X, say to $g_\gamma : X \to Z_\gamma$.

For each $\gamma_0 \in \Gamma$, we will now construct a map from X to $\prod_{\gamma \in \Gamma} Z_\gamma$. Let $G_{\gamma_0} : X \to \prod_{\gamma \in \Gamma} Z_\gamma$ be defined as follows. Let $x \in X$. If $\gamma \le \gamma_0$, then $p_\gamma(G_{\gamma_0}(x)) = p_{\gamma\gamma_0}(g_{\gamma_0}(x))$. Otherwise, if $\gamma \not\le \gamma_0$, set $p_\gamma(G_{\gamma_0}(x)) = g_\gamma(x)$. Clearly, for each $\gamma_0 \in \Gamma$, $G_{\gamma_0} : X \to \prod_{\gamma \in \Gamma} Z_\gamma$ is a map.

Moreover, $G_{\gamma_0}|_A = f$. Indeed, for $x \in A$, and $\gamma \le \gamma_0$, $p_\gamma(G_{\gamma_0}(x)) =$

$p_{\gamma\gamma_0}(g_{\gamma_0}(x)) = p_{\gamma\gamma_0}(p_{\gamma_0}f(x)) = p_\gamma(f(x))$. If $\gamma \nleq \gamma_0$, then $p_\gamma(G_{\gamma_0}(x)) = g_\gamma(x) = p_{\gamma_0}(f(x))$.

We claim that there exists a $\gamma' \in \Gamma$ such that $G_{\gamma'} \subseteq U$.

Since Z is compact, there exists a finite collection $\{U_i | 1 \leq i \leq m\}$ of basic open sets in the product topology which covers Z, and whose union is contained in U. Since U_i is a basic open set, it is of the form $\prod_{\gamma\in\Gamma} V_\gamma$ where $V_\gamma \neq Z_\gamma$ only finitely many times. Let $\Gamma_i = \{\gamma \in \Gamma | V_\gamma \neq Z_\gamma\}$. Then $\bigcup_{i=1}^m \Gamma_i$ is a finite set. Pick $\gamma' \in \Gamma$ such that $\gamma' > \gamma$ for all $\gamma \in \bigcup_{i=1}^m \Gamma_i$.

We now wish to show that $G_{\gamma'}(X) \subseteq U$. Let $x \in X$. Since each $p_{\gamma\gamma'}$ is surjective, $Z_\gamma = p_\gamma(Z)$ for all $\gamma \in \Gamma$. So, since $g_{\gamma'}(X) \subseteq p_{\gamma'}(Z)$, there exists some $z \in Z$ such that $p_{\gamma'}(z) = g_{\gamma'}(x)$. The collection $\{U_i | 1 \leq i \leq m\}$ covers Z, so there exists a k such that $z \in U_k$. Because of our choice of γ', $U_k = \prod_{\gamma\in\Gamma} V_\gamma$ where $V_\gamma = Z_\gamma$ for all γ such that $\gamma \nleq \gamma'$. Since $z \in U_k$, we have $p_\gamma(z) \in V_\gamma$ for all $\gamma \in \Gamma$. We will now show that $G_{\gamma'}(x) \in U_k$. For $\gamma \leq \gamma'$, $p_\gamma(G_{\gamma'}(x)) = p_{\gamma\gamma'}(g_{\gamma'}(x)) = p_{\gamma\gamma'}(p_{\gamma'}(z)) = p_\gamma(z)$. And so, for $\gamma \leq \gamma'$, $p_\gamma(G_{\gamma'}(x)) \in V_\gamma$. If $\gamma \nleq \gamma'$, $V_\gamma = Z_\gamma$, and so of course $p_\gamma(G_{\gamma'}(x)) \in V_\gamma$. We have shown that $G_{\gamma'}(x) \in U_k \subseteq U$. We set $F = G_{\gamma'}$ to conclude the proof. $\square$

