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HAROLD VERNON HUNEKE

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APPROVED BY

Casper Goffman

John B. Greever

John E. Driver

C. E. Springer

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DISSERTATION COMMITTEE

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CHAPTER I

INTRODUCTION

The theory of divergent series is a study of the methods of assigning a "sum" to a divergent series, or of assigning a "limit" to a divergent sequence. It is more practical, for the most part, to deal with sequences than with series and in this thesis only sequences, $\sigma = \{s_n\}$, of real numbers will be considered.

The methods of limiting a sequence $\{s_n\}$ usually consist of forming an auxiliary sequence of linear means of $\{s_n\}$. One very important class of limitation methods is the class of matrix methods and only these will be studied in this thesis.

Consider now an infinite matrix $A = (a_{mn})$, m, n = 0, 1, $\cdots$; that is,

$$A = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n} & \cdots \\ a_{10} & a_{11} & \cdots & a_{1n} & \cdots \\ & & \cdots & & \\ a_{m0} & a_{m1} & \cdots & a_{mn} & \cdots \end{pmatrix}$$
$$\cdots$$

The transformation which consists of premultiplying a sequence $\sigma = \{s_n\}$ by $\underline{A}$ is given by

$$A\sigma = \{t_m\} = \{\sum_{n=0}^{\infty} a_{mn}s_n\}.$$

If the transformed sequence $\{t_m\}$ converges to $\underline{s}$ in the Cauchy sense then the sequence $\sigma = \{s_n\}$ is said to be $\underline{A}$ limitable and $\underline{s}$ is called the $\underline{A}$ limit of $\sigma = \{s_n\}$. The notation, $A \lim_{n\to\infty} s_n = s$ will sometimes be used.

Until recently most of the studies in this field have been concerned with special methods, often for special purposes. Thus, for example, the Cesàro methods [1][1] were introduced in order to obtain definitive results for the convergence of the product of convergent series, the Riesz methods [2] were introduced in connection with problems in the theory of Dirichlet series, the Borel methods [3] were developed in connection with analytic extension problems, and the Hausdorff methods [4] were studied because of their deep relation with the moment problem.

In recent years, attention has been shifted, at least in part, to a study of the class of all regular methods. In this connection one must mention the work of Brudno, but first some definitions need to be given.

A matrix $\underline{B}$ is said to be stronger (b. stronger) than a matrix $\underline{A}$ if every (every bounded) sequence limited by $\underline{A}$ is also limited by $\underline{B}$. Matrices $\underline{A}$ and $\underline{B}$ are said to be equivalent (b. equivalent) if they limit the same set of sequences (bounded sequences). If a matrix $\underline{B}$ is stronger (b. stronger) than a matrix $\underline{A}$ but not equivalent (b. equivalent) to $\underline{A}$ then the matrix $\underline{B}$ is strictly stronger (strictly b. stronger) than the matrix $\underline{A}$.

Matrices $\underline{A}$ and $\underline{B}$ are consistent (b. consistent) if for all sequences (bounded sequences), $\sigma = \{s_n\}$, which are both $\underline{A}$ limitable and

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[1]Numbers in brackets refer to the list of references. Whenever two numbers are used the second number is the page number in the reference.

$\underline{B}$ limitable, $A \lim_{n\to\infty} s_n = B \lim_{n\to\infty} s_n$. A matrix $\underline{A}$ is regular if for every sequence $\sigma = \{s_n\}$ which converges to $\underline{s}$ in the Cauchy sense, $A \lim_{n\to\infty} s_n = \lim_{n\to\infty} s_n = s$.

Since regular matrices do not have to be consistent, the following theorem of Brudno [5] is of great interest: If $\underline{A}$ and $\underline{B}$ are regular matrices and $\underline{B}$ is b. stronger than $\underline{A}$ then $\underline{A}$ and $\underline{B}$ are b. consistent. Moreover Brudno [5] has proven there exist sequences $\{A_n\}$ of regular matrices such that no regular matrix is b. stronger than all of them.

Petersen has recently shown [6] that for every pair $\underline{A}$ and $\underline{B}$ of regular matrices such that $\underline{B}$ is strictly b. stronger than $\underline{A}$, there is a regular matrix $\underline{C}$ such that $\underline{C}$ is strictly b. stronger than $\underline{A}$ and $\underline{B}$ is strictly b. stronger than $\underline{C}$.

Goffman and Petersen [7] have found a class of regular methods which are b. consistent and such that every bounded sequence is limited by one of the methods in the class.

A related problem is the following: does every finite number of consistent regular matrix methods have a matrix method stronger than all of them? Petersen has answered this question in the negative for the case of three methods, and for the case of two methods when all methods considered are positive (i.e., all matrix elements are positive). For the case of any two methods, the question is unanswered.

The preceding results, for the most part, deal with the set of regular matrix methods as an ordered set with respect to the inclusion relation. It should be remarked that a considerable amount of research is yet to be done on the inclusion properties of these general methods.

In addition to the set of all regular methods, it seems worthwhile to study, in a more intensive way than heretofore, various of the classical special classes of methods as ordered sets. The Nörlund methods [8] form a particularly elegant general class of methods which contain the Cesàro methods and retain some of their properties. This thesis deals primarily with the inclusion relationships of the Nörlund methods, in particular for sequences of Nörlund methods.

A Nörlund method, denoted by (N, p_n) is a triangular matrix method in which the matrix $\underline{P}$ of the transformation is determined, or generated by, a sequence of positive constants in the following manner. If $\{p_n\}$ is any sequence of numbers such that $p_0 \neq 0$, $p_n \geq 0$, $n = 1, 2, \cdots$, then the matrix $\underline{P}$ is given by

$$P = \begin{pmatrix} \frac{p_0}{P_0} & 0 & 0 & \cdots & 0 & \cdots \\ \frac{p_1}{P_1} & \frac{p_0}{P_1} & 0 & \cdots & 0 & \cdots \\ \frac{p_2}{P_2} & \frac{p_1}{P_2} & \frac{p_0}{P_2} & \cdots & 0 & \cdots \\ & & \cdots & & & \\ \frac{p_m}{P_m} & \frac{p_{m-1}}{P_m} & \frac{p_{m-2}}{P_m} & \cdots & \frac{p_0}{P_m} & \cdots \\ & & \cdots & & & \end{pmatrix}$$

where $P_m = p_0 + p_1 + \cdots + p_m$, $m = 0, 1, 2, \cdots$.

If a sequence $\sigma = \{s_n\}$ is transformed by the above matrix, the transformed sequence $P\sigma = \{t_n\}$ is such that

$$t_n = \frac{p_0 s_n + p_1 s_{n-1} + \cdots + p_n s_0}{P_n}$$

The Cesàro methods denoted by (C, α), $\alpha > 0$, are the Nörlund methods for which $p_n = \dfrac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)}$, where $\Gamma(\alpha)$ represents the usual

Gamma function.

Any Nörlund method is regular if $\lim_{n\to\infty} \frac{p_n}{P_n} = 0$ and all regular Nörlund methods are consistent. Necessary and sufficient conditions for one regular Nörlund method to be stronger than another were first stated by Riesz [9].

The results of greatest interest in the thesis are the following:

(a) The set of regular Nörlund methods under the operation of symmetric product and the inclusion order relation, form an ordered semi-group.

(b) For any $\alpha > 0$ the set of Cesàro methods weaker than or equivalent to (C, α) and the set of methods strictly stronger than (C, α) determine a cut in the ordered set of Cesàro methods and there is a non-denumerable set of non-equivalent Nörlund methods in this cut.

(c) Given any sequence of regular Nörlund methods, there is a regular Nörlund method stronger than or not comparable with any method in the sequence.

This thesis consists of four chapters. Chapter I is the introduction. Chapter II includes the basic theorems concerning the regularity, consistency, and inclusion properties of the Nörlund methods. It is shown that the regular Nörlund methods form an ordered semi-group under the operation of symmetric product and the inclusion order relation.

In Chapter III it is pointed out that an iteration process consisting of taking the symmetric product of any regular Nörlund method with itself generates a sequence of regular Nörlund methods. In particular the Cesàro methods (C, α), α an integer, form such a sequence.

Hardy [10, 110] gives an example of a regular Nörlund method weaker than all the Cesàro methods (C, α), $\alpha > 0$. The question is raised as to the "number" of regular Nörlund methods with this property. It is shown that there is a non-denumerable set of regular non-equivalent Nörlund methods stronger than convergence yet weaker than all the Cesàro methods. Furthermore, consider the cut in the Cesàro set defined by the set of all Cesàro methods weaker than or equivalent to (C, α) and the set of Cesàro methods strictly stronger than (C, α). It is shown there is a non-denumerable set of regular non-equivalent Nörlund methods in this cut.

Hill [11] and Hardy [10, 109] give an example of a regular Nörlund method stronger than all the Cesàro methods. In Chapter IV, the following question is raised. Given any sequence of regular Nörlund methods is it possible to find a regular Nörlund method stronger than any method of the sequence? It is pointed out that a topological proof of the existence of such a regular Nörlund method is not feasible since the space of regular Nörlund methods with the natural metric is not complete. The best answer obtained is the following theorem. For any sequence of regular Nörlund methods there is a regular Nörlund method which is stronger than or not comparable to any method of the sequence.

