THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

ANALYSIS OF OPTIMAL COMPOSITE FEEDBACK-FEEDFORWARD CONTROL

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

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Norman, Oklahoma

1966

ANALYSIS OF OPTIMAL COMPOSITE FEEDBACK-

FEEDFORWARD CONTROL

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DISSERTATION COMMITTEE

ACKNOWLEDGEMENTS

I wish to express my sincere gratitude to my advisor, Dr. M. L. McGuire, for his advice and assistance throughout this work. I likewise cannot overstate my thanks to my colleague, Dr. M. Heymann, whose many stimulating discussions contributed measurably to my understanding of this entire research area.

Thanks are due also to Dr. G. M. Ewing for his insistence of many important points of rigour in mathematical proofs and to Dr. C. M. Sliepcevich for his help and advice during preparation of this manuscript.

The assistance of other personnel of the Process Dynamics Laboratory, R. A. Sims, M. Mofeez and R. L. Loeffler, was also greatly appreciated. Especially during the experimental phase of this work, their assistance was invaluable.

The successful completion of this investigation required a large amount of high speed digital computation. The assistance of the entire Osage Computer Center is appreciated. In particular, I thank Mr. Paul Johnson for his patient and valuable advice and Mr. Arlin Lee for his response above and beyond the call of duty to help with computer operation.

The financial support of the National Science Foundation, Conoco Petroleum Company and the Oklahoma Research Institute is gratefully acknowledged.

Finally, I must thank my wife, Mary Ann, for her patience and the many younger members of my family who provided the impetus for this labor.

ABSTRACT

Analytic design methods for combination feedback-feedforward control systems are developed and evaluation is made of systems yielding optimal performance while subject to constraints commonly encountered in the chemical industry. The optimization is based on minimization of the mean square output of a system subject to a random disturbance utilizing the mathematical techniques pioneered by Wiener for the solution of the design equations. Side conditions of constraint on mean square control effort, signal-to-noise ratio in the feedback system, and minimization of error output caused by misidentification of plant parameters were found to be necessary to give physically realizable and meaningful designs. The analytic design methods were found useful for analysis of control system performance and capabilities under a variety of constraints but the optimal designs are marginally superior to "ideal" or "invariant" feedforward controllers coupled with tuned proportional feedback controls. The principal improvement in the optimal design was in conservation of control effort when compromises in system performance were necessary because of this restriction.

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CHAPTER I

INTRODUCTION

There is constant need for improvement of process control in the chemical industry. Technical advances in many fields make possible new processes for which good control is required at conditions where measurement itself is difficult. At the other end of the spectrum, competitive pressures usually dictate that older processes must approach near-optimal operation if acceptable profit margins are to be maintained.

Development of optimal control theory has proceeded to a very high plane, especially for problems in electronic- or aerospace-related fields. However actual use of sophisticated optimal control laws in chemical processes has been limited. The present work describes an application of an existing control optimization technique to a class of systems having control capacity constraints and information sources typical of chemical processes.

General Control Considerations

Automatic controllers for chemical processes can be divided into three classes based on the source of the signal used to initiate control action. The type most commonly encountered is primary feedback control in which control action is determined by the difference between the value of the output or controlled variable and some reference or "ideal" output. Another type is feedforward control in which inputs or disturbances to the system are monitored and control action is computed to offset the effect of the disturbances on the output. The third type of control is secondary feedback in which control action is based on the value of secondary or uncontrolled system state variables which are related to the controlled output. However, in the present work, the class of systems will be restricted to those in which secondary variables are not monitored so that this third type of control will not be considered.

Feedback and feedforward control each have relative advantages and disadvantages. The principal advantage of feedback control as compared to feedforward control, and the primary reason for the greater popularity of the former, is that feedback has a high degree of tolerance for design inaccuracies. If, for some reason, a feedback controller produces inexact control compensation, the inexact system output which results is sensed by the controller and becomes the basis for further control action. If inexact control action occurs in a system having feedforward control only, the undesirable output persists indefinitely because feedforward control action is initiated by deviations of the system inputs only.

Feedback control possesses some limitations however, that are absent in feedforward control. When a high degree of control is attained, the difference between actual output and the reference (set point) becomes small so that signal-to-noise ratio problems become important in the feedback amplification system. This problem does not occur with feedforward control since input disturbances containing sufficient energy to cause significant system output generally produce acceptable controller signal-to-noise ratios. Additional difficulties are often encountered in feedback systems because of dead times which occur in the output sensing system. Both feedback and feedforward control are affected adversely by dead time in the controller but with the former, the sensing dead time of feedback circuit must be added to the controller dead time so that the problem is intensified. For example a system which is temperature controlled by cooling water flow may have controller dead time due to factors such as pneumatic transmission losses and valve hysteresis. If feedback control of the output temperature is used, additional transport delay may occur if, for example, the sensor is located downstream. This sensing circuit dead time must be added to that in the controller in computing feedback control capacity. As feedforward control is based on values of input flows or temperatures, it would not be affected by lags in the output measurement.

The various qualities of feedback and feedforward control tend to complement each other. When the best possible control is desired, a composite controller consisting of a combination of feedback and feedforward is indicated. Here it would be expected that a controller basing its operation on two sources of process information is capable of control performance that is superior to that of a controller with but a single source of information.

Optimization techniques to determine the most efficient implementation of this information are used in the present work to develop design equations for composite control systems. Emphasis has been focused on optimization conditions that would be encountered in a typical chemical process.

Statement of General Problem

It is assumed that the objective of the control design and operation is to attenuate the output response of chemical process systems which are activated by random disturbances. In addition it will be assumed that the process system or plant can be adequately described by linear differential equations with constant coefficients; in fact, specific design consideration will be given to controllers which cause the plant to operate in the linear range for a large fraction of the time. Only single variable plants are treated, i.e., plants which have but one disturbance variable and one controlled output variable. The generalization to n disturbances is a trivial multiplication of controllers but the generalization to n outputs is far from trivial.

In setting up the differential equations, process dead times will be taken into consideration by allowing the argument of the time functions to be shifted along the time axis. All times will be referred arbitrarily to the process times, i.e., a disturbance entering a process will be denoted $d(t)$ while the output variable which is sensed τ_C time units later is given as $c(t + \tau_C)$. The value of the controlling or manipulative variable must be specified by the controller τ_M time units before it is used and hence becomes $m(t - \tau_M)$.

Under these conditions the general process dynamics can be described by the differential equation

$$\frac{d^n}{dt^n} c(t+\tau_C) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} c(t+\tau_C) + \ldots + a_0 c(t+\tau_C) =$$
$$= b_j \frac{d^j}{dt^j} m(t-\tau_M) + \ldots + b_0 m(t-\tau_M) + g_k \frac{d^k}{dt^k} d(t) + \ldots + g_0 d(t) , \tag{1-1}$$

where the a's, b's and g's are constants, $c(t + \tau_C)$ is the output variable measured τ_C time units after the process time, $m(t - \tau_M)$ is the manipulative variable specified by the controller τ_M time units before the process time, and $d(t)$ is the disturbance variable that enters the process at "process" time.

It is most convenient to work with the Laplace transformation of this equation that results from assuming zero initial conditions. Transforming (1-1) into the Laplace domain yields

$$C(s) = P_D(s)D(s) + P_M(s)M(s) \quad , \tag{1-2}$$

where $P_D(s)$ and $P_M(s)$ are the "plant transfer functions." If dead times are absent, these functions are rational functions (i.e., ratios of polynomials) in the Laplace transform variable s, while if dead times are present they are rational functions multiplied by an exponential factor in s. The functions $C(s)$, $D(s)$ and $M(s)$ are Laplace transformations of the output variable $c(t)$, disturbance variable $d(t)$, and manipulative variable $m(t)$, respectively. The design objective will be to define transfer functions of controllers, $Q_C(s)$ and $Q_D(s)$, so that a minimum $C(s)$ will occur when the manipulative variable is defined as

$$M(s) = Q_D(s)D(s) - Q_C(s)C(s) \quad . \tag{1-3}$$

The function $Q_C(s)$ is the feedback transfer function since its control action is based on information from the output while $Q_D(s)$ is the feedforward transfer function since its control action is based on information from the input. A block diagram of the system with its controls is shown in Figure 1-1.

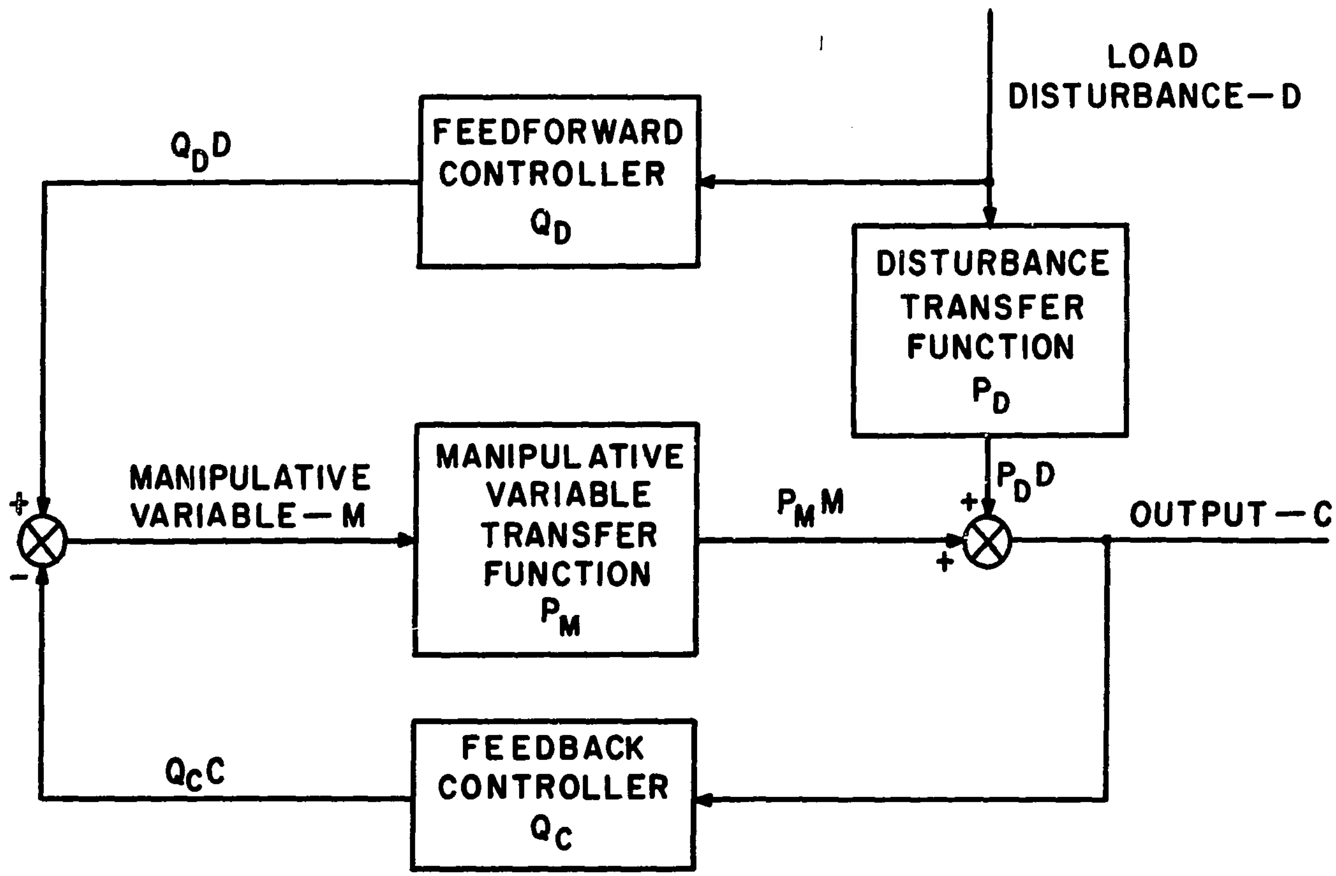

Figure 1-1.--Block Diagram of Plants and Controllers under Consideration

It is obvious that the minimum absolute value of system response occurs if the output variable can be made identically zero, in which case it follows from (1-2) that

$$M(s) = -\frac{P_D(s)}{P_M(s)} D(s) . \tag{1-4}$$

This relation can be realized formally if the feedforward control function is defined as

$$Q_D = -\frac{P_D(s)}{P_M(s)} . \tag{1-5}$$

There are a number of reasons why this ideal solution to the optimal control problem can seldom be achieved in practice. If the transfer function, $P_M(s)$, has a larger exponential dead time than does the function $P_D(s)$, the ratio given in (1-5)* will have a net positive exponential factor which requires that the controller, $Q_D(s)$, react to a disturbance before it occurs. Further if the system is bound by constraints such as a maximum of the magnitude of the manipulative variable, M(s), the use of (1-4) will result in overloading the control system with a corresponding degradation of response. Finally, if there is any error in the plant describing functions, $P_D(s)$ or $P_M(s)$, there will be a "model error output" for which the feedforward controller cannot compensate.

These factors can be formulated in mathematical terms and then incorporated into an explicit analytic optimization technique developed by N. Wiener [W2].* It will be shown that

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*Throughout this work, equations will be referred to by the equation number in parenthesis and brackets will denote references to the bibliography.

in the absence of dead times, this formulation leads to the specification of a feedback controller with infinite gain. Feedback is not only insensitive to model error but is, in fact, the only effective way to control "model error output." If signal-to-noise ratios and dead times in the output sensing and amplification systems are considered in the optimization equations along with the previously mentioned factors of controller dead time, control effort constraint and model error output, then physically realizable composite control designs are obtained that seem to be real optimums. The present work is devoted to the development of the design equations and analysis of the resultant controls.

Control systems synthesized using Wiener's technique minimize the mean square of the value of some state variable of a system under excitation by a random disturbance. Explicit evaluation of the mean square value of this variable is achieved by use operational transform techniques based on Parseval's theorem. Using these techniques, the necessary and sufficient conditions can be determined for the existence of a minimum. The mathematical implications of these conditions lead to the Wiener-Hopf integral equation which can be solved explicitly for the control function by the complex variable techniques described by Wiener. Output minimization in the presence of constraints is achieved by minimizing a weighted sum of several state variables after adjusting the value of the weighting factors (or Lagrangian multipliers) so that the secondary variables satisfy the side conditions when the system is excited by a statistically characterized random disturbance. The constraints must be carefully chosen so that all important factors of the physical problem are represented while still allowing mathematical solution of the equations.

Review of Previous Work

The art and science of feedforward controller design in the process industry began about 15 years ago with the study of cascade control. Although feedback still dominates the field, interest in feedforward has increased steadily as process control engineers have realized that the existence of a continuous monitor of disturbances makes feedforward control particularly suitable for many chemical process problems [C1]. While some investigators have implied that feedforward alone may be sufficient for control purposes, most have concluded that a combination of feedback and feedforward is needed in the majority of cases. Undoubtedly feedforward control will be "encountered more frequently in the future as an essential aspect of composite control" [B10].

Most of the previous work on feedforward control has not been concerned with explicit optimization methods. The reason is that if the exact mathematical model is known, a controller can be constructed which is the mirror image of the plant so that when its output is added to that of the plant, disturbance attenuation is perfect. Harris and Schecter [H2] and Bollinger and Lamb [B7,B8] in their work on chemical reactor controls specified mirror images of linear approximations to the system describing functions for the feedforward portion of their controllers. The feedback section of these control systems, chosen in both cases by cut and try procedures, compensated only for the plant nonlinearities and for inaccuracies of analog computer programming. Haskins and Sliepcevich [H3] in their study of the invariance principle used nominal feedforward controllers with no primary feedback but showed how compensation of analytic non-linearities may also be programmed in the feed-forward section of the controller.

Studies of control of distillation columns are complicated by the fact that it is often impossible to monitor continuously those primary output variables that need to be controlled. Hence transfer functions of disturbances to secondary variables are found and these variables are made invariant by use of "mirror image" control [L6,M1]. Feedback here usually consists of manual adjustment of drift although several authors have advocated development of more sophisticated feedback techniques [L7,M1].

Analytic design methods based on a calculus of variations approach for feedback control systems were investigated by Newton, Gould and Kaiser [N2]. They found an optimal open loop control function and then computed the feedback controller necessary to give the desired open loop function. This optimization method was based on minimization of the mean square error between an ideal reference and the actual output in the presence of constraints, and in particular constraints on control effort. The equations defining necessary conditions for the minimum were solved by use of advanced complex variable theory (as demonstrated by Wiener [W2]) to yield an explicit analytic solution for the control function. The feedback controllers resulting from this treatment were ordinarily not physically realizable so that an artificial "band-pass" constraint was added to limit high frequency gain [N3]. The use of this constraint and its results are not readily evaluated for many process industry applications.

In the present work, this same fundamental mathematical technique is employed; the main difference is in the type of constraints which are used and the fact that two control functions are computed - one relating to feedback alone and another overall control function for both feedback and feedforward. Instead of the "band-pass" constraint, a feedback signal corruption or noise factor is added which is essentially a signal to noise limitation on the feedback gain.

Chang [C2] noted that continuous measurement of the reference signal as well as the plant output was sufficient to specify two control functions separately - one an open loop function compensating for reference set point changes and a feedback control function compensating for results of an external or load disturbance. More generally, Horowitz [H8] showed that as many separate control functions can be specified as there are independent measurements of system variables. In the present work, the ideal reference is assumed to be identically zero so that the open loop reference point control of Chang becomes trivial.

Note that the monitored quantities, i.e., the load disturbance and the output, are not independent in the statistical sense of having a zero covariance. Indeed, if the system transfer functions are known exactly, then one of these quantities is perfectly predictable from a knowledge of the other. In such a situation there exists an infinite number of sets of combination controllers which give identical performance. However if error exists in the mathematical model, measurement of the second variable adds "independent" information not available from the first variable so that unique combinations of feedback and feedforward are specified to yield given performance.

The use of Wiener's methods for optimal control design as extended by others [K1,L1,P2,M2] define only single control functions for each output. Although general discussions of constraints are presented, the cases studied use constraints which yield controllers often not physically realizable under conditions of measurement and model accuracy that exist in chemical process systems. A very important aspect of analytic control design for these systems by Wiener's method is the development of constraints that lead to realistic controllers on one hand and still allow solution of the complicated equations on the other.

Horowitz [H9] presents a comprehensive account of modern non-analytic design methods for control systems, i.e., designs in terms of rise time and overshoot, phase and gain margins, root locus techniques, etc. Although this present work is indebted to much of the design philosophy developed there, analytic methods are used here so that families of solutions for general situations can be investigated more easily and so that general control capacity for these systems can be explicitly evaluated.

CHAPTER II

FUNDAMENTAL MATHEMATICAL BACKGROUND

Since development of these control design techniques is based on some mathematical theorems not frequently encountered in chemical engineering, it is appropriate to review some of them here. Proofs of many of these results are presented in Appendix A.

Integral Transforms of Mean Square Values

The basic objective of the controllers to be optimized is to attenuate system output. In order to preserve mathematical tractability, the measure of merit is taken as the mean square value of the output. Constraints also will be considered in terms of mean square values. It will be necessary to express these mean square values of time functions in terms of Laplace transforms in order to solve the optimization equations. As a preliminary to exposition of the optimization techniques, the general transformation of quadratic time functions into the s-domain will be presented. Furthermore, since the disturbance has been assumed to be a random function, expression of these results in terms of statistical parameters will be required.

The transformation of quadratic time functions is made with the use of a result of Parseval's theorem:

If $c^*(t)$ is an arbitrary function of time which (i) is zero for all $t < 0$; (ii) approaches zero at least as fast as $e^{-\epsilon t}$ for all $\epsilon > 0$ as $t \to \infty$ and (iii) is bounded for all t; then

$$\int_0^\infty c^*(t)^2 dt = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} C^*(s)C^*(-s)ds \ , \qquad (2\text{-}1)$$

where $C^*(s)$ is the Laplace transform of $c^*(t)$. The proof of (2-1) is presented in Appendix A.

If $c^*(t)$ is the output of a linear system having the transfer function, $P_D(s)$, then the system may be described in the Laplace domain (cf. (1-1) and (1-2)) as

$$C^*(s) = P_D(s)D(s) \ . \qquad (2\text{-}2)$$

Thus the integral square output of this system is expressed as follows:

$$\int_0^\infty c^*(t)^2 dt = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} P_D(s)P_D(-s)D(s)D(-s)ds \qquad (2\text{-}3)$$

provided that the response meets the various convergence requirements of Parseval's theorem.

The foregoing equation, which gives the integral square output of a system for a deterministic disturbance, cannot be used if the disturbance is a random time function for which only the statistical properties are available. However the mean square output can be functionally related to the system transfer function and the statistically described disturbance. As a first step in this development, the mean value of the disturbance is assumed to be zero. Clearly it is no problem to design a control system to eliminate any permanent bias. The statistical property of random signals that is of greatest interest here is the so-called correlation function, $\varphi_{A*B*}(t_1,t_2)$. This function is simply the non-normalized covariance between two signals at specified times, t_1 and t_2. That is,

$$\varphi_{A*B*}(t_1,t_2) \triangleq \langle a*(t_1)b*(t_2)\rangle \ , \qquad (2\text{-}4)$$

where $\langle\ldots\rangle$ indicates the mean value. It is further assumed in this work that all random signals are ergodic, i.e., that the mean value of time averages is constant and not a function of the particular period over which the average is taken. This characteristic implies that the correlation function is a function of $t_1 - t_2$ rather than of the individual times. Hence $\varphi_{A*B*}(t_1,t_2)$ becomes $\varphi_{A*B*}(\tau)$ where $\tau = t_1 - t_2$.

In order to represent random signals in the Laplace domain, the cross spectral density, $\Phi_{AB}(s)$ of a random signal is defined

$$\Phi_{A*B*}(s) \triangleq \lim_{T\to\infty} \frac{1}{T} \langle A_T^*(s)B_T^*(-s)\rangle \ , \qquad (2\text{-}5)$$

where the function $A_T^*(s)$ and the function $B_T^*(s)$ are the Laplace transformations of bounded random signals $a*(t)$ and $b*(t)$ for the region $0 \leq t \leq T$ and are equal to zero elsewhere. It is shown in Appendix A that (2-4) and (2-5) may be related by

$$\varphi_{A*B*}(\tau) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{A*B*}(s)e^{s\tau}ds \ . \qquad (2\text{-}6)$$

This result can be used to determine the response of the system described in (2-2) to a random disturbance whose statistics are described by the self- or auto-correlation function, $\varphi_{DD}(\tau)$.

Substitution of $c*(t)$ for both $a*(t)$ and $b*(t)$ in

(2-4), (2-5), and (2-6) and using the relation,

$$\Phi_{C^*C^*}(s) = P_D(s)P_D(-s)\Phi_{DD}(s) \ , \qquad (2\text{-}7)$$

gives the result

$$\varphi_{C^*C^*}(0) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} P_D(s)P_D(-s)\Phi_{DD}(s)ds \ . \qquad (2\text{-}8)$$

Since from (2-4) it follows that

$$\varphi_{C^*C^*}(0) = \langle c^*(t)^2\rangle \ , \qquad (2\text{-}9)$$

it can be seen that (2-8) is the mean square value of $c^*(t)$ as evaluated in the Laplace domain.

Note from definition (2-5) that $\Phi_{DD}(s)$ is symmetric, i.e.,

$$\Phi_{DD}(s) = \Phi_{DD}(-s) \ . \qquad (2\text{-}10)$$

Therefore it is possible to factor $\Phi_{DD}(s)$ into two symmetric parts,

$$\Phi_{DD}(s) = D(s)D(-s) \ , \qquad (2\text{-}11)$$

allowing (2-8) to be made formally identical to (2-3).

Integral Conditions for Optimization

The equations of the previous section related the mean square output and the transfer function of a time invariant linear system to the spectral density of its random disturbance. Optimal control is coincident with the minimum of this mean square output consistent with the existence of any side conditions. In this section, development of Wiener's technique [W2] continues as the necessary and sufficient conditions are developed that allow the minimization of integrals of the type (2-8). It will be seen later that explicit solution of the final control equations is possible if and only if the integrand of the quantity to be minimized is linear in the unknown control operator. In order to simplify the mathematical development, this specialization is introduced at this point. Thus the integral of (2-3) or (2-8) may be rewritten as

$$J[B(s)P_1(s) + P_2(s)] \triangleq \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [B(s)P_1(s) + P_2(s)]\cdot[B(-s)P_1(-s) + P_2(-s)]ds\ , \qquad (2\text{-}12)$$

where B(s) is the Laplace transform of the unknown control operator b(t) and $P_1(s)$ and $P_2(s)$ are Laplace transforms of known or given operators. To reduce the writing, define the following for the arbitrary function, $\mathbf{Q}(s)$,

$$\mathbf{Q} \triangleq \mathbf{Q}(s) \text{ and } \overline{\mathbf{Q}} \triangleq \mathbf{Q}(-s) \qquad (2\text{-}13)$$

where $\mathbf{Q}(s)$ is some transfer function as in (2-12). Equation (2-12) may then be rewritten

$$J(BP_1 + P_2) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} (BP_1 + P_2)(\overline{BP_1} + \overline{P_2})ds \qquad (2\text{-}14)$$

When $\mathbf{Q}(s)$ is discussed as an integrand such as in (2-12) or (2-14), $\overline{\mathbf{Q}}$ is the complex conjugate of $\mathbf{Q}$ since s takes on imaginary values only in this integral.

The search for the unknown function, B, must always be restricted to functions which are physically realizable. This restriction shall mean that B must be non-predictive, i.e., as an operator on a state variable, it can cause no control action until after the disturbance has affected it. Accordingly, b(t) <u>must be zero for negative time</u> and in the transform space, this requirement becomes that all of the poles of the function, B, are in the left half plane (l.h.p.).

In Appendix A, application of the calculus of variations is utilized to show that the necessary and sufficient condition that B(s) provide an extremum for $J[B(s)P_1(s) + P_2(s)]$ is

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}(\overline{B_1P_1})(B_0P_1 + P_2)ds = 0 \qquad (2\text{-}15)$$

This equation would be satisfied if the integrand were identically zero. Since B_1 is an arbitrary function within its class and P_1 is a given non-zero function, the integrand can be zero if and only if

$$B_0P_1 + P_2 = 0 \qquad (2\text{-}16)$$

However, general solution of this equation can lead to a B_0 which is not physically realizable. For instance, if P_1 contains r.h.p. zero while P_2 does not, then B_0P_1 can equal $-P_2$ only if B_0 is unstable.

A more general valid solution of (2-15) can be found by the use of complex variable theory and Cauchy's residue theorem. The complete development of this solution is given in Appendix A when it is shown that necessary and sufficient conditions defining a B_0 so that (2-15) is satisfied is

$$\overline{P_1}(B_0P_1 + P_2) = X \qquad (2\text{-}17)$$

where X is a function with r.h.p. poles only. The left hand side of (2-17) is formally identical with the result obtained by the partial differentiation of the integrand of (2-15) with respect to $\overline{B_1}$. This convenient way to arrive at (2-17) will be used in the subsequent development.

Wiener's Explicit Solution

In order to find an explicit solution for the control function defined by (2-17), consider the expanded form of this equation

$$B_0P_1\overline{P_1} = P_2\overline{P_1} = X \ . \qquad (2\text{-}17)$$

Since the left hand side was derived from an integrand along the imaginary axis, the argument s is always imaginary. It follows that $\overline{P_1}$ is the complex conjugate of P_1 and that the function $P_1\overline{P_1}$ is symmetric with respect to the imaginary axis. If it has a pole or a zero in the r.h.p., then there is a corresponding pole or zero (with the same imaginary part) in the l.h.p. Hence $P_1\overline{P_1}$ can be factored into two parts - one factor with all of its poles and zeros in the l.h.p. and the other with poles and zeroes only in the r.h.p.:

$$P_1\overline{P_1} = Y\overline{Y} \ , \qquad (2\text{-}18)$$

where Y has poles and zeros in the l.h.p. only. Note that Y is identical to the non-exponential part of P_1 if P_1 has no r.h.p. zeros or poles.

Dividing (2-17) by $\overline{Y}$ gives

$$B_0Y + \frac{P_2\overline{P_1}}{\overline{Y}} = \frac{X}{\overline{Y}} \ . \qquad (2\text{-}19)$$

The term B_0Y has no r.h.p. poles and hence is physically realizable. The term $X/\overline{Y}$ has r.h.p. poles only and therefore is not physically realizable. The inverse transform of a r.h.p. pole can be considered as a stable function of negative time or an unstable function of positive time; neither can be physically constructed. The expression $\frac{P_2\overline{P_1}}{\overline{Y}}$ also can be divided into two parts - one that is physically realizable and one that is not. If all of the functions are ratios of polynomials, they may be expressed as a sum of partial fractions. Those fractions with residues in the r.h.p. are transforms of functions which represent response of the system to disturbances before the disturbance occurs. The part of the fraction not physically realizable is designated as $\left[\frac{P_2\overline{P_1}}{\overline{Y}}\right]_-$. The part of the fraction corresponding to the roots of the denominator with negative real parts, i.e., the physically realizable part, is designated as $\left[\frac{P_2\overline{P_1}}{\overline{Y}}\right]_+$.

By subtraction, (2-10) becomes

$$B_0Y + \left[\frac{P_2\overline{P_1}}{\overline{Y}}\right]_+ = \frac{X}{\overline{Y}} - \left[\frac{P_2\overline{P_1}}{\overline{Y}}\right]_- . \qquad (2\text{-}20)$$

The right hand side of this last equation is analytic (i.e., no poles) in the entire l.h.p.; the left hand side is analytic in the entire r.h.p. They are both equal, therefore, to the same function which is analytic in the entire plane. Liouville's theorem in complex variable theory states that functions analytic in the entire plane are constants [H7]. Thus

$$B_0Y + \left[\frac{P_2P_1}{\overline{Y}}\right]_+ = \text{constant} \qquad (2\text{-}21)$$

Evaluation of this constant proceeds by noting that the quantities displayed in (2-21) are, in general, functions of the complex Laplace transform variable, s. Since both sides of this equation are equal to some constant, obviously it is not possible to "solve" this equation to find a numerical value for s. Therefore to evaluate the constant it can be reasoned that the numerator and the denominator of both terms on the left hand side of (2-21) have the same order or degree as each of the factors of the integrand of (2-15) - the only change is the possibility of different signs of some of the roots (i.e., Y is of the same order as P_1). It has been previously shown that this function approaches zero as $s \to \infty$ (Appendix A). Since (2-21) is an identity in s, it must hold for all values of s including that for which s becomes infinitely large. Thus both sides of (2-21) are equal to zero and (2-21) may be solved explicitly for B_0 as follows

$$B_0 = -\frac{1}{Y}\left[\frac{P_2\overline{P_1}}{\overline{Y}}\right]_+ . \qquad (2\text{-}22)$$

In cases where the functions in (2-20) are not rational functions, a similar procedure can still be used - i.e., the term $P_2\overline{P_1}/\overline{Y}$ is divided into portions that are zero for $t < 0$ and portions that are zero for $t > 0$. It will be shown in Chapter III that the former is the principal part of the term at l.h.p. poles. (The principal part of a function at a single pole is the residue at that pole divided by the pole itself.) The balance of the term which is zero for $t > 0$, is subtracted from both sides of this equation and, formally, the steps leading to (2-22) are the same as before.

It may now be seen why it was necessary to restrict the integrand of (2-12) to linearity in the unknown operator B. If B_0 were present in a non-linear form, say as a quadratic factor in (2-17), the solution could proceed in the same way except that $\overline{B_0}$ and B_0^2 would be included in (2-20). Even though both sides of the altered (2-20) would be zero as before, evaluation of the unknown $\overline{B_0}$ at l.h.p. poles of other functions would be required. It would become necessary also to find the square root of functional forms on the right hand side of this equation in order to solve for the unknown function. The only possible alternative would be a very arduous iterative solution for B_0 in the function space.

Utilization of Constraints - Lagrange Multipliers

When several integrals of the type of (2-14) are to be kept within specified limits while minimizing another, a form of the Lagrange multiplier rule is used. Suppose that two functions, $J(BP_1 + P_2)$ and $J(BQ_1 + Q_2)$, are given with J defined as in (2-14). The first of these is to be minimized while maintaining the second below some given positive value (the value of the integral is non-negative; cf. (A-12). Formally the problem is to find a B_0 such that

$$J(B_0P_1 + P_2) = \min , \qquad (2\text{-}23)$$

and

$$J(B_0Q_1 + Q_2) \leq \mathcal{L}^* \qquad (2\text{-}24)$$

$$\text{for } \mathcal{L}^* > 0 .$$

There is, in general, an infinite set of functions, B_1, which satisfy the last equation. Many of these could be considered and then the neighborhood of those giving the least value for (2-23) could be explored further. However the function space is quite a bit larger than the point space and successful search methods for all but trivial problems are nonexistent.

This problem can, however, be solved in an indirect way. Consider the function

$$F(A,\lambda) = J(BP_1 + P_2) + \lambda^2 J(BQ_1 + Q_2) , \qquad (2\text{-}25)$$

where λ is some real constant. This function is a linear combination of integrals of the form of (2-14) and the condition for a minimum can be found by formal differentiation of the integrand with respect to $\overline{B}$ and solving (2-17) to obtain (2-22). This explicit solution for B_0 will be in terms of the unknown, λ. If the function, $B_0(\lambda)$, is substituted into (2-24), a set of numerical λ's can be found satisfying that inequality. Normally, this function is monotonic in both λ^2 and $|\mathcal{E}^*|$, and a unique largest λ can be found in this set which satisfies (2-24) where the inequality has been replaced by equality.

For any arbitrary choice of λ, $F(B_0(\lambda),\lambda)$ will have a minimum value. If the particular λ which satisfies (2-24) is chosen (where the relationship is monotonic and the inequality has been made an equality), then $F(B,\lambda)$ will have assumed its minimum value consistent with the constraint of (2-24) provided this result is interpreted statistically to mean that the constraint is satisfied a certain percentage of the time. Of course once a numerical λ is found, $B_0(\lambda)$ can be computed in numerical terms.

The sequence of solving these equations is usually different in actual computations. A numerical choice is made for λ and substituted into (2-25) which is then solved for B_0 by Wiener's techniques. This function is substituted into integrals of the form of (2-14) giving values of the mean square error and mean square constraint. Other values for the parameter λ are chosen and this process repeated until a relationship of error vs. constraint has been found. Such a parametric set of solutions has, of course, far greater value than that of a single numerical answer.

CHAPTER III

DEVELOPMENT OF DESIGN EQUATIONS

The basic mathematical techniques have now been developed which are capable, under a practical set of conditions, of yielding an optimizing controller function. Three principal problems remain:

(i) Casting the real control design problem and associated constraints into the proper form such that the mathematical machinery can process the data;

(ii) Development of the proper set of constraints so that the procedure yields a control design which gives a real and practical optimum;

(iii) Investigation of some simplifications of the procedure which give control designs not importantly different from the optimum.

(i) Problem Format

Although portions of the mathematical procedure could have been developed for more general systems, use of the entire chain of methods necessitates that the system describing equations be linear and time invariant. Usually this requirement will not be a serious restriction since an optimum control design is to be developed, and deviations from steady state operating conditions might be expected to be small enough so that a linear approximation to the dynamics is reasonably valid. Another limitation dictated by mathematical expedience is that the constraints and side conditions for optimization must all be expressed in terms of quadratic functions. This restriction is actually an attractive feature to the chemical engineer who often works on processes in terms of variances and who will usually be readily able to interpret his problem in terms of mean square deviations. Traditional constraints such as rise time, overshoot, band width, phase margin, etc., are often much more difficult to cast quantitatively into intuitively important terms in process industry problems.

It is important to recognize, however, that mean square deviation may not be the ideal measure of controller performance in all cases. It weights large errors much more than the smaller ones, and this characteristic may or may not be appropriate for a given situation. At times, such as in a chemical reaction where yield is usually a higher order function of the control variables, the weighting for large errors may be insufficient. For random inputs, Chang [C2] has shown that if the cost function is a monotonic increasing function of the absolute value of error, then a quadratic constraint is at least monotone with the cost function. For deterministic inputs, even this modest result is not always true.

The mean square criteria is not used only as a measure of the output error. In general, there will be constraints given which limit the size of a given control variable. For instance, if control is achieved by variation of some flow rate, a certain maximum and minimum flow rate is available; alternatively the rate of change of flow rate may be limiting. If the controller calls for control effort which exceeds the capacity of the plant, the controller is said to be "saturated" and no longer obeys the linear equations of the original model. In principle it would be possible to set up a new non-linear model which accounts for the saturation tendencies of the plant but then the design techniques described would no longer be applicable. Controller design methods for non-linear systems do not have, at the present at least, the scope or generality of the techniques described here. Hence the approach will be to limit the mean square value of various control efforts and, in doing so, limit the fraction of time that the system will operate outside its linear range. For instance if a given signal has a normal or gaussian probability distribution, then the control effort will be within the mean square value about 68% of the time.

(ii) Constraints

The general plan used in the development of the controller design equations is first to perform the optimizations using a minimum number of constraints. Conflicts are shown to exist between the predictions of the resulting control laws and generally known controller performance so that the design bases are reexamined. Constraints are added to the problem based on physical reasons so that realistic optimal control laws are obtained. One of the principal difficulties of this study is the problem of definition of proper realistic constraints in a form that still permits solution of the mathematical problem.

There are a number of implicit constraints contained in the traditional methods which must be extracted and stated explicitly by this variational procedure. Proper constraint definition seems to be one of the reasons why dynamic programming methods and Pontryagin's Maximum Principle often give "optimum" controllers that are different from the traditional designs. These situations of implicit constraints are complicated and subtle and will be discussed in the sample problems as they are used as vehicles to develop the overall design procedure.

(iii) Computational Simplifications

While the mathematical procedures described will, in principle, yield an explicit solution to the optimizing problem, in fact it turns out that, for even moderately complicated problems, the algebraic detail becomes all but prohibitive. In the following a simple but general example will be pursued in detail with introduction and discussion of simplifications at various points which will not significantly affect the final results.

Specific Transfer Functions for Demonstration Purposes

The transfer functions for a simple physical system will be developed in this section for use throughout this chapter so that some subsequent discussions can be made at the physical, intuitive level.

The plant to be modelled is an elementary first order system but does possess all of the important classes of elements found in more complex units: gain, poles and non-minimum phase dead time. Shown in Figure 3-1, it consists of a perfectly stirred tank containing a heating coil through which heat transfer media is circulated. It is assumed that the temperature of this media is constant throughout the coil and can be used as the manipulative, control variable. The material entering the tank is subject to the disturbance of varying temperature, and it is desired to maintain the output temperature of the tank constant. The only accessible point

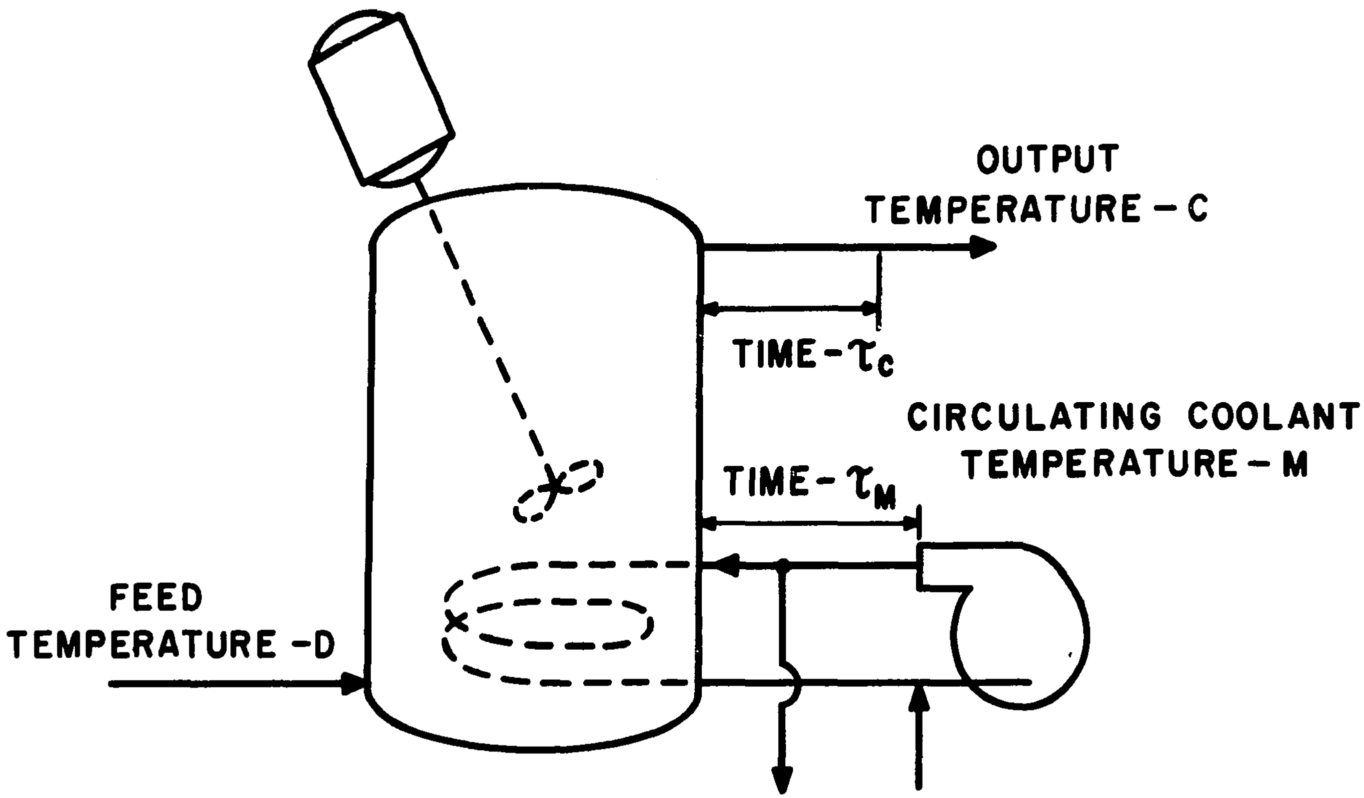

Figure 3-1.--System Used to Demonstrate Design Equations

of measurement of the output is some point downstream so that a pure time delay may exist between a change in the actual system temperature and its measurement. The manipulative variable itself need not react instantly; a dead time may also exist between the time that the controller gives an order and its execution.

The equation describing the dynamics of this system is found by making an energy balance:

$$\rho V C_p \frac{dc(t+\tau_C)}{dt} = FC_p[d(t) - c(t+\tau_C)] + U_H A_H[m(t-\tau_M) - c(t+\tau_C)] , \quad (3\text{-}1)$$

where $d(t)$ is the feed temperature;

$c(t + \tau_C)$ is the output temperature observed at a time τ_C after leaving the vessel;

$m(t - \tau_M)$ is the coil temperature as set by the controller at a time τ_M before entering the vessel;

ρ is the density of the flows;

V is the volume of the tank;

C_p is the specific heat of the flows;

F is the flow rate;

U_H is the overall heat transfer coefficient;

A_H is the heat transfer area.

The gradients or driving forces are the differences between the externally accessible values of the variables which are then corrected for the time lags. It is desired to find the linear operators which will permit construction of the controller equation defining the optimal manipulative variable:

$$L_M m(t) = L_D d(t) - L_C c(t) . \quad (3\text{-}2)$$

These equations can be identified with the general forms previously considered by first subtracting the steady state components and then transforming the resulting equation into the s-domain. The Laplace transform of the perturbation equations are

$$Ce^{\tau_C s} = \frac{K_D}{s + \alpha} D + \frac{K_M e^{-\tau_M s}}{s + \alpha} M \,, \tag{3-3}$$

$$M = Q_D D - Q_C C \,, \tag{1-2}$$

where

$$K_D = \frac{FC_p}{V\rho C_p} \,, \quad K_M = \frac{U_H A_H}{\rho V C_p} \,, \quad \alpha = K_D + K_M \,, \tag{3-4}$$

and C, D, and M are the Laplace transform of $c(t)-c_{ss}$, $d(t)-d_{ss}$, and $m(t)-m_{ss}$ respectively, and Q_D and Q_C are the Laplace transforms of L_D/L_M and L_C/L_M respectively.

Comparison of (3-1) with (1-2) shows that the following definitions are appropriate

$$\frac{K_D e^{-\tau_C s}}{s + \alpha} \triangleq P_D \,, \tag{3-5}$$

$$\frac{K_M e^{-(\tau_C + \tau_M) s}}{s + \alpha} \triangleq P_M \,. \tag{3-6}$$

For future convenience, the following definitions are also made:

$$\frac{K_D}{s + \alpha} \triangleq P_D^* \,, \tag{3-7}$$

$$\frac{K_M}{s + \alpha} \triangleq P_M^* \,. \tag{3-8}$$

The functions P_D^* and P_M^* are the so-called minimum phase portion of the transfer function. Since all of the constants are non-negative, notice, for this general class of systems, that the exponential time lag for the plant transfer function, P_M, is always equal to or larger than the lag of the load transfer function P_D.

Calculation of the optimal control function requires the values of plant transfer functions as well as the statistical characterization of the random disturbance. For a particular problem, disturbance parameters may be measured experimentally or calculated from theoretical considerations [B2,C2,L1,N2,S2]. It has been shown [L1,S2] that the spectral density of any random disturbance can be represented by the sum of terms of the form $\mu^2/(-s^2 + \sigma^2)$, and often one of these terms dominate. It will be assumed here that the spectral density of the disturbance can be adequately represented as

$$\Phi_{DD}(s) = \frac{2\mu^2\sigma}{\sigma^2 - s^2} \quad . \qquad (3\text{-}9)$$

This equation is an exact form of that representing (a) constant magnitude square waves whose sign changes as a random function of time, (b) square waves with randomly changing signs and amplitude, and (c) a gaussian noise produced by passing "white" noise through a first order filter. In general, signals with identical spectral density may have quite different time behavior.

Normally the exact form of the random disturbance does not exert an overpowering influence on the configuration of the optimal controller so that some liberties may be taken in specification of these parameters. If a more complicated spectral density is employed, additional difficulties may be expected in the solution of the design equations.

Canonical Form of Equations for Minimization

In order to proceed with the optimization, equations must be developed relating the disturbance to the controlled output which is to be minimized. Equations will also be developed relating the disturbance to the manipulative variable since this latter is an important state variable that will be subject to constraint.

The overall controlled system response to the disturbance is found by algebraic elimination of the manipulative variable from between (1-2) and (1-3) giving

$$C = P_D D + P_M M \, , \tag{1-2}$$

$$M = Q_D D + Q_C C \, , \tag{1-3}$$

$$C = \frac{P_D + P_M Q_D}{1 + P_M Q_C} D \, . \tag{3-10}$$

The output variable can be eliminated from between the same two equations giving the functional dependence of the manipulative variable on the disturbance,

$$M = \frac{Q_D - P_D Q_C}{1 + P_M Q_C} D \, . \tag{3-11}$$

It is seen that although the system was described by linear differential equations, the above expressions are not linear in the unknown control operators, Q_D and Q_C, so that solution of the equations shown in Chapter II would be very difficult at best. Therefore the general approach described by Newton, Gould, and Kaiser [N2] will be used. The relationships (3-10) and (3-11) will be written in terms of unknown operators that do form linear equations relating the output and manipulative variable to the disturbance. Thus after the optimization has been completed, the unknown operators, Q_C and Q_D, can be determined algebraically as functions of the intermediate operators.

The unknown intermediate operators must be chosen in such a way so that their physical realizability implies physical realizability of the real control functions. In other words, it would be undesirable to define the intermediate functions in such a way that, after solving for them while carefully excluding the possibility of predictive elements or unstable poles, it were found that the algebraic manipulations leading back to the real controllers introduced these undesirable elements. As a first step, define an overall control function, T_D, such that

$$\frac{C}{D} = T_D + P_D \ . \qquad (3\text{-}12)$$

This choice of T_D is somewhat arbitrary; several other forms could have been chosen but all lead to the same type of results. From (3-12) it is seen that C is a nonpredictive and stable function of D if both T_D and P_D are nonpredictive and stable. Optimization conditions have already been defined so that T_D is not predictive, and since P_D is a real plant transfer function, it cannot be predictive. P_D could be unstable however, and if (3-12) is to be used, unstable poles would have to be removed by internal feedback prior to the optimization. In principle, an unknown function could be defined permitting cancellation of an unstable pole with a r.h.p. zero but this procedure would lead to an unstable controller that would saturate due to stray residual noise. Thus a further restriction placed on both P_D and P_M of (1-2) is that unstable poles have been removed by feedback prior to optimization [N2].

Although the definition (3-12) is satisfactory from the standpoint of the C/D relationship, the M/D relationship must also be examined since obviously the manipulative variable may not be a predictive function of the disturbance. Comparison of (3-11) and (3-12) shows that

$$T_D = \frac{P_M(Q_D - P_D Q_C)}{1 + P_M Q_C} , \tag{3-13}$$

from which it follows

$$\frac{M}{D} = \frac{T_D}{P_M} . \tag{3-14}$$

Equation (3-14) presents no problems if P_M contains no non-minimum phase elements, i.e., r.h.p. zeros or pure time delays. If P_M does contain factors of the form $(s - z)e^{-\tau s}$, then the ratio of T_D/P_M will contain factors of the form $e^{\tau s}/(s - z)$ which implies that M could act in anticipation of the disturbance and/or is not a stable function of the disturbance. To prevent predictiveness or instability in M/D, T_D may be redefined by the following modification of (3-12):

$$\frac{C}{D} = B_M T_D + P_D , \tag{3-15}$$

where B_M is the product of non-minimum phase factors, $(s - z)e^{-\tau s}$, of P_M. This definition changes (3-13) to the form

$$T_D = \frac{P_M}{B_M} \frac{(Q_D - P_D Q_C)}{(1 + P_M Q_C)} , \tag{3-16}$$

and yields the following expression for the M/D ratio

$$\frac{M}{D} = \frac{B_M T_D}{P_M} = \frac{T_D}{P_M^*} , \tag{3-17}$$

in which the non-minimum phase elements have been cancelled. Both C/D and M/D are now nonpredictive and stable if T_D is.

The function T_D alone does not uniquely define either Q_D or Q_C; it merely determines a relationship between infinite sets of Q_D's and Q_C's any of which would give the same manipulative variable and output relationship with the disturbance. This result does not stem from the particular definition made for T_D in (3-16); basically the relationships of the manipulative variable and the output with the disturbance can be used to find only one unknown function. If two independent intermediate functions had been defined, say one primarily dependent on Q_C and the other primarily dependent on Q_D, then (3-16) could be used to find some particular combination of these two functions which would appear by itself in both of the equations defining C/D and M/D. Any optimization procedure dependent only on these two ratios could only define a form for the (3-16) relationship and not the individual components. For example, if T_1 and T_2 are defined as follows

$$T_1 \triangleq \frac{P_M}{B_M} \frac{Q_D}{1 + P_M Q_C} , \qquad (3\text{-}18)$$

$$T_2 \triangleq \frac{P_M}{B_M} \frac{P_D Q_C}{1 + P_M Q_C} , \qquad (3\text{-}19)$$

then from (3-16)

$$T_D = T_1 - T_2 , \qquad (3\text{-}20)$$

and

$$\frac{C}{D} = B_M(T_1 - T_2) + P_D , \qquad (3\text{-}21)$$

$$\frac{M}{D} = \frac{T_1 - T_2}{P_M^*} . \qquad (3\text{-}22)$$

Any solution of equations involving only C/D and M/D cannot be specific in T_1 or T_2, but only in the difference, $T_1 - T_2$. Interestingly enough, this result implies that the optimal feedback and feedforward controllers are not individually unique functions of the simple input-output relationships. Another unique intermediate function will be developed after additional quantities have been found which are to be optimized by the control system.

Minimization of Mean Square Error - No Constraints

Equations will be developed in this section to define the overall transfer function, T_D, which when substituted into (3-6), will give minimum mean square output. Optimal controls in the absence of constraints will be considered in some detail since they represent the simplest class of limiting cases for complicated situations that will be studied later. The mean square output may be computed from (2-8) and (3-15) as follows:

$$\varphi_{CC}(0) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} (B_M T_D + P_D)(\overline{B_M T_D} + \overline{P_D})\Phi_{DD} ds \, . \qquad (3\text{-}23)$$

The condition for optimal T_D corresponds to a minimum of this integral which is found by formal partial differentiation of the integrand, $\Phi_{CC}(s)$, with respect to $\overline{T_D}$ and setting the result equal to X:

$$\overline{B_M}(B_M T_D + P_D)\Phi_{DD} = X \, , \qquad (3\text{-}24)$$

where X has poles in the r.h.p. only. It has been seen that Φ_{DD} may be factored into two parts,

$$\Phi_{DD} = D\overline{D} \, , \qquad (2\text{-}17)$$

where D has only l.h.p. poles and zeros and $\overline{D}$ has only r.h.p. poles and zeros. Thus (3-23) may be expanded to give

$$\overline{B_M}B_M T_D D\overline{D} + \overline{B_M} P_D D\overline{D} = X . \qquad (3\text{-}25)$$

This equation will be solved first under the assumption that both P_D and P_M are minimum phase, i.e., τ_M and τ_C of (3-5) and (3-6) are both zero. Thus

$$B_M = B_D = 1 , \qquad (3\text{-}26)$$

and

$$T_D D\overline{D} + P_D^* D\overline{D} = X , \qquad (3\text{-}27)$$

where

$$\begin{aligned} P_D &= B_D P_D^* ; \\ P_M &= B_M P_M^* ; \end{aligned} \qquad (3\text{-}28)$$

and the B's are the non-minimum phase factors of the respective P's. (Note that if dead time exists only in the output sensing circuit, i.e., $\tau_M = 0$, $\tau_C > 0$, then (3-5) and (3-6) show that

$$B_D = B_M = e^{-\tau_C s} . \qquad (3\text{-}29)$$

Use of (3-29) with (3-28) in (3-25) also leads to (3-27) so that the following results are valid whenever there is no dead time in the controller action itself.)

Equation (3-27) is solved by dividing through by $\overline{D}$ and setting both sides of the equation equal to zero, giving

$$DT_D = -P_D^* D , \qquad (3\text{-}30)$$

or, since only l.h.p. poles and zeros are present,

$$T_D = -P_D^* . \qquad (3\text{-}31)$$

Substitution of this result into (3-16) gives

$$-\frac{P_D}{P_M} = \frac{Q_D - P_D Q_C}{1 + P_M Q_C} . \qquad (3\text{-}32)$$

As (3-32) only defines a relationship between Q_D and Q_C, an arbitrary form of one control function must be chosen in order to find the other.

A significant case to examine is the one in which feedback is absent, i.e., Q_C is zero, so that (3-32) gives

$$Q_D = - \frac{P_D}{P_M} . \qquad (3\text{-}33)$$

Substitution of $Q_C = 0$ and the value for Q_D from (3-33) into (3-6) shows that the "optimal" output for this design is identically zero. This relationship given by (3-33) is the "ideal" or "nominal" feedforward controller such as would be developed by ordinary feedforward design methods or by "invariance theory" [H3]. When a control system for a minimum phase plant is designed without constraints and when the feedback portion of such a design is arbitrarily made zero, the optimal control becomes ideal feedforward control.

It is of interest to examine another important specialization of (3-32). If the feedforward control is prohibited, i.e., Q_D is arbitrarily made zero, then it follows:

$$- \frac{P_D}{P_M} = - \frac{P_D Q_C}{1 + P_M Q_C} . \qquad (3\text{-}34)$$

This equation can be satisfied by a non-predictive control function, Q_C, only if $B_M = 1$, i.e., only if dead times τ_C and τ_M are zero, and then only by letting Q_C be infinitely large. Theoretically it appears that the perfect feedforward control of a minimum phase plant can be matched only by unreal infinite gain feedback control. This question will merit further discussion but it is clear here that not all of the physically important constraints have been included in the problem statement. Both the ideal feedforward control and the specification of infinite gain feedback control have resulted from this source.

Before proceeding to the development of constraints, the case of an unconstrained plant containing dead times will be considered using the example system described by (3-5) and (3-6). This case is studied separately because there are significant differences in both the solution method and result when dead times are present. Equation (3-25) may be rearranged

$$T_D D + \overline{B_M} P_D D = \frac{X}{\overline{D}} \; , \tag{3-35}$$

or by particularizing and using (3-22)

$$\frac{T_D \mu}{\sigma + s} + \frac{K_D e^{\tau_M s}}{\alpha + s} \cdot \frac{\mu}{\sigma + s} = \frac{X}{\overline{D}} \; . \tag{3-36}$$

The second term on the right hand side of (3-36) must be split into its physically realizable and unrealizable parts. Although this term has only l.h.p. poles, the exponential factor in the numerator makes part of it non-zero for $t < 0$. To see this point, consider a general case of a sum of fractions of the type $\frac{ke^{\tau s}}{\beta + s}$ where $\beta, \tau > 0$. (It is assumed throughout that the roots of the denominator are distinct. If they are not, the method of finding residues is different but the results are the same.) The inverse transform of this factor is $ke^{-\beta(t+\tau)}u(t+\tau)$ where $u(t)$ is the unit step function. A plot of this result is shown in Figure 3-2. The control function T_D cannot compensate for error which occurs for $t < 0$; hence the only part of Figure 3-2 that is useful for design purposes lies to the right of $t = 0$. This function is described by $ke^{-\beta(t+\tau)}u(t)$ which has the transform of $\frac{ke^{-\beta\tau}}{\beta + s}$ which is the principal part of the original term at this l.h.p. pole.