5.3 Approximate Inverse Systems

Theorem 5.3.1. *Let $\mathbf{Z} = \{Z_\gamma, \varepsilon_\gamma, p_{\gamma\gamma'}, \Gamma\}$ be an approximate inverse system of metric compacta with surjective bonding maps and limit Z. For a space X, if $X\tau Z_\gamma$ for each $\gamma \in \Gamma$, then for any neighborhood U of Z in the topological product $\prod_{\gamma\in\Gamma} Z_\gamma$, for any closed subspace A of X, for any map $f : A \to Z$, and for any $\delta > 0$ there exists a map (i) $F : X \to U$ such that (ii)*

$d(p_\gamma F(x), p_\gamma f(x)) < \delta$ *for all* $x \in A$ *and* $\gamma \in \Gamma$.

Proof. Fix a neighborhood U of Z in the product $\prod_{\gamma\in\Gamma} Z_i$, a closed subspace A of X, a map $f : A \to Z$, and $\delta > 0$. For each $\gamma \in \Gamma$, let p_γ denote the projection map to Z_γ. We will construct a map $F : X \to U$ such that $d(p_\gamma F(x), p_\gamma f(x)) < \delta$ for all $x \in A$ and $\gamma \in \Gamma$.

For each $z \in Z$, let U_z be a basic open set in the product topology such that $z \in U_z \subseteq U$. Since U_z is a basic open set, it is of the form $\prod_{\gamma\in\Gamma} V_\gamma$ where each V_γ is open in Z_γ and $V_\gamma \neq Z_\gamma$ only finitely many times. Let $\Gamma_z = \{\gamma \in \Gamma | V_\gamma \neq Z_\gamma\}$. For each $\gamma \in \Gamma_z$, since $p_\gamma(z) \in V_\gamma$, there exists a $\delta(z,\gamma) > 0$ such that $B(p_\gamma(z), \delta(z,\gamma)) \subseteq V_\gamma$.

Set

$$U_z^* = \left(\prod_{\gamma\in\Gamma_z} B\left(p_\gamma(z), \frac{\delta(z,\gamma)}{3}\right)\right) \times \left(\prod_{\gamma\in\Gamma\setminus\Gamma_z} Z_\gamma\right).$$

Then $z \in U_z^* \subseteq U_z \subseteq U$.

The collection $\{U_z^* | z \in Z\}$ covers Z. So since Z is compact there exists a finite set of points $\{z_1, z_2, ...z_m\}$ such that $\{U_{z_i}^* | 1 \le i \le m\}$ covers Z. Set $\delta^* = \min\left(\min\{\frac{\delta(z_i,\gamma)}{3} | \ 1 \le i \le m\}, \delta\right)$.

The set $\Gamma_0 = \bigcup_{i=1}^m \Gamma_{z_i}$ is finite. Using 3.1.6, pick $\gamma' \in \Gamma$ such that $\gamma' > \gamma$ for all $\gamma \in \Gamma_0$, with the property that if $\gamma'' \ge \gamma'$, then we have $d(p_{\gamma\gamma''} p_{\gamma''}, p_\gamma) < \delta^*$.

For each $\gamma \in \Gamma$, $p_\gamma f : A \to Z_\gamma$ is a map. So by our assumption that $X \tau Z_\gamma$, the map $p_\gamma f$ extends over X, say to $g_\gamma : X \to Z_\gamma$.

We will now construct a map $G_{\gamma'}$ from X to $\prod_{\gamma\in\Gamma} Z_\gamma$. Let $G_{\gamma'} : X \to \prod_{\gamma\in\Gamma} Z_\gamma$ be defined as follows. Let $x \in X$. If $\gamma \in \Gamma_0$, then $p_\gamma(G_{\gamma'}(x)) = p_{\gamma\gamma'}(g_{\gamma'}(x))$. Otherwise, if $\gamma \notin \Gamma_0$, set $p_\gamma(G_{\gamma'}(x)) = g_\gamma(x)$. Clearly $G_{\gamma'} : X \to$

$\prod_{\gamma\in\Gamma} Z_\gamma$ is a map.