CHAPTER II

BASIC THEOREMS

As noted previously, any sequence of positive numbers $\{p_n\}$ with $p_0 \neq 0$ determines a Nörlund method denoted by (N, p_n). Similarly, the method determined by the sequence $\{q_n\}$ will be denoted by (N, q_n). If a sequence $\sigma = \{s_n\}$ is transformed by a Nörlund method (N, p_n), the transformed sequence $\{t_n\}$ is

$$P\sigma = \{t_n\} = \left\{\frac{p_0 s_n + p_1 s_{n-1} + \cdots + p_n s_0}{P_n}\right\}$$

where $P_n = p_0 + p_1 + \cdots + p_n$, $n = 0, 1, 2, \cdots$.

It is obvious from the expression for t_n that the Nörlund method (N, p_n) is unchanged if one considers the sequence $\left\{\frac{p_n}{p_0}\right\}$ as generating the method. Hence there is no loss of generality in assuming $p_0 = 1$ and this convention will be followed in this thesis. Theorems on the regularity, consistency, and inclusion properties of the Nörlund methods will now be stated.

Toeplitz [12] first stated necessary and sufficient conditions that a triangular matrix method be regular. Schur [13] extended the conditions to include any matrix method and the theorem is stated here without proof.

Theorem 2.1. A matrix method $C = (c_{mn})$; $m, n = 0, 1, 2, \cdots$ is regular if and only if:

(a) For every n, $\lim_{m\to\infty} c_{mn} = 0$.

(b) For every m, $\sum_{n=1}^{\infty} c_{mn}$ converges and $\lim_{m\to\infty} \sum_{n=1}^{\infty} c_{mn} = 1$.

(c) There is an $\underline{H}$ such that $\sum_{n=1}^{\infty} |c_{mn}| < H$ for every $\underline{m}$.

The regular matrices are sometimes referred to as Toeplitz matrices.

The condition for the regularity of a Nörlund method is quite simple and is easily obtained by applying the conditions of Theorem 2.1 to the matrix of the Nörlund method (N, p_n) [10, 64].

Theorem 2.2. A Nörlund method (N, p_n) is regular if and only if $\lim_{n\to\infty} \frac{p_n}{P_n} = 0$.

However the following theorem, simple as it is, does not seem to be mentioned in the literature and is needed here.

Theorem 2.3. If $\lim_{n\to\infty} \frac{p_{n+1}}{p_n}$ exists for a Nörlund method (N, p_n), the method is regular if $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} \leqq 1$ and is not regular if $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} > 1$.

Proof: Let $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = r$. If $r < 1$, then $\lim_{n\to\infty} p_n = 0$ and the method is regular.

If $r = 1$, for every $\underline{q}$ there is an $\underline{M}$ such that $n > M$ implies $\frac{p_{n-k}}{p_n} > \frac{1}{2}$ for $k = 1, 2, \cdots, q$.

Hence, $$\frac{p_n}{P_n} = \frac{p_n}{p_0 + p_1 + \cdots + p_{n-q} + \cdots + p_n} \leqq \frac{p_n}{p_{n-q} + \cdots + p_n}$$

$$= \frac{1}{\frac{p_{n-q}}{p_n} + \cdots + \frac{p_{n-1}}{p_n} + 1} \leq \frac{1}{\frac{1}{2}(q+1)} = \frac{2}{q+1} .$$

Therefore, $\lim\limits_{n\to\infty} \frac{p_n}{P_n} = 0$ and the method is regular.

If $r > 1$ there is an $\underline{\alpha}$ such that $1 < \alpha < r$ and an $\underline{M}$ such that for $n > M$, $\frac{p_{n+1}}{p_n} > \alpha$. For $n > M$,

$$\frac{p_n}{P_n} > \frac{p_n}{p_0 + p_1 + \cdots + p_M + p_n\left(\alpha^{\frac{1}{n-M-1}} + \cdots + \frac{1}{\alpha} + 1\right)}$$

Since $\lim\limits_{n\to\infty} \frac{p_{n+1}}{p_n} > 1$ the $\lim\limits_{n\to\infty} p_n = +\infty$. Let $A = p_0 + p_1 + \cdots + p_M$ and $B = \sum_{n=0}^{\infty} \alpha^{\frac{1}{n}}$, then $\frac{p_n}{P_n} > \frac{p_n}{A+Bp_n} = \frac{1}{\frac{A}{p_n} + B}$ for $n > m$. Therefore $\lim\limits_{n\to\infty} \frac{p_n}{P_n} > \frac{1}{B} > 0$ and the method is not regular.

Before stating the theorem on the consistency of the Nörlund methods it is convenient to introduce an operation defined by Silverman [14] which plays a very important role in the thesis. Given two regular Nörlund methods (N, p_n) and (N, q_n) the symmetric product of the two methods is the Nörlund method $(N, g_n) = (N, p_n) \circ (N, q_n)$ where

$$g_n = p_0 q_n + p_1 q_{n-1} + \cdots + p_n q_0 .$$

The method (N, g_n) is a Nörlund method since $p_0 = 1$, $q_0 = 1$, $p_n \geqq 0$, $q_n \geqq 0$ for $n > 0$ implies that $g_0 = 1$ and $g_n \geqq 0$ for $n > 0$.

The proof of the next theorem, although known [10, 65], is given because the proof itself has important implications not stated in the theorem.

<u>Theorem 2.4</u>. Any two regular Nörlund methods (N, p_n) and (N, q_n) are consistent.

Proof: Let $\sigma = \{s_n\}$ be a given sequence and $P\sigma = \{t_m\}$ and $Q\sigma = \{r_m\}$ be the transforms of $\sigma = \{s_n\}$ by the methods (N, p_n) and (N, q_n) respectively. Then

$$t_m = \frac{p_m s_0 + p_{m-1}s_1 + \cdots + p_0 s_m}{P_m} \quad \text{and} \quad r_m = \frac{q_m s_0 + q_{m-1}s_1 + \cdots + q_0 s_m}{Q_m}.$$

Let $\{u_m\}$ be the transform of $\{s_n\}$ by the method $(N, g_n) = (N, p_n) \circ (N, q_n)$. Then

$$u_m = \frac{(p_0 q_m + \cdots + p_m q_0)s_0 + \cdots + (p_0 q_1 + p_1 q_0)s_{m-1} + p_0 q_0 s_m}{p_0 q_0 + (p_0 q_1 + p_1 q_0) + \cdots + (p_0 q_m + \cdots + p_m q_0)}.$$

$$u_m = \frac{p_0(q_m s_0 + \cdots + q_0 s_m) + \cdots + p_{m-1}(q_1 s_0 + q_0 s_1) + p_m q_0 s_0}{p_0(q_0 + \cdots + q_m) + \cdots + p_{m-1}(q_0 + q_1) + p_m q_0}.$$

$$u_m = \frac{p_0 Q_m r_m + \cdots + p_{m-1}Q_1 r_1 + p_m Q_0 r_0}{p_0 Q_m + \cdots + p_{m-1}Q_1 + p_m Q_0}.$$

Thus the sequence $\{u_m\}$ is the transform of the sequence $\{r_m\}$ by the matrix $C_1 = (c_{mn})$, $c_{mn} = \dfrac{p_{m-n}Q_n}{p_0 Q_m + \cdots + p_m Q_0}$ if $n \leq m$ and $c_{mn} = 0$ if $n > m$.

The matrix C_1 is regular since:

(a) $$c_{mn} = \frac{p_{m-n}Q_n}{p_0 Q_m + \cdots + p_m Q_0} \leq \frac{p_{m-n}Q_n}{(p_0 + \cdots + p_m)Q_0} = \frac{p_{m-n}}{P_m} \cdot \frac{Q_n}{Q_0}.$$

But (N, p_n) regular implies $\lim\limits_{m\to\infty} \dfrac{p_{m-n}}{P_m} = 0$, therefore for every <u>n</u>,

$\lim\limits_{m\to\infty} c_{mn} = 0.$

(b) For every $\underline{m}$, $\sum_{n=1}^{\infty} c_{mn} = 1$.

(c) There is an $H = 1$ such that $\sum_{n=1}^{\infty} |c_{mn}| = H$ for every $\underline{m}$.

Thus $G\sigma = \{u_m\} = C_1 Q\sigma$. Similarly there exists a regular matrix C_2 such that $G\sigma = \{u_m\} = C_2 P\sigma$. It follows from the regularity of the matrices C_1 and C_2 that if the sequences $P\sigma$ and $Q\sigma$ converge they must have the same limit. Hence the consistency of the two methods is established.

A very important theorem follows immediately from the proof of Theorem 2.4.

Theorem 2.5. Given two regular Nörlund methods (N, p_n) and (N, q_n). The Nörlund method $(N, g_n) = (N, p_n) \circ (N, q_n)$ is regular and stronger than both (N, p_n) and (N, q_n).

Proof: Using the notation from the proof of Theorem 2.4 there exist regular matrices C_1 and C_2 such that

$$G\sigma = \{u_m\} = C_1 Q\sigma = C_2 P\sigma.$$

Therefore any sequence limitable (N, p_n) or (N, q_n) is also limitable (N, g_n) and (N, g_n) is regular and stronger than both (N, p_n) and (N, q_n).

The study of the inclusion properties of the Nörlund methods is primarily a study of the relationships between certain power series. These relationships are clarified by the following four theorems found in Hardy [10, 65-70].

Theorem 2.6. If (N, p_n) is a regular Nörlund method, the series $\sum_{n=0}^{\infty} p_n x^n$ and $\sum_{n=0}^{\infty} P_n x^n$ are convergent for $|x| < 1$.

