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INTEGRATION THEORY IN A BANACH SPACE AND APPLICATIONS

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TABLE OF CONTENTS

CHAPTER IV

GENERALIZED DIFFERENTIAL EQUATIONS

INTEGRATION THEORY IN A BANACH SPACE AND APPLICATIONS

TO GENERALIZED DIFFERENTIAL EQUATIONS

CHAPTER I

INTEGRATION THEORY IN BANACH SPACES

1. Introduction. In 1955 MacNerney [6] presented a theory of integration in a Banach space patterned after that first introduced by Gowurin [3] and proceeded to develop a theory of harmonic operators which was a generalization of the theory of harmonic matrices presented by Wall [9] in 1954. Again generalizing the results in [9], MacNerney used harmonic operators to examine generalized differential equations in Banach spaces. One of his principal results was a local existence theorem for the solution of a generalized non-linear Riccati differential equation (see [6; Theorem 4.1]).

The principal objective of this paper is to obtain results concerning generalized differential equations in a Banach space and examine the global behavior of the solution of a certain non-linear Riccati generalized differential equation. Many of the theorems on generalized differential equations in a Banach space are generalizations of results in [6]. A theory of integration of point-interval functions is presented in the beginning of the paper in order to obtain a more general concept of integration in a Banach space and to provide a firm foundation for the subsequent theory. The concept of integration presented contains as special cases the integrals presented by Gowurin [3], MacNerney [6], Bochner and Taylor [1], and Hille and Phillips [5; pp. 62-67].

Since substitution theorems play a very important role in the study of generalized differential equations, Chapter I contains three substitution theorems, while two more substitution theorems are presented in Chapter II. A very useful integration by parts theorem is also introduced in Chapter II. Iterated integrals in a Banach space are discussed in Chapter III, and finally in Chapter IV much of the theory developed in the first three chapters is used in considering generalized differential equations in a Banach space. One of the principal results in Chapter IV is a theorem concerning the global behavior of the solution of a non-linear Riccati generalized differential equation.

2. General theory of integration in a Banach space. Following McShane [7], a partition of an interval $[a,b]$ will be any collection $\{(\chi_i,\{x_{i-1},x_i\})_{i=1}^n\}$ of ordered pairs $(\chi_i,\{x_{i-1},x_i\})$, where $x_{i-1} \in [a,b]$, $x_i \in [a,b]$, $x_{i-1} \leq x_i$, and $\chi_i \in [x_{i-1},x_i]$. If $\pi = \{(\chi_i,\{x_{i-1},x_i\})_{i=1}^n\}$ is a partition of $[a,b]$, then

$$\|\pi\| = \max_{i=1}^{n}(x_i - x_{i-1})$$

will be called the norm of π. Let $\mathcal{P}$ be the collection of all partitions of a closed interval $[a,b]$. Two relations, $\geq$ and $\succeq$, may now be defined on $\mathcal{P}$ as follows. If $\pi_1 = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^n\} \in \mathcal{P}$ and $\pi_2 = \{(\sigma_j,\{s_{j-1},s_j\})_{j=1}^m\} \in \mathcal{P}$, then

(a) $\pi_1 \geq \pi_2$ if $\|\pi_1\| \leq \|\pi_2\|$,

and

$$\text{(b)}\quad \pi_1 \gtrsim \pi_2 \text{ if } \{t_i\}_{i=0}^{n} \supseteq \{s_j\}_{j=0}^{m}.$$

Both $(\mathcal{P},\geq)$ and $(\mathcal{P},\gtrsim)$ are directed sets, but since $(\mathcal{P},\geq)$ will occur most frequently in this paper, it will be referred to as <u>the partition collection on</u> $[a,b]$.

If X is a Banach space, then a <u>point-interval function</u> on $[a,b]$ into X will be any function which assigns a value in X to each ordered pair $(\tau,\{s,t\})$ such that $[s,t]$ is a closed subinterval of $[a,b]$ and $\tau \in [s,t]$. The concept of the integral of a point-interval function may now be introduced. If $(\mathcal{P},>)$ is the partition collection on a closed interval $[a,b]$ and f is a point-interval function on $[a,b]$ into a Banach space X, then f is <u>integrable</u> over $[a,b]$ if

$$\lim_{\pi \in \mathcal{P}} \Sigma_{\pi} f(\chi_i,\{x_{i-1},x_i\})$$

exists, and in this case, we write

$$\int_a^b f = \lim_{\pi \in \mathcal{P}} \Sigma_{\pi} f(\chi_i,\{x_{i-1},x_i\}),$$

where $\int_a^b f$ is read "the integral from a to b of f" or "the integral of f over $[a,b]$".

Of course, for a given point-interval function f on a closed interval $[a,b]$ into a Banach space X, the integral $\int_a^b f$ exists if and only if the net N defined on the partition collection $(\mathcal{P},\geq)$ on $[a,b]$ by

$$\text{(2.1)}\qquad N(\pi) = \textstyle\sum_{i=1}^{n} f(\tau_i,\{t_{i-1},t_i\})$$

for each $\pi = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^{n}\} \in \mathcal{P}$, is a Cauchy net in X; that is, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\pi_\alpha \in \mathcal{P}$, $(\alpha = 1,2)$, $\|\pi_1\| < \delta$, and $\|\pi_2\| < \delta$, then

$$\|N(\pi_1) - N(\pi_2)\| < \varepsilon.$$

However, many proofs can be simplified considerably using the following modification of the Cauchy condition for convergence.

THEOREM 2.1. Suppose f is a point-interval function on a closed interval [a,b] into a Banach space X and $(\mathcal{P},>)$ is the partition collection on [a,b]. Let N be the net on $(\mathcal{P},>)$ defined by (2.1) above. Then $\int_a^b f$ exists if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $\pi_1 \in \mathcal{P}$, $\|\pi_1\| < \delta$, $\pi_2 \in \mathcal{P}$, and $\pi_2 \gtrsim \pi_1$, then

$$|N(\pi_1) - N(\pi_2)| < \varepsilon.$$

Without making any additional assumptions, one can prove a great number of results for integration of point-interval functions with values in a Banach space which hold for the integrals of so-called real-valued "interval" functions. Burkill [2] first considered the possibility of taking integrals of real-valued interval functions. A thorough treatment of integrals of real-valued interval functions may be found in Hildebrandt [4], and corresponding theorems for integrals of point-interval functions having values in a Banach space will be used in many cases without further comment.

Since many point-interval functions on a fixed interval [a,b] into a given Banach space X may be independent of the point τ in $(\tau,\{s,t\})$, in such cases a partition $\{(\tau_i,\{t_{i-1},t_i\})_{i=1}^n\}$ of [a,b] will sometimes be denoted merely by $\{t_i\}_{i=0}^n$.

3. A special type of integral. Suppose now that X and Y are Banach spaces and [X,Y] is the Banach space consisting of all continuous linear mappings from X into Y with the usual norm being employed, and suppose f

is a point-interval function on [a,b] into [X,Y]. Then one has two choices in defining the integral of f over [a,b]: (a) The integral can be required to exist in the sense that

$$\lim_{\pi \in \mathcal{P}} \Sigma_{\pi} f(\chi_i,\{x_{i-1},x_i\})$$

exists as a limit taken in the space [X,Y] with its uniform topology; or, (b) For each $x \in X$, let f_x be the point-interval function on [a,b] into Y defined by

$$f_x(\tau,\{s,t\}) = f(\tau,\{s,t\})(x).$$

Now say that $\int_a^b f$ exists if $\int_a^b f_x$ exists for each $x \in X$. The function $\int_a^b f\colon X \to Y$ defined by $(\int_a^b f)(x) = \int_a^b f_x$ can then be shown to be an element of [X,Y] using the principal of uniform boundedness (see [8, pp. 204-205]).

It can be shown that if $\int_a^b f$ exists in the sense of (a), then the same integral exists in the sense of (b) and the two integrals are equal. But it is possible to have $\int_a^b f$ exist in the sense of (b) and yet not exist in the sense of (a). Therefore, from this point on, the expression "$\int_a^b f$ exists" will be used to designate that the integral exists in the sense of (b), while "$\int_a^b f$ exists in [X,Y]" will be used to denote the fact that the integral exists in the sense of (a).

4. A more classical form of integral. Now terminology will be introduced which will remain fixed throughout the remainder of this paper.

S will denote a Banach space and B will denote the Banach space of all continuous linear transformations on S. E will always denote the identity transformation on S, and the symbol $\| \;\|$ will be used to denote the norms of functions in B, as well as the norms of points in S. A function ϕ on a topological space X into B (or S) will be said to be continuous if it is continuous relative to the topology of B (or S). If ϕ: [a,b] $\to$ B (or S), then ϕ will be said to be of bounded variation on [a,b] if

$$\sup_\pi \sum_{i=1}^n \|\phi(t_i) - \phi(t_{i-1})\| < \infty,$$

where the supremum above is taken over all partitions $\pi = \{t_i\}_{i=0}^n$ of [a,b]. The symbol $V(\phi;a,b)$ will be used to denote the quantity $\sup_\pi \sum_{i=1}^n \|\phi(t_i) - \phi(t_{i-1})\|$ and will be called the variation of ϕ over [a,b]. The symbol $V(\phi;b,a)$ will be defined as equal to $V(\phi;a,b)$. It should be noted that Hille and Phillips (see [5, p. 59]) define a function ϕ:[a,b] $\to$ S (or B) to be of strong bounded variation on [a,b] if it is of bounded variation on [a,b] in the sense above, while the term bounded variation is reserved for a different concept of variation. However, the terminology introduced here agrees with that employed by MacNerney [6].

A function f on a topological space X into B (or S) will be said to be bounded on X if

$$\sup_{x \in X} \|f(x)\| < \infty. \tag{4.1}$$

The symbol $\|f\|_X$ will be used to denote the quantity on the left in (4.1), and in cases where there is no ambiguity, the symbol $\|f\|$ will be used in place of $\|f\|_X$.

Two special types of point-interval functions will now be defined. Suppose ϕ:[a,b] $\to$ B (or S). ϕ may now be thought of as a point-interval function by defining

$$\phi(\tau,\{s,t\}) = \phi(\tau),$$

and a second point-interval function may be defined by setting

$$d\phi(\tau,\{s,t\}) = \phi(t) - \phi(s).$$

Of course, there is some ambiguity involved in saying that ϕ may be thought of as either a point function or a point-interval function, but this is unimportant compared to the convenience in notation. If F is a point-interval function on $[a,b]$ into B and g is a point-interval function on $[a,b]$ into B or S, then the <u>product</u> $F \cdot g$ of the point-interval functions F and g is defined by

$$F \cdot g(\tau,\{s,t\}) = F(\tau,\{s,t\})g(\tau,\{s,t\}).$$

$F \cdot g$ will also be written as Fg. Taking products of point-interval functions gives rise to a multitude of point-interval functions on $[a,b]$ which can be constructed from point functions on $[a,b]$. For example, if $F:[a,b] \to B$, $G:[a,b] \to B$, and $g:[a,b] \to S$, then such point-interval functions as $F[dG]$, $[dF]G$, $F[dG]g$, $[dF]Gg$, and $FG[dg]$ all arise in a natural way. Integrals of such point-interval functions are discussed in MacNerney [6] and much of the remainder of this paper will be concerned with integrals of this type.

Because of the way in which the existence of the integral of a point-interval function on $[a,b]$ into B has been defined (see section I.3), in a great deal of the theory to be developed the point-interval functions involved will be constructed so that their values lie in S.

5. <u>Substitution theorems</u>. For real-valued functions f, g, and h on a closed interval $[a,b]$ it is known that if f is bounded on $[a,b]$ and $k(x) = \int_a^x g[dh]$, then $\int_a^b f[dk]$ exists if and only if $\int_a^b fg[dh]$ exists and if the two integrals exist then they are equal (see Hildebrandt [4, p. 53]). However, the situation for the integrals of point-interval functions such as discussed in section I.4 is somewhat different. Therefore, theorems such as Theorems 2.5, 2.6, and 2.7 presented in MacNerney [6] appear.

Three substitution theorems similar to those just mentioned in [6] will be given now, while in the next Chapter two substitution theorems will be presented which are in some ways superior to any presented in [6] or to be presented here.

Proofs for the first two substitution theorems will be omitted since they are similar to that given for the third. The following lemma is inserted to be used in the proof of the third substitution theorem.

LEMMA 5.1. _If_ $D: [a,b] \to B$ _is of bounded variation_, $P: [a,b] \to B$ _is continuous and of bounded variation, and_ $h: [a,b] \to S$ _is of bounded variation, then_

$$\int_a^b D[dP]h$$

exists.

Let $(\mathcal{P},\geq)$ be the partition collection on $[a,b]$, and for the point-interval function $D[dP]h$ on $[a,b]$ into S, let $N: \mathcal{P} \to S$ be the net defined by (2.1). Suppose $\pi_1 = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^n\} \in \mathcal{P}$, $\pi_2 = \{(\sigma_j,\{s_{j-1},s_j\})_{j=1}^m\} \in \mathcal{P}$, $\pi_2 \geq \pi_1$, and for each $i = 1,2,\cdots,n$, suppose that $\{s_{j_i-1},s_{j_i},\cdots,s_{j_i'}\}$ is the set of s_j's contained in the interval $[t_{i-1},t_i]$. Then,

$$\begin{aligned}
&\|N(\pi_1) - N(\pi_2)\| \\
&= \|\textstyle\sum_{i=1}^n D(\tau_i)[P(t_i) - P(t_{i-1})]h(\tau_i) - \sum_{j=1}^m D(\sigma_j)[P(s_j) - P(s_{j-1})]h(\sigma_j)\| \\
&\leq \textstyle\sum_{i=1}^n \sum_{j=j_i}^{j_i'} \{\|[D(\tau_i) - D(\sigma_j)][P(s_j) - P(s_{j-1})]h(\tau_i)\| \\
&\qquad\qquad + \|D(\sigma_j)[P(s_j) - P(s_{j-1})][h(\tau_i) - h(\sigma_j)]\|\} \\
&\leq \sup_{|t-s| \leq \|\pi_1\|} V(P;s,t)\{\|h\|\cdot V(D;a,b) + \|D\|\cdot V(h;a,b)\},
\end{aligned}$$

where $\|h\| = \|h\|_{[a,b]}$ and $\|D\| = \|D\|_{[a,b]}$. Since the continuity of P on [a,b] insures the continuity of V(P;a,t) for $t \in [a,b]$, the quantity above must tend to 0 as $\|\pi_1\| \to 0$, and hence Theorem 2.1 of this Chapter guarantees the existence of the integral $\int_a^b D[dP]h$.