To prove (ii), let $x \in A$ and $\gamma \in \Gamma_0$. Then, $p_\gamma(G_{\gamma'}(x)) = p_{\gamma\gamma'}(g_{\gamma'}(x)) = p_{\gamma\gamma'}p_{\gamma'}(f(x))$. By our choice of γ', we have $d(p_{\gamma\gamma'}p_{\gamma'}(f(x)), p_\gamma(f(x))) < \delta^* \leq \delta$, and so $d(p_\gamma(G_{\gamma'}(x)), p_\gamma f(x)) < \delta$. On the other hand, if $\gamma \notin \Gamma_0$, then $p_\gamma(G_{\gamma'}(x)) = g_\gamma(x) = p_\gamma f(x)$, and so of course, $d(p_\gamma(G_{\gamma'}(x)), p_\gamma f(x)) < \delta$.

We claim that $G_{\gamma'}(X) \subseteq U$. Let $x \in X$. Since each $p_{\gamma\gamma'}$ is surjective, $Z_\gamma = p_\gamma(Z)$ for all $\gamma \in \Gamma$. So, since $g_{\gamma'}(X) \subseteq p_{\gamma'}(Z)$, there exists some $z \in Z$ such that $p_{\gamma'}(z) = g_{\gamma'}(x)$. The collection $\{U^*_{z_i} | 1 \leq i \leq m\}$ covers Z, and so there exists a k such that $z \in U^*_{z_k}$.

We wish to show that $G_{\gamma'}(x) \in U_{z_k} \subseteq U$. We note that $U_{z_k} = \prod_{\gamma\in\Gamma} V_\gamma$, where for each $\gamma \in \Gamma$, V_γ is open in Z_γ. To accomplish this, we need to show that for every $\gamma \in \Gamma$, $p_\gamma(G_{\gamma'}(x)) \in V_\gamma$. First consider $\gamma \in \Gamma_{z_k} \subseteq \Gamma_0$. Then by our choice of γ', we have $d(p_{\gamma\gamma'}p_{\gamma'}, p_\gamma) < \delta^*$. And so,

$$
\begin{aligned}
&d(p_\gamma(G_{\gamma'}(x)), p_\gamma(z_k)) \\
=&d(p_{\gamma\gamma'}(g_{\gamma'}(x)), p_\gamma(z_k)) \\
=&d(p_{\gamma\gamma'}p_{\gamma'}(z), p_\gamma(z_k)) \\
\leq&d(p_{\gamma\gamma'}p_{\gamma'}(z), p_\gamma(z)) + d(p_\gamma(z), p_\gamma(z_k)) \\
<&\delta^* + \frac{\delta(z_k,\gamma)}{3} \\
<&\frac{\delta(z_k,\gamma)}{3} + \frac{\delta(z_k,\gamma)}{3} \\
<&\delta(z_k,\gamma).
\end{aligned}
$$

Thus, $p_\gamma G_{\gamma'}(x) \in B(p_\gamma(z_k), \delta(z_k,\gamma)) \subseteq V_\gamma$.

For $\gamma \notin \Gamma_{z_k}$, we have that $V_\gamma = Z_\gamma$, and so of course $p_\gamma(G_{\gamma'}(x)) \in V_\gamma$. And so, $G_{\gamma'}(x) \in U_{z_k} \subseteq U$. Thus, $G_{\gamma'}(X) \subseteq U$.

To conclude the proof, we set $F = G_{\gamma'}$. $\square$

The following definition and corollary can be found in [1].

Definition 5.3.2. A directed set Γ is said to be ω-*complete* if for each countable chain Γ_0 of elements of Γ there exists an element $\sup(\Gamma_0)$ in Γ.

