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THE INFLUENCE OF ELASTIC BOUNDARY CONDITIONS ON THE RESONANT RESPONSE OF THIN CYLINDRICAL SHELLS

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DISSERTATION COMMITTEE

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TABLE OF CONTENTS

Page

Appendices

LIST OF TABLES

LIST OF FIGURES

Figure		Page

Figure		Page

LIST OF SYMBOLS

a_{ij}	Flexibility influence functions of boundary structure, in./lb.
$\bar{a}_{ij}$	Nondimensional form of a_{ij}
A	Area, $in.^2$
D	Bending stiffness, lb.-in.
E	Modulus of elasticity, psi
E*	Complex modulus of elasticity
F_1, F_2, F_3	Force components per unit length at interface, lb./in.
G	Shear modulus, psi
h	Shell thickness, in.
i	$\sqrt{-1}$
K	Extensional stiffness, lb./in.
ℓ	Shell length, in.
m	Number of axial half-waves
M_x, M_θ	Bending moment per unit length, in.-lb./in.
$M_{x\theta}$	Twisting moment per unit length, in.-lb./in.
M_1	Bending moment per unit length at interface, in.-lb./in.
n	Circumferential wave number
N	Nondimensional frequency parameter
N_x	Axial normal force per unit length, lb./in.
$N_{x\theta}$	Tangential shearing force per unit length, lb./in.
q	Applied normal force per unit area, psi

q_{mn}	Fourier component of applied force
Q_0	Actual concentrated load, lb.
Q	Nondimensional load parameter
r	Shell radius, in.
R^2	Nondimensional geometric parameter
s, s^*	Transform variable
t	Time, sec.
T	Nondimensional time; total kinetic energy, in.-lb.
$u, \bar{u}$	Axial displacement component, in.
U_n	Normal mode form of u
$\mathcal{U}_n$	Laplace transform of U_n
$v, \bar{v}$	Tangential displacement component, in.
V_n	Normal mode form of v
$\mathcal{V}_n$	Laplace transform of V_n
V_x	Transverse shearing force per unit length, lb./in.
V	Potential energy, in.-lb.
$w, \bar{w}$	Normal displacement component, in.
w_{mn}	Generalized normal coordinate
w^*	Dimensionless normal displacement
W_n	Normal mode form of w
W	Work done by applied and boundary forces, in-lb.
$\mathcal{W}_n$	Laplace transform of W_n
x	Axial coordinate, in.
$\bar{x}$	Dimensionless axial coordinate
x_0	Point of application of load, in.

x_m	Maximum value of axial coordinate, in.
α	Phase angle
β	$w_{,x}$
γ	Structural damping parameter
δ	Impulse function, variation operator
δ_{mn}	Kronecker delta
ε	Strain
θ, ξ	Circumferential coordinates
λ	Transform variable
μ	Poisson's ratio
ρ	Mass density, lb.-sec^2/in^4
σ	Stress, psi
ω	Circular frequency
ω_n	Natural circular frequency
Ω_s	Resonant frequency with simple support, Hertz
Ω_e	Resonant frequency with elastic support, Hertz
ϕ^u_{mn} , ϕ^v_{mn} , ϕ^w_{mn}	Normal mode functions
∇^2	Laplacian operator = $\frac{\partial^2}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$
$(\ )'$	Differentiation with respect to its argument
$(\dot{\ })$	Differentiation with respect to time
$(\ )_{,x}$	Partial differentiation with respect to x

THE INFLUENCE OF ELASTIC BOUNDARY CONDITIONS ON THE RESONANT RESPONSE OF THIN CYLINDRICAL SHELLS

CHAPTER I

INTRODUCTION

An important step in the design of a complex structure which is to be subjected to a dynamic load environment is the determination of its dynamic response. This is necessary so that design specifications may be written to ensure its own structural integrity as well as that of components which may be attached internally. Uncertainty in these specifications could result in overdesign, excessive weight and cost and increased probability of failure. Thus, the determination of the dynamic response is very necessary but this is also quite difficult since the response depends upon several factors - such as boundary conditions, the nature of the applied excitation, and "loading due to the attachment of internal components.

For certain types of boundary conditions, called "natural" boundary conditions[1] the forced vibration response

[1]Natural boundary conditions are those for which no flow of energy across the boundary occurs. These are commonly called simple, fixed or free.

of most structural elements (including cylindrical shells) has been studied extensively and is well documented. However, the natural boundary conditions do not usually exist in reality, the true boundary conditions falling somewhere between free and fixed as the two limiting cases.

If the origin of the excitation is aerodynamic or acoustic in nature the exciting forces will be distributed over the surface and usually will be random, both spatially and temporally.

The response of a structure with items of equipment attached is not quite the same as without such items. This alteration of motion is called "loading". This factor is commonly approached in terms of mechanical impedance which in practical problems is a rather complicated complex function of frequency.

The problem of the determination of dynamic response of a complex structure is thus a very formidable one. One may take the approach that an analytical prediction of dynamic response is not warranted and rely on an experimental determination by tests using a prototype model or simply experimental verification of structural integrity. But short of a full scale test, it is not possible to simulate exactly the dynamic environment. In the laboratory, it is usually necessary to separate the substructure in question from its parent structure and attach it to a fixture which has completely different structural characteristics. One must then try to assess the effects of this change in boundary conditions on the dynamic response of the structure.

The problem of determining the effects of elastic boundary conditions on dynamic response cannot be avoided, even by turning to experimental methods. Therefore, this study is directed toward a determination of the effects of elastic boundary conditions on the dynamic response of a complex structure which is subjected to concentrated forces having a harmonic time history. The distributed and random aspects of the excitation could be treated later by the process of superposition. The problem of loading by attachment of components, whether involving single or multiple attachment points, could also be treated later by the superposition technique.

Of all structural elements, one of the most widely used is the thin circular cylindrical shell. In more practical applications it is of composite, sandwich or stiffened construction. However, only the thin isotropic, single layer shell is considered here, taking the point of view that its behavior forms the foundation on which could be based a study of the more practical shells with their complicating features.

A substantial body of literature exists on the forced vibration of shell structures. This has recently been summarized by Bert and Egle [1]. Although their primary interest was in the more practical composite, sandwich and stiffened shells, they cite many references to simple, isotropic shells as they trace the development of the various shell theories from their beginning in 1888 by A. E. H. Love [2]. Rather than trying to outline this development here, the reader is simply referred to books by Kraus [3] and Flügge [4] which are devoted to the derivation and application of shell theories. Only a representative list of the more recent and more pertinent works is given here [5-20]. The influence of natural boundary conditions in free vibration of cylindrical shells has been studied by Forsberg [5] and Vronay and Smith [6]. The response of a cylindrical shell to transient loads has been studied by Kraus and Kalnins [7], Sheng [8], Haywood [9] and Brogan [10]. Berger [11] studied the response of a semi-infinite shell to a general excitation, using a double Laplace transform of the Timoshenko-Love equations.

Steady state response to harmonic excitation of the point and line type were studied by Heckl [12], and by Baron and Bleich [13]. Several papers dealing with the effects of loading are given in [14-18]. The random response has been studied by Hwang and Pi [19] and by Kana [20].

These papers have dealt with shell vibration due to various types of excitation but always have used the natural boundary conditions. One exception to this is a paper by Gaymon [21] in which the effect of interface flexibility is considered in the case of excitation which is applied also at the interface. Two other papers by Egle [22] and Henry and Egle [23] have dealt with the effects of elastic boundary conditions but only in the response of beams and plates. Also Chin and Grossman [24] have studied flexural vibrations of circular plates with elastic support. Nothing else has been found to date concerning elastic boundary conditions with forced vibration.

The structural element to be studied here is the thin, isotropic, single layered, circular cylindrical shell. It is simply supported without axial constraint at one end and has passive elastic supports at the other. It is subjected to a single concentrated force which is normal to the shell and has a simple harmonic time history. The boundary structure is modeled as a distribution of closely spaced independent linear elastic springs. Their effect on the resonant frequency and mode shape are to be determined.

CHAPTER II

DEVELOPMENT OF THE SOLUTION

2.1 Equations of Motion

The present analysis is bas[illegible] the following equations of motion (2-1) which [illegible]unterpart of the Donnell equations. T[illegible] arbitrarily, because of their relativ[illegible], say, the Flügge equations. The d[illegible]ons and corresponding boundary condi[illegible]ix A.

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\mu}{2r^2}\frac{\partial^2 u}{\partial \theta^2} + \text{[illegible]}$$

$$- \frac{1-\mu^2}{E}\rho\frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{1+\mu}{2r}\frac{\partial^2 u}{\partial x \partial \theta} + \frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial w}{\partial \theta}$$

$$- \frac{1-\mu^2}{E}\rho\frac{\partial^2 v}{\partial t^2} = 0$$

$$\frac{\mu}{r}\frac{\partial u}{\partial x} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{w}{r^2} + \frac{h^2}{12}\nabla^4 w + q\frac{1-\mu^2}{Eh}$$

$$+ \frac{1-\mu^2}{E}\rho\frac{\partial^2 w}{\partial t^2} = 0 \qquad (2\text{-}1)$$

The coordinate system and the sign convention for the internal forces and moments is shown in Figure 2.1.

These equations are based on the usual assumptions of

CHAPTER II

DEVELOPMENT OF THE SOLUTION

2.1 Equations of Motion

The present analysis is based on the following equations of motion (2-1) which are the dynamic counterpart of the Donnell equations. These were chosen, quite arbitrarily, because of their relative simplicity, rather than, say, the Flügge equations. The derivation of these equations and corresponding boundary conditions is given in Appendix A.

$$\frac{\partial^2 u}{\partial x^2} + \frac{1-\mu}{2r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1+\mu}{2r}\frac{\partial^2 v}{\partial x \partial \theta} + \frac{\mu}{r}\frac{\partial w}{\partial x} - \frac{1-\mu^2}{E}\rho\frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{1+\mu}{2r}\frac{\partial^2 u}{\partial x \partial \theta} + \frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial w}{\partial \theta} - \frac{1-\mu^2}{E}\rho\frac{\partial^2 v}{\partial t^2} = 0$$

$$\frac{\mu}{r}\frac{\partial u}{\partial x} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{w}{r^2} + \frac{h^2}{12}\nabla^4 w + q\frac{1-\mu^2}{Eh} + \frac{1-\mu^2}{E}\rho\frac{\partial^2 w}{\partial t^2} = 0 \qquad (2\text{-}1)$$

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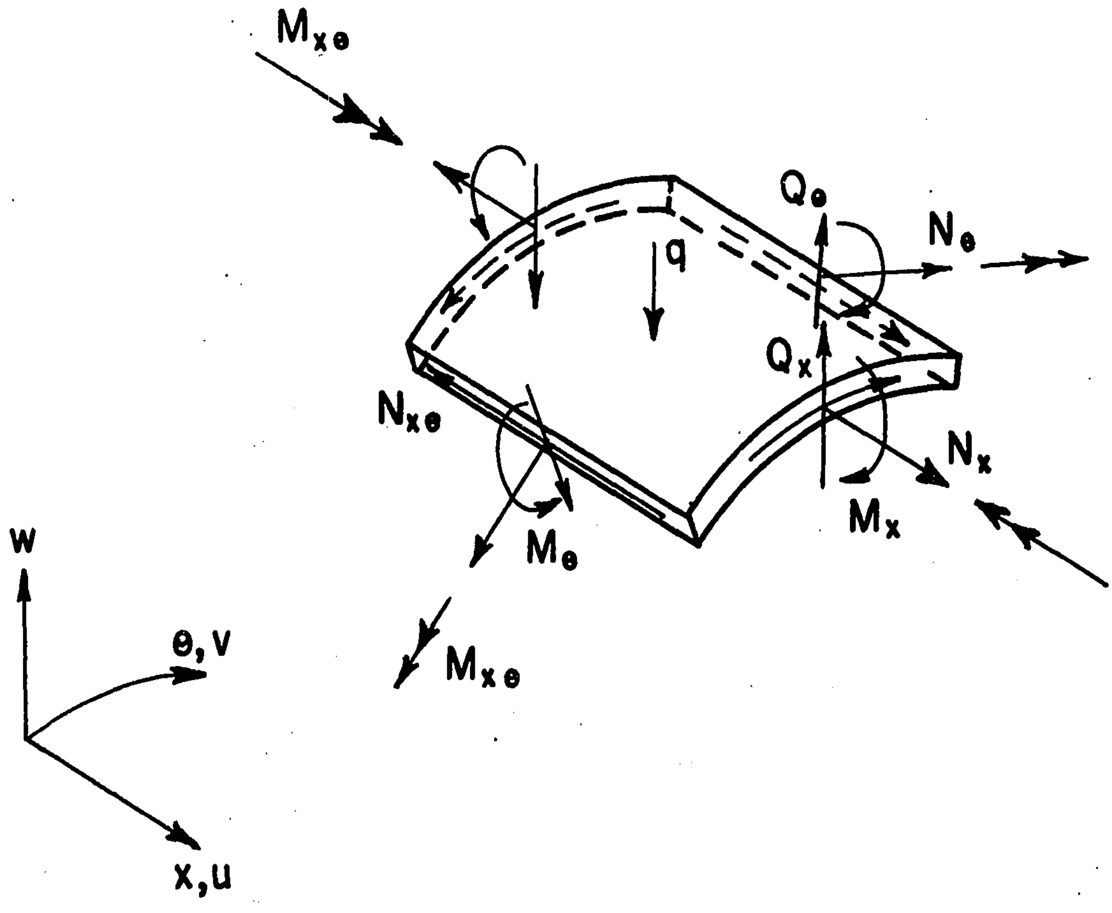

Figure 2.1 Coordinate System and Stress Resultants Acting on Differential Element.

linear shell theory;

1. The shell is thin, of uniform thickness, and of a linear, homogeneous, isotropic material.
2. The deflections of the shell are small.
3. The transverse normal stress is negligible.
4. Normals to the reference surface of the shell remain normal to it and undergo no change in length during deformation.

These postulates constitute what is usually called Love's first approximation. These require the neglect of shear deformation and rotatory inertia. In addition to these, the simplifications due to Donnell are:

1. The transverse shearing force makes a negligible contribution to equilibrium in the circumferential direction.
2. The expressions for the changes in curvature and twist of the shell are the same as those of a flat plate.

These equations of motion are first expressed in terms of nondimensional parameters by introducing the transformations

$$\bar{x} = \frac{x}{r} ,$$

$$T = \omega t ,$$

$$w(x,\theta,t) = \bar{w}(\bar{x},\theta,T) . \qquad (2\text{-}2)$$

The axial coordinate is nondimensionalized with respect to radius instead of length, quite arbitrarily, because it simplified the equations of motion as much as possible. The form of the equations becomes:

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \left(\frac{1-\mu}{2}\right) \frac{\partial^2 \bar{u}}{\partial \theta^2} + \left(\frac{1+\mu}{2}\right) \frac{\partial^2 \bar{v}}{\partial \bar{x} \partial \theta} + \mu \frac{\partial \bar{w}}{\partial \bar{x}} - N \frac{\partial^2 \bar{u}}{\partial T^2} = 0 \ ,$$

$$\left(\frac{1+\mu}{2}\right) \frac{\partial^2 \bar{u}}{\partial \bar{x} \partial \theta} + \left(\frac{1-\mu}{2}\right) \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \theta^2} + \frac{\partial \bar{w}}{\partial \theta} - N \frac{\partial^2 \bar{v}}{\partial T^2} = 0 \ ,$$

$$\mu \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \theta} + \bar{w} + R^2 \left(\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + \frac{2\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \theta^2} + \frac{\partial^4 \bar{w}}{\partial \theta^4}\right)$$

$$+ Q + N \frac{\partial^2 \bar{w}}{\partial T^2} = 0 \ , \tag{2-3}$$

where

$$Q = \text{load parameter} = \frac{qr^2}{Eh} (1-\mu^2) \ ,$$

$$N = \text{frequency parameter} = (1-\mu^2) \frac{\rho \omega^2 r^2}{E} \ ,$$

$$R^2 = \frac{h^2}{12r^2} \ . \tag{2-4}$$

2.2 Assumed Form of the Solution

For a closed cylinder which is subjected to a point harmonic excitation of frequency ω, it is reasonable to assume that the response of the shell is of the form:

$$\bar{u}(\bar{x},\theta,T) = \sum_{n=0}^{\infty} U_n(\bar{x}) \cos n\theta \, e^{iT} \ ,$$

$$\bar{v}(\bar{x},\theta,T) = \sum_{n=1}^{\infty} V_n(\bar{x}) \sin n\theta \, e^{iT} \ ,$$

$$\bar{w}(\bar{x},\theta,T) = \sum_{n=0}^{\infty} W_n(\bar{x}) \cos n\theta \, e^{iT} \ , \tag{2-5}$$

where the $U_n(\bar{x})$, $V_n(\bar{x})$, $W_n(\bar{x})$ are complex functions because of the presence of damping.

Since the excitation is a point load, applied at $(\bar{x}_0, \theta)$, the distributed load q appearing in the equations of motion must be expressed in terms of the impulse function as

$$q = \frac{Q_0}{r} \delta(\bar{x}-\bar{x}_0)\ \delta(\theta) e^{iT} ,$$

where $Q_0 \equiv$ magnitude of the load. The r in the denominator is required since $\int_A q\ dA = Q_0 e^{iT}$. The load parameter Q thus becomes

$$Q_0 = \frac{qr^2(1-\mu^2)}{Eh} = \frac{Q_0 r}{Eh} (1-\mu^2)\ \delta(\bar{x}-\bar{x}_0)\ \delta(\theta) e^{iT} . \tag{2-6}$$

The excitation is assumed to be applied at θ=0. There is no loss in generality in this because of the radial symmetry of the shell. Since the δ(θ) is a periodic quantity with interval 2π its Fourier series expansion is

$$\delta(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos n\theta . \tag{2-7}$$

Substitution of equations (2-5), (2-6) and (2-7) into the equations of motion (2-3) results in, using the third of equations (2-3) for example,

$$\sum_{n=0}^{\infty} \{\mu U_n' + nV_n + W_n + R^2(W_n^{IV} - 2n^2 W_n'' + n^4 W_n)$$

$$-NW_n + Q_n \delta(\bar{x}-\bar{x}_0)\} \cos n\theta\ e^{iT} = 0 .$$

Since this must hold true for all values of θ, the coefficient of each cos nθ must be zero. The argument is the same for the remaining two equations of motion and equations (2-3) are reduced to sets of three coupled ordinary differential equations of the form:

$$U_n'' - \frac{1-\mu}{2} n^2 U_n + \frac{1+\mu}{2} nV_n' + \mu W_n' + NU_n = 0,$$

$$-n \frac{1+\mu}{2} U_n' + \frac{1-\mu}{2} V_n'' - n^2 V_n - nW_n + NV_n = 0,$$

$$\mu U_n' + nV_n + W_n + R^2 (W_n^{IV} - 2n^2 W_n'' + n^4 W_n)$$

$$-NW_n + Q_n \delta(\bar{x}-\bar{x}_0) = 0, \qquad (2\text{-}8)$$

for n = 1, 2, 3 ...

and

$$U_0'' + \mu W_0' + NU_0 = 0$$

$$\mu U_0' + W_0 + R^2 W_0^{IV} - NW_0 + \frac{Q}{2} \delta(\bar{x}-\bar{x}_0) = 0 \qquad (2\text{-}9)$$

for n = 0.