THEOREM 5.1. *Suppose* $G:[a,b] \to B$ *is bounded,* $h:[a,b] \to S$ *is continuous and of bounded variation,* $F:[a,b] \to B$ *is of bounded variation,* $\int_a^b G[dh]$ *exists, and* $q:[a,b] \to S$ *is defined by*

$$q(t) = \int_a^t G[dh].$$

Then, if $\int_a^b FG[dh]$ *exists, it follows that*

$$\int_a^b FG[dh] = \int_a^b F[dq].$$

THEOREM 5.2. *Suppose* $F:[a,b] \to B$ *is bounded,* $G:[a,b] \to B$ *is continuous and of bounded variation,* $h:[a,b] \to S$ *is bounded,* $K:[a,b] \to B$ *is of bounded variation,* $\int_a^b F[dG]h$ *exists, and* $p:[a,b] \to S$ *is defined by*

$$p(t) = \int_a^t F[dG]h.$$

Then, if $\int_a^b KF[dG]h$ *exists, it follows that*

$$\int_a^b KF[dG]h = \int_a^b K[dp].$$

THEOREM 5.3. *Suppose* $K[a,b] \to B$ *and* $G:[a,b] \to B$ *are bounded,* $F:[a,b] \to B$ *is of bounded variation and continuous,* $D:[a,b] \to B$ *and* $h:[a,b] \to S$ *are both of bounded variation,* $\int_a^b K[dF]G$ *exists, and* $P:[a,b] \to B$ *is defined by*

$$P(t) = \int_a^t K[dF]G.$$

Then, if $\int_a^b KF[dG]h$ *exists, it follows that*

$$\int_a^b DK[dF]Gh = \int_a^b D[dP]h.$$

To prove this, note that P(t) is continuous and of bounded variation on [a,b], (See [6; Theorem 2.7]), so by Lemma 5.1 of this Chapter and the fact that D and h are both of bounded variation, it follows that $\int_a^b D[dP]h$ exists. Let $(\mathcal{P},\geq)$ be the partition collection on [a,b], and suppose $\pi = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^n\} \in \mathcal{P}$. Then,

$$\begin{aligned}\int_a^b DK[dF]Gh &= \textstyle\sum_{i=1}^n \int_{t_{i-1}}^{t_i} DK[dF]Gh \\ &= \textstyle\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \{D(s) - D(\tau_i)\}K(s)[dF(s)]G(s)h(s) \\ &\quad + \textstyle\sum_{i=1}^n \int_{t_{i-1}}^{t_i} D(\tau_i)K(s)[dF(s)]G(s)\{h(s) - h(\tau_i)\} \\ &\quad + \textstyle\sum_{i=1}^n \int_{t_{i-1}}^{t_i} D(\tau_i)K(s)[dF(s)]G(s)h(\tau_i).\end{aligned}$$

Now,

$$\left\|\textstyle\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \{D(s) - D(\tau_i)\}K(s)[dF(s)]G(s)h(s)\right\|$$

$$\leq V(D;a,b)\cdot\|K\|\cdot\|G\|\cdot\|h\|\cdot \sup_{|t-s|\leq\|\pi\|} V(F;s,t)$$

and

$$\left\|\textstyle\sum_{i=1}^n \int_{t_{i-1}}^{t_i} D(\tau_i)K(s)[dF(s)]G(s)\{h(s) - h(\tau_i)\}\right\|$$

$$\leq V(h;a,b)\cdot\|D\|\cdot\|K\|\cdot\|G\|\cdot \sup_{|t-s|\leq\|\pi\|} V(F;s,t),$$

where $\|K\| = \|K\|_{[a,b]}$, $\|G\| = \|G\|_{[a,b]}$, $\|h\| = \|h\|_{[a,b]}$, and $\|D\| = \|D\|_{[a,b]}$.

Hence the uniform continuity of $V(F;a,t)$ in t on $[a,b]$ shows that both quantities above tend to 0 as $\|\pi\| \to 0$. Consequently, noting that

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} D(\tau_i)K(s)[dF(s)]G(s)h(\tau_i) = \sum_{i=1}^{n} D(\tau_i)\{P(t_i) - P(t_{i-1})\}h(\tau_i),$$

it follows that

$$\int_a^b DK[dF]Gh = \int_a^b D[dP]h.$$

CHAPTER II

ω-INTEGRATION

1. Introduction. It becomes apparent when studying the integration of point-interval functions on [a,b] into a Banach space X that even though a large number of theorems which hold for the special case when X is the set of real numbers also hold when X is a general Banach space, there are still a number of results which are lost. The concept of ω-integrability to be introduced shortly can be used to regain a number of important theorems for integration in general Banach spaces. Only a few of the most important theorems connected with this paper on the whole will be presented here. It is interesting to note that for finite dimensional inner product spaces integrability and ω-integrability are equivalent concepts (see Hildebrandt [4; p. 30]), while as will be pointed out later, a number of point-interval functions in a Banach space whose integrals can be asserted to exist are actually ω-integrable.

2. The concept of ω-integration. Let ϕ be a point-interval function on [a,b] into B (or S). Define a function $\Sigma\phi$ on the set of all partitions of closed subintervals [c,d] of [a,b] by setting

$$(\Sigma\phi)(\pi) = \textstyle\sum_{i=1}^{n} \phi(I_i), \tag{2.1}$$

for each partition $\pi = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^{n}\} = \{I_i\}_{i=1}^{n}$ of [c,d]. For brevity, the notation used in (2.1) will be simplified to

(2.2) $$(\Sigma\phi)(\pi) = \Sigma_{\pi}\phi(I).$$

Then the <u>oscillation</u> of $\Sigma\phi$ on $[c,d]$ will be defined as the extended real number

$$\omega(\Sigma\phi;c,d) = \sup_{(\pi_1,\pi_2)} \|(\Sigma\phi)(\pi_1) - (\Sigma\phi)(\pi_2)\|,$$

where the supremum is taken over all partitions π_1 and π_2 of $[c,d]$. On the other hand, the <u>oscillation</u> of ϕ on $[c,d]$ will be defined as the extended real number

$$\Omega(\phi;c,d) = \sup\{\|\phi(t) - \phi(s)\| \mid (t,s) \in [c,d] \times [c,d]\}.$$

Notice that $\omega(\Sigma\phi;c,d)$ may be considered as a point-interval function on $[a,b]$ into the extended reals taking a pair $(\tau,\{s,t\})$ to the extended real number $\omega(\Sigma\phi;s,t)$. Denote this point-interval function by $\omega(\Sigma\phi)$. It now makes sense to talk about the existence of the integral

$$\int_a^b \omega(\Sigma\phi).$$

Suppose f is a point-interval function on $[a,b]$ into S. Then f will be said to be ω-<u>integrable</u> on $[a,b]$ if

$$\int_a^b \omega(\Sigma f) = 0.$$

On the other hand, if F is a point-interval function on $[a,b]$ into B, then F will be termed ω-<u>integrable</u> <u>in</u> B on $[a,b]$ if

$$\int_a^b \omega(\Sigma F) = 0.$$

Also, if for each $x \in S$, we have that F_x is the point-interval function on $[a,b]$ into S defined by

$$F_x(t) = F(t)(x)$$

for each $t \in [a,b]$, then F will be said to be ω-integrable on [a,b] if F_x is ω-integrable on [a,b] for each $x \in S$. It is now easy to demonstrate that if f is ω-integrable on [a,b], then f is integrable on [a,b]; if F is ω-integrable in B on [a,b], then F is integrable in B on [a,b]; if F is ω-integrable in B on [a,b], then F is ω-integrable on [a,b]. If S is the set of all real numbers, then ω-integrability may be shown to be equivalent to integrability in the same manner as Hildebrandt [4; p. 30] does for the integration of real-valued interval functions.

The following theorem serves to indicate the large variety of point-interval functions on a closed interval [a,b] into S (or B) which are actually ω-integrable (in B) on [a,b].

THEOREM 2.1. *If* $F:[a,b] \to B$, $G:[a,b] \to B$, *and* $h:[a,b] \to S$ $\{H:[a,b] \to B\}$, *then the point-interval function* $F[dG]h$ $\{F[dG)H\}$ *will be* ω-*integrable* {*in* B} *on* [a,b] *if either*

(a) F *is continuous*, G *is of bounded variation*, *and* h *is continuous* {H *is continuous*};

or,

(b) F *is of bounded variation on* [a,b], G *is continuous and of bounded variation*, *and* h *is of bounded variation* {H *is of bounded variation*}.

Proofs for (a) and (b) will be given for the point-interval function F[dG]H, with the proofs for F[dG]h being similar. Let $\pi_1 = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^n\}$ and $\pi_2 = \{(\sigma_j,\{s_{j-1},s_j\})_{j=1}^m\}$ be any two partitions of a closed subinterval [c,d] of [a,b]. Suppose $\pi_3 = \{(\chi_k,\{x_{k-1},x_k\})_{k=1}^r\}$ is a partition of [c,d] such that $\pi_3 \gtrsim \pi_1$ and $\pi_3 \gtrsim \pi_2$, and for each

$i = 1,2,\cdots,n$ suppose that $\{x_{k_i-1}, x_{k_i}, \cdots, x_{k_i'}\}$ is the set of x_k's contained in $[t_{i-1}, t_i]$. Then, with $\phi = F[dG]H$,

$$\|\Sigma_{\pi_1} \phi(I) - \Sigma_{\pi_2} \phi(I)\| \leq \|\Sigma_{\pi_1} \phi(I) - \Sigma_{\pi_3} \phi(I)\| + \|\Sigma_{\pi_3} \phi(I) - \Sigma_{\pi_2} \phi(I)\|.$$

But,

$$\|\Sigma_{\pi_1} \phi(I) - \Sigma_{\pi_3} \phi(I)\|$$

$$\leq \sum_{i=1}^{n} \sum_{k=k_i}^{k_i'} \{\|[F(\tau_i) - F(\chi_k)][G(x_k) - G(x_{k-1})]H(\chi_k)\|$$

$$+ \|F(\tau_i)[G(x_k) - G(x_{k-1})][H(\chi_k) - H(\tau_i)]\|\}$$

$$\text{(2.3a)} \qquad \leq V(G;c,d)\{\Omega(F;c,d) \cdot \|H\| + \|F\| \cdot \Omega(H;c,d)\},$$

$$\text{(2.3b)} \qquad \leq V(G;c,d)\{\|H\| \cdot V(F;c,d) + \|F\| \cdot V(H;c,d)\},$$

where $\|F\| = \|F\|_{[a,b]}$ and $\|H\| = \|H\|_{[a,b]}$. Also, $\|\Sigma_{\pi_3} \phi(I) - \Sigma_{\pi_2} \phi(I)\|$ satisfies the same inequality, so it follows that $\omega(\Sigma\phi;c,d)$ is less than or equal to twice the quantity in either (2.3a) or (2.3b) above. Now let $\delta > 0$ be arbitrary, and let $\pi = \{t_i\}_{i=0}^{n}$ be any partition of $[a,b]$ with norm less than δ. Then from (2.3a) and (2.3b) it follows that

$$0 \leq \sum_{i=1}^{n} \omega(\Sigma\phi; t_{i-1}, t_i)$$

$$\leq 2V(G;a,b]\{\sup_{|t-s| \leq \delta} \Omega(F;s,t) \cdot \|H\| + \|F\| \cdot \sup_{|t-s| \leq \delta} \Omega(H;s,t)\}$$

and

$$0 \leq \sum_{i=1}^{n} \omega(\Sigma\phi; t_{i-1}, t_i)$$

$$\leq 2 \sup_{|t-s| \leq \delta} V(G;s,t)\{\|H\| \cdot V(F;a,b) + \|F\| \cdot V(H;a,b)\}.$$

Consequently, if F, G, and H satisfy the conditions in (a), then the first set of inequalities above imply $\int_a^b \omega(\Sigma\phi) = 0$, while if F, G, and H satisfy the conditions specified in (b), then the second set of inequalities above imply $\int_a^b \omega(\Sigma\phi) = 0$. Therefore $\phi = F[dG]H$ is ω-integrable in B on [a,b].

Notice that the previous theorem can be applied immediately to show that if $F:[a,b] \to B$ is continuous and $G:[a,b] \to B$ is of bounded variation, then [dG]F and F[dG] are ω-integrable in B on [a,b]. However, the theorem does not show directly that if F is continuous and G is of bounded variation, then [dF]G and G[dF] are ω-integrable on [a,b]. Since this case is also of some interest, the following theorem is presented.

THEOREM 2.2. <u>If</u> $F:[a,b] \to B$, $G:[a,b] \to B$, <u>and</u> $h:[a,b] \to S$, <u>then</u> F[dG] <u>is</u> ω-<u>integrable in</u> B <u>on</u> [a,b] <u>if and only if</u> [dF]G <u>is</u> ω-<u>integrable in</u> B <u>on</u> [a,b], <u>while</u> F[dh] <u>is</u> ω-<u>integrable on</u> [a,b] <u>if and only if</u> [dF]h <u>is</u> ω-<u>integrable on</u> [a,b].