Given any two regular Nörlund methods (N, p_n) and (N, q_n) there are from Theorem 2.6 four associated series, $p(x) = \sum_{n=0}^{\infty} p_n x^n$, $P(x) = \sum_{n=0}^{\infty} P_n x^n$, $q(x) = \sum_{n=0}^{\infty} q_n x^n$, and $Q(x) = \sum_{n=0}^{\infty} Q_n x^n$ all of which converge for $|x| < 1$.

Let $k(x) = \sum_{n=0}^{\infty} k_n x^n = \frac{q(x)}{p(x)}$ and $\ell(x) = \sum_{n=0}^{\infty} \ell_n x^n = \frac{p(x)}{q(x)}$. These series converge for small values of x. From these definitions the following relationships between the coefficients of the series are obtained.

$$q_n = k_0 p_n + k_1 p_{n-1} + \cdots + k_n p_0$$

$$Q_n = k_0 P_n + k_1 P_{n-1} + \cdots + k_n P_0$$

$$p_n = \ell_0 q_n + \ell_1 q_{n-1} + \cdots + \ell_n q_0$$

$$P_n = \ell_0 Q_n + \ell_1 Q_{n-1} + \cdots + \ell_n Q_n .$$

Also $k_0 \ell_0 = 1$ and $k_0 \ell_n + k_1 \ell_{n-1} + \cdots + k_n \ell_0 = 0$ for $n > 0$. The uniqueness of the sequences $\{k_n\}$ and $\{\ell_n\}$ is assured from the uniqueness theorem for power series.

<u>Theorem 2.7</u>. If (N, p_n) and (N, q_n) are two regular Nörlund methods then (N, q_n) is stronger than (N, p_n) if and only if there is an H such that for every n

$$\frac{|k_0|P_n + |k_1|P_{n-1} + \cdots + |k_n|P_0}{Q_n} < H$$

and $\lim_{n\to\infty} \frac{k_n}{Q_n} = 0$.

<u>Corollary 1</u>. If $\lim_{n\to\infty} P_n = +\infty$ then the condition $\lim_{n\to\infty} \frac{k_n}{Q_n} = 0$

can be omitted.

Corollary 2. If $k_n \gtreqless 0$ for all n the condition

$$\frac{|k_0|P_n + |k_1|P_{n-1} + \cdots + |k_n|P_0}{Q_n} < H$$

can be omitted.

Corollary 2 is not given in Hardy but is very useful.

Theorem 2.8. The regular Nörlund methods (N, p_n) and (N, q_n) are equivalent if and only if the associated series $\sum_{n=0}^{\infty} |k_n|$ and $\sum_{n=0}^{\infty} |\ell_n|$ converge.

Theorem 2.9. If the regular Nörlund methods (N, p_n) and (N, q_n) satisfy the conditions:

(a) $\frac{p_{n+1}}{p_n} \geq \frac{p_n}{p_{n-1}}$, $n = 1, 2, \cdots$;

(b) $q_n > 0$ for all n;

(c) there is an r such that for $n > r$, $\frac{q_{n+1}}{q_n} \geq \frac{p_{n+1}}{p_n}$;

then the method (N, q_n) is stronger than the method (N, p_n).

The use of Theorem 2.9 is restricted to those methods for which the generating sequences have no zero terms. Furthermore it follows from Theorem 2.3 that for the method (N, p_n), the sequence $\{p_n\}$ must be non increasing.

Since only the methods strictly stronger than convergence are of importance the next two theorems are of some interest.

Theorem 2.10. The regular Nörlund method (N, q_n) is strictly stronger than convergence if either:

(a) $q(x)$ diverges at $x = 1$ or

(b) $\frac{1}{q(x)}$ diverges absolutely at $x = 1$.

Proof: The method (N, p_n) with $p_n = 0$ for $n > 0$, is equivalent to convergence. Let (N, q_n) be any regular Nörlund method. Then $k(x) = \frac{q(x)}{p(x)} = q(x)$ and $k_n = q_n \geqq 0$ for all $\underline{n}$.

If $q(x)$ diverges at $x = 1$ then $\lim_{n\to\infty} Q_n = +\infty$ and $\sum_{n=0}^{\infty} |k_n| = \sum_{n=0}^{\infty} q_n = +\infty$ and from Theorem 2.8 the method (N, q_n) is strictly stronger than convergence.

If $q(x)$ converges at $x = 1$ but $\frac{1}{q(x)}$ diverges absolutely at $x = 1$ then $\sum_{n=0}^{\infty} |\ell_n| = +\infty$ since $\ell(x) = \frac{1}{k(x)} = \frac{1}{q(x)}$. Therefore from Theorem 2.8 the method (N, q_n) is strictly stronger than convergence.

Theorems 2.3 and 2.9 point out, to a certain extent, the role played by the ratio $\frac{p_{n+1}}{p_n}$ with respect to the regularity and inclusion properties of the Nörlund methods. Further insight is given by the following theorems.

Theorem 2.11. The Nörlund method (N, q_n) with $q_n = r^n$, $n = 0, 1, 2, \cdots$ is equivalent to convergence if $r < 1$, is the Cesàro method $(C,1)$ if $r = 1$, and is not regular if $r > 1$.

Proof: The Nörlund method with $p_n = 0$ for $n > 0$, is equivalent to convergence. On comparing this method with the method (N, q_n), $q_n = r^n$, the series for $k(x)$ is

$$k(x) = \frac{q(x)}{p(x)} = q(x) = 1 + rx + r^2x^2 + \cdots + r^nx^n + \cdots .$$

If $r < 1$, $\sum_{n=0}^{\infty} |k_n| = \sum_{n=0}^{\infty} q_n$ which converges. Also

$$\ell(x) = \frac{1}{k(x)} = \frac{1}{q(x)} = \frac{1}{1 + rx + \cdots + r^nx^n + \cdots} = 1 - rx \quad \text{and} \quad \sum_{n=0}^{\infty} |\ell_n|$$

converges. Therefore the method (N, q_n) for $r < 1$ is equivalent to convergence.

If $r = 1$ the method (N, q) is $(C,1)$ and if $r > 1$ it follows from Theorem 2.3 that the method is not regular.

As has already been shown the operation of symmetric product, with respect to the regular Nörlund methods, preserves regularity and yields a method stronger than either method in the product. Another very important property of the symmetric product is that it preserves the inclusion relationship.

Theorem 2.12. Let (N, p_n) and (N, q_n) be two regular Nörlund methods such that (N, q_n) is stronger than (N, p_n) and let (N, r_n) be any other regular Nörlund method. The Nörlund method $(N, v_n) = (N, q_n) \circ (N, r_n)$ is stronger than the method $(N, u_n) = (N, p_n) \circ (N, r_n)$.

Proof: Since the method (N, q_n) is stronger than the method (N, p_n) there is an H such that

$$\frac{|k_0|P_n + |k_1|P_{n-1} + \cdots + |k_n|P_0}{Q_n} < H$$

for all n, and $\lim_{n\to\infty} \frac{k_n}{Q_n} = 0$, where the k_n are the coefficients of the series $k(x) = \frac{q(x)}{p(x)}$.

Since (N, u_n) and (N, v_n) are regular Nörlund methods there exist sequences of constants $\{K_n\}$ and $\{L_n\}$ such that $K(x) = \frac{v(x)}{u(x)}$ and $L(x) = \frac{u(x)}{v(x)}$. But $(N, v_n) = (N, q_n) \circ (N, r_n)$ therefore $v(x) = q(x)r(x)$ and similarly $u(x) = p(x)r(x)$. Hence $K(x) = \frac{q(x)r(x)}{p(x)r(x)} = k(x)$ and $K_n = k_n$ for all $\underline{n}$. Similarly $L_n = \ell_n$ for all $\underline{n}$.

To show that (N, v_n) is stronger than (N, u_n) consider

$$\frac{|K_0|U_n + |K_1|U_{n-1} + \cdots + |K_n|U_0}{V_n} = \frac{|k_0|U_n + |k_1|U_{n-1} + \cdots + |k_n|U_0}{V_n}$$

$$= \frac{|k_0|(r_0P_n+r_1P_{n-1}+\cdots+r_nP_0) + |k_1|(r_0P_{n-1}+\cdots+r_{n-1}P_0) + \cdots + |k_n|r_0P_0}{V_n}$$

$$= \frac{r_0(|k_0|P_n+|k_1|P_{n-1}+\cdots+|k_n|P_0) + r_1(|k_0|P_{n-1}+\cdots+|k_{n-1}|P_0)+\cdots+r_n|k_0|P_0}{V_n}$$

$$= \frac{H(r_0Q_n + r_1Q_{n-1} + \cdots + r_nQ_0)}{V_n} = H \quad \text{for all } \underline{n}.$$

Also $\lim_{n\to\infty} \frac{k_n}{Q_n} = 0$ implies $\lim_{n\to\infty} \frac{K_n}{V_n} = \lim_{n\to\infty} \frac{k_n}{V_n} < \lim_{n\to\infty} \frac{k_n}{r_0Q_n} = 0$. Therefore from Theorem 2.7 the method (N, v_n) is stronger than the method (N, u_n).

The following theorem follows easily from the preceding theorem and is very important in the development of Chapter III.

<u>Theorem 2.13</u>. If (N, q_n) is strictly stronger than (N, p_n) the method $(N, v_n) = (N, q_n) \circ (N, r_n)$ is strictly stronger than the method $(N, u_n) = (N, p_n) \circ (N, r_n)$.