2.3 Laplace Transformation

The equations of motion are now of a form (2-8) which is suitable for application of the Laplace transformation technique. The transformed equations of motion are:

$$\mathcal{U}_n(s)[s^2 + N - (\frac{1-\mu}{2})n^2] + \mathcal{V}_n(s)[(\frac{1+\mu}{2})ns]$$

$$+ \mathcal{W}_n(s)[\mu s] = sU_n(0) + U_n'(0)$$

$$+ \frac{1+\mu}{2} V_n(0) + \mu W_n(0) ,$$

$$\mathcal{U}_n(s)[-(\frac{1+\mu}{2})ns] + \mathcal{V}_n(s)[(\frac{1-\mu}{2})s^2 + N - n^2]$$

$$+ \mathcal{W}_n(s)[-n] = (\frac{1-\mu}{2})sV_n(0)$$

$$+ \left(\frac{1-\mu}{2}\right) V_n'(0) - \left(\frac{1+\mu}{2}\right) nU_n(0) ,$$

$$\mathcal{U}_n(s)[\mu s] + \mathcal{V}_n(s)[n] + \mathcal{W}_n(s)[R^2 s^4 - 2n^2 R^2 s^2 + n^4 R^2 + 1$$

$$- N] = R^2 s^3 W_n(0) + R^2 s^2 W_n'(0) + R^2 s W_n''(0) + R^2 W_n'''(0)$$

$$- 2n^2 R^2 s W_n(0) - 2n^2 R^2 W_n'(0) + \mu U_n(0) - Q_n e^{-s\bar{x}_0} . \quad (2\text{-}10)$$

There are eight "initial" conditions or boundary conditions involved in these transformed equations, $U_n(0)$, $U_n'(0)$, $V_n(0)$, $V_n'(0)$, $W_n(0)$, $W_n'(0)$, $W_n''(0)$, $W_n'''(0)$. Only four boundary conditions can be expressed at the end $\bar{x}=0$. The others will be treated as constants ultimately to be determined by looking to the end $\bar{x} = \bar{x}_{max}$ for four independent equations resulting from consideration of four boundary conditions at that end.

Assuming a simple support without axial constraint at $\bar{x}=0$, four of the boundary conditions are expressed as follows:

$$\begin{Bmatrix} N_x(\bar{x},\theta,T) \\ \bar{v}\ (\bar{x},\theta,T) \\ \bar{w}\ (\bar{x},\theta,T) \\ M_x(\bar{x},\theta,T) \end{Bmatrix}_{\bar{x}=0} = 0 . \quad (2\text{-}11)$$

These imply that

$$\begin{Bmatrix} \frac{\partial \bar{u}}{\partial \bar{x}}\ (\bar{x},\theta,T) \\ \bar{v}\ (\bar{x},\theta,T) \\ \bar{w}\ (\bar{x},\theta,T) \\ \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}(\bar{x},\theta,T\_ \end{Bmatrix}_{\bar{x}=0} = 0 . \quad (2\text{-}12)$$

Using the assumed form of the solution, equations (2-5), the first boundary condition becomes

$$\left.\frac{\partial \bar{u}}{\partial \bar{x}}\right|_{\bar{x}=0} = 0 = \sum_{n=0}^{\infty} U_n'(0)\cos n\theta \, e^{iT} .$$

Since this is true for all values of θ, the $U_n'(0)$ must all be zero. The argument is similar for the other three equations of the set (2-12) and the boundary conditions at $\bar{x}=0$ become

$$\begin{Bmatrix} U_n'(0) \\ V_n(0) \\ W_n(0) \\ W_n''(0) \end{Bmatrix} = 0 . \tag{2-13}$$

After substituting the known boundary conditions, equations (2-13), into the transformed equations (2-10), the transformed displacements can be determined using Cramer's rule. For example:

$$\mathcal{W}_n(s) = \frac{\text{Num}}{\text{Denom}} , \tag{2-14}$$

where

$$\text{Num} = \begin{vmatrix} s^2+N-\left(\frac{1-\mu}{2}\right)n^2 & \frac{1+\mu}{2}\,ns & sU_n(0) \\ -\left(\frac{1+\mu}{2}\right)ns & \frac{1-\mu}{2}\,s^2+N-n^2 & \frac{1-\mu}{2}\,V_n'(0)-\frac{1+\mu}{2}\,nU_n(0) \\ \mu s & n & \begin{matrix} R^2\left((s^2-2n^2)W_n'(0)+W_n'''(0)\right) \\ +\mu U_n(0)-Q_n e^{-s\bar{x}_0} \end{matrix} \end{vmatrix} \tag{2-15}$$

$$\text{Denom} = \begin{vmatrix} s^2+N-\left(\frac{1-\mu}{2}\right)n^2 & \left(\frac{1+\mu}{2}\right)ns & \mu s \\ -\left(\frac{1+\mu}{2}\right)ns & \left(\frac{1-\mu}{2}\right)s^2+N-n^2 & -n \\ \mu s & n & R^2(s^2-n^2)^2+1-N \end{vmatrix} \tag{2-16}$$

with similar expressions for $U_n(s)$ and $V_n(s)$, both with the same denominator as $W_n(s)$.

Denom is an eighth order polynomial in s, having complex coefficients since damping is present. The determination of its eight roots will be discussed later. These roots will in general all be complex and will be distinct, not appearing in complex conjugate pairs because the coefficients of the polynomial are not real.

2.4 Inverse Transformation

Once the roots of Denom have been determined, $U_n(s)$, $V_n(s)$, $W_n(s)$ can be expanded in partial fractions. In terms of its roots Denom is reconstructed as

$$\text{Denom} = (s-s_1^*)(s-s_2^*) \dots (s-s_8^*) \ . \tag{2-17}$$

The partial fraction expansion is then of the form

$$W_n(s) = \frac{\text{Num}}{\text{Denom}} = \sum_{i=1}^{8} \frac{C_i}{s-s_i^*}$$

where

$$C_i = \frac{\text{Num}}{\text{Denom}} (s-s_i^*)\Big|_{s=s_i^*} \tag{2-18}$$

Since the determination of the C_i requires evaluation of Num at a given root s_i^*, for convenience the symbols T_{ij} are introduced. They represent the term in row i and column j of Denom. In the expansion for $\mathcal{W}_n(s)$, then, the numerator becomes

$$\text{Num} = \begin{vmatrix} T_{11} & T_{12} & sU_n(0) \\ T_{21} & T_{22} & \frac{1-\mu}{2} V_n'(0) - \frac{1+\mu}{2} nU_n(0) \\ T_{31} & T_{32} & R^2\left((s^2-2n^2)W_n'(0)+W_n'''(0)\right)+U(0)-Q_n e^{-s\bar{x}_0} \end{vmatrix} \quad (2\text{-}19)$$

When expanded this is

$$\begin{aligned} \text{Num} = {} & U_n(0)[s(T_{21}T_{32}-T_{31}T_{22})+\mu(T_{11}T_{22}-T_{21}T_{12}) \\ & -\tfrac{1+\mu}{2}(n)(T_{11}T_{32}-T_{31}T_{12})] \\ & +V_n'(0)[\tfrac{1-\mu}{2}(T_{11}T_{32}-T_{31}T_{12})] \\ & +W_n'(0)[(s^2-2n^2)(R^2)(T_{11}T_{22}-T_{21}T_{12})] \\ & +W_n'''(0)[R^2(T_{11}T_{22}-T_{21}T_{12})] \\ & -Q_n e^{-s\bar{x}_0}[T_{11}T_{22}-T_{21}T_{12}] \ . \end{aligned} \quad (2\text{-}20)$$

The expression for Num has been broken up into five different functions of s, one associated with each of the four unknown boundary conditions and one associated with the applied load. The boundary conditions will hereafter be referred to as BC_j following the order in which they appear in equation (2-20). Each of these five functions of s is expanded in a partial fraction expansion of the form of equation (2-18) so that the transformed displacement $\mathcal{W}_n(s)$ is now of the form

$$\mathcal{W}_n(s) = \sum_{j=1}^{4} \sum_{i=1}^{8} FSW_{ij} BC_j + \sum_{i=1}^{8} QSW_i e^{s_i^* \bar{x}_0} \ . \quad (2\text{-}21)$$

The terms FSW_{ij} appearing in equation (2-21) are the constants resulting from the partial fraction expansion, the subscript j referring to the particular function of s from equation (2-20), the subscript i referring to a particular root of Denom. The terms QSW_i are the constants resulting from the expansion of the fifth function of equation (2-20). Written out, these constants are:

$$FSW_{i1} = [s(T_{21}T_{32}-T_{31}T_{22})+\mu(T_{11}T_{22}-T_{21}T_{12}) - \left(\frac{1+\mu}{2}\right)n(T_{11}T_{32}-T_{31}T_{12})](s-s_i^*)/\text{Denom}\Big|_{s=s_i^*} ,$$

$$FSW_{i2} = \left(\frac{1-\mu}{2}\right)(T_{11}T_{32}-T_{31}T_{12})(s-s_i^*)/\text{Denom}\Big|_{s=s_i^*} ,$$

$$FSW_{i3} = R^2(s^2-2n^2)(T_{11}T_{22}-T_{21}T_{12})(s-s_i^*)/\text{Denom}\Big|_{s=s_i^*} ,$$

$$FSW_{i4} = R^2(T_{11}T_{22}-T_{21}T_{12})(s-s_i^*)/\text{Denom}\Big|_{s=s_i^*} ,$$

$$QSW_i = Q_n(T_{11}T_{22}-T_{21}T_{12})(s-s_i^*)/\text{Denom}\Big|_{s=s_i^*} .$$

In a similar manner, the transformed displacements $U_n(s)$ and $V_n(s)$ can be expanded into the forms:

$$U_n(s) = \sum_{j=1}^{4}\sum_{i=1}^{8} FSU_{ij}BC_j + \sum_{i=1}^{8} QSU_i e^{-s_i^*\bar{x}_0},$$

$$V_n(s) = \sum_{j=1}^{4}\sum_{i=1}^{8} FSV_{ij}BC_j + \sum_{i=1}^{8} QSV_i e^{-s_i^*\bar{x}_0} . \tag{2-22}$$

The constants which appear in equations (2-22) are defined as follows:

$$FSU_{i1} = [s(T_{22}T_{33}-T_{32}T_{23})-\left(\frac{1+\mu}{2}\right)n(T_{32}T_{13}-T_{12}T_{33})$$

$$+\ \mu(T_{12}T_{23}-T_{22}T_{13})](s-s_i^*)/\text{Denom}\Big|_{s=s_i^*}\ ,$$

$$FSU_{i2} = \left(\frac{1-\mu}{2}\right)(T_{32}T_{13}-T_{12}T_{33})(s-s_i^*)/\text{Denom}\Big|_{s=s_i^*}\ ,$$

$$FSU_{i3} = R^2(s^2-2n^2)(T_{12}T_{23}-T_{22}T_{13})(s-s_i^*)/\text{Denom}\Big|_{s=s_i^*}\ ,$$

$$FSU_{i4} = R^2(T_{12}T_{33}-T_{22}T_{33})(s-s_i^*)/\text{Denom}\Big|_{s=s_i^*}\ ,$$

$$QSU_i = -Q_n(T_{12}T_{23}-T_{22}T_{13})(s-s_i^*)/\text{Denom}\Big|_{s=s_i^*}\ ,$$

$$FSV_{i1} = [s(T_{31}T_{23}-T_{21}T_{33})-\left(\frac{1+\mu}{2}\right)n(T_{11}T_{33}-T_{31}T_{13})$$

$$+\mu(T_{21}T_{13}-T_{11}T_{23})](s-s_i^*)/\text{Denom}\Big|_{s=s_i^*}\ ,$$

$$FSV_{i2} = \left(\frac{1-\mu}{2}\right)(T_{11}T_{33}-T_{31}T_{13})(s-s_i^*)/\text{Denom}\Big|_{s=s_i^*}\ ,$$

$$FSV_{i3} = R^2(s^2-2n^2)(T_{21}T_{13}-T_{11}T_{23})(s-s_i^*)/\text{Denom}\Big|_{s=s_i^*}\ ,$$

$$FSV_{i4} = R^2(T_{21}T_{13}-T_{11}T_{23})(s-s_i^*)/\text{Denom}\Big|_{s=s_i^*}\ ,$$

$$QSV_i = -Q_n(T_{21}T_{13}-T_{11}T_{23})(s-s_i^*)/\text{Denom}\Big|_{s=s_i^*}\ .$$

The inverse transform can now be taken, using the two transform pairs [25],

$$\frac{1}{s+a} \quad e^{-ax}\ , \text{ and } \quad e^{-as}F(s) \quad f(x-a)u(x-a)\ , \qquad (2\text{-}23)$$

where $u(x-a) \equiv$ unit step function. The displacement functions then become

$$W_n(\bar{x}) = \sum_{j=1}^{4}\sum_{i=1}^{8} FSW_{ij}BC_j e^{-s_i^*\bar{x}} + \sum_{i=1}^{8} QSW_i e^{-s_i^*(\bar{x}-\bar{x}_0)} u(\bar{x}-\bar{x}_0)\ ,$$

$$U_n(\bar{x}) = \sum_{j=1}^{4} \sum_{i=1}^{8} FSU_{ij} BC_j e^{-s_i^* \bar{x}} + \sum_{i=1}^{8} QSU_i e^{-s_i^*(\bar{x}-\bar{x}_0)} u(\bar{x}-\bar{x}_0) \ ,$$

$$V_n(\bar{x}) = \sum_{j=1}^{4} \sum_{i=1}^{8} FSV_{ij} BC_j e^{-s_i^* \bar{x}} + \sum_{i=1}^{8} QSV_i e^{-s_i^*(\bar{x}-\bar{x}_0)} u(\bar{x}-\bar{x}_0) \ . \qquad (2\text{-}24)$$

The solution, equations (2-24), still involves the four unknown boundary conditions $U_n(0)$, $V_n'(0)$, $W_n'(0)$, $W_n'''(0)$ which must be determined by a consideration of the elastic boundary conditions at $\bar{x}=\bar{x}_{max}$. Once this has been done the solution is simply an evaluation of equations (2-24), which are at this point exact within the framework of Love's first approximation and the Donnell assumptions - no other approximations have been made.

2.5 Boundary Conditions

At $\bar{x}=\bar{x}_{max}$ the boundary conditions are expressed as the internal forces acting at the interface between the substructure (cylindrical shell) and the parent structure. These forces (in-plane normal force, in-plane shearing force, transverse shearing force, and bending moment) are related to the displacement in the substructure by the equations (see Appendix A)

$$\bar{F}_1 = \frac{1}{r}\left[\frac{\partial \bar{u}}{\partial \bar{x}} + \mu\left(\frac{\partial \bar{v}}{\partial \theta} + \bar{w}\right)\right],$$

$$\bar{F}_2 = \left(\frac{1-\mu}{2}\right)\left(\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \theta}\right)\frac{1}{r} \ ,$$

$$\bar{F}_3 = \frac{-R^2}{r}\left[\frac{\partial^3 \bar{w}}{\partial \bar{x}^3} + (2-\mu)\frac{\partial^3 \bar{w}}{\partial \bar{x}\,\partial \theta^2}\right] \ ,$$

$$\bar{M}_1 = \frac{-R^2}{r}\left(\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \mu \frac{\partial^2 \bar{w}}{\partial \theta^2}\right) \quad . \tag{2-25}$$

These internal force components are related to the elastic properties of the parent structure by introducing its force-deflection influence functions which are assumed to be known functions. These expressions are as follows:

$$\begin{Bmatrix} du(\theta,t) \\ dv(\theta,t) \\ dw(\theta,t) \\ d\beta(\theta,t) \end{Bmatrix} = \begin{bmatrix} a_{11}(\theta,\xi,\omega) & 0 & a_{13} & a_{14} \\ 0 & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} F_1(\xi,t) \\ F_2(\xi,t) \\ F_3(\xi,t) \\ M_1(\xi,t) \end{Bmatrix} r d\xi \quad . \tag{2-26}$$

These equations relate the displacements at position θ to the forces acting at position ξ. The influence functions a_{ij} are complex functions of θ,ξ, and perhaps also frequency, complex to allow for dissipation in the parent structure, frequency dependent if the parent structure has mass.

By a process of superposition the displacement $\bar{u}(\bar{x}_m,\theta,T)$ for example, is given as

$$\bar{u}(\bar{x}_m,\theta,T) = \int_0^{2\pi} [\bar{a}_{11}(\theta,\xi,\omega)\bar{F}_1(\xi,T)+\bar{a}_{13}\bar{F}_3+\bar{a}_{14}\bar{M}_1] r d\xi \quad , \tag{2-27}$$

where the $\bar{a}_{ij}$ are the nondimensional form of the a_{ij} (see Appendix A). After substituting the force-displacement relationships for the substructure, equations (2-25), into equation (2-27) it becomes

$$\bar{u}(\bar{x}_m,\theta,T) = \int_0^{2\pi} \left[\bar{a}_{11} \frac{\partial \bar{u}}{\partial \bar{x}} + \mu \bar{a}_{11} \frac{\partial \bar{v}}{\partial \theta} \right.$$

$$+\{\mu\bar{a}_{11}\bar{w}-R^2\bar{a}_{13}\left(\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + (2-\mu)\frac{\partial^3\bar{w}}{\partial\bar{x}\partial\theta^2}\right)$$

$$-R^2\bar{a}_{14}\left(\frac{\partial^2\bar{w}}{\partial\bar{x}^2} + \mu\,\frac{\partial^2\bar{w}}{\partial\theta^2}\right)\}\Bigg]d\xi\ . \tag{2-28}$$

The assumed solutions, equations (2-5), are then substituted into equation (2-28), the latter becoming

$$\sum_{n=0}^{\infty} U_n(\bar{x}_m)\cos n\theta\ e^{iT} = \int_0^{2\pi}\sum_{n=0}^{\infty}\Big(\bar{a}_{11}U_n'(\bar{x}_m)+\mu\bar{a}_{11}nV_n(\bar{x}_m)$$
$$+\mu\bar{a}_{11}W_n(\bar{x}_m)+R^2[\mu n^2\bar{a}_{14}W_n(\bar{x}_m)$$
$$+(2-\mu)n^2\bar{a}_{13}W_n'(\bar{x}_m)-\bar{a}_{14}W_n''(\bar{x}_m)$$
$$-\bar{a}_{13}W_n'''(\bar{x}_m)]\Big)\cos n\xi\ e^{iT}d\xi\ . \tag{2-29}$$

This can be readily integrated if the $\bar{a}_{ij}$ are of the form

$$\bar{a}_{ij} = \bar{\bar{a}}_{ij}(\theta,\omega)\delta(\xi-\theta)\ . \tag{2-30}$$

If this is the case equation (2-29) becomes:

$$\sum_{n=0}^{\infty} U_n(\bar{x}_m)\cos n\theta\ e^{iT} = \sum_{n=0}^{\infty}\Big(\bar{\bar{a}}_{11}U_n'(\bar{x}_m)+\mu\bar{\bar{a}}_{11}nV_n(\bar{x}_m)$$
$$+\mu\bar{\bar{a}}_{11}W_n(\bar{x}_m)+R^2[\mu n^2\bar{\bar{a}}_{14}W_n(\bar{x}_m)+(2-\mu)n^2\bar{\bar{a}}_{13}W_n'(\bar{x}_m)$$
$$-\bar{\bar{a}}_{14}W_n''(\bar{x}_m)-\bar{\bar{a}}_{13}W_n'''(\bar{x}_m)]\Big)\cos n\theta\ e^{iT}\ .$$

This equation must hold for all values of θ; therefore the coefficients of cos $n\theta$ must be equal on each side of the equation.