Proposition 5.3.3. *For each countable subset Γ_0 of an ω-complete set Γ there exists an element $\gamma' \in \Gamma$ such that $\gamma' \geq \gamma$ for every $\gamma \in \Gamma_0$.*

Corollary 5.3.4. *Let $\mathbf{Z} = \{Z_\gamma, \varepsilon_\gamma, p_{\gamma\gamma'}, \Gamma\}$ be an approximate inverse system of metric compacta with surjective bonding maps and limit Z, and let Γ be ω-complete. For a space X, if $X \tau Z_\gamma$ for each $\gamma \in \Gamma$, then for any neighborhood U of Z in the topological product $\prod_{\gamma\in\Gamma} Z_\gamma$, for any closed subspace A of X, and for any map $f : A \to Z$, there exists a map $F : X \to U$ such that $F|_A = f$.*

Proof. Using the notation from the proof of 5.3.1, we know that for each $n \in \mathbb{N}$ there exists a map $G_{\gamma_n} : X \to U$ such that for every $\gamma \in \Gamma$ and $x \in A$, $d(p_\gamma G_{\gamma_n}(x), p_\gamma f(x)) < \frac{1}{n}$. Further, we had that for every $\gamma' \geq \gamma_n$ there existed a map $G_{\gamma'} : X \to U$ such that for every $\gamma \in \Gamma$ and $x \in A$, $d(p_\gamma G_{\gamma'}(x), p_\gamma f(x)) < \frac{1}{n}$.

Let $\gamma' \in \Gamma$ be greater than or equal to γ_n for every $n \in \mathbb{N}$. Such an element γ' exists because Γ is ω-complete.

Clearly $G_{\gamma'}(X) \subseteq U$. We claim that $G_{\gamma'}|_A = f$. Let $\gamma \in \Gamma$ and $x \in A$. Then we have that $d(p_\gamma G_{\gamma'}(x), p_\gamma f(x)) < \frac{1}{n}$ for every $n \in \mathbb{N}$. Thus $p_\gamma G_{\gamma'}(x) = p_\gamma f(x)$, and so $G_{\gamma'}|_A = f$. $\square$

Chapter 6

The Dimension (m, n)-dim

In this chapter spaces are assumed to be normal and T_1. In a recent paper, [3], Fedorchuk has defined a new generalization of covering dimension. It is based on the following well-known theorem from classical dimension theory.

Theorem 6.0.5. *(Theorem 1.7 from [3]) A space X satisfies the inequality $\dim X \leq n$ if and only if for every sequence (A_i, B_i), $i = 1, 2, ..., n+1$, of pairs of disjoint closed subsets of X there exists for each $i = 1, ..., n+1$ a partition P_i between A_i and B_i such that $P_1 \cap P_2 \cap ... \cap P_{n+1} = \emptyset$.*

6.1 Introduction to (m, n)-dim

Definition 6.1.1. (Definition 2.1 from [3]) Let $u = (U_1, U_2, ..., U_m)$ be a cover of X and let $\Phi = (F_1, F_2, ..., F_m)$ be a family of closed subsets of X

such that

$$F_j \subset U_j, \quad j = 1, ..., m;$$

$$\text{ord}\Phi \leq 1.$$

Then (u, Φ) is said to be an m**-pair in** X.

Definition 6.1.2. (Definition 2.5 from [3]) Let (u, Φ) be an m-pair of X. A closed set $P \subseteq X$ is said to be an n**-partition** of (u, Φ) if there exists a family of open sets $v = (V_1, V_2, ..., V_m)$ of X such that $F_i \subseteq V_i \subseteq U_i$, for $i = 1, 2, ..., m$; $\text{ord}\, v \leq n$; and $X \setminus P = \bigcup v$.

Definition 6.1.3. (Definition 2.7 from [3]) For each $i = 1, ..., r$ let (u_i, Φ_i) be an m-pair of X. The sequence $((u_1, \Phi_1), (u_2, \Phi_2), ..., (u_r, \Phi_r))$ is called n**-inessential in** X if for each $i = 1, ..., r$ there exists an n-partition P_i of (u_i, Φ_i) such that $P_1 \cap P_2 \cap ... \cap P_r = \emptyset$.