Proof: The method (N, q_n) strictly stronger than (N, p_n) implies by Theorem 2.8 that either $\sum_{n=0}^{\infty} |k_n| = +\infty$ or $\sum_{n=0}^{\infty} |\ell_n| = +\infty$. If $\sum_{n=0}^{\infty} |k_n| = +\infty$ then $\sum_{n=0}^{\infty} |K_n| = +\infty$ or if $\sum_{n=0}^{\infty} |\ell_n| = +\infty$ then

$\sum_{n=0}^{\infty} |L_n| = +\infty$. Therefore the method (N, v_n) is strictly stronger than the method (N, u_n).

In considering the regular Nörlund methods as an ordered set one is lead to the concept of an ordered abelian semi-group. The concept of a semi-group has become of increasing importance in the literature of the last several years.

A semi-group is a set of elements which is closed under a single valued binary operation, denoted here by $^\circ$, which is associative. The set may or may not have an identity element.

In this thesis an ordered semi-group is defined as a semi-group with an order relation $a < b$ such that:

(α) If $a < b$ and $b < c$ then $a < c$.

(β) If $a < b$ then $a \circ c < b \circ c$.

One property of ordered abelian semi-groups used in the thesis is the following.

Theorem 2.14. In any ordered abelian semi-group if $a < b$ and $c < d$ then $a \circ c < b \circ d$.

Proof: If $a < b$ then $a \circ c < b \circ c$ and if $c < d$ then $b \circ c < b \circ d$. Therefore $a \circ c < b \circ d$.

The following theorem unifies, to a certain extent, the properties of the Nörlund methods developed in this chapter.

Theorem 2.15. The regular Nörlund methods under the operation of symmetric product and the order relation "strictly stronger than" form an ordered abelian semi-group.

Proof: It follows from Theorem 2.5 that the set of regular

Nörlund methods is closed under the operation of symmetric product.

That the product is associative can be established easily by writing out the expressions for $[(N, p_n) \circ (N, q_n)] \circ (N, r_n)$ and $(N, p_n) \circ [(N, q_n) \circ (N, r_n)]$ and showing they are equal. The associativity also follows from the associativity under multiplication of the associated power series $p(x)$, $q(x)$, and $r(x)$.

From the definition of "strictly stronger than" the property (α) of an ordered semi-group is satisfied. It follows from Theorem 2.13 that the symmetric product preserves the inclusion property and thus property (β) is satisfied.

The abelian character of the group follows directly from the definition of the operation of symmetric product and is important since Theorem 2.14 will be used in the next chapter.

CHAPTER III

THE CESÀRO METHODS AS RELATED TO THE SET OF REGULAR NÖRLUND METHODS

It is natural to generate a sequence of regular Nörlund methods by repeating the operation of taking the symmetric product of any regular Nörlund method with itself. The mth iterate will be denoted by $(N^{(m)}, p_n)$. Thus

$$(N^{(m)}, p_n) = (N, p_n) \circ (N, p_n) \circ \cdots \circ (N, p_n)$$

the product being repeated m times.

If the regular method (N, q_n) is strictly stronger than the method (N, p_n) with $p_n = 0$ for $n > 0$, it follows from Theorem 2.14 on semi-groups that each method is strictly stronger than its predecessor.

The best known Nörlund methods, the Cesàro methods (C, α) for α integral, are defined by the sequence $\{p_n\}$ with $p_n = \binom{n+\alpha-1}{\alpha-1}$ $n = 0, 1, 2, \cdots$. From the binomial theorem it follows that for a given method (C, α) the p_n's are the coefficients in the expansion of $(1-x)^{-\alpha}$, that is $p(x) = (1-x)^{-\alpha}$.

For $\alpha = 1$ the method $(C, 1) = (N, q_n)$ is the method with $q_n = 1$ for all n. It follows immediately that the Cesàro methods can be interpreted, for α integral, as a sequence generated from $(C, 1)$ by the indicated iteration process since for the method $(N^{(m)}, q_n) =$ $(N, q_n) \circ (N, q_n) \circ \cdots \circ (N, q_n)$, $q(x) = \frac{1}{(1-x)^m}$. That the Cesàro methods (C, α), for α integral, form a strictly stronger sequence of regular methods is thus easily seen.

Questions arise as to the relative strength of various iterates of two given methods. In this connection consider the methods $(C, 1) = (N, q_n)$ and (N, p_n) with $p_1 = 1$, $p_n = 0$ for $n \geq 2$. The method (N, q_n) is strictly stronger than the method (N, p_n) since

$$k(x) = \frac{q(x)}{p(x)} = \frac{1 + x + x^2 + \cdots + x^n + \cdots}{1 + x} = 1 + x^2 + \cdots + x^{2n} + \cdots,$$

$$\lim_{n\to\infty} \frac{k_n}{Q_n} = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} |k_n| = +\infty .$$

However the method $(N^{(2)}, p_n)$ is not comparable to $(C, 1) = (N, q_n)$. To show this consider $k(x)$ for the two methods (N, q_n) and $(N^{(2)}, p_n)$.

$$k(x) = \frac{1 + x + \cdots + x^n + \cdots}{1 + 2x + x^2} = 1 - x + 2x^2 - 2x^3 + 3x^4 - 3x^5 + \cdots$$

with $k_{2n} = (n + 1)$ and $k_{2n+1} = -(n + 1)$. Thus

$$\frac{|k_0|P_{2n+1} + |k_1|P_{2n} + \cdots + |k_n|P_0}{Q_{2n+1}} > \sum_{m=0}^{2n+1} \frac{|k_m|}{Q_{2n+1}} = \frac{(n+1)\left[1+\frac{n+2}{2}\right]}{2n+1}$$

which is unbounded and (N, q_n) is not stronger than $(N^{(2)}, p_n)$. Since $\lim_{n\to\infty} \frac{Q_n}{P_n} = +\infty$ the method $(N^{(2)}, p_n)$ cannot be stronger than (N, q_n) and the two methods are not comparable.

To illustrate the preceding results a sequence will now be constructed which is limitable by the method $(N^{(2)}, p_n)$ and is not limitable by the method (C, 1). The matrix of (N_2, p_n) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & 0 & \cdots \\ 0 & \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & \cdots \\ & & \cdots & & \end{pmatrix}$$

Starting with any three numbers a_1, a_2, a_3 not equal to zero determine a_4 such that

$$\frac{a_1 + 2a_2 + a_3}{4} = \frac{a_2 + 2a_3 + a_4}{4} .$$

Thus $a_4 = a_1 + a_2 - a_3$ and in general determine a_n such that $a_n = a_{n-3} + a_{n-2} - a_{n-1}$. This sequence converges to $\frac{a_1 + 2a_2 + a_3}{4}$ by the method $(N^{(2)}, p_n)$.

If $\{t_n\}$ is the transform of the sequence $\{a_n\}$ by (C, 1) then

$$t_{2n} = \frac{a_1 + n(a_2 + a_3)}{2n} \quad \text{and} \quad t_{2n-1} = \frac{a_1 + a_2 + (n-1)(a_1+a_2)}{2n-1} .$$

Since $\lim_{n\to\infty} t_{2n} = \frac{a_2 + a_3}{2}$ and $\lim_{n\to\infty} t_{2n-1} = \frac{a_1 + a_2}{2}$ the sequence is not limitable (C, 1) if $a_1 \neq a_3$.

However from the properties of ordered semi-groups it follows that $(N^{(2)}, q_n)$ is strictly stronger than $(N^{(2)}, p_n)$ and in general $(N^{(m)}, q_n)$ is strictly stronger than $(N^{(m)}, p_n)$. Whether a method exists such that all of its iterates are weaker than (C, 1) is an open question. The remainder of the chapter will deal with the relations between the set of Cesàro methods and the set of all regular Nörlund methods.

The Cesàro methods are generalized to have meaning for all values of $\alpha > 0$ by letting $p_n = \dfrac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)}$ for $n = 0, 1, 2, \cdots$. As in the case of α integral, $p(x)$ is still given by $p(x) = (1-x)^{-\alpha}$. Some of the inclusion properties of the general Cesàro methods will be considered next.

The next two theorems are well known but the proofs, which are based on the theorems in Chapter II, differ from the proofs found in the literature. First a lemma is needed.

Lemma 1: $$\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} = \sum_{\nu=0}^{n} \frac{\Gamma(\nu+\alpha)}{\Gamma(\nu+1)\Gamma(\alpha)}, \quad \alpha > 0.$$

Proof: The lemma is true for $n = 1$ since

$$\sum_{\nu=0}^{1} \frac{\Gamma(\nu+\alpha)}{\Gamma(\nu+1)\Gamma(\alpha)} = \frac{\Gamma(\alpha)}{\Gamma(1)\Gamma(\alpha)} + \frac{\Gamma(\alpha+1)}{\Gamma(2)\Gamma(\alpha)} = \frac{(1+\alpha)\Gamma(\alpha+1)}{\alpha\Gamma(\alpha)}$$

$$= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)} = \frac{\Gamma(\alpha+2)}{\Gamma(2)\Gamma(\alpha+1)}.$$

Assuming the relation is true for some value of n then:

$$\sum_{\nu=0}^{n+1} \frac{\Gamma(\nu+\alpha)}{\Gamma(\nu+1)\Gamma(\alpha)} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} + \frac{\Gamma(n+\alpha+1)}{\Gamma(n+2)\Gamma(\alpha)}$$

$$= \frac{(n+1+\alpha)\Gamma(n+\alpha+1)}{\Gamma(n+2)\Gamma(\alpha+1)} = \frac{\Gamma(n+\alpha+2)}{\Gamma(n+2)\Gamma(\alpha+1)}.$$

Theorem 3.1. The Cesàro methods are regular for $\alpha > 0$.