The other three equations of the set (2-26) are integrated in the same manner and all four results are repeated here:

$$\begin{Bmatrix} U_n \\ V_n \\ W_n \\ W_n' \end{Bmatrix}_{\bar{x}=\bar{x}_m} = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \\ L_{41} & L_{42} & L_{43} \end{bmatrix} \begin{Bmatrix} U_n \\ V_n \\ W_n \end{Bmatrix}_{\bar{x}=\bar{x}_m} \tag{2-31}$$

where

$$L_{11} = \bar{\bar{a}}_{11} \frac{d}{dx} ,$$

$$L_{12} = \mu n \bar{\bar{a}}_{11} ,$$

$$L_{13} = \mu \bar{\bar{a}}_{11} + R^2 \left(\mu n^2 \bar{\bar{a}}_{14} + (2-\mu) n^2 \bar{\bar{a}}_{13} \frac{d}{dx} - \bar{\bar{a}}_{14} \frac{d^2}{dx^2} - \bar{\bar{a}}_{13} \frac{d^3}{dx^3}\right) ,$$

$$L_{21} = -n \left(\frac{1-\mu}{2}\right) \bar{\bar{a}}_{22} ,$$

$$L_{22} = \frac{1-\mu}{2} \bar{\bar{a}}_{22} \frac{d}{dx} ,$$

$$L_{23} = 0 ,$$

$$L_{31} = \bar{\bar{a}}_{31} \frac{d}{dx} ,$$

$$L_{32} = \mu n \bar{\bar{a}}_{31} ,$$

$$L_{33} = \mu \bar{\bar{a}}_{31} + R^2 \left(\mu n^2 \bar{\bar{a}}_{34} + (2-\mu) n^2 \bar{\bar{a}}_{33} \frac{d}{dx} - \bar{\bar{a}}_{34} \frac{d^2}{dx^2} - \bar{\bar{a}}_{33} \frac{d^3}{dx^3}\right) ,$$

$$L_{41} = \bar{\bar{a}}_{41} \frac{d}{dx} ,$$

$$L_{42} = \mu n \bar{\bar{a}}_{41} ,$$

$$L_{43} = \mu \bar{\bar{a}}_{41} + R^2 \left(\mu n^2 \bar{\bar{a}}_{44} + (2-\mu) n^2 \bar{\bar{a}}_{43} \frac{d}{dx} - \bar{\bar{a}}_{44} \frac{d^2}{dx^2} - \bar{\bar{a}}_{43} \frac{d^3}{dx^3}\right) .$$

The assumption that $\bar{a}_{ij} = \bar{\bar{a}}_{ij} \delta(\theta - \xi)$ is physically equivalent to the assumption that the parent structure can be modeled as a distribution of independent springs, as shown in Figure 2.2(a). The a_{ij} are defined as follows

a_{11} = axial deflection of boundary structure due to a unit axial force applied to the boundary,

a_{22} = tangential deflection of boundary structure due to a unit tangential force applied to the boundary,

or in general

a_{ij} = deflection of boundary structure in i direction due to unit force applied at the boundary in j direction.

The positive sense of the internal forces at the interface is as shown in Figure 2.2(b). Consequently, positive forces acting on the parent structure are equal and opposite to these and a_{11}, a_{22} and a_{33} are negative quantities.

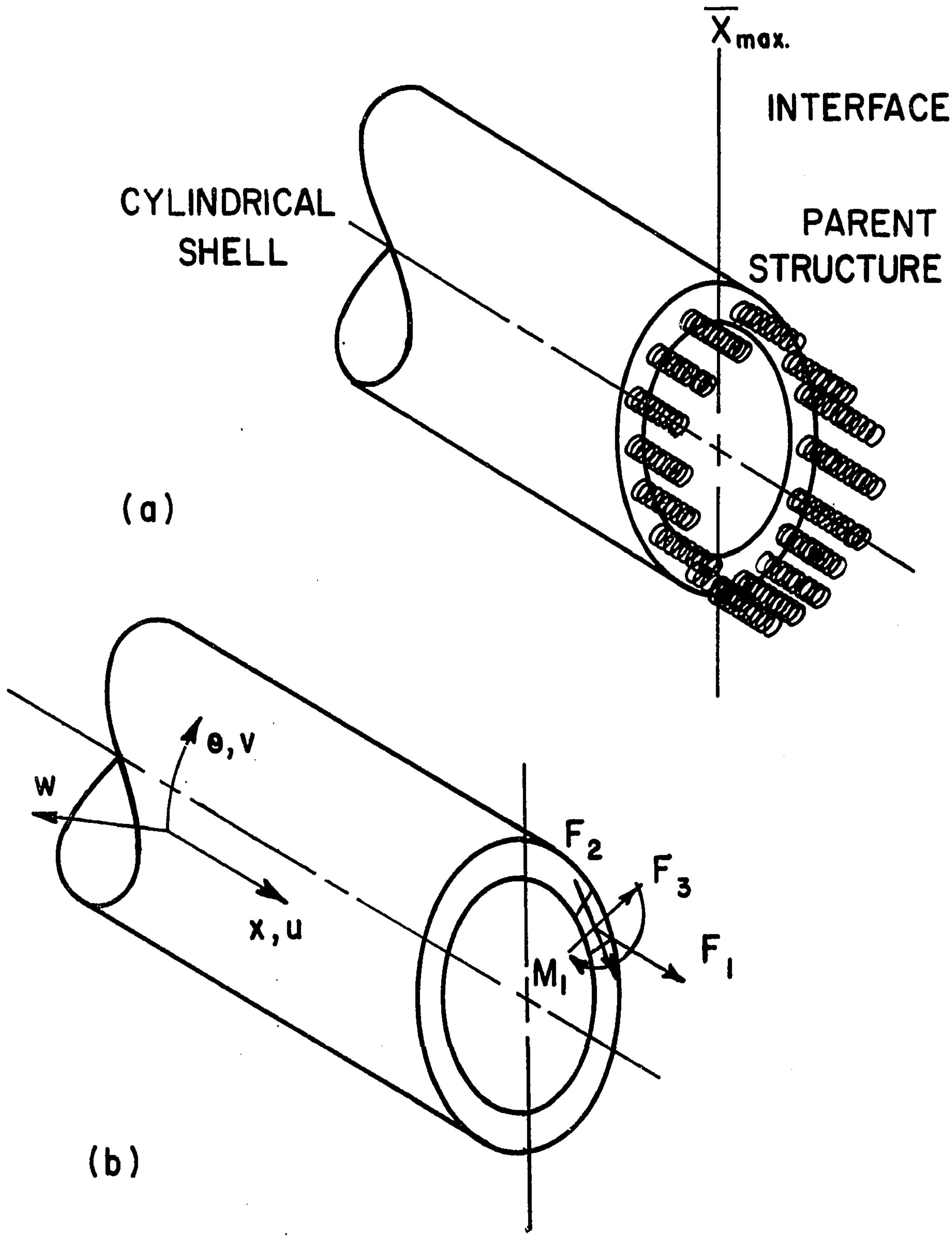

Figure 2.2 (a) Physical Model for Parent Structure
(b) Sign Convention for Internal Forces at Interface.

Equations (2-31) represent four homogeneous boundary conditions at $\bar{x}=\bar{x}_{max}$ and they can be solved for the four unknown "initial" boundary conditions. For the sake of completeness, these four equations are written out here as four simultaneous non-homogeneous algebraic equations in the four unknown boundary conditions. These are the results of substitution of equations (2-24) into (2-31) and rearranging.

Let

$$A_j = \sum_{i=1}^{8} FSU_{ij} e^{s_i^* \bar{x}_m} ,$$

$$B_j = \sum_{i=1}^{8} FSV_{ij} e^{s_i^* \bar{x}_m} ,$$

$$C_j = \sum_{i=1}^{8} FSW_{ij} e^{s_i^* \bar{x}_m} ,$$

$$UXQ = \sum_{i=1}^{8} QSU_i e^{s_i^* (\bar{x}_m - \bar{x}_0)} ,$$

$$VXQ = \sum_{i=1}^{8} QSV_i e^{s_i^* (\bar{x}_m - \bar{x}_0)} ,$$

$$WXQ = \sum_{i=1}^{8} QSW_i e^{s_i^* (\bar{x}_m - \bar{x}_0)} . \qquad (2\text{-}32)$$

Then, equations (2-31) become

$$\sum_{i=1}^{4} BC_i \Big(-A_i + \bar{\bar{a}}_{11} A_i' + \mu n \bar{\bar{a}}_{11} B_i + (\mu \bar{\bar{a}}_{11} + \mu n^2 R^2 \bar{\bar{a}}_{14}) C_i$$
$$+ (2-\mu) n^2 R^2 \bar{\bar{a}}_{13} C_i' - R^2 \bar{\bar{a}}_{14} C_i'' - R^2 \bar{\bar{a}}_{13} C_i'''\Big) = (UXQ) - \bar{\bar{a}}_{11} (UXQ)'$$
$$- \mu n \bar{\bar{a}}_{11} (VXQ) - (\mu \bar{\bar{a}}_{11} + \mu n^2 R^2 \bar{\bar{a}}_{14}) (WXQ) - (2-\mu) n^2 R^2 \bar{\bar{a}}_{13} (WXQ)'$$
$$+ R^2 \bar{\bar{a}}_{14} (WXQ)'' + R^2 \bar{\bar{a}}_{13} (WXQ)''' , \qquad (2\text{-}33)$$

$$\sum_{i=1}^{4} BC_i \left[-B_i - \left(\frac{1-\mu}{2}\right) n\bar{\bar{a}}_{22} A_i + \left(\frac{1-\mu}{2}\right)\bar{\bar{a}}_{22} B_i'\right] = (VXQ)$$

$$+\left(\frac{1-\mu}{2}\right) n\bar{\bar{a}}_{22}(UXQ) - \left(\frac{1-\mu}{2}\right)\bar{\bar{a}}_{22}(VXQ)' \quad , \qquad (2\text{-}34)$$

$$\sum_{i=1}^{4} BC_i \Big[-C_i + \bar{\bar{a}}_{31} A_i' + \mu n \bar{\bar{a}}_{31} B_i + (\mu\bar{\bar{a}}_{31} + \mu n^2 R^2 \bar{\bar{a}}_{34}) C_i$$

$$+(2-\mu) n^2 R^2 \bar{\bar{a}}_{33} C_i' - R^2 \bar{\bar{a}}_{34} C_i'' - R^2 \bar{\bar{a}}_{33} C_i'''\Big] = (WXQ) - \bar{\bar{a}}_{31}(UXQ)'$$

$$-\mu n \bar{\bar{a}}_{31}(VXQ) - (\mu\bar{\bar{a}}_{31} + \mu n^2 R^2 \bar{\bar{a}}_{34})(WXQ) - (2-\mu) n^2 R^2 \bar{\bar{a}}_{33}(WXQ)'$$

$$+R^2 \bar{\bar{a}}_{34}(WXQ)'' + R^2 \bar{\bar{a}}_{33}(WXQ)''' \quad , \qquad (2\text{-}35)$$

$$\sum_{i=1}^{4} BC_i \big(-C_i' + \bar{\bar{a}}_{41} A_i' + \mu n \bar{\bar{a}}_{41} B_i + (\mu\bar{\bar{a}}_{41} + \mu n^2 R^2 \bar{\bar{a}}_{44}) C_i$$

$$+(2-\mu) n^2 R^2 \bar{\bar{a}}_{43} C_i' - R^2 \bar{\bar{a}}_{44} C_i'' - R^2 \bar{\bar{a}}_{43} C_i'''\big) = (WXQ)' - \bar{\bar{a}}_{41}(UXQ)'$$

$$-\mu n \bar{\bar{a}}_{41}(VXQ) - (\mu\bar{\bar{a}}_{41} + \mu n^2 R^2 \bar{\bar{a}}_{44})(WXQ) - (2-\mu) n^2 R^2 \bar{\bar{a}}_{43}(WXQ)'$$

$$+R^2 \bar{\bar{a}}_{44}(WXQ)'' + R^2 \bar{\bar{a}}_{43}(WXQ)''' \quad . \qquad (2\text{-}36)$$

Numerical computation is quite obviously required for the evaluation of this solution. These computations were performed on an IBM 360/50 system. A complete list of the program is included in Appendix C.

2.5 Determination of Roots of the Characteristic Polynomial

Since the primary interest of this study lies in the resonant response of the structure, some mechanism for energy dissipation must be included. The concept of a complex elastic modulus will be used, where

$$E^* = E(1+i\gamma) \ , \qquad (2\text{-}37)$$

E = normal elastic modulus,

γ = structural damping factor .

The modulus of elasticity appears in the frequency parameter N, equation (2-4), and subsequently in the expression for Denom, equation (2-16), making the coefficients of the polynomial complex.

All of the standard machine subroutines for the determination of the roots of a polynomial require real coefficients. However, since γ is small it is possible to compute the roots with a real modulus and to use a Taylor series expansion of the roots in terms of E^* to include the effect of a non-zero structural damping parameter, γ. This Taylor series expansion is as follows:

$$s(E^*) = s(E) + \left.\frac{ds}{dE^*}\right|_{E^*=E} (E^*-E) - \frac{1}{2}\left.\frac{d^2s}{dE^{*2}}\right|_{E^*=E} (E^*-E)^2 + \ldots \qquad (2\text{-}38)$$

The required derivatives are computed as follows. Suppose the polynomial is of the form:

$$as^8+bs^6+cs^4+ds^2+e = 0 \ .$$

Then

$$a's^8+8as^7s'+b's^6+6bs^5s'+c's^4+4cs^3s'+d's^2+2dss'+e' = 0 \ ,$$

from which

$$s' = -\frac{a's^8+b's^6+c's^4+d's^2+e'}{8as^7+6bs^5+4bs^3+2ds} \ , \qquad (2\text{-}39)$$

where $(\ )' = \left.\frac{d(\ )}{dE^*}\right|_{E^*=E}$.

The second derivative is computed in a similar manner and the roots with a complex modulus are then determined by using only the first two terms from the above expansion, equation (2-38).

CHAPTER III

EVALUATION OF THE THEORY

3.1 Simple Support

In order to demonstrate the validity of the solution and associated computer program, the flexibility influence functions were set as shown in matrix (3-1) to simulate a simple support without axial constraint.

$$\begin{bmatrix} (-1\times10^{20},0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (1\times10^{20},0) \end{bmatrix} \qquad (3\text{-}1)$$

The functions $\bar{\bar{a}}_{11}$ relating axial displacement to in-plane normal force and $\bar{\bar{a}}_{44}$ relating rotation to bending moment should be infinitely large--all others should be zero. The solution exhibited the correct behavior in the following respects.

(1). Static deflection

With zero frequency and $\bar{x}_0 = \frac{1}{2}\bar{x}_m$, the displacement is as shown in Figure 3.1 for a typical value of circumferential number. The "exact" solution was obtained, according to Kraus [3], by solving the Donnell equation,

$$\frac{h^2}{12}\nabla^8 w + \frac{1-\mu^2}{r^2}\frac{\partial^4 w}{\partial x^4} = \frac{1-\mu^2}{Eh}\nabla^4 q \, , \tag{3-2}$$

after expanding both w and q in double Fourier series as follows:

$$q = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn} \sin\frac{m\pi x}{\ell} \sin n\theta,$$

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn} \sin\frac{m\pi x}{\ell} \sin n\theta. \tag{3-3}$$

Substitution of (3-3) into equation (3-2) results in

$$\left\{\frac{h^2}{12}\left[\left(\frac{m\pi}{\ell}\right)^2 + \left(\frac{n}{r}\right)^2\right]^4 + \left(\frac{1-\mu^2}{r^2}\right)\left(\frac{m\pi}{\ell}\right)^4\right\} w_{mn}$$
$$= \left[\left(\frac{m\pi}{\ell}\right)^2 + \left(\frac{n}{r}\right)^2\right]^2 \frac{(1-\mu^2)}{Eh} q_{mn} \, , \tag{3-4}$$

from which w_{14} and w_{34} are calculated and plotted in Figure 3.1. The solution obtained here contains all m values and as shown in Figure 3.1 agrees quite well with the "exact" solution to the Donnell equation which uses only two terms, w_{14} and w_{34}.