Proof for the point-interval function F[dG] will be given, while the proof for the point-interval function F[dh] is similar. If $[c,d] \subseteq [a,b]$ and $\pi_1 = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^n\}$ and $\pi_2 = \{(\sigma_j,\{s_{j-1},s_j\})_{j=1}^m\}$ are any two partitions of [c,d], then set $x_o = a$, $x_{n+1} = b$, $\chi_{n+1} = t_n$, $y_o = a$, $y_{m+1} = b$, $\gamma_{m+1} = b$, and for $i = 1,2,\cdots,n$, set $x_i = \tau_i$ and $\chi_i = t_{i-1}$, while for $j = 1,2,\cdots,m$, set $y_j = \sigma_j$ and $\gamma_j = s_{j-1}$. Now form the partitions

$$\pi_1' = \{(\chi_i,\{x_{i-1},x_i\})_{i=1}^{n+1}\} \text{ and } \pi_2' = \{(\gamma_j,\{y_{j-1},y_j\})_{j=1}^{m+1}\}$$

of [c,d]. Then

$$\begin{aligned}&\|\Sigma_{\pi_1}([dF_1]F_2)(I) - \Sigma_{\pi_2}([dF_1]F_2)(I)\|\\ &\quad= \|\Sigma_{\pi_1'}(F_1[dF_2])(I) - \Sigma_{\pi_2'}(F_1[dF_2])(I)\|\\ &\quad\le \omega(\Sigma F_1[dF_2];c,d),\end{aligned}$$

so that $\omega(\Sigma[dF_1]F_2;c,d) \leq \omega(\Sigma F_1[dF_2];c,d)$. A similar argument shows that the reverse inequality holds and thus

$$\omega(\Sigma[dF_1]F_2;c,d) = \omega(\Sigma F_1[dF_2];c,d).$$

Consequently, $F_1[dF_2]$ is ω-integrable in B on [a,b] if and only if $[dF_1]F_2$ is ω-integrable in B on [a,b].

The following notation will be employed from this point on. If ϕ is a point-interval function on [a,b] into S or B which is integrable on [a,b], and if $I = (\tau,\{s,t\})$ is such that $\tau \in [s,t] \subseteq [a,b]$, then $\int_I \phi$ will be defined as equal to $\int_s^t \phi$. An approximation theorem is presented now.

THEOREM 2.3. _If_ f _is a point-interval function on_ [a,b] _into_ S _and_ f _is_ ω-_integrable on_ [a,b], _then for each_ $\varepsilon > 0$ _there is a_ $\delta > 0$ _such that if_ $\pi = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^n\} = \{I_i\}_{i=1}^n$ _is any partition of_ [a,b] _with_ $\|\pi\| < \delta$, _then_

$$\sum_{i=1}^n \left\| \int_{t_{i-1}}^{t_i} f - f(I_i) \right\| < \varepsilon.$$

In particular, for I's _of the form_ $I = (\tau,\{s,t\})$ _and_ $\ell(I) = (t-s)$,

$$\lim_{\ell(I)\to 0} \left\| \int_I f - f(I) \right\| = 0.$$

Let $\varepsilon > 0$ be given, and choose $\delta = \delta_\varepsilon > 0$ such that if $\pi = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^n\}$ is any partition of [a,b] with $\|\pi\| < \delta$, then

$$\sum_{i=1}^n \omega(\Sigma f;t_{i-1},t_i) < \varepsilon.$$

Since

$$\left\| \int_{t_{i-1}}^{t_i} f - f(I_i) \right\| \leq \omega(\Sigma f;t_{i-1},t_i)$$

for each i, it then follows that

$$(2.4) \qquad \sum_{i=1}^{n} \left\| \int_{t_{i-1}}^{t_i} f - f(I_i) \right\| \le \sum_{i=1}^{n} \omega(\Sigma f; t_{i-1}, t_i) < \varepsilon.$$

To prove $\lim_{\ell(I)\to 0} \left\| \int_I f - f(I) \right\| = 0$, remember that if $I = \{\tau, \{s,t\}\}$, then by definition $\int_I f = \int_s^t f$ so that the result follows immediately from (2.4) above.

The substitution theorems promised in Chapter I will now be supplied. The reader should compare these theorems with Theorems 5.1, 5.2, and 5.3 in Chapter I and Theorems 2.4, 2.5, and 2.6 in MacNerney [6]. For the meaning of point-interval functions constructed from point functions and the products of point-interval functions see Section I.4.

THEOREM 2.4. <u>If</u> f <u>is</u> <u>a</u> <u>point-interval</u> <u>function</u> <u>on</u> [a,b] <u>into</u> S <u>which</u> <u>is</u> ω-<u>integrable</u> <u>on</u> [a,b], g <u>is</u> <u>the</u> <u>point-interval</u> <u>function</u> <u>on</u> [a,b] <u>into</u> S <u>defined</u> <u>by</u>

$$g(I) = \int_I f$$

<u>for</u> <u>each</u> $I = (\tau, \{s,t\})$ <u>such</u> <u>that</u> $\tau \in [s,t] \subseteq [a,b]$, <u>and</u> $H:[a,b] \to B$ <u>is</u> <u>bounded</u>, <u>then</u> $\int_a^b H \cdot f$ <u>exists</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> $\int_a^b H \cdot g$ <u>exists</u>, <u>and</u> <u>the</u> <u>two</u> <u>integrals</u> <u>are</u> <u>equal</u> <u>when</u> <u>they</u> <u>do</u> <u>exist</u>.

If $\pi = \{(\tau_i, \{t_{i-1}, t_i\})_{i=1}^{n}\} = \{I_i\}_{i=1}^{n}$ is any partition of [a,b], then

$$\begin{aligned} &\left\| \sum_{i=1}^{n} (H \cdot f)(I_i) - \sum_{i=1}^{n} (H \cdot g)(I_i) \right\| \\ &\qquad \le \sum_{i=1}^{n} \|H(\tau_i)\| \left\| f(I_i) - \int_{I_i} f \right\| \\ &\qquad \le \|H\| \cdot \sum_{i=1}^{n} \left\| f(I_i) - \int_{t_{i-1}}^{t_i} f \right\|, \end{aligned}$$

where $\|H\| = \|H\|_{[a,b]}$. Theorem 2.3 of this chapter then guarantees that the right hand member in the above inequality tends to 0 as $\|\pi\| \to 0$.

THEOREM 2.5. <u>If</u> F <u>is</u> <u>a</u> <u>point-interval</u> <u>function</u> <u>on</u> [a,b] <u>into</u> B

<u>which is</u> ω-<u>integrable in</u> B, G <u>is the point-interval function on</u> [a,b] <u>into</u> B <u>defined by</u>

$$G(I) = \int_I F,$$

<u>for each</u> $I = (\tau,\{s,t\})$ <u>such that</u> $\tau \in [s,t] \subseteq [a,b]$, <u>and if</u> $K:[a,b] \to B$ <u>and</u> $h:[a,b] \to S$ <u>are bounded, then</u> $\int_a^b K \cdot G \cdot h$ <u>exists if and only if</u> $\int_a^b K \cdot F \cdot h$ <u>exists, and when the integrals do exist they are equal.</u>

If $\pi = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^n\} = \{I_i\}_{i=1}^n$ is any partition of [a,b], then

$$\begin{aligned}\|\textstyle\sum_{i=1}^n (K \cdot G \cdot h)(I_i) - \sum_{i=1}^n (K \cdot F \cdot h)(I_i)\| \\ &\le \|K\| \cdot \|h\| \cdot \{\textstyle\sum_{i=1}^n \|\int_{I_i} F - F(I_i)\|\} \\ &\le \|K\| \cdot \|h\| \cdot \{\textstyle\sum_{i=1}^n \omega(\Sigma F;t_{i-1},t_i)\},\end{aligned}$$

where $\|K\| = \|K\|_{[a,b]}$ and $\|h\| = \|h\|_{[a,b]}$. But the quantity on the right in the inequality above approaches 0 as $\|\pi\| \to 0$ since F is ω-integrable in B on [a,b].

As an example of how the preceding theorem might be applied, the following corollary is given. The proof of the corollary can be obtained quite simply using Theorem 2.1 and Theorem 2.5.

COROLLARY. <u>If</u> $F:[a,b] \to B$ <u>and</u> $H:[a,b] \to B$ <u>are of bounded variation</u>, $G:[a,b] \to B$ <u>is continuous and of bounded variation, and if</u> $K:[a,b] \to B$ <u>and</u> $h:[a,b] \to S$ <u>are bounded, then</u> $\int_a^b KF[dG]Hh$ <u>exists if and only if</u> $\int_a^b K(s)[d_s(\int_a^s F[dG]H)]h(s)$ <u>exists, and if the two integrals do exist, then they are equal.</u>

The substitution theorems now become powerful tools in proving further results such as those given below.

THEOREM 2.6. <u>If</u> $F_i:[a,b] \to B$, $(i = 1,2,3)$, <u>and</u> $\phi:[a,b] \to S$ <u>are such</u>

that F_1 and ϕ are bounded, if the point-interval function $F_2[dF_3]$ is ω-integrable in B on [a,b], and if $\int_a^b F_1[dF_2]F_3\phi$ and $\int_a^b F_1[d(F_2F_3)]\phi$ both exist, then $\int_a^b F_1F_2[dF_3]\phi$ exists and

$$\int_a^b F_1[d(F_2F_3)]\phi = \int_a^b F_1[dF_2]F_3\phi + \int_a^b F_1F_2[dF_3]\phi.$$

Note that Theorem 2.2 of this Chapter, together with the fact that $F_2[dF_3]$ is ω-integrable in B, shows that $[dF_2]F_3$ is ω-integrable in B. Also, $d(F_2F_3)$ is clearly ω-integrable in B. So, upon applying Theorem 2.5 twice, we have

$$\int_a^b F_1[dF_2]F_3\phi - \int_a^b F_1[d(F_2F_3)]\phi$$

$$= \int_a^b F_1(s)[d_s(\int_a^s [dF_2]F_3 - F_2(s)F_3(s))]\phi(s)$$

$$= \int_a^b F_1(s)[d_s(-\int_a^s F_2[dF_3])]\phi(s)$$

$$= -\int_a^b F_1F_2[dF_3]\phi.$$

THEOREM 2.7. If $F_i:[a,b] \to B$ $(i = 1,2)$ and $f_3:[a,b] \to S$ are such that the point-interval function $F_2[df_3]$ is ω-integrable on [a,b], and $\int_a^b F_1F_2[df_3]$ and $\int_a^b [dF_1]F_2f_3$ both exist, then $\int_a^b F_1[dF_2]f_3$ exists, and

$$F_1(t)F_2(t)f_3(t)\Big|_{t=a}^{t=b} = \int_a^b [dF_1]F_2f_3 + \int_a^b F_1[dF_2]f_3 + \int_a^b F_1F_2[df_3].$$

Applying Theorem 2.4, we have the equation

$$\int_a^b F_1F_2df_3 = \int_a^b F_1(s)d_s(\int_a^s F_2[df_3]) \tag{2.5}$$

$$= \int_a^b F_1(s)d_s(F_2(s)f_3(s) - \int_a^s [dF_2]f_3).$$

The existence of the above integral, together with the existence of the integral $\int_a^b [dF_1]F_2F_3$, and hence of $\int_a^b F_1 d(F_2f_3)$, insures that the integral $\int_a^b F_1(s)d_s(\int_a^s [dF_2]f_3)$ exists. But the fact that $F_2[df_3]$ is ω-integrable on $[a,b]$ implies by Theorem 2.2 of this Chapter that $[dF_2]f_3$ is ω-integrable on $[a,b]$ and hence by Theorem 2.4, the integral $\int_a^b F_1[dF_2]f_3$ exists and equals $\int_a^b F_1(s)d_s(\int_a^s [dF_2]f_3)$. Then using (2.5) and integration by parts on $\int_a^b F_1 d(F_2f_3)$, it follows that

$$\int_a^b F_1F_2[df_3] = F_1(s)F_2(s)f_3(s)\Big|_{s=a}^{s=b} - \int_a^b [dF_1]F_2f_3 - \int_a^b F_1[dF_2]f_3.$$

CHAPTER III

ITERATED INTEGRALS

1. Introduction. Iterated integrals of the forms to be considered in this chapter have not been discussed in the literature previous to this paper as far as is known. Two distinct types of iterated integrals will be discussed. The first may be a mere curiosity, but the second will be shown to have useful applications and several valuable theorems will be derived for it.

In this chapter it will be necessary to consider the partition collections on two distinct closed intervals [a,b] and [c,d]. To distinguish these partition collections from one another we will use $\mathcal{P}$ to denote the set of all partitions on [a,b] and $\mathcal{O}$ to denote the set of all partitions on [c,d], while the symbol $\geq$ will be used to denote the ordering in both $\mathcal{P}$ and $\mathcal{O}$ as defined in Chapter I. If $(\mathcal{P},\geq)$ is the partition collection on [a,b] and $(\mathcal{O},\geq)$ is the partition collection on [c,d], then the directed product of $(\mathcal{P},\geq)$ and $(\mathcal{O},\geq)$ will be the directed set $(\mathcal{S},\geq)$, where $\mathcal{S} = \mathcal{P} \times \mathcal{O}$ and $(x,y) \geq (z,w)$ if $x \geq z$ and $y \geq w$. The use of one symbol $\geq$ to denote the ordering in all of the directed sets above may be somewhat objectionable, but it is felt that no confusion will arise as a result of this convention in notation.

The following two theorems will be found useful in the sections which follow. The proof of the first theorem is very simple, while the proof of the second theorem is similar to that for the case when real-valued nets are being considered; consequently, both proofs are omitted.