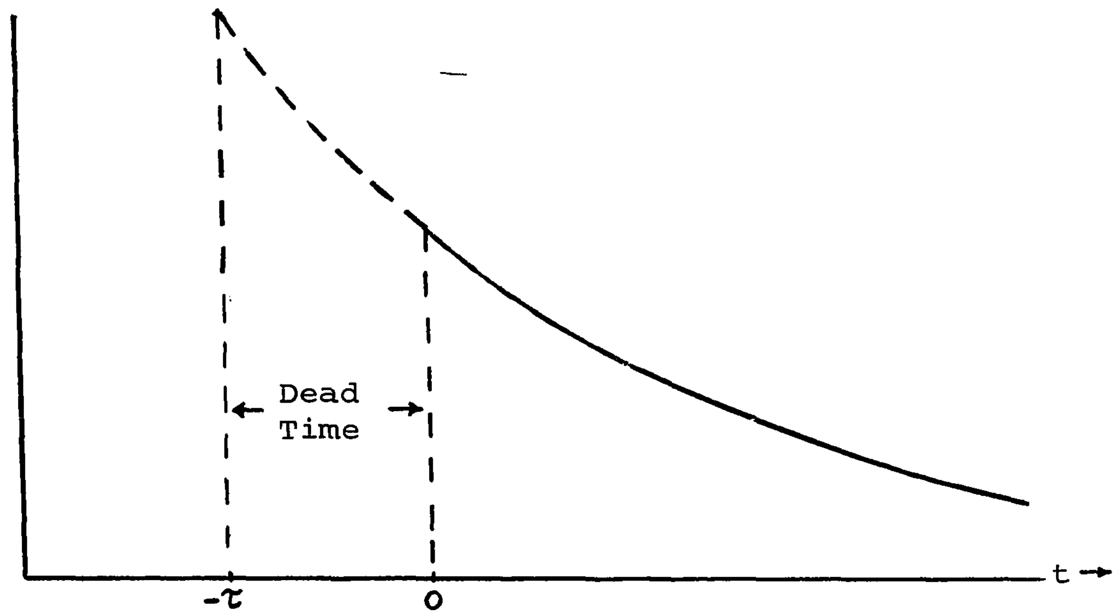

Figure 3-2. Inverse Transform of $\frac{ke^{\tau s}}{\beta + s}$

Solution of (3-35) now proceeds by retaining only the principal parts of the second term at l.h.p. poles which gives

$$\frac{T_D \mu}{\sigma + s} + \frac{K_D \mu}{\sigma - \alpha}\left[\frac{e^{-\tau_M \alpha}}{\alpha + s} - \frac{e^{-\tau_M \sigma}}{\sigma + s}\right] = 0 , \qquad (3\text{-}37)$$

from which it follows that

$$T_D = -P_D^*(T_{D0} + T_{D1}s) , \qquad (3\text{-}38)$$

where

$$T_{D0} \triangleq \frac{\alpha e^{-\tau_M \sigma} - \sigma e^{-\tau_M \alpha}}{\alpha - \sigma} ,$$

and

$$T_{D1} \triangleq \frac{e^{-\tau_M \sigma} - e^{-\tau_M \alpha}}{\alpha - \sigma} .$$

The relation between the specific controller functions, Q_C and Q_D, is found by substitution of (3-38) into (3-16):

$$-P_D^*(T_{D0} + T_{D1}s) = P_M^* \frac{Q_D - P_D Q_C}{1 + P_M Q_C} \quad . \tag{3-39}$$

This controller is the limiting case for all systems to be studied containing dead time and thus will be pursued further. First consider the controller for no feedback, i.e., $Q_C = 0$, so that it follows from (3-39) that

$$Q_D = -\frac{P_D^*}{P_M^*}(T_{D0} + T_{D1}s) \quad . \tag{3-40}$$

This result is the classical solution to the "predictor" problem [N2]. If a control system is asked to anticipate a random signal and maintain a minimum mean square deviation from the reference, the control action is attenuated to compensate only for that part of the signal it knows about. Differentiation of the signal is used to attempt to "predict" trends based on average frequencies.

The situation for the existence of feedback control only is found by allowing the feedforward function Q_D to be zero. Then from (3-39)

$$Q_C = \frac{e^{\tau_C s}\left(T_{D0} + T_{D1}s\right)}{P_M^*[1 - e^{-\tau_M s}(T_{D0} + T_{D1}s)]} \quad . \tag{3-41}$$

This solution is valid and Q_C is physically realizable if $\tau_C = 0$. Thus a feedback controller has been found which controls with the same degree of effectiveness as feedforward control. This equivalence is theoretically possible if there is a finite dead time in the controller but none in the output sensing loop so that both feedback and feedforward are subject to the same dead time.

This result can be clarified by consideration of the physical situation. If there is no dead time in the controller, the output response can theoretically be reduced to zero by feedforward control. Finite feedback control cannot do this well since the control action cannot be keyed to a variable that is identically zero. If pure time delays exist, however, a zero output cannot be achieved even by a perfect feedforward controller. In such a case, there is an output to which a feedback controller can be keyed. All that is necessary is to determine the functionality relating input and output and simply design the feedback controller to give the same signal as the feedforward controller gives for the same disturbance. Of course, limits of gain and noise will probably prevent realization of this controller in actual practice.

Control System with Constraint on Control Effort

The mathematics of optimization in the presence of a constraint have already been presented in Chapter II. Implementation of these techniques where a constraint is placed on the magnitude of available control effort is considered in the following discussion.

In general, the control effort may be constrained in several ways: (i) by the maximum level which it can attain; (ii) by some maximum rate of change; or (iii) by some maximum value of a linear combination of these or other operators on the manipulative variable. Thus let

$$A \stackrel{\Delta}{=} P_A M \, , \tag{3-42}$$

where P_A is some linear operator in the Laplace transform space, and let the constraining condition be

$$|a(t)| \leq \mathcal{L} \, , \tag{3-43}$$

where a(t) is the inverse transform of A.

The objective will be to find a T_D which minimizes the mean square output but in addition requires that (3-43) be valid a large fraction of the time. This dual goal will be achieved by minimizing the integral corresponding to (2-25) whose integrand is the sum

$$F(T_D,\lambda) = \Phi_{CC} + \lambda^2 \Phi_{AA} \ . \qquad (3\text{-}44)$$

The quantity, Φ_{CC}, is the Fourier transform of the mean square output as computed by (3-23) and Φ_{AA} is the integrand of

$$\varphi_{AA}(0) = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \frac{P_A \overline{P_A}}{P_M \overline{P_M}} T_D \overline{T_D} \Phi_{DD} ds \ . \qquad (3\text{-}45)$$

Substituting (3-23) and (3-45) into (3-44) and then formally differentiating the integrand with respect to $\overline{T_D}$ and setting the result equal to X leads to

$$[\overline{B_M}(B_M T_D + P_D) + \lambda^2 \frac{P_A \overline{P_A}}{P_M \overline{P_M}} T_D]\Phi_{DD} = X \ . \qquad (3\text{-}46)$$

Rearranging and using (2-10) gives

$$(1 + \lambda^2 \frac{P_A \overline{P_A}}{P_M \overline{P_M}})DT_D + P_D D\overline{B_M} = \frac{X}{\overline{D}} \ . \qquad (3\text{-}47)$$

The function contained in parentheses of the first term of (3-47) must be symmetric with respect to the imaginary axis - every pole or zero with a positive real part has a counterpart with a negative real part. Thus this term may be written as follows

$$1 + \lambda^2 \frac{P_A \overline{P_A}}{P_M \overline{P_M}} = Y\overline{Y} \ . \qquad (3\text{-}48)$$

where Y has l.h.p. poles and zeros only, and $\overline{Y}$ has r.h.p. poles and zeros only. With this substitution, (3-47) becomes

$$T_D YD + \frac{P_D D\overline{B_M}}{\overline{Y}} = \frac{X}{\overline{DY}} \; . \qquad (3\text{-}49)$$

The principal parts of the second term are found at l.h.p. poles and the physically realizable part of the left hand side of (3-49) is set equal to zero, giving

$$T_D = -\frac{1}{YD}\left[\frac{P_D D\overline{B_M}}{\overline{Y}}\right]_+ \; , \qquad (3\text{-}50)$$

where, as before, $[\ldots]_+$ means the sum of principal parts in the r.h.p. All quantities on the right hand side of (3-49) are known except the parameter, λ, which can be evaluated by substituting $T_D(\lambda)$ into (3-45) and then requiring that

$$\varphi_{AA}(0) = \mathcal{L} \; . \qquad (3\text{-}51)$$

If (3-51) is valid, then $\mathcal{L}$ is the variance of a(t). By proper selection of $\mathcal{L}$, (3-43) can be made true for an arbitrary fraction of the time provided that the distribution function for a(t) is known. Of course this distribution function is seldom known so that this condition may be approximated by using the normal (or any other suitable) distribution. A very conservative approach could be used in the form of the Chebychev inequality [W4].

To illustrate the foregoing general discussion, the specific solution will be obtained for the system described by (3-3). Assume that the constraint of (3-43) is taken to be only on the magnitude of control effort so that P_A is unity. Thus (3-48) becomes

$$Y\overline{Y} = 1 + \frac{\lambda^2}{K_M^2}(\alpha^2 - s^2) \; , \qquad (3\text{-}52)$$

and from (3-49) it follows that

$$T_D = \frac{-K_M^2}{\lambda^2}\frac{\sigma + s}{\beta + s}\left[\frac{K_D e^{\tau_M s}}{(\sigma + s)(\alpha + s)(\beta - s)}\right]_+ , \qquad (3\text{-}53)$$

where

$$\beta^2 = \alpha^2 + \frac{K_M^2}{\lambda^2} .$$

The principal parts of the bracketed expression are found at the l.h.p. poles giving explicitly

$$T_D = \frac{-K_M^2}{\lambda^2(\sigma - \alpha)} \frac{\left[\frac{\sigma e^{-\tau_M \alpha}}{\beta + \alpha} - \frac{\alpha e^{-\tau_M \sigma}}{\beta + \sigma} + \left(\frac{e^{-\tau_M \alpha}}{\beta + \alpha} - \frac{e^{-\tau_M \sigma}}{\beta + \sigma}\right) s\right]}{(\alpha + s)(\beta + s)} . \qquad (3\text{-}54)$$

The only unknown on the right hand side of this equation is λ which appears explicitly in the equation as well as implicitly in β. A numerical choice for λ fixes the functional form and numerical constants of T_D so that it can be substituted into (3-45) to find a numerical value for $\varphi_{AA}(0)$. Adjustments in the value chosen for λ could then be made until (3-51) is satisfied. With T_D fixed, the nominal output is computed using (3-23) and the relationship between the feedback and feedforward controllers (3-16) is defined.

It is of interest to investigate the form of the specific control functions, Q_D and Q_C, when only one of the two are permitted. To reduce writing, define constants so that (3-54) may be written

$$T_D = \frac{-T_{D0} - T_{D1} s}{(\alpha + s)(\beta + s)} . \qquad (3\text{-}55)$$

If Q_C is made zero, then using (3-55) in (3-16) gives

$$Q_D = \frac{-T_{D0} - T_{D1} s}{K_M(\beta + s)} . \qquad (3\text{-}56)$$

Conversely, if Q_D is made zero, then the same equations yield

$$Q_C = \frac{\dfrac{(+T_{D0} + T_{D1}s)(\alpha + s)}{K_M K_D(\beta + s)} e^{\tau_C s}}{1 - \dfrac{(T_{D0} + T_{D1}s)}{K_D(\beta + s)} e^{-\tau_M s}} \quad . \qquad (3\text{-}57)$$

It is noted here as with (3-41) that if the feedback dead time, τ_C, is zero, it is possible to define a feedback controller that theoretically performs as effectively as feedforward in output attenuation. Again the explanation lies in the fact that perfect output attenuation is not expected of the feedforward controller so that finite output exists to which feedback can be keyed. In (3-57) as in (3-41) the numerator of the feedback controller is one degree higher in s than the equivalent feedforward controller thus making it a differentiator. However, the feedforward controller operates directly on the input disturbance while the feedback controller operates on the partially integrated and smoothed output. The net result can intuitively be seen to be equivalent.

Effect of Error in the Mathematical Model

In the previous development, optimization goals have been achieved using only one of the two available degrees of freedom which was accomplished by arbitrarily choosing one of the two control functions as zero. In a particular application, circumstances may make it necessary to choose one control function to be zero but in general, the "best" combination of Q_C and Q_D will be sought. It has been already pointed out that other constraints will be required to specify the individual functions, Q_C and Q_D, uniquely. An additional constraint will be defined through consideration of feedforward control limitations. These limitations have made it secondary in use to feedback even though the previous equations seem to indicate the equality if not superiority of feedforward.

One of the principal weaknesses of feedforward control is its sensitivity to error in the mathematical model used in its design. If feedforward control is based on an inaccurate model, it will produce partially inappropriate control action. No knowledge of its error is transmitted back to the controller since feedforward compensation is based only on the value of the input.

Model error can be divided into two classes: (i) permanent error, i.e., a value of a parameter which has been assumed or measured inaccurately; or (ii) transient error, i.e., parameter values which change with time. The former type of error will succumb to development of accurate model identification techniques, or can be "tuned out" by adaptive control practices. However the variability of transient error will cause model error output indefinitely. Examples of the latter range from scaling of heat transfer surfaces and ambient temperature cycling to variations of catalyst concentration or activities in systems where other variables are the measured forcing functions. In short, this type of error is a form of unmeasured disturbance which is regarded here in relation to the transfer functions of the monitored variables. An optimal controller must take this type of error into consideration.

Computation of the model error factor will be made by determination of incremental response due to incremental changes to the system parameters. Thus, from (3-6)

$$C + \Delta C = (T_D^* + \Delta T_D^* + P_D + \Delta P_D)(D + \Delta D) \qquad (3\text{-}58)$$

where ΔC, ΔT_D^*, ΔP_D and ΔD are errors and $T_D^* = B_M T_D$. From definition (2-5), the spectral density of this output is

$$\Phi_{C+\Delta C, C+\Delta C} = \lim_{T\to\infty} \frac{1}{T} \langle (C + \Delta C)_T \overline{(C + \Delta C)_T} \rangle . \qquad (3\text{-}59)$$

The error terms may be restricted to factors that are uncorrelated with system time constants by inclusion of the known, correlatable portion in these parameters. This inclusion would occur automatically, for instance, in empirical experimental system identification techniques. Thus, assume

$$\lim_{T\to\infty} \frac{1}{T} \langle C_T \Delta C_T \rangle = \lim_{T\to\infty} \frac{1}{T} \langle \Delta C_T C_T \rangle = 0 , \qquad (3\text{-}60)$$

and (3-59) becomes

$$\Phi_{C+\Delta C, C+\Delta C} = \lim_{T\to\infty} \frac{1}{T} \langle C_T \overline{C_T} + \Delta C_T \overline{\Delta C_T} \rangle = \Phi_{CC} + \Phi_{\Delta C \Delta C} . \qquad (3\text{-}61)$$

As shown by (A-13), the contour integral of the last term in this equation is positive unless $\Delta c(t)$ is identically zero.

The mean value of the "error output," ΔC, may be found by subtraction of (3-15) from (3-58) which yields

$$\Delta C = (\Delta T_D^* + \Delta P_D) D + (T_D^* + P_D) \Delta D + (\Delta T_D^* + \Delta P_D) \Delta D . \qquad (3\text{-}62)$$

The following covariances are assumed to be zero,

$$\langle D \Delta D \rangle = \langle (\Delta T_D^* + \Delta P_D)(T_D^* + P_D^*) \rangle = \langle \Delta D (\Delta T_D^* + \Delta P_D) \rangle = 0 , \qquad (3\text{-}63)$$

and the definition of spectral density is used again so that the mean square value of the error output is

$$\langle \Delta C \overline{\Delta C} \rangle = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} [\langle (\Delta T_D^* + \Delta P_D) \overline{(\Delta T_D^* + \Delta P_D)} \rangle D\overline{D} + + (T_D^* + P_D)(T_D^* + P_D) \langle \Delta D \overline{\Delta D} \rangle] ds . \qquad (3\text{-}64)$$

Equation (3-64) gives the value of output which is to be expected because of error in the mathematical model used to design the control system. A simple way to include the effect of model error in the design procedure would be to minimize the integral of the sum in (3-15). Somewhat more generally, a minimization of $\langle \Delta c^2 \rangle$ with a constraint on the value of $\langle c^2 \rangle$ will be considered so that the effect of emphasizing reduction of either $\langle c^2 \rangle$ or $\langle \Delta c^2 \rangle$ on the optimal design can be evaluated.

Explicit Determination of Parameter Error

If the output defined by (3-64) is to be included in the optimization procedure, the Δ's must be related to the primary parameter variations. These relationships will be developed in the following discussion.

Assume that the various functions can be expanded into Maclaurin's series with respect to the parameter variations:

$$P(\xi) = P(0) + \left.\frac{\partial P}{\partial \xi}\right|_{\xi=0} \xi + \frac{1}{2!}\left.\frac{\partial^2 P}{\partial \xi^2}\right|_{\xi=0} \xi^2 + \ldots , \qquad (3\text{-}65)$$

where P is a transfer function and ξ is error in a particular parameter. P(0) is the estimate of $P(\xi)$ used for design; the term $\frac{\partial P}{\partial \xi}$ is called the "error coefficient" [T1]. As a first approximation to $P(\xi)$, only the linear term of this series will be retained assuming that the higher order terms are small with respect to the first:

$$\Delta P = P(\xi) - P(0) = \left.\frac{\partial P}{\partial \xi}\right|_{\xi=0} \xi \, . \qquad (3\text{-}66)$$

In all of the problems considered here, the operators are rational functions in s multiplied by an exponential time lag. Therefore representation of the general transfer function with its errors is

$$P(\xi) = (K + \Delta K) \frac{\prod_i (z_i + \Delta z_i + s)}{\prod_j (p_j + \Delta p_j + s)} e^{-(E + \Delta E)s} , \qquad (3\text{-}67)$$

where ξ is the vector $[\Delta K, \Delta z_1, \ldots, \Delta z_n, \Delta p_1, \ldots, \Delta p_m, \Delta E]$. The following equations are derived from the definitions in (3-67):

$$\left.\frac{\partial P}{\partial \Delta K}\right|_{\Delta K=0} = \frac{P}{K} ,$$

$$\left.\frac{\partial P}{\partial \Delta z_i}\right|_{\Delta z_i=0} = \frac{P}{z_i + s} ,$$

$$\left.\frac{\partial P}{\partial \Delta p_j}\right|_{\Delta p_j=0} = \frac{-P}{p_j + s} , \qquad (3\text{-}68)$$

$$\left.\frac{\partial P}{\partial \Delta E}\right|_{\Delta E=0} = -sP .$$

Therefore the error due to parameter variation may be approximated

$$\Delta P = \left[\frac{\Delta K}{K} + \sum_i \frac{\Delta z_i}{z_i + s} - \sum_j \frac{\Delta p_j}{p_j + s} - \Delta E_s \right] P . \qquad (3\text{-}69)$$

Equation (3-16) shows that T_D^* is affected by errors in the plant functions as well as errors in the controller functions. Normally however, any variations in the controller will be at least an order of magnitude less than those in the plant functions. Time constants of transmission lines and control valve reactions would be included in the plant transfer functions (possibly as an equivalent dead time). For these reasons, it will be assumed that

$$\Delta Q_C = \Delta Q_D = 0 \ . \tag{3-70}$$

Therefore,

$$\Delta T_D^* = \frac{\partial T_D^*}{\partial P_M} \Delta P_M + \frac{\partial T_D^*}{\partial P_D} \Delta P_D \ . \tag{3-71}$$

Evaluation of the partial derivatives using (3-16) gives

$$\Delta T_D^* = \frac{T_D^*}{1 + P_M Q_C} \frac{\Delta P_M}{P_M} - \frac{P_D P_M Q_C}{1+P_M Q_C} \frac{\Delta P_D}{P_D} \ , \tag{3-72}$$

from which it follows that

$$\Delta T_D^* + \Delta P_D = \frac{T_D^*}{1 + P_M Q_C} \frac{\Delta P_M}{P_M} + \frac{P_D}{1 + P_M Q_C} \frac{\Delta P_D}{P_D} \ . \tag{3-73}$$

In (3-73) there again appears an undefined function, Q_C, to be minimized, which is not linear in describing a "state" quantity, ΔC. No amount of rearranging will allow ΔC to be described by a single unknown function of Q_C and Q_D which is linear in describing both C and ΔC; hence a new unknown function must be introduced. A number of choices are possible but the one chosen here will be defined as

$$T_C \triangleq \frac{1}{B_M B_D} \frac{P_M P_D Q_C}{1 + P_M Q_C} \ , \tag{3-74}$$

where B_D is the non-minimum phase portion of P_D. Other choices of the form of T_C would not affect the end result. Using this definition in (3-73) gives

$$\Delta T_D^* + \Delta P_D = \left(T_D^* \frac{\Delta P_M}{P_M} + \Delta P_D \right) \left(1 - \frac{B_M B_D T_C}{P_D} \right) , \tag{3-75}$$

from which it follows that

$$\Phi_{\Delta C,\Delta C} =$$

$$= \left(T_D^* \frac{\Delta P_M}{P_M} + \Delta P_D \right) \overline{\left(T_D^* \frac{\Delta P_M}{P_M} + \Delta P_D \right)} \left(1 - \frac{B_M B_D T_C}{P_D}\right) \overline{\left(1 - \frac{B_M B_D T_C}{P_D}\right)} \Phi_{DD} . \tag{3-76}$$

Thus the spectral density of a third state variable has been defined adding another dimension to the optimization problem. The previously described optimization procedure can be used to find the control function for a minimum of the mean square value of one of these quantities while the variances of the other two equal the respective constants. There are six permutations of completely equivalent ways to formulate this problem. The one used here is given as follows:

Assume that the control effort is constrained by

$$\langle a(t)^2 \rangle \leq \mathfrak{L} , \tag{3-77}$$

and that the system output is constrained by

$$\langle c(t)^2 \rangle \leq \mathfrak{H} , \tag{3-78}$$

then find the operators T_D^* and T_C such that $\langle \Delta c(t)^2 \rangle$ is minimized.

Resorting to the use of Lagrange multipliers, the problem becomes that of minimizing a contour integral whose integrand is:

$$F(T_D, T_C, \lambda_1, \lambda_2) = \Phi_{\Delta C,\Delta C} + \lambda_1^2 \Phi_{CC} + \lambda_2^2 \Phi_{AA} ; \tag{3-79}$$

where the contour integrals of Φ_{AA} and Φ_{CC} satisfy the constraints of (3-77) and (3-78) respectively. The minimum is found by formal differentiation with respect to $\overline{T_D}$ and $\overline{T_C}$

and setting the results equal to X_1 and X_2 which are functions with r.h.p. poles only. Explicitly

$$\frac{\partial F}{\partial \overline{T_D}} = \overline{B_M}\,\frac{\overline{\Delta P_M}}{\overline{P_M}}\left(B_M T_D \frac{\Delta P_M}{P_M} + \Delta P_D\right)\left(1 - \frac{B_M B_D T_C}{P_D}\right)\overline{\left(1 - \frac{B_M B_D T_C}{P_D}\right)} D\overline{D} +$$

$$+ (\Delta D \overline{\Delta D} + \lambda_1^2 D\overline{D})\overline{B_M}(B_M T_D + P_D) + \lambda_2^2 \frac{P_A \overline{P_A}}{P_M \overline{P_M}} T_D D\overline{D} = X_1 \,, \tag{3-80}$$

and

$$\frac{\partial F}{\partial \overline{T_C}} = \overline{\left(\frac{B_M B_D}{P_D}\right)}\left(1 - \frac{B_M B_D T_C}{P_D}\right)\left(\frac{B_M T_D}{P_M}\Delta P_M + \Delta P_D\right)\overline{\left(\frac{B_M T_D}{P_M}\Delta P_M + \Delta P_D\right)} D\overline{D}$$

$$= X_2 \,. \tag{3-81}$$

Solution of (3-81) proceeds by noting that the last two factors form a symmetric function which is factorable into parts with l.h.p. poles and zeros, and r.h.p. poles and zeros. Let these factors be Y and $\overline{Y}$ respectively. Then,

$$\left(1 - \frac{B_M B_D T_C}{P_D}\right)\overline{\left(\frac{B_M B_D}{P_D}\right)} Y = \frac{X_2}{\overline{Y}} \,. \tag{3-82}$$

If the transfer functions are all minimum phase, i.e., if B_M and B_D are both unity, (3-82) can be satisfied if $T_C = P_D$ or if $Y = 0$. If $T_C = P_D$ then (3-80) can be solved for T_D in a manner very similar to that already investigated. If $Y = 0$, the first term of (3-80) is also zero and it would be nothing more than coincidence if T_D were defined so that it also satisfied the second part of (3-80). Therefore, the solution $T_C = P_D$ will be examined. The more complicated problem of non-minimum phase transfer functions would require evaluation of physically realizable parts as in (3-37).

From (3-74), $T_C = P_D$ if and only if Q_C approaches infinity at all frequencies, a solution which has been met earlier. Here, however, the situation is different; Q_D has not been arbitrarily set equal to zero; indeed, there is no arbitrariness whatsoever in this solution. After (3-80) has been solved, as previously, yielding a value for T_D, then (3-16) may be solved for Q_D. For a very large Q_C, (3-16) may be rearranged

$$Q_D = \frac{1}{P_M} (T_D + P_D) \ . \tag{3-83}$$

Thus, although Q_C approaches infinity, the feedforward function, Q_D, remains a finite function balancing the feedback so that the overall control function, T_D, becomes such that control effort and output attenuation constraints are met. This result contradicts the notion that is sometimes implied that infinite gain feedback theoretically causes infinite control effort. The infinite gain feedback in this computation achieves near-zero output by adding "mirror images" of the measured and unmeasured disturbance to the plant. Of course this feedback is physically unrealizable despite the fact that it is within previous constraints so that a further examination of design bases and constraints is in order.

The result which pitted an infinite feedback system against the feedforward to achieve a finite, optimal overall transfer function seems to be the result, at least in part, of assuming zero error in the controller. The results which call for infinite gain of the feedback controller stem from a more subtle source. The basic problem is that continuous equations are describing a physical phenomena which has, in effect, a limited threshold. Consider the following analogous situation from the field of diffusion.

If a salt shaker were emptied into the Mississippi River, ordinary continuous diffusion equations would indicate that if a chemist in Tokyo took a series of samples, in principle, he could detect a change in salt concentration due to this disturbance even though the level of the change were infinitesimally small. Likewise, the feedback controller in the above design is being called upon to make corrections based upon output changes which are very small. If a zero mean square output is not required, the feedback controller attempts to maintain the variance of the output as near to the constraint limit as possible to minimize other values, even in the face of parameter variations. To achieve this result, maximum gain or a very narrow proportional band is employed. While the design method has been developed with constraints against instability (cf. (3-15)), a feedback gain of infinity can be no better than borderline stable. Infinite gain will be applied not only to the signal but also to any noise in the controller and in the sensing element. Control action based on this noise introduces error in the output, as well as contributing to saturation of the controller. An addition to the equations will be introduced in the next section to counteract the tendencies discussed here.

Design Equations for Finite Feedback Control

In control systems of the type considered here, the controller measures the value of some state variable and then computes a control effort that depends on the value of this state variable. In a real situation, it processes not only the value of the given state variable but also any error that is present in the sensed value of the variable and any noise generated in the controller. The system output is therefore not the result of only the primary disturbance and the ideal computed control effort but in addition is corrupted by control effort erroneously computed from controller noise. This excess control effort also contributes to saturation of the controller capacity. These two effects limit the degree of control effort that can be keyed to a given variable. From a slightly different point of view, the signal-to-noise ratio of the quantity which ultimately activates the manipulative variable must be sufficiently high so that there is more disturbance attenuation than output corruption.

These effects are usually much more important in the feedback portion of the control system than the feedforward. It is seldom desirable to initiate a great deal of feedforward control effort because of a small disturbance. Disturbance signals important enough to affect a plant output significantly are generally large enough to give high signal-to-noise ratios in a controller. On the other hand, feedback controllers are particularly susceptible to these difficulties. The very success of any control effort adds to the problem - as the output becomes closer to the ideal of zero, the signal-to-noise ratio of the feedback diminishes. In the limit, the feedback controller will be attempting to compensate for signals generated within itself and can tend to produce more output than it eliminates. This behavior is an instability that is somewhat different from that traditionally produced by poles in the r.h.p. Both however, tend to occur at high amplification factors and prevention of this effect is one of several concealed constraints in the "gain margin" and similar factors built into the traditional design methods. For the above reasons, a signal-to-noise type constraint will be imposed only on the feedback system.

The formulation of a gain constraint on the feedback presents somewhat of a problem in order that it be in a form amenable to solution by Wiener's techniques. It would also be desirable to avoid introduction of another Lagrange multiplier since each new variable introduces another dimension to be explored by numerical calculation and Bellman's "Curse of Dimensionality" [B1] becomes especially acute.

The method to be used here will be to alter (1-3) so that the feedback function, Q_C, operates not only on the output, C, but also on a random noise factor, δ. (The same Greek letter will be used for the factor and its transform; the context should indicate which is meant if the argument is not stated.) The factor δ tends to mask the effect of small outputs so that most feedback control effort is delayed until the output exceeds δ. Thus the response somewhat resembles that of a system with a small dead time. The manipulative variable then becomes

$$M = Q_D D - Q_C C(C + \delta) \ . \tag{3-84}$$

The other equations remain unchanged but will be repeated here for convenience while deriving the final design equations:

$$C = P_D D + P_M M \ , \tag{1-2}$$

$$T_D = P_M^* \frac{Q_D - Q_C P_M}{1 + P_M Q_C} \ , \tag{3-7}$$

$$A = P_A M \ , \tag{3-42}$$

$$\Delta C = (\Delta[B_M T_D] + \Delta P_D)D + (B_M T_D + P_D)\Delta D \ , \tag{3-62}$$

$$T_C = \frac{P_D}{B_M B_D} \cdot \frac{P_M Q_C}{1 + P_M Q_C} . \tag{3-64}$$

From the above it follows

$$M = \frac{(Q_D - Q_C P_D) D - Q_C \delta}{1 + P_M Q_C} = \frac{B_M}{P_M} \left(T_D D + \frac{B_D}{P_D} T_C \delta \right) , \tag{3-85}$$

and

$$C = \frac{(P_D + P_M Q_D) D - P_M Q_C \delta}{1 + P_M Q_C} = (B_M T_D + P_D) D + \frac{B_M B_D}{P_D} T_C \delta , \tag{3-86}$$

and

$$\Delta C = \left(\frac{B_M T_D}{P_M} \frac{\Delta P_M}{1 + P_M Q_C} + \frac{\Delta P_D}{1 + P_M Q_C} \right) D + (B_M T_D + P_D) \Delta D$$

$$= \left(\frac{B_M T_D}{P_M} \Delta P_M + \Delta P_D \right) \left(1 - \frac{B_M B_D T_C}{P_D} \right) D + (B_M T_D + P_D) \Delta D . \tag{3-87}$$

The term δ does not appear in (1-2), but is introduced into (3-86) by (3-84) and is neglected in finding ΔC.

These equations are used to find the optimizing T_D and T_C the same way as in the foregoing developments. The function,

$$F(T_D, T_C, \lambda_1, \lambda_2) = \Phi_{\Delta C, \Delta C} + \lambda_1^2 \Phi_{CC} + \lambda_2^2 \Phi_{AA} , \tag{3-88}$$

is the integrand of the contour integral which is to be minimized. From (3-85) to (3-87),

$$\Phi_{\Delta C, \Delta C} = \left(B_M T_D \frac{\Delta P_M}{P_M} + \Delta P_D\right) \overline{\left(B_M T_D \frac{\Delta P_M}{P_M} + \Delta P_D\right)} \left(1 - \frac{B_M B_D T_C}{P_D} \right) \cdot$$

$$\cdot \overline{\left(1 - \frac{B_M B_D T_C}{P_D} \right)} \Phi_{DD} + (B_M T_D + P_D) (B_M T_D + P_D) \Phi_{\Delta D \Delta D} , \tag{3-89}$$

$$\Phi_{CC} = (B_M T_D + P_D)\overline{(B_M T_D + P_D)}\Phi_{DD} + \frac{B_M \overline{B_M} B_D \overline{B_D} T_C \overline{T_C}}{P_D \overline{P_D}} \Phi_{\delta\delta} \qquad (3\text{-}90)$$

$$\Phi_{AA} = B_M \overline{B_M} \frac{P_A \overline{P_A}}{P_M \overline{P_M}} \left(T_D \overline{T_D} \Phi_{DD} + \frac{B_D \overline{B_D} T_C \overline{T_C}}{P_D \overline{P_D}} \Phi_{\delta\delta}\right) . \qquad (3\text{-}91)$$

In these equations it was assumed that no correlation exists between D and ΔD or δ.

Necessary and sufficient conditions for the minimum of (3-88) are found as before by formal partial differentiation of the integrand with respect to $\overline{T_D}$ and $\overline{T_C}$ and setting the two results equal to X_1 and X_2, functions with r.h.p. poles only.

$$\frac{\partial F}{\partial \overline{T_D}} = \frac{\overline{B_M \Delta P_M}}{\overline{P_M}} \left(B_M T_D \frac{\Delta P_M}{P_M} + \Delta P\right) \left(1 - \frac{B_M B_D T_C}{P_D}\right) \overline{\left(1 - \frac{B_M B_D T_C}{P_D}\right)} \Phi_{DD} +$$
$$+ \overline{B_M}(B_M T_D + P_D)\Phi_{\Delta D \Delta D} + \lambda_1^2 \overline{B_M}(B_M T_D + P_D)\Phi_{DD}$$
$$+ \lambda_2^2 B_M \overline{B_M} \frac{P_A \overline{P_A}}{P_M \overline{P_M}} T_D \Phi_{DD}$$
$$= X_1 , \qquad (3\text{-}92)$$

$$\frac{\partial F}{\partial \overline{T_C}} = - \frac{B_M \overline{B_M}}{\overline{P_D}} \left(B_M T_D \frac{\Delta P_M}{P_M} + \Delta P_D\right) \overline{\left(B_M T_D \frac{\Delta P_M}{P_M} + \Delta P_D\right)} \left(1 - \frac{B_M B_D T_C}{P_D}\right) \cdot$$
$$\cdot \Phi_{DD} + \lambda_1^2 \frac{B_M \overline{B_M} B_D \overline{B_D}}{P_D \overline{P_D}} T_C \Phi_{\delta\delta} + \lambda_2^2 \frac{B_M \overline{B_M} P_A \overline{P_A} B_D \overline{B_D}}{P_M \overline{P_M} P_D \overline{P_D}} T_C \Phi_{\delta\delta}$$
$$= X_2 . \qquad (3\text{-}93)$$

The spectral density of δ appears only in the second of these equations which, as will be seen, determines T_C. The noise factor, δ, affects the overall transfer function only insofar as it affects T_C thereby changing the amount of identification uncertainty that must be allowed for in the feedforward control.

Since there are two equations with two unknowns, in principle it is possible to reduce by elimination to one equation which can be solved for one unknown. Since the unknowns are functions however, and functions with differing arguments at that, this elementary approach will not work. Therefore, recourse must be taken to successive numerical approximations to obtain a solution. Although convergence of the solution has not been shown, there are reasons to believe that the numerical process will always converge rapidly to a solution and the examples worked herein show convergence. Although this situation is a rather unsatisfactory state of affairs, it has the saving grace that it seems to work.

The general method of solution is based on the idea that the principal reason for existence of the control system is the elimination of output, that is, to make $C \equiv 0$. It was seen previously that a successful design for $C \doteq 0$ is $B_M T_D \doteq -P_D$. On the other hand, if the control effort is very small because of constraints, then $B_M T_D \doteq 0$. Although a number of other important factors have been introduced to the equations, an approximation to be used in (3-85) to the overall control effort would be some constant, γ, times P_D, i.e., let

$$B_M T_D \doteq -\gamma P_D \ , \quad 0 \leq \gamma \leq 1 \ . \tag{3-94}$$

Equations (3-92), (3-93) and (3-94) could then be solved by successive approximations adjusting the value for γ until

$$\langle c(t)^2 \rangle_{\text{controlled}} \doteq (1 - \gamma)^2 \langle c(t)^2 \rangle_{\text{uncontrolled}} \; . \qquad (3\text{-}95)$$

The resulting control functions would then be very near optimal.

Actual calculations later indicate that the control design is quite insensitive to the value used for γ in (3-94). This result is logical since often the error factors themselves are little more than shrewd guesses and seldom are functions known with a high degree of accuracy. A choice of $\gamma = 1$ was found to be quite satisfactory for reasonably effective controllers.

With the aid of (3-95), equation (3-93) may be solved explicitly for T_C. Using (3-95) in (3-93) and rearranging the result gives

$$T_C \left\{ \frac{B_M \overline{B_M}}{P_D^* \overline{P_D^*}} \left[P_D \overline{P_D} \left(\frac{\Delta P_D}{P_D} - \gamma \frac{\Delta P_M}{P_M} \right) \overline{\left(\frac{\Delta P_D}{P_D} - \gamma \frac{\Delta P_M}{P_M} \right)} \Phi_{DD} + \right.\right.$$
$$\left.\left. + \left(\lambda_1^2 + \lambda_2^2 \frac{P_A \overline{P_A}}{P_M \overline{P_M}} \right) \Phi_{\delta\delta} \right] \right\} - \left\{ \overline{B_M B_D} P_D \left(\frac{\Delta P_D}{P_D} - \gamma \frac{\Delta P_M}{P_M} \right) \cdot \right.$$
$$\left. \cdot \overline{\left(\frac{\Delta P_D}{P_D} - \gamma \frac{\Delta P_M}{P_M} \right)} \Phi_{DD} \right\} = X_2 \; . \qquad (3\text{-}96)$$

The function enclosed in braces in the first term does not change value when (-s) is substituted for s and may be factored into parts with only l.h.p. or r.h.p. poles and zeros. Defining new functions and repeating (3-96)

$$T_C Y\overline{Y} - Z = X_2 \; ; \qquad (3\text{-}97)$$

solving as in previous sections:

$$T_C = \frac{1}{Y} \left[\frac{Z}{\overline{Y}} \right]_+ \; . \qquad (3\text{-}98)$$

With T_C known, T_D is found by rearranging (3-84) to give

$$T_D \left\{ B_M \overline{B_M} \frac{\Delta P_M \overline{\Delta P_M}}{P_M \overline{P_M}} \left(1 - \frac{B_M B_D T_C}{P_D} \right) \overline{\left(1 - \frac{B_M B_D T_C}{P_D} \right)} \Phi_{DD} + \right.$$

$$+ B_M \overline{B_M} \left[\Phi_{\Delta D \Delta D} + \left(\lambda_1^2 + \lambda_2^2 \frac{P_A \overline{P_A}}{P_M \overline{P_M}} \right) \Phi_{DD} \right] \Big\} +$$

$$+ \left\{ \overline{\frac{B_M \Delta P_M}{P_M}} \Delta P_D \left(1 - \frac{B_M B_D T_C}{P_D} \right) \overline{\left(1 - \frac{B_M B_D T_C}{P_D} \right)} \Phi_{DD} + \right.$$

$$\left. + \overline{B_M} P_D [\Phi_{\Delta D, \Delta D} + \lambda_1^2 \Phi_{DD}] \right\} = X_1 \ ; \qquad (3\text{-}99)$$

or

$$T_D U \overline{U} + W = X_1 \ . \qquad (3\text{-}100)$$

and this equation is solved as before. There are <u>no direct approximations</u> in this equation <u>except</u> for that included in <u>T_C</u> from (3-94) through (3-98). The assumption for T_D made in that equation has the effect here of influencing the degree of desirable "detuning" of T_D because of model mis-identification. Again it is intuitive that the solution to this set of equations should be fast converging.

With T_C and T_D both defined, arbitrariness discussed earlier has been removed from Q_C and Q_D. Solving (3-64) for Q_C gives

$$Q_C = \frac{\dfrac{B_M B_D}{P_M P_D} T_C}{1 - \dfrac{B_M B_D}{P_D} T_C} \ . \qquad (3\text{-}101)$$

The non-minimum phase factors in the numerator B_M and B_D are cancelled by their counterparts in P_M and P_D from which these factors arose. In the denominator, the second term vanishes as $s \to \infty$ and since Q_C cannot be infinite at any frequency, the second term must be less than one at all frequencies. Furthermore the second term is positive because Q_C must not be negative - otherwise the factor $1/(1 + P_M Q_C)$ would not attenuate disturbances (cf. (3-64)). For these reasons the denominator may be expressed in a power series as follows:

$$\frac{1}{1 - \frac{B_M B_D}{P_D} T_C} = 1 + \frac{B_M B_D}{P_D} T_C + \left(\frac{B_M B_D}{P_D} T_C \right)^2 + \ldots , \qquad (3\text{-}102)$$

and is physically realizable since the terms can contain only dead times and/or r.h.p. zeros (B_D cancels positive exponentials and r.h.p. poles arising from r.h.p. zeros of P_D).

The physical significance of this solution is interesting. The control action at the time of sensing the output is given by the numerator of (3-101). After a period of time corresponding to the dead time in the system (from sensing until control action begins), the control effort is modified by the second term in the above expansion. After another similar lag, a third modification occurs as a result of the third term above, and so on. Thus the controller applies what it considers to be ideal corrective action due to any disturbance. This action is modified when information about the results of the original change reach the controller.

With T_C, Q_C and T_D known, rearrangement of (3-16) can be solved for Q_D using definition (3-64):

$$B_M T_D = \frac{P_M Q_D}{1 + P_M Q_C} - B_M B_D T_C , \qquad (3\text{-}103)$$

or

$$Q_D = \frac{B_M}{P_M} (T_D + B_D T_C)(1 + P_M Q_C) \quad ; \tag{3-104}$$

and since

$$1 + P_M Q_C = \frac{1}{1 - \frac{B_M B_D}{P_D} T_C} , \tag{3-105}$$

Q_D then becomes

$$Q_D = \frac{\frac{B_M}{P_M} (T_D + B_D T_C)}{(1 - \frac{B_M B_D}{P_D} T_C)} . \tag{3-106}$$

The numerator of this function is realizable since B_M cancels non-minimum phase elements in P_M and the denominator is expandable as in (3-102).

For the situation of high quality nominal control, i.e., when T_D approaches $- P_D$, it is of interest to consider a rearranged version of (3-106), namely,

$$Q_D = \frac{B_M T_D}{P_M} \left(\frac{1 - \left(\frac{-P_D}{T_D} \right) B_D \frac{T_C}{P_D}}{1 - B_M B_D \frac{T_C}{P_D}} \right) , \tag{3-107}$$

from which it is seen that

$$\lim_{T_D \to -P_D} Q_D = B_M \frac{T_D}{P_M} . \tag{3-108}$$

Actually the approximation of (3-108) is valid whenever reasonably good output attenuation is found from the design equations.

Detailed algebraic solution of the foregoing design equations is given in Appendix B for a system similar to that of (3-3) except that it is generalized to third order in the denominator and first order in the numerator. Digital computer programs written in Osage Algol* providing numerical solutions to these equations are listed in Appendix D.

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*The Osage Algol language is a modification of Algol 60 [W2,N1] used on the Osage high speed computer at the University of Oklahoma Computer Center.

CHAPTER IV

ANALOG COMPUTER SIMULATIONS

In this chapter, analog computer simulation for the various controllers which were developed in previous chapters will be considered. The analog computer is rather idealized compared to those systems encountered in practice. The mathematical model is known with a precision and accuracy usually unattainable in a process plant. Non-linearities are virtually absent unless deliberately programmed and the extraneous noise level is much lower than normally encountered in a real system. These factors can give a misleading picture of controller efficacy unless some steps are taken to consider them deliberately. Nonetheless analog computer results can often give a valuable intuitive insight to the statistical quantities which have been heretofore drily computed.

The primary disturbance used for these tests was derived from a custom built signal generator in which, at intervals determined by an adjustable time base, the regulated output voltage was switched electronically in a random fashion depending on whether the number of electrical impulses collected on the plate of a thyratron tube has been odd or even during the time base interval. The instrument used generated square waves of constant absolute magnitude but with random sign changes whose average frequency was variable over a range of 0.5 to 500 zero crossings per second.

The desired spectral density of the input disturbance is obtained by filtering the noise generator output. Thus if Φ_{DD} is the desired spectral density and $\Phi_{D^*D^*}$ is the spectral density of the actual output where

$$\Phi_{DD} = \frac{\mu^2}{\sigma^2 - s^2} , \qquad (4\text{-}1)$$

and

$$\Phi_{D^*D^*} = \frac{\mu^{*2}}{\sigma^{*2} - s^2} , \qquad (4\text{-}2)$$

then a filter of the form $\frac{\mu}{\mu^*} \frac{\sigma^* + s}{\sigma + s}$ will give the desired signal since

$$\Phi_{DD} = \left| \frac{\mu}{\mu^*} \frac{s + \sigma^*}{s + \sigma} \right|^2 \Phi_{D^*D^*} . \qquad (4\text{-}3)$$

This type of transformation is limited only by differentiator accuracy and saturation when "stepping up" the frequency and by power content of the signal when "stepping down" the frequency. The resulting output signal is, of course, no longer a series of randomly distributed, constant magnitude square waves but (when stepping down) more closely approaches a signal with gaussian amplitude variation as the degree of filtration is increased (cf. (3-22) and [B2,L1,N2,S2]).

Some difficulties occur in analog programs for transfer functions with very small coefficients of high order derivatives. When these equations are "solved" for the highest order derivative, very high feedback gains result which use an excessive number of amplifiers and impair computing accuracy. To alleviate this problem, the small roots can be factored from the denominator of the rational function and the partial fractions corresponding to these factors are programmed separately. The output from the small terms is added to that of the principal function to form the entire response.

In order to conserve the number of amplifiers and to preserve computational accuracy, passive circuitry was extensively used so that several time constants could be computed in one amplifier [B3]. The results from this arrangement were accurate so long as the order of the denominator of the transfer function exceeded that of its numerator. When the two were equal, long division was performed so that, as a consequence of superposition, a constant term could be added to a stable transfer function.

$$\frac{\sum_{0}^{n} a_j s^j}{\sum_{0}^{n} b_j s^j} x = \left[\frac{\sum_{0}^{n-1} (a_j - \frac{a_n}{b_n} b_j) s^j}{\sum_{0}^{n} b_j s^j} + \frac{a_n}{b_n} \right] x \, . \qquad (4\text{-}4)$$

In some cases, the controller is required to differentiate the input signal one or more times so that the order of the right hand side of (4-4) may be higher than the left. A control law with differentiation can result when control prediction is desired. In this case the controller uses the differentiation to indicate the trend of the signal and bases control action on the trend as well as the magnitude. Rearrangement and polynomial division as before gives an equation similar to (4-4) but with positive powers of s present as well as the constant k to be coefficients of x. Again these can be added together from a separate treatment because of superposition.

Differentiation on the analog computer was accomplished by the "implicit differentiator" shown in Figure 4-1. If the potentiometer in the lower loop has the value of one-tenth, then the equation described by this model is

$$x + y - \int y dt = y \,, \tag{4-5}$$

so that

$$y = \frac{dx}{dt} \,. \tag{4-6}$$

Variation of the value of the potentiometer allows close approximation to perfect differentiation. If the pot setting is less than one-tenth, the circuit is stable while it approaches perfect differentiation as the value is raised to one-tenth. The best setting in practice may be somewhat different from one-tenth because of slight variations of amplifier input resistances and hence the circuit can be tuned in order to cancel out these effects.

In practice, differentiation is often unsatisfactory since noise and inaccuracies are magnified by infinite differencing. Actual differentiators are usually only approximations to true differentiation; in some cases the approximations may be quite poor [B10].

When part or all of the required transfer function contained a dead time, the proper signal was fed through a time delay generator which used a third order Pade approximation [B3,B10]. Thus any transfer function of the type previously considered in this work can be programmed.

Figure 4-2 represents the overall analog computer program with blocks representing the various transfer functions. The transfer functions of some of the control functions are zero for specific situations.

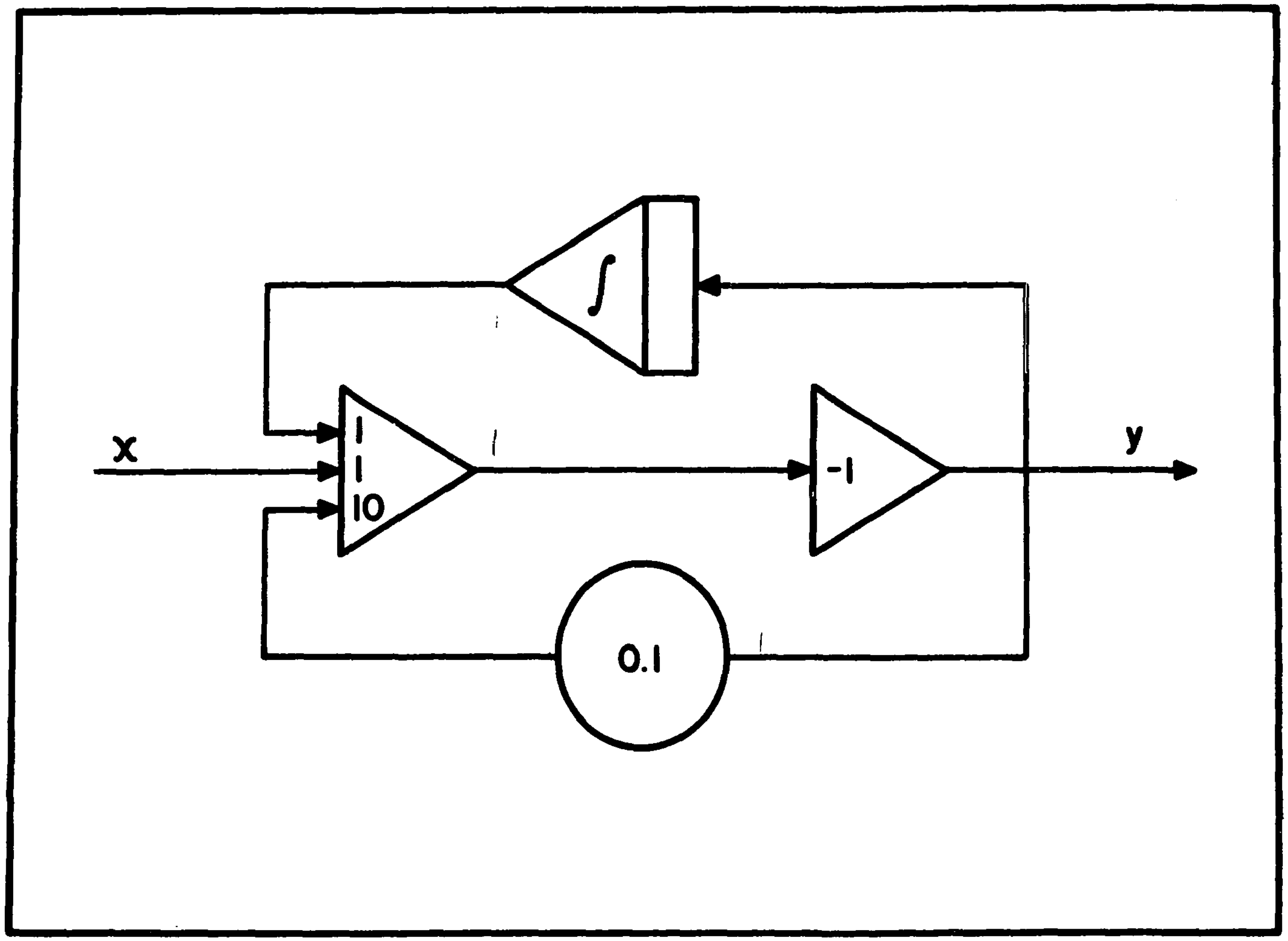

Figure 4-1.--Implicit Differentiation Circuit for Analog Computer

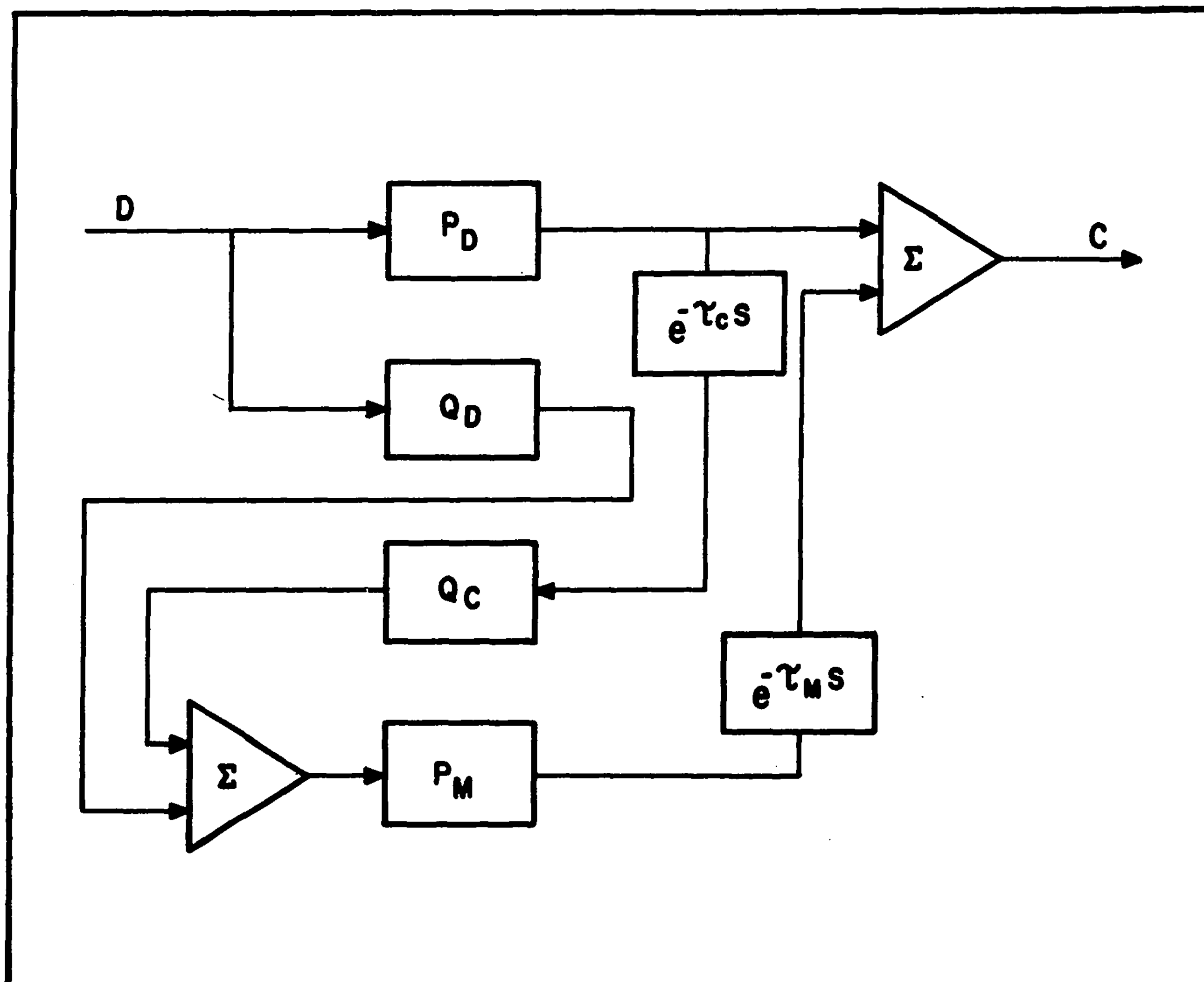

Figure 4-2.--Block Diagram of Plant and Controller for Analog Computation

CHAPTER V

ANALYSIS OF OPTIMAL CONTROL SYSTEMS

The design techniques presented in previous chapters have been used to generate controllers for various plant models. The effectiveness of these controllers was studied by extensive digital computation and analog simulation. The results of the digital studies are presented here primarily in the form of graphical correlations of optimal controller performance. The performance diagrams depict the relationships between available control effort and the level of output attenuation for various values of disturbance frequencies, model dead times, model error and feedback noise. The analog computer was used to generate plant responses illustrating pertinent points of design and control effectiveness.

The arrangement used in presentation of these results is similar to that used to present the development of the design equations, i.e., the simpler situations with few constraints are studied first. As more complex factors are included, the previous correlations and analog simulations serve as standards and limiting cases. This method of presentation has the advantage that the "undiluted" effect of the several variables can be studied without being masked by other factors.