Proof: From Lemma 1 for $\alpha > 0$,

$$\frac{p_n}{P_n} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \cdot \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} = \frac{\alpha}{n+\alpha}.$$

Therefore $\lim_{n\to\infty} \frac{p_n}{P_n} = 0$ and the Cesàro methods are regular for $\alpha > 0$.

Theorem 3.2. Given the Cesaro methods $(C, \alpha) = (N, p_n)$ and $(C, \beta) = (N, q_n)$. If $\beta > \alpha$ the method (C, β) is strictly stronger than the method (C, α).

Proof: For the methods (N, p_n) and (N, q_n), $p(x) = (1-x)^{-\alpha}$ and $q(x) = (1-x)^{-\beta}$. Therefore, $k(x) = \dfrac{q(x)}{p(x)} = (1-x)^{-(\beta-\alpha)}$ and

$$k_n = \frac{\Gamma(n+\beta-\alpha)}{\Gamma(n+1)\Gamma(\beta-\alpha)}.$$

Since $(\beta-\alpha) > 0$ all the Gamma functions and hence all the k_n's are positive. Therefore

$$\frac{|k_0|P_n + |k_1|P_{n-1} + \cdots + |k_n|P_0}{Q_0} = 1.$$

From Lemma 1, $$\lim_{n\to\infty} \frac{k_n}{Q_n} = \lim_{n\to\infty} \frac{\dfrac{\Gamma(n+\beta-\alpha)}{\Gamma(n+1)\Gamma(\beta-\alpha)}}{\displaystyle\sum_{\nu=0}^{n} \frac{\Gamma(\nu+\beta)}{\Gamma(\nu+1)\Gamma(\beta)}}$$

$$\lim_{n\to\infty} \frac{k_n}{Q_n} = \lim_{n\to\infty} \frac{\Gamma(n+\beta-\alpha)}{\Gamma(n+1)\Gamma(\beta-\alpha)} \cdot \frac{\Gamma(n+1)\Gamma(\beta+1)}{\Gamma(n+\beta+1)}$$

$$= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)} \cdot \lim_{n\to\infty} \frac{\Gamma(n+\beta-\alpha)}{\Gamma(n+\beta+1)}$$

But $\dfrac{\Gamma(n+\beta-\alpha)}{\Gamma(n+\beta+1)} \leq \dfrac{\Gamma(n+\beta)}{\Gamma(n+\beta+1)} = \dfrac{1}{n+\beta}$. Therefore $\lim\limits_{n\to\infty} \dfrac{k_n}{Q_n} = 0$ and (C, β) is stronger than (C, α).

The series $$\sum_{n=0}^{\infty} |k_n| = \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta-\alpha)}{\Gamma(n+1)\Gamma(\beta-\alpha)} = \lim_{n\to\infty} \frac{\Gamma(n+\beta-\alpha+1)}{\Gamma(n+1)\Gamma(\beta-\alpha+1)}$$

$= +\infty$ and therefore the method (C, β) is not equivalent to the method (C, α) and the theorem is established.

It was shown in Chapter 2 that the operation of symmetric product preserved the order relation. Another important property of this operation with respect to the Cesàro methods will now be established.

Theorem 3.3. Let $(C, \alpha) = (N, p_n)$ and $(C, \beta) = (N, q_n)$ be given Cesàro methods. The method $(N, r_n) = (N, p_n) \circ (N, q_n)$ is the method $(C, \alpha+\beta)$.

Proof: For the methods (N, p_n) and (N, q_n), $p(x) = (1-x)^{-\alpha}$ and $q(x) = (1-x)^{-\beta}$. For the method $(N, p_n) \circ (N, q_n)$

$$r_n = p_0q_n + p_1q_{n-1} + \cdots + p_nq_0.$$

Therefore

$$r(x) = p(x)\cdot q(x) = (1-x)^{-(\alpha+\beta)} \quad \text{and} \quad r_n = \frac{\Gamma(n+\alpha+\beta)}{\Gamma(n+1)\,\Gamma(\alpha+\beta)} .$$

Thus the method $(N, r_n) = (C, \alpha+\beta)$.

Hardy [10, 110] gives an example of a method weaker than all the Cesàro methods (C, α) with $\alpha > 0$. The method given is the method with $p_n = \frac{1}{n+1}$, $n = 0, 1, 2, \cdots$. The existence of a non-denumerable set of non-equivalent methods weaker than all the Cesàro methods (C, α), $\alpha > 0$, will now be established. The following theorem needed in this development is of interest in its own right.

Theorem 3.4. If (N, p_n) and (N, q_n) are regular Nörlund methods such that:

(a) (N, q_n) is stronger than (N, p_n);

(b) $\lim\limits_{n\to\infty} \dfrac{Q_n}{P_n} = +\infty$;

then (N, q_n) is strictly stronger than (N, p_n).

Proof: For any two regular Nörlund methods there exists a sequence $\{k_n\}$ such that

$$Q_n = k_0P_n + k_1P_{n-1} + \cdots + k_nP_0$$

$$Q_n \leqq |k_0|P_n + |k_1|P_{n-1} + \cdots + |k_n|P_0 .$$

Since the sequence P_n is non-decreasing

$$\frac{Q_n}{P_n} < |k_0| + |k_1| + \cdots + |k_n| .$$

But $\lim_{n\to\infty} \frac{Q_n}{P_n} = +\infty$ implies $\sum_{n=0}^{\infty} |k_n| = +\infty$ and the method (N, q_n) is not equivalent to (N, p_n) hence must be strictly stronger than (N, p_n).

Theorem 3.5. The set of Nörlund methods $(N, p_n^{(\delta)})$, $p_n^{(\delta)} = \frac{2}{n+2}\left[\frac{\log 2}{\log(n+2)}\right]^{\delta}$, $0 \leqq \delta \leqq 1$, $n = 0, 1, 2, \cdots$, are regular and such that if $\delta > r$, $(N, p_n^{(r)})$ is strictly stronger than $(N, p_n^{(\delta)})$.

Proof: The methods are regular for

$$\lim_{n\to\infty} p_n^{(\delta)} = \lim_{n\to\infty} \frac{2}{n+2}\left[\frac{\log 2}{\log(n+2)}\right]^{\delta} = 0 \text{ for } 0 \leqq \delta \leqq 1.$$

The sequences $\{p_n^{(\delta)}\}$ and $\{p_n^{(r)}\}$, $\delta > r$, satisfy the conditions of Theorem 2.9. To establish this consider

$$\frac{p_n^{(\delta)}}{p_{n-1}^{(\delta)}} = \frac{n+1}{n+2}\left[\frac{\log(n+1)}{\log(n+2)}\right]^{\delta} \qquad \text{and}$$

$$\frac{p_{n+1}^{(\delta)}}{p_n^{(\delta)}} = \frac{n+2}{n+3}\left[\frac{\log(n+2)}{\log(n+3)}\right]^{\delta} .$$

But $\frac{n+2}{n+3} > \frac{n+1}{n+2}$ since $n^2 + 4n + 4 > n^2 + 4n + 3$ and

$\frac{\log(n+2)}{\log(n+3)} > \frac{\log(n+1)}{\log(n+2)}$ since

$$\frac{\log^2(n+2)}{\log(n+3)\log(n+1)} = \frac{\log^2(n+2)}{\left[\log(n+2) + \log\left(1+\frac{1}{n+2}\right)\right]\left[\log(n+2) + \log\left(1-\frac{1}{n+2}\right)\right]}$$

$$= \frac{\log^2(n+2)}{\log^2(n+2) + \left[\log(n+2) + 1\right]\left[\log\left(1-\frac{1}{(n+2)^2}\right)\right]}$$

Therefore for $n \geqq 0$, $0 \leqq \delta \leqq 1$

$$\frac{p_{n+1}^{(\delta)}}{p_n^{(\delta)}} \geqq \frac{p_n^{(\delta)}}{p_{n-1}^{(\delta)}} .$$

Similarly for $n \geqq 0$, $0 \leqq \gamma \leqq 1$, $\gamma < \delta$, $p_n^{(\gamma)} > 0$ and

$$\frac{p_{n+1}^{(\gamma)}}{p_n^{(\gamma)}} = \frac{n+2}{n+3}\left[\frac{\log(n+2)}{\log(n+3)}\right]^{\gamma} \geqq \frac{n+2}{n+3}\left[\frac{\log(n+2)}{\log(n+3)}\right]^{\delta} = \frac{p_{n+1}^{(\delta)}}{p_n^{(\delta)}} .$$

Therefore the method $(N, p^{(\gamma)})$ is stronger than the method $(N, p^{(\delta)})$. To show the method $(N, p^{(\gamma)})$ is strictly stronger consider

$$P_n^{(\delta)} = \sum_{m=0}^{n} \frac{2}{m+2}\left[\frac{\log 2}{\log(m+2)}\right]^{\delta} .$$