THEOREM 1.1. *If* $(\mathcal{P},\geq)$ *and* $(\mathcal{O},\geq)$ *are the partition collections on* $[a,b]$ *and* $[c,d]$ *respectively,* $(\mathcal{D},\geq)$ *is the directed product of* $(\mathcal{P},\geq)$ *and* $(\mathcal{O},\geq)$, *and* $N:\mathcal{D} \to S$, *then*

$$\lim_{\pi_1 \in \mathcal{P}} N(\pi_1,\pi_2)$$

exists uniformly for $\pi_2 \in \mathcal{O}$ *if and only if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $\pi_i \in \mathcal{P}$, $(i = 1,2)$, $\|\pi_1\| < \delta$, *and* $\pi_2 \geq \pi_1$, *then*

$$\|N(\pi_1,\pi_3) - N(\pi_2,\pi_3)\| < \varepsilon$$

for each $\pi_3 \in \mathcal{O}$.

THEOREM 1.2. *If* $(\mathcal{P},\geq)$ *and* $(\mathcal{O},\geq)$ *are directed sets,* $(\mathcal{D},\geq)$ *is the directed product of* $(\mathcal{P},\geq)$ *and* $(\mathcal{O},\geq)$, *and if* $f:\mathcal{D} \to S$ *is a net such that* $\lim_p f(p,q)$ *exists for each* $q \in \mathcal{O}$ *and* $\lim_q f(p,q)$ *exists uniformly for* $p \in \mathcal{P}$, *then*

$$\lim_p \lim_q f(p,q),\ \lim_q \lim_p f(p,q),\ \text{and}\ \lim_{p,q} f(p,q)$$

all exist and are equal.

THEOREM 1.3. *If* $F:[a,b] \times [a,b] \to B$ *is continuous,* $G:[a,b] \to B$ *is continuous and of bounded variation, and* $g:[a,b] \to S$ *is continuous and of bounded variation, then for* $\tau \in [a,b]$ *we have that the integrals*

$$\int_\tau^s [dG(u)]F(u,s) \quad \text{and} \quad \int_\tau^s F(s,u)[dg(u)]$$

are continuous functions of s *on* $[a,b]$.

Suppose $s_0 \in [a,b]$ and let $\varepsilon > 0$ be given. The function $V(G;a,s)$ is a continuous function of s on $[a,b]$ since the same is true of G, while F is continuous on the compact set $[a,b] \times [a,b]$, and is therefore uniformly continuous on $[a,b] \times [a,b]$. Hence there exists a $\delta = \delta_\varepsilon > 0$

such that if $|s - s_o| < \delta$ and $s \in [a,b]$ then

$$|V(G;a,s) - V(G;a,s_o)| < \varepsilon,$$

and for each $u \in [a,b]$ we have

$$\|F(u,s) - F(u,s_o)\| < \varepsilon.$$

Then, for $|s - s_o| < \delta$ it follows that

$$\left\|\int_\tau^s [dG(u)]F(u,s) - \int_\tau^{s_o} [dG(u)]F(u,s_o)\right\|$$

$$\leq \left\|\int_\tau^s [dG(u)](F(u,s) - F(u,s_o))\right\| + \left\|\int_{s_o}^s [dG(u)]F(u,s_o)\right\|$$

$$\leq \varepsilon\{V(G;a,b) + \|F\|\},$$

where $\|F\| = \|F\|_{[a,b] \times [a,b]}$. Thus, $\int_\tau^s [dG(u)]F(u,s)$ is continuous in s on [a,b]. The continuity of $\int_\tau^s F(s,u)[dg(u)]$ can be established in a similar manner.

2. <u>A peculiar form of iterated integral</u>. Suppose $F:[a,b] \times [c,d] \to B$, $G:[a,b] \to B$, $g:[c,d] \to S$, and the integral $\int_c^d F(x',y)G(x)dg(y)$ exists for each $(x',x) \in [a,b] \times [a,b]$. Let $(\mathcal{P},\geq)$ be the partition collection on [a,b], and let $N: \mathcal{P} \to S$ be defined by

$$N(\pi) = \sum_{i=1}^n \int_c^d F(\chi_i,y)(G(x_i) - G(x_{i-1}))dg(y)$$

for each $\pi = \{(\chi_i,\{x_{i-1},x_i\})_{i=1}^n\} \in \mathcal{P}$. If the limit

$$\lim_{\pi \in \mathcal{P}} N(\pi)$$

exists, then by definition, the symbol $\int_a^b \int_c^d F(x,y)[dG(x)]dg(y)$ will be used to denote this limit.

THEOREM 2.1. <u>Suppose</u> $F:[a,b] \times [c,d] \to B$ <u>is continuous</u>, $G:[a,b] \to B$

is of bounded variation, and $g:[c,d] \to S$ *is of bounded variation. Then the integrals*

$$\int_a^b \int_c^d F(x,y)[dG(x)]dg(y) \text{ and } \int_c^d [\int_a^b F(x,y)dG(x)]dg(y)$$

both exist and are equal.

Note that the integral

$$\int_a^b F(s,t)dG(s)$$

exists for each $t \in [c,d]$ by Theorem 2.3 in MacNerney [6], and if $s_i \in [a,b]$ $(i = 1,2)$, then the function

$$R(t) = F(s_1,t)G(s_2)$$

defined for $t \in [c,d]$ is continuous on $[c,d]$. Therefore, by Theorem 2.3 in [6], the integral

$$\int_c^d F(s_1,y)G(s_2)dg(y)$$

exists for each $(s_1,s_2) \in [a,b] \times [a,b]$.

Now let $(\mathcal{P},\geq)$ be the partition collection on $[a,b]$, and let $(\mathcal{O},\geq)$ be the partition collection on $[c,d]$. Let $(\mathcal{D},\geq)$ be the directed product of $(\mathcal{P},\geq)$ and $(\mathcal{O},\geq)$, and define a net $N:\mathcal{D} \to S$ by setting

$$N(\pi_1,\pi_2) = \sum_{j=1}^m [\sum_{i=1}^n F(\tau_i,\sigma_j)\{G(t_i) - G(t_{i-1})\}]\{g(s_j) - g(s_{j-1})\}$$

for each $\pi_1 = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^n\} \in \mathcal{P}$ and each $\Pi_2 = \{(\sigma_j,\{s_{j-1},s_j\})_{j=1}^m\} \in \mathcal{O}$. Note that

$$\lim_{\pi_1} N(\pi_1,\pi_2) = \sum_{j=1}^m \{\int_a^b F(t,\sigma_j)dG(t)\}\{g(s_j) - g(s_{j-1})\},$$

and

$$\lim_{\pi_2} N(\pi_1,\pi_2) = \sum_{i=1}^n \int_c^d F(\tau_i,s)\{G(t_i) - G(t_{i-1})\}dg(s).$$

Furthermore, $\lim_{\pi_1} N(\pi_1,\pi_2)$ is a uniform limit for $\pi_2 \in \mathcal{O}$. For if $\varepsilon > 0$ is given, choose $\delta = \delta_\varepsilon > 0$ such that if $(t_i,s_i) \in [a,b] \times [c,d]$, $(i = 1,2)$, and $|(t_1,s_1) - (t_2,s_2)| < \delta$ then

$$\|F(s_1,t_1) - F(s_2,t_2)\| < \varepsilon.$$

Suppose $\pi_1 = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^n\} \in \mathcal{P}$, $\|\pi_1\| < \delta$, $\pi_1' = \{(\chi_k,\{x_{k-1},x_k\})_{k=1}^r\} \in \mathcal{P}$, $\pi_1' \geqq \pi_1$, and for each $i = 1,2,\cdots,n$ let $\{x_{k_i-1}, x_{k_i}, \cdots, x_{k_i'}\}$ be the set of x_k's contained in the interval $[t_{i-1},t_i]$. Then, for each $\pi_2 = \{(\sigma_j,\{s_{j-1},s_j\})_{j=1}^m\} \in \mathcal{O}$ we have

$$\|N(\pi_1,\pi_2) - N(\pi_1',\pi_2)\|$$

$$= \|\textstyle\sum_{i=1}^n \sum_{k=k_i}^{k_i'} \sum_{j=1}^m [F(\tau_i,\sigma_j) - F(\chi_k,\sigma_j)][G(x_k) - G(x_{k-1})][g(s_j) - g(s_{j-1}]\|$$

$$\leq \varepsilon\, V(G;a,b)V(g;a,b),$$

and therefore Theorem 1.1 of this Chapter implies that $\lim_{\pi_1} N(\pi_1,\pi_2)$ is a uniform limit for $\pi_2 \in \mathcal{O}$. Consequently, by Theorem 1.2, it follows that

$$\lim_{\pi_2} \lim_{\pi_1} N(\pi_1,\pi_2) = \int_c^d [\int_a^b F(x,y)dG(x)]dg(y)$$

and

$$\lim_{\pi_1} \lim_{\pi_2} N(\pi_1,\pi_2) = \int_a^b \int_c^d F(x,y)[dG(x)]dg(y)$$

both exist and are equal.

3. <u>Totally uniform bounded variation and a form of continuous variation</u>. Suppose X is a normed linear space, Q is a subset of the real numbers, and $F:[a,b] \times Q \to X$. Then $F(t,s)$ will be said to be <u>of totally uniform bounded variation</u> in t on $[a,b]$ for $s \in Q$ if

(3.1) $$\sup_\pi [\textstyle\sum_{i=1}^n\{\sup_{q \in Q}\|F(t_i,q) - F(t_{i-1},q)\|\}] < \infty,$$

where the symbol $\sup_\pi$ indicates the supremum over all partitions $\pi = \{t_i\}_{i=0}^n$ of $[a,b]$ of the quantity in brackets. A similar definition applies if $F: Q \times [a,b] \to X$.

THEOREM 3.1. _If_ $F(s,t)$ _is a mapping from_ $[a,b] \times [a,b]$ _into_ B _which is of totally uniform bounded variation in_ s _on_ $[a,b]$ _for_ $t \in [a,b]$, $g:[a,b] \to S$ is of bounded variation, and $f:[a,b] \to S$ is defined by

$$f(s) = \int_a^s F(s,u)dg(u),$$

then f _is of bounded variation on_ $[a,b]$.

If $s \in [a,b]$ and $t \in [a,b]$, then

$$\left\|\int_a^s F(s,u)dg(u) - \int_a^t F(t,u)dg(u)\right\|$$

$$\leq \left\|\int_a^s (F(s,u) - F(t,u))dg(u)\right\| + \left\|\int_s^t F(t,u)dg(u)\right\|$$

$$\leq (\sup_{u \in [a,b]}\|F(s,u) - F(t,u)\|) \cdot V(g;a,b) + \|F\| \cdot V(g;s,t),$$

where $\|F\| = \|F\|_{[a,b] \times [a,b]}$. Therefore, if $\{t_i\}_{i=0}^n$ is a partition of $[a,b]$, the inequality above yields

$$\sum_{i=1}^n \|f(t_i) - f(t_{i-1})\|$$

$$\leq \{\sum_{i=1}^n V(g;a,b) \cdot \sup_{u \in [a,b]}\|F(t_i,u) - F(t_{i-1},u)\|\} + \|F\|V(g;a,b)$$

$$\leq V(g;a,b)\{M + \|F\|\},$$

where $M = \sup_\pi [\sum_{i=1}^n \{\sup_{q \in [a,b]}\|F(t_i,q) \quad F(t_{i-1},q\|\}]$, and consequently, $V(f;a,b) < \infty$.

Suppose X is a normed linear space and $F(s,t)$ is a mapping from $[a,b] \times [c,d]$ into X which is such that for each $s_o \in [a,b]$ we have that $F(s_o,t)$ is of bounded variation on $[c,d]$. Then $F(s,t)$ will be said to be _of bounded variation in_ t _continuously_ for $s \in [a,b]$ if for each

$\varepsilon > 0$ and each $s_o \in [a,b]$ there is a $\delta_{s_o,\varepsilon} > 0$ such that if $s \in [a,b]$ and $|s - s_o| < \delta_{s_o,\varepsilon}$ then

$$V[F(s,-) - F(s_o,-);c,d] < \varepsilon.$$

A similar definition is given when the roles played by the first and second variables are interchanged. It can now be proved that the continuity described in the first definition given above is uniform in that for each $\varepsilon > 0$ there is a $\delta = \delta_\varepsilon > 0$ such that if $s_1 \in [a,b]$, $s_2 \in [a,b]$, and $|s_1 - s_2| < \delta$ then

$$V[F(s_1,-) - F(s_2,-);c,d] < \varepsilon,$$

with a similar result holding when the roles of the first and second variables are interchanged.

THEOREM 3.2. _If_ $F(s,t)$ _is a mapping of_ $[a,b] \times [a,b]$ _into_ B _which is continuous on_ $[a,b] \times [a,b]$ _and of bounded variation in_ t _on_ $[a,b]$ _continuously for_ $s \in [a,b]$, $g:[a,b] \to S$ _is continuous, and_ $f:[a,b] \to S$ _is defined by_

$$f(s) = \int_a^s F(s,u)dg(u)$$

for each $s \in [a,b]$, _then_ f _is continuous on_ $[a,b]$.

Note that

$$\int_a^s F(s,u)dg(u) = F(s,s)g(s) - F(s,a)g(a) - \int_a^s [d_uF(s,u)]g(u),$$

and $F(s,s)g(s) - F(s,a)g(a)$ is a continuous function of s on $[a,b]$. The function $\int_a^s [d_uF(s,u)]g(u)$ is also continuous for $s \in [a,b]$. For let $s_o \in [a,b]$ and let $\varepsilon > 0$ be given. The continuity of $F(s_o,t)$ for $t \in [a,b]$ insures that $V[F(s_o,-);s_o,t]$ is also continuous for $t \in [a,b]$. Hence there is a $\delta > 0$ such that if $s \in [a,b]$ and $|s - s_o| < \delta$ then

$$V[F(s,-) - F(s_o,-);a,b] < \varepsilon$$

and

$$V[F(s_o,-);s_o,s] < \varepsilon.$$

Then if $s \in [a,b]$ and $|s-s_o| < \delta$, we have

$$\left\|\int_a^s [d_uF(s,u)]g(u) - \int_a^{s_o} [d_uF(s_o,u)]g(u)\right\|$$

$$\leq \left\|\int_a^s [d_u\{F(s,u) - F(s_o,u)\}]g(u)\right\| + \left\|\int_{s_o}^s [d_uF(s_o,u)]g(u)\right\|$$

$$\leq \|g\| \cdot \{V[F(s,-) - F(s_o,-);a,b] + V[F(s_o,-);s_o,s]\}$$

$$\leq 2\varepsilon \cdot \|g\|,$$

where $\|g\| = \|g\|_{[a,b]}$. It then follows that the integral $\int_a^s F(s,u)dg(u)$ is continuous for $s \in [a,b]$.