The system to be considered in the following discussion is the model as developed in Chapter III:

$$C = \frac{K_D e^{-\tau_C s}}{\alpha + s} D + \frac{K_M e^{-(\tau_C + \tau_M)s}}{\alpha + s} M . \tag{3-3}$$

To simplify the presentation the following parameter values are assumed:

$$K_M = K_D = 1 ,$$
$$\alpha = K_M + K_D = 2 ,$$

where α has the units of 1/unit time. Various values for τ_C and τ_M are to be considered. The disturbance, D, is of the form previously considered with spectral density

$$\Phi_{DD} = \frac{2\mu^2\sigma}{\sigma^2 - s^2} , \tag{3-9}$$

where μ^2 is the mean square amplitude and σ is the mean frequency. The manipulative variable, M, is defined by the control law

$$M = Q_D D - Q_C (C + \delta) , \tag{3-84}$$

where δ is the feedback noise with constant spectral density

$$\Phi_{\delta\delta} = \delta^2 . \tag{5-1}$$

Q_D and Q_C are the feedback and feedforward control functions respectively as defined by (3-101) and (3-106).

Systems Without Parameter Errors or Dead Times

As a basis for later comparisons and in order to study certain effects in the absence of as many complications as possible, the system described by (3-3) will first be considered with the dead times, τ_C and τ_M, both equal to zero. In addition, it is assumed that there exists negligible error in the system parameters. The digitally computed performance diagram for this model (Figure 5-1) shows the variation of mean square output, $\langle c^2 \rangle$, as a function of mean square control effort, $\langle a^2 \rangle$, for parametric values of mean disturbance frequency, σ. These optimal controls were the result of minimizing the sum $\langle c^2 \rangle + \lambda^2 \langle a^2 \rangle$ and parametric values of the weighting factor, λ, are also shown. The mean square value of the disturbance for all cases was 25.

The value of parameters is given in reduced units referred to the output variable. Thus a constant disturbance having a [mean] square amplitude of 25, would, if uncontrolled, produce a mean square displacement in the output of $25/\alpha^2$ or 6.25. The actual uncontrolled mean square output computed for the above random disturbance was 3.571. These units may be multiplied by any factor for a particular problem.

Figure 5-1 shows that the mean square output decreases as control effort increases approaching zero as the control effort approaches a finite asymptote. The fact that a mean square control effort of 25 is required asymptotically for a disturbance whose mean square value is 25 is the more or less fortuitous result of having identical plant and control gains ($K_M = K_D = 1$). This equality of limiting control effort and disturbance magnitude can always be arranged for models of this type by judicious use of scaling factors on the disturbance and manipulative variable since the "ideal" feedforward control ratio, M/D, is the ratio of gains, (cf. (3-33)).

While there is a broad variation in control effort-output relationships as a function of disturbance frequency, there is much less variation in the parameters of the optimal controllers. For the simple case considered here, the overall control function, T_D, is

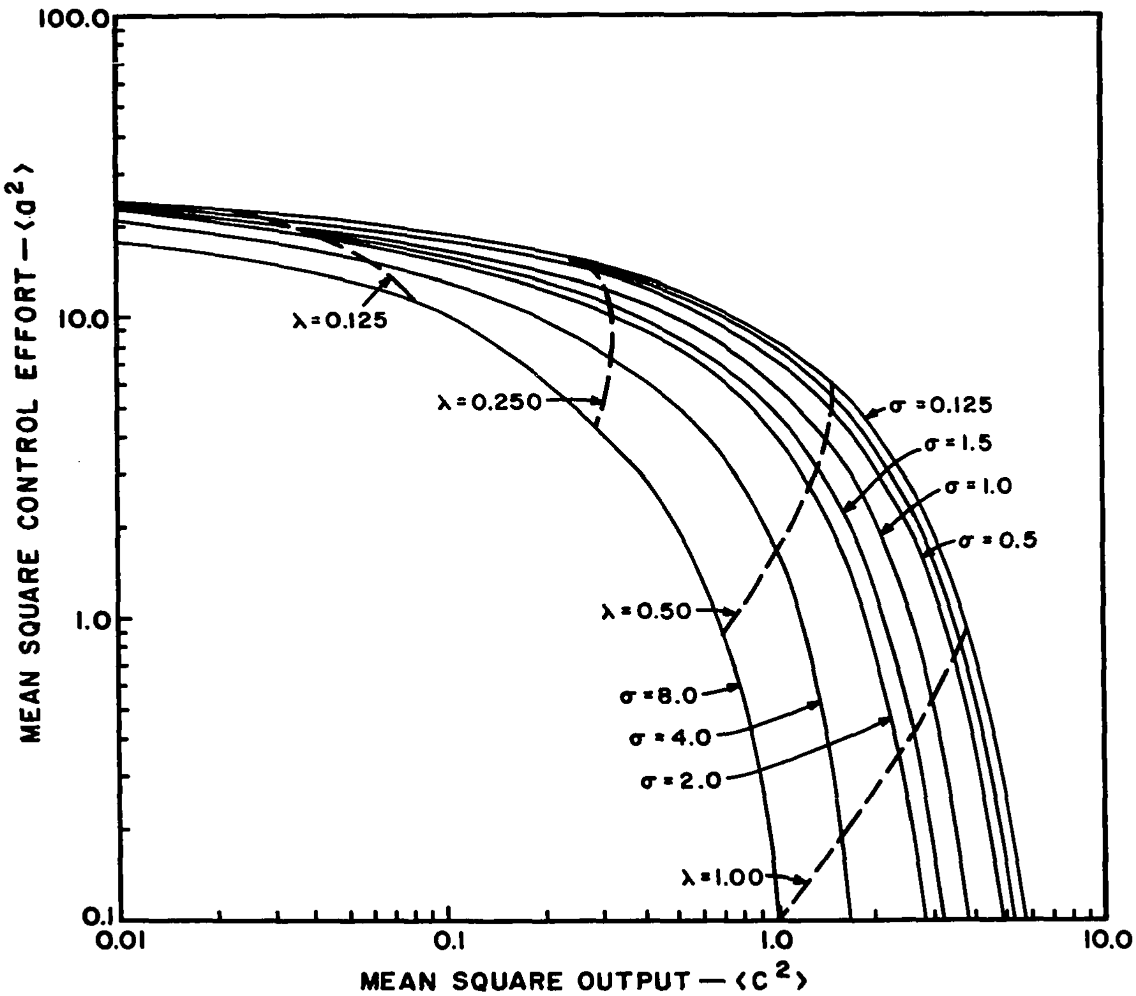

Figure 5-1.--Response Characteristics of Optimal Feedforward Control of First Order System for Various Disturbance Frequencies, σ. Results are from minimization of sum $\langle c^2\rangle + \lambda^2\langle a^2\rangle$.

$$T_D = - \frac{T_{D1,0} + T_{D1,1}s}{(1 + s/\alpha)(1 + s/\beta)} \qquad (5-2)$$

(cf. 3-55)). The disturbance frequency σ does not enter into the calculation of the poles of this overall control transfer function.

The two non-zero constants of the numerator of T_D are plotted on Figure 5-2 as functions of disturbance frequency for constant values of the Lagrange multiplier λ. The coefficient of the first derivative $T_{D1,1}$ is essentially constant over the entire frequency range while the constant term $T_{D1,0}$ does decrease somewhat at higher frequencies, reflecting the fact that these frequencies produce less output and thus require less gain to attenuate their effect. Overall there is only modest controller variation for a broad disturbance frequency change and it is evident that a controller designed for the system natural frequency (2 radians per unit time) would be only moderately different from the optimum controller based on a unit step function disturbance, i.e., as the disturbance frequency approached zero. For the rest of these example calculations, the control systems will be evaluated for a mean disturbance frequency of 1.5.

The model under consideration is assumed to be free of parameter error. As shown in Chapter III, this assumption leads to specification of feedforward control without feedback if there exists finite noise or dead time in the feedback circuit. The feedforward controller, Q_D, that is thus specified has one pole and one zero (Table 5-1). As the amount of control effort increases and output decreases, the time constants of the pole and zero both decrease and the two become almost equal as they get smaller. The small time

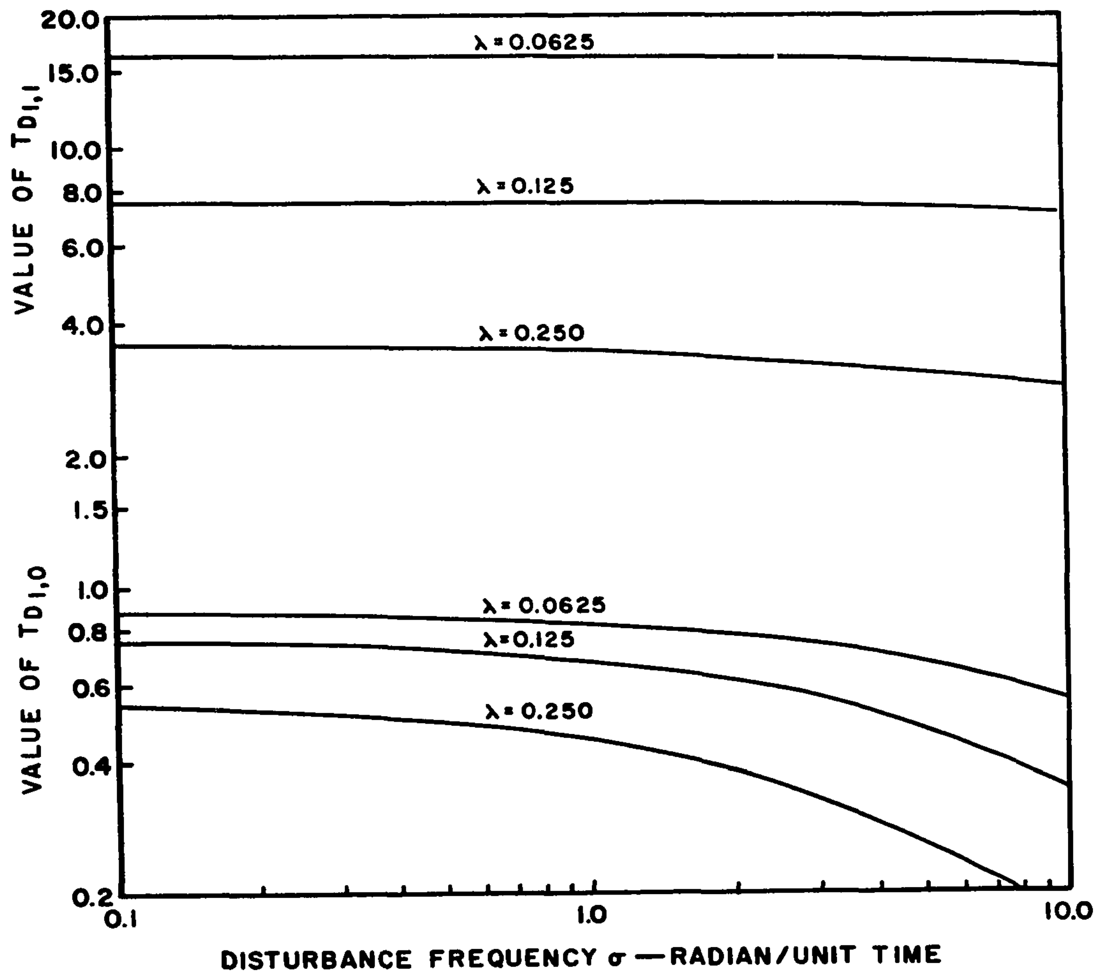

Figure 5-2.--Variation of Optimal Controller Parameters as a Function of Disturbance Frequency, σ. Values of Lagrange multiplier, λ, are same as on Figure 5-1.

TABLE 5-1

PARAMETERS OF FEEDFORWARD CONTROL FUNCTION, $Q_D = k\,\frac{1 + zs}{1 + ps}$,
FOR SYSTEM WITHOUT DEAD TIME OR MODEL ERROR

Mean Square Output, $\langle c^2 \rangle$	Mean Square Control Effort, $\langle a^2 \rangle$	Gain, k	Time Constant of Zero, z	Time Constant of Pole, p	Lagrange Multiplier, λ, of the minimized sum $\langle c^2 \rangle + \lambda^2 \langle a^2 \rangle$
$.4402 \times 10^{-8}$	24.998	.9999	.00194	.0020	0.0019531
$.6999 \times 10^{-7}$	24.991	.9999	.00385	.0039	0.0039063
$.1101 \times 10^{-5}$	24.964	.9996	.00760	.0078	0.0078125
$.1699 \times 10^{-4}$	24.859	.9984	.01481	.0156	0.015625
$.2526 \times 10^{-3}$	24.469	.9935	.02812	.0312	0.03125
$.3469 \times 10^{-2}$	23.126	.9755	.0510	.0620	0.0625
$.4011 \times 10^{-1}$	19.209	.9128	.0851	.1210	0.125
.3215	11.270	.7372	.1254	.2235	0.25
1.335	3.313	.4280	.1580	.3535	0.5
2.626	0.433	.1621	.1743	.4473	1.0

constants tend to make both less important individually, and further since they approach a common value, they tend to cancel so that the controller becomes a simple proportioning device (cf. (3-33)).

In Figure 5-3 the Bode diagrams of the mirror images of the overall controller transfer functions corresponding to some of the controllers in Table 5-1 are shown along with the Bode diagram of the plant transfer function. As implied by (3-17), the ideal overall control transfer function, T_D, from the stand-point of output attenuation would be the negative of the plant transfer function, P_D. As the degree of control effort increases with decreasing values of Lagrange multipliers, λ, the difference between P_D and $-T_D$ becomes negligible so that the sum of $T_D + P_D$ vanishes - the perfect feedforward controller.

Several optimal controllers were studied on an analog computer and compared with the "ideal" or invariance feed-forward controller of (3-33) under conditions such that the manipulative variable M was constrained by electrically clipping the controller output. This clipping is equivalent to controller saturation in a chemical plant and as such is a non-linearity so that the responses cannot be computed analytically by the methods heretofore described. The input disturbance was that described by (3-9) where $\sigma = 1.5$ and was obtained by filtering a square wave having a random average frequency about 20 times higher. The resulting noise signal is approximately gaussian with respect to the model.

In Figures 5-4 to 5-6, sample responses of an "ideal" feedforward controller are compared with those of two optimal controllers. The latter are designed so that the mean square control effort is 88% and 67% respectively of that of the

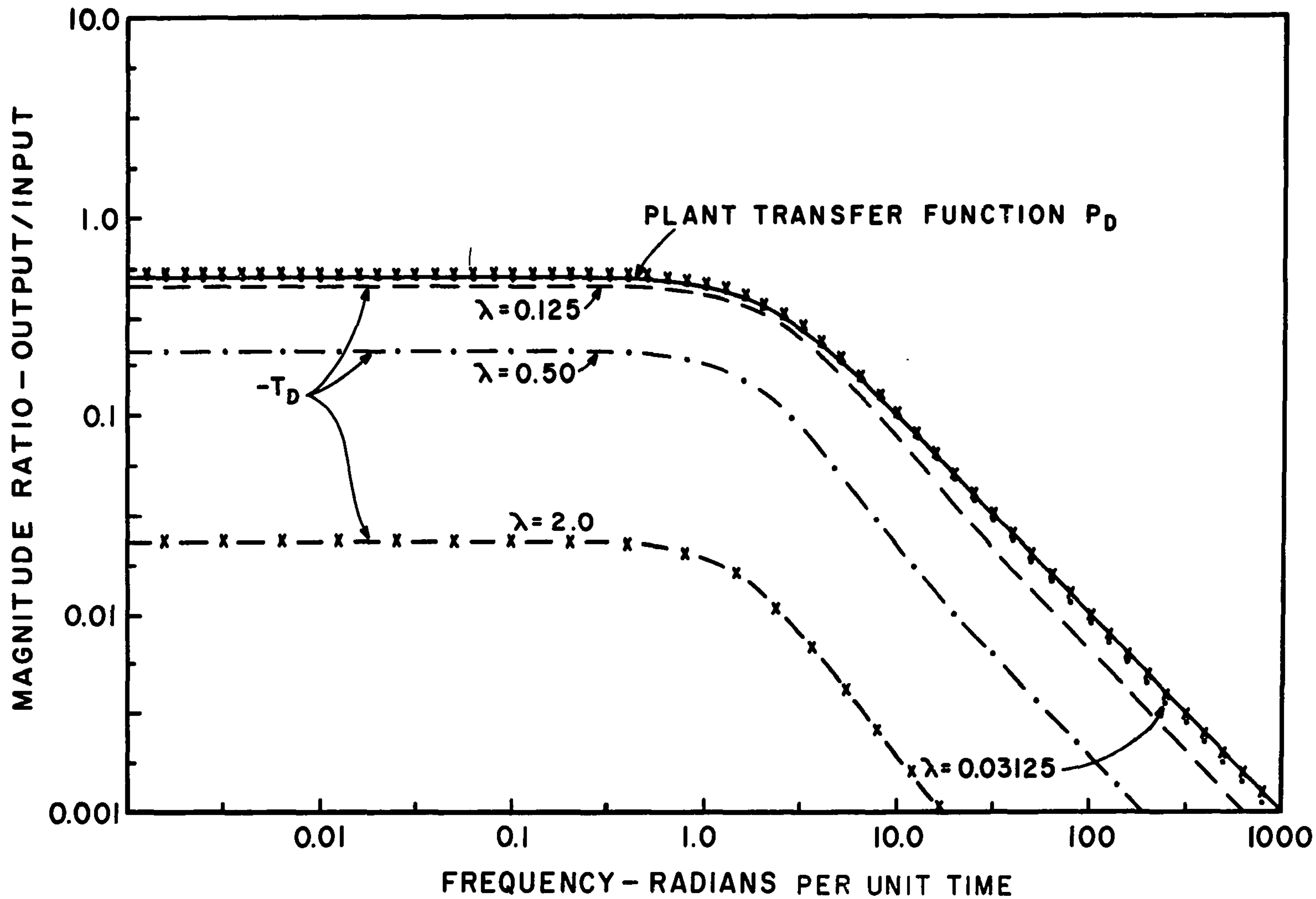

Figure 5-3.--Bode Diagram of Optimal Feedforward Controllers
Listed in Table 5-1

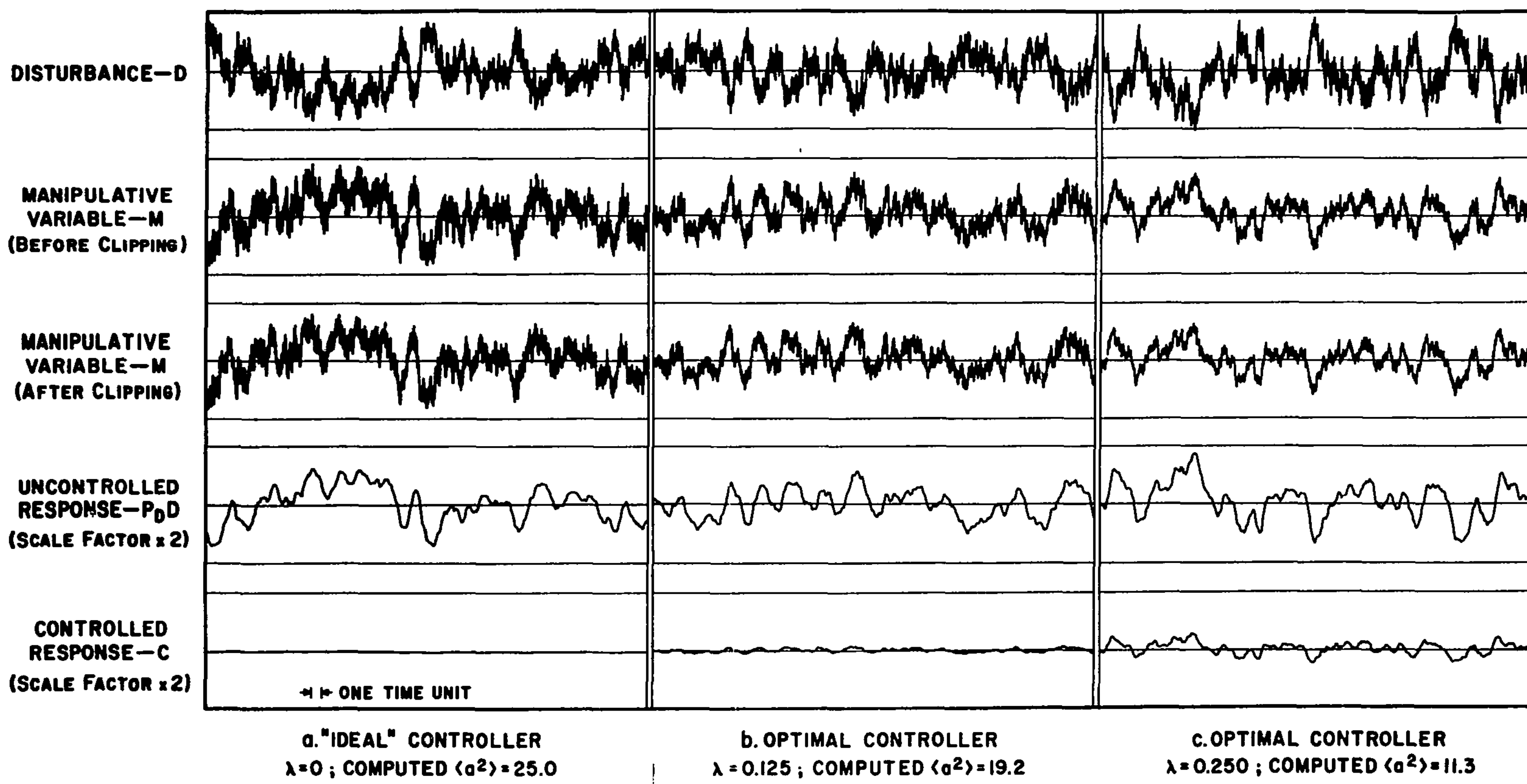

Figure 5-4.--Response of Feedforward Controlled System to a Random Disturbance Without Control Effort Constraint. Values of Lagrange multiplier, λ, are from Table 5-1.

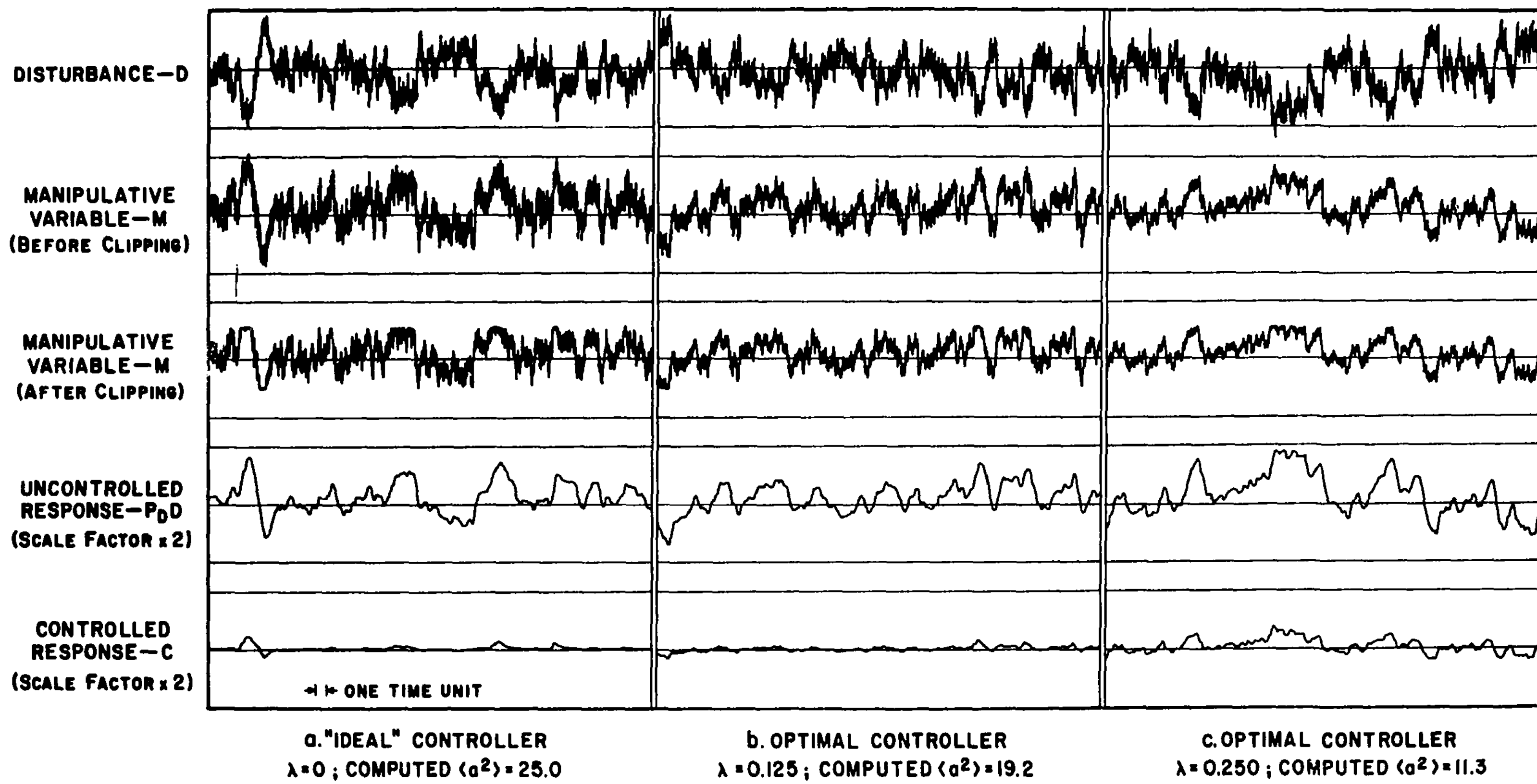

Figure 5-5.--Response of Feedforward Controlled System to a Random Disturbance with Clipping at 67% of Maximum Disturbance Level. Values of Lagrange multiplier, λ, are from Table 5-1.

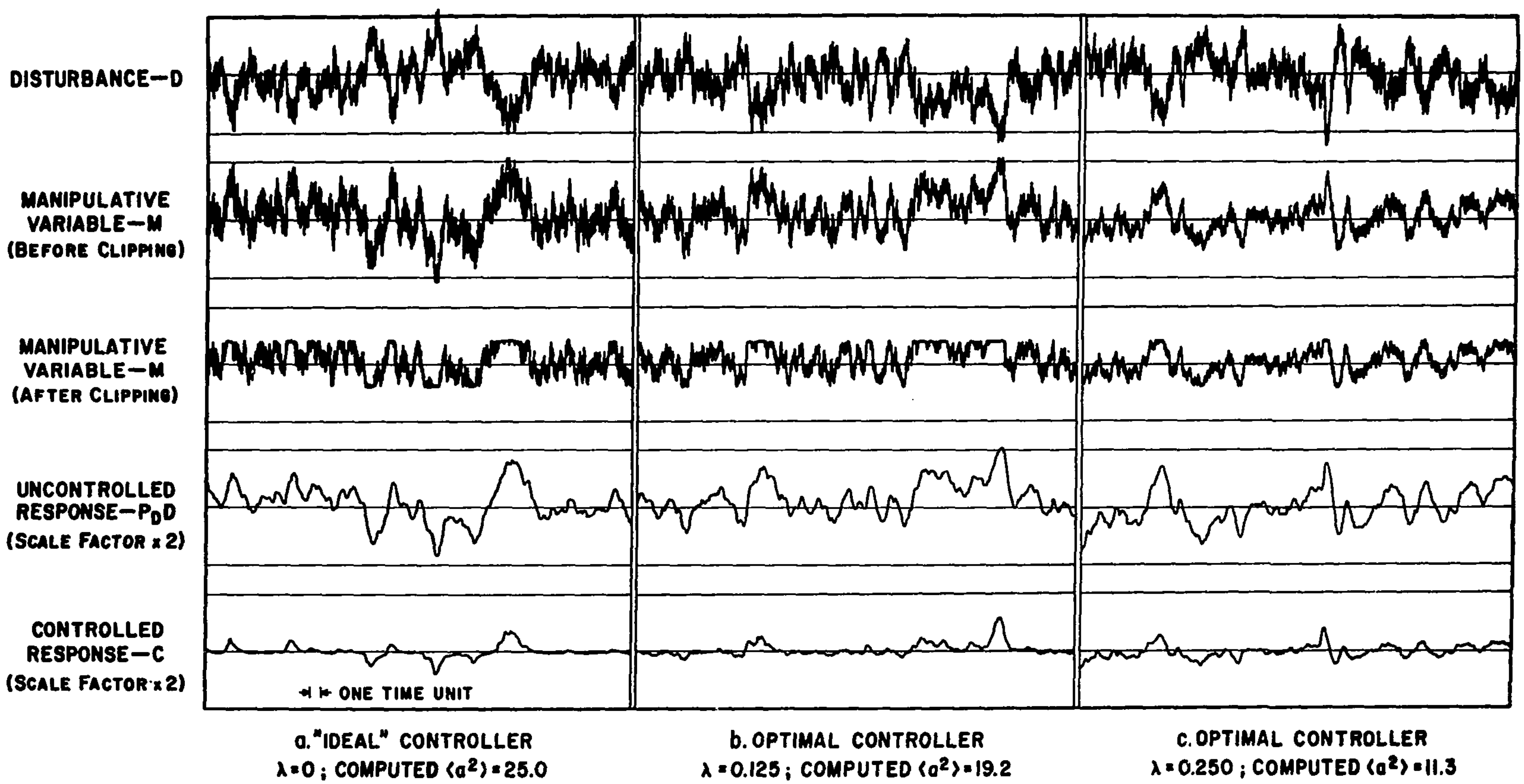

Figure 5-6.--Response of Feedforward Controlled System to a Random Disturbance with Clipping at 50% of Maximum Disturbance Level. Values of Lagrange multiplier, λ, are from Table 5-1.

"ideal" controller. The uppermost channel on these figures is a record of the disturbance D. The second represents the action of the manipulative variable M as called for by the controller. In the case of the "ideal" controller, this channel is a mirror image of the disturbance while for the optimal controllers, there is attenuation of the magnitude of M. The third channel shows the manipulative variable after clipping occurs. It is seen that there was far more saturation of the signal from the "ideal" controller than that from the optimal controllers. The "ideal" controller used more mean square control effort than the others but, of course, the maximum could not exceed the level of clipping for any of the designs.

The second channel from the bottom shows the uncontrolled output $P_D D$ and the lowest channel is the controlled output C. The variation of the controlled output as a function of clipping level is given in Figure 5-7 for the above series of tests. Even though all tests were run for a time period exceeding 250 system time constants, the random nature of the disturbance caused considerable variation in the results, especially in the data from the system having poorest control. However it is clear that the greatest output attenuation was achieved by the "ideal" controller even under conditions where considerable clipping of the manipulative variable has occurred. When the controller outputs were clipped to relatively low levels, the output from all three systems approach the same value since all were saturated most of the time.

The above discussion shows that the optimal controller does not perform its prime function - disturbance attenuation - better than the simple "ideal" controller even under

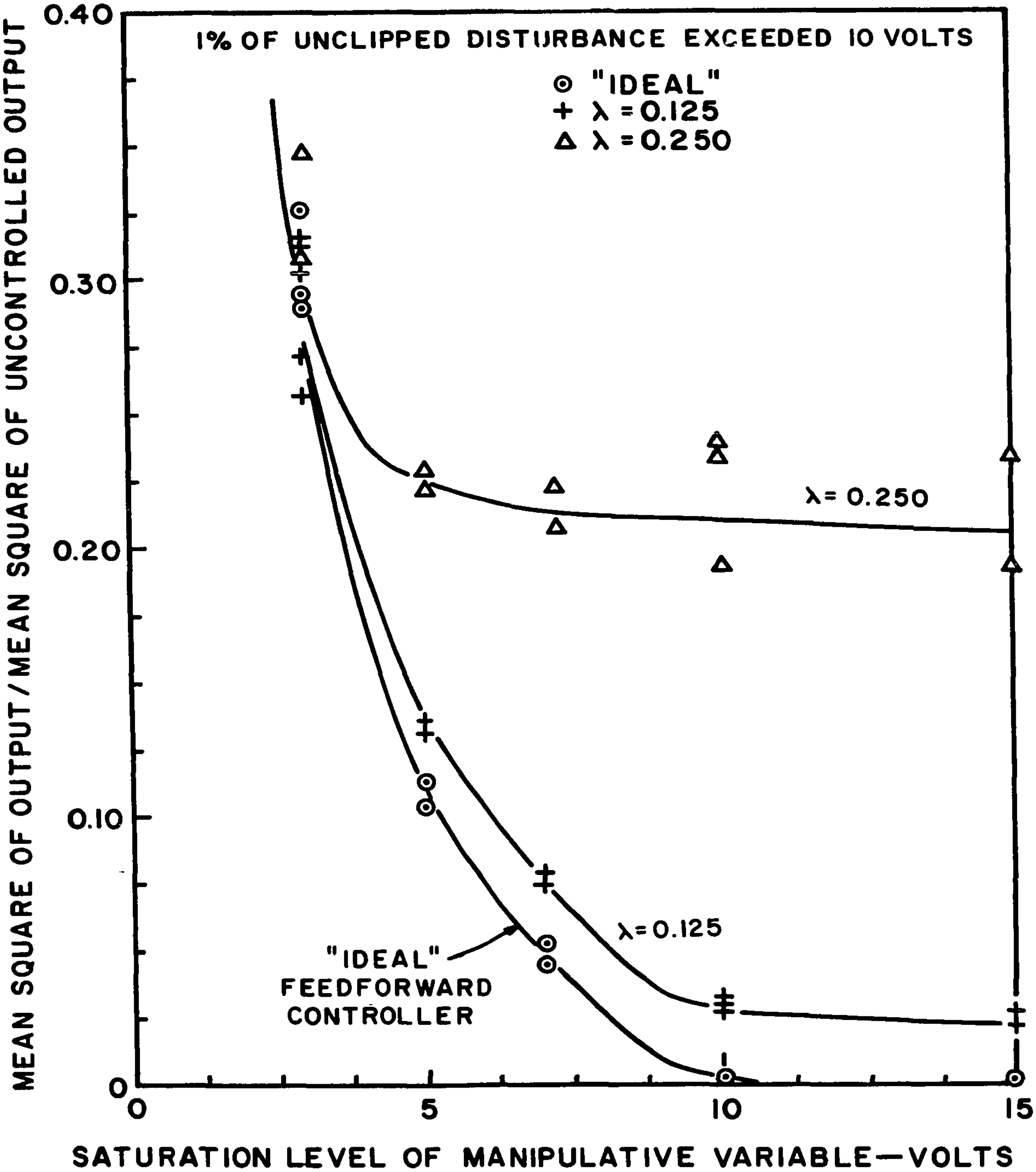

Figure 5-7.--Analog Computer Response Results for Various Feedforward Control Systems as a Function of Controller Saturation. Values of Lagrange multiplier, λ, are from Table 5-1.

conditions of constrained control effort. The principal improvement is that the optimal controller saturates the constrained control boundaries a smaller fraction of the time. Although control saturation may not be of importance for some applications, such behavior frequently implies other undesirable nonlinear effects such as overloading and hysteresis. These effects will cause further important degradation of control efficiency. The responses shown above where the control effort has been clipped cleanly may tend to underemphasize this beneficial aspect of the optimal controller. Nonetheless, it is obvious that even if controller saturation is to be avoided, a sanguine estimate of the amount of available control effort should be used for design purposes. An actual optimal controller will not use more control effort than its design value thus some disturbance attenuation will be sacrificed in using the optimum design if greater control effort is available than had been assumed in the design calculations.

Systems Without Parameter Error but with Dead Time

Introduction of non-zero dead time into the model described by (3-3) has an important effect on the performance of the controlled system. It is assumed that the time delay occurs in the response of the controller, i.e., $\tau_C = 0$, $\tau_M > 0$. Since this time delay applies equally to both feedforward and feedback control, the assumption of no parameter error dictates that feedforward control without feedback is to be used again for this case. In Figure 5-8 the relationship between available control effort and output attenuation is plotted for optimal controllers of the same system as before but with various parameters of dead time. The lines

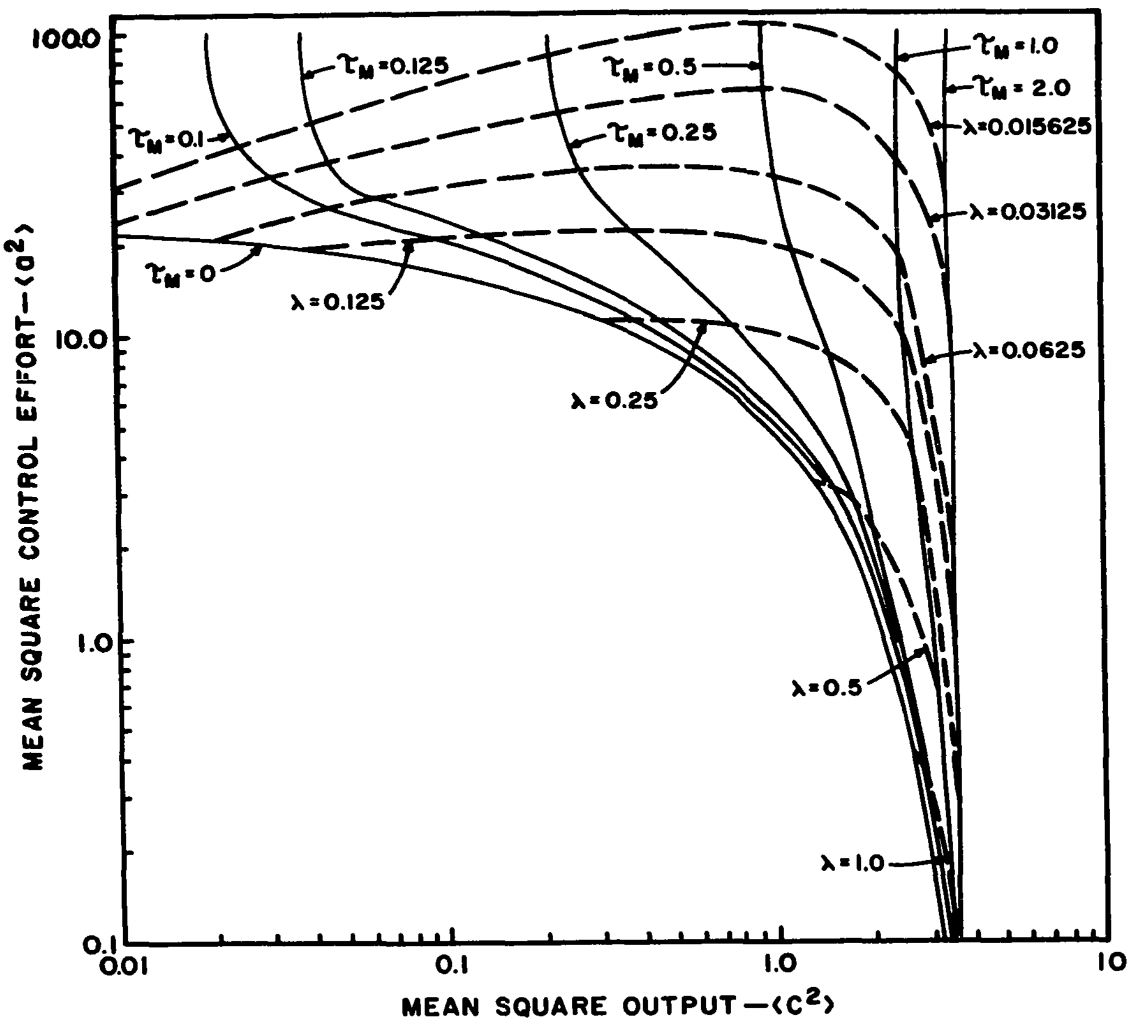

Figure 5-8.--Response Characteristics of Optimal Feedforward Control of First Order System for Various Controller Dead Times, τ_M. Results are from minimization of sum $\langle c^2 \rangle + \lambda^2 \langle a^2 \rangle$.

of constant values of the Lagrange multiplier, λ, used in the optimization are also shown. The size of the time delay limits the degree of output attenuation that can be attained regardless of the amount of control effort applied. It is interesting to note, however, that even if the dead time is equal to the system time constant (the system time constant in this case is the reciprocal of the natural frequency and equals one half), the mean square output may still be reduced to 30% of the uncontrolled value.

As in the previous case, the optimal feedforward controller has one pole and one zero but there is an important difference. As shown in Table 5-2, the time constant of the pole decreases and hence becomes less important as the available control effort increases. The time constant of the zero however, approaches a finite limit so that the controller approaches the ideal, differentiating predictor of (3-40) as the control effort constraint is relaxed. The Bode diagrams of the overall control function shown in Figures 5-9 and 5-10 reflect the influence of this zero by the slope change of the magnitude ratio just above the system natural frequency. It is especially evident for the case when the dead time is 0.1 in Figure 5-9. The differentiation of the disturbance causes the mean square of the control effort to approach infinity in an attempt to achieve better control.

The responses shown in Figure 5-11 are analog computer tests of the same system as was used previously except a pure dead time τ_M of 0.10 time units was included. The controller used in the test shown in the top channel was the "ideal" or invariant controller that was so successful in attenuation of output in the minimum phase system previously considered. In this case, however, the design gives

TABLE 5-2

POLE AND ZERO FREQUENCY OF FEEDFORWARD CONTROL FUNCTION, $Q_D = k \frac{(1 + zs)}{(1 + ps)}$,

FOR FIRST ORDER SYSTEM WITH DEAD TIME = 0.125, PARAMETER ERROR = 0

Mean Square Output, $\langle c^2 \rangle$	Mean Square Control Effort, $\langle a^2 \rangle$	Gain, k	Time Constant of Zero, z	Time Constant of Pole, p	Lagrange Multiplier, λ, of the Minimized Sum $\langle c^2 \rangle + \lambda^2 \langle a^2 \rangle$
.0357	415.6	.978	.00098	.1032	.0009765
.0360	221.6	.978	.00195	.1039	.0019531
.0368	128.8	.977	.00390	.1051	.0039063
.0383	76.08	.975	.00780	.1077	.0078125
.0413	51.35	.972	.01560	.1125	.015625
.0479	38.04	.965	.03118	.1216	.03125
.0655	29.46	.942	.0620	.1374	.0625
.1367	21.20	.871	.1210	.1614	.125
.4778	11.22	.695	.2235	.1903	.25
1.4992	3.11	.399	.3535	.2143	.5
2.710	0.40	.150	.447	.2265	1.0

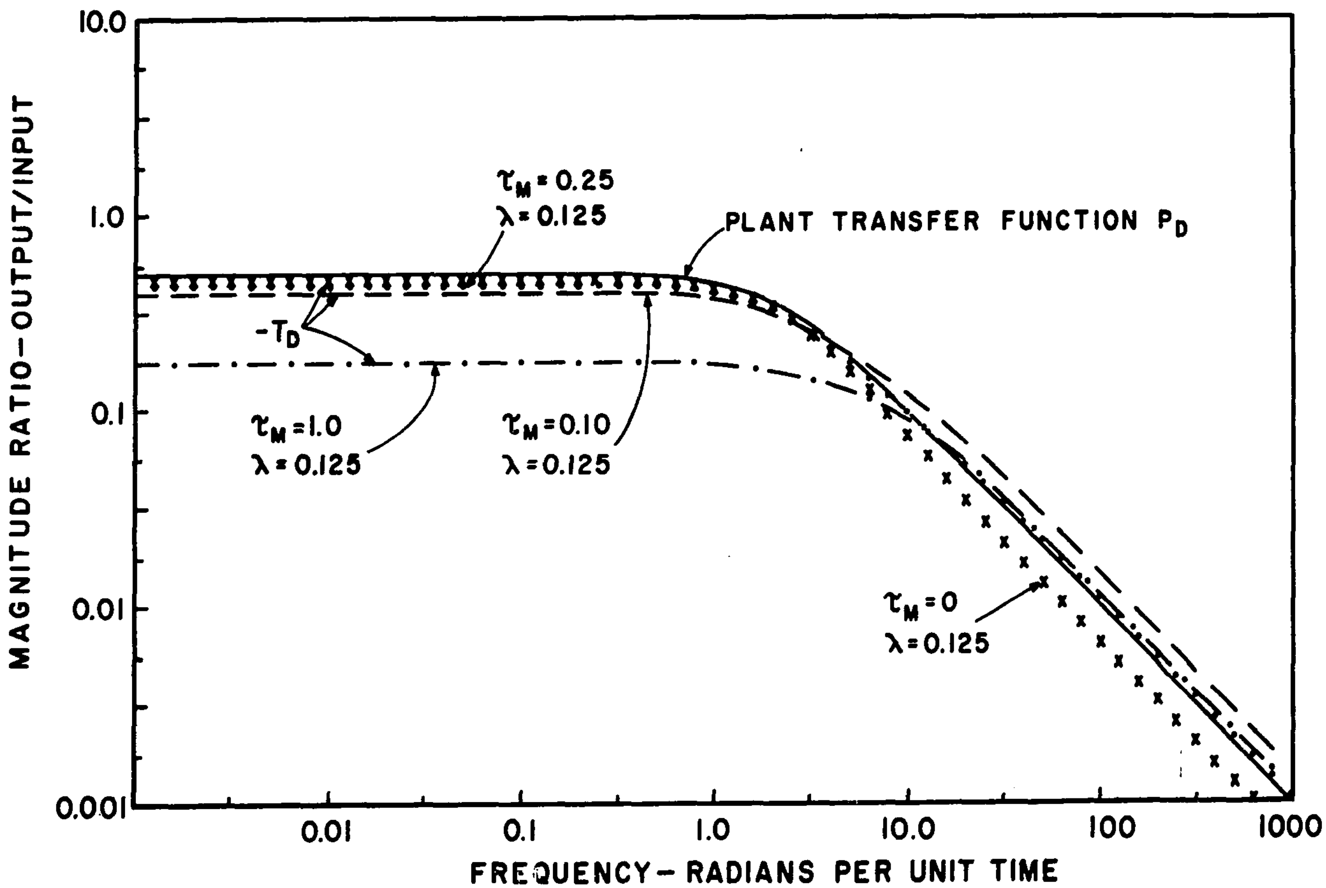

Figure 5-9.--Bode Diagram of Optimal Feedforward Controllers for System with Dead Time.

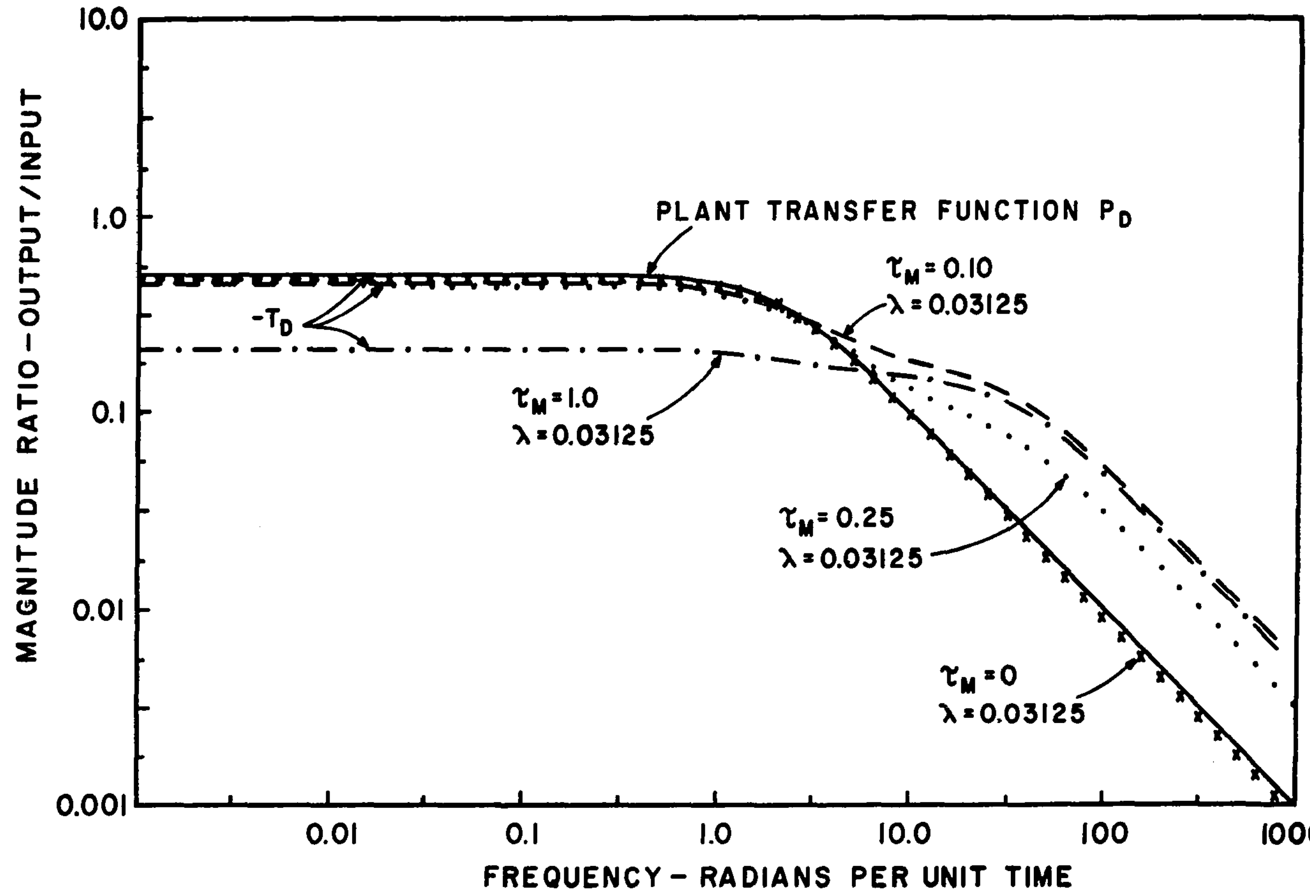

Figure 5-10.--Bode Diagram of Optimal Feedforward Controllers for System with Dead Time.

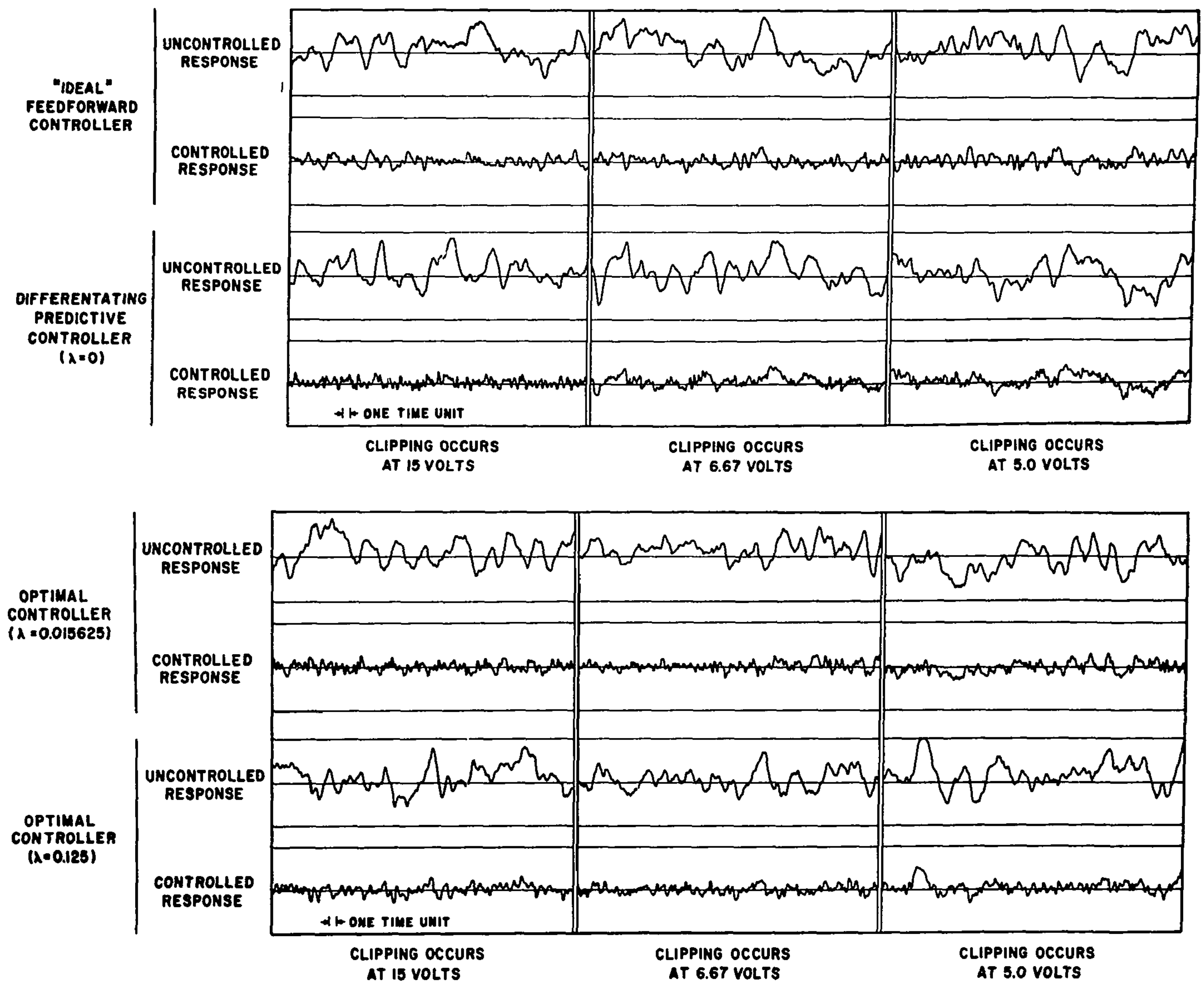

Figure 5-11.--Response of Feedforward Controlled System with Dead Time, τ_M, of 0.1 for a Random Disturbance. Values of Lagrange multiplier, λ, are from Table 5-2.

relatively poor control. The redeeming feature of this controller is that the performance does not seem to deteriorate rapidly as the degree of manipulative variable clipping is increased. The results in Figure 5-12 show that the "ideal" controller is one of the poorest for very little clipping, but becomes one of the better controllers as the saturation level is reduced.

The second set of curves on Figure 5-11 shows the response of the differentiating predictor of (3-40) where the weighting factor for control effort is zero; i.e., there is no constraint on control effort. It can be seen that an improvement in response is obtained over the nominal controller in the region of least clipping. The manipulative variable has not been shown for these cases because the controller response for this differentiator was simply a series of high frequency fluctuations from one control limit to the other. (The same is true to a lesser degree for the imperfect differentiators to be discussed in the following.) Not only does this controller use a large amount of control effort but, as can be seen from the above figures, clipping of control effort exerts a relatively large effect on its efficiency. Although this unconstrained optimal controller is easily the best considered at the highest level of control effort, it falls off to be the poorest when the ability of the controller to respond is restricted (Figure 5-12).

The lower channels on Figure 5-11 show the controlled response of this system (which contains dead time) where controllers are used that were optimized in the presence of constraints on control effort. These controllers correspond to those implied in the construction of the performance diagram, Figure 5-8, resulting from minimization of the sum,

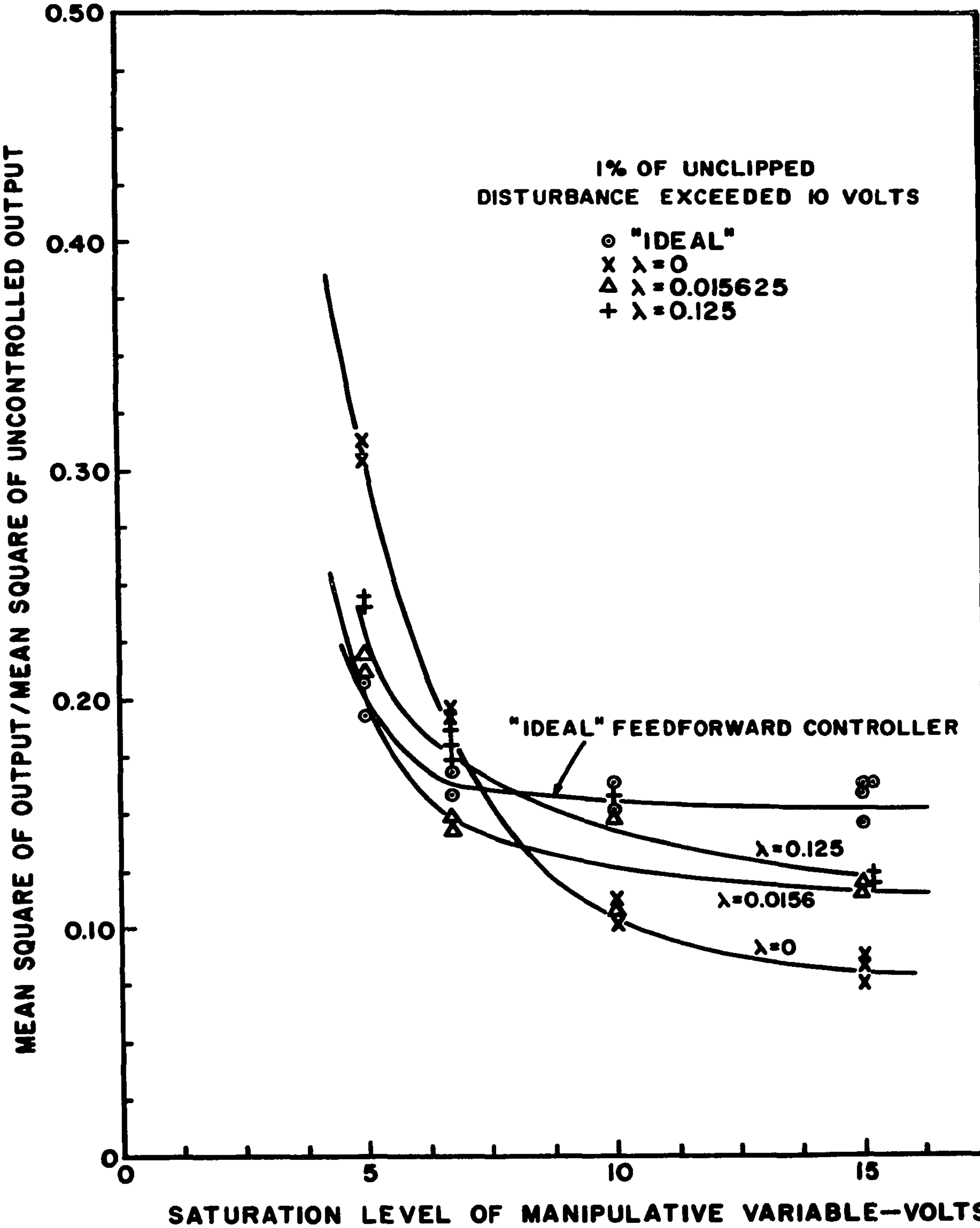

Figure 5-12.--Analog Computer Response Results for Various Feedforward Control Systems with Dead Time, τ_M, of 0.125 as a Function of Controller Saturation. Values of Lagrange multiplier, λ, are from Table 5-2.