But $p_n^{(\delta)}$ is a monotonic decreasing function therefore

$$\int_1^n \frac{2}{t+2}\left[\frac{\log 2}{\log(t+2)}\right]^{\delta} dt < P_n^{(\delta)} < \int_0^n \frac{2}{t+2}\left[\frac{\log 2}{\log(t+2)}\right]^{\delta} dt.$$

$$\frac{2(\log 2)^{\delta}}{1-\delta}\left\{[\log(n+2)]^{1-\delta} - [\log 3]^{1-\delta}\right\} < P_n^{(\delta)}$$

$$< \frac{2(\log 2)^{\delta}}{1-\delta}\left\{[\log(n+2)]^{1-\delta} - [\log 2]^{1-\delta}\right\}.$$

Thus $\displaystyle\lim_{n\to\infty} \frac{(1-\delta)P_n^{(\delta)}}{2(\log 2)^{\delta}[\log(n+2)]^{1-\delta}} = 1$. Similarly

$$\lim_{n\to\infty} \frac{(1-\gamma)P_n^{(\gamma)}}{2(\log 2)^{\gamma}[\log(n+2)]^{1-\gamma}} = 1 \quad \text{and}$$

$$\lim_{n\to\infty} \frac{P_n^{(\gamma)}}{P_n^{(\delta)}} = \lim_{n\to\infty} \frac{1-\delta}{1-\gamma}(\log 2)^{\gamma-\delta}[\log(n+2)]^{\delta-\gamma} = +\infty$$

for $\gamma < \delta$.

Therefore from Theorem 3.4 the method $(N, p_n^{(\gamma)})$ is strictly stronger than the method $(N, p_n^{(\delta)})$.

It will now be shown that the set of methods exhibited in the preceding theorem are weaker than all the Cesàro methods (C, α) for $\alpha > 0$.

Theorem 3.6. The Cesàro methods $(C, \alpha) = (N, q_n)$, $\alpha > 0$, $q_n = \dfrac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)}$ are stronger than the regular Nörlund methods $(N, p_n^{(\delta)})$, $p_n^{(\delta)} = \dfrac{2}{n+2}\left[\dfrac{\log 2}{\log(n+2)}\right]^{\delta}$, $0 \leqq \delta \leqq 1$.

Proof: It follows from the preceding theorem that it is sufficient to prove that $(C, \alpha) = (N, q_n)$ is stronger than $(N, p_n^{(0)})$.

For $\delta = 0$, $\dfrac{p_{n+1}^{(\delta)}}{p_n^{(\delta)}} > \dfrac{p_n^{(\delta)}}{p_{n-1}^{(\delta)}}$ and $\dfrac{p_{n+1}^{(\delta)}}{p_n^{(\delta)}} = \dfrac{n+2}{n+3}$. For all $\underline{n}$, $q_n > 0$ and

$$\frac{q_{n+1}}{q_n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+2)\Gamma(\alpha)} \cdot \frac{\Gamma(n+1)\Gamma(\alpha)}{\Gamma(n+\alpha)} = \frac{n+\alpha}{n+1}.$$

For any α there is an $r > \frac{2}{\alpha}$ such that for $n > r$, $\dfrac{n+2}{n+3} < \dfrac{n+\alpha}{n+1}$ since $2 < \alpha n + 3\alpha$. Thus the methods (C, α) and $(N, p_n^{(0)})$ satisfy the conditions of Theorem 2.9 and (C, α) is stronger than $(N, p_n^{(0)})$ for all $\alpha > 0$.

The non-denumerable set of regular Nörlund methods $(N, p_n^{(\delta)})$, $p_n^{(\delta)} = \dfrac{2}{n+2}\left[\dfrac{\log 2}{\log(n+2)}\right]^{\delta}$, $0 \leqq \delta \leqq 1$, thus lies in the cut between convergence and the set of Cesàro methods. In fact much more can be said.

Theorem 3.7. The set of regular Nörlund methods $(N, p_n^{(\delta)}) \circ (C, k)$ for a fixed k, $p_n^{(\delta)} = \frac{2}{n+2}\left[\frac{\log 2}{\log(n+2)}\right]^{\delta}$, $0 \leq \delta \leq 1$, is stronger than (C, k) yet weaker than the set of methods $(C, k+\alpha)$ for all $\alpha > 0$.

Proof: The proof is immediate from Theorems 3.3 and 2.11. If $(C, 0)$ is defined as convergence, that is, $p_0 = 1$, $p_n = 0$ for $n > 0$, then from Theorem 3.3

$$(C, 0) \circ (C, k) = (C, k)$$
$$(C, \alpha) \circ (C, k) = (C, k+\alpha).$$

But from Theorem 2.11 the symmetric product preserves the "strictly stronger than" relationship and since the methods $(N, p_n^{(\delta)})$ lie in the cut between $(C, 0)$ and (C, α) for all $\alpha > 0$ then the methods $(N, p_n^{(\delta)}) \circ (C, k)$ must lie in the cut between the method (C, k) and the methods $(C, k+\alpha)$ for all $\alpha > 0$.

Since the Cesàro methods have always been considered as the important Nörlund methods the existence of non-denumerable sets of regular Nörlund methods in cuts in the set of the Cesàro methods is very interesting. Whether regular Nörlund methods can be found in the cut in the Cesàro methods defined by the set of methods stronger than (C, α) and the set not stronger than $(C, \alpha-\delta)$ for $\delta > 0$ is an open question.

CHAPTER IV

SEQUENCES OF NÖRLUND METHODS

Hardy [10, 109] and Hill [11] give an example of a regular Nörlund method stronger than all the Cesàro methods. This leads to an interesting question. Can any sequence of regular Nörlund methods be asymptotically stronger than the set of regular Nörlund methods with respect to the inclusion property? That is, does a sequence of regular methods exist such that every Nörlund method is weaker than some method from the sequence of methods. The primary purpose of this chapter is to answer negatively the above question.

It was first hoped the problem could be solved by topological methods in the following manner. Let $\underline{P}$ be the class of all regular Nörlund methods. Let a sequence of methods $\{(N, p_{mn})\}$ in $\underline{P}$ be said to converge to the method (N, q_n) if for every n, $q_n = \lim_{m \to \infty} p_{mn}$. With this metric $\underline{P}$ is a metrizable topological space.

It seems plausible that one could show that:

(a) For every method (N, p_m) in $\underline{P}$, the set of methods weaker than (N, p_m) is closed;

(b) For every method (N, p_m) in $\underline{P}$, the set of methods stronger than (N, p_m) is everywhere dense.

If $\underline{P}$ were complete, it would follow that the set $\underline{P}$ is nowhere dense and the set of methods weaker than the methods of a sequence of methods $\{(N, p_{mn})\}$ would therefore be of first category. Since any complete metric space is of second category the desired result would follow.

However the space with the above metric is not complete as is seen from the following example. Let the sequence of methods (N, p_{mn}), m, n = 0, 1, 2, ⋯ be such that

$$p_{mn} = \frac{1 - \left(\frac{2^n - 1}{2^n}\right)^m}{1 - \left(\frac{2^n - 1}{2^n}\right)} = 2^n\left[1 - \left(1 - \frac{1}{2^n}\right)^m\right] .$$

These methods are regular for every $\underline{m}$ since

$$\lim_{n\to\infty} \frac{p_{m,n+1}}{p_{mn}} = \lim_{n\to\infty} \frac{2^{n+1}\left[1 - \left(1 - \frac{1}{2^{n+1}}\right)^m\right]}{2^n\left[1 - \left(1 - \frac{1}{2^n}\right)^m\right]}$$

$$= \lim_{n\to\infty} 2 \frac{-m\left[1 - \frac{1}{2^{n+1}}\right]^{m-1} (2^{-(n+1)}\log 2)}{-m\left[1 - \frac{1}{2^n}\right]^{m-1} (2^{-n}\log 2)} = 1.$$

But $q_n = \lim_{m\to\infty} p_{mn} = \lim_{m\to\infty} 2^n\left[1 - (1 - \frac{1}{2^n})^m\right] = 2^n$. Hence $\lim_{n\to\infty} \frac{q_{n+1}}{q_n} = 2$ and from Theorem 2.2 the method is not regular.

Since the space of regular Nörlund methods is not complete with this rather natural metric, topological methods were not investigated further.

The original problem may be replaced by a simpler problem in the following manner. Given any sequence of regular Nörlund methods $\{(N, p_{mn})\}$, m, n = 0, 1, 2, ⋯. Form a new sequence of regular Nörlund methods $\{(N, q_{mn})\}$, m, n = 0, 1, 2, 3, ⋯ such that

$$(N, q_{mn}) = (N, p_{1n}) \circ (N, p_{2n}) \circ \cdots \circ (N, p_{mn}).$$

From the properties of the symmetric product it follows that for every $\underline{m}$ the methods (N, q_{mn}) are regular, the method (N, q_{mn}) is stronger than the method (N, p_{mn}) for all $\underline{m}$, and for $r > s$ the method (N, q_{rn}) is stronger than the method (N, q_{sn}). The new sequence also has the property that for any $\underline{m}$, $q_{m+1,n} \geqq q_{mn}$ since $p_{m+1,0} = 1$ and

$$q_{m+1,n} = p_{m+1,0}\, q_{mn} + p_{m+1,1}\, q_{m,n-1} + \cdots + p_{m+1,n}\, q_{m0}.$$

Consider now the sequence of methods $\{(N, Q'_{mn})\}$ where $(N, Q'_{mn}) = (N, q_{mn}) \circ (C, 1) \circ (C, 1)$. These methods are regular and possess the following properties. First, the method (N, Q'_{mn}) is stronger than the method (N, q_{mn}); second, $Q'_{rn} \geqq Q'_{sn}$ for $r > s$ since $q_{m+1,n} \geqq q_{mn}$ for all $\underline{m}$; third, for any $\underline{m}$, $\lim\limits_{n\to\infty} \dfrac{Q'_{m,n+1}}{Q_{mn}} = 1$ since for any regular Nörlund method (N, p_n) the method (N, P_n), where $P_n = p_0 + \cdots + p_n$, has the property that $\lim\limits_{n\to\infty} \dfrac{P_{n+1}}{P_n} = 1$; fourth, $Q'_{m,n+1} > Q'_{mn}$ for all $\underline{m}$ and for $n > 0$ since the sequence generating the method $(N, q_{mn}) \circ (C, 1)$ has no zero terms; fifth, $\lim\limits_{n\to\infty} Q'_{mn} = +\infty$ for all $\underline{m}$ since $Q_{mn} \geqq 1$ for all $\underline{m}$ and $\underline{n}$.