4. <u>A more classical form of iterated integral</u>. In this section integrals of the form

$$\int_c^d \{\int_a^b [dG(x)]F(x,y)\}dg(y) \tag{4.1}$$

and

$$\int_a^b [dG(x)]\{\int_c^d F(x,y)dg(y)\} \tag{4.2}$$

will be discussed, where $F:[a,b] \times [c,d] \to B$, $G:[a,b] \to B$, and $g:[c,d] \to S$.

THEOREM 4.1. <u>If</u> $F:[a,b] \times [c,d] \to B$ <u>is continuous</u>, $G:[a,b] \to B$ <u>is of bounded variation, and</u> $g:[c,d] \to S$ <u>is of bounded variation, then the integrals</u>

$$\int_a^b [dG(x)]\{\int_c^d F(x,y)dg(y)\} \quad \text{and} \quad \int_c^d \{\int_a^b [dG(x)]F(x,y)\}dg(y)$$

<u>both exist and are equal</u>.

Let $(\mathcal{P},\geq)$ be the partition collection on $[a,b]$, let $(\mathcal{Q},\geq)$ be the partition collection on $[c,d]$, and let $(\mathcal{D},\geq)$ be the directed produce of $(\mathcal{P},\geq)$ and $(\mathcal{O},\geq)$. Consider the net $N:\mathcal{D} \to S$ defined by

$$N(\pi_1,\pi_2) = \sum_{j=1}^{m} \{\sum_{i=1}^{n} [G(t_i) - G(t_{i-1})]F(\tau_i,\sigma_j)\}[g(s_j) - g(s_{j-1})]$$

$$= \sum_{i=1}^{n} [G(t_i) - G(t_{i-1})]\{\sum_{j=1}^{m} F(\tau_i,\sigma_j)[g(s_j) - g(s_{j-1})]\}$$

for each $\pi_1 = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^{n}\} \in \mathcal{P}$ and each $\pi_2 = \{(\sigma_j,\{s_{j-1},s_j\})_{j=1}^{m}\} \in \mathcal{O}$. It is clear that $\lim_{\pi_1} N(\pi_1,\pi_2)$ and $\lim_{\pi_2} N(\pi_1,\pi_2)$ exist for each fixed $\pi_2 \in \mathcal{O}$ and each fixed $\pi_1 \in \mathcal{P}$ respectively. Furthermore, $\lim_{\pi_1} N(\pi_1,\pi_2)$ exists uniformly for $\pi_2 \in \mathcal{O}$. For let $\varepsilon > 0$ be given and choose $\delta = \delta_\varepsilon > 0$ such that if $(t_i,s_i) \in [a,b] \times [c,d]$, $(i = 1,2)$, and $|(t_1,s_1) - (t_2,s_2)| < \delta$ then

$$\|F(t_1,s_1) - F(t_2,s_2)\| < \varepsilon.$$

Suppose $\pi_1 = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^{n}\} \in \mathcal{P}$, $\|\pi_1\| < \delta$, while $\pi_1' = \{(\chi_k,\{x_{k-1},x_k\})_{k=1}^{\ell}\} \in \mathcal{P}$ with $\pi_1' \geq \pi_1$, and for each $i = 1,2,\cdots,n$ let $\{x_{k_i-1},x_{k_i},\cdots,x_{k_i'}\}$ denote the set of x_k's in $[t_{i-1},t_i]$. Then if $\pi_2 = \{(\sigma_j,\{s_{j-1},s_j\})_{j=1}^{m}\} \in \mathcal{O}$ it follows that

$$\|N(\pi_1,\pi_2) - N(\pi_1',\pi_2)\|$$

$$\leq \|\sum_{j=1}^{m} \{\sum_{i=1}^{n} \sum_{k=k_i}^{k_i'} [G(x_k) - G(x_{k-1})][F(\tau_i,\sigma_j) - F(\chi_k,\sigma_j)][g(s_j) - g(s_{j-1})]\|$$

$$\leq \varepsilon\, V(G;a,b)\cdot V(g;a,b).$$

Theorems 1.1 and 1.2 of this Chapter then imply that

$$\lim_{\pi_1} \lim_{\pi_2} N(\pi_1,\pi_2) = \lim_{\pi_2} \lim_{\pi_1} N(\pi_1,\pi_2),$$

which is equivalent to the equation

$$\int_a^b [dG(x)]\{\int_c^d F(x,y)dg(y)\} = \int_c^d \{\int_a^b [dG(x)]F(x,y)\}dg(y).$$

THEOREM 4.2. *Suppose* $F:[a,b] \times [a,b] \to B$ *is continuous*, $G:[a,b] \to B$ *is continuous and of bounded variation, and* $g:[a,b] \to S$ *is continuous and of bounded variation. Then,*

$$\int_a^b [dG(x)]\{\int_a^x F(x,y)dg(y)\} = \int_a^b \{\int_y^b [dG(x)]F(x,y)\}dg(y).$$

A proof for this theorem will not be given since it is very similar to the proof of the next theorem.

THEOREM 4.3. *Suppose* $F(s,t)$ *is a function on* $[a,b] \times [a,b]$ *into* B *which is continuous on* $[a,b] \times [a,b]$ *and of totally uniform bounded variation in* s *on* $[a,b]$ *for* $t \in [a,b]$, $G:[a,b] \to B$ *is of bounded variation and continuous on* $[a,b]$, *and* $g:[a,b] \to B$ *is of bounded variation. Then,*

$$\int_a^b [dG(x)]\{\int_a^x F(x,y)dg(y)\} = \int_a^b \{\int_y^b [dG(x)]F(x,y)\}dg(y).$$

Theorem 3.1 shows $\int_a^x F(x,y)dg(y)$ is a function of x which is of bounded variation on $[a,b]$, and since G is continuous on $[a,b]$ it follows that the integral

$$\int_a^b [dG(x)]\{\int_a^x F(x,y)dg(y)\}$$

does exist (see MacNerney [6; Theorems 2.2, 2.3]). Also, Theorem 1.3 guarantees that $\int_y^b [dG(x)]F(x,y)$ is a continuous function of y on $[a,b]$, so that the fact that g is of bounded variation on $[a,b]$ insures that the integral

$$\int_a^b \{\int_y^b [dG(x)]F(x,y)\}dg(y)$$

exists (see [6; Theorem 2.3]). Now define a function $K:[a,b] \times [a,b] \to B$ by setting

$$K(x,y) = \begin{cases} F(x,y) & \text{if} \quad x \geq y \\ F(y,y) & \text{if} \quad x \leq y. \end{cases}$$

It may be verified readily that K is continuous on $[a,b] \times [a,b]$. Then applying Theorem 4.1, and noting that $K(x,y) = F(y,y)$ if $x < y$, it follows that

$$\int_a^b [dG(x)]\{\int_a^x F(x,y)dg(y)\}$$
$$= \int_a^b [dG(x)]\{\int_a^x K(x,y)dg(y)\}$$
$$= \int_a^b [dG(x)]\{\int_a^b K(x,y)dg(y)\} - \int_a^b [dG(x)]\{\int_x^b K(x,y)dg(y)\}$$
$$= \int_a^b \{\int_a^b [dG(x)]K(x,y)\}dg(y) - \int_a^b [dG(x)]\{\int_x^b F(y,y)dg(y)\}.$$

But

$$\int_a^b \{\int_a^b [dG(x)]K(x,y)\}dg(y)$$
$$= \int_a^b \{\int_a^y [dG(x)]K(x,y)\}dg(y) + \int_a^b \{\int_y^b [dG(x)]K(x,y)\}dg(y),$$

and

$$\int_a^b \{\int_a^y [dG(x)]K(x,y)\}dg(y) = \int_a^b [G(y) - G(a)]F(y,y)dg(y).$$

Furthermore,

$$\int_a^b [dG(x)]\{\int_x^b F(y,y)dg(y)\}$$
$$= -G(a)\int_a^b F(y,y)dg(y) - \int_a^b G(x)d_x[\int_x^b F(y,y)dg(y)]$$
$$= \int_a^b [G(y) - G(a)]F(y,y)dg(y).$$

The conclusion now follows by combining these results.

THEOREM 4.4. <u>If</u> $G:[a,b] \to B$ <u>is of bounded variation</u>, $F(s,t)$ <u>is a function on</u> $[a,b] \times [a,b]$ <u>into</u> B <u>which is continuous in</u> s <u>on</u> $[a,b]$ <u>for each fixed</u> $t \in [c,d]$ <u>and is of uniformly bounded variation in</u> t <u>for</u> $s \in [a,b]$, <u>and</u> $g:[c,d] \to S$ <u>is continuous</u>, <u>then</u>

$$(4.3) \qquad \int_a^b [dG(x)]\{\int_c^d F(x,y)dg(y)\} = \int_c^d \{\int_a^b [dG(x)]F(x,y)\}dg(y).$$

Suppose that it can be shown that

$$(4.4) \qquad \int_a^b [dG(x)]\{\int_c^d [d_yF(x,y)]g(y)\} = \int_c^d [d_y\{\int_a^b [dG(x)]F(x,y)\}]g(y),$$

in which process, of course, the existence of these integrals has been established. Then,

$$(4.5) \qquad \int_c^d [d_y\{\int_a^b [dG(x)]F(x,y)\}]g(y)$$
$$= \int_a^b [dG(x)]F(x,y)g(y))\Big|_{y=c}^{y=d} - \int_c^d \{\int_a^b [dG(x)]F(x,y)\}dg(y),$$

and

$$(4.6) \qquad \int_a^b [dG(x)]\{\int_c^d [d_yF(x,y)]g(y)\}$$
$$= \int_a^b [dG(x)]\{F(x,d)g(d) - F(x,c)g(c) - \int_c^d F(x,y)dg(y)\}.$$

Therefore, $\int_c^d \{\int_a^b [dG(x)]F(x,y)\}dg(y)$ exists, and since $F(x,d)g(d) - F(x,c)g(c)$ is a continuous function of x on $[a,b]$ and G is of bounded variation on $[a,b]$, we have that the integral

$$\int_a^b [dG(x)]\{F(x,d)g(d) - F(x,c)g(c)\}$$

exists (see MacNerney [6; Theorem 2.4]). The existence of the two integrals just mentioned and the relation (4.6) then proves that $\int_a^b [dG(x)]\{\int_c^d F(x,y)dg(y)\}$ exists. Using (4.4), (4.5), and (4.6) it now follows that (4.3) holds. To establish (4.4), let $(\mathcal{P},\geq)$ be the partition collection on $[a,b]$, let $(\mathcal{Q},\geq)$ be the partition collection on $[c,d]$, and let $(\mathcal{D},\geq)$ be the directed product of $(\mathcal{P},\geq)$ and $(\mathcal{Q},\geq)$. Consider the net $N:\mathcal{D} \to S$ defined by

$$N(\pi_1,\pi_2) = \sum_{i=1}^{n} [G(t_i) - G(t_{i-1})]\{\sum_{j=1}^{m} [F(\tau_i,s_j) - F(\tau_i,s_{j-1})]g(\sigma_j)\}$$

$$= \sum_{j=1}^{m} \{\sum_{i=1}^{n} [G(t_i) - G(t_{i-1})][F(\tau_i,s_j) - F(\tau_i,s_{j-1}]\}g(\sigma_j),$$

for each $\pi_1 = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^{n}\} \in \mathcal{P}$ and each $\pi_2 = \{(\sigma_j,\{s_{j-1},s_j\})_{j=1}^{m}\}$ in $\mathcal{O}$. It is clear that $\lim_{\pi_1} N(\pi_1,\pi_2)$ and $\lim_{\pi_2} N(\pi_1,\pi_2)$ both exist. Furthermore, $\lim_{\pi_2} N(\pi_1,\pi_2)$ exists uniformly for $\pi_1 \in \mathcal{P}$. For let $\varepsilon > 0$ be given and choose $\delta = \delta_\varepsilon > 0$ such that if $|y_1 - y_2| < \delta$ and $y_i \in [c,d]$, $(i = 1,2)$, then

$$\|g(y_1) - g(y_2)\| < \varepsilon.$$

Suppose $\pi_2 = \{(\sigma_j,\{s_{j-1},s_j\})_{j=1}^{m}\} \in \mathcal{O}$, $\|\pi_2\| < \delta$,

$\pi_2' = \{(\chi_k,\{x_{k-1},x_k\})_{k=1}^{\ell}\} \in \mathcal{O}$, and $\pi_2' \geq \pi_2$, and for each $j = 1,2,\cdots,n$ let $\{x_{k_j-1},x_{k_j},\cdots,x_{k_j'}\}$ be the set of all x_k's in $[s_{j-1},s_j]$. Then if $\pi_1 = \{(\tau_i,\{t_{i-1},t_i\})_{i=1}^{n}\} \in \mathcal{P}$ it follows that

$$\|N(\pi_1,\pi_2) - N(\pi_1,\pi_2')\|$$

$$= \|\sum_{i=1}^{n}[G(t_i) - G(t_{i-1})]\{\sum_{j=1}^{m} \sum_{k=k_j}^{k_j'} [F(\tau_i,x_k) - F(\tau_i,x_{k-1})][g(\sigma_j) - g(\chi_k)]\}\|$$

$$\leq \varepsilon\, M \cdot V(G;a,b),$$

where $M = \sup_{s \in [c,d]} V[F(s,-);a,b]$. Thus, by Theorem 1.1, $\lim_{\pi_2} N(\pi_1,\pi_2)$ is a uniform limit for $\pi_1 \in \mathcal{P}$. Now Theorem 1.2 shows that

$$\lim_{\pi_1} \lim_{\pi_2} N(\pi_1,\pi_2) = \lim_{\pi_2} \lim_{\pi_1} N(\pi_1,\pi_2).$$

But

$$\lim_{\pi_1} \lim_{\pi_2} N(\pi_1,\pi_2) = \lim_{\pi_1} \sum_{i=1}^{n}[G(t_i) - G(t_{i-1})]\{\int_c^d [d_yF(\tau_i,y)]g(y)\}$$

$$= \int_a^b [dG(x)]\{\int_a^b [d_yF(x,y)]g(y)\},$$

and

$$\lim_{\pi_2} \lim_{\pi_1} N(\pi_1,\pi_2) = \lim_{\pi_2} \sum_{j=1}^{m}\{\int_a^b [dG(x)][F(x,s_j) - F(x,s_{j-1})]\}g(\sigma_j)$$

$$= \int_c^d [d_y\{\int_a^b [dG(x)]F(x,y)\}]g(y),$$

and therefore (4.4) is established.