Definition 6.1.4. (Definition 2.8 from [3]) Let $m, n \in \mathbb{N}$ with $n \leq m$. To every space X one assigns the **dimension** (m, n)-dim$X$, which is an integer ≥ 1 or ∞. The dimension function (m, n)-dim is defined in the following way:

1. (m, n)-dim$X = -1$ if and only if $X = \emptyset$;

2. (m, n)-dim$X \leq k$, where $k \geq 0$, if every sequence $((u_1, \Phi_1), (u_2, \Phi_2), ..., (u_{k+1}, \Phi_{k+1}))$, where for each $i = 1, 2, ..., k + 1$ (u_i, Φ_i) is an m-pair of X, is n-inessential in X;

3. (m, n)-dim$X = \infty$, if (m, n)-dim$X \leq k$ is false for each $k \geq -1$.

Theorem 6.1.5. *(Theorem 2.9 from [3]) For every space X we have*

$$(2,1)\text{-dim}X = \dim X.$$

Proposition 6.1.6. *(Proposition 2.19 from [3]) Let $f : X \to Y$ be a map and let a sequence (u_i, Φ_i) of m-pairs of Y be n-inessential in Y. Then the sequence $(f^{-1}u_i, f^{-1}\Phi_i)$ is n-inessential in X.*

Proposition 6.1.7. *(Proposition 2.20 from [3]) Let (u_i^1, Φ_i^1) and (u_i^2, Φ_i^2), $1 \leq i \leq r$, be two sequences of m-pairs of X. Let $u_i^1 = ({}^1U_1^i, ..., {}^1U_m^i)$, $\Phi_i^1 = ({}^1\Phi_1^i, ..., {}^1\Phi_m^i)$, $u_i^2 = ({}^2U_1^i, ..., {}^2U_m^i)$, and $\Phi_i^2 = ({}^2\Phi_1^i, ..., {}^2\Phi_m^i)$. Assume that*

$${}^1F_j^i \subseteq {}^2F_j^i \subseteq {}^2U_j^i \subseteq {}^1U_j^i, \quad i = 1, ..., r; \quad j = 1, ..., m.$$

If the sequence (u_i^2, Φ_i^2), $i = 1, ..., r$ is n-inessential in X, then the sequence (u_i^1, Φ_i^1), $1 \leq i \leq r$, is n-inessential in X.

6.2 Approximate Inverse Limits and (m, n)-dim

In [3], Fedorhcuk shows that (m, n)-dim is preserved by inverse limits.

Theorem 6.2.1. *(Theorem 2.21 from [3]) Let $S = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces X_a with (m, n)-dim$X_a \leq k$, and let $X = \lim S$. Then (m, n)-dim$X \leq k$.*

In 6.2.8, we will prove that (m, n)-dim is preserved by limits of approximate inverse systems. First, we must establish several facts concerning approximate inverse systems.

In this section, let $\mathbf{X} = \{X_a, \varepsilon_a, p_{aa'}, A\}$ be an approximate inverse system of metric compacta, with directed set $(A, \leq)$, as described in chapter three. Let $X = \lim \mathbf{X}$.

The following theorem is pulled from the proof of Theorem 3 of [10].

Theorem 6.2.2. *Let $\mathcal{U} = \left(p_{a_1}^{-1}(U_1), ..., p_{a_m}^{-1}(U_m)\right)$ be a covering of X, where $a_i \in A$ and $U_i \subseteq X_{a_i}$ is open in X_{a_i}, $i = 1, ..., m$. Then there exists an index $a \in A$ and an open covering $\mathcal{V} = (V_1, ..., V_m)$ of X_a such that $p_a^{-1}(V_i) \subseteq U_i$ for $i = 1, ..., m$.*