$\langle c^2 \rangle + \lambda^2 \langle a^2 \rangle$ for the given values of Lagrange multiplier, λ. As indicated on Figures 5-11 and 5-12, efficiency of output attenuation and sensitivity to control effort clipping using these controllers is a compromise between the performance of the nominal controller and the unconstrained optimal controller. When sufficient control effort is available, a control design similar to the constrained optimal would seem most reasonable. However, it is clear that a considerable reduction in performance must be expected from the control of a system containing pure time delay regardless of the design used.

Sometimes the net dead time can be reduced or eliminated entirely by anticipatory measurement of the disturbance before it enters the system. It is evident from the foregoing results that premeasurement is desirable whenever it is possible. In that way, the system would be converted into one similar to that represented in Figure 5-1 provided the disturbance could be anticipated a length of time corresponding to the dead time of the controller. However premeasurement usually introduces or intensifies model error so that a more comparable situation will be treated below.

Systems with Model Error but without Dead Time

The next class of systems to be considered is one in which model error is present but dead time is assumed absent. Again the system is that described by (3-3) and (3-9) but mutually independent mean square errors of 0.25 was assumed for each of the parameters. Strictly speaking it is not possible to have mean square errors of 0.25 in the dead times while the mean value of $\tau_C = \tau_M = 0$. However, for the present, interest is to be focused on optimum system performance in the absence of dead time and the output resulting from this factor may be considered merely as a separated unmonitored system input. The cases for which the mean value of dead times are not zero will be analyzed in a later section.

The output-control effort relationship for various levels of output due to model error is shown in Figures 5-13, 5-14, 5-15, and 5-16. Each graph is constructed for a different value of feedback noise level. A very low level of feedback noise is considered in Figure 5-13 where the average of the absolute value of the feedback noise δ is 10^{-4}, i.e., 0.0053% of the average absolute value of the uncontrolled output. Lines of constant "error output" $\langle \Delta c^2 \rangle$ are nearly horizontal over the full range of interest showing that the nominal output is a weak function of control effort for a constant error output. Figure 5-14 is the same as Figure 5-15 except that the level of noise and error in the feedback system is assumed to have an absolute value 100 times greater, i.e., the absolute value of the feedback noise is 0.53% of the uncontrolled output. (Expressed in terms of the specific model developed in Chapter III, if the input temperature variation is 5°, then the mean absolute value of the uncontrolled output would be 1.88° and the feedback noise assumed for the present case is 0.01°.)

Some of the optimal control functions computed for construction of this graph are given in Table 5-3. Groups A, B and C show the parameters of the overall control function, T_D, and feedback control function, T_C, for broad ranges of output $\langle c^2 \rangle$ but at relatively constant sensitivity or model error output $\langle \Delta c^2 \rangle$. Within each group the feedback transfer

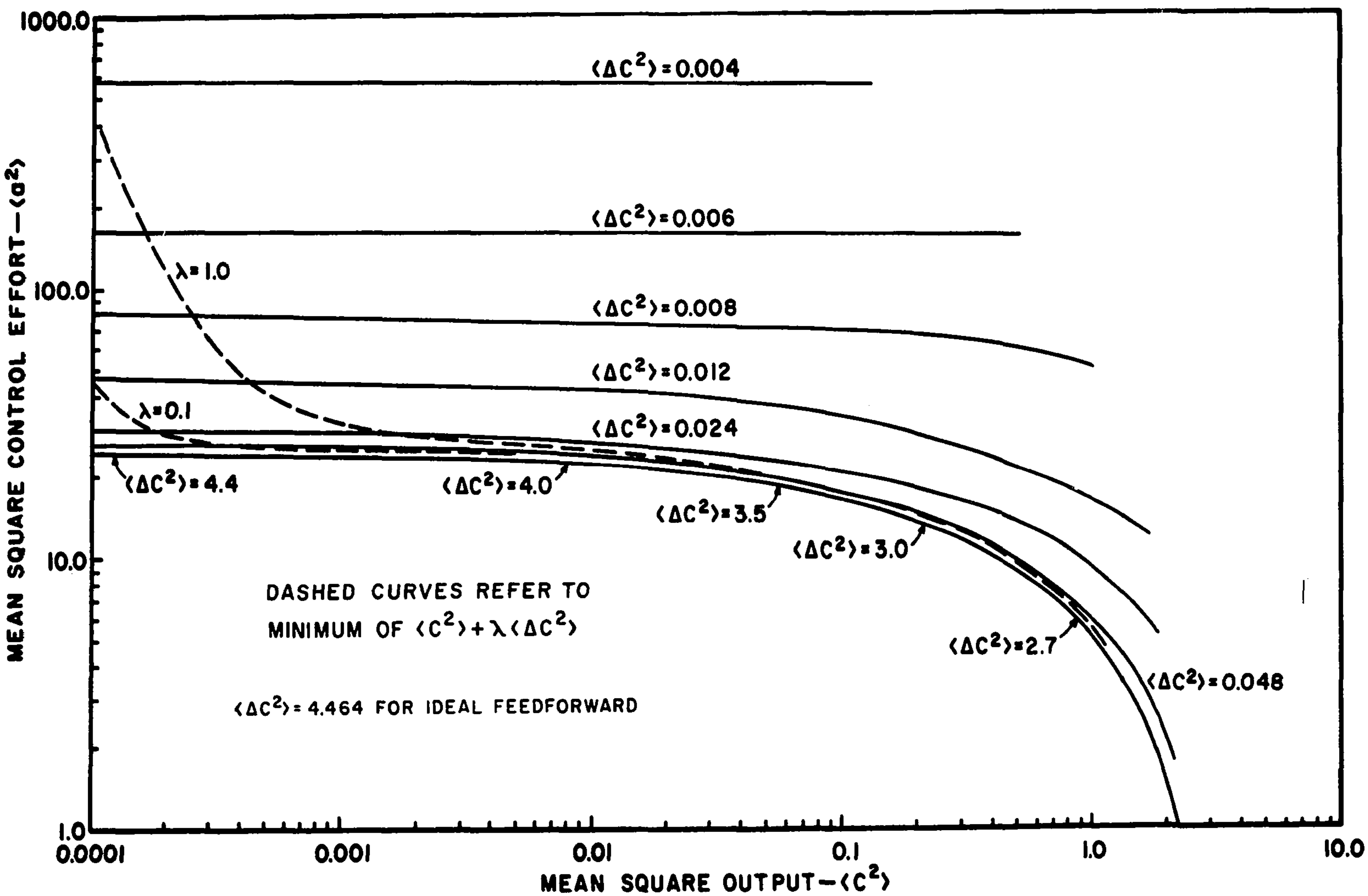

Figure 5-13.--Response Characteristics for Optimal Composite Control of First Order System with Parameters of Model Error Output, $\langle \Delta c^2 \rangle$. Mean square value of feedback noise is 10^{-8}.

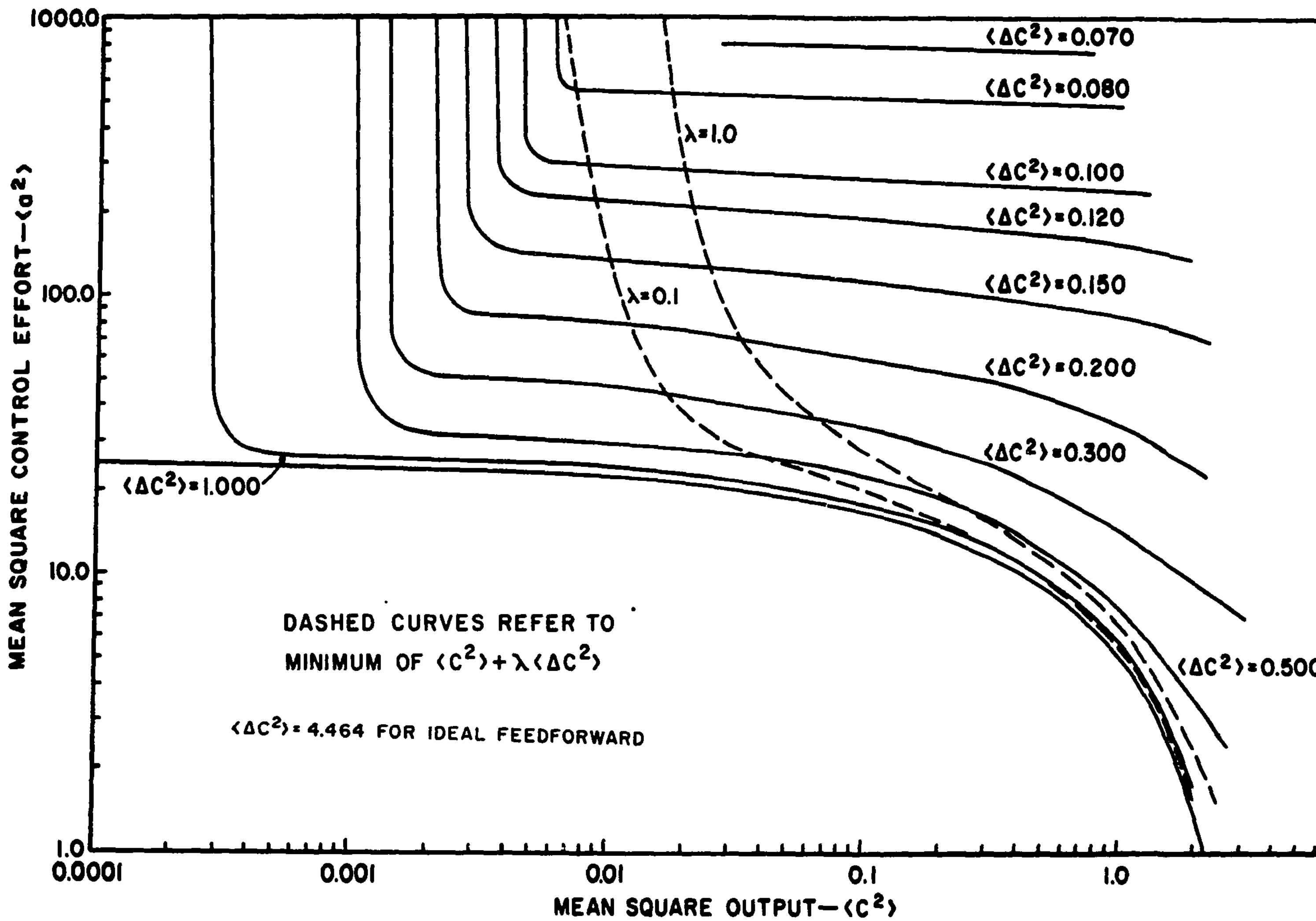

Figure 5-14.--Response Characteristics for Optimal Composite Control of First Order System with Parameters of Model Error Output, $\langle \Delta c^2 \rangle$. Mean square value of feedback noise is 10^{-4}.

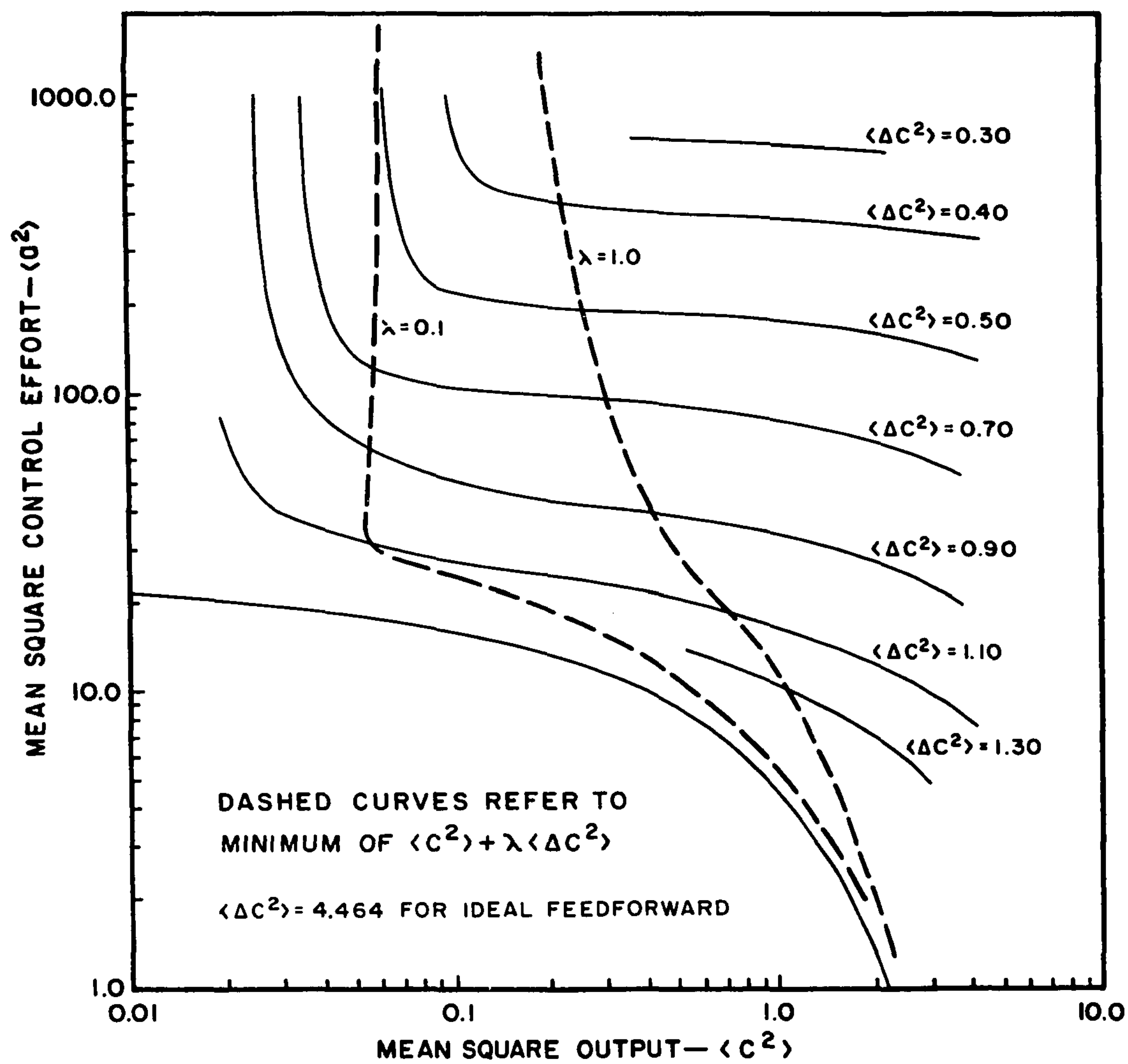

Figure 5-15.--Response Characteristics for Optimal Composite Control of First Order System with Parameters of Model Error Output, $\langle \Delta c^2 \rangle$. Mean square value of feedback noise is 10^{-2}.

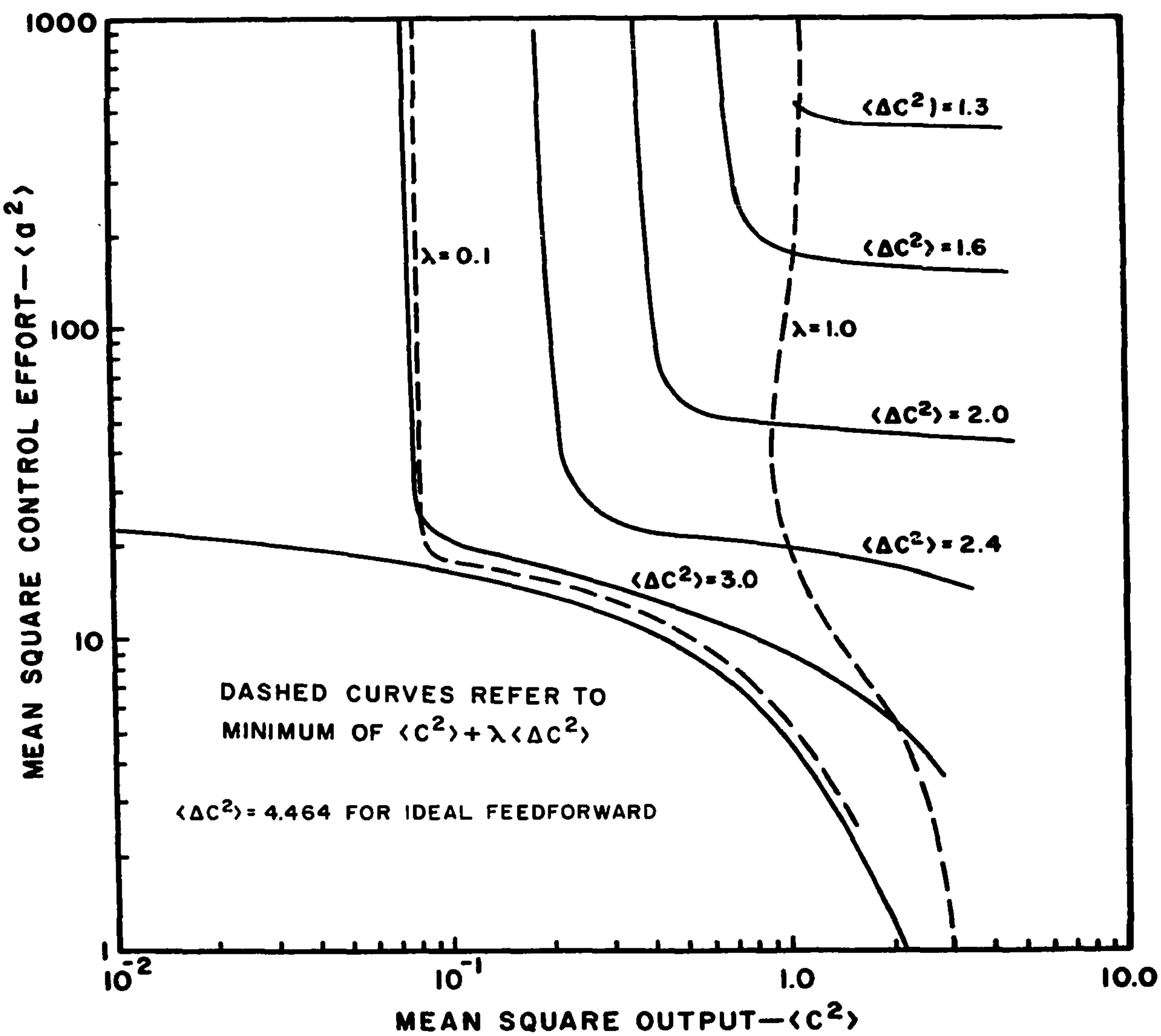

Figure 5-16.--Response Characteristics for Optimal Composite Control of First Order System with Parameters of Model Error Output, $\langle \Delta c^2 \rangle$. Mean square value of feedback noise is 1.0.

TABLE 5-3

OPTIMAL CONTROL LAWS FOR SYSTEMS DESCRIBED IN FIGURE 5-12

Group	Outputs			Control Functions					
				Feedback Function - T_C			Overall Control Function - T_D		
	$\langle c^2 \rangle$	$\langle a^2 \rangle$	$\langle \Delta c^2 \rangle$	Gain	Zeros	Poles	Gain	Zeros	Poles
A	.2334	1250	.059	.9909	.50	.00215 ± .00215j	.428	.1790 ± .1425j	.50, .2323 ± .1872j
	.0452	1253	.060	.9909	.50	.00216 ± .00214j	.483	.0934 ± .0583j	.50, .1098 ± .1040j
	.0151	1192	.065	.9904	.50	.00227 ± .00202j	.496	.0501 ± .0248j	.50, .0545 ± .0537j
	.0071	943	.063	.9854	.50	.00295 ± .00203j	.498	.0310 ± .0190j	.50, .0319 ± .0315j
B	.8290	158.3	.116	.9819	.50	.00430 ± .00430j	.276	.2774 ± .2222j	.50, .3327 ± .1703j
	.1152	168.3	.119	.9819	.50	.00430 ± .00429j	.450	.1293 ± .0933j	.50, .1629 ± .1348j
	.0171	172.3	.128	.9817	.50	.00436 ± .00423j	.491	.0689 ± .0391j	.50, .0776 ± .0738j
	.0067	289.9	.125	.9792	.50	.00437 ± .00429j	.498	.0337 ± .0189j	.50, .0353 ± .0348j
C	2.201	20.6	.226	.9642	.50	.00860 ± .00859j	.087	.4836 ± .2383j	.50, .2357, .4538
	.8220	24.7	.225	.9640	.50	.00860 ± .00859j	.277	.1918 ± .1611j	.50, .2529 ± .0824j
	.0556	37.0	.245	.9643	.50	.00863 ± .00856j	.462	.0966 ± .0606j	.50, .1186 ± .0910j
	.0046	64.4	.231	.9648	.50	.00843 ± .00731j	.496	.0494 ± .0284j	.50, .0535 ± .0500j
D	.2334	1250	.059	.9909	.50	.00215 ± .00215j	.428	.1790 ± .1425j	.50, .2323 ± .1872j
	.1152	168.3	.119	.9819	.50	.00430 ± .00429j	.450	.1293 ± .0933j	.50, .1629 ± .1348j
	.2602	17.6	.350	.9466	.50	.01292 ± .01286j	.385	.1194 ± .0804j	.50, .1646 ± .0646j
	.2462	13.2	.675	.8954	.50	.02603 ± .02555j	.387	.0981 ± .0411j	.50, .0922, .1899
	.1742	14.2	3.13	0	-	-	.406	.1125	.50, .1857
E	.0151	1192	.065	.9904	.50	.00227 ± .00202j	.496	.0501 ± .0248j	.50, .0545 ± .0537j
	.0171	172.3	.128	.9817	.50	.00436 ± .00423j	.491	.0689 ± .0391j	.50, .0776 ± .0738j
	.0264	22.5	.551	.9285	.50	.01747 ± .01692j	.468	.0758 ± .0396j	.50, .0818 ± .0650j
	.0203	21.0	1.230	.8556	.50	.03650 ± .03209j	.470	.0703 ± .0249j	.50, .0839 ± .0266j
	.0188	18.1	3.77	0	-	-	.471	.0730	.50, .0981

function remains constant but there is considerable variation in the overall transfer function. Groups D and E show the parameters of the two transfer functions over a broad range of sensitivities but at relatively constant output. Here the situation is reversed from that of the first three groups. There is little variation in the overall transfer function, but there are variations in the feedback transfer functions showing that changes in sensitivity have been achieved almost entirely by the use of feedback. (Note here that the overall transfer function includes both the feedback and feedforward; cf. (3-16); hence there is no implication that output attenuation is achieved by feedforward alone.) In Table 5-4 the error output for several optimal controllers is compared

TABLE 5-4

MODEL ERROR OUTPUT OF OPTIMAL CONTROLLERS WITH AND WITHOUT FEEDBACK

Controls Designed to Minimize Model Error Output				Controllers Designed without Consideration of Model Error	
Operating as Designed		Operating with Feedback Removed			
$\langle \Delta c^2 \rangle$	$\langle c^2 \rangle$	$\langle \Delta c^2 \rangle$	$\langle c^2 \rangle$	$\langle \Delta c^2 \rangle$	$\langle c^2 \rangle$
.059	.2334	3.07	.222	3.03	.222
.060	.0451	3.92	.0338	3.62	.0338
.065	.0151	4.35	.0043	4.16	.0043
.063	.0071	4.47	.0004	4.33	.0004
.116	.8290	2.52	.823	2.76	.823
.119	.1152	3.38	.110	3.23	.110
.128	.0171	4.16	.012	3.93	.012
.125	.0067	4.47	.0004	4.33	.0004

to the error output from the same controllers when feedback is removed. The error output of the "optimal" controllers without feedback increases to approximately the same level as that of a controller designed for an errorless model. Thus the earlier conclusion that sensitivity reduction is achieved solely by the use of feedback has been confirmed.

A phenomenon occurs in Figure 5-14 that was not observed in Figure 5-13. At low levels of nominal error there is a limit to the degree of "desensitizing" that is possible with feedback. Once this limit has been reached, the "error output" is almost unaffected by further increases in the allowable mean square control effort. Figure 5-17, a cross plot of this data, illustrates the situation more clearly. Use of feedback consuming only small additional amounts of control effort initially effects a marked reduction in system sensitivity. For relatively large values of mean square output, further sensitivity reduction is possible but gives a high relative cost in control effort. At the smaller values of mean square output, the initial sensitivity reduction, although considerable, is less than at high values of mean square output. In this latter case, it is virtually impossible to achieve very low levels of error output. Thus there is a lower limit on output attenuation for a given sensitivity which arises almost entirely from the second term of (3-90); i.e., it is the response caused by amplification of noise and sensing error in the feedback system. Any attempt to decrease the sensitivity when there is a low nominal output must result in a greater portion of the overall control being performed by feedback and use of more feedback would cause an increase in output due to feedback signal corruption.

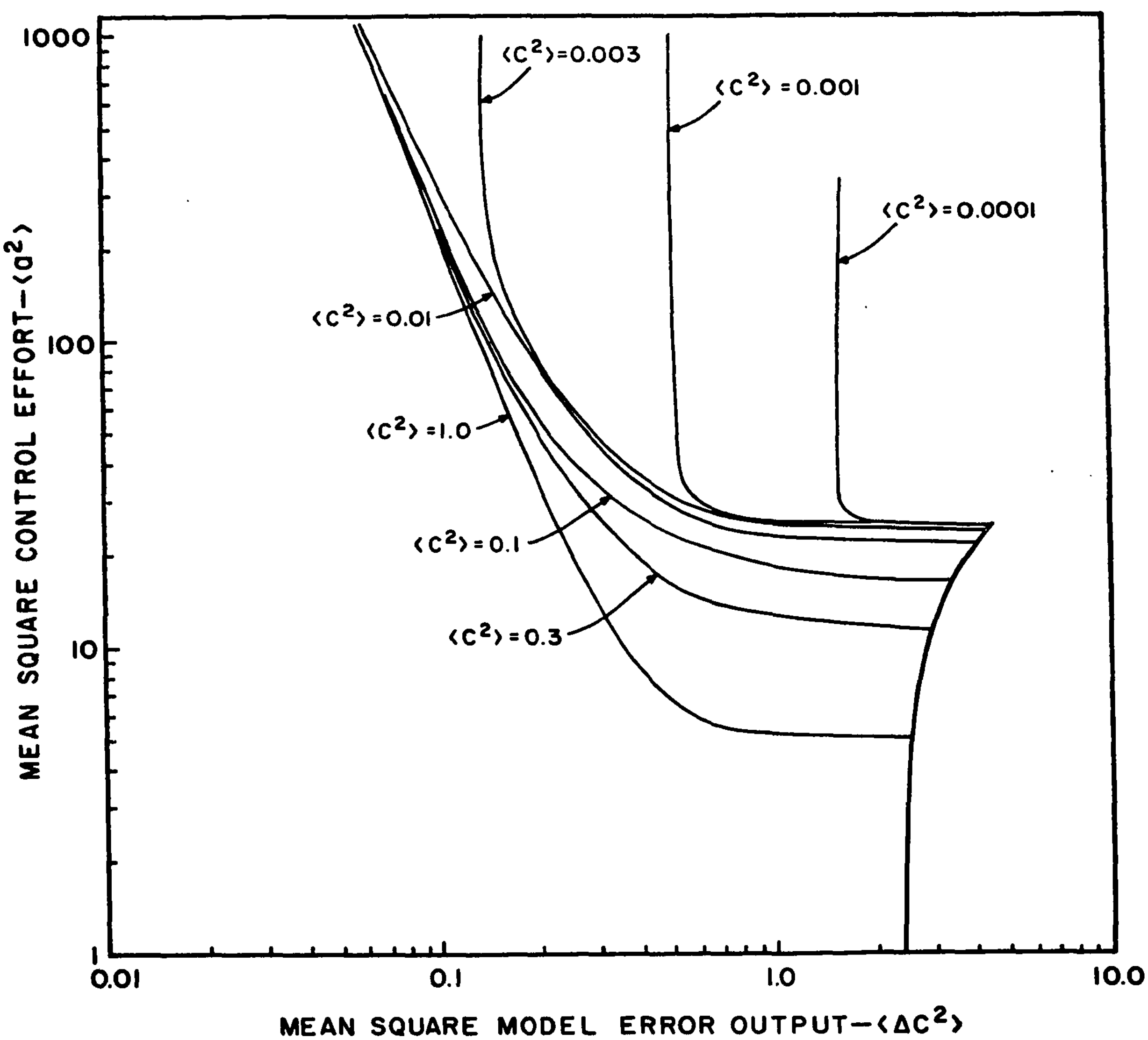

Figure 5-17.--Response Characteristics for Optimal Composite Control of First Order System with Parameters of Nominal Output, $\langle c^2 \rangle$. Mean square value of feedback noise is 10^{-4}.

The presence of factors in the system that impair feedback performance cause a pronounced increase of system sensitivity to model error. Figures 5-15 and 5-16 show the relationship between control effort, nominal output and model error output for cases where the average feedback noise has been increased by factors of 10 over the previous example to 5.3% and 53% of the average absolute value of the uncontrolled output. Although the curves are topographically similar to those before, the parametric values of the error output show that there is a significant loss of overall sensitivity reduction. Very little reduction can be achieved in the latter case even for large amounts of control effort.

In principle, the conditions for an optimum controller for a specific application could be found on these graphs by having estimates available for the feedback noise, error in model parameters, maximum available control effort and either maximum allowable output or maximum allowable model error output. Knowing four of these quantities would define a point on one of these graphs which in turn would define the achievable level of the fifth quantity. Computation of the quantities implies a definition of the optimal control used to compute the point in question. From a different point of view, this point would correspond to values of the Lagrange multipliers λ_1 and λ_2 of (3-88) which, when substituted into the design equations, would give a controller design whose performance is described by the coordinates of the point. Estimates for the feedback noise, model error and available control effort are usually available from design considerations or actual measurements. The two output quantities $\langle c^2 \rangle$ and $\langle \Delta c^2 \rangle$ however, should both be as small as possible. In some cases, an allowable upper limit on one or both may exist but normally it will be simpler to think of minimizing a weighted sum of the two.

On each of the foregoing figures describing overall system performance, dashed lines are shown where the sum of $\langle c^2 \rangle + \lambda \langle \Delta c^2 \rangle$ is minimized for values of the weighting factor λ equal to 0.1 and 1.0. These lines generally lie close to the base curve (where the feedback gain is low) for moderate levels of nominal output and then rise sharply at some fairly low level of nominal output. From the previous discussions it can be seen that at moderate levels of nominal output, the optimal controller uses feedforward control to attenuate the principal portion of transient disturbances. The feedback compensates only a smaller portion, mainly the drift. At some point, depending on the particular situation, the amount of feedback is drastically increased but a corresponding increase in system performance is quite limited by the time this point is reached. Only in cases of very low feedback noise or relatively poor output attenuation would the optimal controller consist primarily of feedback compensation.

To illustrate the response of composite control systems, a model similar to the one discussed above was programmed on the analog computer except that error is assumed to exist only in the plant gain, K_D. A mean square feedback noise level of 10^{-2} corresponding to about 5.3% of the uncontrolled output was used for design purposes. The output control effort diagram of this system is quite similar to those shown previously except that the level of error output is considerably lower since there is only one error source (Figure 5-18). The optimal controller that was programmed corresponded to conditions represented by the starred point on Figure 5-18. This particular combination was chosen for

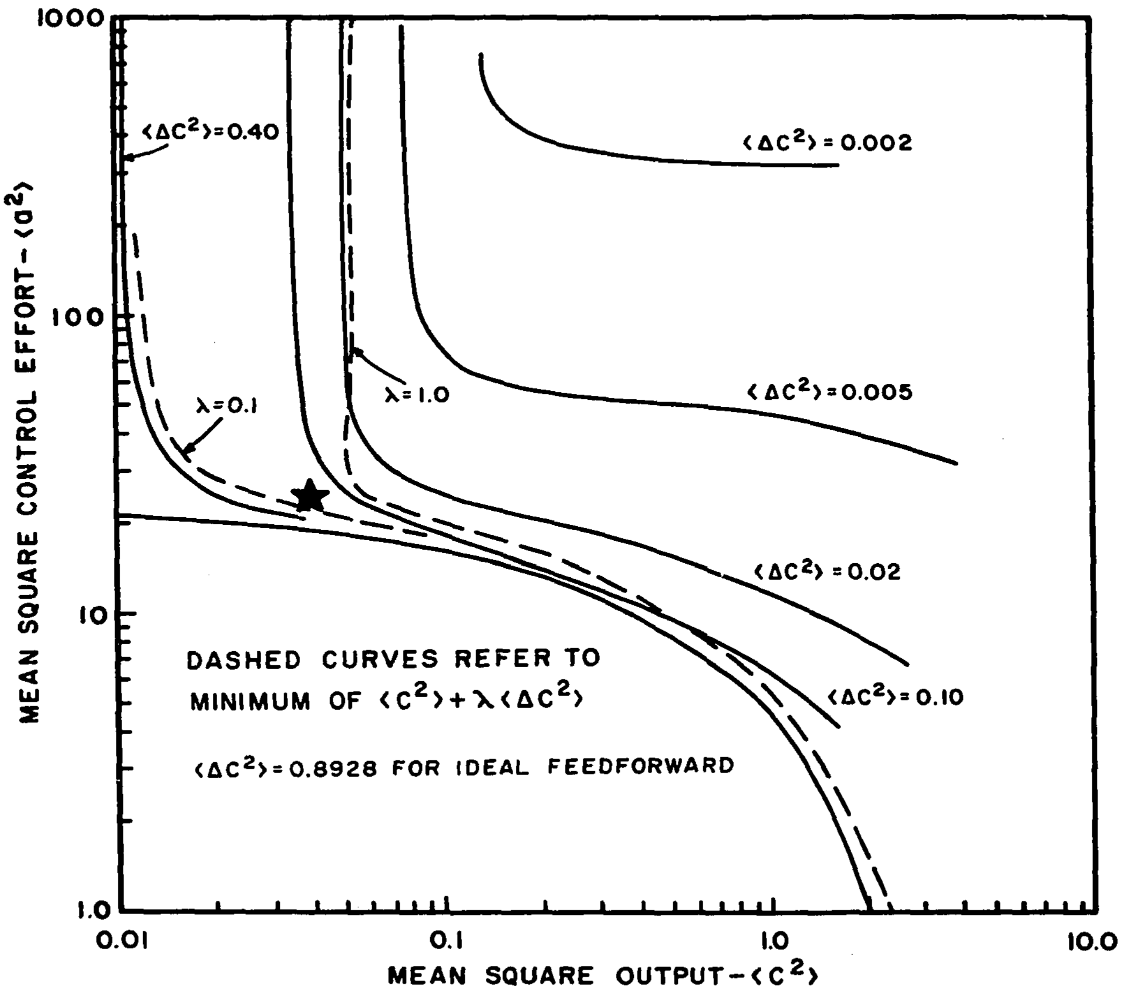

Figure 5-18.--Response Characteristics for Optimal Composite Control of First Order System with Parameters of Model Error Output, $\langle \Delta c^2 \rangle$. Same conditions as for Figure 5-15 except that model error is in plant gain only.

illustrative purposes because it represents a compromise between control effort and sensitivity while retaining good nominal output attenuation. Feedforward control yielding this nominal output would have a mean square error output approximately 20 times greater than the optimum. The parameters of the optimal control system are given in Table 5-5.

TABLE 5-5

OPTIMAL CONTROLLER TRANSFER FUNCTIONS FOR SYSTEM WITH ERROR IN PLANT GAIN

Overall Control Function, T_D:

$$T_D = 0.973 \frac{(1 + .07s)}{(1 + .101s)(2 + s)} .$$

Feedback Control Function, T_C:

$$T_C = 0.951 \frac{(1 + .245s)}{(1 + .100s)(1 + [.206 \pm 176j]s^2)(2 + s)} .$$

Feedforward Controller Function, Q_D:

$$Q_D = 0.865 \frac{(1 + .064s)(1 + .790s)}{(1 + .083s)(1 + .667s)} .$$

Feedback Controller Function, Q_C:

$$Q_C = 6.48 \frac{(1 + .245s)}{(1 + .083s)(1 + .667s)} .$$

Samples of system response under various conditions are shown in Figures 5-19, 5-20, 5-21 and 5-22. If there is no parameter error, perfect control can be achieved by feed-forward control alone. Feedforward control of the perfect

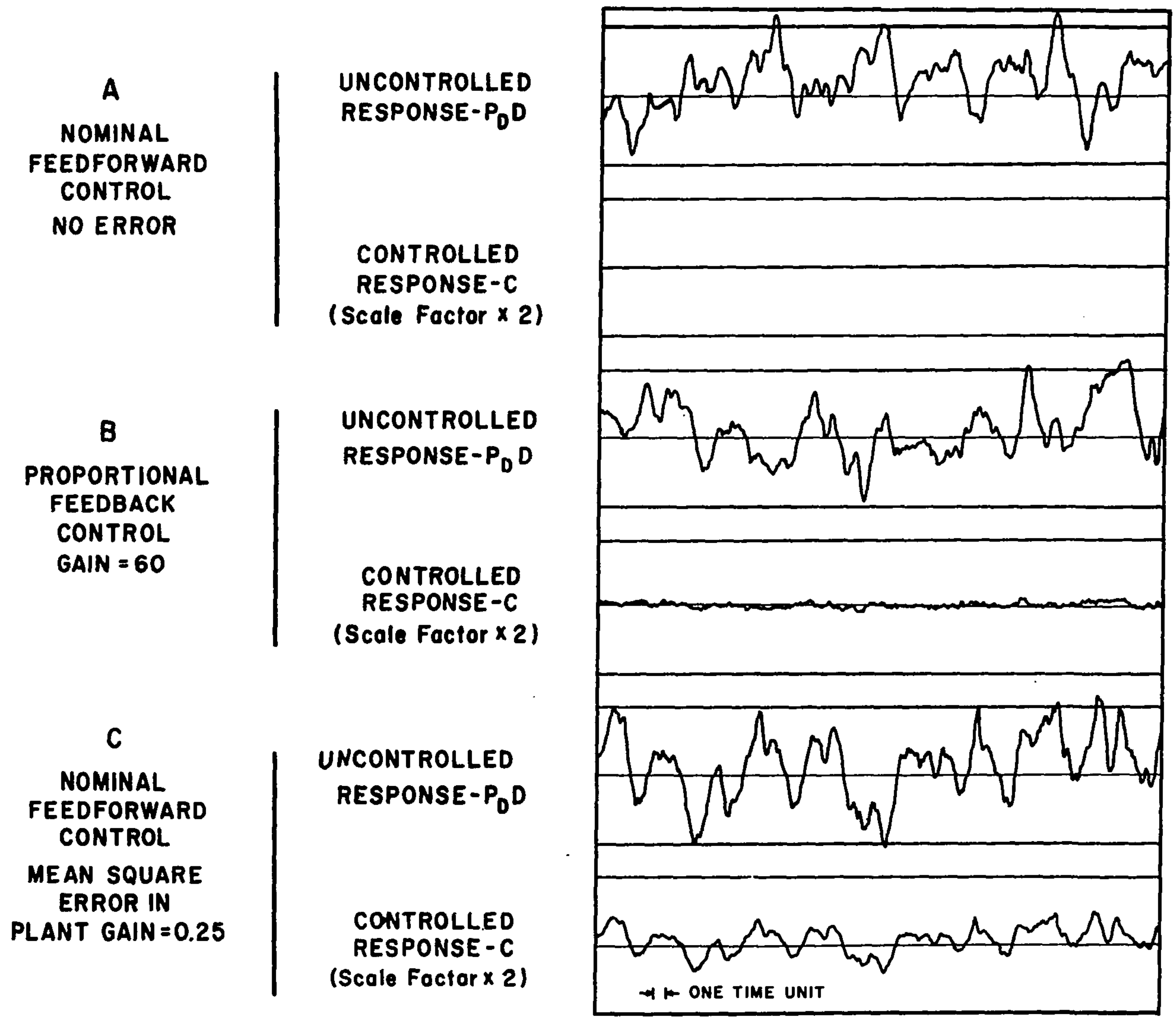

Figure 5-19.--Response of Controlled System to a Random Disturbance

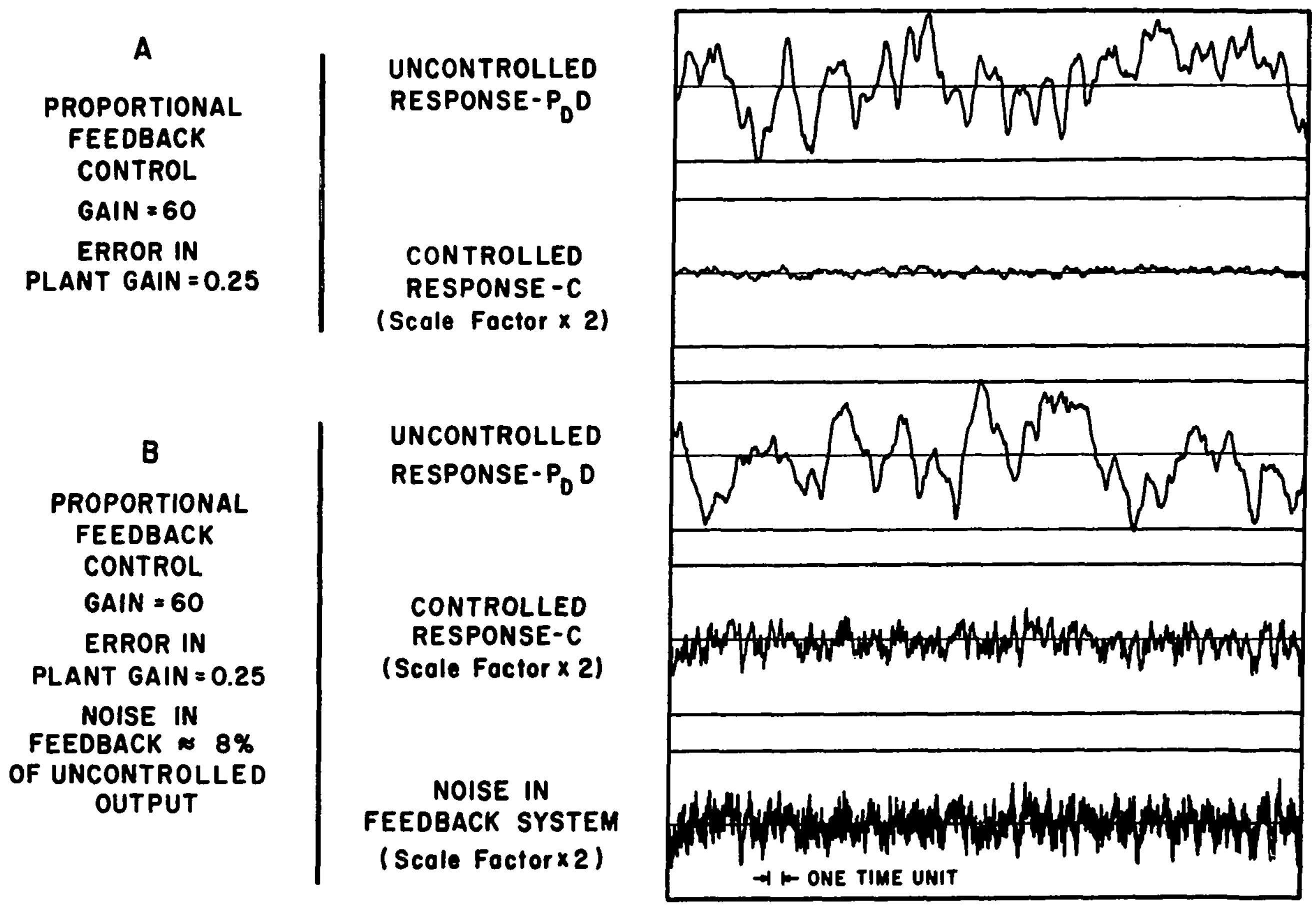

Figure 5-20.--Response of Controlled System to a Random Disturbance

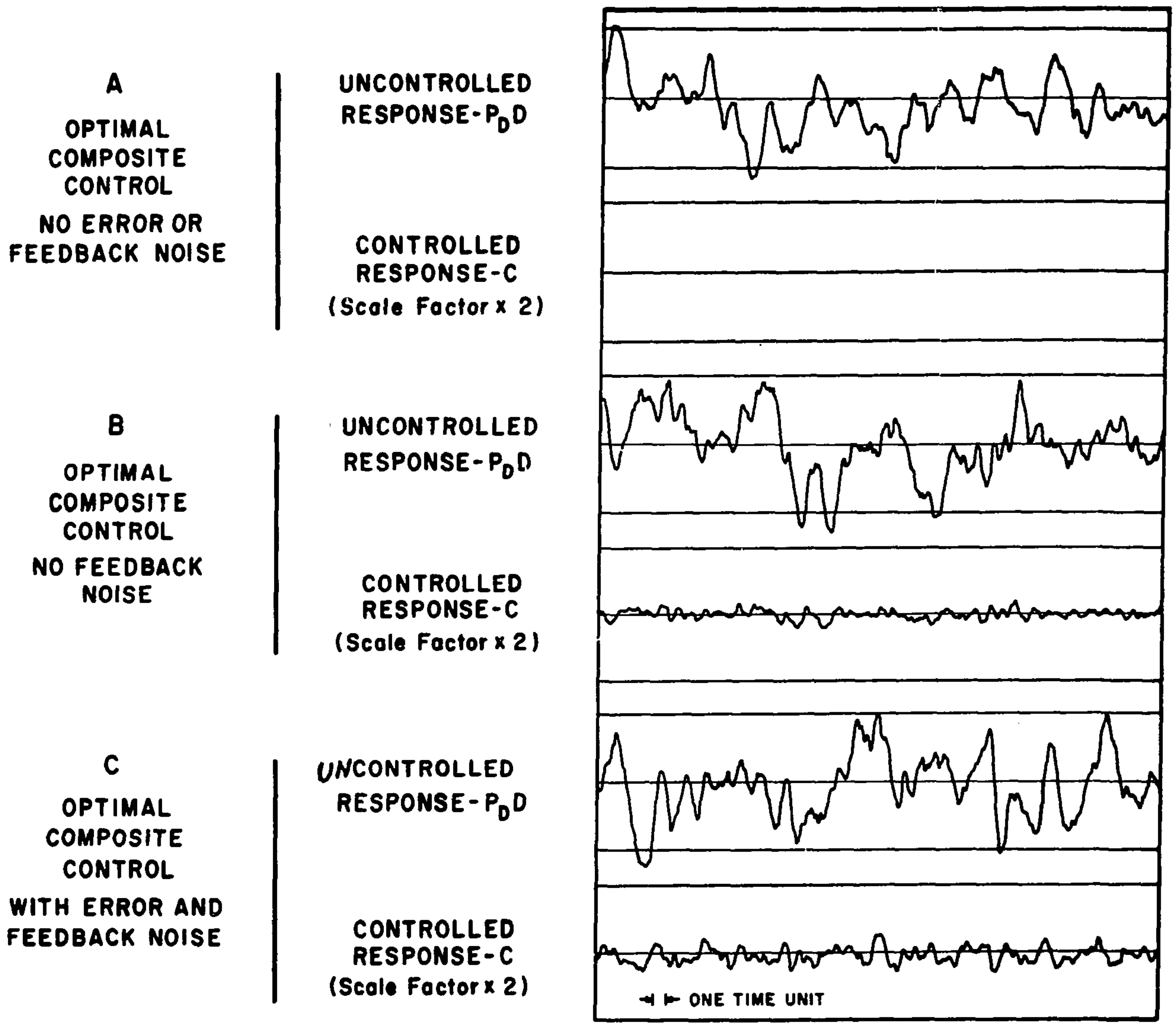

Figure 5-21.--Response of Controlled System to a Random Disturbance

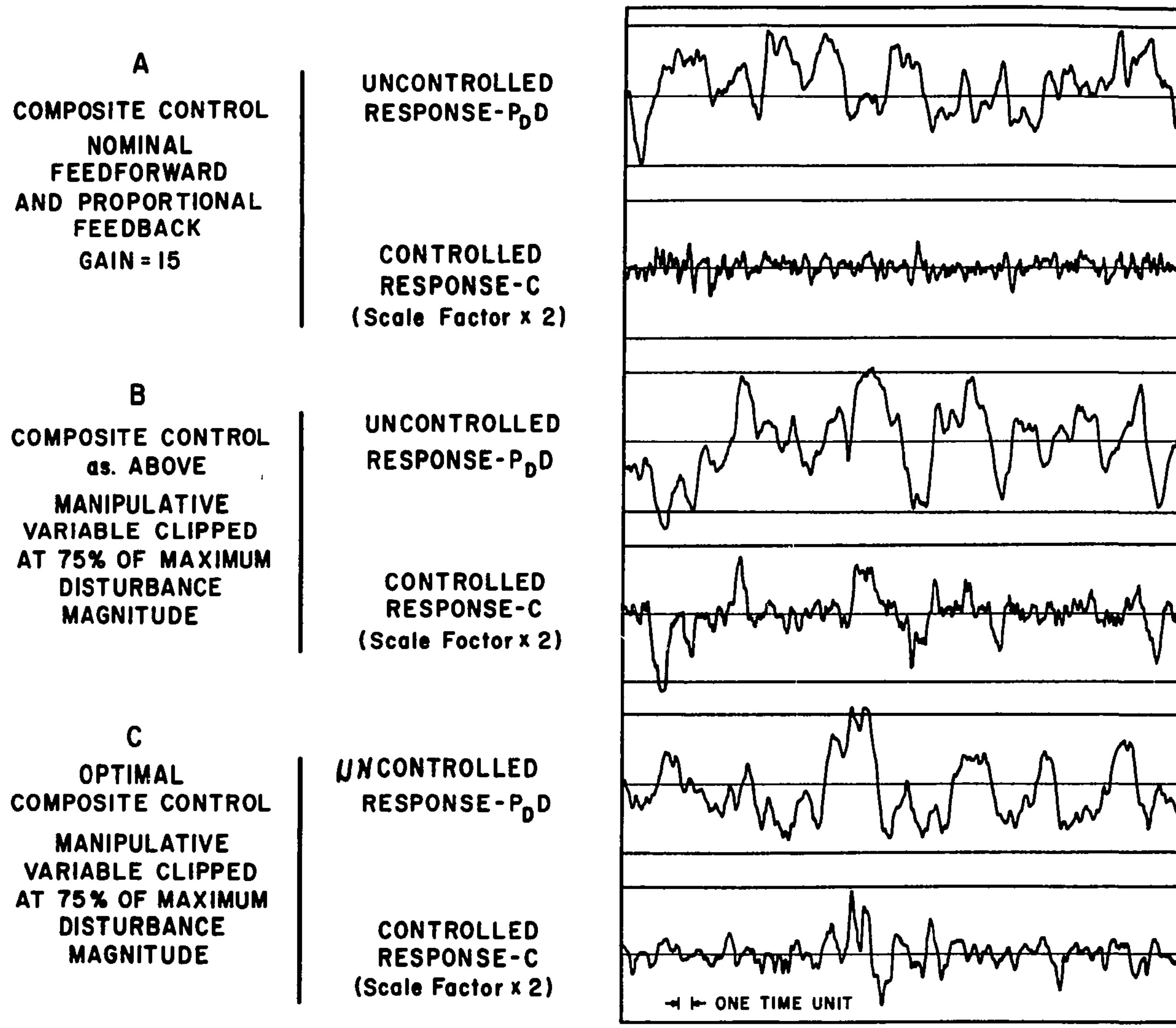

Figure 5-22.--Response of Controlled System to a Random Disturbance

model in Figure 5-19A shows constant near-zero output despite the scale factor of two between the uncontrolled and controlled response shown on this series of figures. Theoretically, control by feedback for this system can be made arbitrarily close in every way to control by feedforward by use of very high gain and, although it is not realizable in practice, perfect output attenuation by feedback can be very nearly achieved on an analog computer. Figure 19B shows the same system controlled by a proportional feedback controller with a gain of 60 - a practical maximum for most process applications. Although the control shown here is significantly poorer than with feedforward, it is still quite good (remembering the scale factor of two between the two representations).

Error in the gain factor of the plant transfer function drastically impairs feedforward control performance (Figure 5-19C). If this error were constant and permanent, it could be "tuned" out by adjusting feedforward gain but it is assumed here that such effects are transitory and cannot be permanently tuned out. The effect of this error on performance of the feedback system is almost negligible (Figure 5-20A). As a matter of fact the feedback controlled output remains a constant fraction of the uncontrolled output; the controlled output is larger in Figure 5-20A than 5-19B simply because the plant gain error was assumed in a direction such that the actual output is greater than that calculated for a given disturbance. Feedback is sensitive to two types of constraints however - dead time and noise injection into the feedback detection and gain system. This latter is shown in Figure 5-20B where the noise depicted in the lowest channel was added to the signal entering the feedback control system.

In actual practice it would not be possible to isolate the noise as shown here; part would be generated in the feedback system and part would exist as faulty measurement.

Optimal control performance is shown in Figure 5-21. Figure 5-21A shows the response of the optimal system when error and feedback noise are absent. Close examination is required to determine the difference between the response of this case and that of ideal feedforward under similar conditions; the values for the optimal feedforward control parameters in Table 5-5 are not greatly different from those of ideal feedforward. In Figure 5-21B, the effect of the addition of the model error is shown. The feedback portion keeps the controller relatively efficient although not so efficient as the high gain feedback of Figure 5-20A. When feedback noise is added, however, the results shown in Figure 5-21C show that the optimal composite control is superior to all previously tested. Furthermore, this system consumes far less control effort than the feedback of Figure 5-20B, its nearest competitor in terms of output attenuation.

This question remains however: "How does optimal composite control compare with a composite system designed by non-analytic methods?" It was observed earlier (Figure 5-7) that in the absence of dead times, output attenuation of ideal feedforward control is as good or better than that of optimal feedforward and that the major control improvement with the optimal control is reduction of control effort. In the present case, the fact that the overall control function, T_D, approaches $-P_D$ guarantees that the optimal Q_D approaches the nominal Q_D (cf. (3-108) and Table 5-5) so that the principal difference for comparison will be in the feedback.

Ideal feedforward control formed part of the non-analytic composite controller which was compared to the optimal controller. Proportional feedback was added and the gain adjusted until the output was approximately minimized for the system operating with all side conditions. This minimized output, which occurred at feedback gain of 10 to 15, is shown in Figure 5-22A to be somewhat different in character but about the same in magnitude as that of the optimal system (Figure 5-21C). As in the earlier case with feedforward alone, the main difference was that somewhat less control effort was required by the optimal. When saturation of control effort was simulated by clipping the manipulative variable at a level corresponding to about 75% of the maximum disturbance magnitude, performance of both systems deteriorated but the output response of the nominal feedforward with proportional feedback was not significantly different from that of the optimal system (Figure 5-22B, C).

Thus composite control is significantly superior to feedback or feedforward alone but output attenuation of the optimal composite control system is not better than that of a nominal non-analytic system even when subjected to side conditions very close to those for which the optimal has been designed.