THEOREM 4.5. <u>Suppose</u> $G:[a,b] \to B$ <u>is of bounded variation and continuous on</u> $[a,b]$, $F(s,t)$ <u>is a function on</u> $[a,b] \times [a,b]$ <u>into</u> B <u>which is continuous on</u> $[a,b] \times [a,b]$, <u>of totally uniform bounded variation in</u> t <u>on</u> $[a,b]$ <u>for</u> $s \in [a,b]$, <u>and of bounded variation in</u> t <u>on</u> $[a,b]$ <u>continuously for</u> $s \in [a,b]$, <u>and</u> $g:[a,b] \to S$ <u>is continuous on</u> $[a,b]$. <u>Then,</u>

$$\int_a^b [dG(x)]\{\int_a^x F(x,y)dg(y)\} = \int_a^b \{\int_y^b [dG(x)]F(x,y)\}dg(y).$$

Theorem 3.2 shows $\int_a^x F(x,y)dg(y)$ is a continuous function of x on $[a,b]$, so the fact that G is of bounded variation on $[a,b]$ guarantees that the integral

$$\int_a^b [dG(x)]\{\int_a^x F(x,y)dg(y)\}$$

exists (see MacNerney [6; Theorem 2.4]). Also, by an argument similar to that used in the proof of Theorem 3.1, the function $\int_y^b [dG(x)]F(x,y)$ is of bounded variation in y on $[a,b]$, and since g is continuous, the integral

$$\int_a^b \{\int_y^b [dG(x)]F(x,y)\}dg(y)$$

exists (see [6; Theorems 2.4, 2.1]). Now define $K:[a,b] \times [a,b] \to B$ by setting

$$K(x,y) = \begin{cases} F(x,x) & \text{if } x \le y \\ F(x,y) & \text{if } x \ge y. \end{cases}$$

Then, as is easily verified, K is continuous on $[a,b] \times [a,b]$ and $K(s,t)$ is of uniformly bounded variation in t on $[a,b]$ for $s \in [a,b]$ so that by Theorem 4.4,

$$\int_a^b [dG(x)]\{\int_a^x F(x,y)dg(y)\}$$

$$= \int_a^b [dG(x)]\{\int_a^x K(x,y)dg(y)\}$$

$$= \int_a^b [dG(x)]\{\int_a^b K(x,y)dg(y)\} - \int_a^b [dG(x)]\{\int_x^b K(x,y)dg(y)\}$$

$$= \int_a^b \{\int_a^b [dG(x)]K(x,y)\}dg(y) - \int_a^b [dG(x)]\{\int_x^b F(x,x)dg(y)\}.$$

But

$$\int_a^b \{\int_a^y [dG(x)]F(x,x)\}dg(y)$$

$$= \int_a^b [dG(x)]F(x,x)g(b) - \int_a^b [d_y\{\int_a^y [dG(x)]F(x,x)\}]g(y)$$

$$= \int_a^b [dG(x)]F(x,x)g(b) - \int_a^b [dG(y)]F(y,y)g(y)$$

$$= \int_a^b [dG(x)]\{\int_x^b F(x,x)dg(y)\},$$

where the fact that the interval function $[dG(x)]F(x,x)$ is ω-integrable in B together with Theorem 2.5 of Chapter II has been used to assert that

$$\int_a^b [d_y\{\int_a^y [dG(x)]F(x,x)\}]g(y) = \int_a^b [dG(y)]F(y,y)g(y).$$

Finally since

$$\int_a^b \{\int_a^b [dG(x)]K(x,y)\}dg(y)$$

$$= \int_a^b \{\int_y^b [dG(x)]F(x,y)\}dg(y) + \int_a^b \{\int_a^y [dG(x)]F(x,x)\}dg(y),$$

the theorem follows.

CHAPTER IV

GENERALIZED DIFFERENTIAL EQUATIONS

1. <u>Introduction</u>. The present chapter will include some generalizatios of the terminology and results connected with generalized differential equations which are presented in MacNerney [6; Sections 3,4], and a discussion of the global behavior of the solution of a certain nonlinear Riccati generalized differential equation will be given. Throughout this chapter, J will denote a non-degenerate subinterval of the reals, and H_J will denote the set of all functions $M(s,t)$ on $J \times J$ into B satisfying the following four conditions:

(i) for each $t \in J$, $M(s,t)$ is of bounded variation in s on each compact subinterval of J and continuous in s on J;

(ii) for each $s \in J$, $M(s,t)$ is bounded in t on each compact subinterval of J and continuous in t on J in the strong topology on B; that is, for each $x \in S$, the function $M(s,t)(x)$ is a continuous function in t;

(iii) for each $s \in J$, $M(s,s) = E$;

(iv) for arbitrary points $(r,s,t) \in J \times J \times J$, $M(s,t)M(t,r) = M(s,r)$.

The elements of H_J will be called the <u>harmonic operators</u> on J. Harmonic matrices were introduced by Wall [9; p. 160] and subsequently the more general idea of a harmonic operator was given by MacNerney [6]. The definition of harmonic operator given in this paper differs from that of MacNerney in that he assumed the function $M(s,t)$ to have as its domain $R \times R$, where R denotes the set of all real numbers. The symbol ϕ_J will be used to denote the set of all functions $F:J \to B$ which are continuous and of bounded variation on each compact subinterval of J.

The following two theorems can now be established in essentially the same manner as Theorems 3.1 and 3.2 presented in [6], and therefore the proofs will be omitted. The principal difference between these theorems and those given in [6] is that MacNerney took J as the set of all real numbers and required that $F(0) = 0$. It should be noted that Theorem 1.2 below shows that the condition (ii) given in the definition of harmonic operators may be replaced by a much stronger condition; namely, for each $s \in J$, the function $M(s,t)$ is of bounded variation in t on each compact subinterval of J and continuous in t on J.

THEOREM 1.1. *For each* $F \in \phi_J$ *there is one and only one* $M \in H_J$ *such that the relation*

$$M(s,t) = E + \int_s^t [dF(u)]M(u,t) \tag{1.1}$$

holds for each $(s,t) \in J \times J$. *If* $\tau \in J$ *and* $T \in B$, *then for each* $M \in H_J$ *there is one and only one* $F \in \phi_J$ *such that* $F(\tau) = T$ *and the relation* (1.1) *holds on* $J \times J$.

If $F \in \phi_J$ and M is the member of H_J such that (1.1) holds on $J \times J$, then we will write $M \sim F$.

THEOREM 1.2. *If* $F \in \phi_J$, $M \in H_J$ *is such that* $M \sim F$, $T \in B$, $\tau \in J$, *and* $F(\tau) = T$, *then each of the following statements is true:*

(i) *The function* M *is continuous on* $J \times J$ *and for each* $(s,t) \in J \times J$,

$$\|M(s,t) - E\| \leq \exp\{V(F;s,t)\} - 1.$$

(ii) *If* $F(a)F(b) = F(b)F(a)$ *for each* $(a,b) \in J \times J$, *and* $(s,t) \in J \times J$, *then*

$$M(s,t) = \exp\{F(t) - F(s)\}.$$

(iii) *For each* $(r,s) \in J \times J$,

$$F(s) = T + \int_s^\tau [d_u M(u,r)]M(r,u) = T + \int_\tau^s M(u,r)[d_u M(r,u)].$$

(iv) *For each* $(s,t) \in J \times J$,

$$M(s,t) = E + \int_s^t M(s,u)dF(u).$$

(v) *If* Q *is a compact subset of* J, *then* $M(s,t)$ *is of bounded variation in* s *on each compact subinterval of* J *uniformly for* $t \in Q$, *and is of bounded variation in* t *on each compact subinterval of* J *uniformly for* $s \in Q$.

The following theorem and its corollary are stronger forms of the Theorems 3.3 and 3.4 presented in MacNerney [6]. In Theorem 3.3 for example, MacNerney required the domain J of F to be the set of all real numbers, F(0) = 0, and the function g in the theorem below was required to be of bounded variation on each compact subinterval of the reals. In the theorem below, J is an arbitrary subinterval of the real numbers, F(0) is not restricted to be 0, and g can be continuous or of bounded variation on each compact subinterval of J. A similar statement holds for the relation between the corollary to Theorem 1.3 and MacNerney's Theorem 3.4 given in [6].

THEOREM 1.3. *Suppose* $r \in J$, $x \in S$, *and* $g:J \to S$ *is of bounded variation (or continuous) on each compact subinterval of* J. *If* $F \in \phi_J$ *and* $M \in H_J$ *is such that* $M \sim F$, *then the only function* f *on* J *into* S *such that*

(i) $f(r) = x$ <u>and</u> f <u>is bounded on each compact subinterval of</u> J, <u>and</u>

(ii) $f(t) - f(s) + \int_s^t [dF]f = g(t) - g(s)$ <u>for each</u> $(s,t) \in J \times J$,

<u>is the function</u> f <u>defined by</u>

$$f(s) = M(s,r)x + \int_r^s M(s,u)dg(u).$$

Note that (i) and (ii) hold if and only if f is of bounded variation (or continuous) on each compact subinterval of J (see [6; Theorem 2.6]) and

$$f(t) = x - \int_r^t [dF]f + \int_r^t dg \tag{1.2}$$

holds for $t \in J$.

An elegant proof of uniqueness may now be obtained as follows. Note that whether f is of bounded variation or continuous on each compact subinterval of J, the fact that F is both of bounded variation and continuous on each compact subinterval of J insures that the interval function $[dF]f$ is ω-integrable on $[r,t]$ for each $t \in J$, and hence using Theorems 2.4 and 2.5 of Chapter II and the relation in Theorem 1.2 (iv), it follows that (1.2) holds for a function f which is of bounded variation (or continuous) on each compact subinterval of J only if for each $t \in J$ we have

$$\begin{aligned}\int_r^t M(r,u)df(u) &= -\int_r^t M(r,u)d_u\{\int_r^u [dF]f\} + \int_r^t M(r,u)dg(u)\\ &= -\int_r^t M(r,u)[dF(u)]f(u) + \int_r^t M(r,u)dg(u)\\ &= -\int_r^t [d_uM(r,u)]f(u) + \int_r^t M(r,u)dg(u).\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}\int_r^t M(r,u)dg(u) &= \int_r^t M(r,u)df(u) + \int_r^t [d_uM(r,u)]f(u)\\ &= M(r,u)f(u)\Big|_{u=r}^{u=t}\\ &= M(r,t)f(t) - f(r),\end{aligned}$$

so that

$$f(t) = M(t,r)[f(r) + \int_r^t M(r,u)dg(u)] \tag{1.3}$$
$$= M(t,r)x + \int_r^t M(r,u)dg(u).$$

This proves uniqueness as well as demonstrating the exact form of the function f.

Using an approach similar to that which Wall employed in [9] to establish a similar theorem in which S was taken as complex n-space, write

$$f(t) = M(t,r)u(t),$$

where

$$u(t) = x + \int_r^t M(r,u)dg(u).$$

Note that

$$u(t) = x + \int_r^t M(r,u)dg(u) = x + M(r,t)g(t) - g(r) - \int_r^t [d_u M(r,u)]g(u),$$

so that u is of bounded variation (or continuous) on each compact subinterval of J if g has this property (see MacNerney [6; Theorems 2.5, 2.6]); the same is also true of f since M(r,t) is continuous and of bounded variation in t on each compact subinterval of J. Then since the point-interval function [dF(s)]M(s,r) is ω-integrable in B and the point-interval function M(r,u)dg(u) is ω-integrable, Theorems 2.4 and 2.5 of Chapter II and relation (1.1) in Theorem 1.1 can be used to show that

$$x - \int_r^t [dF]f$$
$$= x - \int_r^t [dF(s)]M(s,r)u(s)$$
$$= x - \int_r^t [d_s\{\int_r^s [dF(y)]M(y,r)\}]u(s)$$
$$= x - \{\int_r^t [dF(y)]M(y,r)\}u(t) + \int_r^t \{\int_r^s [dF(y)]M(y,r)\}du(s)$$

$$= u(r) + [M(t,r) - E]u(t) + \int_r^t [E - M(s,r)]du(s)$$

$$= f(t) - \int_r^t M(s,r)du(s)$$

$$= f(t) - \int_r^t M(s,r)M(r,s)dg(s) = f(t) - \int_r^t dg(s).$$

COROLLARY. *Suppose that* $r \in J$, $T \in B$, *and* $g:J \to S$ *is of bounded variation (or continuous) on each compact subinterval of* J. *If* $F \in \phi_J$, $M \in H_J$, *and* $M \sim F$, *then the only function* f *on* J *into* S *such that*

(i) f *is bounded on each compact subinterval of* J, *and*

(ii) $f(s) = g(s) + \int_s^r [dF]f$ *for all* $s \in J$,

is the function f *on* J *into* S *defined by*

$$f(s) = g(s) + \int_r^s [d_uM(s,u)]g(u). \tag{1.4}$$

Since (ii) implies $f(r) = g(r)$, with $x = f(r)$, this equation may be written as

$$f(t) = x - \int_r^t [dF]f + \int_r^t dg,$$

which by the previous theorem has the unique solution

$$f(t) = M(t,r)x + \int_r^t M(t,u)dg(u)$$

$$= g(t) + \int_r^t [d_uM(t,u)]g(u).$$

In order to give a second rather interesting proof of the fact that the function $f(t)$ defined in Theorem 1.3 satisfies the relation (1.2) in that theorem, the following is presented.