Proof. Choose closed sets $F_i \subseteq X$, $i = 1, ..., m$, such that $F_i \subseteq p_{a_i}^{-1}(U_i)$, $i = 1, ..., m$, and $(F_1, ..., F_m)$ covers X. Next choose closed sets $H_i \subseteq X_{a_i}$ such that $H_i \subseteq U_i$ and $F_i \subseteq p_{a_i}^{-1}(H_i) \subseteq p_{a_i}^{-1}(U_i)$. Finally, choose numbers $\eta_i > 0$, $i = 1, ..., m$, such that the η_i-neighborhood of H_i is contained in U_i. That is, $N(H_i, \eta_i) \subseteq U_i$.

By 3.1.6, there is an $a \geq a_1, ..., a_m$ such that $d(p_{a_i}, p_{a_i a} p_a) < \eta_i/2$, $i = 1, ..., m$. Now consider the sets $W_i = N(H_i, \eta_i/2)$, $G_i = p_{a_i a}^{-1}(W_i), i = 1, ..., m$. We claim that $p_{a_i}^{-1}(H_i) \subseteq p_a^{-1}(G_i) \subseteq p_{a_i}^{-1}(U_i)$, $i = 1, ..., m$.

Indeed, if $x \in p_{a_i}^{-1}(H_i)$, then $p_{a_i a} p_a(x) \in W_i$, $i = 1, ..., m$. We also have that $p_a(x) \in G_i$, that is, $x \in p_a^{-1}(G_i)$, which establishes that $p_{a_i}^{-1}(H_i) \subseteq p_a^{-1}(G_i)$.

To establish that $p_a^{-1}(G_i) \subseteq p_{a_i}^{-1}(U_i)$, consider $x \in p_a^{-1}(G_i)$. Clearly $p_a(x) \in G_i = p_{p_i a}^{-1}(W_i)$, and thus $p_{a_i a} p_a(x) \in N(H_i, \eta_i/2)$.

We conclude that $p_{a_i}(x) \in N(H_i, \eta_i) \subseteq V_i$. That is, $x \in p_{a_i}^{-1}(U_i)$.

Since $(F_1, ..., F_m)$ is a covering of X, $(p_a^{-1}(G_1), ..., p_a^{-1}(G_m))$ is an open covering of X. Set $V_1 = G_1$, $V_2 = G_2$, ... $V_{m-1} = G_{m-1}$, and $V_m = G_m \cup X_a \; p_a(X)$. Then the collection $\mathcal{V} = (V_1, ..., V_m)$ is an open covering of X_a, and $p_a^{-1}(V_i) \subseteq U_i$ for $i = 1, ..., m$. $\square$

Corollary 6.2.3. *For each finite open covering* $(U_1, ..., U_m)$ *of* X *there exists an index* $a \in A$ *and an open covering* $(V_1, ..., V_m)$ *of* X_a *such that* $p_a^{-1}(V_i) \subseteq U_i$, *for* $i = 1, ..., m$.

Lemma 6.2.4. *For each finite open covering* $(U_1, ..., U_m)$ *of* X *and each index* $a \in A$ *there exists an index* $a' \geq a$ *such that there is an open covering* $(V_1, ..., V_m)$ *of* $X_{a'}$ *with* $p_{a'}^{-1}(V_i) \subseteq U_i$, *for* $i = 1, ..., m$.

Using 3.2.2, one can show the following.

Lemma 6.2.5. *Let* $\Phi = (F_1, F_2, ..., F_m)$ *be a collection of closed sets of* X *such that* $\operatorname{ord}(\Phi) \leq 1$. *Then there exists* $a \in A$ *such that the collection* $p_a(\Phi) = (p_a(F_1), p_a(F_2), ..., p_a(F_m))$ *has order* ≤ 1.

Corollary 6.2.6. *Lemma 2.4 is true for any* $a' \geq a$.