The original problem can thus be replaced by the following equivalent problem. Given any sequence $\{(N, p_{mn})\}$ of regular Nörlund methods with the properties:

(a) $p_{rn} \geqq p_{sn}$ for $r > s$ and for all $\underline{n}$;

(b) $\lim\limits_{n\to\infty} \dfrac{p_{m,n+1}}{p_{mn}} = 1$ for all $\underline{m}$;

(c) $p_{m,n+1} > p_{mn}$ for all $\underline{m}$ and for all n;

(d) $\lim\limits_{n\to\infty} p_{mn} = +\infty$ for every $\underline{m}$;

find a regular Nörlund method stronger than any method of the sequence.

The sequence of Cesàro methods, for integral values of α, possesses the properties listed above and in addition to condition (b) the Cesàro methods possess the property $\frac{p_{m+1,n+1}}{p_{mn}} > \frac{p_{m,n+1}}{p_{mn}}$, that is the ratio approaches one more slowly as $\underline{m}$ increases.

It is now of interest to examine the example given by Hardy and Hill. First, the following theorem is necessary.

<u>Theorem 4.1</u>. If (N, p_n) and (N, q_n) are regular Nörlund methods and:

(a) $\lim_{n\to\infty} P_n = +\infty$;

(b) $\lim_{n\to\infty} \frac{P_n}{Q_n} = 0$;

(c) there is an $\underline{M}$ such that for $n \geq M$, $k_n \geq 0$ where

$$k(x) = \frac{q(x)}{p(x)};$$

then (N, q_n) is stronger than (N, p_n).

Proof: For the methods (N, p_n) and (N, q_n)

$$Q_n = k_0 P_n + k_1 P_{n-1} + \cdots + k_n P_0.$$

There is an $\underline{M}$ such that for $n \geq M$, $k_n \geq 0$ and since $\lim_{n\to\infty} \frac{P_n}{Q_n} = 0$ there is an η such that for $n > \eta$, $\frac{P_n}{Q_n} < 1$. Let

$$A = \max\left[\sum_{i=0}^{M} |k_i|,\ \frac{|k_0|P_m + \cdots + |k_m|P_0}{Q_m} \text{ for } m = 0, 1, \cdots, \eta\right]$$

Then

$$\frac{|k_0|P_n + |k_1|P_{n-1} + \cdots + |k_M|P_{n-M} + |k_{m+1}|P_{n-M-1} + \cdots + |k_n|P_0}{Q_n}$$

$$= \frac{|k_0|P_N + |k_1|P_{n-1} + \cdots + |k_M|P_{n-M}}{Q_n} + \frac{k_{M+1}P_{n-M-1} + \cdots + k_nP_0}{Q_n}$$

$$\leqq 2\,\frac{|k_0|P_N + \cdots + |k_M|P_{n-M}}{Q_n} + \frac{k_0P_n + \cdots + k_nP_0}{Q_n}$$

$$\leqq 2A + 1$$

for every $\underline{n}$. Since $\lim_{n\to\infty} P_n = +\infty$ it follows from Theorem 2.7 that (N, q_n) is stronger than (N, p_n).

A method stronger than all the Cesàro methods $(C, \alpha) = (N, p_n)$ is the method (N, q_n) with $q_n = e^{\sqrt{n}}$. Hardy and Hill show that for these methods $\lim_{n\to\infty} \frac{P_n}{Q_n} = 0$ and that only a finite number of the k_n's are negative.

One attempt to solve the general problem consisted of trying to construct, from a sequence of methods having properties (a), (b), and (c) previously listed, a method satisfying the conditions of Theorem 4.1 relative to each method of the sequence. It was hoped this might be done by controlling the rate of decrease of the ratio $\frac{p_{n+1}}{p_n}$. This approach did not yield a solution but did suggest how a modified form of the problem could be solved. First three lemmas are given which simplify the proof of the main theorem.

<u>Lemma 1</u>: Given two regular Nörlund methods (N, p_n) and (N, q_n). If $\lim_{n\to\infty} \frac{Q_n}{P_n} = +\infty$ then (N, q_n) is stronger than or not comparable to

(N, p_n).

Proof: If (N, p_n) is stronger than (N, q_n) there exists a sequence $\{\ell_n\}$ and an $\underline{H}$ such that

$$\frac{|\ell_0|Q_n + |\ell_1|Q_{n-1} + \cdots + \ell_n Q_0}{P_n} < H \text{ for all } \underline{n}.$$

Therefore $\frac{Q_n}{P_n} < \frac{H}{\ell_0} = H$. But this contradicts the hypothesis $\lim_{n\to\infty} \frac{Q_n}{P_n} = +\infty$ hence the method (N, q_n) is stronger than or not comparable with (N, p_n).

Lemma 2. Given regular Nörlund methods (N, p_n) and (N, q_n) such that $\liminf_{n\to\infty} \frac{Q_n}{P_n} > 0$ then the method (N, Q_n) is stronger than or not comparable with the method (N, p_n).

Proof: Since the method (N, Q_n) is regular, $\lim_{n\to\infty} \frac{Q_n}{\sum_{m=0}^{n} Q_m} = 0$.

Therefore $\lim_{n\to\infty} \frac{\sum_{m=0}^{n} Q_m}{P_n} = \lim_{n\to\infty} \frac{\sum_{m=0}^{n} Q_m}{Q_n} \cdot \frac{Q_n}{P_n} = +\infty$ and from Lemma 1 it follows that the method (N, Q_n) is stronger than or not comparable with the method (N, p_n).

Lemma 3. If the sequence $\{p_n\}$ is such that $\frac{p_{n+1}}{p_n} > 1$ for all $\underline{n}$ and $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1$, then for any positive integer $\underline{m}$ and a constant $A > 1$ there is an $\underline{N}$ such that for $n > N$

$$\sum_{k=n+m+1}^{n+2m} p_k < A \sum_{k=n+1}^{n+m} p_k.$$

Proof: Let $\underline{r}$ be such that $r = \sqrt[m]{A}$. Since $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1$ there is an $\underline{N}$ such that for $n > N$, $\frac{p_{n+1}}{p_n} < r$. Then

$$\sum_{k=m+n+1}^{m+2n} p_k < p_{m+n+1}\left[\frac{r^m-1}{r-1}\right] = rp_{m+n}\left[\frac{r^m-1}{r-1}\right]$$

and

$$\sum_{k=n+1}^{n+m} p_k < p_{n+m}\left[\frac{\frac{1}{r^m}-1}{\frac{1}{r}-1}\right] = \frac{p_{n+m}}{r^{m-1}}\left[\frac{r^m-1}{r-1}\right] .$$

Therefore

$$\sum_{k=m+n+1}^{n+2m} p_k < r^m \sum_{k=n+1}^{n+m} p_k = A \sum_{k=n}^{n+m} p_k .$$

Theorem 4.2. For any sequence of regular Nörlund methods $\{(N, p_{mn})\}$, $m, n = 0, 1, 2, \cdots$, possessing the properties:

(a) $p_{rn} \geqq p_{sn}$ for $r > s$ and for all $\underline{n}$;

(b) $\lim_{n\to\infty} \frac{p_{m,n+1}}{p_{mn}} = 1$ for all $\underline{m}$;

(c) $p_{m,n+1} > p_{mn}$ for all $\underline{n}$ and all $\underline{m}$;

(d) $\lim_{n\to\infty} p_{mn} = +\infty$ for every $\underline{m}$;

there exists a regular Nörlund method (N, Q_n) which is stronger than or not comparable to any method of the sequence.

Proof: The method of approach to the problem will be to construct by a diagonal process a Nörlund method (N, q_n) such that $\lim_{n\to\infty} \frac{q_{n+1}}{q_n} = 1$ and $\liminf_{n\to\infty} \frac{Q_n}{P_{mn}} > 0$ for all $\underline{m}$. Then from Lemma 2 it follows that the method (N, Q_n) is the desired method.

Consider the first two methods of the sequence (N, p_{1n}) and (N, p_{2n}). Since $\lim_{n\to\infty} \frac{p_{m,n+1}}{p_{m,n}} = 1$ for all $\underline{m}$ there exists an η_1 such that for $n > \eta_1$, $\frac{p_{1,n+1}}{p_{1n}} < 1 + \frac{1}{2} = r_1$ and $\frac{p_{2,n+1}}{p_{2n}} < 1 + \frac{1}{2} = r_1$.