Model Error with Pure Dead Time

If there is dead time in the controller ($\tau_M > 0$), then even in the absence of model error and feedback noise, there is a non-zero lower limit on the degree of attainable output attenuation (Figure 5-8). When model error and feedback noise are present along with the dead time a further degradation occurs in controller performance. Figures 5-23 and 5-24 show output vs. control effort for systems with mean

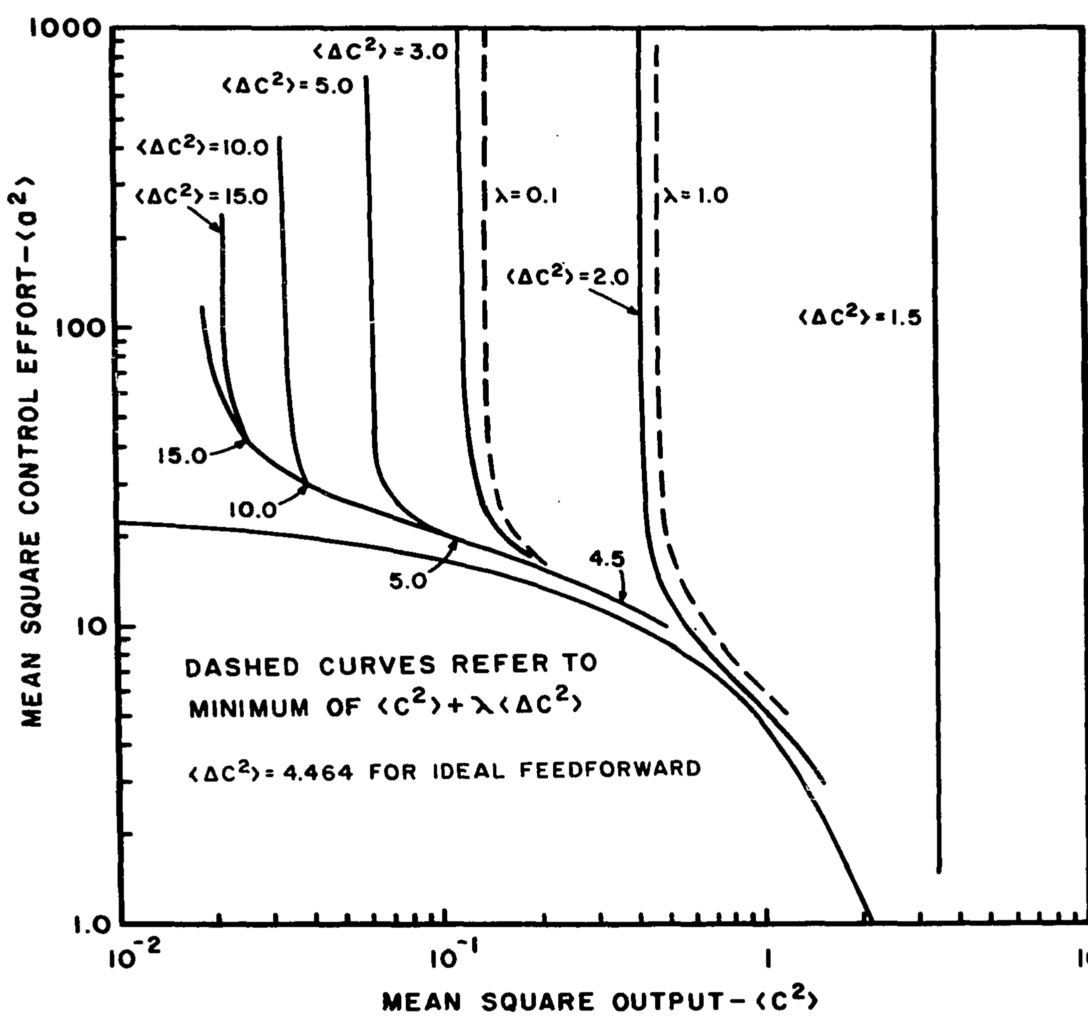

Figure 5-23.--Response Characteristics for Optimal Composite Control of System with Dead Time, τ_M, of 0.10 in the Presence of Model Error. Mean square value of feedback noise is 10^{-8}.

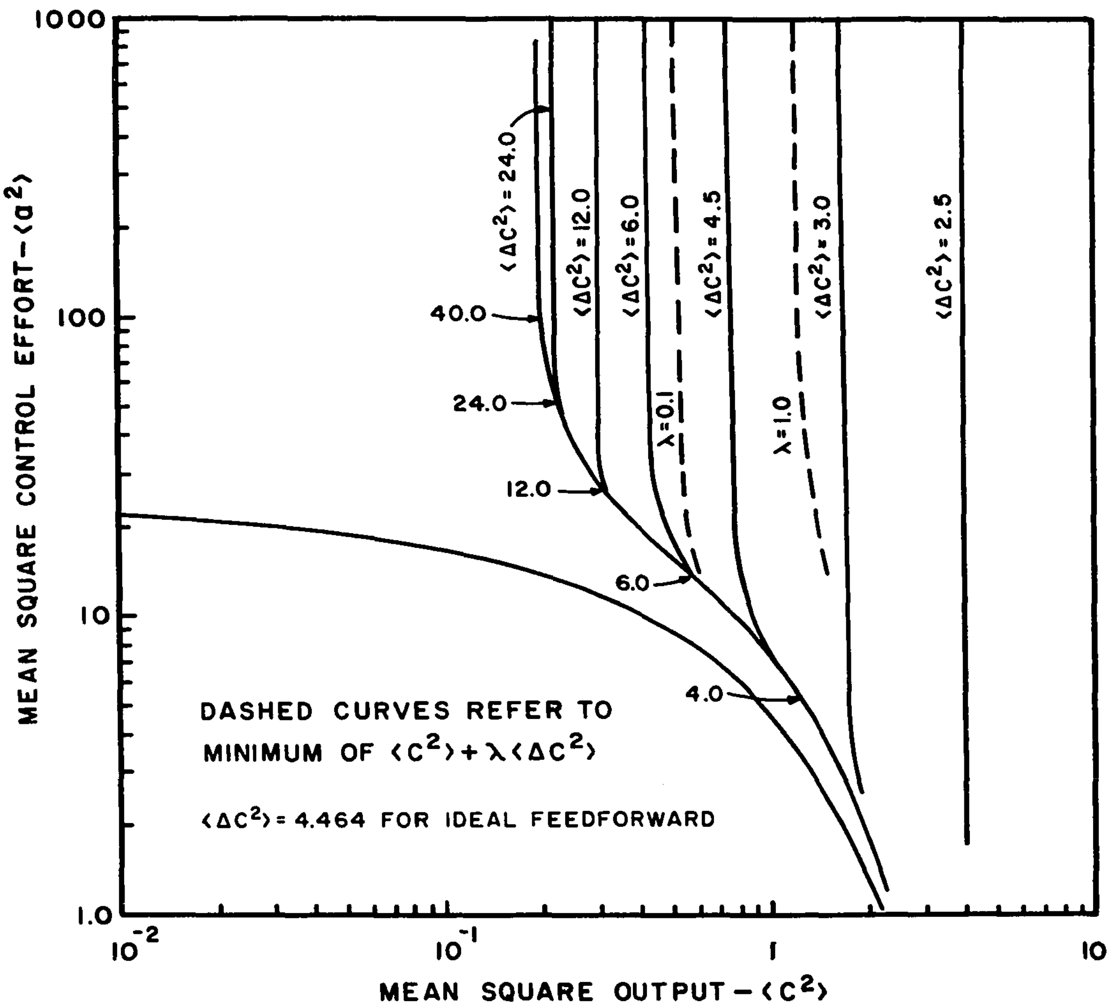

Figure 5-24.--Response Characteristics for Optimal Composite Control of System with Dead Time, τ_M, of 0.250 in the Presence of Model Error. Mean square value of feedback noise is 10^{-8}.

square model error of 0.25 in each of the parameters and in the presence of pure time delays of 0.10 and 0.25 respectively. It should be remembered that a dead time, τ_M, delays the response of the feedforward and feedback controllers by the same amount. Thus only relatively poor sensitivity reduction is possible even when nominal output is relatively high.

Some values of model error output on these figures exceed those for the nominal feedforward control system without feedback given in Table 5-4 and on the "no-feedback" base line on Figure 5-13. Thus, the optimal controller for a system containing dead time is considerably more sensitive to model error than nominal control. This result, coupled with the previously discussed sensitivity of this controller to permissible maximum control effort (Figure 5-12), indicates that caution is necessary in application of this "predictive" control design and underscores the advantage of using feedforward without dead times wherever possible

The situation for dead time in the feedback circuit only ($\tau_C > 0$) is shown in Figure 5-25 for $\tau_C = 0.1$. The high model error output in this case results from the fact that the dead time limits the capability of the feedback to attenuate output. Thus even though the feedback noise level is very low (i.e., the same as considered in Figure 5-13 - 0.0053% of uncontrolled output), the time delay prevents effective sensitivity reduction. This result combined with information from Table 5-4 shows not only that sensitivity reduction is achieved by use of feedback but also that, in the absence of feedback, very little can be done to reduce sensitivity.

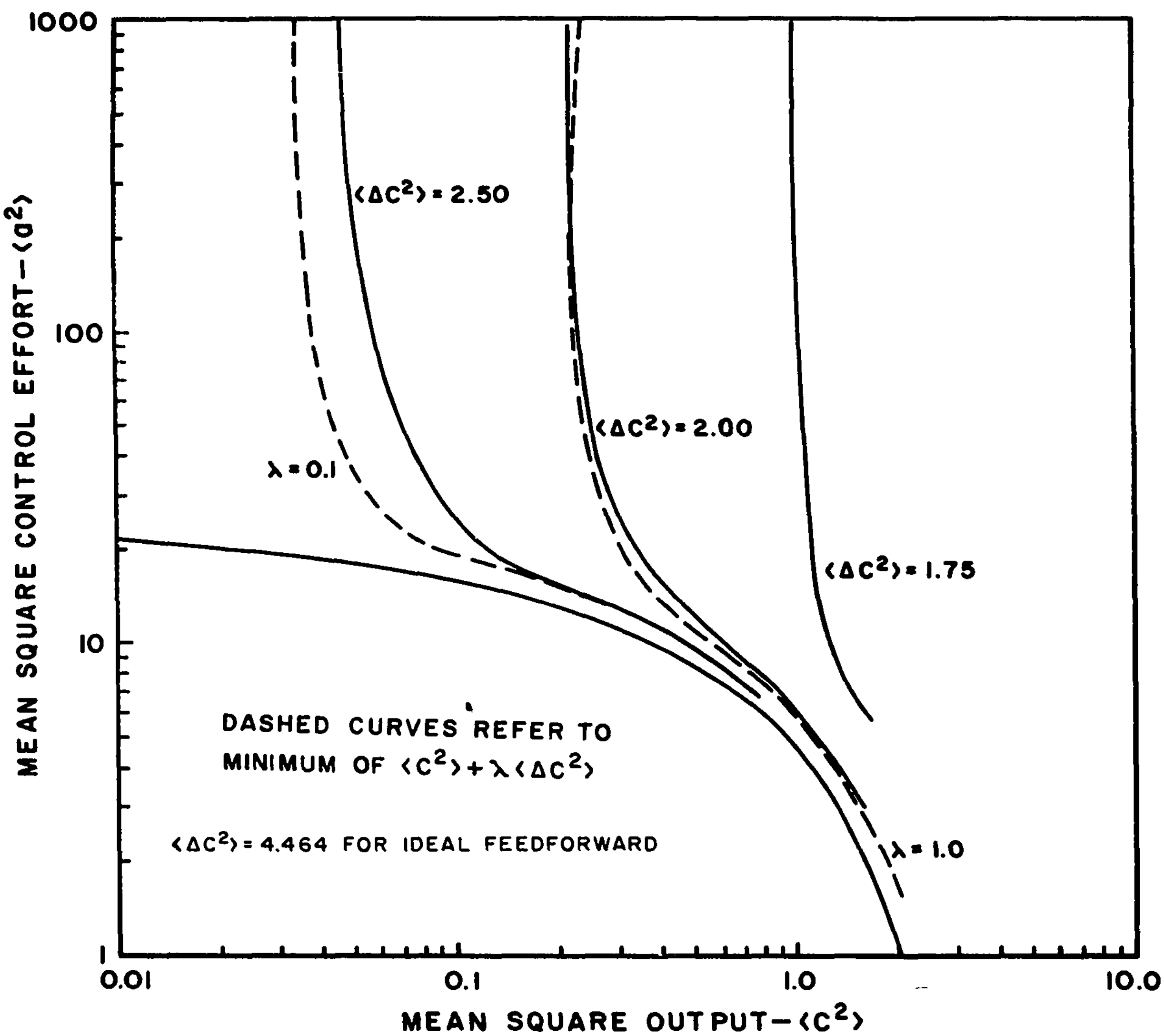

Figure 5-25.--Response Characteristics for Optimal Composite Control of System with Dead Time in Feedback, τ_c, of 0.10 in the Presence of Model Error. Mean square value of feedback noise is 10^{-8}.

Optimal Control as a Function of Model Error

The previous figures were constructed on the basis of mean square errors of 0.25 in each of the parameters of the system. If the error output were "normalized" by dividing it by the variance of the model parameters, i.e., the mean square error of each of the parameters (where all errors are equal), then the given curves change very little for all levels of model error. This effect is shown in Figure 5-26 where mean square control effort is plotted against "normalized" model error output at two levels of nominal output. Points shown for mean square model error of 0.1, 0.25 and 0.8 are seen to lie on the same curve. Furthermore the controllers corresponding to points on these curves were essentially independent of model error level. Within each of the groups of controllers shown in Table 5-6, the only important change was in the value of the model error. All controller parameters are relatively constant within each group. However a change in the level of model error changes the magnitude of model error output and shifts the location of the curve describing optimum for weighted sums, $\langle c^2 \rangle + \lambda \langle \Delta c^2 \rangle$; thus the form of the optimal control for a specific situation of constraints would be affected.

All previous calculations except those associated with Figure 5-18 were made on the basis of uniform error in each of the model parameters. Figure 5-18 showed that similar results are obtained when the model error exists in only one of the parameters, which, in that case, was the plant gain factor, K_D. The magnitudes of the model error output was less, of course, than when error was assumed to exist in all of the parameters.

A comparison of the magnitude of model error output

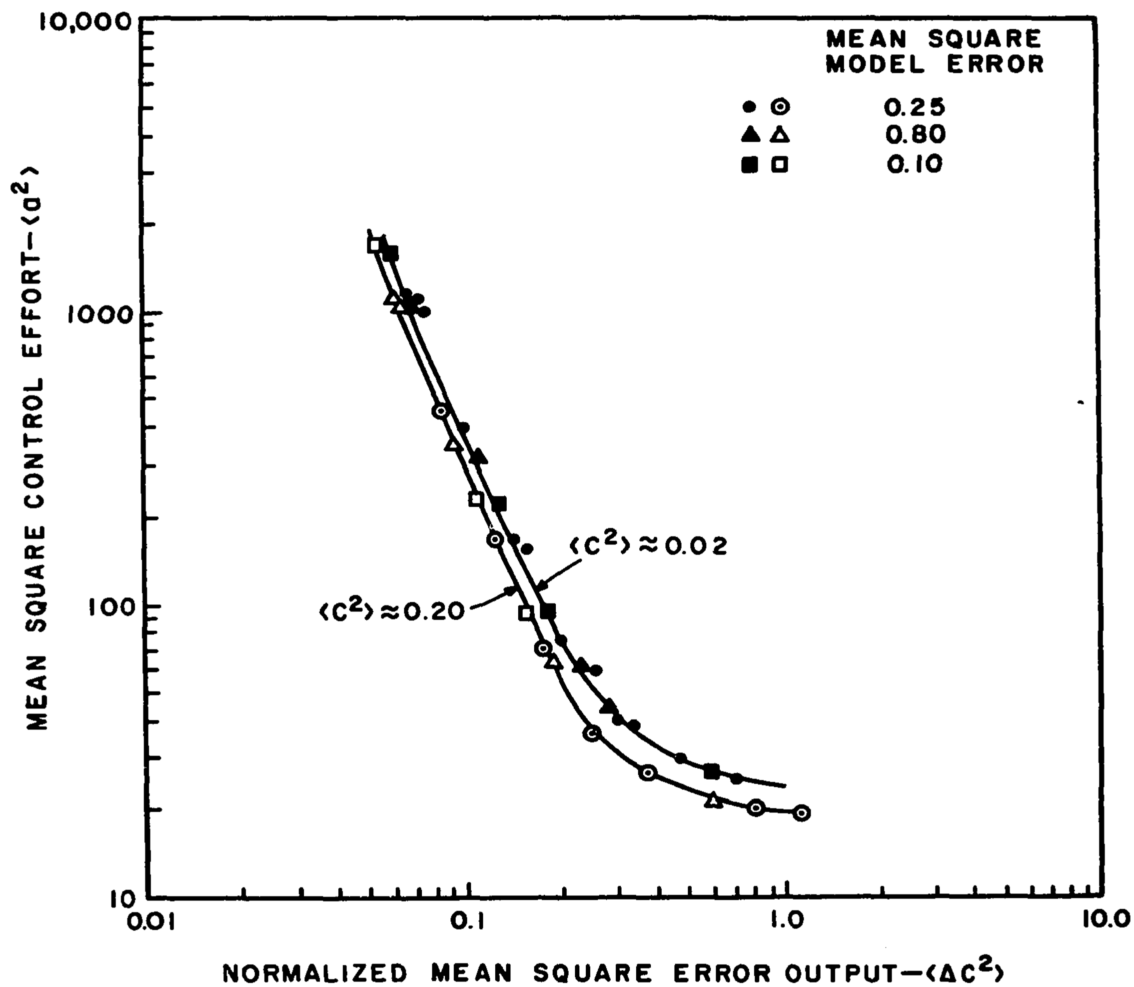

Figure 5-26.--Response Characteristics for Optimal Composite Control of First Order System at Various Levels of Uniform Model Error. Normalized $\langle \Delta c^2 \rangle$ is defined as $\langle \Delta c^2 \rangle$/mean square model error; Feedback noise level is 10^{-4}.

TABLE 5-6

OPTIMAL CONTROL LAWS FOR SYSTEMS AT VARIOUS LEVELS OF MODEL ERROR

Mean Square Model Error $\langle \Delta E^2 \rangle$	Outputs				Control Functions					
					Feedback Function - T_C			Overall Control Function - T_D		
	$\langle c^2 \rangle$	$\langle a^2 \rangle$	$\langle \Delta c^2 \rangle$	$\frac{\langle \Delta c^2 \rangle}{\langle \Delta E^2 \rangle}$	Gain	Zero	Poles	Gain	Zeros	Poles
0.10	.0688	235.2	.0302	.302	.9839	.50	.00382 ± .00382j	.459	.1228 ± .0867j	.50, .1526 ± .1313j
0.25	.0872	169.3	.0846	.339	.9819	.50	.00430 ± .00430j	.450	.1293 ± .0933j	.50, .1629 ± .1348j
0.80	.0526	160.6	.281	.351	.9815	.50	.00441 ± .00439j	.465	.1121 ± .0761j	.50, .1375 ± .1191j
0.10	.0446	37.9	.0698	.698	.9642	.50	.00858 ± .00851j	.461	.0968 ± .0608j	.50, .1189 ± .0914j
0.25	.0444	37.6	.1757	.702	.9640	.50	.00863 ± .00856j	.462	.0966 ± .0606j	.50, .1186 ± .0910j
0.80	.0349	51.8	.4711	.590	.9698	.50	.00722 ± .00716j	.469	.0948 ± .0597j	.50, .1146 ± .0945j
0.10	.0143	950	.0197	.197	.9898	.50	.00241 ± .00226j	.494	.0574 ± .0297j	.50, .0633 ± .0620j
0.25	.0134	1193	.0462	.185	.9904	.50	.00207 ± .00202j	.496	.0501 ± .0248j	.50, .0545 ± .0537j
0.80	.0153	1042	.1516	.190	.9902	.50	.00233 ± .00221j	.494	.0603 ± .0316j	.50, .0669 ± .0654j
0.10	.0136	175.7	.0360	.360	.9818	.50	.00433 ± .00422j	.490	.0707 ± .0404j	.50, .0799 ± .0758j
0.25	.0129	172.8	.0910	.364	.9816	.50	.00436 ± .00423j	.491	.0689 ± .0391j	.50, .0776 ± .0738j
0.80	.0121	196.0	.279	.350	.9825	.50	.00417 ± .00402j	.492	.0659 ± .0369j	.50, .0738 ± .0707j
0.10	.0039	48.0	.0798	.798	.9646	.50	.00846 ± .00673j	.497	.0446 ± .0257j	.50, .0478 ± .0451j
0.25	.0046	41.3	.2137	.854	.9621	.50	.00909 ± .00807j	.495	.0516 ± .0297j	.50, .0562 ± .0517j
0.80	.0043	45.0	.653	.815	.9638	.50	.00866 ± .00736j	.496	.0480 ± .0276j	.50, .0518 ± .0484j
0.10	.0046	116.1	.0514	.514	.975	.50	.00594 ± .00293j	.498	.0358 ± .0195j	.50, .0378 ± .0369j
0.25	.0054	148.8	.1121	.448	.978	.50	.00522 ± .00312j	.498	.0380 ± .0260j	.50, .0398 ± .0390j
0.80	.0062	137.4	.3532	.443	.979	.50	.00506 ± .00405j	.497	.0442 ± .0236j	.50, .0473 ± .0461j
0.10	.0020	210.9	.0780	.780	.9496	.50	.0233, .000438	.499	.0305 ± .0180j	.50, .0311 ± .0263j
0.25	.0020	228.6	.1965	.785	.9491	.50	.0236, .000391	.499	.0305 ± .0180j	.50, .0312 ± .0262j
0.80	.0026	215.6	.486	.607	.9602	.50	.0180, .000715	.499	.0297 ± .0180j	.50, .0305 ± .0276j

for the same relative error in each of the plant parameters is given in Table 5-7 for which a mean square model error of 0.25 was assumed for <u>each</u> of the elements in the following model:

$$C = \frac{K_D(1 + R_D s)e^{-\tau_C} D + K_M(1 + R_M s)e^{-(\tau_C + \tau_M)s} M}{(s + \alpha)(1 + R_1 s)} , \qquad (5\text{-}6)$$

where

$$K_D = K_M = 1 ,$$
$$R_D = R_M = R_1 = 0.5 ,$$
$$\alpha = 2 ,$$
$$\tau_C = \tau_M = 0.5 .$$

The model error output was computed on the basis of a nominal feedforward controller without feedback and for a disturbance whose spectral density is the same as that for the previous correlations.

Model error output was zero for variations in pole locations and feedback dead time since the feedforward controller does not depend on system natural frequencies or output sensing. However when these errors are present along with others, interactions occur producing model error output especially in design of the feedback portion of the control. Note that the product $P_M Q_C$, which occurs throughout the feedback design (cf. (3-64)), is quite dependent on system natural frequencies.

The least positive output resulted from error in the time constants of the zeros of P_D and P_M and was the same for both. Slightly more mean square output resulted from error in the gain factors of the load and manipulative variable transfer functions and again the mean square output was the

TABLE 5-7

MODEL ERROR OUTPUT CAUSED BY PARAMETER VARIATION OF 0.25 FOR INDIVIDUAL ELEMENTS OF MODEL

$$C = \frac{K_D(1 + R_D s)e^{-\tau_C s} D + K_M(1 + R_M s)e^{-\tau_M s} M}{(\alpha + s)(1 + R_1 s)}$$

Term Containing Error	Mean Square Model Error Output, $\langle c^2 \rangle$
K_D	0.8929
K_M	0.8929
R_D	0.7653
R_M	0.7653
α	0
R_1	0
τ_M	2.6784
τ_C	0

same for gain error in either of the two functions. The largest model error output resulted from error in the controller dead time. As illustrated previously in Figures 5-22 and 5-23, the model error output could be considerably higher if the controller were designed as an optimal partial differentiator instead of an ideal controller.

Optimal Controls for Higher Order Plants

The results of optimal control design for first order systems can be extended with few changes to more complicated plants. This simple extension is possible because the optimization is essentially a cancellation technique; i.e., the control function cancels the non-minimum phase plant poles and zeros and adds poles and zeros that give the desired performance.

A significant difference does appear when the net system order is higher however, in that feedback noise and dead time exerts a greater influence on optimal control sensitivity. This effect is seen in the control effort-output diagram shown in Figure 5-27 for the following system

$$C = \frac{M + D}{(s + 2)(1 + 0.45s)(1 + 0.60s)} \qquad (5\text{-}7)$$

The feedback noise is assumed to be 10^{-4} (0.65% of the absolute value of the uncontrolled output) and the model error has a uniform mean square value of 0.25 for all of the parameters. Although the topography of Figure 5-27 is quite similar to that of Figure 5-14 which depicts a first order system with similar constraints, it is seen that the general level of model error output is a much larger fraction of the "no-feedback" base for the third order system.

An explanation of this result lies in consideration

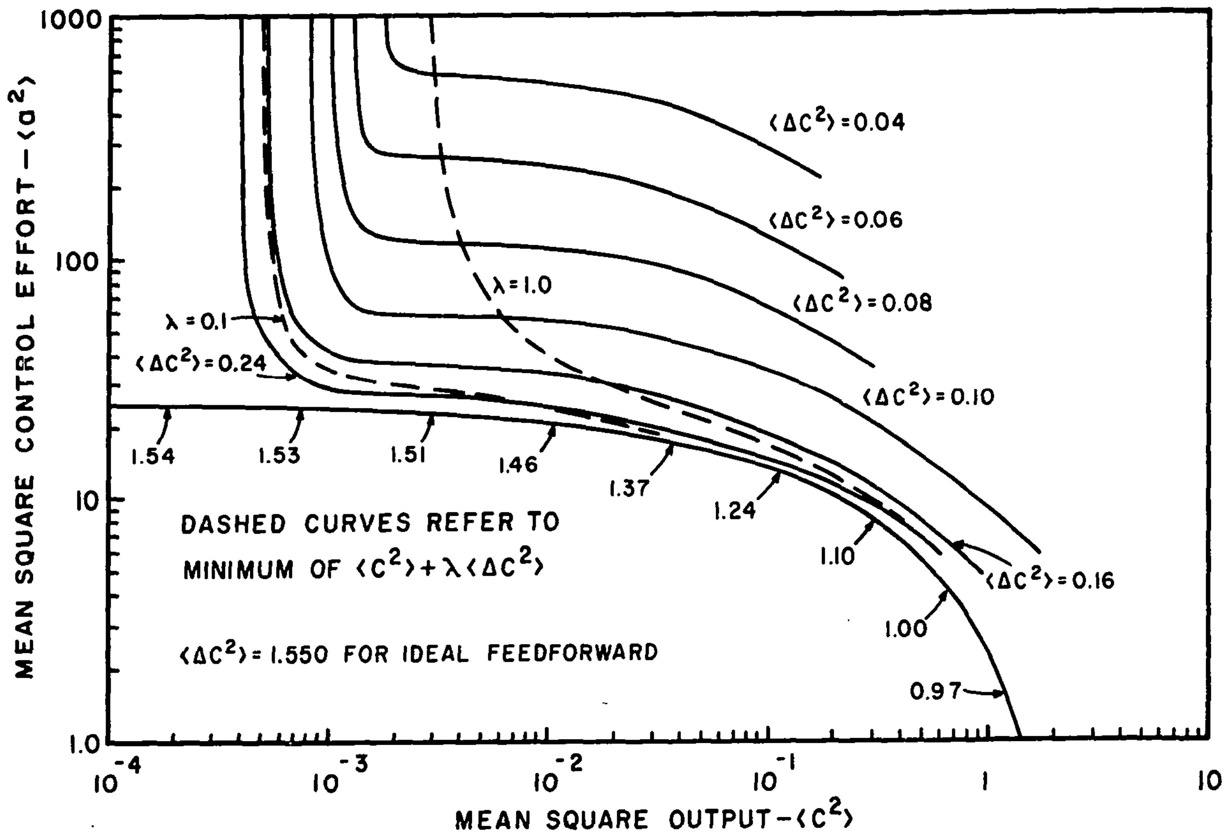

Figure 5-27.--Response Characteristics for Optimal Composite Control of Third Order System with Parameters of Model Error Output, $\langle \Delta c^2 \rangle$. Mean square value of feedback noise is 10^{-4}.

of the third order system response to a step function (Figure 5-28). The s-shaped response curve starts very gradually so that the output is masked by noise for a longer period of time than for first order response. Thus feedback control of higher order systems tends to act as though small dead times were present.

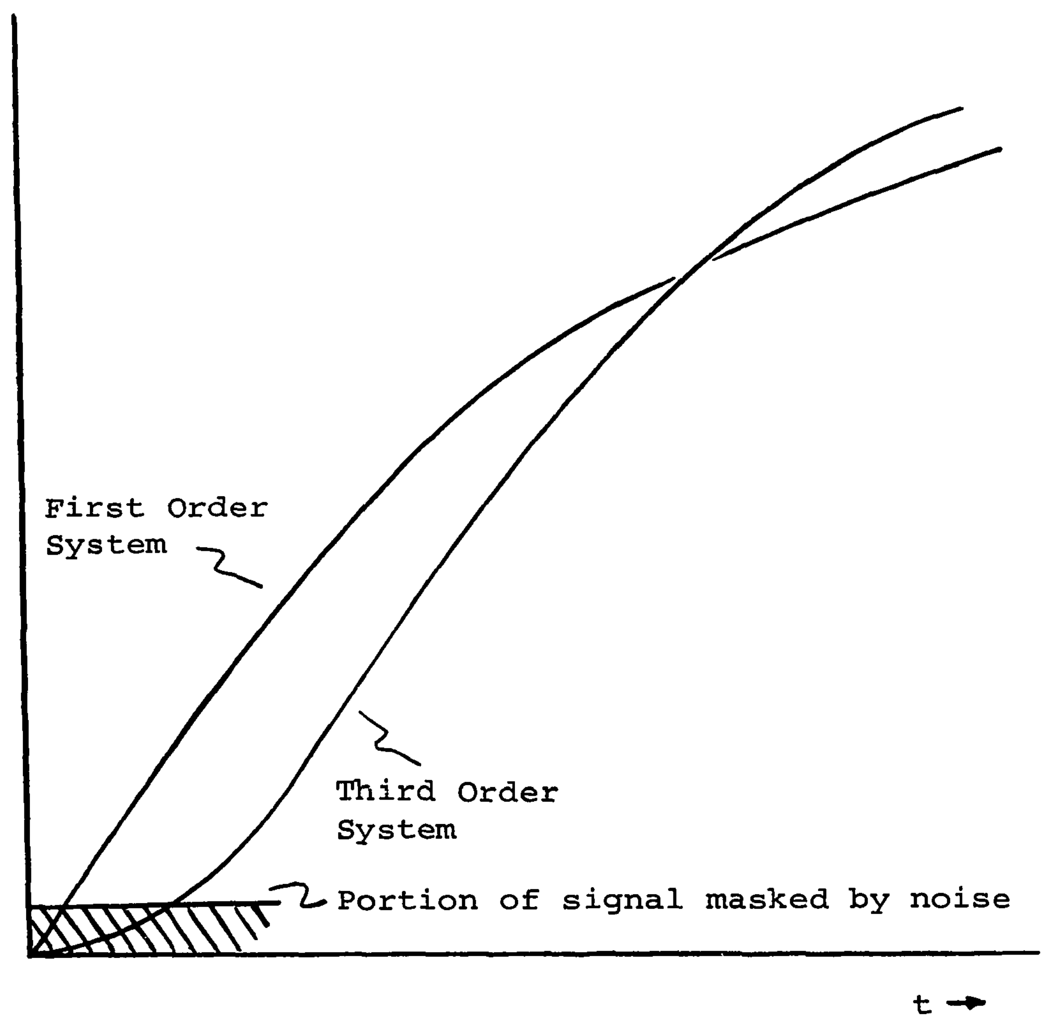

Figure 5-28. System Response to Step Functions

CHAPTER VI

EXPERIMENTAL STUDIES

In this chapter, previously considered controller designs are applied to a physical system. The mathematical procedures leading to the controller designs and the analog computer studies presented in the last chapter are self-contained within the framework of basic assumptions by which the dynamics of chemical process systems were represented. The validity of these assumptions is an important factor under consideration in the experiments described in this chapter.

Process Description

The experimental system used was one which has evolved through five previous doctoral research projects [F1,S3,G1,B5,H3]. At the heart of the process, shown schematically in Figures 6-1 and 6-2, is a perfectly stirred, jacketed vessel with hot and cold fluids entering the stirred center and jacket respectively. The control objective is to vary the flow rate of the hot fluid so that the temperature of the separating wall remains constant despite variations in the flow rate of the cold fluid. Even though this apparatus has become quite artificial and the control objective somewhat impractical, it possesses a number of very desirable characteristics.

(1) The heat transfer which occurs is an important

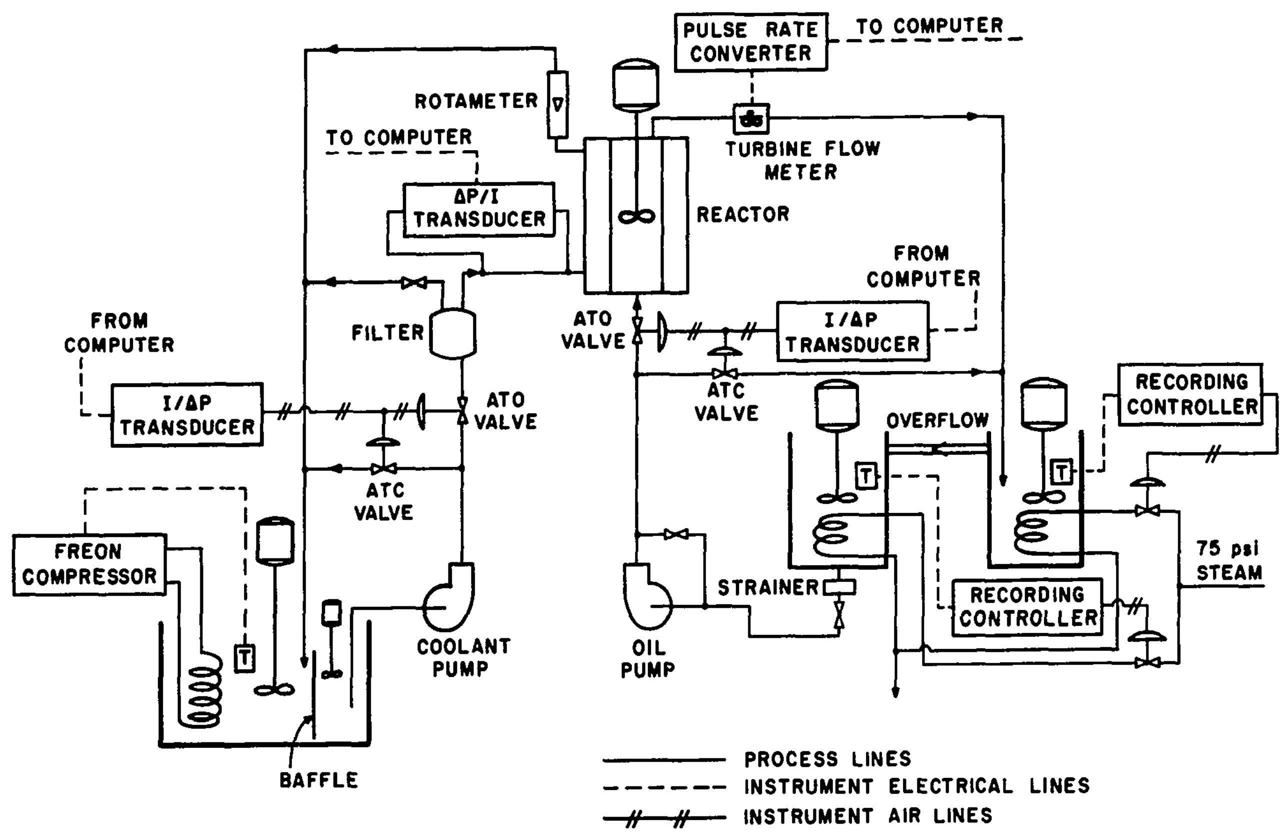

Figure 6-1.--Schematic Flow Sheet of Experimental System

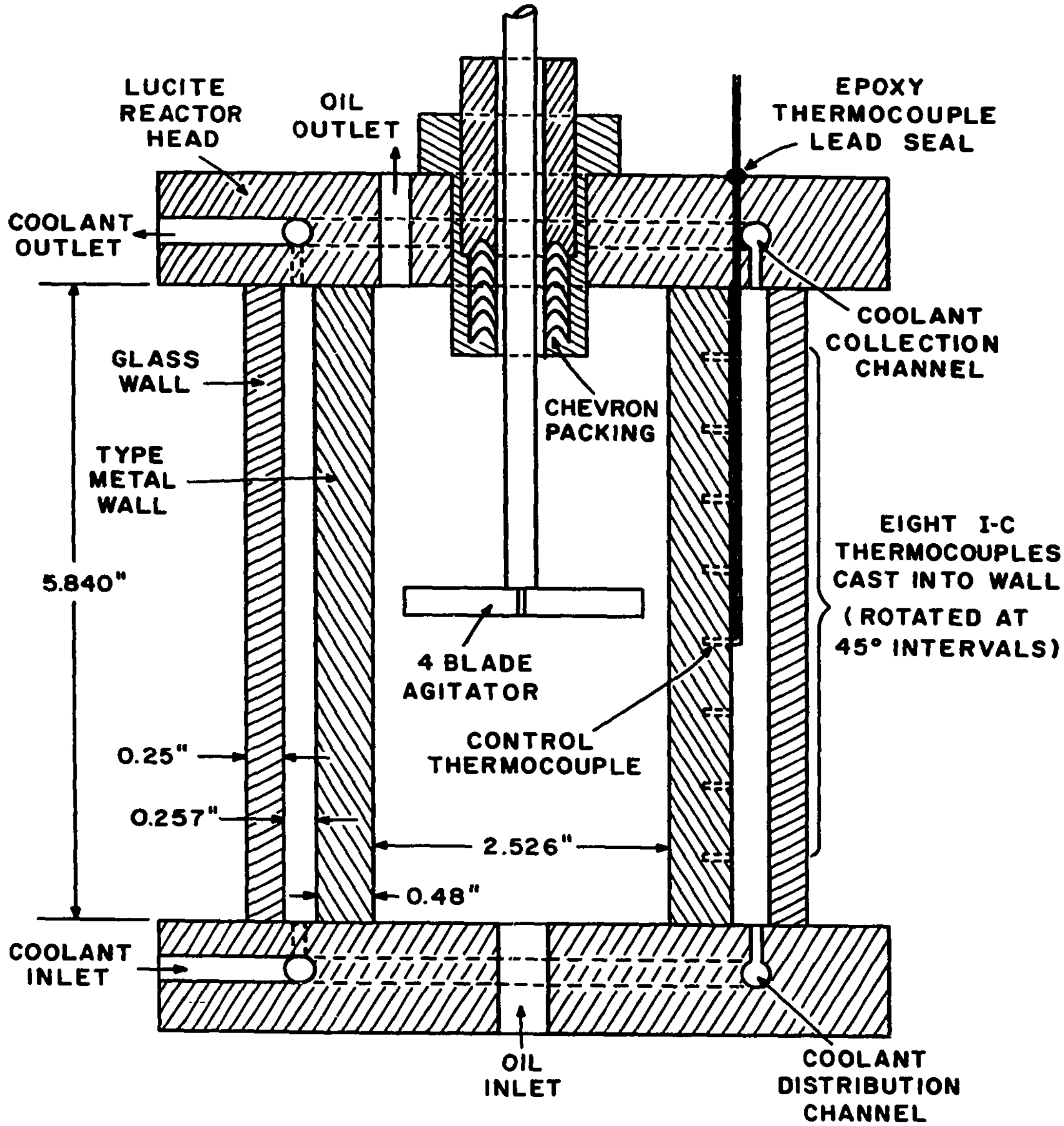

Figure 6-2.--Detail of Heat Exchange Vessel

and typical basic chemical engineering process;

(2) The system dynamics are reasonably sensitive and responsive to control so that significant control is possible;

(3) The system must be described by higher order equations so that some of the more elaborate controller designs can be investigated;

(4) Product-type non-linearities are present so that some dynamic model error always exists regardless of the care in identification and operation of the system;

(5) The system itself is simple in basic design keeping maintenance and operating problems moderate.

System Components

Reactor

The simulated reactor (Figure 6-2) was identical to that used by Stewart [S3], except that the wall between the oil and coolant was replaced. Instead of steel, the wall was molded of type metal and was 0.480 inches thick. Eight thermocouples were located at 3/4 inch spacing approximately 45° apart. Only one, the fourth from the bottom was monitored continuously to be used as the controlled output variable c(t). Hot oil entered the one liter reactor at the center of the bottom. The oil was agitated by a 4 paddle stirrer driven by a 1/15 horsepower 1725 RPM electric motor. The oil left the reactor at the top and off center. The coolant, a 50% solution of ethylene glycol and water, entered through a distributer into the bottom of the annular space surrounding the reactor wall and left from the top.

Constant Temperature Feed Tanks

The oil was maintained at a constant temperature in a 35-gallon tank. The oil was agitated by a 1/8 horsepower 1725 RPM Lightnin Mixer, Model NC2. The tank contained coils for both cold water and steam. The steam flow rate was controlled by a Research control valve (1/2 inch), Model 75S, air-to-close (C_v = 0.8 and a 3-15 pound range spring). The recording controller was a Minneapolis-Honeywell Brown Electronik Potentiometer Pyrometer, Model 152P14P-93-18, with a copper-constantan thermocouple pickup.

The oil from the heated tank then flowed through a 30-gallon insulated, agitated tank where temperature variations were blended out. Under conditions of extreme flow rate changes, the output temperature was observed to drift slowly and uniformly less than 1/2°F per hour.

The glycol-water solution was maintained at constant temperature in a refrigerated cooler with a capacity of approximately 25 gallons. The glycol solution was agitated by a Mixing Equipment Co. 1/4 h.p. "Lightning" mixer Model CV-4 with an adjustable stirring rate of 100 to 1800 RPM. The cooler contained freon coils in which the temperature was controlled by cycling the operation of the freon compressor. A Fenwal thermoswitch (Catalog Number 17552-0), having a temperature range of -100 to 600°F, and a series of relays started the freon compressor whenever the temperature rose above the setpoint of the Fenwal thermoswitch.

A 5 gallon section of the coolant container was baffled away from the main portion and separately agitated by a 20 watt laboratory type mixer. Coolant flow to the reactor was withdrawn from this section with the resulting blending action virtually eliminating temperature variation in this flow.

Flow Systems

The oil was circulated by a California bronze gear pump (1/4 inch pipe connections) driven by a Goulds Number 2 electric motor (3/4 horsepower). The pump discharge pressure was set at 40 psi by a valve on the bypass line. The glycol solution was circulated by a 1/4 horsepower gear pump operating at 40 to 60 psi.

Flow Controllers and Transducers

The reactor inlet flows of both oil and coolant were controlled by a flow splitter arrangement. In each stream the flow was divided so that the fluid passed through an air-to-open control valve into the reactor and through an air-to-close control valve to the constant temperature feed tanks. All four valves were 1/4 inch, Type 75, Research control valves with 3 to 15-pound range springs. The valves each had a C_v of 0.2 for the oil and 0.08 for the coolant.

The pneumatic signal to the valves originated in the Taylor Transet electro-pneumatic transducers 701T which had a range of 3 to 15 psi. The input signals to the transducers were generated at the analog computer usually from a DC amplifier output. Feedback loops were built around both valves to give positive flow control since the valves were sluggish in action and contained considerable hysteresis. Simple proportional control with a small amount of integration to reduce setpoint error was used for these loops. The control was not highly critical; it was simply necessary that the flow rates follow the setpoints if the resulting data was to be meaningfully analyzed. The time constants involved in these systems were 0.1% to 2% of the dominant system characteristics.

Flow Measurements and Transducers

The oil flow rate sensor was a Waugh turbine flowmeter, Model FL-6SB-1, rated at 0.15 to 1.0 gallons per minute. The pulses from this pickup were converted to a continuous voltage by a Waugh, Model FR-111, pulse rate converter. The output of the pulse rate converter was a voltage of 250 millivolts maximum, which was available at the analog computer where it could be amplified.

The coolant flow rate was detected by measuring the pressure drop caused by flow through a 15 ft. length of 1/4" polyethylene tubing partially restricted by #12 copper wire running through it. This arrangement was not only less susceptible to plugging than an orifice giving comparable pressure drop but also yielded a linear flow - ΔP relationship which improved measurement accuracy at lower flow rates. Even so, some difficulties with plugging and air bubble blockage occurred until an automobile oil filter with air bleed on the top was installed. The pressure drop was measured with a strain cell from a Beckman Model 112 Data Logger giving a -3 to 12 mv output for 0 to 15 psi pressure drop. This signal was amplified directly in the analog computer.

Temperature Measurement

The preamplifier for the wall temperature, controlled variable, was a Sanborn, Model 350-1500, low-level DC preamplifier with a Model 350-2 plug-in unit. This instrument allowed an adjustable gain up to 50,000 and an input suppression of $\pm$ 100 millivolts.

Dynamic Mathematic Model

The theoretical dynamic mathematical model of this system is derived in Appendix C. The transfer functions predicted by this derivation are

$$C = \frac{.24(1 + 0.32s)D + .048(1 + 0.38s)M}{(1 + 1.03s)(1 + 0.42s)(1 + 0.22s)} , \qquad (6\text{-}1)$$

where C is wall temperature in °F, D is coolant flow rate and M is oil flow rate both in lb/min. This equation could have been used to design the optimal control systems but the experience of Haskins [H3] indicated that relatively large error factors would be required.

To avoid large error factors which would result in optimal controllers consisting primarily of feedback, direct experimental evaluation of the dynamic parameters was made using an identification technique originated by M. Heymann [H5] and currently being developed by R. A. Sims at the University of Oklahoma Process Dynamics and Control Laboratory. The technique is based on determination of system natural frequencies by time domain integrations of the relaxing response of the system disturbed by sets of linearly independent initial conditions. System response to sets of orthogonal forcing functions is used to evaluate the zeros by time domain integrations using the previously determined homogeneous weighting functions. Complete explanation of both the identification technique and details of the particular identification can be found elsewhere [H5,H6,S1].

The results of the identification were

$$C = \frac{K_D(1 + 1.33s)D + K_M(1 + 0.33s)M}{(1 + 0.60s)(1 + 1.38s)} , \qquad (6\text{-}2)$$

where the unit of time for the time constants is minutes. This system was then activated by a random disturbance formed by filtration of noise from the signal generator previously described. The generated noise had a mean frequency of 1.0 radians per second and was filtered down to a frequency of 1.5 radians per minute. That is,

$$\Phi_{DD} = \frac{\mu^2}{1.5^2 - s^2} . \qquad (6\text{-}3)$$

For estimation of pole error, the system natural frequency nearest this disturbance frequency was chosen as the inaccurate pole. Relatively good reproducibility of pole and zero locations was indicated in the experimental identification where the accuracy of pole and zero locations appear to be about ± 10%. For optimal controller design purposes, pole mean square error corresponding to twice this error was chosen.

Numerical values for the gain factors are not listed in (6-2) since these quantities have little inate significance, and they depend on amplification factors, temperature-to-voltage conversions, flow meter and valve calibrations, etc. The gain of the various experimental controller were set as a percentage of feedforward gain necessary to return steady state offsets to the control point. The identification studies did indicate, however, that plant gains varied considerably as a function of disturbance sign and magnitude. This variation results from the product nonlinearities that appear in the theoretical describing equation (C-1). However, inherent partial compensation of this effect occurred with the controllers because the control objective of constant wall temperature meant that low coolant flow rates were matched by low hot oil flow rates and vice versa. Although the effect of flow rate on gain was the same for both flows, the cancellation is not complete. For example, a flow rate change of ±20% of the steady state caused the gain to change by a factor of nearly two; for the same variation of flows, the ratio of gains changed by approximately 30%. These facts were noted in the computations by assuming gain error of 35% in both gains but assuming also a convariance between them of 50%.

Results during identification tests indicated that noise in the output was about 1 to 3% of the uncontrolled output. For controller design purposes a mean square error corresponding to about 4% of the uncontrolled output was assumed. In addition dead times of 0.035 minutes were assumed in both the controller and feedback circuits. The error assumed in these dead times was approximately equal to their magnitude.

Controller Results

A complete control effort-output diagram, shown in Figure 6-3, was prepared on the basis of (6-2) and the above mentioned side conditions. Control laws which were selected to be tested are defined implicitly by the conditions at the three starred points in Figure 6-3. The characteristics of these controllers are listed in Table 6-1.

In the presence of control effort constraints, an optimal control law similar to that of system "A" would be desired. This control law approaches ideal feedforward control because of the low model error although some feedback is present to reduce long term steady state drift. Good control performance is indicated by this control law. If greater long term drift could be expected, along with control effort constraint, the optimal control design could be made on the basis lower model error output. This would cause

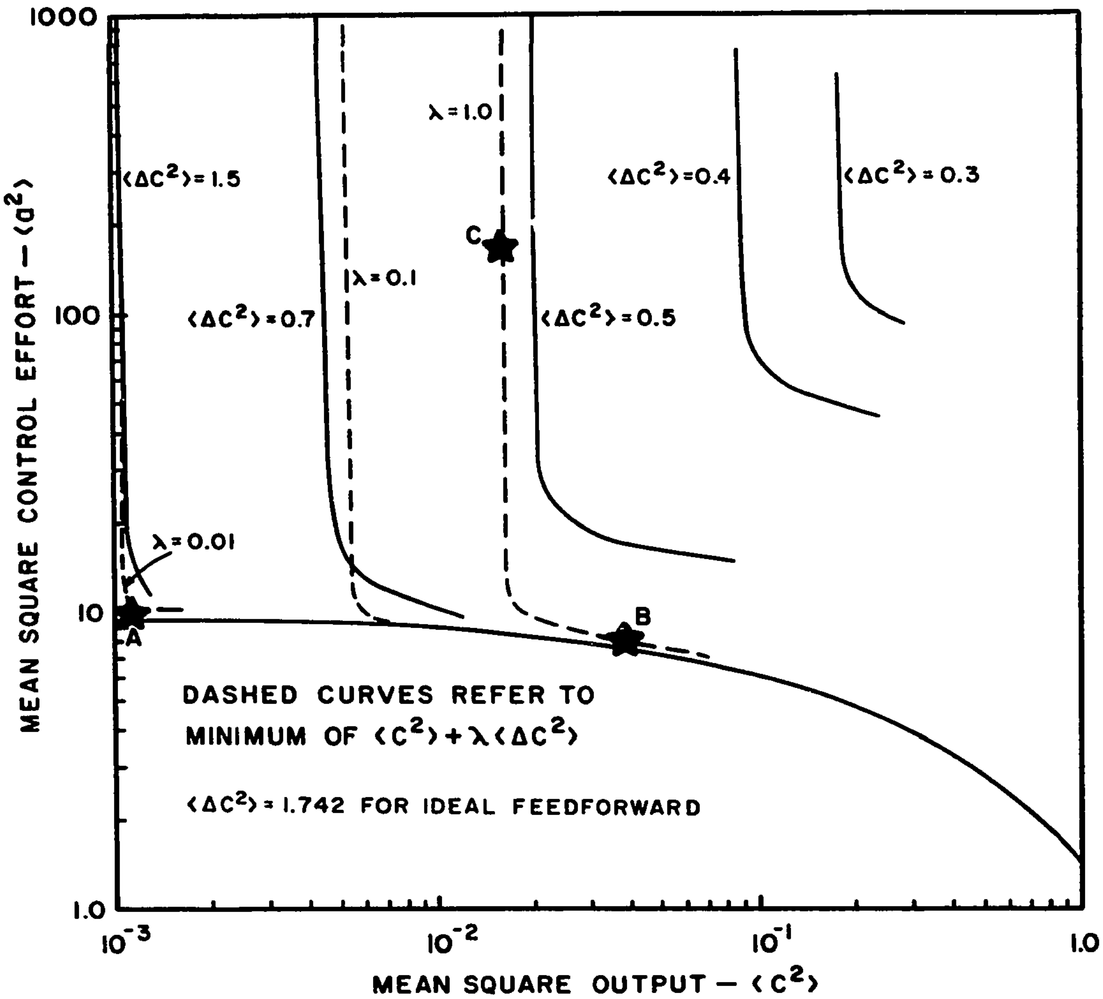

Figure 6-3.--Response Characteristics for Optimal Composite Control of Experimental System (Computed)

TABLE 6-1

OPTIMAL CONTROLLER TRANSFER FUNCTIONS FOR EXPERIMENTAL MODEL:

$$C = \frac{(1 + 0.33s)e^{-.03s}D + (1 + 1.33s)e^{-.07s}M}{(1.68 + s)(1 + 1.38s)}$$

Controller	Overall Control Function - T_D	Overall Feedback Control Function - T_C
Nominal	$T_{DN} = -\frac{(1 + 0.33s)}{(1.68 + s)(1 + 1.38s)}$	--
A	$T_D = 0.995 \frac{(1 + .072s)}{(1 + .038s)} (T_{DN})$	$T_C = 0.558 \frac{(1 + [.457 \pm .133j]s)^{2*}}{(1 + .033s)(1 + .496s)(1 + [.561 \pm .383j]s)^2}$
B	$T_D = 0.908 \frac{(1 + .158s)(1 + .226s)(1 + .281s)}{(1 + .129s)(1 + .194s)(1 + .379s)} (T_{DN})$	$T_C = 0.905 \frac{(1 + [.380 \pm .119j]s)^2}{(1 + .160s)(1 + .356s)(1 + [.183 \pm .200j]s)^2}$
C	$T_D = 0.991 \frac{(1 + .035s)(1 + .120s)}{(1 + .029s)(1 + .083s)} (T_{DN})$	$T_C = 0.939 \frac{(1 + [.347 \pm .106j]s)^2}{(1 + .0004s)(1 + .359s)(1 + [.194 \pm .180j]s)^2}$
	Feedforward Controller Function - Q_D	Feedback Controller Function - Q_C
Nominal	$Q_{DN} = -\frac{(1 + 0.33s)}{(1 + 1.33s)}$	$Q_{CN} = \frac{(1 + 1.38s)(1.68 + s)}{(1 + 1.33s)}$
A, B, C	$Q_D = \frac{Q_{DN}\left[\frac{T_D \cdot Q_C}{Q_{DN}} - T_C \cdot e^{-.035s}\right]}{1 - T_C \cdot Q_{CN} \cdot e^{-.07s}}$	K = 0.937 (A), K = 1.52 (B), K = 1.57 (C); $Q_C = K \frac{T_C \cdot Q_{CN}}{1 - T_C \cdot Q_{CN}e^{-.07s}}$

*$(1 + [a \pm bj]s)^2$ indicates a pair of complex conjugate roots where a is the real part and b is the imaginary part.

controller "B" to be specified but at significant cost in output attenuation.

In the absence of strong control effort constraints, an optimal control law similar to system "C" would be selected. Because of the relatively small dead times, this optimal controller approaches ideal feedforward with proportional feedback. High performance would be expected of this controller from the standpoint of attenuation of both nominal output and model error output.

A preliminary examination of these controllers was made using analog computer simulations of the plant. These results, shown in Figure 6-4, were obtained from the idealized plant, that is, the plant without deliberately added noise and model error; hence they must be interpreted with the results of the previous chapter in mind. The disturbance and manipulative variable are shown in this figure along with the controlled and uncontrolled output so that later comparison is possible since it is not possible to record controlled and uncontrolled response of a real system simultaneously.

The best control shown is that with controller A which approaches ideal feedforward control since only the small dead times produce significant output. Although controller C does not approach ideal feedforward control as closely as does controller A, it possesses a steady state feedback gain of 25.4 compared to only 2.1 for controller A. Thus idealized analog computer performance of controller C is slightly inferior to that of A but C can be expected to perform better in the presence of model error. Controller B has almost no feedforward control and achieves imperfect output attenuation with a feedback controller having a steady

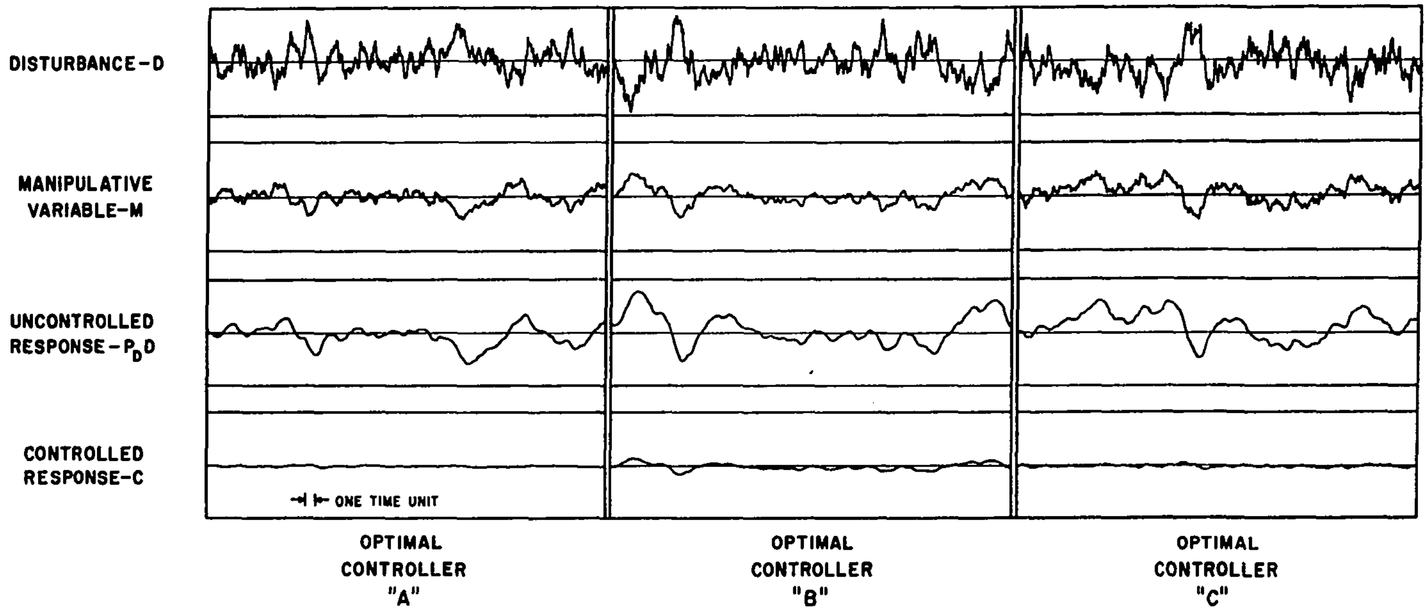

Figure 6-4.--Analog Computer Results for Optimal Composite Control of Experimental System

state gain of 16.0. The results of control effort constraints for A and B appear to have produced only marginal results but the idealized results fail to include any effect of feedback noise.