Theorem 6.2.7. *Let* $(U_1, U_2, ..., U_m)$ *be a finite open covering of* X *and* $\Phi = (F_1, F_2, ..., F_m)$ *be a collection of closed subsets of* X *such that* $\operatorname{ord}(\Phi) \leq 1$ *and* $F_i \subseteq U_i$. *Then for some* $a \in A$ *the collection* $p_a(\Phi) = (p_a(F_1), p_a(F_2), ..., p_a(F_m))$ *has order* ≤ 1, *and there exists a finite covering* $(V_1, V_2, ..., V_m)$ *of* X_a *such that* $p_a^{-1}(V_i) \subseteq U_i$ *and* $p_a(F_i) \subseteq V_i$ *for* $i = 1, 2, ...m$.

Theorem 6.2.8. *Let* $\mathbf{X} = \{X_a, \varepsilon_a, p_{a'a}, A\}$ *be an approximate inverse system with* (m, n)-$\dim X_a \leq k$ *for all* $a \in A$, *and let* $X = \lim \mathbf{X}$. *Then* (m, n)-$\dim X \leq k$.

Proof. Let $((u_i, \Phi_i))$, $i = 1, ..., k+1$ be a sequence of m-pairs of X. We wish to show that this sequence is n-inessential in X. Let $u_i = (U_1^i, U_2^i, ..., U_m^i)$ and $\Phi_i = (F_1^i, F_2^i, ..., F_m^i)$.

Using 6.2.7, for each $i = 1, ..., k+1$ we can find an index $a_i \in A$ such that $u_i^i = (U_1^{(i,i)}, U_2^{(i,i)}, ..., U_m^{(i,i)})$ and $p_{a_i}(\Phi_i) = (p_{a_i}(F_1^i), p_{a_i}(F_2^i), ..., p_{a_i}(F_m^i))$ are an m-pair of X_{a_i} satisfying the condition that $p_{a_i}^{-1}(U_j^{(i,i)}) \subseteq U_j^i$ for $j = 1, ..., m$. Note that from the definition of an m-pair we have $\operatorname{ord} p_{a_i}(\Phi_i) \leq 1$ and $p_{a_i}(F_j^i) \subseteq U_j^{(i,i)}$ for $j = 1, ..., m$.

There exist closed neighborhoods $G_j^{(i,i)}$ of $p_{a_i}(F_j^i)$ and open sets $V_j^{(i,i)}$ of X_{a_i} such that the following conditions hold: (1) $p_{a_i}(F_j^i) \subseteq \operatorname{int} G_j^{(i,i)} \subseteq G_j^{(i,i)} \subseteq V_j^{(i,i)} \subseteq \overline{V_j^{(i,i)}} \subseteq U_j^{(i,i)}$, (2) $v_i^i = (V_1^{(i,i)}, V_2^{(i,i)}, ..., V_m^{(i,i)})$ covers X_{a_i}, (3) the order of $g_i^i = (G_1^{(i,i)}, G_2^{(i,i)}, ..., G_m^{(i,i)})$ is ≤ 1. Then (v_i^i, g_i^i) is an m-pair of X_{a_i}.

(4) For each $i = 1, ..., k+1$, there exists $\delta_i > 0$ so that for all $j = 1, ..., m$ the δ_i neighborhood of $p_{a_i}(F_j^i)$ is contained in $G_j^{(i,i)}$ and the δ_i neighborhood of $\overline{V_j^{(i,i)}}$ is contained in $U_j^{(i,i)}$.

(5) Using 3.1.6, pick $a_0 \in A$ so that for all $i = 1, ..., m$, $a_0 \geq a_i$ and $d(p_{a_i}, p_{a_i a_0} p_{a_0}) < \delta_i$.