THEOREM 1.4. *Suppose that* $M \in H_J$, *and* $[a,b]$ *and* $[c,d]$ *are two compact subintervals of* J. *Then* $M(s,t)$ *is of bounded variation in* t *on* $[c,d]$ *continuously for* $s \in [a,b]$ *and of totally uniform bounded variation in* t

on $[c,d]$ *for* $s \in [a,b]$. *The same type of results holds if the roles of* s *and* t *are interchanged.*

Let $\tau \in J$, $T \in B$, and choose $F \in \phi_J$ such that $F(\tau) = T$ and $M \sim F$. Suppose $\{t_i\}_{i=0}^n$ is an arbitrary partition of $[c,d]$. Then,

$$\begin{aligned}\sum_{i=1}^n \{\sup_{s \in [a,b]} \|M(s,t_i) - M(s,t_{i-1})\|\} &= \sum_{i=1}^n \{\sup_{s \in [a,b]} \|\int_{t_{i-1}}^{t_i} M(s,u)dF(u)\|\} \\ &\leq V(F;c,d)\|M\|,\end{aligned}$$

where $\|M\| = \|M\|_{[a,b] \times [c,d]}$. This proves that $M(s,t)$ is of totally uniform bounded variation in t on $[c,d]$ for $s \in [a,b]$. To prove that $M(s,t)$ is of bounded variation in t on $[c,d]$ continuously for $s \in [a,b]$, let $\varepsilon > 0$ be given. Choose $\delta = \delta_\varepsilon > 0$ such that if $(u,v) \in [a,b] \times [c,d]$, $(z,w) \in [a,b] \times [c,d]$, and $|(u,v) - (z,w)| < \delta$, then

$$\|M(u,v) - M(z,w)\| < \varepsilon.$$

If $x \in [a,b]$, $y \in [a,b]$, and $|x - y| < \delta$, it follows that if $\{t_i\}_{i=0}^n$ is an arbitrary partition of $[c,d]$, then

$$\begin{aligned}\sum_{i=1}^n \|M(x,t_i) - M(y,t_i) - M(x,t_{i-1}) + M(y,t_{i-1})\| &= \sum_{i=1}^n \|\int_{t_{i-1}}^{t_i} (M(x,u) - M(y,u))dF(u)\| \\ &\leq \varepsilon\, V(F;c,d).\end{aligned}$$

Thus, if $x \in [a,b]$, $y \in [a,b]$, and $|x - y| < \delta$ then

$$V[M(x,-) - M(y,-);c,d] \leq \varepsilon\, V(F;c,d).$$

Now a second proof will be presented to show that (1.3) satisfies (1.2) on J in Theorem 1.3 if g is of bounded variation (or continuous) on each compact subinterval of J. Employing Theorem 1.4, relation (1.1) in Theorem 1.1, Theorem 4.3 of Chapter III (or Theorem 4.5 of Chapter III), and assuming f is defined by (1.3) on J, it follows that

$$x - \int_r^t [dF]f + \int_r^t dg$$

$$= x - \int_r^t [dF(u)]\{M(u,r)x + \int_r^u M(u,v)dg(v)\} + \int_r^t dg$$

$$= x + [M(t,r) - E]x - \int_r^t \{\int_v^t [dF(u)]M(u,v)\}dg(v) + \int_r^t dg$$

$$= M(t,r)x + \int_r^t [M(t,v) - E]dg(v) + \int_r^t dg$$

$$= M(t,r)x + \int_r^t M(t,v)dg(v) = f(t).$$

2. <u>A non-linear equation</u>. The three following results are generalizations of results given in section 4 in MacNerney [6].

THEOREM 2.1. *If* $k \in J$, *and for* $i = 1,2$ *we have* $G_i \in \phi_J$, $M_i \in H_J$, *and* $M_i \sim G_i$, *then for* $Q \in B$ *the only continuous function* P *on* J *into* B *such that*

$$P(t) = Q + \int_k^t P[dG_2] - \int_k^t [dG_1]P, \tag{2.1}$$

for $t \in J$ *is the function* $P:J \to B$ *given by*

$$P(t) = M_1(t,k)QM_2(k,t) \tag{2.2}$$

for $t \in J$.

The following is a rather interesting proof of uniqueness. If P(t) is a continuous function on J satisfying (2.1), then the fact that the G_i are continuous and of bounded variation on each compact subinterval of J insures that P will also have this property (see [6, Theorem 2.7]) and hence the integral

$$\int_k^t M_1(k,u)[dP(u)]M_2(u,k)$$

will exist for each $t \in J$. Furthermore, the point-interval functions

$P[dG_2]$ and $[dG_1]P$ are ω-integrable in B over each compact subinterval of J so that upon using Theorem 2.5 of Chapter II, the relation (1.1) of Theorem 1.1, and the relation in Theorem 1.2 (iv), we have

$$\int_k^t M_1(k,u)[dP(u)]M_2(u,k)$$
$$= \int_k^t M_1(k,u)P(u)[dG_2(u)]M_2(u,k) - \int_k^t M_1(k,u)[dG_1(u)]P(u)M_2(u,k)$$
$$= -\int_k^t M_1(k,u)P(u)d_uM_2(u,k) - \int_k^t [d_uM_1(k,u)]P(u)M_2(u,k).$$

But Theorem 2.7 of Chapter II shows that

$$\int_k^t M_1(k,u)[dP(u)]M_2(u,k)$$
$$= M_1(k,u)P(u)M_2(k,u)\Big|_{u=k}^{u=t} - \int_k^t M_1(k,u)P(u)d_uM_2(u,k)$$
$$- \int_k^t [d_uM_1(k,u)]P(u)M_2(u,k).$$

Hence, $M_1(k,t)P(t)M_2(t,k) = P(k) = Q$, or

$$P(t) = M_1(t,k)QM_2(k,t). \tag{2.2}$$

Conversely, if (2.2) holds on J, then upon using Theorem 2.7 of Chapter II we obtain

$$Q + \int_k^t M_1(u,k)QM_2(k,u)dG_2(u) - \int_k^t [dG_1(u)]M_1(u,k)QM_2(k,u)$$
$$= Q + \int_k^t M_1(u,k)Qd_uM_2(k,u) + \int_k^t [d_uM_1(u,k)]QM_2(k,u)$$
$$= Q + M_1(u,k)QM_2(k,u)\Big|_{u=k}^{u=t}$$
$$= M_1(t,k)QM_2(k,t) = P(t).$$

The following corollary to Theorem 2.1 can now be established quite easily.

COROLLARY. _If the hypotheses of Theorem_ 2.1 _are satisfied and in_ *addition* $Q^{-1} \in B$, *then the solution* $P(t)$ *of* (2.1) *is such that* $P^{-1}(t) \in B$ *for* $t \in J$.

The theorem which follows is a generalization of Lemma 2, p. 363 in [6], and can be proved in the same manner as this lemma is established in [6].

THEOREM 2.2. *If* $k \in J$, F_{ij} $(i,j = 1,2)$ *are continuous functions on* J *which are of bounded variation on each compact subinterval of* J, $Q_i \in B$ *for* $i = 1,2$, *and* $P_i : J \to B\ (i = 1,2)$ *are continuous on* J *and*

$$P_i(t) = Q_i + \int_k^t P_i[dF_{21}]P_i + \int_k^t P_i[dF_{22}] - \int_k^t [dF_{11}]P_i - \int_k^t dF_{12}$$

for $t \in J$, *then* $P_1 - P_2$ *satisfies the equation*

$$P_1(t) - P_2(t) = (Q_1 - Q_2) + \int_k^t (P_1 - P_2)[dG_2] - \int_k^t [dG_1](P_1 - P_2)$$

where

$$G_1(t) = F_{11}(t) - \frac{1}{2}\int_k^t (P_1 + P_2)[dF_{21}]$$

and

$$G_2(t) = F_{22}(t) + \frac{1}{2}\int_k^t [dF_{21}](P_1 + P_2).$$

The following existence theorem can now be stated. The proof is exactly the same as the proof of the corresponding Theorem 4.1 in MacNerney's paper [6], where Theorems 2.1 and 2.2 of this Chapter are used in place of MacNerney's Lemma 1 and Lemma 2 given on pp. 362-363 in [6].

THEOREM 2.3. *If* $F_{ij} \in \phi_J$ $(i,j = 1,2)$, $k \in J$, *and* $Q \in B$, *then there is an open subset* K *of* J *containing* k *and such that the only continuous function* $P : K \to B$ *such that*

$$P(t) = Q + \int_k^t P[dF_{21}]P + \int_k^t P[dF_{22}] - \int_k^t [dF_{11}]P - \int_k^t dF_{12} \tag{2.3}$$

for $t \in K$, is the function $P:K \to B$ given by

$$P(t) = L(t)R^{-1}(t),$$

where $(L(t),R(t))$ is the pair of continuous functions on J into B which satisfy the linear integral equations

$$(2.4)\qquad \begin{cases} L(t) = Q - \int_k^t [dF_{11}]L - \int_k^t [dF_{12}]R \\ R(t) = E - \int_k^t [dF_{21}]L - \int_k^t [dF_{22}]R \end{cases}$$

for $t \in J$.

The theorem above is an existence theorem. In order to investigate the global behavior of a continuous function P satisfying (2.3), the following results are presented.

THEOREM 2.4. If $A:[a,b] \to B$ is continuous on $[a,b]$ and $A^{-1}(t) \in B$ for each $t \in [a,b]$, then $A^{-1}:[a,b] \to B$ is continuous on $[a,b]$.

For each $t \in [a,b]$, set

$$m(t) = \inf_{\|x\| = 1}\|A(t)(x)\|.$$

Now, $m(t) > 0$ for each t in $[a,b]$ since $A^{-1}(t) \in B$ for each $t \in [a,b]$. Let $\varepsilon > 0$ be given, and choose $\delta = \delta_\varepsilon > 0$ such that if $|t - s| < \delta$, $t \in [a,b]$, and $s \in [a,b]$ then

$$\|A(t) - A(s)\| < \varepsilon.$$

Then if $x \in S$, $\|x\| = 1$, $t \in [a,b]$, $s \in [a,b]$, and $|t - s| < \delta$, it follows that

$$\Big| \|A(t)(x)\| - \|A(s)(x)\| \Big| \le \|A(t) - A(s)\| < \varepsilon,$$

and therefore,

$$|m(s) - m(t)| \le \varepsilon.$$

Consequently m is a continuous function on $[a,b]$ and since $m(t) > 0$ on this interval there is a real number $m_o > 0$ such that $m(t) \ge m_o$ for

$t \in [a,b]$. Since $\|A^{-1}(t)\| \le 1/m(t) \le 1/m_o$, it now follows that for each s and t in $[a,b]$ we have

$$\|A^{-1}(t) - A^{-1}(s)\| = \|A^{-1}(t)(A(s) - A(t))A^{-1}(s)\|$$
$$\le \frac{1}{m_o^2} \|A(s) - A(t)\|,$$

and hence the continuity of A on $[a,b]$ shows that $A^{-1}(t)$ is continuous as a mapping from $[a,b]$ into B.

THEOREM 2.5. _If_ $A:[a,b] \to B$ _is of bounded variation on_ $[a,b]$, $A^{-1}(t) \in B$ _for_ $t \in [a,b]$, _and if there is a positive constant_ m _such that_ $\|A^{-1}(t)\| \le m$ _on_ $[a,b]$, _then_ $A^{-1}(t)$ _is of bounded variation in_ t _on_ $[a,b]$.

If $\{t_i\}_{i=0}^n$ is an arbitrary partition of $[a,b]$, then

$$\sum_{i=1}^n \|A^{-1}(t_i) - A^{-1}(t_{i-1})\|$$
$$= \sum_{i=1}^n \|A^{-1}(t_i)(A(t_{i-1}) - A(t_i))A^{-1}(t_{i-1})\|$$
$$\le m^2\, V(A;a,b).$$

Consequently, $V(A^{-1};a,b) \le m^2\, V(A;a,b)$, and $A^{-1}(t)$ is of bounded variation in t on $[a,b]$.

As a direct consequence of Theorems 2.4 and 2.5, we have the following result.

THEOREM 2.6. _If_ $A:[a,b] \to B$ _is continuous and of bounded variation on_ $[a,b]$ _and_ $A^{-1}(t) \in B$ _for each_ $t \in [a,b]$, _then_ $A^{-1}:[a,b] \to B$ _is continuous and of bounded variation on_ $[a,b]$.

For convenience, the following terminology is now introduced. Suppose K is a non-degenerate subinterval of J and $F_{ij} \in \phi_J$ $(i,j = 1,2)$. If $Q \in B$ and $k \in K$, then any continuous function $P:K \to B$ which satisfies (2.3) on K will be called a _solution_ of (2.3) on K. If $G_i \in B$ $(i = 1,2)$

and $k \in K$, then any pair of continuous functions $(L;R)$ on K to B which satisfy

$$(2.5)\qquad \begin{cases} L(t) = G_1 - \int_k^t [dF_{11}]L - \int_k^t [dF_{12}]R \\ R(t) = G_2 - \int_k^t [dF_{21}]L - \int_k^t [dF_{22}]R \end{cases}$$

for $t \in K$ will be termed a <u>solution</u> of 2.5 on K.