(2). Frequency Response

The maximum amplitude (at $\bar{x}=\frac{1}{2}\bar{x}_m$) for $\bar{x}_0=\frac{1}{2}\bar{x}_m$ is plotted in Figures 3.2 and 3.3 as a function of frequency for three values of the structural damping parameter. From Figures 3.1 and 3.3 it should be noted that the amplification factor at resonance for $\gamma=0.01$ is exactly $1/\gamma$ (the static displacement $w_{14}=0.142\times10^{-4}$ while the resonant amplitude at the resonant frequency for m=1 was 0.142×10^{-2}). This is as it should be; the solution should behave as if it were a single degree-of-freedom system since there is considerable separation

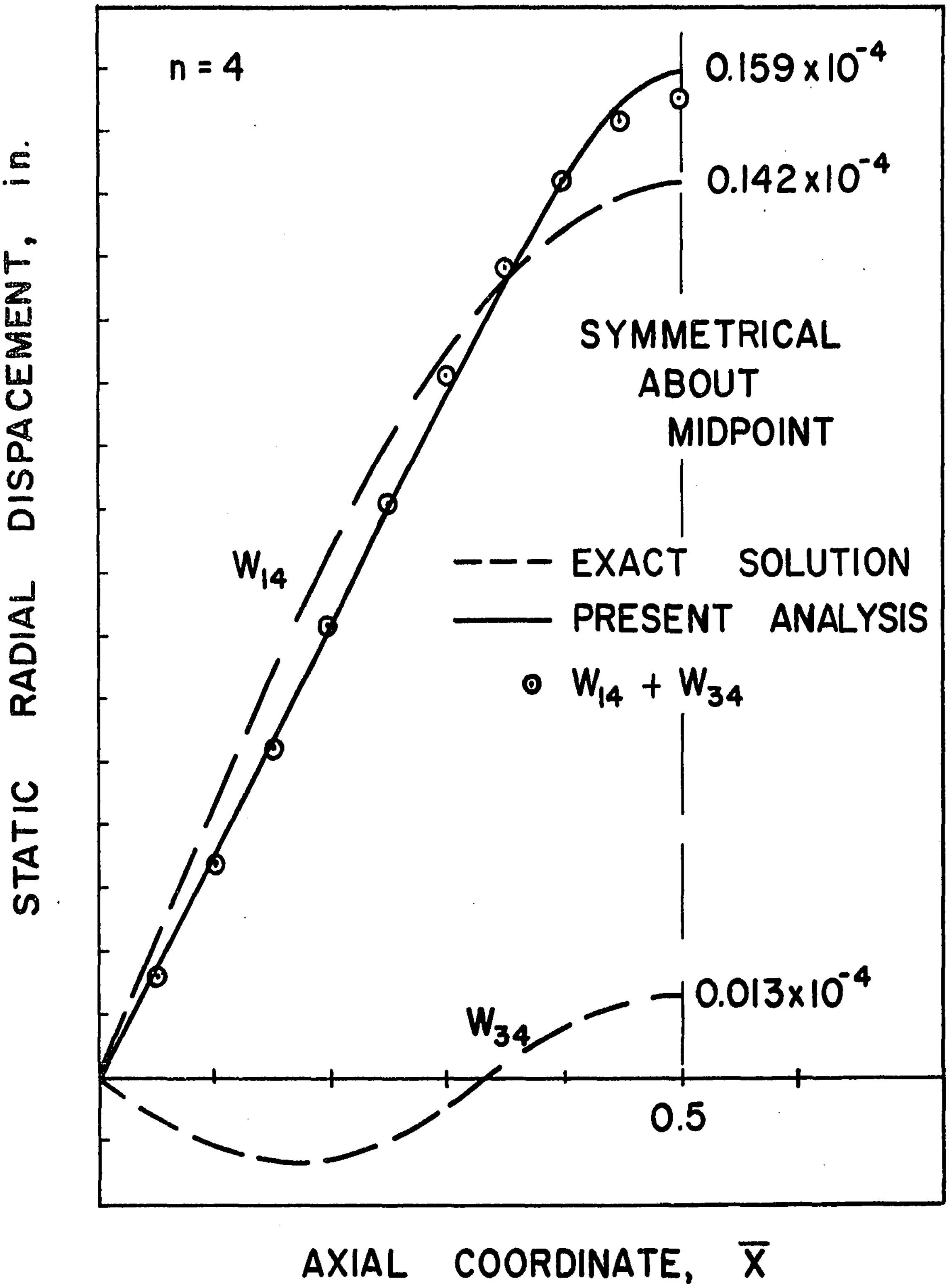

Figure 3.1 Radial Displacement Distribution for Static Point Load at Midpoint, n=4.

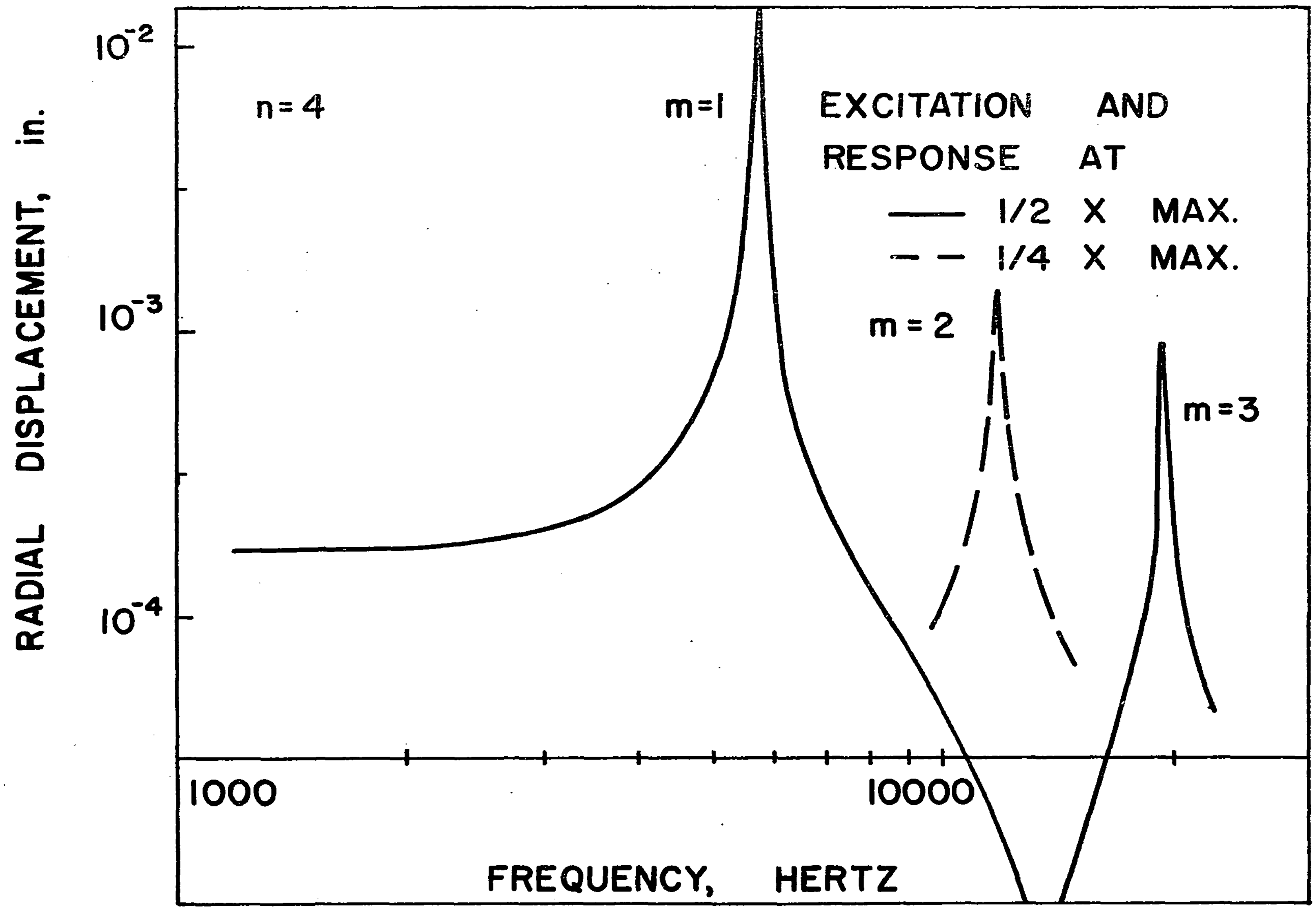

Figure 3.2 Radial Displacement Frequency Response
to Harmonic Point Excitation at Antinode,
n=4, m=1,2,3.

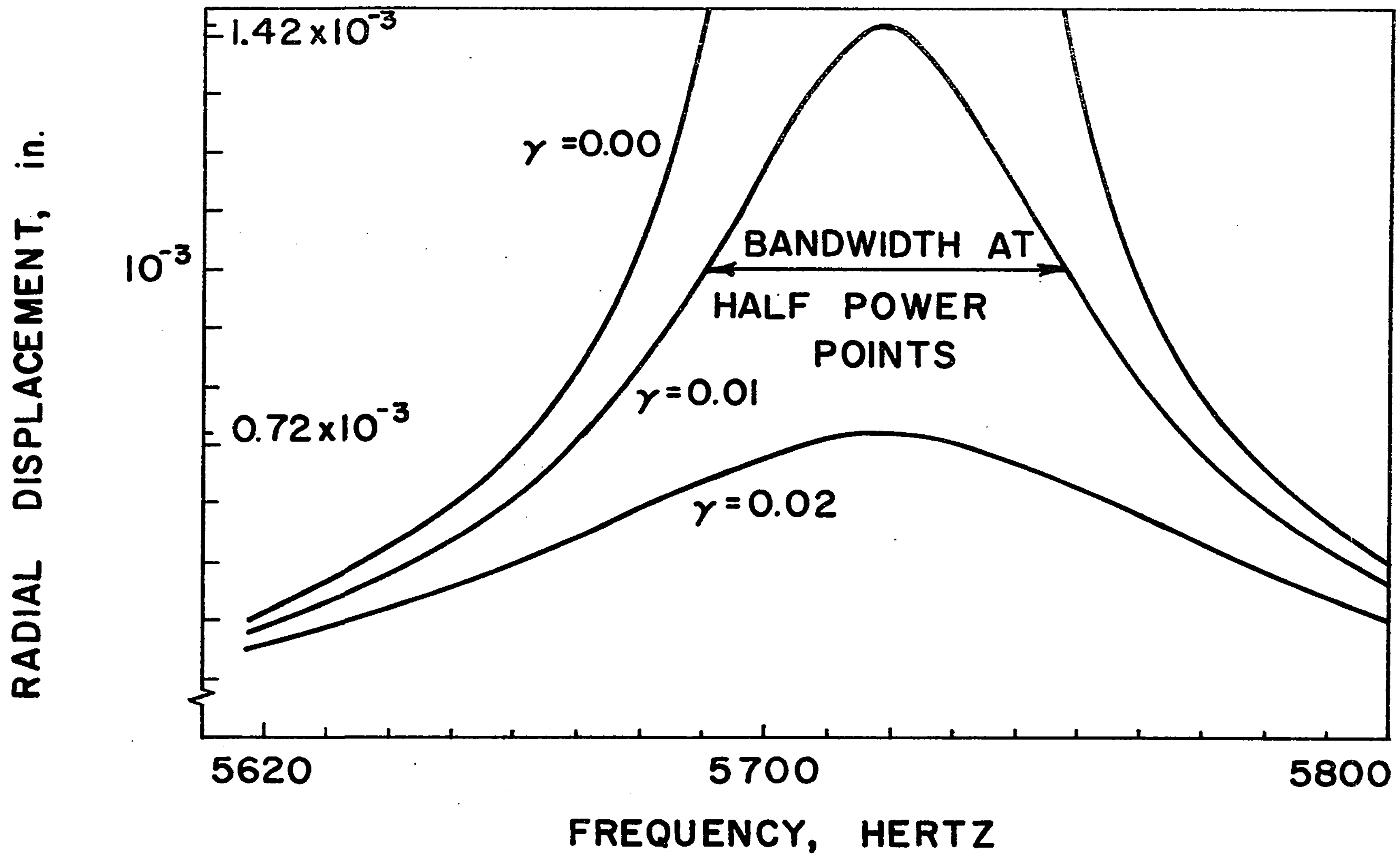

Figure 3.3 Detailed Radial Displacement Frequency Response to Harmonic Point Load at Antinode, n=4, m=1.

between resonant frequencies for the various values of m. The single degree-of-freedom frequency response equation is

$$X = \frac{X_0}{\left([1-(\omega/\omega_n)^2]^2 + \gamma^2\right)^{1/2}} \tag{3-5}$$

which reduces to an amplification of $1/\gamma$ at resonance. The band width at the "half-power" points is also equal to frequency/response ratio as it is for a single degree of freedom system.

(3). Phase Angle

The phase angle is plotted in Figure 3.4 for various values of the structural damping parameter vs. frequency. This frequency variation is proper.

(4). Mode Shape

The mode shape was as shown in Figure 3.5 at a frequency near resonance for m=1. This is very nearly sinusoidal as would be expected because of the predominance of the term corresponding to w_{14}.

(5). Reciprocity

The fact that the displacements satisfy the dynamic counterpart of the Maxwell reciprocity relationship is demonstrated in Figure 3.6. The displacement $W_n(\bar{x})$ was computed for $\bar{x}_0 = \frac{1}{3}\bar{x}_m$, $\frac{1}{2}\bar{x}_m$, $\frac{2}{3}\bar{x}_m$, respectively, at a given frequency near the resonant frequency. The two curves for $\bar{x}_0 = \frac{1}{3}\bar{x}_m$ and $\bar{x}_0 = \frac{2}{3}\bar{x}_m$ are identical. Also

$$W_n(\tfrac{1}{3}\bar{x}_m)\Big|_{\bar{x}=\frac{1}{2}\bar{x}_m} = W_n(\tfrac{1}{2}\bar{x}_m)\Big|_{\bar{x}=\frac{1}{3}\bar{x}_m} ,$$

which satisfies the reciprocal relationship.

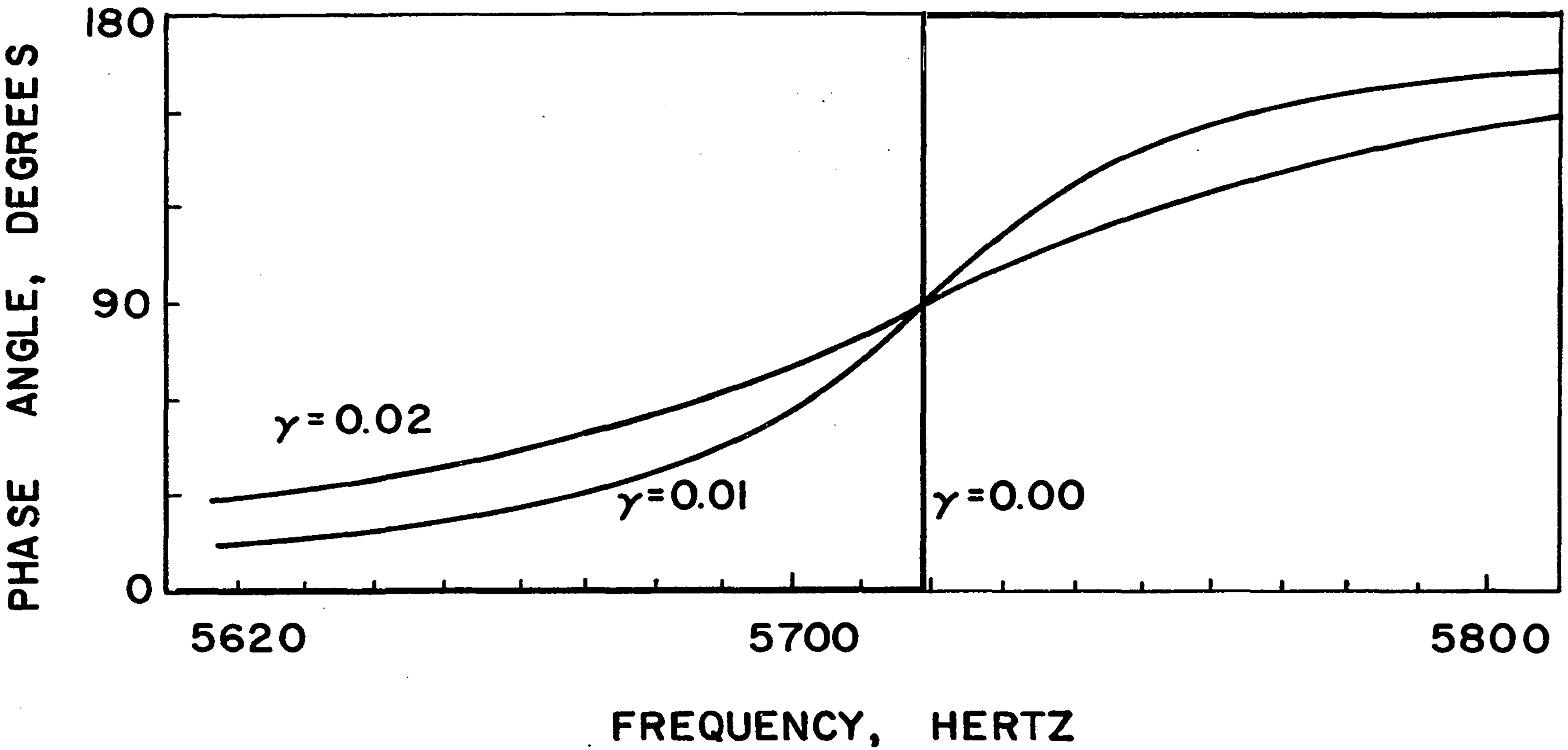

Figure 3.4 Phase Angle Between Excitation and Radial Displacement Response at Midpoint, n=4, m=1.

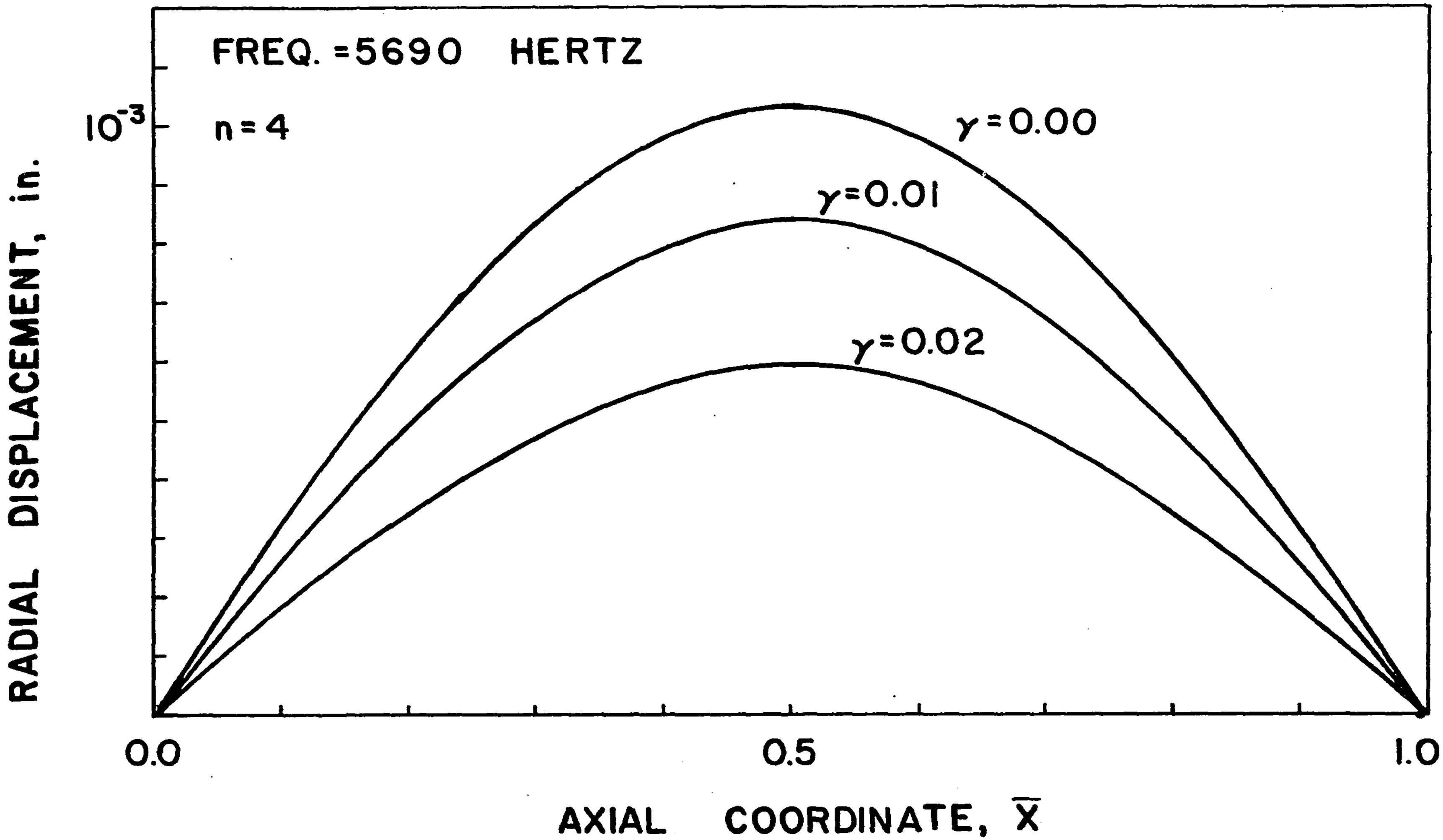

Figure 3.5 Radial Displacement Distribution near Resonance, n=4, m=1, γ=0.0, 0.01, 0.02.