The experimental plant responses shown in Figures 6-5, 6-6 and 6-7 demonstrate first of all that extensive efforts extended to stabilize and define the system bore fruit in terms of plant controllability - the real plant results were nearly as good as analog computer data. Such efforts can be expected to be rewarding in any difficult control problem.

The steady state, open loop plant without measurable disturbance or control is seen in Figure 6-5A to be quite stable with very little drift and a moderate output noise level. In Figure 6-5B, system response to step functions is shown. It is clear that an accurate system identification has been obtained since ideal feedforward control eliminates 93% of the absolute value of the uncontrolled step response (Table 6-2). Figure 6-5C shows the uncontrolled system response to a typical disturbance for later comparison with controlled system response to a similar disturbance.

Optimal control of the physical system is shown in Figure 6-6. Quantitative results are presented in Table 6-2. Because of the extended real system operating time, accuracy of continuous integration of system output is limited. Therefore graphical integration of the absolute value of response deviation from nominal steady state values was made. The ratio of the integrated absolute value of the output to that of the disturbance for various controllers was compared to a similar ratio for the uncontrolled system. Control efficiency based on this ratio is defined as

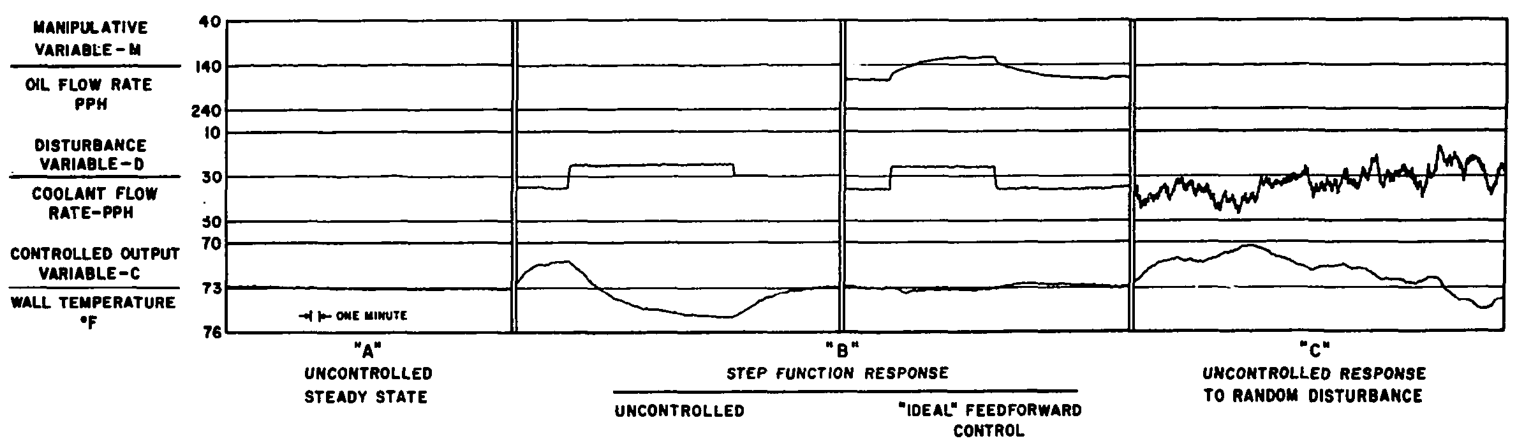

Figure 6-5. Response of Experimental System to Disturbances

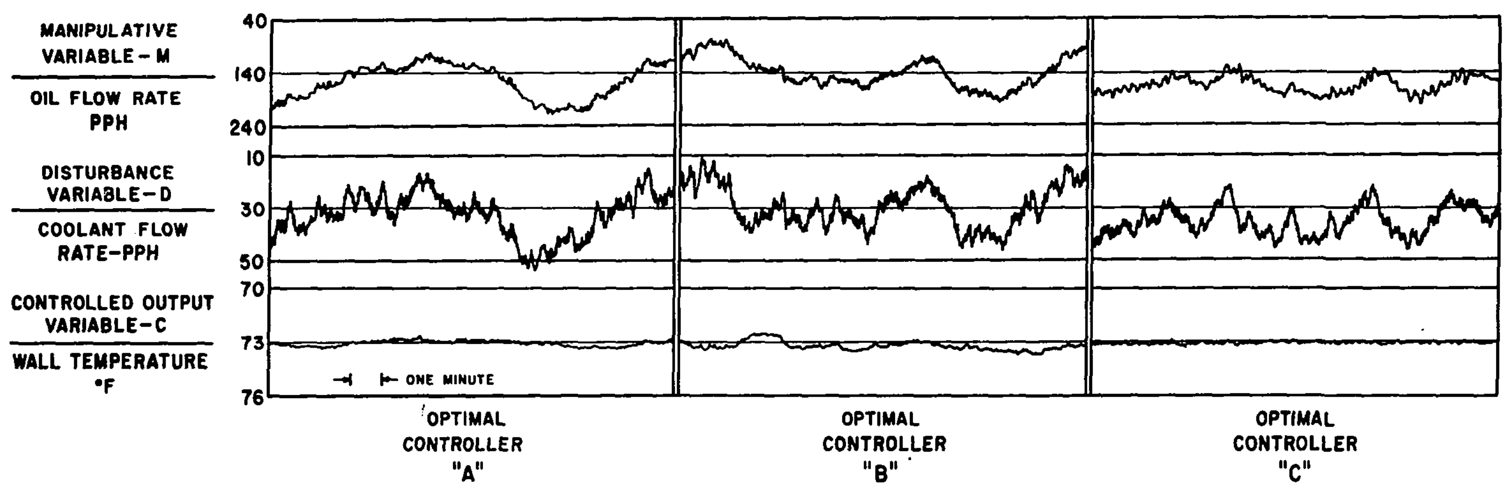

Figure 6-6. Response of Experimental System with Optimal Composite Controllers from Table 6-2

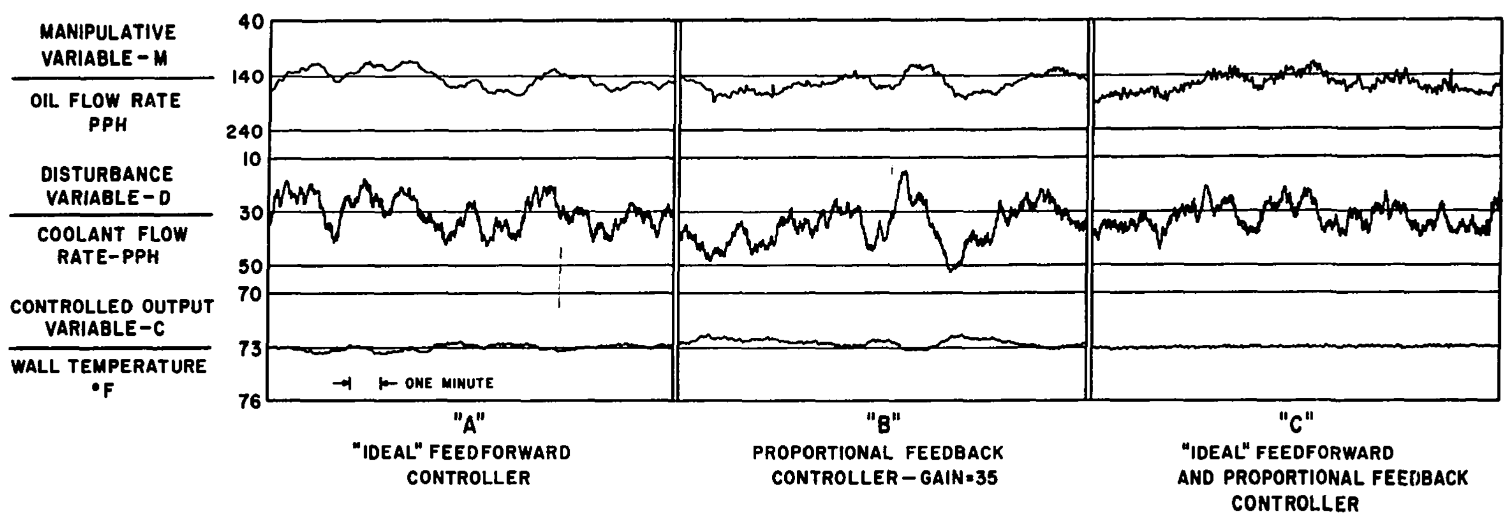

Figure 6-7.--Response of Experimental System with Nominal Controllers

TABLE 6-2

EFFECTIVENESS OF CONTROLLERS FOR EXPERIMENTAL SYSTEM

Type of Controller	Controller Efficiency $1 - \frac{\left(\frac{\int |c| dt}{\int |d| dt} \right)_{controlled}}{\left(\frac{\int |c| dt}{\int |d| dt} \right)_{uncontrolled}}$	Relative Control Effort $\frac{\left(\frac{\int |m| dt}{\int |d| dt} \right)_{controlled}}{\left(\frac{\int |m| dt}{\int |d| dt} \right)_{optimal\ C}}$
Random Disturbance:		
Optimal A	0.922	0.89
Optimal B	0.815	0.72
Optimal C	0.963	1.00
Ideal Feedforward	0.916	0.75
Proportional Feedback	0.843	0.94
Ideal F.F. plus Proportional F.B.	0.973	1.06
Step Input:		
Ideal Feedforward	0.926	--

$$\text{Control efficiency} = 1 - \frac{\left(\dfrac{\int |c|dt}{\int |d|dt}\right)_{\text{controlled}}}{\left(\dfrac{\int |c|dt}{\int |d|dt}\right)_{\text{uncontrolled}}} . \qquad (6\text{-}4)$$

Similar quantities were computed for relative control effort except that the standard for comparison was the value of control effort required by the optimal controller C. Thus, relative control effort is defined as

$$\text{Relative control effort} = \frac{\left(\dfrac{\int |m|dt}{\int |d|dt}\right)_{\text{controlled}}}{\left(\dfrac{\int |m|dt}{\int |d|dt}\right)_{\text{optimal C}}} . \qquad (6\text{-}5)$$

These quantities are given in Table 6-2 for various controllers.

Earlier results shown in Figures 5-7 and 5-13 caution against drawing quantitative conclusions based on short duration, randomly driven tests. However the striking nature of the results in Figure 6-6 leaves little doubt as to the high effectiveness of the optimal controllers A and C. As predicted by analog computer results, performance of the optimal controller B was considerably poorer than that of the other two.

While optimal control performance is encouraging, earlier analog tests indicated that composite nominal feedforward and proportional feedback control is also quite effective. In Figure 6-7A it is seen that feedback control alone with a steady state gain equal to 35 produces fairly effective control. Higher feedback gains produced oscillations probably caused by small dead times in the system. Ideal feedforward control, shown in Figure 6-7B, gave better performance than with feedback control although it is possible that some drift may have occurred if the test had been continued for very long periods of time. The high degree of feedforward control efficiency was made possible by the accuracy of the model identification. Figure 6-7C illustrates the result of ideal feedforward control with the proportional feedback such as described above. The control was excellent and was virtually indistinguishable from that of the optimal composite controller shown in Figure 6-6C.

These results indicate that the design procedures do lead to physically realizable controls that perform much as expected. For this well stabilized and identified plant, these controllers approximated nominal controllers whose response is quite similar to that of the optimal. The constraint on control effort did not greatly influence results partially because of the low noise level and partially because the small dead times prevented extremely high feedback gain in the nominal controllers.

CHAPTER VII

SUMMARY AND CONCLUSIONS

The design equations including all of the constraints yield physically realizable controls which give real performance that corresponds broadly to the intuitive ideas leading to their development. In the absence of significant model error and dead times primarily feedforward controllers are specified. If the system is assumed to be constrained by available control effort magnitude, optimal control laws result which attenuate the output somewhat poorer than an ideal feedforward controller even under conditions of constrained control effort; however the optimal control saturates a smaller fraction of the time so that superior optimal output attenuation would result if saturation were accompanied by other undesirable effects such as hysteresis. It would be recommended that in most cases, however, that implementation of optimal controls should be based on optimistic estimates of available control effort since the optimal controller will not use more control effort for output attenuation even if it is available. If control saturation does not invoke extra penalties, and in the absence of control dead time, ideal feedforward is superior to optimal feedforward.

For systems containing dead times, significant improvements in control performance can be achieved with the optimal design. However this controller is a differentiator which is more sensitive than an ideal controller not only to constraints of control effort but also to error in the mathematical model. Optimal design for these cases should be made with less optimistic estimates of available control effort.

When consideration of model error is included in the design equations, feedback control is specified in addition to feedforward. In fact, attenuation of output resulting from model error can be affected only slightly by modification of the feedforward controller and must therefore be achieved almost entirely by feedback. If noise associated with the feedback system is low and if dead times are absent from the output sensing and amplification circuits, the optimal controller tends to consist primarily of feedback. There is a definite limit, however, to the degree of output attenuation with the use of feedback. Amplification of noise and sensing error produces more output than is being eliminated when extremely low levels of output are sought. Furthermore, small dead times in real systems initiate oscillations when large feedback gains are employed. These same factors limit the degree of sensitivity reduction that can be achieved by feedback.

For the carefully stabilized and identified experimental system, it was found that either feedforward or feedback control yielded very good results. However, the marked superiority of the composite control was not only indicated from the computations, but was also clearly demonstrated by the experimental study. When the very best control is desired, this form is indicated.

The relative independence of system sensitivity from the form of the feedforward control indicates that decoupling of the equations is possible. Thus all effects of model error can be legitimately deleted from the computations of the feedforward control function. This decoupling of design equations results in vast simplifications for investigation of optimal multivariable control systems.

Recommendations for Future Work

Feedforward Control and System Identification

Feedforward control and identification of the gain and zeros of the process system are completely interdependent - any improvement in one also aids the other. During this investigation phenomena were observed during use of feedforward control that would be useful for explicit model identification. The manner in which process gains were evaluated for control purposes by "tuning" out a steady state offset was described in Chapter VI. This same method could provide a basis for accurate determination of product nonlinearities (or first approximations to higher order nonlinearities) by measurement of process gain as a function of disturbance level. This information would be useful for nonlinear feedforward control.

Inaccurate determination of system zeros manifests itself in feedforward control as overshoot or undershoot of the response when steady state gain is accurate. A study of the form and magnitude of step response of a system with feedforward control could be used to locate the zeros with high accuracy.

Optimal Use of Secondary Feedback

In many process systems, variables are or can be continuously monitored which need not be subject to control. These variables can be used as a source of process information to aid in control of critical output variables. The question occurs as to what form the controller should assume for optimal use of this information. The goal of the control problem would be to develop another control function to be used to determine the value of the manipulative variable. Such control would be useful where large dead times prevent use of feedback control. Methods similar to those discussed in the present work should be useful in approaching this problem especially since decoupling of feedback and feed-forward equations can be used to simplify the algebriac detail. The suggested approach is to form the correlation functions between the output and the model error outputs of the other state variables and then to find the proper combination of state variables that minimizes the model error output.

LIST OF REFERENCES

B1. Bellman, R. E. Adaptive Control. Princeton, N.J.: Princeton University Press, 1961.

B2. Bendat, J. S. Principles and Application of Random Noise Theory. New York: John Wiley and Sons, Inc., 1958.

B3. Bishop, K. A. and Sims, R. A. "Analog Computation Using the Modified Donner Model 3100-D," Internal Report at University of Oklahoma Process Dynamics and Control Laboratory, 1963.

B4. Bishop, K. A. and Sims, R. A. "Electronic Instrumentation for Research in Process Control," Paper presented at 1962 Mid-America Electronics Conference, Kansas City, Mo., 1962.

B5. Bishop, K. A. "A Generalized Pulse Testing Technique for Linear System Identification," Ph.D. Dissertation, The University of Oklahoma, 1965.

B6. Bliss, G. A. Lectures on the Calculus of Variations. Chicago, Ill.: University of Chicago Press, 1946.

B7. Bollinger, R. E. and Lamb, D. E. "The Design of a Combined Feedforward-Feedback Control System," 1963 Joint Automatic Control Conference, Preprint XVIII-4, University of Minnesota, 1963.

B8. Bollinger, R. E. and Lamb, D. E. "Multivariable Systems: Analysis and Feedforward Control," I & E C. Fundamentals, Vol. 1, No. 4, p. 245, 1962.

B9. Bolza, O. Lectures on the Calculus of Variations. Chicago, Ill.: University of Chicago Press, 1904.

B10. Buckley, P. Techniques of Process Control. New York: John Wiley and Sons, Inc., 1964.

C1. Calvert, S. and Coulman, G. "Feedforward Control: Its Future Role in the Chemical Industry," C. E. P., Vol. 57, No. 9, p. 45, 1961.

C2. Chang, S. S. L. Synthesis of Optimum Control Systems. New York: McGraw-Hill, Inc., 1961.

D1. Davis, M. C. "Factoring the Spectral Matrix," IEEE Trans., Auto. Control, Vol. AC-8, p. 296, 1963.

E1. Eckman, D. P. Automatic Process Control. New York: John Wiley & Sons, Inc., 1958.

F1. Fanning, R. J. and Sliepcevich, C. M. "The Dynamics of Heat Removal from a Continuous Agitated-Tank Reactor," A.I.Ch.E. Journ., Vol. 5, No. 2, p. 240, 1959.

F2. Franks, R. G. and Worley, C. W. "Quantitative Analysis of Cascade Control," Ind. Eng. Chem., 48, p. 1074, 1956.

G1. Gallier, P. W., Sliepcevich, C. M. and Pucket, T. H. Some Practical Limitations of Correlation Techniques in Determining Frequency Response," C. E. P. Symposium Series, Vol. 57, No. 36, p. 59, 1961.

G2. Gollin, N. W. "Cascade Control Systems," Control Eng., 3, p. 94, 1956.

G3. Grau, A. A. "Solution of Polynomial Equation by Bairstow-Hitchcock Method," Communications ACM, Vol. 3, p. 74, 1960.

H1. Harriot, P. Process Control. New York: McGraw-Hill Book Company, 1964.

H2. Harris, J. T. and Schechter, R. S. "Feedforward Control of a Chemical Reactor," I & E C. Fundamentals, Vol. 2, No. 3, p. 245, 1963.

H3. Haskins, D. E. and Sliepcevich, C. M. "The Invariance Principle of Control for Chemical Processes," I & E C. Fundamentals, Vol. 4, No. 3, p. 241, 1965.

H4. Hazelroek, P. and van der Waerden, B. L. "Theoretical Considerations on the Optimum Adjustment of Regulators," Trans. ASME, 72, p. 309, 1950.

H5. Heymann, M. "A Time Domain Technique for Linear System Identification," Ph.D. Thesis, University of Oklahoma, 1965.

H6. Heymann, M., McGuire, M. L. and Sliepcevich, C. M. "Time Domain Process Identification," Submitted for Publication.

H7. Hille, E. Analytic Function Theory. Vol. I. New York: Ginn and Company, 1959.

H8. Horowitz, I. M. "Fundamental Theory of Automatic Linear Feedback Control Systems," IRE Trans. on Automatic Control, AC-5, p. 5, 1959.

H9. Horowitz, I. M. Synthesis of Feedback Systems. New York: Academic Press, 1963.

H10. Horowitz, I. M. "Synthesis of Linear Multivariable Feedback Control Systems," Trans. AIEE, AC-5, p. 94, 1960.

K1. Kipiniak, W. Dynamic Optimization and Control, A Variational Approach. New York: M.I.T. Press and John Wiley & Sons, 1961.

K2. Kulebakin, V. S. "The Theory of Invariance of Regulating and Control Systems," Automatic and Remote Control, Proceedings IFAC, Moscow, 1960, Buttersworth, London, p. 106, 1961.

L1. Laning, J. H. and Battin, R. H. Random Processes in Automatic Control. New York: McGraw-Hill Book Co., 1956.

L2. Lefkowitz, I. and Eckmann, D. P. "Application and Analysis of a Computer Control System," Trans. of A.S.M.E. Jour. of Basic Engr., p. 569, 1959.

L3. Leitmann, G. Optimization Techniques with Applications to Aerospace Systems. New York and London: Academic Press, 1962.

L4. Lupfer, D. E. and Oglesby, M. W. "The Application of Dead Time Compensation to a Chemical Reactor for Automatic Control of Production Rate," ISA Trans., Vol. 1, No. 1, p. 72, 1962.

L5. Lupfer, D. E. and Oglesby, M. W. "Automatic Control of Distillation Columns," I & E C. Fundamentals, Vol. 53, p. 963, 1961.

L6. Lupfer, D. E. and Parsons, J. R. "A Predictive Control System for Distillation Columns," CEP, Vol. 58, No. 9, p. 37, 1962.

L7. Luyben, W. L. and Gerster, J. A. "Feedforward Control of Distillation Columns," I & E C. Fundamentals, Vol. 3, No. 4, p. 106, 1964.

M1. MacMullan, E. C. and Shinskey, F. G. "Feedforward Analog Computer Control of a Superfractionator," Control Engineering, 1964.

M2. Merriam, C. W. Optimization Theory and the Design of Feedback Control Systems. New York: McGraw-Hill, Inc., 1964.

M3. Mesarović, M. D. The Control of Multivariable Systems. New York: M.I.T. Press and John Wiley & Sons, Inc., 1960.

N1. Naur, P. (Editor). "Revised Report on the Algorithmic Language Algol 60," Communication of the ACM, 6, No. 1, p. 1, 1963.

N2. Newton, G. C., Gould, L. A. and Kaiser, J. F. Analytical Design of Linear Feedback Controls. New York: John Wiley & Sons, Inc., 1957.

N3. Newton, G. C. "Design of Control Systems for Minimum Bandwidth," Trans. AIEE, 74, Part II, p. 161, 1955.

P1. Paige, L. J. and Swift, J. D. Linear Algebra. Boston: Ginn and Company, 1961.

P2. Peterson, E. L. Statistical Analysis and Optimization of Systems. New York: John Wiley & Sons, Inc., 1961.

P3. Petrov, B. N. "The Invariance Principle and the Conditions for Its Application," Automatic and Remote Control, Proceedings IFAC, Moscow, 1960, Buttersworth, London, p. 117, 1961.

P4. Preminger, J. and Rootenberg, J. "Some Considerations Relating to Control Systems Employing the Invariance Principle," IEEE Trans. on Auto Control, Vol. AC-9, p. 209, 1964.

R1. Rozonoer, L. I. "A Variational Approach to the Problem of Invariance of Automatic Control Systems," Automatika and Telemekhanika, Vol. 24, No. 6, p. 680, 1963; No. 7, p. 793, 1963.

S1. Sims, R. A. "Time Domain Identification of Zeros of a Linear System," Ph.D. Dissertation, The University of Oklahoma, To be published.

S2. Solodovnikov, V. V. Introduction to the Statistical Dynamics of Automatic Control Systems. New York: Dover Publications, Inc., 1960. (First published in Russian 1952)

S3. Stewart, W. S., Sliepcevich, C. M. and Pucket, T. H. "Dynamics of Heat Removal from a Jacketed Agitated Vessel," C.E.P. Symposium Series, Vol. 57, No. 36, p. 119, 1961.

T1. Tomović, R. Sensitivity Analysis of Dynamic Systems. New York: McGraw-Hill, Inc., 1963.

W1. Webb, P. U. "Reducing Process Disturbances with Cascade Control," Control Eng., 8, p. 63, 1961.

W2. Wiener, N. Extrapolation, Interpolation and Smoothing of Stationary Time Series. New York: M.I.T. Press and John Wiley & Sons, Inc., 1949. (Paperback Edition 1964)

W3. Worrell, R. B. "The Osage Algorithmic Language - Osage Algol," M.S. Thesis, The University of Oklahoma, 1964.

W4. Wilkinson, J. H. "The Evaluation of the Zeros of Ill-Conditioned Polynomials, Part I," Numerische Mathematik, 1, p. 150, 1959.

W5. Wilks, S. S. Mathematical Statistics. New York: John Wiley and Sons, Inc., 1962.

APPENDICES

APPENDIX A

MATHEMATICAL PROOFS

Proof of (2-1)

The transformation of quadratic time functions from the time domain to the Laplace domain is made by the use of (2-1):

$$\int_0^{\infty} c^*(t)^2 dt = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} C^*(s)C^*(-s)ds \ , \qquad (2\text{-}1)$$

where $c^*(t)$ is an arbitrary piecewise continuous function of time which (i) is zero for all $t < 0$; (ii) approaches zero at least as fast as $e^{-\epsilon t}$ for all $\epsilon > 0$ as $t \to \infty$; and (iii) is bounded for all t; and $C^*(s)$ is the Laplace transform of $c^*(t)$. The proof is straight-forward. By definition

$$C^*(s) = \int_0^{\infty} c^*(t)e^{-st}dt \ , \qquad (A\text{-}1)$$

and

$$c^*(t) = \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} C^*(s)e^{st}ds \ . \qquad (A\text{-}2)$$

In (A-1) the lower limit on the integral may be extended to $-\infty$ because by hypothesis (i) $c^*(t)$ is zero for $t < 0$. Furthermore, since $c^*(t)$ is also bounded for all t, $C^*(s)$ may not have r.h.p. poles allowing the convergence factor, γ, in (A-2) to be taken as zero. Thus from (A-2) and (2-1),

$$\int_0^\infty c^*(t)^2 dt = \frac{1}{2\pi j}\int_0^\infty c^*(t)\int_{-j\infty}^{j\infty} C^*(s)e^{st}ds\,dt\ . \qquad \text{(A-3)}$$

Under conditions of the hypothesis, the order of integration may be reversed:

$$\int_0^\infty c^*(t)^2 dt = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} C^*(s)\int_0^\infty c^*(t)e^{st}dt\,ds\ , \qquad \text{(A-4)}$$

which by virtue of (A-1), is just a restatement of (2-1).

Proof of (2-6)

In order to transform the mean square value of random signals from the time domain into the Laplace domain, first consider two signals $a_T^*(t)$ and $b_T^*(t)$ which are identical with piecewise continuous functions $a^*(t)$ and $b^*(t)$ respectively in the interval $0 \leq t \leq T$ but which are zero elsewhere. If $a^*(t)$ and $b^*(t)$ are bounded, $a_T^*(t)$ and $b_T^*(t)$ are Laplace transformable. Denote these transforms by $A_T^*(s)$ and $B_T^*(s)$ respectively. Then it follows that

$$\frac{1}{T}\int_0^{T-\tau} a^*(t)b^*(t+\tau)dt = \frac{1}{T}\int_0^\infty a_T^*(t)b_T^*(t+\tau)dt\ ,$$

$$= \frac{1}{2\pi jt}\int_0^\infty a_T^*(t)\int_{-j\infty}^{j\infty} B_T^*(s)e^{s(t+\tau)}dsdt\ ,$$

$$= \frac{1}{2\pi jT}\int_{-j\infty}^{j\infty} A_T^*(-s)B_T^*(s)e^{s\tau}ds\ . \qquad \text{(A-5)}$$

Taking the mean value of this last equation yields

$$\frac{T-\tau}{T}\varphi_{A^*B^*}(\tau) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{1}{T}\langle A_T^*(-s)B_T^*(s)\rangle e^{s\tau}ds \;, \qquad \text{(A-6)}$$

since $\langle a^*(t)b^*(t+\tau)\rangle$ is a constant equal to $\varphi_{A^*B^*}(\tau)$, where $\langle\ldots.\rangle$ indicates mean value (cf (2-4)).

The spectral density $\Phi_{A^*B^*}(s)$, of a random signal is defined

$$\Phi_{A^*B^*}(s) \triangleq \lim_{T\to\infty}\frac{1}{T}\langle A_T^*(-s)B_T^*(s)\rangle \;. \qquad \text{(2-5)}$$

By letting T approach infinity, (A-6) becomes

$$\varphi_{A^*B^*}(\tau) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\Phi_{A^*B^*}(s)e^{s\tau}ds \;. \qquad \text{(2-6)}$$

Equation (2-6) shows that the spectral density is the Fourier transform of the correlation function.

Conditions for Minimum of (2-14)

To define a physically realizeable optimum transfer function B that minimizes the following equation,

$$J(BP_1 + P_2) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}(BP_1 + P_2)\overline{(BP_1 + P_2)}ds \;, \qquad \text{(2-14)}$$

assume that the optimum exists and denote it by B_0.

Let B_1 be any other transfer function belonging to the same class. Then the minimum value of $J[(B_0 + \epsilon B_1)P_1 + P_2]$ where ϵ is any real number, will occur at $\epsilon = 0$ and therefore $\frac{d}{dt}J[(B_0 + \epsilon B_1)P_1 + P_2]$ vanishes at $\epsilon = 0$. To find this derivative, expand (2-14) to give

$$J[(B_0 + \epsilon B_1)P_1 + P_2] = J(B_0P_1 + P_2) + \epsilon^2 J(B_1P_1) + \\ + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \epsilon\overline{B_1P_1}(B_0P_1 + P_2)ds + \\ + \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \epsilon B_1P_1\overline{(B_0P_1 + P_2)}ds \ . \quad \text{(A-7)}$$

By substitution of $-\sigma$ for s as the variable of integration, the last two integrals of this equation are proved equal; hence,

$$J[(B_0 + \epsilon B_1)P_1 + P_2] = J(B_0P_1 + P_2) + \epsilon^2 J(B_1P_1) + 2\epsilon I, \quad \text{(A-8)}$$

where
$$I = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \overline{B_1P_1}(B_0P_1 + P_2)ds \ . \quad \text{(A-9)}$$

After differentiation of (A-8) and setting ϵ equal to zero

$$\frac{dJ[(B_0 + \epsilon B_1)P_1 + P_2]}{dt}\bigg|_{\epsilon=0} = 2I \ . \quad \text{(A-10)}$$

Thus, a necessary condition that B_0 provide a minimum for $J(BP_1 + P_2)$ is the vanishing of the integral (A-9) or

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \overline{B_1P_1}(B_0P_1 + P_2)ds = 0 \ , \quad \text{(2-15)}$$

where B_1 is an arbitrary member of the family of admissible functions, B.

Other standard conditions from the calculus of variations showing necessity or sufficiency that a function, B_0, satisfying (2-15) provides a minimum cannot easily be applied. Instead an oblique approach is used to prove sufficiency. Let B be any function in the admissible class and define a B_1 such that:

$$B = B_0 + B_1 \, . \qquad \text{(A-11)}$$

Then from an equation similar to (A-8),

$$J(BP_1 + P_2) - J(B_1P_1) = J(B_0P_1 + P_2) + 2I \, . \qquad \text{(A-12)}$$

Since B contains much arbitrariness within the admissible class, so does B_1 and the term I of (A-12) is not essentially different from that of (A-9).

By making the substitution of $j\omega$ for s in an integral of the form of (2-17), it is found that

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} Q(s)Q(-s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Q(j\omega)|^2 d\omega \geq 0 \, . \qquad \text{(A-13)}$$

Hence $J(B_1P_1)$ is never negative and it is seen from (A-11) that therefore $J(B_0P_1 + P_2) \leq J(BP_1 + P_2)$ if $I = 0$. Thus $I = 0$ is also a sufficient condition for B_0 to be an optimizing function.

Conditions for Satisfaction of (2-15)

The integral along the imaginary axis in (2-15),

$$\frac{1}{2\pi j}\int_{j\infty}^{j\infty} \overline{B_1P_1}(B_0P_1 + P_2)ds = 0 \, , \qquad \text{(2-15)}$$

may be considered as the difference between the contour integral which encircles the l.h.p. including the imaginary axis as a bound and the line integral along the semi-circle only (Figure A-1). The condition imposed for the convergence of the integrals of (2-3) and (2-8) show that each of the factors in (2-15) approach zero as $s \to \infty$. This condition implies that s^2 times the integrand of (2-15) is a bounded number as $s \to \infty$ which is sufficient to ensure that the integral on the line integral on the semi-circle around the l.h.p. is equal to zero [H7]. Therefore

$$\frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \overline{B_1 P_1}(B_0 P_1 + P_2)ds = \frac{1}{2\pi j} \int_C \overline{B_1 P_1}(B_0 P_1 + P_2)ds \ , \quad \text{(A-13)}$$

where $\int_C$ indicates the contour described above. Thus conditions for the vanishing of the contour integral are also conditions for the vanishing of (2-15).

By Cauchy's residue theorem, the value of the contour integral is $2\pi j$ times the sum of residues at the poles in the l.h.p. Since B_1 is arbitrary <u>except</u> that it must have poles in the l.h.p. only, $\overline{B_1}$ must have poles only in the r.h.p.. From this requirement it follows that a sufficient condition for that (2-15) be satisfied is that $\overline{P_1}(B_0 P_1 + P_2)$ have poles in the r.h.p. only. This condition is also necessary. Since $\overline{B_1}$ is arbitrary except for pole locations, it contributes arbitrary values to residues at the poles of the function $\overline{P_1}(B_0 P_1 + P_2)$. Thus if $\overline{P_1}(B_0 P_1 + P_2)$ has any l.h.p. poles, a B_1 could be selected so that the sum of residues does not vanish.

To summarize: necessary and sufficient conditions that B_0 satisfy the condition of (2-15) where B_0 and B_1 belong to a class of linear operators free of poles in the

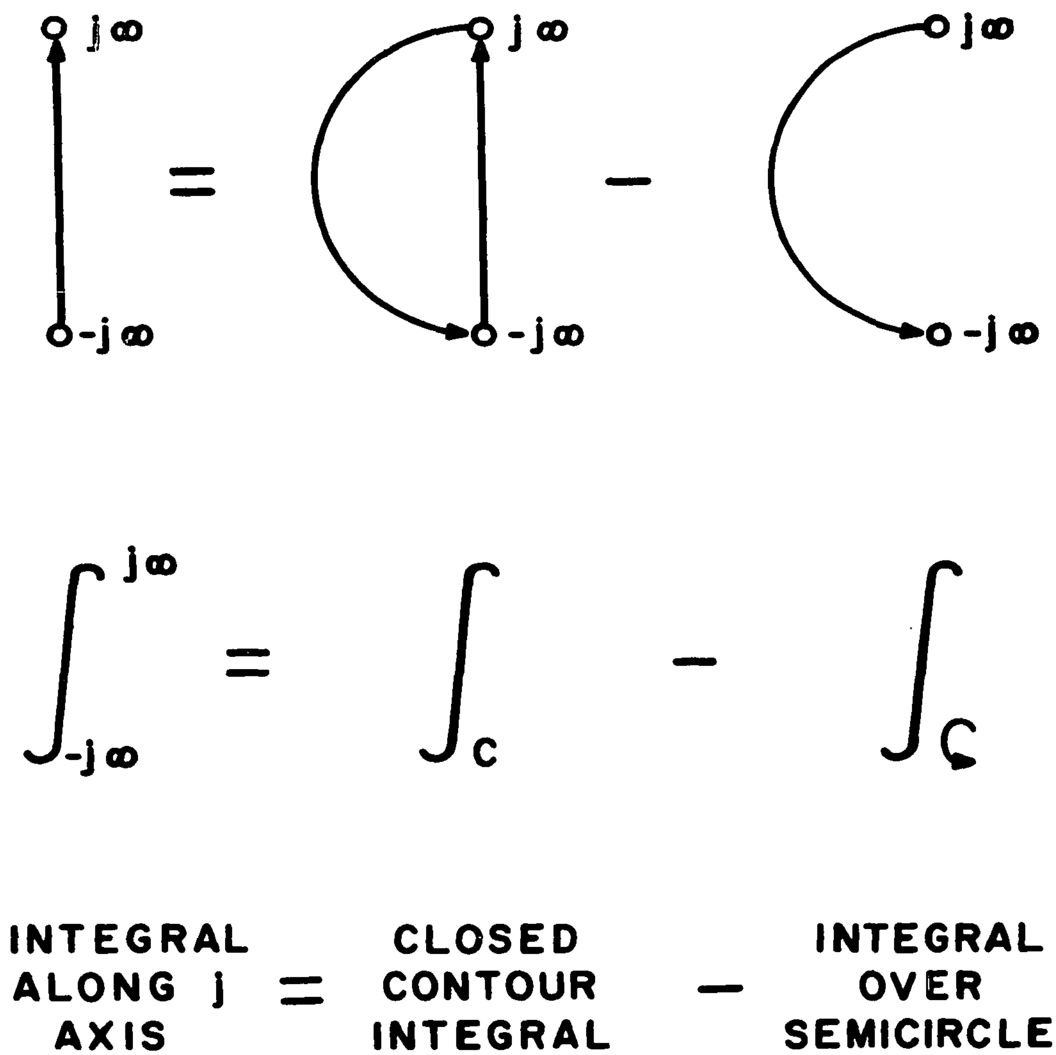

Figure A-1.--Evolution of Integral (2-15) in Complex Plane r.h.p., is that $\overline{P_1}(B_0P_1 + P_2)$ be equal to some function having r.h.p. poles only; that is,

$$\overline{P_1}(B_0P_1 + P_2) = X \tag{2-16}$$

where X is a function with only r.h.p. poles.

APPENDIX B

DETAILED SOLUTION OF DESIGN EQUATIONS--THIRD ORDER SYSTEM

Extensive difficulties were generated by numerical solution of the equations in Chapter III. An initial attempt was made to solve a general system where the plant transfer functions were each fifth degree in the numerator and denominator. Dead times were assumed present along with error in all parameters except three time constants in the numerator and denominator. This attempt was abandonned when it was found that the factor multiplying T_D in (3-99) was a ratio of 30th degree polynomials. Even the solution for the simple example system that was developed near the beginning of Chapter III is not trivial. In this Appendix, control design equations are developed in detail for a higher order generalization of that system. Digital computer programs written for solution of these equations in the Osage Algol compiler language* are listed in Appendix D. The programs follow generally the format and nomenclature presented here.

The general system of (1-2) will be developed in detail here for transfer functions of the form

$$P_D = \frac{K_D(1 + R_D s)e^{-E_D s}}{(s + \alpha)(1 + R_1 s)(1 + R_2 s)}\,, \qquad \text{(B-1)}$$

*The Osage Algol language is a slightly modified version of the Algol 60 language [N1,W3] used on the Osage high speed computer at the University of Oklahoma Computer Center.

$$P_M = \frac{K_M(1 + R_M)e^{-E_M s}}{(s + \alpha)(1 + R_1 s)(1 + R_2 s)} . \qquad \text{(B-2)}$$

There is a threefold reason for this form. First, it illustrates the method; second, the specific results are useful since most chemical processes can be represented satisfactorily by equations of this form; and third, it reduces simply to the system described in Chapter III by making the coefficient R's equal to zero.

Several assumptions needed later in the solution are listed here for convenience:

(1) It is assumed that a relatively "good" overall transfer function will be found and a $\gamma = 1$ is selected for (3-94). Since the denominator of P_D and P_M are equal, one term in the factor $\left(\frac{\Delta P_D}{P_D} - \gamma \frac{\Delta P_M}{P_M} \right)$ of (3-94) is eliminated;

(2) The noise associated with the threshold level of the feedback system is assumed to be "white" noise whose spectral density is a constant, δ^2;

(3) The errors in α, K_D, K_M, E_D and E_M are assumed to be large enough so that all others may be neglected. Thus:

$$\Delta P_D = \left(\frac{\Delta K_D}{K_D} - \frac{\Delta \alpha}{s + \alpha} - s\Delta E_D \right) P_D , \qquad \text{(B-3)}$$

$$\Delta P_M = \left(\frac{\Delta K_M}{K_M} - \frac{\Delta \alpha}{s + \alpha} - s\Delta E_M \right) P_M ; \qquad \text{(B-4)}$$

(4) The variance and covariance of the errors are displayed in the 5 x 5 triangular matrix:

$$[E_{ij}] = \begin{bmatrix} \frac{\Delta K_D^2}{K_D} & - & - & - & - \\ \frac{\Delta K_D}{K_D}\Delta\alpha & \Delta\alpha^2 & - & - & - \\ \frac{\Delta K_D}{K_D}\Delta E_D & \Delta\alpha\ \Delta E_D & \Delta E_D^2 & - & - \\ \frac{\Delta K_D}{K_D}\frac{\Delta K_M}{K_M} & \Delta\alpha\frac{\Delta K_M}{K_M} & \Delta E_D\frac{\Delta K_M}{K_M} & \frac{\Delta K_M^2}{K_M} & - \\ \frac{\Delta K_D}{K_D}\Delta E_M & \Delta\alpha\ \Delta E_M & \Delta E_D\ \Delta E_M & \frac{\Delta K_M}{K_M}\Delta E_M & \Delta E_M^2 \end{bmatrix} \tag{B-5}$$

where all matrix elements are mean values.

(5) P_A of (3-42) is assumed to be one, i.e., the control effort constraint is on the magnitude of the manipulative variable only.

The factors in $\overline{YY}$ of (3-97) will be calculated first. With $\gamma = 1$ and ΔP defined as in (B-4)

$$\frac{\Delta P_D}{P_D} - \gamma\frac{\Delta P_M}{P_M} = \frac{\Delta K_D}{K_D} - \frac{\Delta K_M}{K_M} - s(\Delta E_D - \Delta E_M) , \tag{B-6}$$

The term arising from $\Delta\alpha$ does not appear in this difference. Continuing from (B-5)

$$\left(\frac{\Delta P_D}{P_D} - \gamma\frac{\Delta P_M}{P_M}\right)\overline{\left(\frac{\Delta P_D}{P_D} - \gamma\frac{\Delta P_M}{P_M}\right)} = E_{11} - 2E_{41} + E_{44} - $$
$$- s^2(E_{33} - 2E_{53} + E_{55}) ,$$

$$\left(\frac{\Delta P_D}{P_D} - \gamma \frac{\Delta P_M}{P_M} \right) \overline{\left(\frac{\Delta P_D}{P_D} - \gamma \frac{\Delta P_M}{P_M} \right)} \triangleq E_D - E_K s^2 \quad \text{(B-7)}$$

Now define

$$\lambda_1^2 + \lambda_2^2 \frac{P_A \overline{P_A}}{P_M \overline{P_M}} \triangleq \frac{\sum_{i=0}^{3} \beta_i (-s^2)^i}{1 - R_M^2 s^2} , \quad \text{(B-8)}$$

so that

$$\beta_3 = \lambda_2^2 \gamma_3 / K_M^2 ,$$
$$\beta_2 = \lambda_2^2 \gamma_2 / K_M^2 ,$$
$$\beta_1 = \lambda_2^2 \gamma_1 / K_M^2 + \lambda_1^2 R_M^2 ,$$
$$\beta_0 = \lambda_2^2 \gamma_0 / K_M^2 + \lambda_1^2 ,$$

and

$$\gamma_0 \triangleq \alpha^2 ,$$
$$\gamma_1 \triangleq 1 + \alpha^2 (R_1^2 + R_2^2) ,$$
$$\gamma_2 \triangleq (R_1^2 + R_2^2) + \alpha^2 R_1^2 R_2^2 ,$$
$$\gamma_3 \triangleq R_1^2 R_2^2 ,$$

so that the denominator of $P_M \overline{P_M}$ or $P_D \overline{P_D}$ is

$$(\alpha^2 - s^2)(1 - R_1^2 s^2)(1 - R_2^2 s^2) = \gamma_0 - \gamma_1 s^2 + \gamma_2 s^4 - \gamma_3 s^6 . \quad \text{(B-9)}$$

Using (B-7), (B-8) and (B-9) with (3-96) and (3-97), let

$$Y\overline{Y} \triangleq \frac{\sum_{i=0}^{7} yy_i (-s^2)^i}{(\sigma^2 - s^2)(1 - R_M^2 s^2)(1 - R_D^2 s^2)} , \quad \text{(B-10)}$$

where the coefficients yy_i result from adding the rational functions

$$\frac{\mu^2(E_D - E_K s^2)}{\sigma^2 - s^2} + \frac{\delta^2}{K_D^2}\,\frac{\left(\sum_0^3 \beta_i(-s^2)^i\right)\left(\sum_0^3 \gamma_i(-s^2)^i\right)}{(1 - R_D^2 s^2)(1 - R_M^2 s^2)} \;.$$

In order to factor (B-10) into functions with r.h.p. and l.h.p. poles and zeros, a dummy variable was substituted for s^2 and the seven roots of the numerator were found. This root finding step proved to be a very troublesome one. A digital computer program based on the Bairstow-Hitchcock method of polynomial factorization was employed [G3] but more than half of the cases considered failed to converge. The difficulty was caused by a combination of "ill-conditioned" polynomials [W4] and very small roots. Modifications were made to this program (which are listed in Appendix D) so that about only 5% of the cases of interest did not converge. However, polynomial factorization remains a very important and critical problem in these design procedures.

The factors of Y were the square root of the factors of (B-10) if the factorization was successful. Thus, Y could be reconstructed:

$$Y \triangleq \frac{yy_0 \sum_1^7 y_i s^i}{(\sigma + s)(1 + R_D s)(1 + R_M s)} \;,$$

$$\triangleq \frac{yy_0(1 + yr_1 s)(1 + yr_2 s) \;\ldots\; (1 + yr_7 s)}{(\sigma + s)(1 + R_D s)(1 + R_M s)} \;, \qquad \text{(B-11)}$$

where $y_0 = 1$. The factor yy_0 may be arbitrarily distributed between Y and $\overline{Y}$. The arrangement of (B-11) is chosen for later convenience.

Following (3-96), (3-97) and definition (B-7)

$$Z = \frac{e^{E_M s} K_D(1 + R_D s)(E_D - E_K s^2)\mu^2}{(\alpha + s)(1 + R_1 s)(1 + R_2 s)(\sigma^2 - s^2)} , \qquad \text{(B-12)}$$

so that

$$\frac{Z}{\bar{Y}} = \frac{e^{E_M s} K_D(1 - R_D^2 s^2)(1 - R_M s)(E_D - E_K s^2)\mu^2}{(s+\alpha)(s+\sigma)(1 + R_1 s)(1+R_2 s)(1 - yr_1 s) \ \ldots \ (1 - yr_7 s)} . \qquad \text{(B-13)}$$

Then it follows that

$$\left[\frac{Z}{\bar{Y}} \right]_+ \triangleq \frac{\sum_{i=0}^{3} g_i s^i}{(s + \alpha)(s + \sigma)(1 + R_1 s)(1 + R_2 s)} , \qquad \text{(B-14)}$$

where the g_i result from evaluation of the principal parts of (B-13) at each of the l.h.p. poles and addition of the four resulting fractions in s.

From (3-98), (B-11) and (B-14) the overall feedback function T_C is found:

$$T_C = \frac{(1 + R_D s)(1 + R_M s)\left(\sum_0^3 g_i s^i \right)}{y Y_0 (s + \alpha)(1 + R_1 s)(1 + R_2 s)\left(\sum_0^7 y_i s^i \right)} . \qquad \text{(B-15)}$$

All constants on the right hand side of (B-15) are known except the multipliers λ_1 and λ_2 which were introduced in (B-8). Treating T_C as a known function (which it would be if values for λ_1 and λ_2 have been chosen), solution of (3-100) can now be achieved.

Let $\Phi_{\Delta D \Delta D} \equiv 0$ for the present. Later the solutions can be explored for different values of $\Phi_{\Delta D}$. It will be found that there is not much change in the solution as these parameters vary. This result is as it should be of course; the control system should attentuate all reasonable disturbances well even though tuned to one type considered most probable.

Several other factors of (3-99) must be computed.

$$1 - \frac{B_M B_D T_C}{P_D} = 1 - \frac{e^{-E_M s}(1 + R_M s)\left(\sum_0^3 g_i s^i\right)}{K_D y y_0 \left(\sum_0^7 y_i s^i\right)} . \tag{B-16}$$

In what follows, it will be desirable to be able to combine all of the factors of U including the function of (B-16) into one rational function containing only powers of s. If the exponential time delay E_M is non-zero, this will not be possible. In these cases, a simplifying approximation has been introduced:

$$e^{-E_M s} \doteq \frac{1 - \frac{E_M}{2} s}{1 + \frac{E_M}{2} s} . \tag{B-17}$$

From the power series expansions,

$$e^{-E_M s} = 1 - E_M s + \frac{(E_M s)^2}{2!} - \frac{(E_M s)^3}{3!} + \ldots \tag{B-18}$$

and

$$\frac{1 - \frac{E_M}{2} s}{1 + \frac{E_M}{2} s} = 1 - E_M s + \frac{(E_M s)^2}{2} - \frac{(E_M s)^3}{2^3} + \ldots \tag{B-19}$$

There is only a 50% error in the cubic term for the approximation for values of s less than one. For larger values of s the expansion (B-19) is not even valid and the two sides of

(B-17) approach different limits. From Laplace inversion theorems, the values of s are large for small values of t (i.e., $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$); hence the optimum amount of detuning of T_D because of probable errors in the mathematical model will not be correct for high frequency components of the signal. Since these components generally damp rapidly in chemical processes anyway, this inaccuracy is not serious. (An optimum predictive filter design using the approximation (B-17) is shown in [N2].) Using the approximation changes (B-16) to

$$1 - \frac{B_M B_D T_C}{P_D} = 1 - \frac{(1 + R_M s)\left(\sum_0^3 g_i s^i\right)\left(1 - \frac{E_M}{2} s\right)}{K_D \left(\sum_0^7 y_i s^i\right)\left(1 + \frac{E_M}{2} s\right)} \cdot \qquad \text{(B-20)}$$

The high order of (B-20) seriously complicates subsequent computations for T_D so that an investigation of the first term of (3-89), the progenitor of (B-20), is in order. This term represents the output produced because the model is in error. Some of this output is attenuated by the feedback portion of the "erroneous" design. The entire quantity of (3-89) as it affects (3-96) and subsequent developments has the effect of "detuning" the overall control so that large errors will not occur even if the model is partially wrong but at the expense of best performance if the model is exactly right.

Numerical calculations show that in the absence of feedback, i.e., where T_C is zero, very little change in the model error output is possible with T_D. Therefore the factor, (B-20), which has the effect of "detuning" T_D to compensate for model error, is of only minor importance. For this reason, (B-20) could be approximated by a lower order rational function. The computer program was written so that zeros would cancel poles until the degree of the denominator was no greater than four. A maximum degree of four was chosen as criteria because this required no cancellation in the case of first order systems with dead times as developed in Chapter III. The individual cancellations were made on pairs of roots as close as possible in magnitude. In cases where complex roots were present in both the numerator and denominator of (B-20), a pair was cancelled on the basis of nearness of absolute value. More frequently complex roots were present only in the denominator. In such cases, the complex root was approximated by two real roots - one equal to the real part and one equal to the absolute value of the complex root.

Thus (B-20) becomes

$$1 - \frac{B_M B_D T_C}{P_D} \doteq 1 - \frac{g_0 (1 - \frac{E_M}{2} s)(1 + gr_1^* s) \ldots (1 + gr_3^* s)}{K_D yy_0 (1 + yr_1^* s) \ldots (1 + yr_4^* s)} , \qquad \text{(B-21)}$$

where gr_i^* and yr_i^* are the uncancelled roots discussed above. This is expanded to define

$$1 - \frac{B_M B_D T_C}{P_D} \triangleq \frac{\sum_0^4 H_{1,i} s^i}{\sum_0^4 H_{2,i} s^i} , \qquad \text{(B-22)}$$

and

$$(1 - \frac{B_M B_D T_C}{P_D})\overline{(1 - \frac{B_M B_D T_C}{P_D})} \triangleq \frac{\sum_0^4 HH_{1i} (-s^2)^i}{\sum_0^4 HH_{2i} (-s^2)^i} \qquad \text{(B-23)}$$

The other factors in (3-99) will now be computed. From (B-4)

$$\frac{\Delta P_M}{P_M} = \left(\frac{\Delta K_M}{K_M} - \frac{\Delta\alpha}{\alpha + s} - s\Delta E_M \right)$$

$$= \frac{\left(\frac{\Delta K_M}{K_M}\alpha - \Delta\alpha \right) + \left(\frac{\Delta K_M}{K_M} - \alpha\Delta E_M \right) s - \Delta E_D s^2}{\alpha + s} , \qquad \text{(B-24)}$$

and

$$\frac{\Delta P_D}{P_D} = \frac{\left(\frac{\Delta K_D}{K_D}\alpha - \Delta\alpha \right) + \left(\frac{\Delta K_D}{K_D} - \alpha\Delta E_D \right) s - \Delta E_D s^2}{\alpha + s} . \qquad \text{(B-25)}$$

Use (B-5) and (B-23) to define

$$\frac{\Delta P_M \overline{\Delta P_M}}{P_M \overline{P_M}} \triangleq \frac{\sum_{0}^{2} a_i (-s^2)^i}{\alpha^2 - s^2} , \qquad \text{(B-26)}$$

where

$$a_0 = E_{22} - 2\alpha E_{42} + \alpha^2 E_{44} ,$$

$$a_1 = E_{44} - 2E_{52} + \alpha^2 E_{55} ,$$

$$a_2 = E_{55} ;$$

A similar expression can be computed for $\frac{\Delta P_D \overline{\Delta P_D}}{P_D \overline{P_D}}$ and from (B-24) and (B-25)

$$\frac{\Delta P_D \overline{\Delta P_M}}{P_D \overline{P_M}} = \frac{\sum_{0}^{4} w_i s^i}{\alpha^2 - s^2} , \qquad \text{(B-27)}$$

where

$$w_0 = E_{22} - \alpha(E_{21} + E_{42}) + \alpha^2 E_{41} ,$$

$$w_1 = E_{42} - E_{21} + \alpha(E_{32} - E_{52}) + \alpha^2(E_{51} - E_{43}) ,$$

$$w_2 = E_{52} + E_{32} - E_{41} - \alpha^2 E_{53} ,$$
$$w_3 = E_{43} - E_{51} ,$$
$$w_4 = E_{53} .$$

Now using (3-22), (3-99), (B-8), (B-23) and (B-26) to calculate $U\overline{U}$ of (3-100);

$$U\overline{U} = \frac{\mu^2}{\sigma^2 - s^2} \left[\frac{(\sum_0^4 HH_{1i}(-s^2)^i)(\sum_0^2 a_i(-s^2)^i)}{(\sum_0^4 HH_{2i}(-s^2)^i)(\alpha^2 - s^2)} + \frac{\sum_0^3 \beta_i(-s^2)^i}{1 - R_M^2 s^2} \right] ,$$

$$\triangleq \frac{\mu^2 \sum_0^8 uu_i(-s^2)^i}{(\sigma^2 - s^2)(\alpha^2 - s^2)(1 - R_M^2 s^2)(\sum_0^4 HH_{2i}(-s^2)^i)} . \qquad \text{(B-28)}$$

This expression $U\overline{U}$ must now be divided into functions with r.h.p. and l.h.p. poles and zeros only. The roots of U were found by factorization of the numerator of (B-28) with s^2 replaced by dummy variable. The l.h.p. roots of U are the negative square roots of these eigenvalues. Although the order of the numerator of (B-28) was usually greater than that of (B-10), fewer difficulties were experienced in the factorization of (B-28). Thus U is expressed

$$U = \frac{uu_0 \mu^2 (1 + ur_1 s) \cdot (1 + ur_2 s) \ \ldots \ (1 + ur_8 s)}{\sigma\alpha(1 + \frac{s}{\sigma})(1 + \frac{s}{\alpha})(1 + R_M s)\pi_1^4 (1 + Hr_i s)} , \qquad \text{(B-29)}$$

where the ur_i and Hr_i are the roots of the numerator of (B-28) and denominator of (B-23) respectively.