Set

$$
\begin{aligned}
V_j^{(i,0)} &= p_{a_i a_0}^{-1}(V_j^{(i,i)}); && i = 1, ..., k+1; \quad j = 1, ..., m \\
G_j^{(i,0)} &= p_{a_i a_0}^{-1}(G_j^{(i,i)}); && i = 1, ..., k+1; \quad j = 1, ..., m \\
v_i^0 &= (V_1^{(i,0)}, V_2^{(i,0)}, ..., V_m^{(i,0)}); && i = 1, ..., k+1 \\
g_i^0 &= (G_1^{(i,0)}, G_2^{(i,0)}, ..., G_m^{(i,0)}); && i = 1, ..., k+1.
\end{aligned}
$$

It readily follows from (1), (2) and (3) that for each $i = 1, ..., k+1$, (v_i^0, g_i^0) is an m-pair of X_{a_0}. Since $(m,n)\text{-}\dim X_{a_0} \leq k$ this sequence of m-pairs is

n-inessential in X_{a_0}. Then by 6.1.6, the sequence

$$\left(p_{a_0}^{-1}(v_i^0), p_{a_0}^{-1}(g_i^0)\right), \quad i = 1, ..., k+1 \tag{6.1}$$

is n-inessential in X.

To conclude the proof we must show for each $i = 1, ..., k+1$ and $j = 1, ...m$ that (i) $F_j^i \subseteq p_{a_0}^{-1}(G_j^{(i,0)})$ and (ii) $p_{a_0}^{-1}(V_j^{(i,0)}) \subseteq U_j^i$.

Fix i and j.

To show (i), let $x \in F_j^i$. Then $p_{a_i}(x) \in p_{a_i}(F_j^i)$. By (5) we have $d(p_{a_i}(x), p_{a_i a_0}(x)) < \delta_i$, and so by (4) we have $p_{a_i a_0} p_{a_0}(x) \in G_j^{(i,i)}$. This implies that $p_{a_0}(x) \in p_{a_i a_0}^{-1}(G_j^{(i,i)}) = G_j^{(i,0)}$, giving us $x \in p_{a_0}^{-1}(G_j^{(i,0)})$, and proving (i).

To show (ii), let $x \in p_{a_0}^{-1}(V_j^{(i,0)}) = p_{a_0}^{-1}(p_{a_i a_0}^{-1}(V_j^{(i,i)}))$. Then $p_{a_i a_0} p_{a_0}(x) \in V_j^{(i,i)}$. By (5) we have $d(p_{a_i}(x), p_{a_i a_0}(x)) < \delta_i$, and so by (4) we have $p_{a_i}(x) \in U_j^{(i,i)}$. This implies that $x \in p_{a_i}^{-1}(U_j^{(i,i)}) \subseteq U_j^i$, and proves (ii).

And so by 6.1.7, the sequence (u_i, Φ_i), $i = 1, ..., k+1$, is n-inessential in X. $\square$

Chapter 7

Conclusion

We have established several basic properties of extensional maps. We have shown that extensional maps are preserved by the limit of an inverse system. Moreover, we have shown that extensional maps are preserved by the limit of an approximate inverse system. To accomplish this we established several new results concerning approximate systems. In particular, we have shown that a closed subspace of an approximate limit is itself the limit of some approximate system.

We have also shown that a notion similar to extension dimension can be defined for extensional maps, and we then proved that this extensional map dimension exists.

Finally, we investigated the dimension (m, n)-dim. We showed that (m, n)-dim is preserved by the limit of an approximate inverse system. To accomplish this we established several new results concerning covers of the approximate limit.

There are many interesting questions concerning extensional maps that remain open. Extensional maps obviously preserve extension properties. Are there other properties that extensional maps preserve? Are there conditions on the spaces that would guarantee the existence of an extensional map between them? This paper deals exclusively with compact Hausdorff or compact metric spaces. Could these theorems be extended to non-compact spaces?

There are also many interesting questions involving (m, n)-dim. It is currently even unknown what (m, n)-dim$\mathbb{R}^n$ is equal to. Also, it is known that every compact Hausdorff space with dim $\leq k$ can be written as the limit of an approximate system of compact polyhedra, each having dim $\leq k$. Is the same true for (m, n)-dim?

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