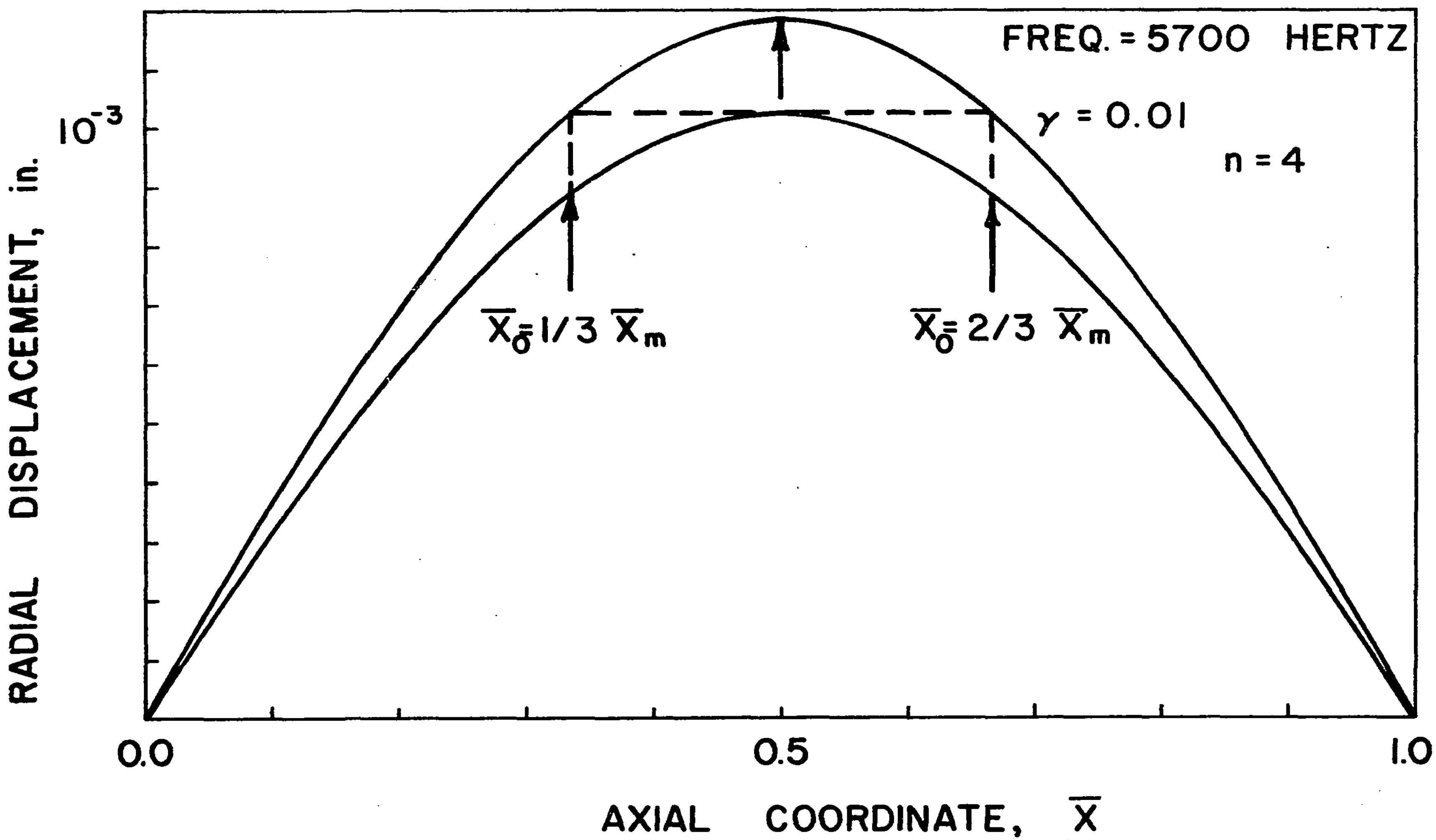

Figure 3.6 Radial Displacement Distribution near Resonance, Demonstrating Reciprocity, n=4, m=1, γ=0.01.

The behavior of the solution as discussed so far has been for a shell whose ratio of length to radius was unity and ratio of radius to thickness was twenty. The material properties were $\mu = 0.3$, $E = 30\text{x}10^6$ psi, $\rho = 0.731\text{x}10^{-3}$ lb.-sec.^{2}/in.^{4} For these geometrical ratios, absolute confidence in the computer evaluation of the solution is warranted.

3.2 Limitations

The solution as presented in equations (2-24) and the technique presented for the determination of the unknown boundary conditions is valid, in principle, for cylindrical shells of any length. There is, however, a limitation in the ability of the 360/50 computing system to evaluate the solution. The limitation is due to a loss of precision that occurs in the computation of the coefficients (2-32) of the four simultaneous equations (2-33) through (2-36) and in their solution. The coefficients involve terms of the form

$$C_j = \sum_{i=1}^{8} (FSW)_{ij} \; e^{s_i^* \bar{x}_m} \qquad (2\text{-}32)$$

The eight terms $(FSW)_{ij}$ are all of the same order of magnitude but the exponential terms are not. The roots s_i^* are typically of the form (if undamped)

$$s_i^* = \pm(a \pm ib), \; \pm ic, \; \pm d. \qquad (3\text{-}6)$$

If damping is present the roots are only slightly different, but are no longer complex conjugates, pure real, or pure imaginary. In equation (3-6) the a and b are of the order 7 to

10, c and d are typically between 1 and 5. The range of the terms involved in the summation such as in equation (2-32) is dependent upon $\bar{x}_m$ as well as s_i^* and is of the order 10^{12} if $\bar{x}_m$ is only of order 2 and even though the computation is done in double precision, the effect of the smaller roots (including $\pm ic$) is lost in the computation. There is not a well defined limitation but when the real part of the product $s_i^*\bar{x}_m$ becomes bigger than 10 to 12 a gradual decay in the validity of the result becomes evident. The magnitude of the roots increases as the circumferential number increases but is not very strongly dependent upon shell thickness or frequency of excitation.

As the length of the shell, $\bar{x}_m$, increases, the minimum natural frequency corresponds to cmaller circumferential numbers and hence to smaller roots [5] but this effect is not significant enough to overcome the multiplication by a large $\bar{x}_m$. As the thickness of the shell decreases the minimum natural frequency corresponds to ever larger circumferential numbers. Thus the limitation is in terms of shell length and also the shell thickness. With the present 360/50 computing system, using double precision, the product $s_i^*\bar{x}_m$ must be less than approximately ten. This limitation in terms of the ratios of length to radius and radius to thickness has not been fully explored but corresponds approximately to the ratios $\ell/r < 1.5$ and $10 < r/h < 100$.

It has been suggested that the non-dimensionalization be done with respect to length instead of radius so that $0 \le \bar{\bar{x}} \le 1$. This, however, leads to new roots λ_i which are exactly $\frac{\ell}{r}\, s_i^*$ and the product $\lambda_i \bar{\bar{x}}_m$ is equal to $s_i^* \bar{x}_m$.

CHAPTER IV

APPLICATION OF THE THEORY

A complete investigation of the region of influence of the boundary flexibility on the resonant response was intended, that is, the dependence of the resonant frequency and mode shape upon the boundary conditions was to be studied in terms of the length to radius ratio, radius to thickness ratio and circumferential wave number. However, due to the limitation imposed by the 360/50 computing system, which was discussed in Chapter III, these effects were studied only for the ratios $\ell/r = 1$ and $r/h = 20$. The shell parameters which were used throughout are: ℓ=3 in., r=3 in., h=0.15 in., Q_0=10 lbs., μ=0.3, E=30×10^6psi, ρ=0.731 $\times10^{-3}$ lb.sec^2/in^4, γ=0.01. The load is always applied at an antinode as indicated by an arrow in the response plots to follow.

Four boundary flexibilities were investigated. These were

a_{11}, axial flexibility,

a_{22}, tangential flexibility,

a_{33}, transverse flexibility, and

a_{44}, rotational flexibility.

The influence of these flexibilities, varied individually, on the mode shapes corresponding to the first three axial modes and their resonant frequencies were determined and are shown in Figures 4.1 through 4.21. In each case the influence functions which were not being investigated were set equal to those for a simple support without axial constraint. The circumferential number corresponding to the minimum natural frequency for a simply supported shell with one axial half wave was used predominately. For the geometrical ratios used here this circumferential number was n=4, see reference [5].

The effect of these flexibilities on the resonant frequency for one axial half wave (m=1) was determined for several circumferential numbers and these effects are illustrated in Figures 4.1 through 4.4. In each case the ratio of the elastic resonant frequency (Ω_e) to the simply supported resonant frequency (Ω_s) is plotted vs. the flexibility. The numbers in parentheses on the figures are the simply supported resonant frequencies.

Forsberg [5]reported that changes in the boundary conditions may alter the value of n associated with the minimum frequency. That this could occur was observed only for a_{22}. This is illustrated in Figure 4.5 from which the threshold of flexibility at which this change occurs can be read.

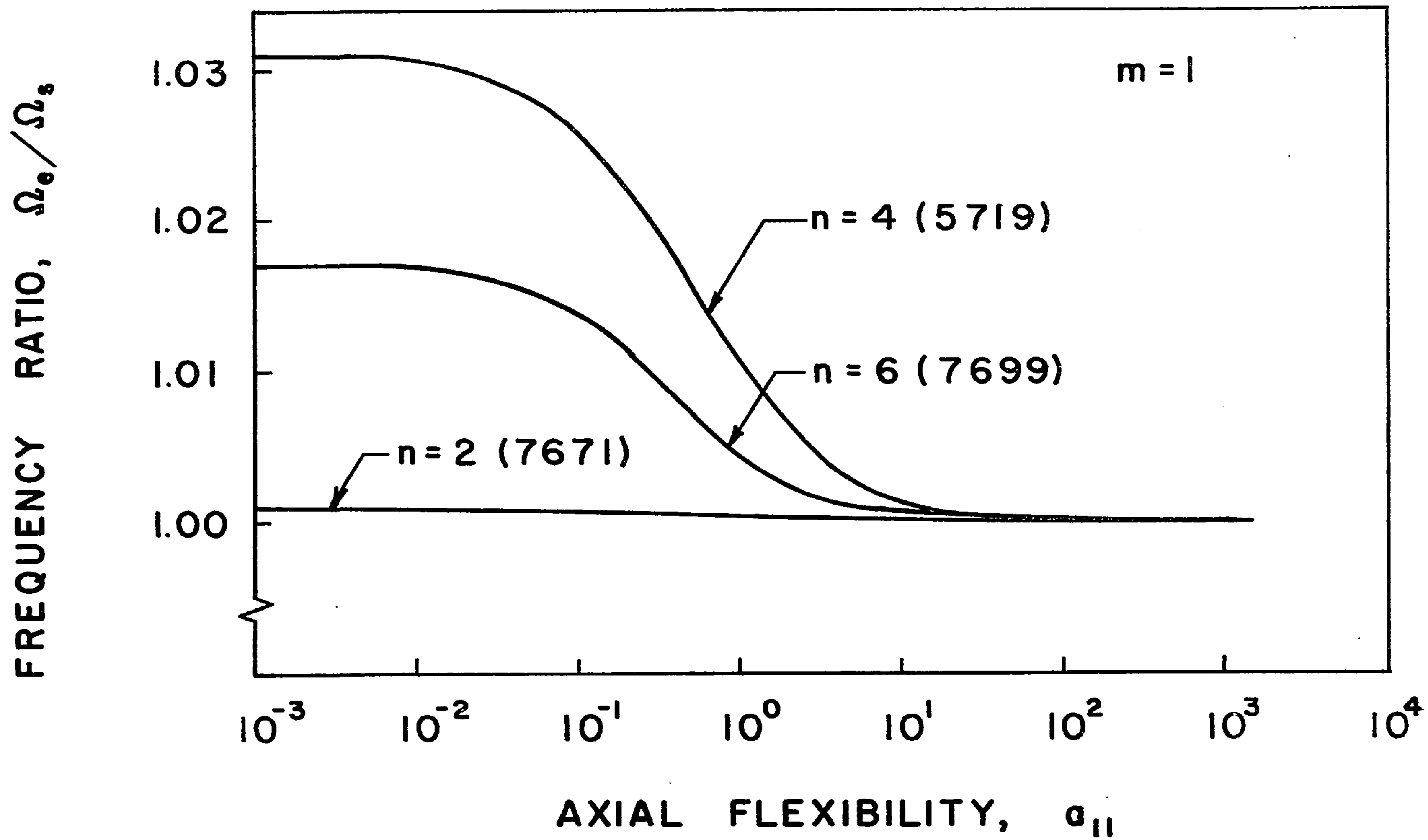

Figure 4.1 Resonant Frequency Ratio vs. Axial Flexibility, $-a_{11}$; $a_{22}=0$ $a_{33}=0$, $a_{44}=10^{20}$, $n=2,4,6$, $m=1$.

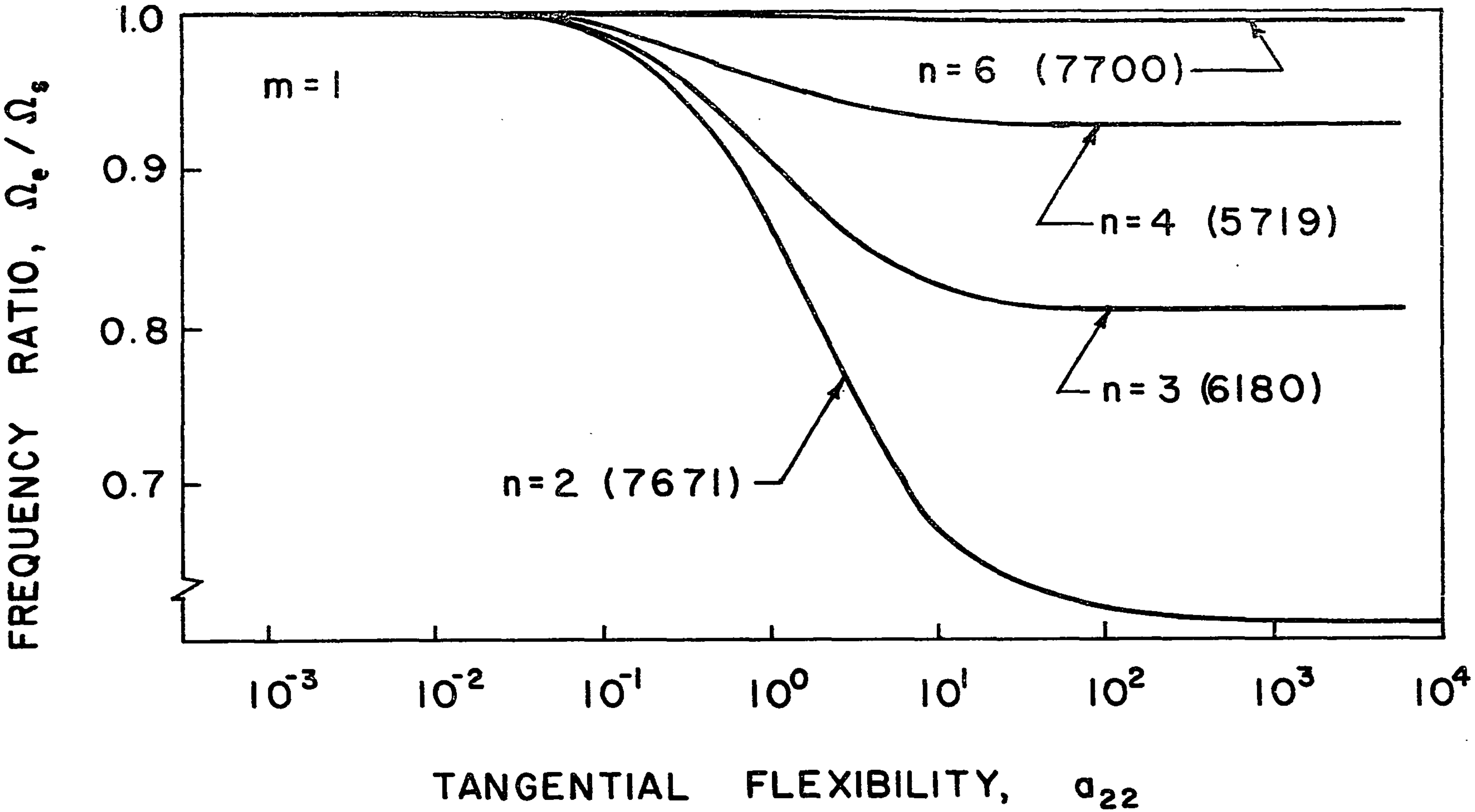

Figure 4.2 Resonant Frequency Ratio vs. Tangential Flexibility, $-a_{22}$; $a_{11}=-10^{20}$, $a_{33}=0$, $a_{44}=10^{20}$, n=2,3,4,6, m=1.