Using the same equations with (B-27) to calculate W of (3-100):

$$W = \overline{B}_M P_D [\lambda_1^2 + \frac{\overline{\Delta P_M \Delta P_D}}{\overline{P_M} P_D} (1 - \frac{B_M B_D T_C}{P_D})\overline{(1 - \frac{B_M B_D T_C}{P_D})} \Phi_{DD}$$

$$= \frac{(1 + R_D s) K_D \mu^2 e^{(E_M - E_D)s}}{(\sigma + s)(\alpha + s)^2 (1 + R_1 s)(1 + R_2 s)(\sigma - s)} \left[\lambda_1^2 (\alpha + s) + \right.$$

$$\left. + \frac{(\sum_0^4 w_i s^i)(\sum_0^4 HH_{1i}(-s^2)^i)}{(\alpha - s) \sum_0^4 HH_{2i}(-s^2)^i} \right] . \quad \text{(B-30)}$$

The l.h.p. singularities of this equation are the plant poles and the negative eigenvalues of $\sum_0^4 HH_{2i}(-s^2)^i$. Let the evaluation of the principal parts at the poles of (B-30) be

$$\left[\frac{W}{\overline{\overline{U}}} \right]_+ \triangleq \frac{\mu^2 \sum_0^8 EE_i s^i}{(1 + \frac{1}{\alpha} s)^2 (1 + \frac{1}{\sigma} s)(1 + R_1 s)(1 + R_2 s) \pi_1^4 (1 + Hr_i s)} . \quad \text{(B-31)}$$

It should be mentioned, perhaps, that such steps as arriving at (B-31) are extremely complicated because of the high order of the system and because of the occurrence of complex roots. Using (3-100) and (B-31)

$$T_D = \frac{H_{40} \alpha \sigma (\sum_0^8 EE_i s^i)(1 + R_M s)}{uu_0 (1 + \frac{1}{\alpha} s)(1 + R_1 s)(1 + R_2 s) \pi_1^8 (1 + ur_i s)}$$

$$\triangleq \frac{\sum_0^9 T_{D1i} s^i}{\sum_0^{11} T_{D2i} s^i} . \quad \text{(B-32)}$$

The desired result at last! As before, T_D is expressed as a function of the parameters λ_1 and λ_2. It will be necessary to evaluate $\langle c^2 \rangle$ and $\langle a^2 \rangle$ to determine these parameters; then $\langle \Delta c^2 \rangle$ can be calculated. From (3-90)

$$\Phi_{CC} = (B_M T_D + P_D)\overline{(B_M T_D + P_D)}\Phi_{DD} + \frac{B_M \overline{B}_M B_D \overline{B}_D T_C \overline{T}_C}{P_D \overline{P}_D}\Phi_{\delta\delta} . \quad \text{(B-33)}$$

The first term in this expression can be expanded

$$(B_M T_D + P_D)\overline{(B_M T_D + P_D)}\Phi_{DD} = (B_M \overline{B}_M T_D \overline{T}_D + \overline{B_M T_D} P_D + B_M T_D \overline{P_D} + P_D \overline{P_D})\Phi_{DD} . \quad \text{(B-34)}$$

As indicated by (3-23), this expression must be integrated to calculate the mean square output. Since the only type of non-minimum phase element considered in this example is the exponential time delay, $B_M \overline{B_M} = 1$ so that the first and the last terms of the right hand expression are polynomials which can be integrated directly by finding residues or by using specially calculated equations for rational functions [N1,S1,C1]. By substitution for the dummy variable of integration

$$\int_{-\infty}^{\infty} B_M T_D \overline{P_D} \Phi_{DD} ds = \int_{-\infty}^{\infty} \overline{B_M T_D} P_D \Phi_{DD} ds^* . \quad \text{(B-35)}$$

Therefore the integral of one of these terms may be evaluated and then doubled. Contour integration of $\overline{B_M T_D} P_D \Phi_{DD}$ is the simplest as there are only four l.h.p. poles - three from P_D and one from D. The contour integration may be made directly by finding residues at each of the l.h.p. poles. The integration of the last term of (B-33) is routine since non-minimum phase elements cancel. T_C is given by (B-15) and $\Phi_{\delta\delta}$ is assumed to be "white" noise with a constant spectral density δ^2.

The value of the mean square control effort is evaluated similarly. Since P_A and $B\overline{B}$ both equal one, it follows from (3-91) that

$$\Phi_{AA} = \frac{T_D\overline{T_D}}{P_M\overline{P_M}}\Phi_{DD} + \frac{T_C\overline{T_C}}{P_D\overline{P_D}P_M\overline{P_M}}\Phi_{\delta\delta} \, . \qquad \text{(B-36)}$$

This is evaluated using (B-2) and (B-32):

$$\frac{T_D D}{P_M^*} = \frac{\mu\alpha}{\sigma} \frac{\sum_0^9 T_{D1,i} s^i}{(1 + \frac{s}{\sigma})(1 + R_M s)\pi_1^8 (1 + ur_i s)} \, , \qquad \text{(B-37)}$$

and from (B-1), (B-2) and (B-15)

$$\frac{T_C}{P_D^* P_M^*} = \frac{(\alpha + s)(1 + R_1 s)(1 + R_2 s)(\sum_0^3 g_i s^i)}{yY_0 K_D K_M (\sum_0^7 y_i s^i)} \, . \qquad \text{(B-38)}$$

Contour integration of these rational functions is also routine.

Computation of the error caused by faulty model identification is accomplished by the use of (3-89).

$$\Phi_{\Delta C \Delta C} = (B_M T_D \frac{\Delta P_M}{P_M} + \Delta P_D)\overline{(B_M T_D \frac{\Delta P_M}{P_M} + \Delta P_D)}(1 - \frac{B_M B_C T_C}{P_D})\overline{(1 - \frac{B_M B_C T_C}{P_D})}\Phi_{DD} \, . \qquad \text{(B-39)}$$

This expression was integrated by the same method described for (B-34). The equation was expanded by multiplication into the sum of rational functions and rational functions multiplied by exponential factors. The former were integrated by polynomial integration formulas while direct evaluation of residues at l.h.p. poles was utilized for the latter.

Having evaluated the error terms, the control functions may now be found explicitly. Referring to (3-102) and defining

$$L \triangleq \frac{B_D T_C}{P_D}$$

$$= \frac{(1 + R_M s)\left(\sum_0^3 g_i s^i\right)}{K_D y y_0 \left(\sum_0^7 y_i s^i\right)}, \qquad \text{(B-40)}$$

the feedback controller Q_C is determined

$$Q_C = \frac{\frac{L}{P_M}}{1 - Le^{-E_M s}},$$

$$= \frac{L}{P_M}\left(1 + Le^{-E_M s} + L^2 e^{-2E_M s} + \ldots\right. \qquad \text{(B-41)}$$

Similarly, using (3-98) gives

$$Q_D = \frac{\frac{P_D}{P_M}\left(\frac{T_D}{P_D} - Le^{-E_D s}\right)}{\left(1 - Le^{-E_M s}\right)},$$

$$= \frac{P_D}{P_M}\left(\frac{T_D}{P_M} - Le^{-E_D s}\right)\left(1 + Le^{-E_M s} + L^2 e^{-2E_M s} + \ldots\right). \qquad \text{(B-42)}$$

As mentioned earlier, the actual computations were made by assuming values for λ_1 and λ_2. Then both control functions, T_C and T_D, were calculated and from these the various outputs evaluated. In principle, it would be possible to adjust values for λ_1 and λ_2 successively until the values of the outputs and constraints corresponded to those desired for the plant. In actual fact, the programs necessary for these calculations exceeded the memory capacity of the computer so that the computation was divided into three portions - part I computed T_C and T_D; part II evaluated the integrals and part III computed the explicit control functions, Q_C and Q_D. Cross plots of various outputs and constraints were constructed so that the values of λ_1 and λ_2 giving the desired performance were found by graphical interpolation. The result of this method of computation was superior to direct calculation of the final values since the answer was found in terms of a family of solutions indicating the effect that a variation in the parametric values would have on the desired control.

APPENDIX C

DERIVATION OF THEORETICAL DYNAMIC MATHEMATICAL MODEL OF EXPERIMENTAL SYSTEM

The theoretical mathematical model for the experimental system described in Chapter VI was derived following Haskins [H3] from energy balances on the oil, coolant and wall:

$$(\rho V C_p)_f \frac{dT_f}{dt} = (hA_H)_i (T_w - T_f) + (C_p W)_f (T_{f_{in}} - T_f) - q \; ,$$

$$(\rho V C_p)_w \frac{dT_w}{dt} = (hA_H)_i (T_f - T_w) - (hA)_o (T_w - T_{cm}) \; , \qquad \text{(C-1)}$$

$$(\rho V C_p)_c \frac{dT_{cm}}{dt} = (hA_H)_o (T_w - T_{cm}) + (C_p W)_c (T_{c_{in}} - T_{co}) \; ,$$

where ρ = density,

V = volume,

C_p = specific heat,

T = temperature,

t = time,

h = film heat transfer coefficient,

A_H = heat transfer area,

q = heat losses

W = bulk flow rate,

and subscripts,

f = hot fluid or oil flow,

c = cool fluid or coolant mixture,

w = wall metal properties,

i = inside wall,

o = outside wall,

co = coolant effluent out of jacket,

cm = arithmetic mean coolant temperature, i.e.,

$$T_{cm} = \frac{T_{c_{in}} + T_{co}}{2} . \qquad \text{(C-2)}$$

This describing equation contains product nonlinearities since the oil flow rate, W_f, multiplies oil temperature, T_f, and coolant flow rate, W_c, multiplies coolant temperature, T_{co}. The "non-linear perturbation model" which results from subtraction of the steady state from (C-1) may be converted to the "linear perturbation model" by Taylor series expansion eliminating the cross-product terms. Thus (C-1) becomes

$$(\rho C_p V)_f \frac{dT_f}{dt} = -[(hA)_i + (W_{ss}C_p)_f]T_f + (hA)_i T_w + [C_p(T_{in} - T_{ss})]_f W_f ,$$

$$(\rho C_p V)_w \frac{dT_w}{dt} = (hA)_i T_f - [(hA)_i + (hA)_o]T_w + (hA)_o T_{cm} , \qquad \text{(C-3)}$$

$$(\rho C_p V)_c \frac{dT_{cm}}{dt} = (hA)_o T_w - [(hA)_o + 2(W_{ss}C_p)_c]T_{cm} + [C_p(T_{in} - T_m)]_c W_c ,$$

where subscript ss refers to steady state values and all other symbols are same as before except the various temperatures which now refer to unsteady state perturbations. In order to abbreviate some of the subsequent algebra, (C-3) is rewritten with implicit definitions as follows

$$C_f \frac{dT_f}{dt} = -f_f T_f + H_i T_w + \Delta T_f W_f \ ,$$

$$C_w \frac{dT_w}{dt} = H_i T_f - (H_i + H_o) T_w + H_o T_{cm} \ , \qquad \text{(C-4)}$$

$$C_c \frac{dT_{cm}}{dt} = H_o T_w - f_c T_{cm} + \Delta T_c W_c \ .$$

The overall transfer function relating the output, T_w, to the independent variables, W_c and W_f, is obtained by Laplace transformation of (C-4) followed by substitution of the first and third into the second giving

$$\{C_w C_f C_c s^3 + [C_w C_f f_c + C_w C_c f_f + C_f C_c (H_o + H_i)] s^2 +$$
$$+ [C_w f_f f_c + (C_f f_f + C_c f_c)(H_o + H_i) - (C_c H_i^2 + C_f H_o^2)] s +$$
$$+ [(H_i + H_o) f_f f_c - (H_o^2 f_f + H_i^2 f_c)]\} T_w =$$
$$= H_o \Delta T_c (C_f s + f_f) W_f + H_i \Delta T_f (C_c s + f_c) W_c \ . \qquad \text{(C-5)}$$

Haskins evaluated these constants by steady state measurements and his results were used to estimate similar values for the altered reactor dimensions and different flow rates used in this experiment. The values, shown in Table C-1, when substituted into (C-5) give

$$C = \frac{.24(1 + 0.32s)D + .048(1 + 0.38s)M}{(1 + 1.03s)(1 + 0.42s)(1 + 0.22s)} \ , \qquad \text{(C-6)}$$

where C is T_w (°F),
D is W_c (lbs/min), and
M is W_f (lbs/min).

TABLE C-1

LIST OF SYSTEM CONSTANTS: NOMENCLATURE, VALUES, UNITS, AND SOURCES

Symbol	Nomenclature	Value	Units	Source
Cp_c	Coolant heat capacity	0.763	BTU/lb°F	Handbook data
Cp_f	Oil heat capacity	0.538	BTU/lb°F	Lab. measurement [H3]
Cp_w	Wall metal heat capacity	0.042	BTU/lb°F	Handbook data
h_i	Oil side heat transfer coefficient	25	BTU/hr-ft^2-°F	Estimate from lab and handbook data
h_o	Coolant side heat transfer coefficient	100	BTU/hr-ft^2-°F	Estimate from lab and handbook data
A_i	Inside area	0.32	ft^2	Lab. measurement
A_o	Outside area	0.44	ft^2	Lab. measurement
T_{ci}	Coolant inlet temperature	21.5	°F	Steady-state data
T_{in}	Reactor oil inlet temperature	165	°F	Steady-state data
Tw_{ss}	Steady-state wall temperature	73	°F	Steady-state data
q	Heat loss term	250	BTU/hr	Steady-state data
V_c	Reactor coolant volume	.0102	cu.ft	Lab. measurement
V_f	Reactor oil volume	.0170	cu.ft	Lab. measurement
V_w	Reactor wall volume	.0162	cu.ft	Lab. measurement
Wf_{ss}	Steady-state oil flow rate	140	lb/hr	Steady-state data
Wc_{ss}	Steady-state coolant flow rate	30	lb/hr	Steady-state data
ρ_c	Coolant density	67.1	lb/cu.ft	Handbook data
ρ_f	Oil density	52.3	lb/cu.ft	Lab. measurement [H3]
ρ_w	Wall metal density	603	lb/cu.ft	Handbook data

APPENDIX D

NOMENCLATURE

$A, A(s)$ = Laplace transform of $a(t)$ (3-42)*

A_H = heat transfer area (3-1), (C-1)

$A^*, A^*(s)$ = Laplace transform of $a^*(t)$

$a(t)$ = linear function of manipulative variable (3-42)

a_i = constant (1-1), (4-4), (B-26)

$a^*(t)$ = continuous random time function (2-4)

B = general unknown transfer function (2-12)

B_D = non-minimum phase portion of disturbance transfer function (3-74)

B_M = non-minimum phase portion of manipulative variable transfer function (3-15)

B_1 = arbitrary member of a class of general unknown transfer functions (2-15)

B_O = general control transfer function yielding optimal control (2-15)

$B^*, B^*(s)$ = Laplace transform of $b(t)$ (2-4)

$b^*(t)$ = continuous random time function (2-4)

b_j = constant in system describing equation (1-1), (4-4)

$C, C(s)$ = Laplace transform of $c(t)$ (1-2), (2-1)

$c(t)$ = output or controlled variable (1-1)

c_p = specific heat (3-1), (C-1)

$D, D(s)$ = Laplace transform of $d(t)$ (1-2)

---

*The numbers in parentheses at the end of the definitions refer to the equation number where the symbol was defined or first appeared.

D = minimum phase portion of spectral density of random disturbance (2-17)

$d(t)$ = disturbance or load variable (1-1)

E = dead time in plant transfer function (3-67)

E_D = dead time in disturbance transfer function (B-1)

E_M = dead time in manipulative variable transfer function (B-2)

E_O, E_K = constants defined by (B-7)

E_{ij} = element of error covariance matrix (B-5)

EE_i = coefficients in (B-31)

F = flow rate (3-1)

$F(A,\lambda)$ = sum of integrals to be minimized (2-25)

g_i = constant in system describing equation (1-1)

g_i = coefficients of T_C (B-14)

gr_i^* = roots defined by (B-21)

h = film heat transfer coefficient (C-1)

$H_{n,m}$ = coefficients defined by (B-22)

$HH_{n,m}$ = coefficients of (B-23)

Hr_i = roots of denominator of (B-23)

$\mathcal{H}$ = maximum value of mean square output (3-78)

I = integral defined by (A-9)

j = imaginary number $\sqrt{-1}$ (2-1)

$J[\quad]$ = integral function of argument (2-12)

K = gain factor of general plant transfer function (3-67)

K_D = gain factor for disturbance transfer function (3-3), (3-4), (B-1)

K_M = gain factor for manipulative variable transfer function (3-3), (3-4), (B-2)

L = transfer function defined by (B-40)

L_μ for μ = C,D,M = linear operator in controller equation (3-2)

$\mathcal{L}^*$ = maximum allowable value of a general system state variable (2-24)

$\mathcal{L}$ = maximum value of mean square control effort (3-77)

$M, M(s)$ = Laplace transform of $m(t)$ (1-2)

$m(t)$ = manipulative or control variable (1-1)

P = general plant transfer function (3-65)

$P_A, P_A(s)$ = linear operator on manipulative variable to determine constraining conditions (3-42)

$P_D(s)$ = transfer function of disturbance in system Laplace domain equation (1-2)

P_D^* = minimum phase portion of disturbance transfer function (3-7)

$P_M(s)$ = transfer function of manipulative variable in system Laplace domain equation (1-2)

P_M^* = minimum phase portion of manipulative variable transfer function (3-8)

$P_1(s)$, $P_2(s)$ = general given or known transfer functions (2-12)

p_i = pole of plant transfer function (3-67)

$Q, Q(s)$ = general transfer function (2-13)

$Q_C(s)$ = transfer function of feedback controller (1-3)

$Q_D(s)$ = transfer function of feedforward controller (1-3)

Q_1, Q_2 = general given or known transfer functions (2-24)

q = heat losses (C-1)

R_D = time constant of zero of disturbance transfer function (B-1)

R_M = time constant of zero of manipulative variable transfer function (B-2)

R_1, R_2 = time constants of poles of plant transfer function (B-1), (B-2)

s = Laplace transform variable (1-2)

T = upper limit on an interval for definition of time functions (2-5)

T = temperature (C-1)

$T_C, T_C(s)$ = feedback control function (3-74)

$T_D, T_D(s)$ = overall control function (3-12),(3-16)

T_{D_n} = coefficients in overall transfer function (3-38), (3-55)

$T_{D_{n,m}}$ = general coefficients in overall control function (5-2)

T_D^* = a form of the overall control transfer function

T_1, T_2 = control transfer functions (3-18),(3-19)

t = time (1-1)

U = transfer function with l.h.p. poles and zeros only (3-99),(3-100),(B-28)

U_H = overall heat transfer coefficient (3-1)

ur_i = roots of numerator of (B-28)

V = tank volume (3-1),(C-1)

W = transfer function defined by (3-99),(3-100),(B-30)

W = bulk flow rates (C-1)

w_i = constants in (B-27)

X, X_1, X_2 = general transfer functions having poles in the r.h.p. only (2-17)

x = time function (4-4)

Y = a function that is equal to a given transfer function except that r.h.p. zeros are replaced by l.h.p. zeros of the same magnitude and exponential factors are absent (2-18),(3-82),(B-11)

y = time function (4-5)

yy_i = coefficients of $Y\overline{Y}$ (B-10)

y_i = coefficients of Y (B-11)

yr_i = roots of Y (B-11)

Z = transfer function defined by (3-96),(3-97),(B-12)

z_i = zero of plant transfer function (3-67)

Greek Letters

α = system natural frequency or pole of transfer function (3-3),(3-4),(B-1)

β = factor defined in (3-53)

β_i = coefficients in (B-8)

γ = weighting factor (3-94)

γ_i = coefficients in (B-8)

$\Delta\alpha$ = error in system natural frequency (B-3)

ΔC = model error output or increment in output due to parameter variation (3-58)

ΔD = error in evaluation of disturbance (3-58)

ΔE = error in exponential dead time of general plant transfer function (3-67)

ΔE_D = error in dead time of disturbance transfer function (B-3)

ΔE_M = error in dead time of manipulative variable transfer function (B-4)

ΔK = error in gain of general plant transfer function (3-67)

ΔK_D = error in gain of disturbance transfer function (B-3)

ΔK_M = error in gain of manipulative variable transfer function (B-4)

ΔP_D = error in disturbance transfer function (3-58),(B-3)

ΔP_M = error in manipulative variable transfer function (B-4)

Δp_i = error in i^{th} pole of general plant transfer function (3-67)

ΔT_D^* = error in overall control transfer function (3-58)

Δz_i = error in i^{th} zero of general plant transfer function (3-67)

δ = random noise in feedback circuit (3-84)

ϵ = arbitrary real number (A-7)

$\lambda, \lambda_1, \lambda_2$ = Lagrange multiplier or weighting factors (2-25), (3-79)

μ = magnitude factor of disturbance (3-9)

μ^* = magnitude factor in generated random noise (4-2)

ξ = vector of parameter errors (3-65),(3-67)

ρ = density (3-1),(C-1)

σ = mean frequency of disturbance (3-9)

σ^* = mean frequency of generated random noise (4-2)

τ = time difference $(t_1 - t_2)$ (2-6)

τ_C = dead time in output circuit (1-1)

τ_M = dead time in controller (1-1)

$\Phi_{AB}(s)$ = cross spectral density of random time functions, a(t) and b(t) (2-5)

$\varphi_{AB}(t_1, t_2)$ = correlation function of random functions a(t) and b(t) (2-4)

$\langle \ldots \rangle$ = mean value (2-4)

APPENDIX E

COMPUTER PROGRAM FOR CONTROLLER DESIGN EQUATIONS

In this appendix, the listing is presented of computer programs which were used to compute optimal control laws examined in this work. These programs result directly from the equations developed in Appendix B and the general nomenclature is similar in both developments.

The three principal programs are listed first: In program 201, the overall control functions, T_C and T_D, are computed. The output from program 201 is punched paper tape containing (in machine language) the initial data along with various intermediate data including the control functions. This tape is reintroduced to program 263 where various values of mean square output are computed and printed. The same intermediate tape is used again as data for program 265 where values for the specific controller functions, Q_C and Q_D, are computed. All of the control functions are factored and listed by this program to facilitate their use. The remainder of the listed program are subroutines required by one or more of these three principal programs.