THEOREM 2.7. <u>Suppose</u> K <u>is a non-degenerate subinterval of</u> J <u>and</u> $F_{ij} \in \phi_J$ $(i,j = 1,2)$. <u>If</u> $Q \in B$ <u>and</u> $k \in K$, <u>then there exists a solution</u> P <u>of the equation</u>

$$(2.3)\qquad P(t) = Q + \int_k^t P[dF_{21}]P + \int_k^t P[dF_{22}] - \int_k^t [dF_{11}]P - \int_k^t dF_{12}$$

<u>on</u> K <u>if and only if for each</u> $N \in B$ <u>such that</u> $N^{-1} \in B$, <u>there is a solution of</u>

$$(2.6)\qquad \begin{cases} L(t) = QN - \int_k^t [dF_{11}]L - \int_k^t [dF_{12}]R \\ R(t) = N - \int_k^t [dF_{21}]L - \int_k^t [dF_{22}]R \end{cases}$$

<u>on</u> K <u>having the property that</u> $R^{-1}(t) \in B$ <u>for each</u> $t \in K$. <u>Furthermore, if</u> $(L;R)$ <u>is a solution of</u> (2.6) <u>on</u> K <u>such that</u> $R^{-1}(t) \in B$ <u>for each</u> $t \in K$, <u>then the function</u> $P:K \to B$ <u>defined by</u>

$$P(t) = L(t)R^{-1}(t)$$

<u>is the solution of</u> (2.3) <u>on</u> K.

Suppose (2.3) has a solution P on K, and $N \in B$ is such that $N^{-1} \in B$. Theorem 2.5 of Chapter II implies that a continuous function $R:K \to B$ will satisfy the equation

$$(2.7)\qquad R(t) = N - \int_k^t [dF_{21}]PR - \int_k^t [dF_{22}]R$$

<u>for</u> $t \in K$ <u>if and only if for each</u> $t \in K$ <u>we have</u>

$$(2.8) \qquad R(t) = N - \int_k^t [d_s\{\int_k^s [dF_{21}]P - F_{22}(s)\}]R(s).$$

Now MacNerney's Theorem 2.7 in [6] shows $\int_k^t [dF_{21}]P$ is continuous for $t \in K$ and of bounded variation in t on each closed subinterval [a,b] of K. Therefore, we have that

$$F(s) = (\int_k^s [dF_{21}]P - F_{22}(s)) \in \phi_K.$$

If $M \in H_K$ and $M \sim F$, then Theorem 2.1 insures the existence of a unique continuous function $R:K \to B$ which satisfies (2.7) on K and which is given by

$$R(t) = M(t,k)N.$$

Since $N^{-1} \in B$, by hypothesis, and $M^{-1}(t,k) = M(k,t) \in B$ for each $t \in K$, it then follows that

$$R^{-1}(t) = N^{-1}M(k,t) \in B$$

for each $t \in K$, and since $M(k,t)$ is continuous and of bounded variation in t on each compact subinterval of K it follows that $R^{-1}(t)$ is continuous and of bounded variation on each compact subinterval of K.

Now set

$$L(t) = P(t)R(t),$$

for $t \in K$. Since (2.7) holds on K, the equation

$$R(t) = N - \int_k^t [dF_{21}]L - \int_k^t [dF_{22}]R$$

must be true for $t \in K$. Also,

$$\begin{aligned}
L(t) &+ \int_k^t [dF_{11}]L + \int_k^t [dF_{12}]R \\
&= P(t)R(t) + \int_k^t [dF_{11}]PR + \int_k^t [dF_{12}]R \\
&= QN + \int_k^t d[PR] + \int_k^t [dF_{11}]PR + \int_k^t [dF_{12}]R \\
&= QN + \int_k^t [dP]R + \int_k^t P[dR] + \int_k^t [dF_{11}]PR + \int_k^t [dF_{12}]R.
\end{aligned}$$

Using (2.7), Theorem 2.5 of Chapter II, and the fact that P is a solution of (2.3) on K, the quantity above becomes

$$QN + \int_k^t [dP]R - \int_k^t P([dF_{21}]PR + [dF_{22}]R) + \int_k^t [dF_{11}]PR + \int_k^t [dF_{11}]R$$

$$= QN + \int_k^t [d_s\tilde{P}(s)]R(s) = QN,$$

where

$$\tilde{P}(s) = P(s) - Q - \int_k^s P[dF_{21}]P - \int_k^s P[dF_{22}] + \int_k^s [dF_{11}]P + \int_k^s dF_{12}.$$

Consequently, (L;R) is a solution of (2.6) on K such that $R^{-1}(t) \in B$ for each $t \in K$.

Conversely, suppose (L;R) is a solution of (2.6) on J such that $R^{-1} \in B$ for each $t \in K$. R(t) is necessarily continuous on K and of bounded variation on each closed subinterval of K, so that Theorem 2.6 guarantees that $R^{-1}(t)$ is continuous on K and of bounded variation on each compact subinterval of K. Set

$$P(t) = L(t)R^{-1}(t)$$

for $t \in K$. Then the equation

$$R(t) = N - \int_k^t [dF_{21}]L - \int_k^t [dF_{22}]R$$

holds and an application of Theorem 2.5 of Chapter II yields the relation

$$\int_k^t P[dF_{21}]P + \int_k^t P[dF_{22}] = -\int_k^t P[dR]R^{-1}.$$

But now the fact that

$$L(t) = QN - \int_k^t [dF_{11}]L - \int_k^t [dF_{12}]R,$$

for $t \in K$ and further applications of Theorem 2.5 of Chapter II gives

$$-\int_k^t P[dR]R^{-1} = -\int_k^t P(s)d_s(\int_k^s [dR]R^{-1})$$

$$= -\int_k^t P(s)d_s(-\int_k^s R[dR^{-1}])$$

$$= \int_k^t PR[dR^{-1}]$$

$$= P(t) - P(k) - \int_k^t [dL]R^{-1}$$

and

$$\int_k^t [dL]R^{-1} = \int_k^t [d_s(QN - \int_k^s [dF_{11}]L - \int_k^s [dF_{12}]R)]R^{-1}(s)$$

$$= - \int_k^t [dF_{11}]P - \int_k^t [dF_{12}].$$

Then noting that $P(k) = Q$ it follows that

$$\int_k^t P[dF_{21}]P + \int_k^t P[dF_{22}] = P(t) - Q + \int_k^t [dF_{11}]P + \int_k^t dF_{12},$$

and hence P is the solution of (2.3) on K.

The following lemma is now an immediate consequence of Theorem 2.7.

LEMMA 2.1. _Suppose_ $R \in \phi_J$, $k \in J$, _and_ $R(k) = E$. _If_ $L(t) \equiv E$ _on_ J _and_ $F(t) = E - R(t)$ _for_ $t \in J$, _then_ $(L;R)$ _is the solution of_

$$L(t) = E \qquad (2.9)$$

$$R(t) = E - \int_k^t [dF]L$$

_on_ J. _Furthermore, if_ J_o _is the maximal subinterval of_ J _containing_ k _and such that_ $R^{-1}(t) \in B$ _for each_ $t \in J_o$, _then_

$$P(t) = R^{-1}(t)$$

is the solution of the equation

$$P(t) = E - \int_k^t P[dF]) \qquad (2.10)$$

_on_ J_o, _and_ J_o _is the maximal subinterval of_ J _containing_ k _on which_ (2.10) _has a solution_.

Using this lemma an example will now be given to illustrate what may happen at an end point of a maximal interval of existence of a solution of 2.3.

Let $S = \ell^2$ and $B = [\ell^2]$, the Banach space of all continuous linear transformations on ℓ^2. Let e_n for $n = 1,2,\cdots$ be the element of ℓ^2 having 0 in all coordinate positions except the n-th, where it has the value 1. Now, for each real value t, define a linear transformation on ℓ^2 by setting

$$T(t)(e_i) = \begin{cases} te_i & \text{if} \quad t \geq \frac{1}{i}, \\ \frac{1}{i} e_i & \text{if} \quad t \leq \frac{1}{i}, \end{cases}$$

and then extending T(t) to all of S by defining

$$T(t)(\textstyle\sum_{i=1}^{\infty} \alpha_i e_i) = \sum_{i=1}^{\infty} \alpha_i T(t)(e_i)$$

for each $\sum_{i=1}^{\infty} \alpha_i e_i \in \ell^2 = s$. It is to be noted that $T(t)(\sum_{i=1}^{\infty} \alpha_i e_i)$ is also in ℓ^2, as asserted, since

$$(2.11) \quad T(t)(\textstyle\sum_{i=1}^{\infty} \alpha_i e_i) = \begin{cases} \sum_{i=1}^{\infty} \frac{1}{i} \alpha_i e_i & , \text{ if } t \leq 0, \\ \sum_{i=1}^{j(t)-1} \frac{1}{i} \alpha_i e_i + t(\sum_{i=j(t)}^{\infty} \alpha_i e_i), & \text{if } 0 < t < 1, \\ t(\sum_{i=1}^{\infty} \alpha_i e_i) & , \text{ if } t \geq 1, \end{cases}$$

where for a given $t \in (0,1)$ the symbol $j(t)$ in the above denotes the least positive integer n such that $\frac{1}{n} \leq t$. From (2.11) it is easy to see that

$$\|T(t)\| = \begin{cases} t & \text{if} \quad t > 1, \\ 1 & \text{if} \quad t < 1, \end{cases}$$

so that $T(t) \in B$ for each real number t. If $t \in (0,\infty)$, then from (2.11) it is clear that if $x \in S$ and $\|x\| = 1$, then

$$\|T(t)(x)\| \geq \begin{cases} t & \text{if} \quad t \geq 1, \\ \frac{1}{j(t)} & \text{if} \quad 0 < t < 1. \end{cases}$$

Also, if $\sum_{i=1}^{\infty} \alpha_i e_i \in \ell^2$, then (2.11) shows that if $0 < t < 1$ then

$$T(t)(\sum_{i=1}^{j(t)-1} i\alpha_i e_i + \frac{1}{t} (\sum_{i=j(t)}^{\infty} \alpha_i e_i)) = \sum_{i=1}^{\infty} \alpha_i e_i,$$

while if $t \geq 1$ we have

$$T(t)(\frac{1}{t} [\sum_{i=1}^{\infty} \alpha_i e_i]) = \sum_{i=1}^{\infty} \alpha_i e_i.$$

Therefore, $T^{-1}(t) \in B$ for each $t \in (0,\infty)$. However, although $T^{-1}(t)$ clearly exists for $t \leq 0$, it is not a continuous linear mapping and its domain is not $S = \ell^2$. This follows from the fact that the bounded sequence $\{e_1, e_2, \cdots, e_n, \cdots\}$ is contained in the domain of $T^{-1}(t)$ and is mapped onto the unbounded sequence $\{e_1, 2e_2, 3e_3, \cdots, ne_n, \cdots\}$, so that $T^{-1}(t)$ is not continuous (as a linear mapping), while if the domain of $T^{-1}(t)$ were equal to S, then Theorem 4.2-H in Taylor [8] would imply $T^{-1}(t) \in B$.

Furthermore, we note that $T(t)$ is continuous and of bounded variation on each compact subinterval of the reals. For if t_1 and t_2 are both real numbers, then

$$\|(T(t_1) - T(t_2))(e_i)\| = \begin{cases} |t_1 - t_2| & \text{if } t_1 \geq \frac{1}{i},\ t_2 \geq \frac{1}{i}, \\ 0 & \text{if } t_1 \leq \frac{1}{i},\ t_2 \leq \frac{1}{i}, \\ |t_1 - \frac{1}{i}| & \text{if } t_1 \geq \frac{1}{i},\ t_2 < \frac{1}{i} \end{cases}$$

$$\leq |t_1 - t_2|.$$

Therefore it follows that if $x = \sum_{i=1}^{\infty} \alpha_i e_i \in \ell^2$ and $\|x\| = 1$, then

$$\begin{aligned} \|(T(t_1) - T(t_2))(x)\| &= \{\sum_{i=1}^{\infty} |\alpha_i|^2 \|(T(t_1) - T(t_2))(e_i)\|^2\}^{\frac{1}{2}} \\ &\leq |t_1 - t_2| \{\sum_{i=1}^{\infty} |\alpha_i|^2\}^{\frac{1}{2}} \\ &= |t_1 - t_2|, \end{aligned}$$

and consequently $T(t)$ is uniformly continuous on the set of all real numbers. Also, if $[a,b]$ is a compact subinterval of the reals and $\{t_i\}_{i=0}^n$ is any partition of $[a,b]$, then

$$\sum_{i=1}^n \|T(t_i) - T(t_{i-1})\| \le \sum_{i=1}^n |t_i - t_{i-1}| = b - a,$$

so that $V(T;a,b) \le b - a$, and $T(t)$ is of bounded variation on $[a,b]$.

Now define a function R on the reals into B by setting

$$R(t) = T(t+1)$$

for each real number t. Then R is uniformly continuous on the set of real numbers, of bounded variation on each compact subinterval of the reals, $R^{-1}(t) \in B$ for $t \in (-1,\infty)$, $R^{-1}(t)$ exists for $t \in (-\infty,-1]$ but is not continuous as a linear mapping and does not have domain S, and $R(0) = E$. Upon applying Lemma 1, it follows that if

$$F(t) = E - R(t)$$

on the set of real numbers and $L(t) = E$ for each real t, then $(L;R)$ is the solution of

$$\begin{cases} L(t) = E, \\ R(t) = E - \int_0^t [dF]L \end{cases}$$

on the set of all real numbers. Also, $P(t) = R^{-1}(t)$ is the solution of

$$P(t) = E - \int_0^t P[dF]P \tag{2.12}$$

on $(-1,\infty)$ and $(-1,\infty)$ is the maximal subinterval of the reals containing 0 and on which (2.12) has a solution.

It is to be noted that the only reason for choosing $R(t) = T(t+1)$ instead of $R(t) = T(t)$ is so that $F(0) = 0$ will hold. This is not necessary, but it is of some interest since many authors such as MacNerney [6] require that this condition does hold true.

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INDEX

TERMS

SYMBOLS