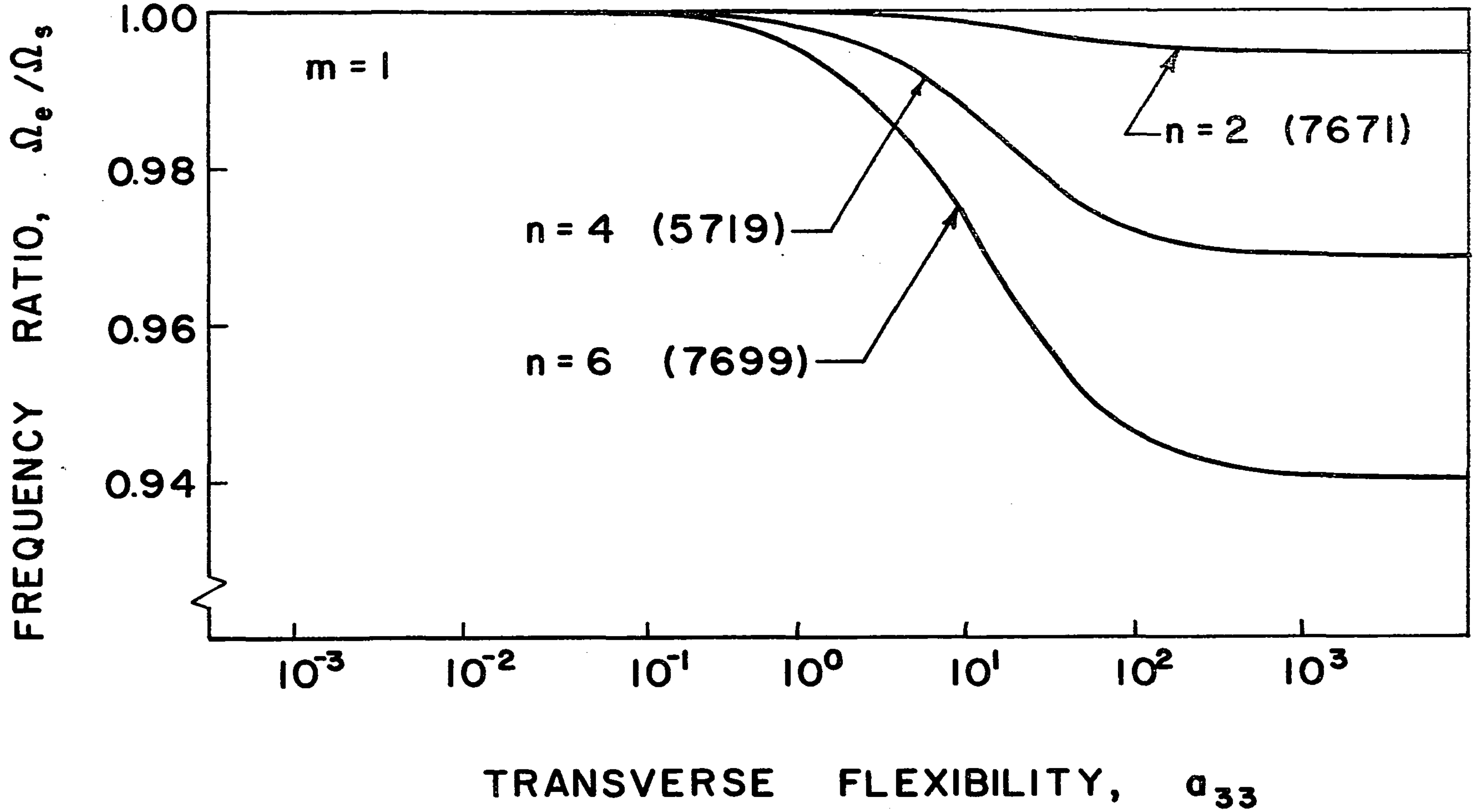

Figure 4.3 Resonant Frequency Ratio vs. Transverse Flexibility, $-a_{33}$; $a_{11}=-10^{20}$, $a_{22}=0$, $a_{44}=10^{20}$, $n=2,4,6$, $m=1$.

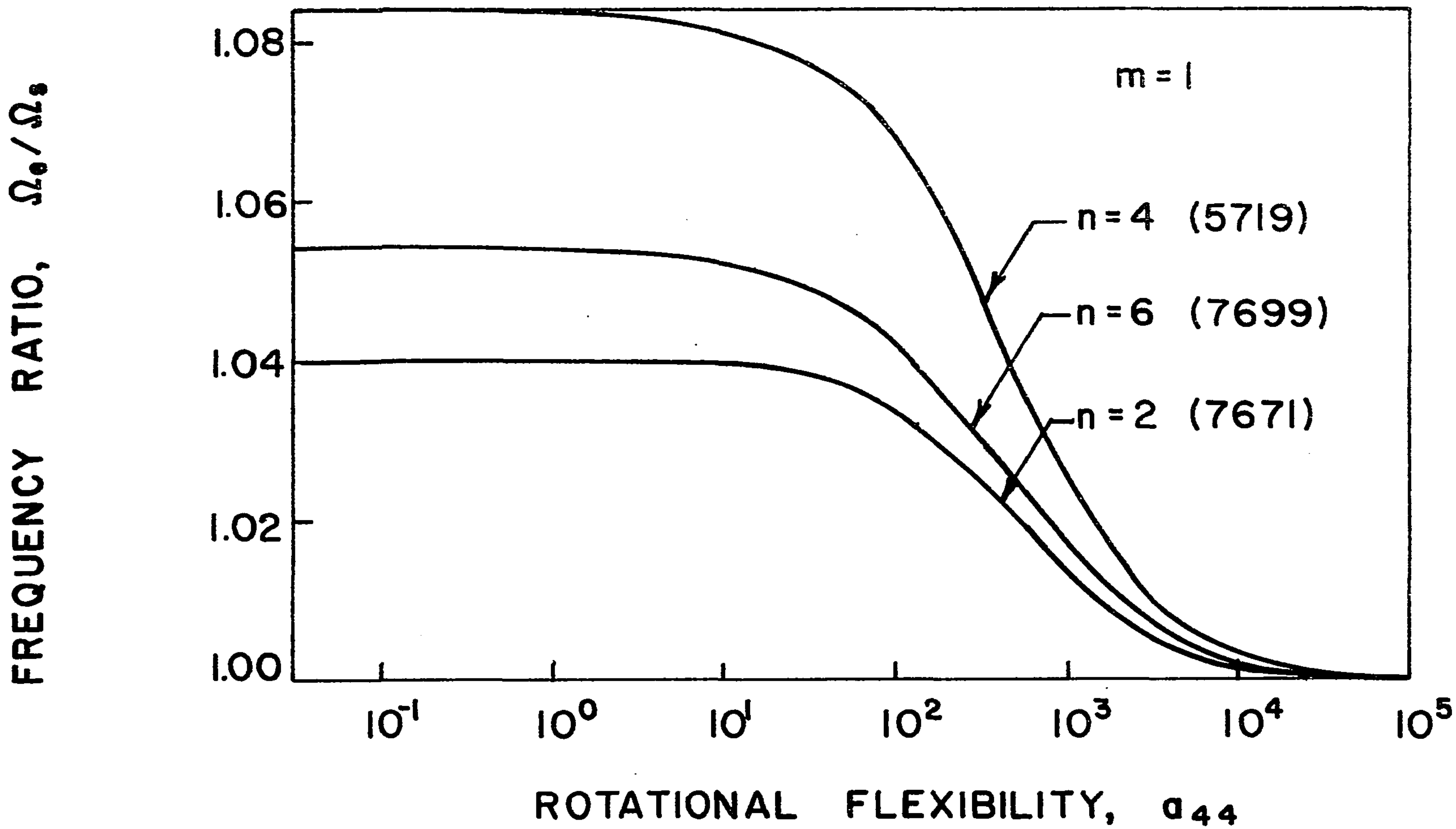

Figure 4.4 Resonant Frequency Ratio vs. Rotational Flexibility, $-a_{44}$; $a_{11}=-10^{20}$, $a_{22}=0$, $a_{33}=10$ $n=2,4,6$, $m=1$.

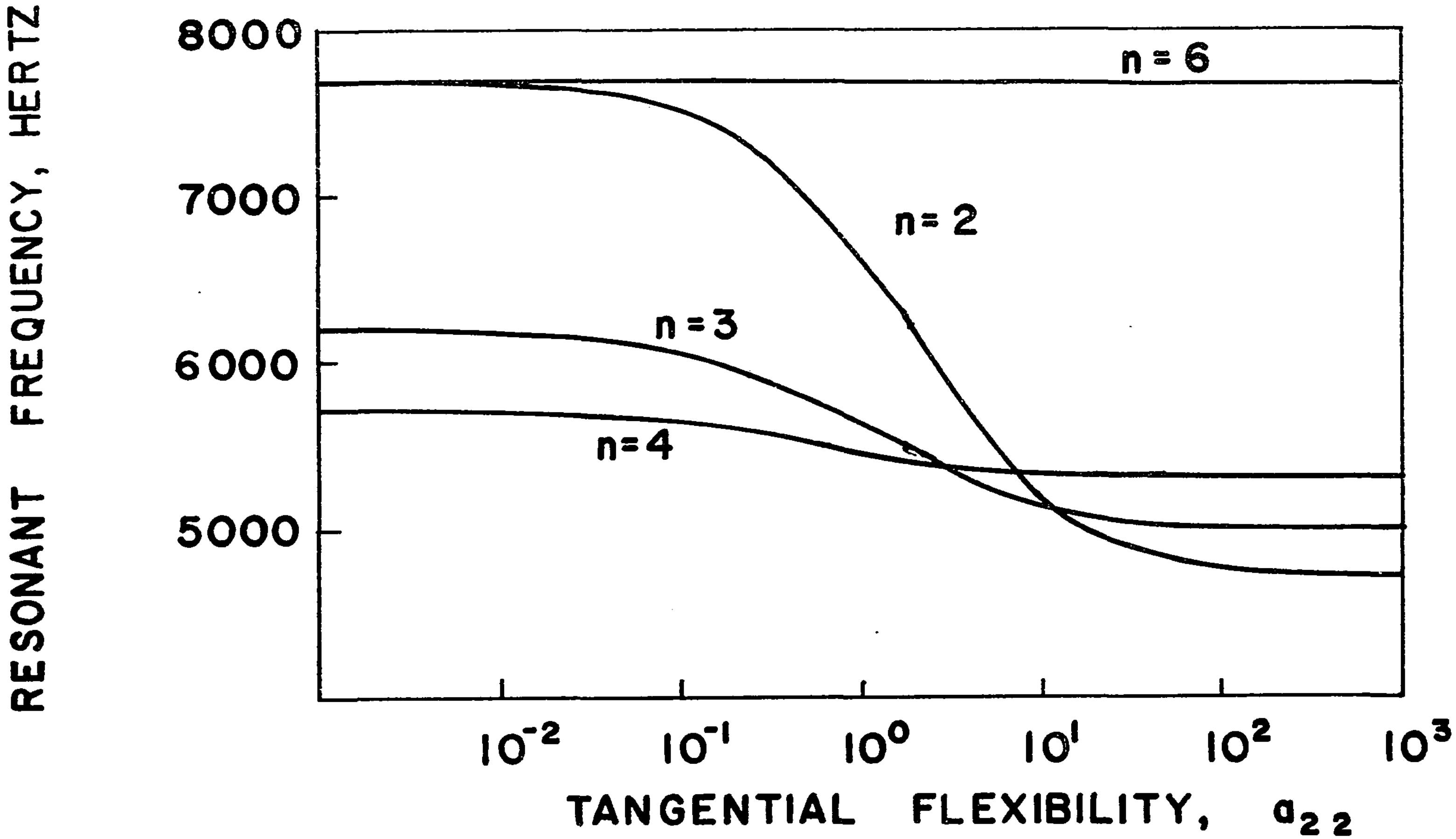

Figure 4.5 Resonant Frequency vs. Tangential Flexibility, $-a_{22}$; $a_{11}=-10^{20}$, $a_{33}=0$, $a_{44}=10$, n=2,3,4,6, m=1.

The effects of these four flexibilities on the resonant frequencies corresponding to the first three axial half waves was determined and is shown in Figures 4.6 through 4.9, again plotted as a ratio with respect to the simply supported resonant frequency. This was done with n=4 in each case although the minimum natural frequency for some of the higher values of m corresponds to a smaller value of n.

The mode shape at resonance for each of the first three axial modes was plotted for three representative values of each of the four influence functions, and are displayed in Figures 4.10 through 4.21. These were all done with n=4.

This same information is displayed in tabular form in Tables 4.1 through 4.4. Included in this tabular information is the value of a nondimensional displacement, w*, which occurs at an antinode. This dimensionless displacement is defined as

$$w^* = \frac{2\pi r \ell \rho \gamma h}{Q_0} \omega^2 w \ .$$

The motivation for the use of this parameter and its derivation are given in Appendix B. This quantity, w*, appears to be relatively insensitive to mode shape, frequency, boundary flexibility or the damping parameter.

An interesting observation can be made from an inspection of Figure 4.18, the effect of transverse flexibility on the third axial mode. That is, the maximum displacement occurs for an intermediate value of the flexibility coeffi-

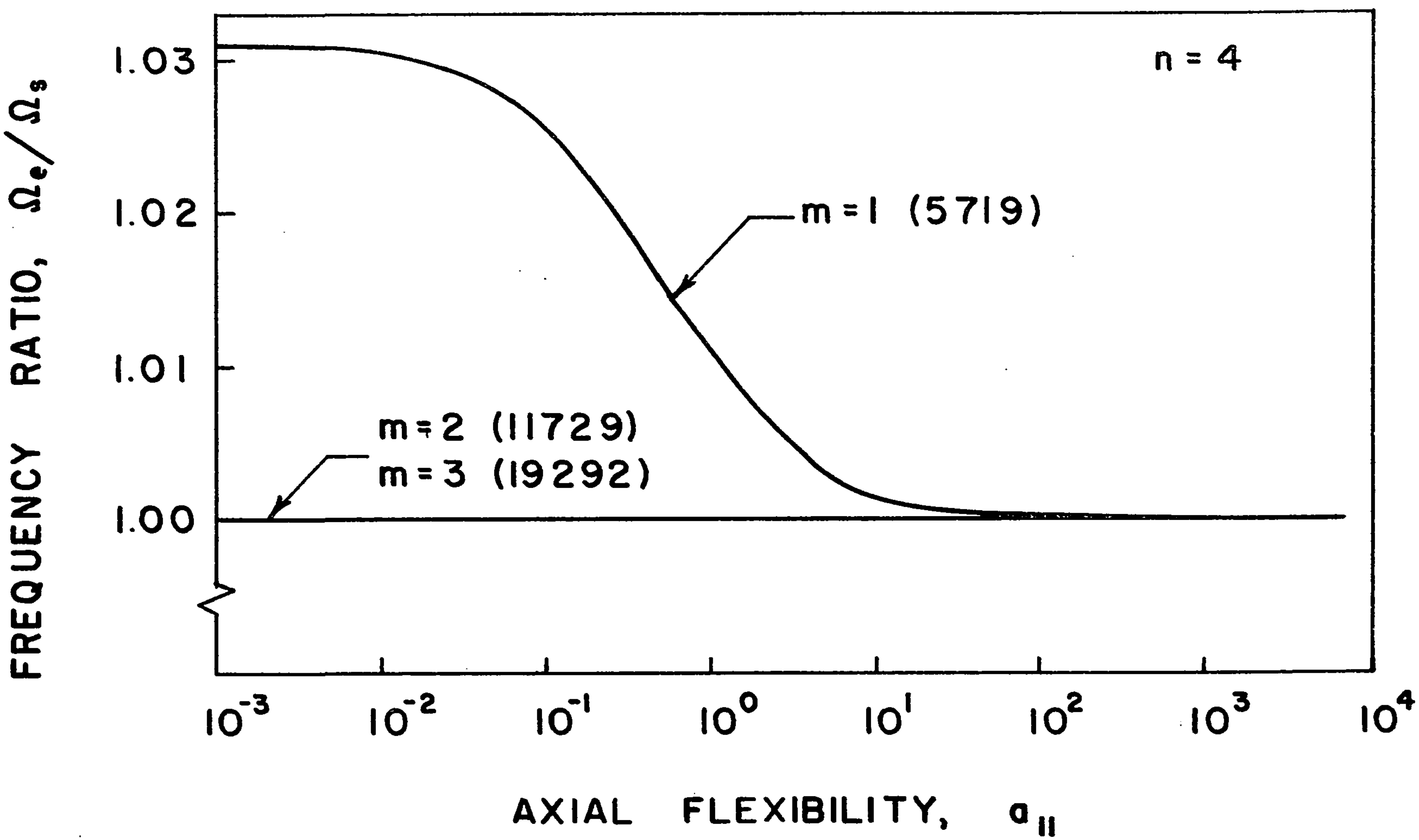

Figure 4.6 Resonant Frequency Ratio vs. Axial Flexibility, $-a_{11}$; $a_{22}=0$, $a_{33}=0$, $a_{44}=10^{20}$, $n=4$, $m=1,2,3$.

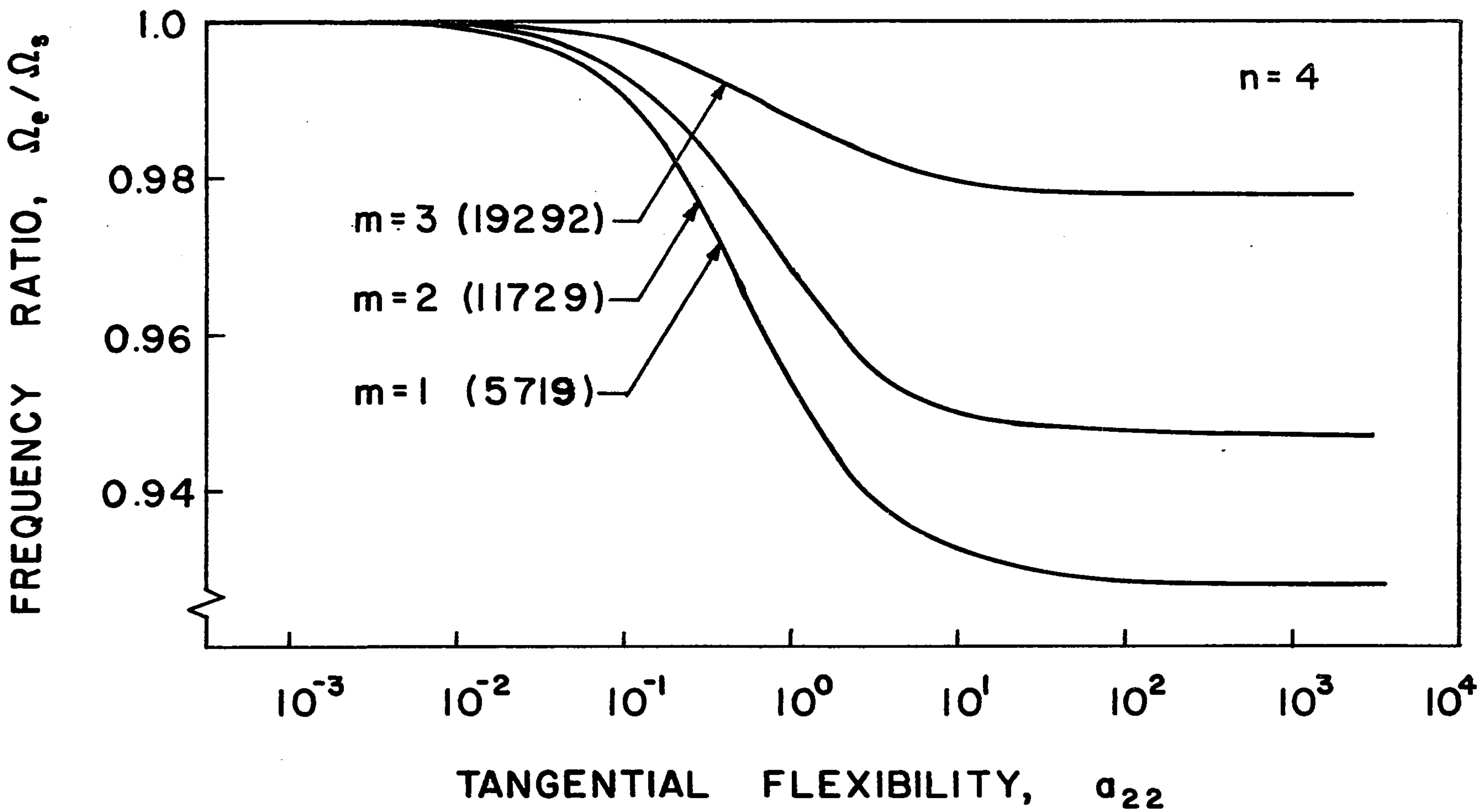

Figure 4.7 Resonant Frequency Ratio vs. Tangential Flexibility, $-a_{22}$; $a_{11}=-10^{20}$, $a_{33}=0$, $a_{44}=10^{20}$, $n=4$, $m=1,2,3$.