Explanation of input variables and computational procedures is given in comments throughout all of the programs. A sample of input data for program 201 is given at the end of the program listings.

```
Program 201;
Comment This program calculates optimum feedback and
        feedforward components of a controller for a
        system described by C=P_D D+P_M M, where
                C is output,
                D is disturbance,
                M is control,
                P_D is EKD*EXP(-EDS)(1+RdS)/(1+R1S)(1+R2S)(S+ALP),
                P_M is EKM*EXP(-EMS)(1+RmS)/(1+R1S)(1+R2S)(S+ALP),
        and where these elements are subject to
        variations displayed in the E[5,5] covariance matrix:
(EEKD/EKD)^2          -              -               -               -
EALP*EEKD/EKD         EALP^2         -               -               -
EED*EEKD/EKD          EED*EALP       EED^2           -               -
EEKD*EEKM/EKD*EKM     EALP*EEK/EKM   EED*EEKM/EKM    (EEKM/EKM)^2    -
EEM*EEKD/EKD          EEM*EALP       EEM*EED         EEM*EEKM/EKM    EEM^2
        and where the limit of measurement of C^2 is EPS.
        The disturbance has the spectral density:
        D(s)D(-s)=EMUS/(SIG^2-s^2).
        Specific inputs are described at the READ statement.
        Output from this program is punched pare tape of HEXADS data to
        be used in Program 263 for integratin of state variables and in
        in Program 265 for determination and listing of the control functions
        T_D and T_C as well as the specific controller functions Q_C and Q_D.
        PART I: PROCEDURES AND FORMATS;
Begin
Integer NT1,NT2,IN,NA,NB,I,J,JA,k,LL,LP,MM,K,
        IM,K1,K2,K3,K4,K5,K6,PR,PM,PT,PI;
Real ALP,EKM,EKD,EMUS,SIG,AMB1,AMB2,ECKM,ECKD,EPS,BET,EEO,EEK,YK,DY2,
     DY1,Y11,Y12,EM,TCK,TCN,YY12,YY14,LIMIT,LAMBA1,LAMBA2,WK,TCKS,
     FACT1,FACT2,ED,HK,HKK,TOT,X,UK,EL,ALPS,BETS,
        RM,RD,R1,R2,R1S,R2S,RDS,RMS,
     AMB1S,AMB2S,EKMS,EKDS,SIGS,FF,FS,FA,EF2,EF1,EFO,TDK,TDS,PDS,IP,EI,
        RAT,eps0,eps1,eps2,eps3,UOS,dex,
       CSQ,ASQ,SAZO,SAZ1,SAZ2,DELCS,RSQ,ESQ,TDKS,ESS,ES,UK,GSQ;
Array E[1:5,1:5],W,sig[0:10],AZ,BZ[0:4];
Integer Array ex[0:8], nat[0:8];
McProcedure product(175,1,10);
McProcedure Reduce(276,1,12);
McProcedure Cancel(202,1,20);
McProcedure Cancelp(206,1,3);

McProcedure Order(200,1,3);
McProcedure ZoverY(271,1,26);
McProcedure Factor(203,1,15);
McProcedure Poly(205,1,3);
McProcedure bairsto(207,1,12);
McProcedure WoverU(275,1,21);
Format  T1(2(J1),S9,'TO',S12,'T1',S12,'T2',S12,'T3',S12,'T4',S12,'T5',S12,'T6'),
        O1(J1,6(S2,R6)),            O6(J1),
        O3(2(J1),6(S2,R6)),         O7(J1,S9,'HO',S12,'H1',S12,'H2',S12,'H3',S12,'H4'),
        O4(J1,7(S2,R6)),            O8(J4),              O9(J7),
        O5(2(J1),S10,5(S2,I4)),
        P1(3(J1),S12,'KM',S12,'KD',S12,'ALP',S12,'RD',S12,
        'RM',J1,S4,5(S2,R6),J1,S12,'R1',S12,'R2',
        S12,'EM',S13,'ED',S12,'EPS',J1,5(S2,R6),2(J1),S4,
        'AZ:',3(S4,R6),J1,S4,'W:',5(S4,R6),J1,
        S2,'EEO=',R6,S2,'EEK=',R6,J1),
        P5(J1,S10,'O8=',R6,J1),
        P2(J1,S2,'SIG=',R6,S4,'MU=',R6,J1),
        P3(3(J1),S2,'LAMBDA 1 =',R8,S4,'LAMBDA 2=',R8,J1);

Comment 201, PART II: CALCULATION OF TC;
rep:
READPT(DECIMAL,E[1,1],...,E[5,5],
        EKD,EKM,RD,RM,R1,R2,EM,ED,
        ALP,SIG,ECKM,ECKD,K2,EPS,
        k,dex,RAT,eps0,eps1,eps2,eps3,
```

```
        sig[1],...,sig[4],PT,
        LAMBA1,LAMBA2,NT1,NT2,FACT1,FACT2);
Comment E[1,1],...,E[5,5] are values of the covariance matrix of errors and
        EKD,EKM,RD,RM,R1,R2,EM,ED,ALP are system parameters shown above,
        Various values of the disturbance frequency may be tested so that SIG
        is SIG1/ECKD,
        ECKM is the amplitude of the disturbance, i.e., EMUS=2*SIG*ECKM,
        ECKD is the above stepping factor for SIG,
        K2 is the number of disturbance frequencies to be tested,
        EPS is the noise level in the feedback,
        k is the number of iterations in BAIRSTO 207,
        dex is the smallest nonzero coefficient accepted in FACTOR 203,
        eps's are convergence factors in BAIRSTO 207,
        sig's  1 - 3 are convergence factors in CANCEL 202 and
        sig[4] is convergence factor in ZoverY 271,
        PT is an unused integer,
        LAMBA's are starting values of Lagrange multipliers,

        NT's are number of increments of the Lagrange multipliers,
        FACT's are the values for increments of Lagrange multipliers;
ALPS←ALP²;
        EEK←E[3,3]-2.0*E[5,3]+E[5,5];
        EEO←E[1,1]-2.0*E[4,1]+E[4,4];
        AZ[0]←ALPS*E[4,4]-2.0*ALP*E[4,2]+E[2,2];
        BZ[0]←ALPS*E[1,1]-2.0*ALP*E[1,2]+E[2,2];
        AZ[1]←-E[4,4]+2.0*E[5,2]-ALPS*E[5,5];
        BZ[1]←-E[1,1]+2.0*E[3,2]-ALPS*E[3,3];
        AZ[2]←E[5,5];
        BZ[2]←E[3,3];
        W[4]←E[5,3];
        W[3]←E[4,3]-E[5,1];
        W[2]←-E[4,1]+E[3,2]+E[5,2]-ALPS*E[5,3];
        W[1]←-E[2,1]+E[4,2]+ALPS*(E[5,1]-E[4,3])+ALP*(E[3,2]-E[5,2]);
        W[0]←ALPS*E[4,1]-ALP*(E[2,1]+E[4,2])+E[2,2];
PRINT(P1,EKM,EKD,ALP,RD,RM,R1,R2,EM,ED,
        EPS,
        AZ[0],...,AZ[2], W[0],...,W[4], EEO,EEK);
RDS←RD²;RMS←RM²;R1S←R1²;R2S←R2²;
For K1←1 Step 1 Until K2 Do Begin SIG←SIG*ECKD;
SIGS←SIG²;EKDS←EKD²;EKMS←EKM²;EMUS←ECKM*2.0*SIG;
PRINT(P2,SIG,EMUS);AMB1←LAMBA1;
For NA←1 Step 1 Until NT1 Do
Begin AMB1←FACT1*AMB1; AMB1S←AMB1²;
AMB2←LAMBA2;
For NB←1 Step 1 Until NT2 Do
Begin  Array H,G[1:2,0:10],U,UR,Y[0:12],DS[1:6,0:4],
        HR,UU,EE,YR,x,DTD,DI,CI,g,gr[0:10],TD,Q[1:4,0:12];
YR[10]←gr[4]←RM;
AMB2←AMB2*FACT2;  AMB2S←AMB2²;
If NB=PT*(NB$PT)+PT$PT Then PM←1 Else PM←0;
Comment |DelPd/Pd-DelPm/Pm|²=EEO-EEK*S²;
PRINT(P3,AMB1,AMB2);
ZoverY(sig[4],AMB1S,AMB2S,ALP,SIG,RD,RM,EEO,EEK,
K3,DS,UR,
R1,R2,EMUS,EKD,EM,EPS,eps0,eps1,eps2,eps3,k,dex,RAT,Y,YR,g,gr,J,EKM);
If J=-4 Then Begin K←J; Go to PL End;

Comment 210, PART III: CALCULATION OF Td;
PRINT(P5,g[8]);
HK←g[0]/(EKD*g[8]);
Reduce(K3,J,DS,gr,YR,H,EM,LIMIT,sig,HK,HR);

If PM=1 Then Begin
PRINT(07);PRINT(03,H[1,0],...,H[1,4]);
        PRINT(01,H[2,0],...,H[2,4]);
PRINT(01,HK,LIMIT,g[8]);
End:
```

```
Comment 1-Tc/Pd=(S^4+H[1,3]S^3+...+H[1,0])/(S^4+H[2,3]S^3+...+H[2,0]);
For I←1,2 Do Begin
For J←0,1,2,3,4 Do DI[J]←(-1)^J*H[I,J];
product(I,4,H,U,0,4,E,DI,I,G,U);
For J←0,1,2,3,4 Do G[I,J]←G[I,2*J];End;
Comment |1-Tc/Pd|^2=(G[1,8]S^8- G[1,6]S^6+...+G[1,0])/(G[2,8]S^8-G[2,6]S^6+...+G[2,0]);
Comment |DelPm/Pm|^2=((AZ4)S^4-(AZ2)S^2+AZ0)/(ALPHA^2-S^2);
Comment ((DelPm/Pm)(DelPd/Pd)(W4S^4+W3S^3+...+W0)/(ALPHA^2-S^2);
CI[0]←ALPS;CI[1]←-1;x[0]←1;x[1]←-RMS;
product(0,2,E,AZ,1,4,G,U,0,E,DTD);
product(0,1,E,CI,2,4,G,U,0,E,DI);
UR[1]←-UR[1];UR[3]←-UR[3];product(0,6,E,DTD,0,1,E,x,0,E,U);
product(0,5,E,DI,0,3,E,UR,0,E,UU);
If PM=1 Then PRINT(01,U[0],UU[0],UR[0],DTD[0],DI[0],CI[0],x[0]);
For I←0,1,2,3,4,5,6,7 Do UU[I]←U[I]+UU[I];
K←8;U[0]←UU[0];
Comment U*UBAR=(UU[12]S^12-UU[10]S^10+...+UU[0])/((S^4+H[2,3]S^3+...H[2,0])(S+ALPHA)*
         (same bar)).  To find U, it is necessary to factor numerator into
         (U[0])(UR[1]^2S^2-1)...(UR[6]^2S^2-1).  The factored numerator of U is then
         (U[0])^1/2(UR[1]S-1)...(UR[6]S-1);
Factor(K,UU,eps0,eps1,eps2,eps3,k,dex,RAT,UR,nat,ex,CI,DTD,I);
If K=-4 Then Go to PL;
For IM←1 Step 1 Until I Do UR[IM]←SQRT(UR[IM]);
For  IM←IM+1 Step 1 Until 8 Do UR[IM]←0;
If PM=1 Then Begin For LP←1,2 Do
PRINT(01,G[LP,0],...,G[LP,4]);
End;
YR[0]←1; For K5←1 Step 1 Until K Do Begin
MM←(I+2*K5+1)$2;Q[K5,2]←1/SQRT(CI[MM]);
Q[K5,1]←SQRT(2*Q[K5,2]+DTD[MM]/CI[MM]);
Q[K5,0]←1;
If PM=1 Then Begin
PRINT(05,MM);
PRINT(03,Q[K5,0],...,Q[K5,2]);End;
End;
If K=1 Then Begin
For K5←1,2 Do YR[K5]←Q[1,K5] End;
If 2<K Then product(1,2,Q,DI,2,2,Q,CI,0,E,YR);
If K=3 Then product(3,2,Q,DI,0,4,E,YR,0,E,YR);
YR[10]←RM;
For I←0,1,2,3,4 Do Begin DTD[I]←H[2,I]; CI[I]←G[2,I]; DI[I]←G[1,I] End;
WoverU(W,DTD,DI,CI,UR,YR,K,HR,AMB1S,SIG,ALP,RD,R1,R2,EKD,EM,ED,EE,x);
HK←-G[2,0]*ALP*SIG/UU[0];
For MM←0,1,2,3,4,5,6,7,8 Do TD[1,MM]←EE[MM]*HK;
PRINT(T1); PRINT(04,TD[1,0],...,TD[1,8]);
UR[9]←DI[1]←1/ALP; UR[10]←DI[2]←R1; UR[11]←DI[3]←R2;
UR[12]←1/SIG; Poly(DI,3,CI); Poly(UR,8,U);
For LP←2,3 Do Begin product(0,3,E,CI,0,8,TD,U,LP,TD,UR);
For K5←1 Step 1 Until K Do Begin
product(LP,9,TD,DI,K5,2,Q,DI,LP,TD,U);End;
CI[0]←1; CI[1]←1/SIG+RM; CI[2]←RM/SIG; CI[3]←0;
PRINT(04,TD[LP,0],...,TD[LP,11]);End;
For I←1,2,3,4,5,6,7,8,9,10,11,12 Do TD[4,I]←UR[I];
PRINT(04,TD[4,1],...,TD[4,12]);
Comment TD[1,6]S^6+TD[1,5]S^5+...+TD[1,0]is numerator of Td and TdD/Pm
         U[0] ALP(TD[2,7]S^7+TD[2,6]S^6+...+1) is the numerator of TD.
         U[0] SIG EKM(TD[3,7]S^7+TD[3,6]S^6+...+1) is the denominator of T_DD/PM.
         END OF PART III;

Comment 210, PART IV: CALCULATION OF MEAN ERRORS;
PL:PUNCH(HEXADS,AMB1,AMB2,NA,NB,K);
If NA=1∧NB=1 Then PUNCH(HEXADS,ALP,ALPS,SIG,SIGS,R1,R2,RD,RM,
         EKDS,EKD,EPS,EMUS,EKMS,NT1,NT2,EM,ED,EEO,EEK,
         W[0],...,W[4],AZ[0],...,AZ[2],BZ[0],...,BZ[2]);
If K=-4 Then Go to ML;
For I←1,2 Do PUNCH(HEXADS,H[I,0],...,H[I,4]);
PUNCH(HEXADS,TD[1,0],...,TD[4,12],g[0],...,g[3],g[8],Y[0],...,Y[7]);
For I←1 Step 1 Until K Do PUNCH(HEXADS,Q[I,0],...,Q[I,2]);

ML:PRINT(08);End End End;PRINT(09);Go to rep;End;.0
```

```
Program 263;
Comment Special purpose program. This program integrates expressions to find the
        mean square values of output from systems with control laws from Program 201.
        The input to this program is the punched HEXADS from 201 and are defined
        there.  The printed output is as follows:
        ASQ=mean square control effort = RSQ+ESQ,
        CSQ=mean square output = TDS+PDS+EI+IP,
        DELCS=estimate of mean suqare error output using TD =-PD,
        DEL=mean square error output.
        Other intermediate data may be obtained by the use of sense lights 13,14, or 15;
Begin
        Real ALP,ALPS,SIG,R1,R2,RD,RM,EKDS,EKD,EPS,EM,ED,EKMS,EEO,
        r1,r2,r3,r4,r5,r6,r7,r8,r9,s,
        EEK,SIGS,TDS,PDS,IP,EI,CSQ,RSQ,ESQ,ASQ,DELCS,AMB1,AMB2,EMUS;
        Array TD[1:4,0:12],H[1:3,0:4],a,b[1:2,0:2],Q,E[1:3,0:12],R,A,B,C,D,G,Y[0:15];
        Integer K,NA,NB,NT1,NT2,I,i,j,k,l;
McProcedure Product(175,1,13);
McProcedure Poly(205,1,3);
Real Mc Procedure Integl(204,1,3);
McProcedure Order(200,1,3);
Switch L←L1,L2;
Format  O1(J1,S12,'RSQ',S12,'ESQ',S12,'TDS',S12,'PDS',S12,
        'IP',S12,'EI',J1,S3,6(S3,R6),2(J1),S3,'ASQ=',
        R8,S3,'CSQ=',R8,S3,'DELCS=',R8,S3,'DEL=',R8,4(J1)),

        O2(2(J1),S3,'LAMBDA1=',R6,S3,'LAMBDA2=',R6,S3,'K=',I4),

        O6(4(J1),S6,'ALP',S9,'SIG',S9,'R1',S10,'R2',S10,'RM',S10,'RD',
        J1,6(S2,R4),2(J1),S6,'KD',S10,'EPS',S9,'MU',S10,'EEO',S10,'EM',
        S10,'ED',J1,6(S2,R4),2(J1)),

        O4(2(J1),S10,'TEST OUTPUT',3(5(S4,R6),J1)),

        O5(J1,7(S2,R6)),

        O3(J7);
Real Procedure Residue;
        Begin
        R[6]←0;
        For j←9,10,11,12 Do Begin
           If TD[4,j]=0 Then R[j]←0 Else Begin
              s←1/TD[4,j];R[7]←0;R[8]←1;
              For i←0,1,2,3,4,5,6,7,8,9,10,11,12 Do Begin
                 R[8]←R[8]*(1+TD[4,i]*s); End;
              For i←0,1,2,3,4,5,6,7,8 Do
                R[7]←R[7]+TD[1,i]*s^i;R[7]←R[7]*(1-RD*s);
              For i←9,10,11,12 Do Begin
                 If ¬(i=j) Then R[8]←R[8]*(1-TD[4,i]*s)
                                       Else R[8]←R[8]*TD[4,i];End;

              For k←1 Step 1 Until K Do
              R[8]←R[8]*(1+Q[k,1]*s+Q[k,2]*s^2);
              R[j]←R[7]*EXP((ED-EM)*s)/R[8] End;
           R[6]←R[6]+R[j] End;
        Residue←R[6]; End;
Procedure Delta;
        Begin
        For i←1,2 Do Begin b[i,0]←SQRT(a[i,0]); b[i,2]←SQRT(a[i,2]);
             b[i,1]←SQRT(2*b[i,0]*b[i,2]-a[i,1]); End i;
        If l=0 Then Begin
             For i←8,7,6,5,4,3,2,1,0 Do C[i]←TD[1,i] End l;
        B[8]←0; For i←7,6,5,4,3,2,1,0 Do Begin
        B[i]←(C[i+1]-B[i+1])*ALP; C[i+1]←B[i+1]; End; B[0]←C[0];
        If 0<RM Then Begin A[15]←1/RM; B[10]←0;
             For i←9,8,7,6,5,4,3,2,1,0 Do B[i]←(TD[3,i+1]-B[i+1])*A[15]; End
        Else For i←0,1,2,3,4,5,6,7,8 Do B[i]←TD[3,i]; Product(0,7,E,B,2,4,H,B,0,E,D);
        A[0]←1; A[1]←1/ALP+R1+R2; A[2]←(R1+R2)/ALP+R1*R2;
        A[3]←R1*R2/ALP;  Product(0,11,E,D,0,3,E,A,0,E,D);
```

```
          Product(0,3,E,C,1,4,H,A,0,E,C);  Product(0,7,E,C,1,2,b,A,0,E,C);
          If ¬(D[11]=0) Then C[9]←0;
          R[6]←EMUS*Integl(10,D,C)/(ALPS*SIG^2);
          A[1]←1.02*ALP; A[2]←0.98*ALP; A[3]←1/R1; A[4]←1/R2; A[5]←SIG; A[6]←1/H[2,1];
          A[15]←1; For i←3,4,6 Do If 0<A[i] Then A[15]←A[15]*A[i];
          R[9]←R[10]←R[12]←0;
          For i←1,2,3,4,5,6 Do Begin If 0<A[i] Then Begin
              For j←1,2,3,4 Do R[j]←0; R[5]←1;
              For j←8,7,6,5,4,3,2,1,0 Do R[1]←TD[1,j]+A[i]*R[1];
              For j←11,10,9,8,7,6,5,4,3,2,1,0 Do R[2]←TD[2,j]+A[i]*R[2];
              For j←4,3,2,1,0 Do R[3]←H[3,j]-A[i]*R[3];
              R[4]←2*(1-RD*A[i])*(H[1,0]-H[1,1]*A[i])*EXP(-A[i]*EM)/((ALP+A[i])*(SIG+A[i]));
          R[4]←R[4]*(H[1,0]+H[1,1]*A[i])/(H[2,0]+H[2,1]*A[i]);
              For j←1 Step 1 Until i-1 ,i+1 Step 1 Until 6 Do If 0<A[j] Then R[5]←R[5]*(A[j]-A[i]);
          R[6+i]←EMUS*A[15]*R[1]*R[3]*R[4]/(R[2]*R[5]); End End i;
          A[0]←b[2,0]; A[1]←RD*b[2,0]+b[2,1]; A[2]←RD*b[2,1]+b[2,2]; A[3]←RD*b[2,2];
          Product(0,3,E,A,1,4,H,A,0,E,C); C[8]←0;
          A[1]←A[2]←1/ALP; A[3]←R1; A[4]←R2; A[5]←1/SIG;
          Poly(A,5,A); Product(2,4,H,B,0,5,E,A,0,E,D);
          R[14]←EMUS*Integl(9,D,C)/(ALPS*SIG)^2;
          R[13]←R[6]+R[7]+R[8]+R[9]+R[10]+R[11]+R[12]+R[14];
          If SLON(15) Then
          PRINT(05,R[6],...,R[14]);
End Delta;
rep:READPT(HEXADS,AMB1,AMB2,NA,NB,K);

If NA=1ΔNB=1 Then READPT(HEXADS,ALP,ALPS,SIG,SIGS,R1,R2,RD,RM,
          EKDS,EKD,EPS,EMUS,EKMS,NT1,NT2,EM,ED,EEO,EEK,
          H[3,0],...,H[3,4],a[1,0],...,a[2,2]);
If NB=1 Then PRINT(06,ALP,SIG,R1,R2,RM,RD,EKD,EPS,EMUS,EEO,EM,ED);
          PRINT(02,AMB1,AMB2,K);
If K=-4 Then Go to PL;
     READPT(HEXADS,H[1,0],...,H[2,4],TD[1,0],...,TD[4,12],
                G[0],...,G[3],G[8],Y[0],...,Y[7]);
For I←1 Step 1 Until K Do READPT(HEXADS,Q[I,0],...,Q[I,2]);
If SLON(14) Then PRINT(05,TD[1,0],...,TD[4,12]);
If 0<TD[2,10] Then Begin
          For i←1,2,3,4,5,6,7,8 Do A[i]←TD[4,i];
          Order(8,A,B);
          j←8; i←9; For i←i-1 While B[i]=0 Do j←j-1;
          For  i←1 Step 1 Until K Do
          If Q[i,2]<B[j]*B[j-1] ∇ j<2 Then Begin
          For k←1,2,3 Do Begin
          For l←0,l+1 While ¬(TD[k,l+3]=0) Do Begin
                    TD[k,l+1]←TD[k,l+1]-Q[i,1]*TD[k,l];
                    TD[k,l+2]←TD[k,l+2]-Q[i,2]*TD[k,l] End;
          TD[k,l+2]←0;
          TD[k,l+1]←0 End; Q[i,1]←Q[i,2]←0; j←j+1; Go to L10 End;
L9:       For k←1,2,3 Do
For i←1,i+1 While ¬(TD[k,i+1]=0) Do
          TD[k,i]←TD[k,i]-B[j]*TD[k,i-1];
          TD[k,i]←0;
          For l←1 Step 1 Until j-1 Do TD[4,l]←B[l];
          For l←j Step1 Until 8  Do TD[4,l]←0;
L10:      If 0<TD[2,10] Δ 1<j Then Begin j←j-1; Go to L9 End;
End;
If SLON(13) Then PRINT(05,TD[1,0],...,TD[4,12]);
          C[1]←1; C[0]←SIG; Product(0,1,E,C,2,11,TD,B,0,E,D);
For I←0,1,2,3,4,5,6,7,8,9 Do C[I]←TD[1,I];
TDS←EMUS*Integl(10,D,C);
          Delta;
D[0]←ALP*SIG;D[1]←ALP+SIG;D[2]←1;
C[0]←1;C[1]←R1+R2;C[2]←R1*R2;
Product(0,2,E,D,0,2,E,C,0,E,B);
C[0]←1; C[1]←RD; C[2]←C[3]←0;
PDS←EMUS*EKDS*Integl(4,B,C);
IP←2*EMUS*Residue*EKD/(ALP*SIGS);
D[0]←1; D[1]←RM;
Product(0,1,E,D,0,3,E,G,0,E,C);C[5]←C[6]←0;
```

```
EI←EPS/(G[8]²*EKDS)*Integl(7,Y,C);
CSQ←TDS+PDS+IP+EI;
For i←0,1,2,3,4,5,6,7,8,9,10 Do Begin
C[i]←TD[1,i]; D[i]←TD[3,i] End;
RSQ←EMUS*ALPS/(EKMS*SIGS)*Integl(10,D,C);
B[0]←ALP;B[1]←1+ALP*(R1+R2);
B[2]←ALP*R1*R2+R2+R1;B[3]←R1*R2;
Product(0,3,E,B,0,3,E,G,0,E,C);
L1:L2: ESQ←EPS/(G[8]²*EKMS*EKDS)*Integl(7,Y,C);

ASQ←RSQ+ESQ;

E[1,0]←SQRT(EEO);  E[1,1]←SQRT(EEK);
E[1,2]←E[1,1]*RD;  E[1,1]←E[1,1]+E[1,0]*RD;
A[0]←SIG; A[1]←1; Product(0,1,E,A,0,3,E,B,2,E,C);
Product(1,2,E,D,1,4,H,D,0,E,C);
Product(2,4,E,C,2,4,H,C,0,E,D);
DELCS←EKDS*EMUS*Integl(8,D,C);

PRINT(01,RSQ,ESQ,TDS,PDS,IP,EI,ASQ,CSQ,DELCS,R[13]);
PL: If NA=NT1∧NB=NT2 Then PRINT(03);
Go to rep; End;0000
```

```
Program 265;
Comment This program computes and factors T_D, T_C, Q_D, and Q_C from input of
        punched HEXADS from 201.
        If SLON(10) Then procedure will attempt to factor transfer functions
        until convergence criteria in BAIRSTO 207 is 0.001,
        If SLON(8) Then procedure will divide root at ALP out from all TDi,
        If SLON(9) Then procedure will divide root at 1/RM out from all TDi,
        If SLON(12) Then procedure will not factor tD and TC,
        If SLON(14) Then procedure will not find or factor QD,QC and LC;
Begin
        Real ALP,ALPS,SIG,SIGS,R1,R2,RD,RM,KDS,KD,EPS,EMUS,F,
             KMS,EM,ED,EEO,EEK,AMB1,AMB2,KDG8,KMG8,KMA,
             KM,eps,ee0,em,ed,amb1,amb2;
        Integer NA,NB,K,NT1,NT2,total,i,j,k;
        Array W,AZ,BZ[0:4],a[1:2,0:2],g[0:8]; Integer I;
        Array TD[1:4,0:12],H[1:3,0:4],Q,E[1:3,0:12],Y,G,A,B,LCN,
             QDD1,QDN2,QDN1,QCN,QDD,QDN,QCD[0:20],
             aeps,aee0,aem,aed,aamb1,aamb2[0:30];
McProcedure Bairsto(207,1,12);
Format  00(J1,S9,'EPS',S15,'EEO',S15,'EM',S15,
        'ED',S14,'LAMBDA 1', S10,'LAMBDA 2',J1,6(S3,R8),J1),
        01(J7),
        F1(2(J1),'  TD',I1,J1,7(S2,R7),J1,S15,6(S2,R7)),
        F2(2(J1),'  G',J1,S10,4(S2,R7)),
        F3(2(J1),'  Y',J1,8(S1,R6));
McProcedure Product(175,1,1);
McProcedure Poly(205,1,3);
Format  001(J1,S20,'0',S14,'1',S14,'2',S14,'3',S14,'4',S14,'5',
                2(J1),S4,'QDN1',S4,6(S2,R7),J1,S27,3(S2,R7),
                2(J1),S4,'QDD1',S4,6(S2,R7),J1,S27,4(S2,R7),
                2(J1),S4,'QDN2',S4,5(S2,R7),
                2(J1),S4,'QDD2',S4,6(S2,R7),J1,S27,2(S2,R7),
                2(J1),S4,'QCN',S5,6(S2,R7),J1,S27,R7,
                2(J1),S4,'QCD',S5,6(S2,R7),J1,S27,2(S2,R7),
                2(J1),S4,'LCN',S5,5(S2,R7),J7),
        002(J1,S20,'0',S14,'1',S14,'2',S14,'3',S14,'4',S14,'5',
                2(J1),S4,'QDN',S5,6(S2,R7),J1,2(S27,5(S2,R7),J1),
                J1,S4,'QDD',S5,6(S2,R7),J1,2(S27,5(S2,R7),J1),S27,S2,R7,
                2(J1),S4,'QCN',S5,6(S2,R7),J1,S29,R7,
                2(J1),S4,'QCD',S5,6(S2,R7),J1,S27,2(S2,R7),
                2(J1),S4,'LCN',S5,5(S2,R7),J7);
```

```
Procedure Root(A,L);
        Array A; IntegerL;
        Begin    Real eps,q; Integer i,j,I,n,m,K; Array e,f[0:10];
                 Integer Array g,h[0:10];
        Format   P1(2(J1),'   EXIT:',S9,5(I1,S18)),
                 P2(J1,' NATURE:',S9,5(I1,S18)),
                 P3(J1,'  ROOTS:',5(S2,R10)),
                 P4(2(J1));
        k←L; For q←|A[k]| While q<1D-30 Do k←k-1;
        eps←1D-10;  K←50;
        k← If 1<k Then k Else 1;
AGAIN:  Bairsto(k,A,eps,eps,eps,eps,K,n,e,f,h,g);
        PRINT(P1,g[1],...,g[n]);
        PRINT(P2,h[1],...,h[n]);
        PRINT(P3,e[1],...,e[n]);
        PRINT(P3,f[1],...,f[n]); PRINT(P4);
        If g[n]=4 ∧ eps<0.001 ∧ ¬( SLON(10)) Then Begin eps←100*eps;
        For i←1 Step 1 Until n-1 Do Begin

        k←2*((k+1)$2)-2;
        If g[i]=-1 Then Begin B[1]←-2*e[i]; B[2]←e[i]²+f[i]² End
                 Else Begin B[1]←-e[i]-f[i]; B[2]←e[i]*f[i] End;
        For j←0 Step 1 Until k Do Begin
                 A[j+1]←A[j+1]-B[1]*A[j]; A[j+2]←A[j+2]-B[2]*A[j] End End I;
        Go to AGAIN End ex;
        End of Root;
rep:    READPT(HEXADS,AMB1,AMB2,NA,NB,K);
        If NA=1 ∧ NB=1 Then READPT(HEXADS,ALP,ALPS,SIG,SIGS,R1,R2,RD,RM,
                 KDS,KD,EPS,EMUS,KMS,NT1,NT2,EM,ED,EEO,EEK,
        H[3,0],...,H[3,4],a[1,0],...,a[2,2]);
        If K=-4 Then Go to rep; READPT(HEXADS,
                 H[1,0],...,H[2,4],TD[1,0],...,TD[4,12],
                 G[0],...,G[3],G[8],Y[0],...,Y[7]);
        For I←1 Step 1 Until K Do READPT(HEXADS,Q[I,0],...,Q[I,2]);
        If SLON(15) Then Begin
        READPT(HEXADS,A[1],...,A[4],K,E[1,0],...,E[3,12],
                 E[1,0],...,E[3,12],E[1,0],...,E[3,0]);
        For I←1 Step 1 Until K Do READPT(HEXADS,E[I,0],...,E[I,2]);
        End;
        PRINT(OO,EPS,EEO,EM,ED,AMB1,AMB2);
If SLON(9) ∨ SLON(8) Then Begin
        F←If SLON(8) Then ALP Else 1/RM;
NOW:    For k←1,2,3 Do Begin
        For i←11,i-1 While 1<i Do Begin
        A[i-1]←(TD[k,i]-A[i])*F;
        TD[k,i]←A[i] End; TD[k,0]←A[0] End;
        If SLON(9) ∧ F=ALP Then Begin F←1/RM; Go to NOW End End;
If SLON(12) Then Begin If SLON(13) Then EXHLT(777);  Go to REPEAT End;
        For i←1,2 Do Begin k←-1;
        For j←0,j+1 While ¬(TD[i,j]=0) Do
                 Begin A[j]←TD[i,j]; k←k+1 End;
        PRINT(F1,i,TD[i,0],...,TD[i,k]);
        Root(A,k) End i;
        For i←0,1,2,3 Do Begin A[i]←G[i];
                 A[i]←A[i]/G[8] End i;
        PRINT(F2,A[0],...,A[3]);
        k←If G[3]<1D-30 Then 2 Else 3;
        Root(A,k);
        For i←7,i-1 While Y[i]<1D-30 Do k←i-1;
        For i←k Step -1 Until 0 Do A[i]←Y[i];
        PRINT(F3,Y[0],...,Y[k+1]);
        Root(A,k);
REPEAT:  If SLON(14) Then Begin PRINT(O1); Go to rep End;
         PRINT(OO,EPS,EEO,EM,ED,AMB1,AMB2);
                 KM←SQRT(KMS);KDG8←KD*G[8]; KMG8←KM*G[8]; KMA←ALP/KM;
                 A[0]←1/KDG8; A[1]←A[0]*RM; Product(0,1,E,A,0,3,E,G,0,E,LCN); Comment Eq.16-42;
                 A[0]←1/KMG8; A[1]←A[0]*RD; Product(0,1,E,A,0,3,E,G,0,E,QDN2); Comment Eq.16-44;
```

```
                              QDD1[9]←0; For i←8,7,6,5,4,3,2,1,0 Do Begin L3:
                                      QDD1[i]←(TD[3,i+1]-QDD1[i+1])*SIG;
                                      QDN1[i]←TD[1,i]*KMA; EndL3; Comment Eq.16-44;
                              A[1]←1/ALP; A[2]←R1; A[3]←R2; Poly(A,3,B);
                              Product(0,3,E,B,0,3,E,G,0,E,QCN); F←KMA/KDG8;
                              For i←0,1,2,3,4,5,6 Do QCN[i]←F*QCN[i]; Comment Eq.16-44;
        If 1D-3<ED Then Begin L4:
                Root(QDN1,9); Root(QDD1,9);
                Root(QDN2,7); Root(Y,7);
        Root(QCN,7); Root(LCN,7);
                PRINT(001,QDN1[0],...,QDN1[8],QDD1[0],...,QDD1[9],
                              QDN2[0],...,QDN2[4],Y[0],...,Y[7],
                              QCN[0],...,QCN[6],Y[0],...,Y[7],
                              LCN[0],...,LCN[4]);
        Go to rep End L4;
        Product(0,9,E,QDD1,0,4,E,QDN2,0,E,A); A[14]←A[15]←0;
        Product(0,8,E,QDN1,0,7,E,Y,0,E,B);
        For i←0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15 Do QDN[i]←A[i]+B[i]; Comment Eq.17-16 and 18;
        If 1D-3<EM Then Begin L5:
                Product(0,9,E,QDD1,0,7,E,Y,0,E,QDD);
                Root(QDN,16); Root(QDD,16);
                Root(QCN,7); Root(Y,7);
                Root(LCN,7);
                PRINT(002,QDN[0],...,QDN[15],QDD[0],...,QDD[16],
                              QCN[0],...,QCN[6],Y[0],...,Y[7],LCN[0],...,LCN[4]);
End L5  Else Begin L6:
                For i←0,1,2,3,4,5,6,7 Do QCD[i]←Y[i]-LCN[i];
                Product(0,7,E,QCD,0,9,E,QDD1,0,E,QDD);
                Root(QDN,16); Root(QDD,16);
                Root(QCN,7); Root(QCD,7);
                PRINT(002,QDN[0],...,QDN[15],QDD[0],...,QDD[16],
                              QCN[0],...,QCN[6],QCD[0],...,QCD[7]);
End L6;  Go to rep End of 265;
0
```

```
Program 175;
Procedure product(p,n,U,X,q,m,V,Y,r,W,Z);
Comment This procedure multilpies polynomial If p=0 Then X[i] Else U[p,i]
        with polynomial If q=0 Then Y[i] Else V[q,i] to give the
        polynomial If r=0 Then Z[i] Else W[r,i]+  In all cases, the
        lowest index coefficient represents the constant term;
Value p,n,q,m,r,X,Y;
Array U,V,W,X,Y,Z;
Integer n,m,p,q,r;
Begin
        Real Array x[0:15],y[0:15],z[0:15];
        Integer i,j,k,a,b,c;
a←p;b←q;c←r;
If 0<a Then Begin For i←0 Step 1 Until n Do x[i]←U[a,i] End
        Else For i←0 Step 1 Until n Do x[i]←X[i];
If 0<b Then Begin For i←0 Step 1 Until m Do y[i]←V[b,i] End
        Else For i←0 Step 1 Until m Do y[i]←Y[i];
For k←0 Step 1 Until n+m Do z[k]←0;
For i←0 Step 1 Until n Do Begin
For j←0 Step 1 Until m Do Begin
k←i+j; z[k]←x[i]*y[j]+z[k];End End;
If 0<c Then Begin For i←0 Step 1 Until n+m Do W[c,i]←z[i];End
        Else For i←0 Step 1 Until n+m Do Z[i]←z[i]; End;
```

```
Program 200;
Procedure Order(p,x,y);
Comment This program orders p objects- x[1:p] to y[1:p]
        where y[1] is the largest;
Value p,x;Real Array y,x;  Integer p;
Begin Integer k,n;
        For n←p Step -1 Until 1 Do
        For k←1 Step 1 Until n Do
        If x[k]<x[n] Then Begin y[k]←x[k];x[k]←x[n];x[n]←y[k] End;
        For n←1 Step 1 Until p Do y[n]←x[n];
        End;
```

```
Program 204;
Real Procedure Integl(k,D,C);

Comment  Inputs: k - order of denominator,
                 D - coefficients of denominator,
                 C - coefficients of numerator,
         Output: Value of the contour integral of the rational function

                 C(s)*C(-s)/(D(s)*D(-s)) around l.h.p.

         A check is made first so that the
         coefficient of the highest power of D is not less than 10-18,
         Equations from Newton, Gould,and Kaiser.;
Value k,D,C;
Real Array D,C;
Integer k;
 Begin Real DEL5,DEL6,DEL7,DEL8,DEL9,DEL10,
D0,D1,D2,D3,D4,D5,D6,D7,D8,D9,D10,
C0,C1,C2,C3,C4,C5,C6,C7,C8,C9,
M0,M1,M2,M3,M4,M5,M6,M7,M8,M9,
A1,A2,A3,A4,A5,A6,A7,A8,A9,A10,
B1,B2,B3,B4,B5,I9,I10;
 Integer j,m;
Format ok(J1,S2,'INTERMEDIATE OUTPUT',J1,3(6(S2,R6),J1));
Real Array M[0:8];
Switch L←L1,L2,L3,L4,L5,L6,L7,L8,L9,L10;
m←k;
For j←m While |D[j]|<1D-18 Do m←m-1; Go to L[j];
L9:L10:D0←D[0];D1←D[1];D2←D[2];D3←D[3];D4←D[4];
D5←D[5];D6←D[6];D7←D[7];D8←D[8];D9←D[9];D10←D[10];
C0←C[0];C1←C[1];C2←C[2];C3←C[3];C4←C[4];C5←C[5];
C6←C[6];C7←C[7];C8←C[8];C9←C[9];
If m=9 Then Go to LL9; Go to LL10;
L1:Integl←0;Go to L11;
   L2:Integl←(C[1]^2*D[0]+C[0]^2*D[2])/(2.0*D[0]*D[1]*D[2]);Go toL11;
L3:Integl ←(C[2]^2*D[0]*D[1]+(C[1]^2-2.0*C[0]*C[2])*D[0]*D[3]+ C[0]^2*D[2]
   *D[3])/(2.0*D[0]*D[3]*(-D[0]*D[3]+D[1]*D[2]));
Go to L11;
L4: Integl←(C[3]^2*(-D[0]^2*D[3]+D[0]*D[1]*D[2])+(C[2]^2-2.0*C[1]*C[3])*
    D[0]*D[1]*D[4]+(C[1]^2-2.0*C[0]*C[2])*D[0]*D[3]*D[4]+C[0]^2*(-D[1]
    *D[4]^2+D[2]*D[3]*D[4]))/(2.0*D[0]*D[4]*
    (-D[0]*D[3]^2-D[1]^2*D[4]+ D[1]*D[2]*D[3])); Goto L11;
L5:M[1]←-D[0]*D[3]+D[1]*D[2];  M[2]←-D[0]*D[5]+D[1]*D[4];

M[3]←(D[2]*M[2]-D[4]*M[1])/D[0]; M[4]←(D[2]*M[3]-D[4]*M[2])/D[0];
M[0]←(D[3]*M[1]-D[1]*M[2])/D[5]; DEL5←(D[1]*M[4]-D[3]*M[3]+D[5]*M[2])*D[0];
Integl←(C[4]^2*M[0]+(C[3]^2-2.0*C[2]*C[4])*M[1]+(C[2]^2-2.0*C[1]*C[3]+2.0*C[0]*C[4])*
M[2]+(C[1]^2-2.0*C[0]*C[2])*M[3]+C[0]^2*M[4])/(2.0*DEL5);Go to L11;
L6:M[1]←-D[0]*D[1]*D[5]+D[0]*D[3]^2+D[1]^2*D[4]-D[1]*D[2]*D[3];
M[2]←D[0]*D[3]*D[5]+D[1]^2*D[6]-D[1]*D[2]*D[5]; M[3]←D[0]*D[5]^2
+D[1]*D[3]*D[6]-D[1]*D[4]*D[5]; M[4]←(D[2]*M[3]-D[4]*M[2]+D[6]*M[1])/D[0];
M[5]←(D[2]*M[4]-D[4]*M[3]+D[6]*M[2])/D[0]; M[0]←(D[4]*M[1]-D[2]*M[2]+
```

```
D[0]*M[3])/D[6]; DEL6←(D[1]*M[5]-D[3]*M[4]+D[5]*M[3])*D[0];
Integl←(C[5]^2*M[0]+(C[4]^2-2.0*C[3]*C[5])*M[1]+(C[3]^2-2.0*C[2]*C[4]+
   2.0*C[1]*C[5])*M[2]+(C[2]^2-2.0*C[1]*C[3]+2.0*C[0]*C[4])*M[3]+
(C[1]^2-2*C[0]*C[2])*M[4]+C[0]^2*M[5])/(2*DEL6); Go to L11;
L7: M[1]←-(D[1]*D[4]-D[0]*D[5])^2+(D[0]*D[3]-D[1]*D[2])*(D[0]*D[7]-D[1]*D[6]
     +D[2]*D[5]-D[3]*D[4]);
    M[2]←(D[0]*D[7]-D[1]*D[6])*(-D[0]*D[5]+D[1]*D[4])+(D[0]*
      D[3]-D[1]*D[2])*(D[2]*D[7]-D[3]*D[6]);
    M[3]←-(D[0]*D[7]-D[1]*D[6])^2+(D[0]*D[3]-D[1]*D[2])*(D[4]*D[7]-D[5]*D[6]);
    M[0]←(D[5]*M[1]-D[3]*M[2]+D[1]*M[3])/D[7];
    M[4]←(D[2]*M[3]-D[4]*M[2]+D[6]*M[1])/D[0];
    M[5]←(D[2]*M[4]-D[4]*M[3]+D[6]*M[2])/D[0];
    M[6]←(D[2]*M[5]-D[4]*M[4]+D[6]*M[3])/D[0];
    DEL7←D[0]*(D[1]*M[6]-D[3]*M[5]+D[5]*M[4]-D[7]*M[3]);
    Integl←(C[6]^2*M[0]+(C[5]^2-2.0*C[4]*C[6])*M[1]+(C[4]^2-2.0*C[3]*C[5]+2.0*
      C[2]*C[6])*M[2]+(C[3]^2-2.0*C[2]*C[4]+2.0*C[1]*C[5]-2.0*C[0]* C[6])*
      M[3]+(C[2]^2-2 ·*C[1]*C[3]+2.0*C[0]*C[4])*M[4]+(C[1]^2-2.0*C[0]*
      C[2])*M[5]+C[0]^2*M[6])/(2.0*DEL7); Goto L11;
L8:M[1]←(D[0]*D[7]+D[2]*D[5])*(-D[0]*D[1]*D[7]+D[0]*D[3]*D[5]+2.0*
    D[1]^2*D[6])+(D[3]*D[7]-D[5]^2)*(D[0]^2*D[5]+D[1]*D[2]^2)+D[1]*D[3]*D[8]*(D[0]*
     D[3]-D[1]*D[2])-D[1]^2*D[8]*(D[0]*D[5]-D[1]*D[4])+(-D[2]*D[7]+D[3]*D[6]-
    D[4]*D[5])*(D[0]*D[3]^2+D[1]^2*D[4])-D[1]*D[6]*(D[1]^2*D[6]+3.0*D[0]*
    D[3]*D[5])-D[1]*D[2]*D[3]*(D[3]*D[6]-D[4]*D[5])+2.0*D[0]*D[1]*D[4]*D[5]^2;
   M[2]←(D[0]*D[3]-D[1]*D[2])*(D[0]*D[7]^2-D[1]*D[5]*D[8]-D[1]*D[6]*D[7]+
    D[2]*D[5]*D[7])+(D[3]*D[8]-D[4]*D[7])*(-D[0]*D[1]*D[5]+D[0]*D[3]^2
    -D[1]*D[2]*D[3]+D[1]^2*D[4])-D[0]*D[5]*D[7]*(D[0]*D[5]-
    D[1]*D[4])+D[1]^2*D[8]*(D[0]*D[7]-D[1]*D[6]);
   M[3]←-D[1]*(D[1]*D[8]-D[2]*D[7])^2+(-D[5]*D[8]+D[6]*D[7])*
    (D[0]*D[1]*D[5]-D[0]*D[3]^2+D[1]*D[2]*D[3]-D[1]^2*D[4])+
    D[0]*D[7]^2*(-D[0]*D[5]+D[1]*D[4]+D[2]*D[3])-2.0*D[0]*
    D[1]*D[3]*D[7]*D[8];
   M[4]←(-D[5]*D[8]+D[6]*D[7])*(2.0*D[0]*D[1]*D[7]-D[0]*D[3]*D[5]
    +D[1]*D[2]*D[5]-D[1]^2*D[6])+(-D[3]*D[8]+D[4]*D[7])*(D[0]*D[3]
    *D[7]-D[1]*D[2]*D[7]+D[1]^2*D[8])-D[0]^2*D[7]^3;
   M[0]←(D[6]*M[1]-D[4]*M[2]+D[2]*M[3]-D[0]*M[4])/D[8];

   M[5]←(D[2]*M[4]-D[4]*M[3]+D[6]*M[2]-D[8]*M[1])/D[0];
   M[6]←(D[2]*M[5]-D[4]*M[4]+D[6]*M[3]-D[8]*M[2])/D[0];
   M[7]←(D[2]*M[6]-D[4]*M[5]+D[6]*M[4]-D[8]*M[3])/D[0];
   DEL8←D[0]*(D[1]*M[7]-D[3]*M[6]+D[5]*M[5]-D[7]*M[4]);
   Integl←(C[7]^2*M[0]+(C[6]^2-2.0*C[5]*C[7])*M[1]+(C[5]^2-2.0*C[4]*
    C[6]+2.0*C[3]*C[7])*M[2]+(C[4]^2-2.0*C[3]*C[5]+2.0*C[2]*C[6]-
    2.0*C[1]*C[7])*M[3]+(C[3]^2-2.0*C[2]*C[4]+2.0*C[1]*C[5]-2.0
    *C[0]*C[6])*M[4]+(C[2]^2-2.0*C[1]*C[3]+2.0*C[0]*C[4])*M[5]+
    (C[1]^2-2.0*C[0]*C[2])*M[6]+C[0]^2*M[7])/(2.0*DEL8);
Goto L11;
LL9:      A1←D1*D2-D0*D3;
A2←D1*D4-D0*D5;
A3←D3*D4-D2*D5;
A4←D1*D6-D0*D7;
A5←D3*D6-D2*D7;
A6←D5*D6-D4*D7;
A7←D1*D8-D0*D9;
A8←D3*D8-D2*D9;
A9←D5*D8-D4*D9;
A1 .←D7*D8-D6*D9;
M1←A1*(A1*A10-A2*A9+A3*A6+A3*A8+2.*A4*A6-A5*A5-A5*A7-A7*A7)+A2*(-A2*A6-A3*A7+A4*A5+2*A4*A7)-A4*A4*A4;
M2←A1*(A3*A9+A4*A9-A5*A8+A6*A7-A7*A8)+A2*(-A2*A9+A4*A8+A7*A7)-A4*A4*A7;
M3←A1*(A3*A10·A4*A10+A7*A9-A8*A8)+A2*(-A2*A10+A7*A8)-A4*A7*A7;
M4←A1*(A5*A10+2.*A7*A10-A8*A9)+A2*(A7*A9-A4*A10)-A7*A7*A7;
M5←(D2*M4-D4*M3+D6*M2-D8*M1)/D0;
M6←(D2*M5-D4*M4+D6*M3-D8*M2)/D0;
M7←(D2*M6-D4*M5+D6*M4-D8*M3)/D0;
M8←(D2*M7-D4*M6+D6*M5-D8*M4)/D0;
DEL9←(D1*M8-D3?M7+D5*M6-D7*M5+D9*M4)*D0;
M0←(D7*M1-D5*M2+D3*M3-D1*M4)/D9;
I9←((C8*C8*M0+(C7*C7-2.*C6*C8)*M1+(C6*C6-2.*C5*C7+2.*C4*C8)*M2+(C5*C5-2.*C4*C6+2.*C3*C7-2.*C2*C8)*M3
        +(C4*C4-2.*C3*C5+2.*C2*C6-2.*C1*C7+2.*C0*C8)*M4+(C3*C3-2.*C2*C4+2.*C1*C5-2.*C0*C6)*M5
```

```
        +(C2*C2-2.*C1*C3+2.*C0*C4)*M6+(C1*C1-2.*C0*C2)*M7+C0*C0*M8))/(2.*DEL9);
Integl←I9; Goto L11;
LL10:   A1←D1*D2-D0*D3;
A2←D1*D4-D0*D5;
A3←D3*D4-D2*D5;
A4←D1*D6-D0*D7;
A5←D3*D6-D2*D7;
A6←D5*D6-D4*D7;
A7←D1*D8-D0*D9;
A8←D3*D8-D2*D9;
A9←D5*D8-D4*D9;
A10←D7*D8-D6*D9:
B1←D1*D10;
B2←D3*D10;
B3←D5*D10;
B4←D7*D10;
B5←D9*D10;
M1←A1*((D1*(-A2*B5+A4*B4-A5*A10-A6*B2-2.*A7*A10+A8*A9+A9*B1)
        +D3*(A1*B5+A3*A10+A4*A10-A4*B3+A5*B2+A7*A9-A8*A8-A8*B1-B1*B1)
        +D5*(-A1*B4+A2*B3-A3*A9-A3*B2-A4*A9+A5*A8-A6*A7+A7*A8+A7*B1)
        +D7*(A1*A10-A2*A9+A3*A6+A3*A8+A3*B1+2.*A4*A6+A4*B1-A5*A5-A5*A7-A7*A7)))
        +A2*((D1*(A4*A10+A6*B1-A7*A9+B1*B1)+D3*(-A2*A10-A5*B1+A7*A8+A7*B1)
        +D5*(A2*A9+A3*B1-A4*A8-A4*B1-A7*A7)+D7*(-A2*A6-A2*B1-A3*A7+A4*A5+2.*A4*A7)))
        +A4*((A7*(-D3*A7+D5*A4-2.*D1*B1)+A4*(D3*B1-D7*A4)))+D1*A7*A7*A7;
M2←A1*((D1*(-A5*B4-A7*B4+A8*B3-A10*B1+B1*B3)+D3*(A3*B4+A4*B4-A8*B2+A9*B1-B1*B2)
        +D5*(-A3*B3-A4*B3+A5*B2-A6*B1+A7*B2)
        +D9*(A1*A10-A2*A9+A3*A6+A3*A8+A3*B1+2.*A4*A6+A4*B1-A5*A5-A5*A7-A7*A7)))
        +A2*((D1*(A4*B4-A7*B3)+D3*(-A2*B4+A7*B2+B1*B1)+D5*(A2*B3-A4*B2-A7*B1)
        +D9*(-A2*A6-A2*B1-A3*A7+A4*A5+2.*A4*A7)))
        +A4*((B1*(-D3*A7+D5*A4-D1*B1)-D9*A4*A4))+D1*A7*A7*B1;
M3←A1*((D1*(-A5*B5-A7*B5-B1*B4+B2*B3)+D3*(A3*B5+A4*B5+B1*B3-B2*B2)
        +D7*(-A3*B3-A4*B3+A5*B2-A6*B1+ A7*B2)
        +D9*(A1*B4-A2*B3+A3*B2+A3*A9+A4*A9-A5*A8+A6*A7-A7*A8-A7*B1)))
        +A2*((D1*(A4*B5-B1*B3)+D3*(-A2*B5+B1*B2)+D7*(A2*B3-A4*B2-A7*B1)
        +D9*(-A2*A9-A3*B1+A4*A8+A4*B1+A7*A7)))+A4*((A4*(D7*B1-D9*A7)-B1*B1*D3))+A7*B1*B1*D1;
M4←A1*((D1*(-A8*B5-2.*B1*B5+B2*B4)+D5*(A3*B5+A4*B5+B1*B3-B2*B2)
        +D7*(-A3*B4-A4*B4+A8*B2-A9*B1+B1*B2)
        +D9*(A1*B5+A3*A10+A4*A10-A4*B3+A5*B2+A7*A9-A8*A8-A8*B1-B1*B1)))
        +A2*((D1*(A7*B5-B1*B4)+D5*(B1*B2-A2*B5)+D7*(A2*B4-A7*B2-B1*B1)
        +D9*(-A2*A10-A5*B1+A7*A8+A7*B1)))+A4*((D9*(-A7*A7+A4*B1)+B1*(D7*A7-D5*B1)))+D1*B1*B1*B1;
M5←A1*((D3*(-A8*B5-2.*B1*B5+B2*B4)+D5*(A5*B5+A7*B5+B1*B4-B2*B3)
        +D7*(-A5*B4-A7*B4+A8*B3-A10*B1+B1*B3)
        +D9*(A2*B5-A4*B4+A5*A10+A6*B2+2.*A7*A10-A8*A9-A9*B1)))
        +A2*((D3*(A7*B5-B1*B4)+D5*(B1*B3-A4*B5)+D7*(A4*B4-A7*B3)
        +D9*(-A4*A10-A6*B1+A7*A9-B1*B1)))+A7*((D9*(2.*A4*B1-A7*A7)+B1*(D7*A7-D5*B1)))
        +B1*B1*(D3*B1-D7*A4);
M6←(D2*M5-D4*M4+D6*M3-D8*M2+D10*M1)/D0;
M7←(D2*M6-D4*M5+D6*M4-D8*M3+D10*M2)/D0;
M8←(D2*M7-D4*M6+D6*M5-D8*M4+D10*M3)/D0;
M9←(D2*M8-D4*M7+D6*M6-D8*M5+D10*M4)/D0;
DEL10←D0*(D1*M9-D3*M8+D5*M7-D7*M6+D9*M5);
M0←(D8*M1-D6*M2+D4*M3-D2*M4+D0*M5)/D10;
I10←((C9*C9*M0+(C8*C8-2.*C7*C9)*M1+(C7*C7-2.*C6*C8+2.*C5*C9)*M2+(C6*C6-2.*C5*C7+2.*C4*C8-2.*C3*C9)*M3
        +(C5*C5-2.*C4*C6+2.*C3*C7-2.*C2*C8+2.*C1*C9)*M4+(C4*C4-2.*C3*C5+2.*C2*C6-2.*C1*C7+2.*C0*C8)*M5
        +(C3*C3-2.*C2*C4+2.*C1*C5-2.*C0*C6)*M6+(C2*C2-2.*C1*C3+2.*C0*C4)*M7+(C1*C1-2.*C0*C2)*M8+C0*C0*M9))/(

2.*DEL10);
Integl←I10; Goto L11;
L11:End;
```

```
Program 205;
Procedure Poly(UR,JJ,TD);
Comment Poly multiplies factored polynomials to produce an
        unfactored polynomial of order up to 10 (= JJ)
```

```
        ie.(1+a[1]x)...(1+a[JJ]x) → TD[0] + ... + TD[JJ]*x^JJ;
Value UR,JJ; Real Array UR,TD; Integer JJ;
Begin Integer I,J,LL,MM,JA,IM,IA,IB,IC; Real Array DTD[0:10];
        For I←I While |UR[JJ]|<1D-20 Do Begin TD[JJ]←0;JJ←JJ-1 End;
        For I←1 Step 1 Until JJ Do TD[I]←0.0;
        TD[0]←1.0; TD[10]←1.0;
        For I←1 Step 1 Until JJ Do Begin TD[1]←TD[1]+UR[I];
                                   TD[10]←TD[10]*UR[I];
        For J←I+1 Step 1 Until JJ Do Begin DTD[2]←UR[I]*UR[J];
        For LL←J+1 Step 1 Until JJ Do Begin DTD[3]←UR[LL]*DTD[2];
        For MM←LL+1 Step 1 Until JJ Do Begin DTD[4]←UR[MM]*DTD[3];
        For JA←MM+1 Step 1 Until JJ Do Begin DTD[5]←UR[JA]*DTD[4];
        For IM←JA+1 Step 1 Until JJ Do Begin DTD[6]←UR[IM]*DTD[5];
        For IA←IM+1 Step 1 Until JJ Do Begin DTD[7]←UR[IA]*DTD[6];
        For IB←IA+1 Step 1 Until JJ Do Begin DTD[8]←UR[IB]*DTD[7];
        For IC←IB+1 Step 1 Until JJ Do Begin DTD[9]←UR[IC]*DTD[8];
        TD[9]←TD[9]+DTD[9] End;
        TD[8]←TD[8]+DTD[8] End;
        TD[7]←TD[7]+DTD[7] End;
        TD[6]←TD[6]+DTD[6] End;
        TD[5]←TD[5]+DTD[5] End;
        TD[4]←TD[4]+DTD[4] End;
        TD[3]←TD[3]+DTD[3] End;
        TD[2]←TD[2]+DTD[2] End;End;
        EndEnd;
```

```
Program 207;
Procedure bairsto(n,a,eps0,eps1,eps2,eps3,K,m,x,y,nat,ex);
Value n; Integer n,K,m; Real eps0,eps1,eps2,eps3;
Integer Array ex,nat; Real Array a,x,y;
Comment This Bairstow Hitchcock iteration is used to find
        successively pairs of roots of a polynomial equation
        of degree n with coefficients a[i] (i=0,1,...,n)
        where a[n] is the constant term. On exit from the
        procedure, m is the number of pairs of roots found,
        x[i] and y[i] (i=1,...,m) are a pair of real roots
        if nat[i]=1, the real and imaginary parts of a complex
        pair if nat[i]=-1, and ex[i] indicates which of the
        following conditions was met to exit from the iteration
        loop in finding this pair
          (1) Remainders r1,r0, become absolutely less than eps1.
          (2) Corrections, incrp, incrq, become absolutely less
              than eps2
          (3) The ratios, incrp/p, incrq/q, become absolutely
              less than eps3
          (4) The number of iterations become K. In the last case,
              the pair of roots found is not reliable and no further
              effort to find additional roots is made. The quantity
              eps0 is used as a lower bound fmr the denominator in
              the expressions from which incrp and incrq are found;
Begin Integer i,j,k,n1,n2,m1;
Array b,c[0:n+1]; Real p,q,r0,r1,s0,s1,v0,v1,det0,det1,det2,incrp,
                  incrq,S,T;
If a[0]=0 Then Go to final;
For i ← 0 Step 1 Until n Do
b[i]←a[i]/a[0]; b[n+1]←0; n2←(n+1)$2; n1←2*n2;
For m1 ← 1 Step 1 Until n2 Do Begin
If n1<3 Then Begin ex[m1]←1; p←b[1]; q←b[2]; Go to next End;
p ← 0; q ← 0;
For k ← 1 Step 1 Until K Do Begin
step:    For i ← 0 Step 1 Until n1 Do
c[i] ← b[i];
For j ← n1-2, n1-4 Do Begin
```

```
For i ← 0 Step 1 Until j Do Begin
c[i+1] ← c[i+1]-p*c[i]; c[i+2] ← c[i+2]-q*c[i] End End;
r0 ← c[n1]; r1 ← c[n1-1]; s0 ← c[n1-2]; s1 ← c[n1-3];
v0 ← -q*s1; v1 ← s0-s1*p; det0 ← v1*s0-v0*s1;
If |det0' < eps0 Then Begin p ← p+1; q ← q+1;
Goto step End;
det1 ← s0*r1-s1*r0; det2 ← r0*v1-r1*v0; incrp ← det1/det0;
incrq ← det2/det0; p ← p+incrp; q ← q+incrq;
If |r0' < eps1 Then Begin
If |r1' < eps1 Then Begin ex[m1] ← 1; Goto next End End;
If |incrp' < eps2 Then Begin
If |incrq' < eps2 Then Begin ex[m1] ← 2; Goto next End End;
If |incrp/p' < eps3 Then Begin
If |incrq/q' < eps3 Then Begin ex[m1] ← 3; Goto next End End End;
ex[m1] ← 4;
next:     S ← -p/2; T ← S^2-q;
If ¬(T<0) Then Begin
T ← SQRT(T); nat[m1] ← 1; x[m1] ← S+T; y[m1] ← S-T End;
If T<0 Then Begin
nat[m1] ← -1; x[m1] ← S; y[m1] ← SQRT(-T) End;
If ex[m1]=4 Then Goto out;
For j ← 0 Step 1 Until n1-2 Do Begin
b[j+1] ← b[j+1]-p*b[j]; b[j+2] ← b[j+2]-q*b[j] End;
n1 ← n1-2;
If n1<1 Then Begin
out:     m ← m1; Goto final End;
End; final: End;
```

```
Program 207;
Procedure bairsto(n,a,eps0,eps1,eps2,eps3,K,m,x,y,nat,ex);
Value a,n; Integer n,K,m; Real eps0,eps1,eps2,eps3;
Integer Array ex,nat; Real Array a,x,y;
Comment  This modification 'SR' of bairsto adds a root at .001
         to odd order polynomials to eliminate convergence
         problems when these polynomials have a very small
         root.  Other comments for the above bairsto apply;
Begin Integer i,j,k,n1,n2,m1;
Array b,c[0:n+1]; Real p,q,r0,r1,s0,s1,v0,v1,det0,det1,det2,incrp,
                       incrq,S,T;
Integer Index;  Index←0;  RETURN:
If a[0]=0 Then Go to final;
n2←(n+1)$2; n1←2*n2; b[0]←1;
If n1=n Then For i←1,i+1 While i<n Do b[i]←a[i]/a[0] Else Begin
         For i←1,i+1 While i<n1 Do b[i]←(a[i]+0.001*a[i-1])/a[0];
                Index←1 End;
For m1 ← 1 Step 1 Until n2 Do Begin
If n1<3 Then Begin ex[m1]←1; p←b[1]; q←b[2]; Go to next End;
p ← 0; q ← 0;
For k ← 1 Step 1 Until K Do Begin
step:     For i ← 0 Step 1 Until n1 Do
c[i] ← b[i];
For j ← n1-2, n1-4 Do Begin
For i ← 0 Step 1 Until j Do Begin
c[i+1] ← c[i+1]-p*c[i]; c[i+2] ← c[i+2]-q*c[i] End End;
r0 ← c[n1]; r1 ← c[n1-1]; s0 ← c[n1-2]; s1 ← c[n1-3];
v0 ← -q*s1; v1 ← s0-s1*p; det0 ← v1*s0-v0*s1;
If |det0' < eps0 Then Begin p ← p+1; q ← q+1;
Goto step End;
det1 ← s0*r1-s1*r0; det2 ← r0*v1-r1*v0; incrp ← det1/det0;
incrq ← det2/det0; p ← p+incrp; q ← q+incrq;
If b[n1]=0 Then q←incrq←0;
If |r0' < eps1 Then Begin
If |r1' < eps1 Then Begin ex[m1] ← 1; Goto next End End;
If |incrp' < eps2 Then Begin
```

```
If |incrq' < eps2 Then Begin ex[m1] ← 2; Goto next End End;
If |incrp/p' < eps3 Then Begin
If |incrq/q' < eps3 Then Begin ex[m1] ← 3; Goto next End End End;
ex[m1] ← 4;
next:     S ← -p/2; T ← S²-q;
If ¬(T<0) Then Begin
T ← SQRT(T); nat[m1] ← 1; x[m1] ← S+T; y[m1] ← S-T End;
If T<0 Then Begin
nat[m1] ← -1; x[m1] ← S; y[m1] ← SQRT(-T) End;
If ex[m1]=4 Then Goto out;
For j ← 0 Step 1 Until n1-2 Do Begin
b[j+1] ← b[j+1]-p*b[j]; b[j+2] ← b[j+2]-q*b[j] End;
n1 ← n1-2;
If n1<1 Then Begin
out:    m ← m1; Goto final End;
End; final:
If Index=1 Then Begin
        For i←1 Step 1 Until m Do Begin
        If nat[i]=1 Then Begin
        p←x[i]+0.001;  If |p|<0.0001 Then Begin x[i]←0; Go to L1 End;
        q←y[i]+0.001; If |q|<0.0001 Then Begin y[i]←0; Go to L1 End End End End; L1:End;
```

```
Program 207;
Procedure bairsto(n,a,eps0,eps1,eps2,eps3,K,m,x,y,nat,ex);
Value a,n;  Integer n,K,m;  Real eps0,eps1,eps2,eps3;
Integer Array ex,nat; Real Array a,x,y;
Comment  This modification 'SS' of bairsto returns to pick a new
         p and q if convergence fails and ex = 4; thus it finds
         a different set of roots and then returns to find the
         smallest set.  This modification was the most success-
         ful form of bairsto to find roots under difficult conditions
         but it was very time consuming since K iteerations were
         required before it tried a second set of roots.  The other
         comments in the above bairsto apply;
Begin Integer i,j,k,n1,n2,m1;
Array b,c[0:n+1]; Real p,q,r0,r1,s0,s1,v0,v1,det0,det1,det2,incrp,
                       incrq,S,T;
Integer Index;  Index←0;  RETURN:
If a[0]=0 Then Go to final;
For i ← 0 Step 1 Until n Do
b[i]←a[i]/a[0]; b[n+1]←0; n2←(n+1)$2; n1←2*n2;
For m1 ← 1 Step 1 Until n2 Do Begin
If n1<3 Then Begin ex[m1]←1; p←b[1]; q←b[2]; Go to next End;
p←q←0; NOW:
For k ← 1 Step 1 Until K Do Begin
step:     For i ← 0 Step 1 Until n1 Do
c[i] ← b[i];
For j ← n1-2, n1-4 Do Begin
For i ← 0 Step 1 Until j Do Begin
c[i+1] ← c[i+1]-p*c[i]; c[i+2] ← c[i+2]-q*c[i] End End;
r0 ← c[n1]; r1 ← c[n1-1]; s0 ← c[n1-2]; s1 ← c[n1-3];
v0 ← -q*s1; v1 ← s0-s1*p; det0 ← v1*s0-v0*s1;
If |det0' < eps0 Then Begin p ← p+1; q ← q+1;
Goto step End;
det1 ← s0*r1-s1*r0; det2 ← r0*v1-r1*v0; incrp ← det1/det0;
incrq ← det2/det0; p ← p+incrp; q ← q+incrq;
If |r0|<eps1 ∧ |r1|<eps1 Then Begin ex[m1]←1; Go to next End;
If |incrp|<eps2 ∧ |incrq|<eps2 Then Begin ex[m1]←2; Go to next End;
If |incrp/p|<eps3 ∧ |incrq/q|<eps3 Then Begin ex[m1]←3; Go to next End End;
If Index=0 Then Begin p←2*b[1]/(n-1); q←(p/2)²; Index←1; Go to NOW End;
ex[m1] ← 4;
next:     S ← -p/2; T ← S²-q;
If ¬(T<0) Then Begin
T ← SQRT(T); nat[m1] ← 1; x[m1] ← S+T; y[m1] ← S-T End;
```

```
If T<0 Then Begin
nat[m1] ← -1; x[m1] ← S; y[m1] ← SQRT(-T) End;
If ex[m1]=4 Then Goto out;
For j←0 Step 1 Until n1-2 Do Begin
b[j+1] ← b[j+1]-p*b[j]; b[j+2] ← b[j+2]-q*b[j] End;
n1←n1-2; If Index=1 Then Begin Index←0; p←q←0 End;
If n1<1 Then Begin
out:     m ← m1; Goto final End;
End ; final: End;
OO
```

```
Program 202;
Procedure Cancel(N,M,X,Y,p,q,sig,tau,m,n,s,k,u,j,v);
Comment  (Special purpose program) This program sets up
         criteria for the cancellation of poles and zeros
         for Program 276.
         Input is fraction (X[1]*X[2]*...*X[N])/(Y[1]*Y[2]*...*Y[M].
           Desired output is (u[1]*u[2]*...*u[k])/(v[1]*v[2]*...*v[j]).
        Input data:
        N,M,x[1:N],y[1:M] are self explanatory, (N<M)
        p←desired (maximum) number of output poles,
(2)     q←index of a certain poles used below,
(0.01)  sig1←ratio of y[i]/y[q],i=1,...,M for which y[i] is small
        enough to be neglected and summed into an exponential time lag.
(0.5)   sig2←fraction by which similar small x[i] may exceed y[i] and
        still be cancelled against them.
        Criteria on cancellation is that the ratio of cancelling poles
        and zeros be near to one.  The initial allowed deviation from
        one is sig3.  Subsequent criteria are (1+sig3)*minimum ratio.
        Outputs:
        tau←exponential time lag, positive for lag,
        m,n←number of poles and zeros after combination of small time
        constants into tau,
        s←m-p←number of poles to be cancelled by same number of zeros,
        u[1:k],v[1:j]←non-cancelled poles and zeros- the desired function;
Value N,M,X,Y,p,q,sig;
Integer m,n,p,q,s,N,M,j,k;              Real tau;
Real Array X,Y,u,v,sig;
Begin Integer e,f,g,h,i,l,t;            Integer Array beta[0:16,0:17];
Real test;                              Real Array x[0:15],y[0:15],gamma[0:16,0:17];
McProcedure Order(200,1,3);
McProcedure Cancelp(206,1,11);
Procedure check(crit);
Real crit; Begin
        test←1D8;i←1;For g←1 Step 1 Until m Do Begin
                If beta[i,g]=1 Then i←i+1 Else Begin
                If beta[i,g]=0 Then Begin If g=m Then Begin
                If |gamma[i,g]|<test Then test←|gamma[i,g]| End End Else Begin
                If ¬(beta[i-1,g-1]=1)∧|gamma[i,g-1]|<test∧2<g Then
                test←|gamma[i,g-1]|;
                For f←i While f<n∧¬(beta[f+1,g]=0) Do i←i+1;
                If ¬(beta[i,g]=1)∧|gamma[i,g]|<test Then
                test←|gamma[i,g]|; i←i+1;
                End;crit←test*0.99;End;
                If i<n Then Else g←m+1;End End;
Order(N,X,x);Order(M,Y,y);
j←N+1;tau←0;m←p;   For k←M Step -1 Until p+1 Do Begin
If y[k]/y[q] <sig[1] Then Begin tau←tau+y[k];j←j-1 End Else Begin
        m←k; Goto next1; End End;
next1:  t←M;   For l←N Step -1 Until j Do Begin
        If l<0 Then Begin n←0; Goto next2; End;
        If x[l]<y[t]*(1+sig[2]) Then Begin n←l-1;
        tau←tau-x[l];t←t-1End Else Begin
```

```
        n←1; Go to next2;End End;
next2:  M←m;N←n; s←m-p;
        e←3;
repeat:Cancelp(N,x,M,y,sig[3],u,k,v,j,beta,gamma);
        If e<3Δm-j Then Begin sig[3]←sig[3]*1.06;
                e←e+1; Go to repeat;End;
        If j<p Then Go to out;
        If k=0 Then Go to out; check(sig[3]);N←k;M←j;
n←N;m←M;e←1;
        For i←1 Step 1 Until j Do Begin x[i]←u[i];y[i]←v[i] End;
        Go to repeat; out:End;
```

```
Program 203;
Procedure Factor(K,U,eps0,eps1,eps2,eps3,k,dex,RAT,UR,nat,ex,CI,DTD,I);
Comment (Special purpose program for Program 201).
        This procedure factors polynomials of order K with coefficients
        U where U[K] is the coeficient of the highest term and
        u[0] is the constant term.  It automatically checks
        to see if the highest order term is greater than dex -
        otherwise it returns a lower order polynomial.   I is the
        number of real roots. The answers are as (s-a) etc. for CI
        and DTD, or as (as-1) etc.for UR. For real roots that are
        near to one another the bairsto routine
        often gives a small imaginary part.  If these are small
        ie. if Im/Real < RAT, then these are eliminated in the
        procedure for finding UR.
        Complex roots are given as coefficients of a quadratic factor
        ie, 1+DTD*S+CI*S^2 where there are K sets of complex roots.
        Bairsto 207 is called to perform the actual factorization;
Value U,eps0,eps1,eps2,eps3,RAT,k;   Real eps0,eps1,eps2,eps3,RAT,dex;
Real Array CI,DTD,U,UR;              Integer K,k,I;     Integer Array nat,ex;
Begin Real ESQ;                      Real Array DI[0:10];            Integer J,MM,LP,IM;
        Mc Procedure bairsto(207,1,12);
I←-1;For J←K While |U[J]|<dex Do Begin K←K-1;If K<0 Then U[K]←1End;
        For J←K Step -1 Until 0 Do Begin I←I+1;DI[I]←U[J];End;J←K;
        If I<0 Then Begin I←0;MM←0;Go to rep; End;
        bairsto(I,DI,eps0,eps1,eps2,eps3,k,MM,CI,DTD,nat,ex);
rep:
For K←1 Step 1 Until MM Do Begin If ex[K]=4 Then Begin K←-4;Go to PL End End;
nat[MM+1]←2;K←0;k←1;T:For J←k Step 1 Until MM Do Begin
        If nat[J]=-1 Then Begin
        If RAT<|DTD[J]/CI[J]| Then Begin
        K←K+1;CI[MM+1]←CI[J]^2+DTD[J]^2;DTD[MM+1]←2*CI[J];I←I-2;
        For IM←J Step 1 Until MM Do Begin
        CI[IM]←CI[IM+1];DTD[IM]←DTD[IM+1];nat[IM]←nat[IM+1] End;k←J;Goto T End
        Else DTD[J]←CI[J] End End; ESQ←DTD[MM-K];
        LP←1;
        For IM←1 Step 1 Until MM-K Do Begin If |ESQ'<|CI[IM]'Then U[LP]←CI[IM]Else Begin
        U[LP]←ESQ; ESQ←CI[IM] End;
        If |ESQ'<|DTD[IM]' Then U[LP+1]←DTD[IM] Else Begin U[LP+1]←ESQ; ESQ←DTD[IM]End;
            LP←LP+2 End; U[2*(MM-K)]←ESQ;
        For IM←1 Step 1 Until I Do UR[IM]←1.0/U[IM];
        For IM←I+1 Step 1 Until 2*MM Do UR[IM]←0.0;
        PL:End End;
```

```
Program 206;
Procedure Cancelp(p,N,q,D,crit,nn,jn,dd,jd,beta,gamma);
Comment  (Special purpose program for Program 201).
         .This procedure cancels poles and zeros for
         which the ratio of |N/D -1|< crit.
         N is the list of p zeros- D is the list of
         q poles: nn is the list of not cancelled jn
         poles and dd is the list of not cancelled
         jd poles;
Integer p,q,jn,jd; Array N,D,nn,dd,gamma; Real crit; Integer Array beta;
Begin
Integer i,j,k; Array n[1:15],d[1:15],alpha[0:16,0:17];
McProcedure Order(200,1,3);
Order(p,N,n);Order(q,D,d);
For i←0 Step 1 Until p+2 Do Begin
For j←0 Step 1 Until q+2 Do Begin
alpha[i,j]←0;beta[i,j]←0;gamma[i,j]←1 End; End;i←p;
L1:For j←q Step -1 Until 1 Do Begin If beta[i+1,j]=0 Then Begin
If d[j]<1D-30 Then alpha[i,j]←n[i]/1D-30 Else
alpha[i,j]←n[i]/d[j];gamma[i,j]←alpha[i,j]-1;
If crit<gamma[i,j] Then beta[i,j]←-1 Else Begin
If |gamma[i,j]|<crit Then beta[i,j]←1;i←i-1;
If 0<i Then Go to L1 Else Go to L2;End End
          Else beta[i,j]←-1;End;L2:
For i←1 Step 1 Until p Do nn[i]←n[i];
For j←1 Step 1 Until q Do dd[j]←d[j];
For i←1 Step 1 Until p Do Begin For j←1 Step 1 Until q Do Begin
If 0<beta[i,j] Then Begin
          nn[i]←0;dd[j]←0 End;End;End;
jn←0; For i←1 Step 1 Until p Do Begin If nn[i]=0 Then Else
Begin jn←jn+1;nn[jn]←nn[i] End;End;
jd←0; For i←1 Step 1 Until q Do Begin If dd[i]=0 Then Else
Begin jd←jd+1;dd[jd]←dd[i] End;End;
 End;7
```

```
Program 271;
Procedure Z over Y (sig4,AMB1S,AMB2S,ALP,SIG,
          RD,RM,EEO,EEK,N,D,bet,R1,R2,EMUS,EKD,EM,
          EPS,eps0,eps1,eps2,eps3,k,dex,RAT,
          y,yr,gg,gr,J,EKM);
Comment  (Specialized program to find function [Z/Y]+ in Program 201);
Value sig4,AMB1S,AMB2S,ALP,SIG,RD,EKM,
      RM,EEO,EEK,R1,R2,EPS,eps0,EMUS,EKD,EM,
      eps1,eps2,eps3,dex,RAT,k;
Real sig4,AMB1S,AMB2S,ALP,SIG,RD,
     RM,EEO,EEK,R1,R2,EPS,eps0,eps1,EMUS,EKD,EM,
          eps2,eps3,dex,RAT,EKM;
Array y,yr,gg,gr,D,bet;
Integer k,J,N;
Begin
Array CI,DI,DTD,gam[0:15],X[0:1,0:1];
McProcedure Product(175,1,13);

Integer Array nat,ex[0:10];
Real gal,gig,gr1,gr2,a,ALPS,SIGS,RDS,RMS,R1S,R2S;
Integer I,MM,P;
Real Procedure g(S);
          Real S;
          Begin Real denom; Integer l;
          denom←1.0;
          For l←1 Step 1 Until 7 Do denom←denom*(1-yr[l]*S);
          For I←1 Step 1 Until N Do Begin
          denom←denom*(D[I,2]*S^2-D[I,1]*S+1) End;
          g←EMUS*EXP(EM*S)*EKD*(1-RDS*S^2)*(1-RM*S)*
```

```
        (EEO-EEK*S^2)/denom;
       End;
McProcedure Factor(203,1,12);
McProcedure Poly(205,1,3);
ALPS←ALP^2; SIGS←SIG^2;RDS←RL^2;RMS←RM^2;R1S←R1^2;R2S←R2^2;
EPS←EPS/EKD^2;AMB2S←AMB2S/EKM^2;
gam[0]←ALPS;gam[1]←1.0+ALPS*(R1S+R2S);
gam[2]←R1S+R2S+ALPS*R1S*R2S;gam[3]←R1S*R2S;
bet[0]←AMB2S*gam[0]+AMB1S;bet[1]←AMB2S*gam[1]+AMB1S*RMS;
bet[2]←AMB2S*gam[2];bet[3]←AMB2S*gam[3];
Comment LAMBA1^2+LAMBA2^2/PmPm=(bet0-bet1 S^2+bet2 S^4-bet3 S^6)/(1-RM^2S^2);
CI[1]←RMS; CI[2]←RDS; CI[3]←EEK/EEO; Poly(CI,3,CI);
Product(0,3,X,gam,0,3,X,bet,0,X,DI); gal←1/SIGS;
For I←1,2,3,4,5,6,7 Do DTD[I]←DI[I]+DI[I-1]*gal; DTD[0]←DI[0];

gal←EMUS*EEO; gig←SIGS*EPS;
For I←0,1,2,3,4,5,6,7 Do y[I]←gal*CI[I]+gig*DTD[I];
J←7;gg[8]←y[0];
Factor(J,y,eps0,eps1,eps2,eps3,k,dex,RAT,yr,nat,ex,CI,DTD,MM);
N←J;
For  I←MM+1 Step 1 Until 7 Do yr[I]←0;
If J=-4 Then Goto PL;
For J←1,2,3,4,5,6,7 Do Begin
If 0<yr[J] Then yr[J]←0.00001 Else Begin a←-yr[J]; yr[J]←SQRT(a) End End; Poly(yr,7,y);
         For I←1 Step 1 Until N Do Begin
         P←(MM+2*I+1)$2;  D[I,0]←1; gal←1/CI[P];
         If gal<0 Then Begin D[I,1]←D[I,2]←0 End Else Begin
         D[I,2]←SQRT(gal); gig←2*D[I,2]-DTD[P]*gal;
         If 0<gig Then D[I,1]←SQRT(gig) Else Begin D[I,1]←D[I,2]←0 End End;
         Product(0,7,X,y,I,2,D,DI,0,X,y) End;
gal←g(-ALP)/((SIG-ALP)*(1-R1*ALP)*(1-R2*ALP));
gig←g(-SIG)/((ALP-SIG)*(1.0-R1*SIG)*(1.0-R2*SIG));
If R1<sig4 Then gr1←0 Else
Begin a←1/R1; gr1←g(-a)/((ALP-a)*(SIG-a)*(1-R2*a)) End;
If R2<sig4 Then gr2←0 Else
Begin a←1/R2;gr2←g(-a)/((ALP-a)*(SIG-a)*(1-R1*a)) End;
R1S←R1+R2; R2S←R1*R2; ALPS←ALP*SIG; SIGS←ALP+SIG;
gg[0]←gal*SIG+gig*ALP+(gr1+gr2)*ALPS;
gg[1]←gal*(SIG*R1S+1)+gig*(ALP*R1S+1)+gr1*(R2*ALPS+SIGS)+gr2*(R1*ALPS+SIGS);
gg[2]←gal*(SIG*R2S+R1S)+gig*(ALP*R2S+R1S)+gr1*(R2*SIGS+1)+gr2*(R1*SIGS+1);
gg[3]←(gal+gig)*R2S+gr1*R2+gr2*R1;
J←3; Factor(J,gg,eps0,eps1,eps2,eps3,k,dex,RAT,gr,nat,
ex,CI,DTD,MM);
For I←MM+1 Step 1 Until 3 Do gr[I]←0;
For I←1,2,3 Do gr[I]←-gr[I];
If J=1 Then Begin D[6,1]←CI[2];D[6,2]←DI[2];End;
PL:      End;0


Program 275;
Procedure WoverU(w,h2,h3,h4,ur,yr,m,hr,lam,sig,alp,rd,r1,r2,kd,em,ed,ee,x);
Comment  (Special purpose program computes [W/U]+ for Program 201)
         output:
                   ee[0:8]←coefficients of numerator polynomial,
                   x[1:9]←time constants of denominator,
                   res[1:9]←residues at each time constant;
Value w,h2,h3,h4,ur,hr,lam,sig,alp,rd,r1,r2,em,ed;
Real Array w,h2,h3,h4,ur,hr,ee,x,yr;
Integer m;
Real lam,sig,alp,rd,r1,r2,em,ed,kd;
Begin Real Array v[0:9],tot[0:11],y[0:10],res[1:9],z[1:9,0:8];
Real s,sq,a;
Integer i,j,k,n;
McProcedure Poly(205,1,3);
```

```
For j←1,2,3,4,5,6,7,8 Do Begin
        If i=9 Then y[j]←z[1,j];
If y[j+1]=0 Then z[i,j]←0 Else z[i,j]←y[j]-x[i]*z[i,j-1] End End;
End;
For i←0,1,2,3,4,5,6,7,8 Do Begin ee[i]←0;
For j←1,2,3,4,5,6,7,8,9 Do ee[i]←res[j]*z[j,i]+ee[i] End;
ee[9]←0;  If ¬(yr[10]=0) Then
Begin
For j←1,2,3,4,5,6,7,8,9 Do tot[j]←ee[j]+yr[10]*ee[j-1];
For j←1,2,3,4,5,6,7,8,9 Do ee[j]←tot[j] End;
End;
0
```

```
Program 276;
Procedure Reduce(N,J,d,gr,yr,H,EM,tau,sig,HK,hr);
Comment   (Special purose program).  This program reduces the
          order of the rational function 1-sumG1,i/sumG2,j
          so that the order of the denominator is ≤4.
          Parameters:
          N← No. of complex roots in denominator,
          J← No. of complex root sets in numerator,
          d← Coefficients of complex roots as a quad-
          ratic term - d[i,2]s^2+d[i,1]s+1
                    i is 6 for numerator
                    i is 1,...,5 for denominator
          gr← real roots of numerator
          yr← real roots of denominator
          H← polynomial coefficients
                    i=1 for numerator
                    i=2 for denominator
          EM← time lag
          tau←  added time delay from small time constants
          sig← limits for Cancel
          HK← constant = TCK/H[2,0]
          hr← factors of H[2,i];
Value d,gr,yr,EM;
          Array d,gr,hr,yr,H,sig;
          Real EM,tau,HK;
          Integer N,J;
Begin     Array DI,X,u,v[0:10],D[0:6,0:2];
          Integer i,p,R,kk,k,jj,j,m,n,s,r,q;
          McProcedure Cancel(202,1,20);
          McProcedure Order(200,1,3);
          McProcedure Poly(205,1,3);
          McProcedure product(175,1,13);
If |HK|<.005 Then Begin For i←1,2,3,4 Do
          H[1,i]←H[2,i]←hr[i]←0;
          H[1,0]←H[2,0]←1;Go to PL;End;
R←0; For i←1,2,3,4,6 Do For p←0,1,2 Do D[i,p]←d[i,p];
If¬(N=0)∧¬(J=0) Then Begin R←2; J←0;gr[2]←gr[4];
          For i←1 Step 1 Until N Do
          X[i]←|1-|D[i,2]/D[6,2]||; X[0]←1D6;
          For i←1 Step 1 Until N Do Begin
          If X[i]<X[i-1] Then q←i; End; N←N-1;

          For j←q Step 1 Until N Do Begin
          For k←1,2 Do D[j,k]←D[j+1,k]; End End;
If (N=0)∧¬(J=0) Then gr[2]←gr[3]←SQRT(D[6,2]);
D[0,2]←0; D[5,0]←1; p←0;
For i←1 Step 1 Until N Do Begin p←2;
          If D[i-1,2]<D[i,2] Then Begin
                    D[5,2]←D[i,2]; D[5,1]←D[i,1];
                    yr[9-i]←yr[8-i]←SQRT(D[i-1,2]) End
          Else yr[9-i]←yr[8-i]←SQRT(D[i,2]) End;
```

```
Procedure v123(v,ss);
        Real ss;                    Real Array v;
        Begin
        s←ss; sq←s²; tot[1]←1; tot[2]←tot[3]←tot[4]←tot[5]←tot[11]←0;
For i←1 Step 1 Until 8-2*m Do tot[1]←tot[1]*(1-ur[i]*s);
For i←m*2 Step -1 Until 0 Do tot[11]←yr[i]-s*tot[11];
tot[1]←tot[1]*tot[11];
        For i←4,3,2,1,0 Do Begin
        tot[2]←h2[i]-s*tot[2];
        tot[3]←h3[i]+sq*tot[3];
        tot[4]←h4[i]+sq*tot[4];
        tot[5]←w[i]+s*tot[5] End;
        v[1]←kd*(1+rd*s)*(alp-s)*EXP((em-ed)*s)*tot[2]/tot[1];
        v[1]←v[1]*(1-s*yr[10]);
        v[2]←tot[3]*tot[5]/((alp-s)); v[3]←tot[4];
        v[7]←lam+v[2]/(v[3]*(alp+s)); End;
Procedure v459(v,ss);
        Real ss; Real Array v;
        Begin Real s;
        s←ss;  v[4]←1;
        v[5]←(s+alp)²*(sig+s)*(1+r1*s)*(1+r2*s)*tot[2];
        For i←1,2,3,4 Do
                If ¬(i=n) Then v[4]←v[4]*(1+hr[i]*s) End;
Procedure v67(v,ss);
        Real ss;                    Real Array v;
        Begin Real s,ii;
        s←ss;  sq←s*s;  For i←1,6,7,8,9,10 Do tot[i]←0;
        For i←1 Step 1 Until 8-m*2 Do tot[1]←tot[1]+ur[i]/(1-ur[i]*s);
        For i←m*2 Step -1 Until 1 Do tot[10]←i*yr[i]-s*tot[10];
        tot[1]←tot[1]+tot[10]/tot[11];
        tot[1]←tot[1]-r1/(1+r1*s)-r2/(1+r2*s)-1/(sig+s);
        For i←4,3,2,1 Do Begin
        tot[6]←i*h2[i]-s*tot[6];
        tot[7]←i*w[i]+s*tot[7];
        tot[8]←i*h3[i]+sq*tot[8];

        tot[9]←i*h4[i]+sq*tot[9] End;
        tot[10]←em-ed+rd/(1+rd*s)-1/(alp-s)-yr[10]/(1-yr[10]*s)
                -tot[6]/tot[2]+tot[1];
        tot[11]←tot[7]/tot[5]+2*s*tot[8]/tot[3]
             +1/(alp-s)-2*s*tot[9]/tot[4];
        v[6]←v[1]*(lam+(tot[10]+tot[11])*v[2]/v[3]);
        v[7]←(sig+s)*(1+r1*s)*(1+r2*s); End;
v123(v,-sig);
res[8]←v[1]*v[7]/((alp-sig)*(1-r1*sig)*(1-r2*sig)*sig);
If 0<r1 Then Begin a←-1/r1; v123(v,a);
res[2]←v[1]*v[7]/((alp+a)*(sig+a)*(1.+r2*a)); End
Else res[2]←0;
If 0<r2 Then Begin a←-1/r2; v123(v,a);
res[3]←v[1]*v[7]/((alp+a)*(sig+a)*(1+r1*a))End
Else res[3]←0;
If hr[3]=0 ∧ hr[4]=0 Then Begin
s←h2[1]-hr[1];  sq←h2[2]-hr[1]*s;
s←s-hr[2];  sq←sq-hr[2]*s;
hr[3]←s/2;  hr[4]←SQRT(sq) End;
For n←1,2,3,4 Do Begin
If 0<hr[n] Then Begin a←-1/hr[n]; v123(v,a); v459(v,a);
res[n+3]←v[1]*v[2]/(v[4]*v[5]);
End Else res[n+3]←0; End;
v123(v,-alp); v67(v,-alp);
res[9]←v[1]*v[2]/(v[3]*v[7]*alp²);
res[1]←v[6]/(v[7]*alp);
x[1]←x[9]←1/alp; x[8]←1/sig;
x[2]←r1; x[3]←r2; x[4]←hr[1];
x[5]←hr[2]; x[6]←hr[3]; x[7]←hr[4];
Poly(x,9,y);
For i←2,3,4,5,6,7,8,1,9 Do Begin z[1,0]←1;
If ¬(res[i]=0) Then Begin
```

```
yr[8-p-R]←EM/2;  q←0;
For i←1,2,3,4 Do Begin
        If gr[i]<0 Then Begin
        gr[i+5]←gr[i]; gr[i]←0; q←q+1 End Else
                gr[i+5]←0; End;
Cancel (4-R,8-p-R,gr,yr,4-p,2,sig,tau,m,n,s,k,u,j,v);
For kk←k+1 Step 1 Until 4 Do u[kk]←0;
For jj←j+1 Step 1 Until 4 Do v[jj]←0;
X[0]←1; X[1]←-EM/2;
Poly(u,3,DI); product(0,3,D,DI,0,1,D,X,0,D,X);
For i←1 Step 1 Until q Do Begin gr[1]←gr[i+5];
gr[0]←1;product(0,4,D,X,0,1,D,gr,0,D,X);End;
If p=2 Then Begin v[3]←D[5,1]/2; v[4]←SQRT(D[5,2]) End;
Poly(v,4,DI);
For i←0,1,2,3,4 Do Begin hr[i]←v[i];
        H[2,i]← DI[i]; H[1,i]←H[2,i]-HK*X[i] End; PL:End;
```

Some sample data for Program 201:

0.25	0.0	0.0	0.0	0.0			
0.0	0.25	0.0	0.0	0.0			
0.0	0.0	0.25	0.0	0.25			
0.0	0.0	0.0	0.25	0.0			
0.0	0.0	0.25	0.0	0.50			
1.	1.	0.	0.	0.	0.	0.0	0.0
2.0	1.	25.	1.5	1	1D-4		
100	1D-30	.02	1D-8	1D-8	1D-8	1D-8	
.001	.5	.05	1D-10	12			
.01	.001	7	5	4.	4.		