Since $p_{1\eta_1} \leqq p_{2\eta_1}$, $p_{1n} < p_{1,n+1}$ for all $\underline{n}$ and $\lim_{n\to\infty} p_{1n} = +\infty$ there is a $\delta_1 > \eta_1$ such that

$$p_{1,\delta_1-1} \leqq p_{2\eta_1} \leqq p_{1,\delta_1}.$$

Then

$$\frac{p_{2\eta_1}}{p_{1,\delta_1-1}} \leqq \frac{p_{1\delta_1}}{p_{1,\delta_1-1}} < r_1.$$

Let $q_n = p_{1n}$ for $n = 0, 1, 2, \cdots, (\delta_1 - 1)$

$$q_{\delta_1} = p_{2\eta_1}.$$

This diagonal process allows one to pick terms from the next sequence while still controlling the value of $\frac{q_{n+1}}{q_n}$.

Let $\underline{A}$ be a constant greater than $\underline{1}$. Let $a_1 = \delta_1 - \eta_1$ and $r_2 = \min (1 + \frac{1}{2^2}, \sqrt[a_1]{A})$. For the methods (N, p_{2n}) and (N, p_{3n}) there is an η_2 such that:

(a) For $n > \eta_2$, $\frac{p_{2,n+1}}{p_{2n}} < r_2$

(b) For $n > \eta_2$, $\frac{p_{3,n+1}}{p_{3n}} < r_2$

$$\text{(c)} \quad \sum_{i=\eta_1}^{\eta_2} p_{2i} > \frac{1}{A} \sum_{i=\eta_2+1}^{\eta_2+a_1} p_{2i} \geqq \frac{1}{A} \sum_{i=\eta_2+1}^{\eta_2+a_1} p_{1i}$$

Since $\lim_{n\to\infty} \frac{p_{m,n+1}}{p_{mr}} = 1$ for all $\underline{m}$, relations (a) and (b) are valid.

Relation (c) follows from the given condition that $p_{rn} \geqq p_{sn}$ for $r > s$ and from Lemma 3 with $\eta_2 - a_1$ greater than the $\underline{N}$ of Lemma 3.

The significance of the a_1 should be noted here. a_1 is the number of overlapping terms as one proceeds from sequence $\{p_{1n}\}$ to $\{p_{2n}\}$ and the η_2 is chosen so that the $\sum_{i=\eta_1}^{\eta_2} p_{2i}$ shall be of the same "size" as the sum of a_1 terms beyond $p_{1\eta_2}$ from the sequence $\{p_{1n}\}$.

Since $p_{3\eta_2} \geqq p_{2\eta_2}$, $p_{2n} < p_{2,n+1}$ for all $\underline{n}$ and $\lim_{n\to\infty} p_{2n} = +\infty$ there is a $\delta_2 > \eta_2$ such that

$$p_{2,\eta_1+\delta_2-1} \leqq p_{3\eta_2} \leqq p_{2,\eta_1+\delta_2}.$$

Then

$$\frac{p_{3\eta_2}}{p_{2,\eta_1+\delta_2-1}} \leqq \frac{p_{2,\eta_1+\delta_2}}{p_{2,\eta_1+\delta_2-1}} < r_2.$$

Let $q_{\delta_1+n} = p_{2,\eta_1+n}$ for $n = 0, 1, 2, \cdots, (\delta_2-1)$.

Continuing this process $\underline{k}$ times one obtains for all $n \leqq k$: a set of integers $[a_n]$ such that $a_n = (\delta_1 - \eta_1) + \sum_{i=2}^{n-1}\left[\delta_i - (\eta_i - \eta_{i-1})\right]$; a set of ratios $[r_n]$ with $r_n = \min\left(1+\frac{1}{2^n}, A^{\frac{1}{a_{n-1}}}\right)$; a set $[\eta_n]$ such that

$$\sum_{i=\eta_{m-1}}^{\eta_m} p_{mi} > \frac{1}{A} \sum_{i=\eta_m+1}^{\eta_m+a_{m-1}} p_{mi} \geqq \frac{1}{A} \sum_{i=\eta_m+1}^{\eta_m+a_{m-1}} p_{si} \quad \text{for all } s < m;$$

sets $[\delta_m]$ and $[q_m]$ such that

$$q_{\delta_1+\delta_2+\cdots+\delta_{m-1}+n} = p_{m,\eta_{m-1}+n} \quad \text{for } n = 0, 1, \cdots, (\delta_m - 1)$$

and

$$\frac{q_{n+1}}{q_n} < r_m \quad \text{for } \delta_1 + \delta_2 + \cdots + \delta_{m-1} < n < \delta_1 + \cdots + \delta_m.$$

For $a_{k+1} = \delta_1 - \eta_1 + \sum_{i=2}^{k} \left[\delta_i - (\eta_i - \eta_{i-1})\right]$ let

$r_{k+1} = \min\left(1 + \frac{1}{2^{k+1}}, \sqrt[a_k]{A}\right)$. For the methods $(N, p_{k+1,n})$ and $(N, p_{k+2,n})$ there is an η_{k+1} such that

(a) For $n > \eta_{k+1}$, $\frac{p_{k+1,n+1}}{p_{k+1,n}} < r_{k+1}$

(b) For $n > \eta_{k+1}$, $\frac{p_{k+2,n+1}}{p_{k+1,n}} < r_{k+1}$

(c) $\sum_{i=\eta_k}^{\eta_{k+1}} p_{k+1,i} > \frac{1}{A} \sum_{i=\eta_{k+1}+1}^{\eta_{k+1}+a_k} p_{k+1,i} \geqq \frac{1}{A} \sum_{i=\eta_{k+1}+1}^{\eta_{k+1}+a_k} p_{si}$

for all $s < k + 1$.

Since $p_{k+2,\eta_{k+1}} \geqq p_{k+1,\eta_{k+1}}$, $\lim_{n\to\infty} p_{k+1,n} = +\infty$, and the sequence $\{p_{k+1,n}\}$ is increasing there is a δ_{k+1} such that

$$p_{k+1,\eta_k+\delta_{k+1}-1} \leqq p_{k+2,\eta_{k+1}} \leqq p_{k+1,\eta_k+\delta_{k+1}}.$$

Then

$$\frac{p_{k+2,\eta_{k+1}}}{p_{k+1,\eta_k+\delta_{k+1}-1}} \leqq \frac{p_{k+1,\eta_k+\delta_{k+1}}}{p_{k+1,\eta_k+\delta_{k+1}-1}} < r_{k+1}.$$

Let $q_{\delta_1+\cdots+\delta_k+n} = p_{k+1,\eta_k+n}$ for $n = 0, 1, \cdots, (\delta_{k+1}-1)$.

Thus one obtains:

(α) a sequence of integers $\{a_n\}$ with

$$a_n = (\delta_1 - \eta_1) + \sum_{i=2}^{n} [\delta_i - (\eta_i - \eta_{i-1})];$$

(β) a sequence $\{r_n\}$ with $\lim_{n\to\infty} r_n = 1$; a sequence $\{\eta_n\}$

such that

$$\sum_{i=\eta_{m-1}}^{\eta_m} p_{mi} > \frac{1}{A} \sum_{i=\eta_m+1}^{\eta_m+a_m-1} p_{mi} \geqq \frac{1}{A} \sum_{i=\eta_m+1}^{\eta_m+a_{m-1}} p_{si} \text{ for all } s < m;$$

(γ) sequences $\{\delta_m\}$ and $\{q_m\}$ such that

$$\frac{q_{n+1}}{q_n} < r_m \quad \text{for} \quad \delta_1 + \cdots + \delta_{m-1} < n < \delta_1 + \cdots + \delta_m.$$

The method (N, q_n) is regular since by construction $\lim_{n\to\infty} \frac{q_{n+1}}{q_n} = 1$.

Consider now the relation between the method (N, q_n) and any method (N, p_{mn}) of the sequence of methods.

$$\frac{P_{mn}}{Q_n} = \frac{\sum_{i=0}^{\eta_m} p_{mi} + \sum_{i=\eta_m}^{n-a_{m-1}} p_{mi} + \sum_{i=n-a_{m-1}}^{n} p_{mi}}{Q_n}.$$

Since for any fixed $\underline{m}$, η_m is a constant and $\lim_{n\to\infty} Q_n = +\infty$ then

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{\eta_m} p_{mi}}{Q_n} = 0.$$

For $\underline{s}$ such that $\eta_m < s < n-a_{m-1}$ there is for every term p_{ms} from the sequence $\{p_{mn}\}$ a term q_{s_1} from the sequence $\{q_n\}$ such that $q_{s_1} \geq p_{ms}$.

Hence $\sum_{i=\eta_m}^{n-a_{m-1}} p_{mi} < Q_n$.

From the construction of the sequence $\{q_n\}$ it follows that $Q_n > \frac{1}{A} \sum_{i=n-a_{m-1}}^{n} p_{mi}$. Therefore $\limsup_{n\to\infty} \frac{P_{mn}}{Q_n} < 1 + A$ and it follows from Lemma 2 that the method (N, Q_n) is stronger than or not comparable to the method (N, p_{mn}) for any $\underline{m}$ and the theorem is established.

Whether a stronger theorem than the above can be established is an open question. Some thought was given to the sequence of methods consisting of the iterates of a given method but no stronger theorem was found for this case.

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