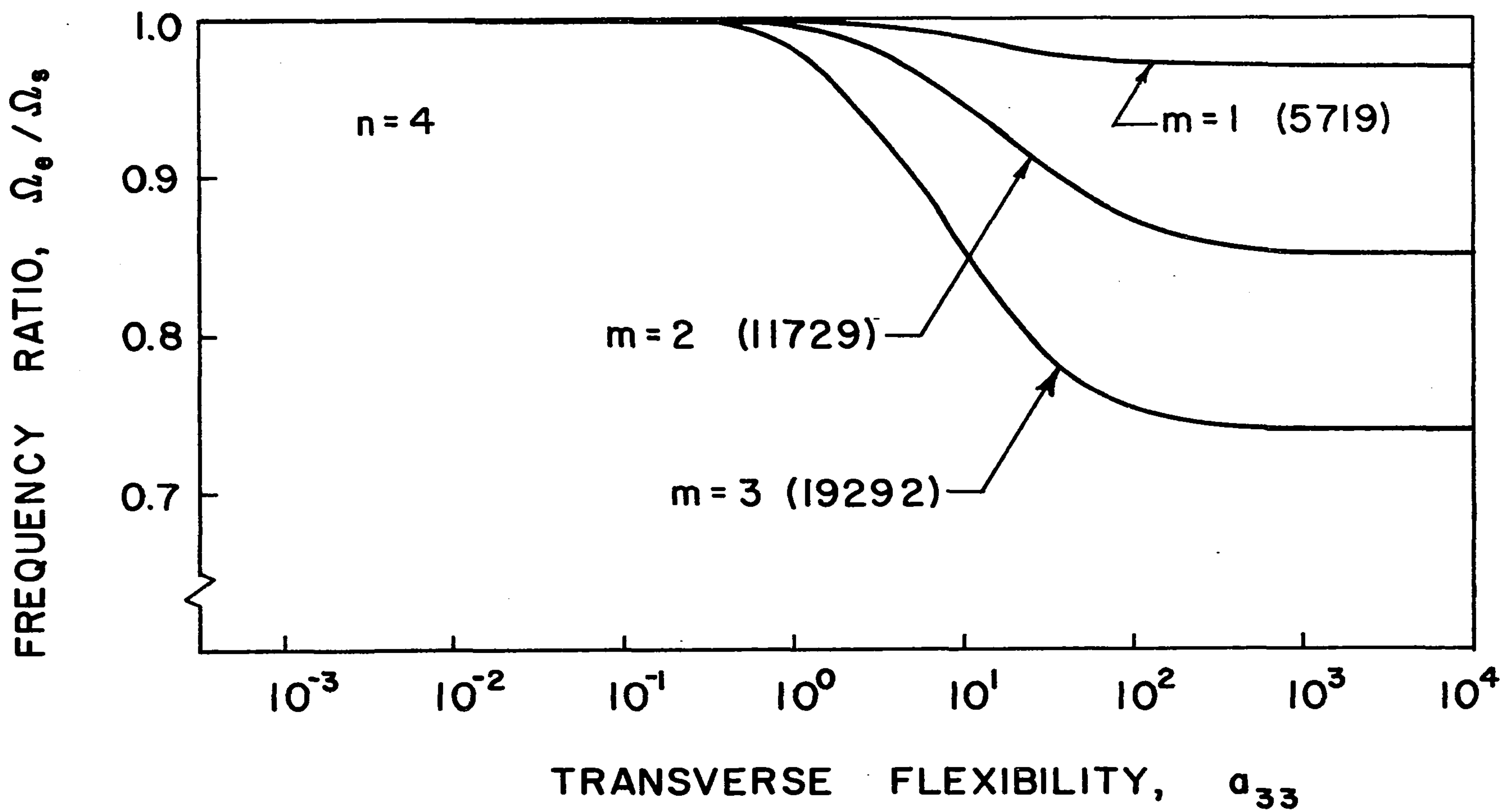

Figure 4.8 Resonant Frequency Ratio vs. Transverse Flexibility, $-a_{33}$; $a_{11}=-10^{20}$, $a_{22}=0$, $a_{44}=10^{20}$, $n=4$, $m=1,2,3$.

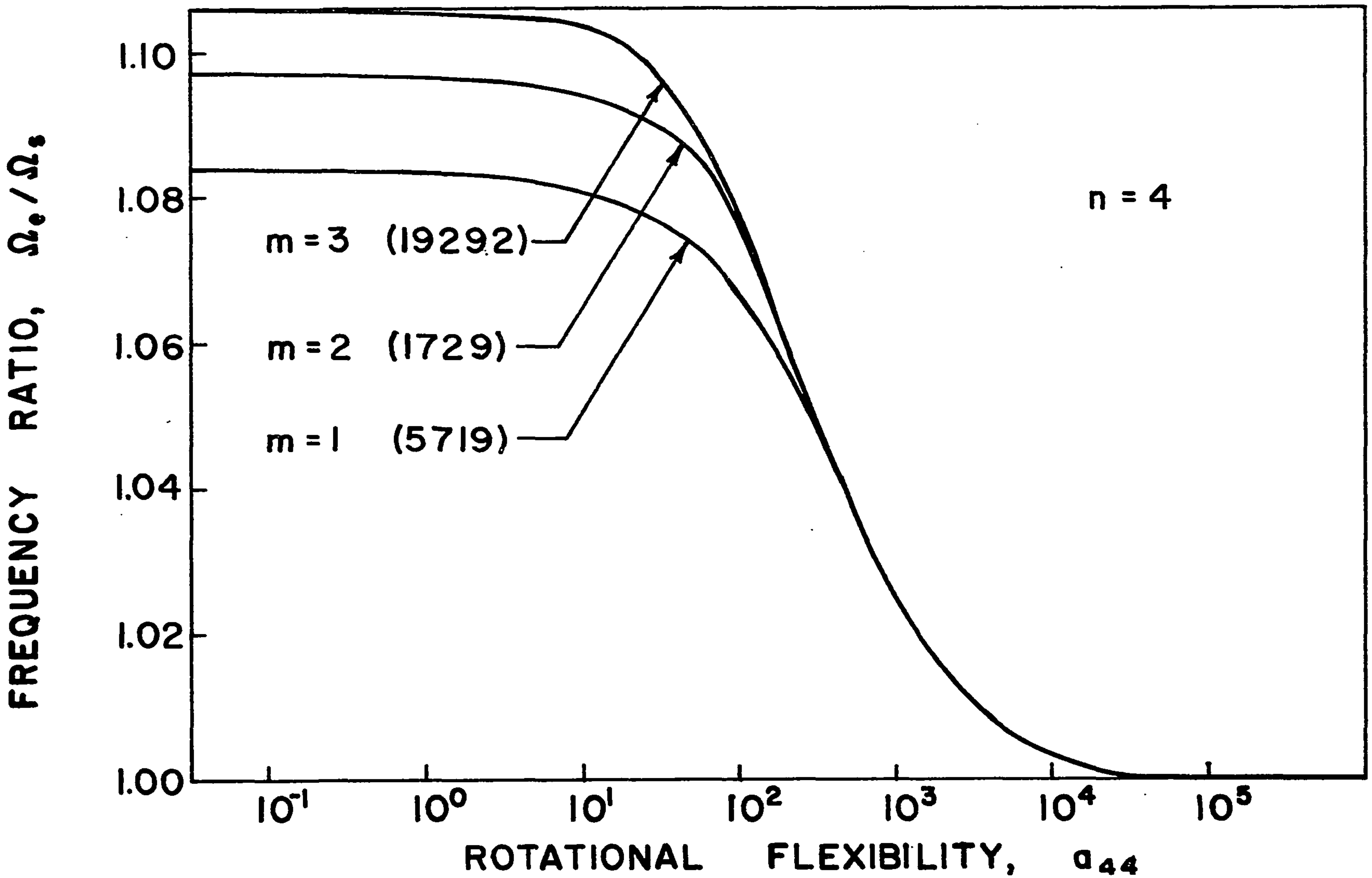

Figure 4.9 Resonant Frequency Ratio vs. Rotational Flexibility, a_{44}; $a_{11}=-10^{20}$, $a_{22}=0$, $a_{33}=0$, $n=4$, $m=1,2,3$.

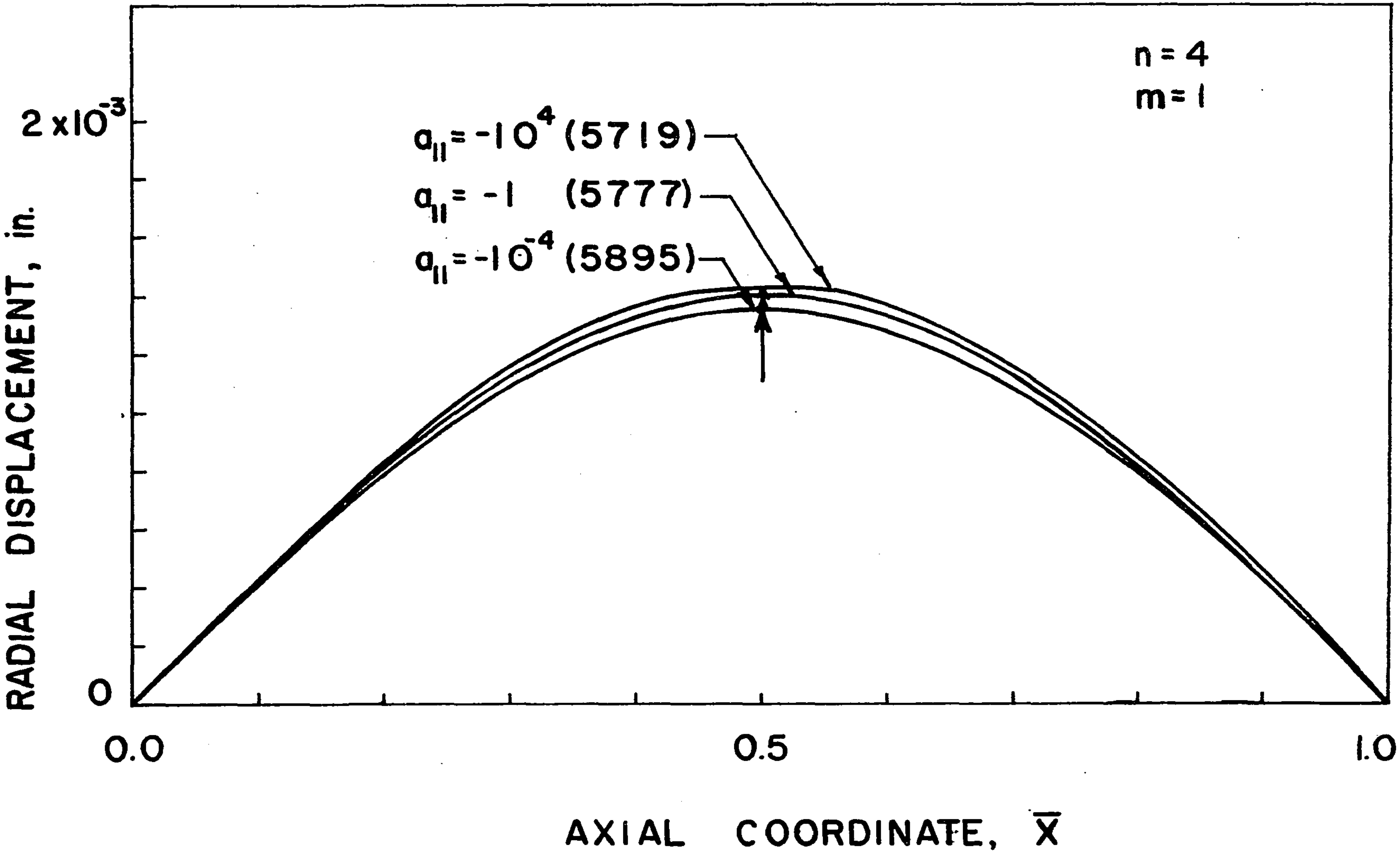

Figure 4.10 Radial Displacement Distribution at Resonance, $a_{11}=-10^{-4}$, -1, -10^4, $n=4$, $m=1$.

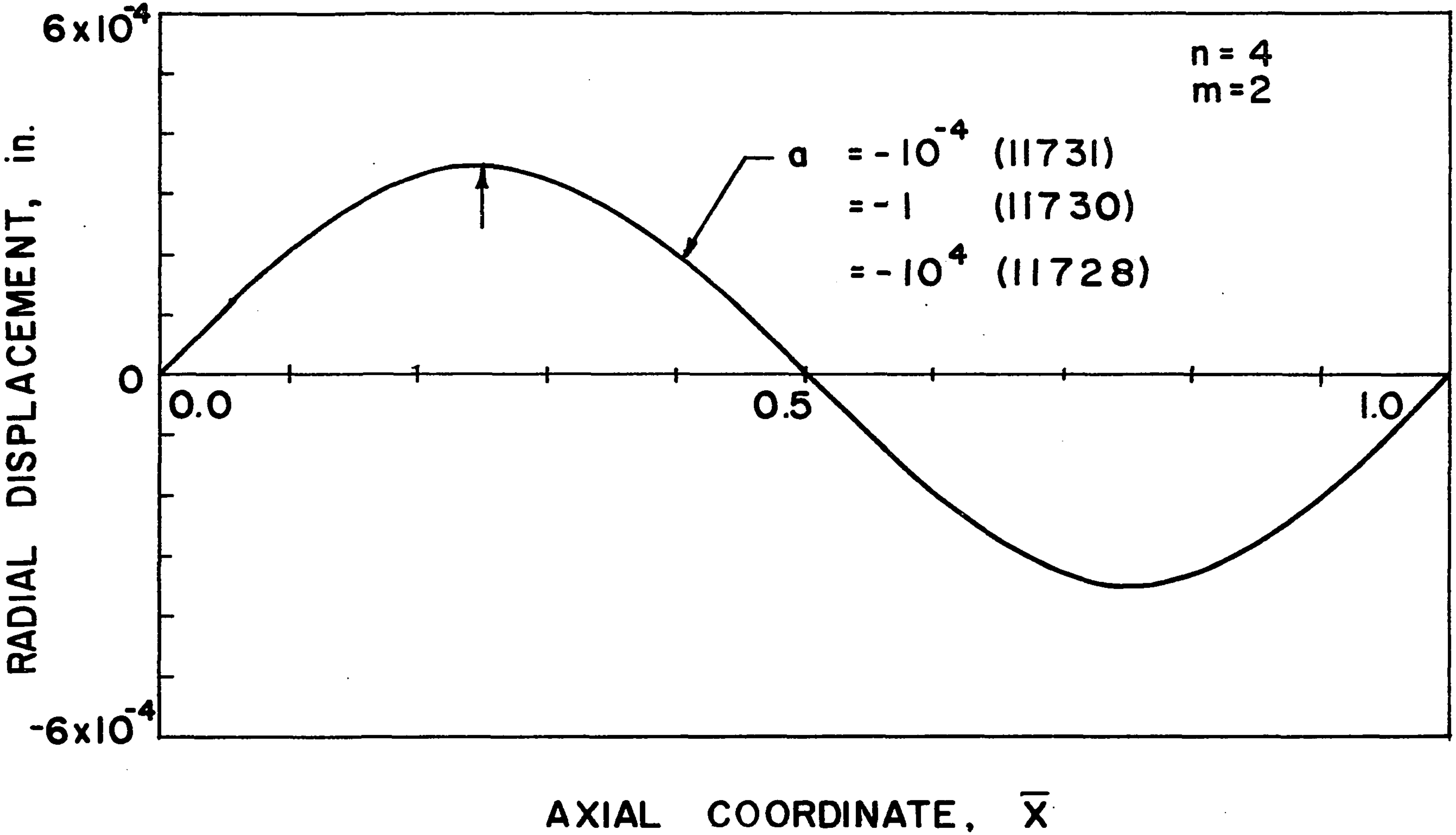

Figure 4.11 Radial Displacement Distribution at Resonance, $a_{11}=-10^{-4}$, -1, -10^{4}, n=4, m=2.

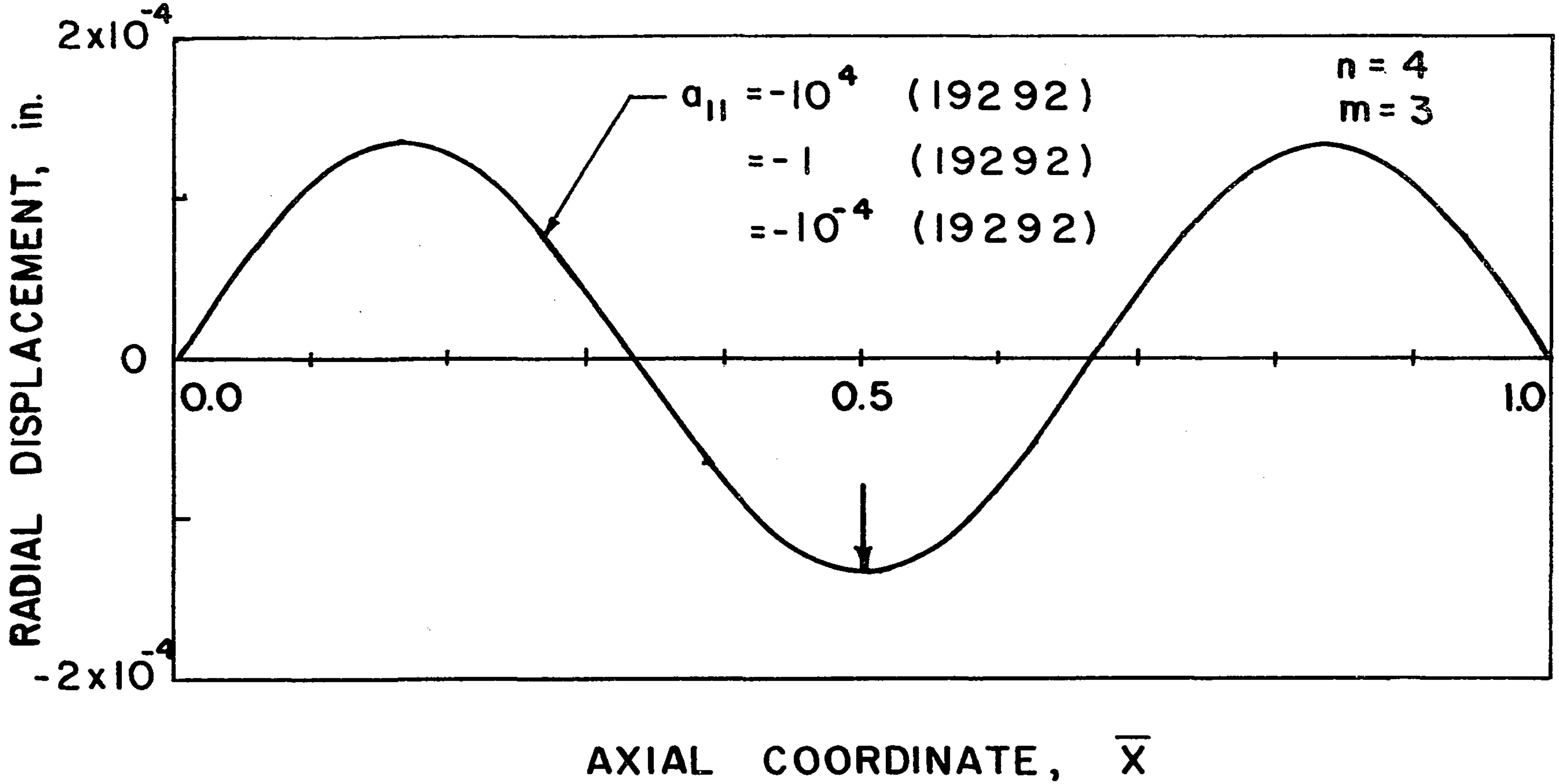

Figure 4.12 Radial Displacement Distribution at Resonance, $a_{11}=-10^{-4}$, -1, -10^{4}, $n=4$, $m=3$.

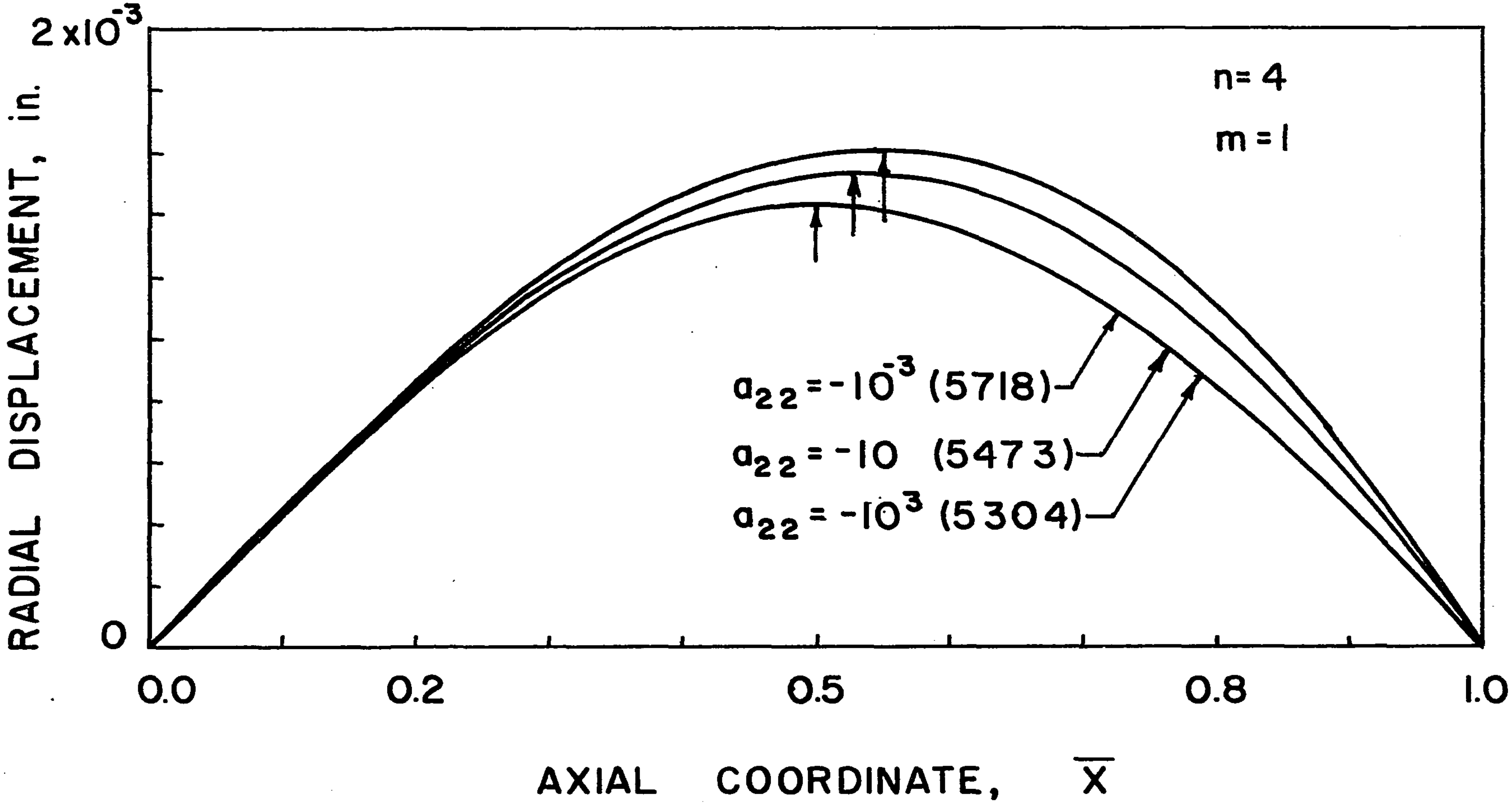

Figure 4.13 Radial Displacement Distribution at Resonance, a_{22}=-10^{-3}, -10, -10^{3}, n=4, m=1.

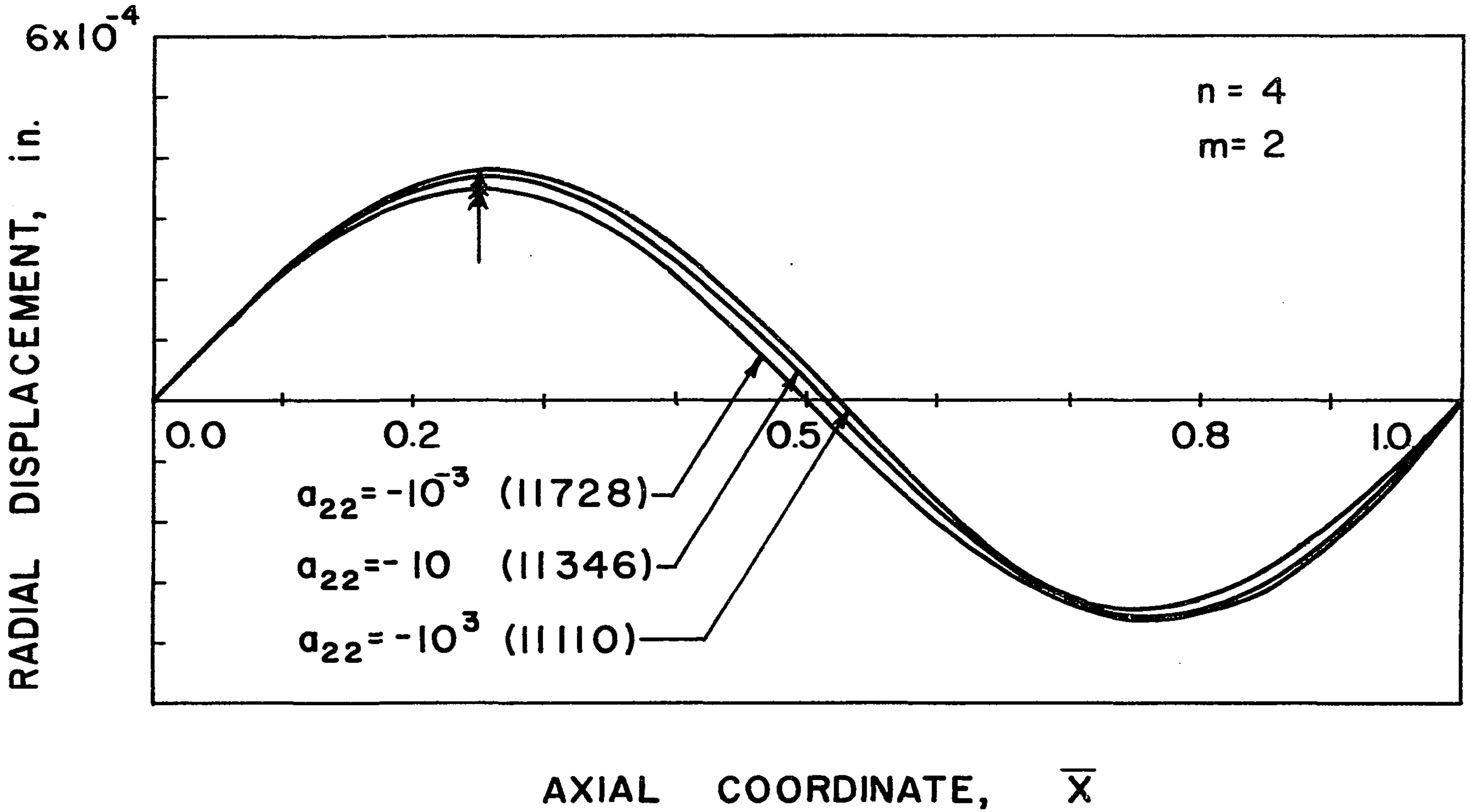

Figure 4.14 Radial Displacement Distribution at Resonance, $a_{22}=-10^{-3}$, -10, -10^{3}, n=4, m=2.

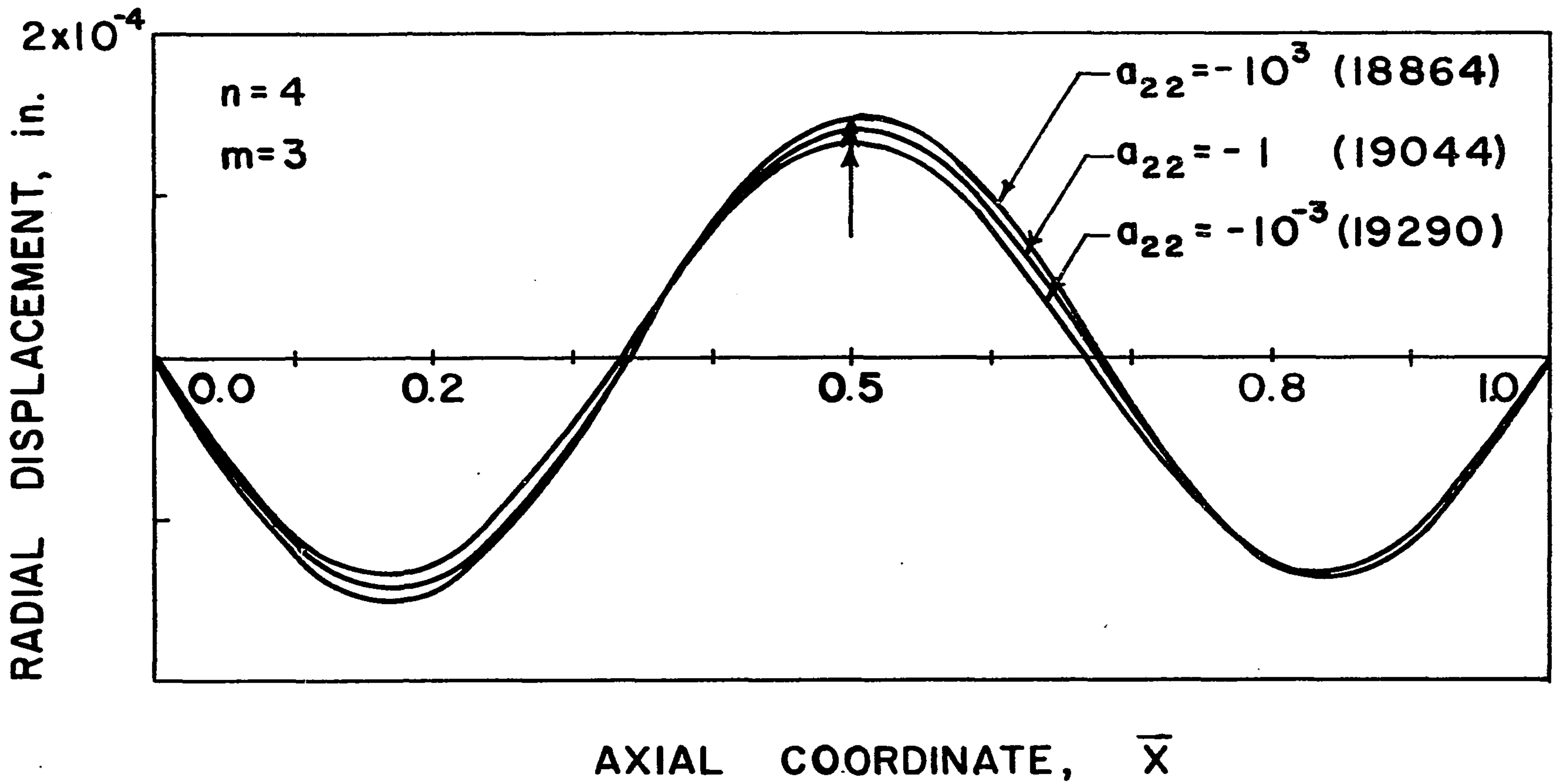

Figure 4.15 Radial Displacement Distribution at Resonance, $a_{22}=-10^{-3}$, -10, -10^{5}, n=4, m=3.

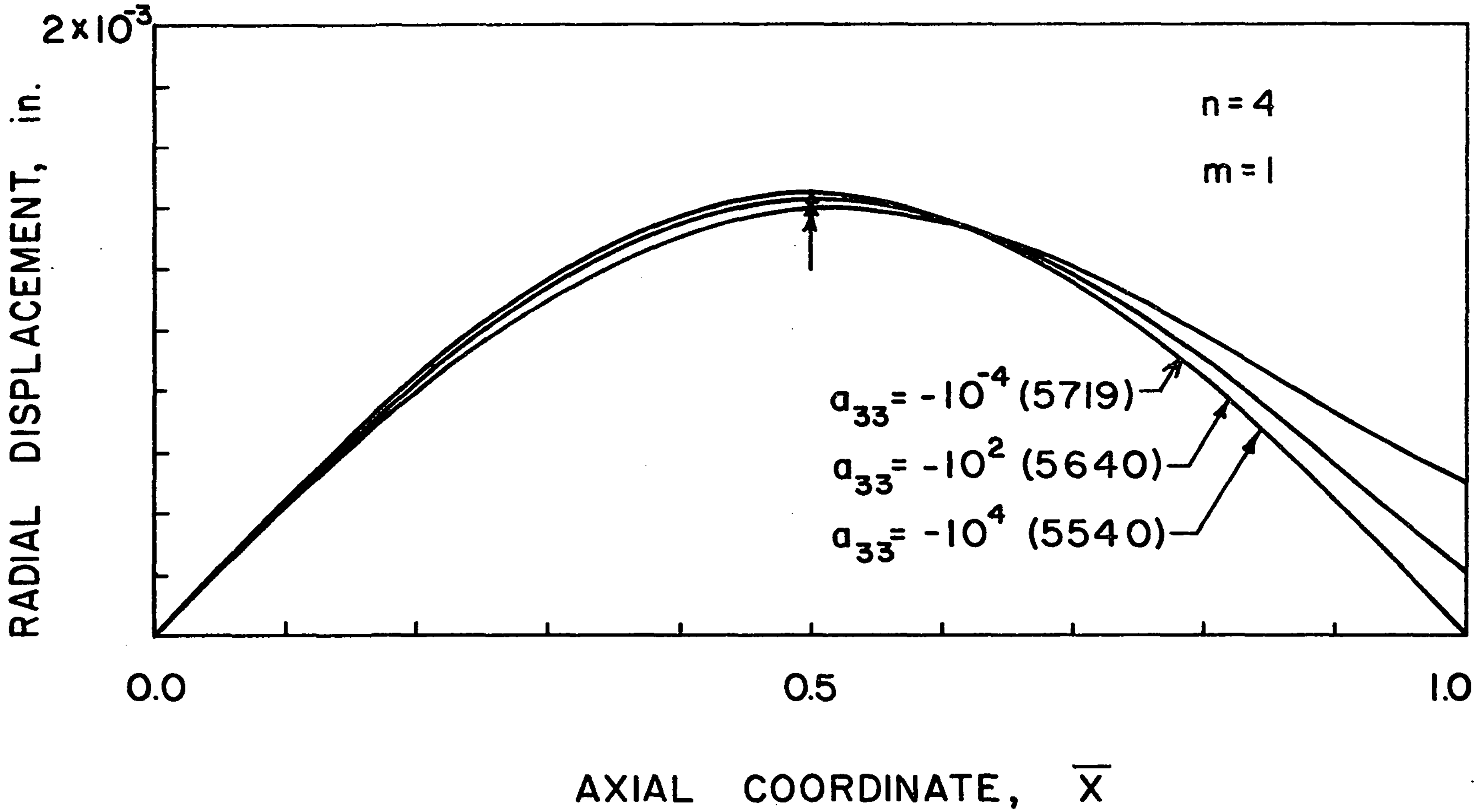

Figure 4.16 Radial Displacement Distribution at Resonance, a_{33}=-10^{-4}, -10^{3}, -10^{4}, n=4, m=1.

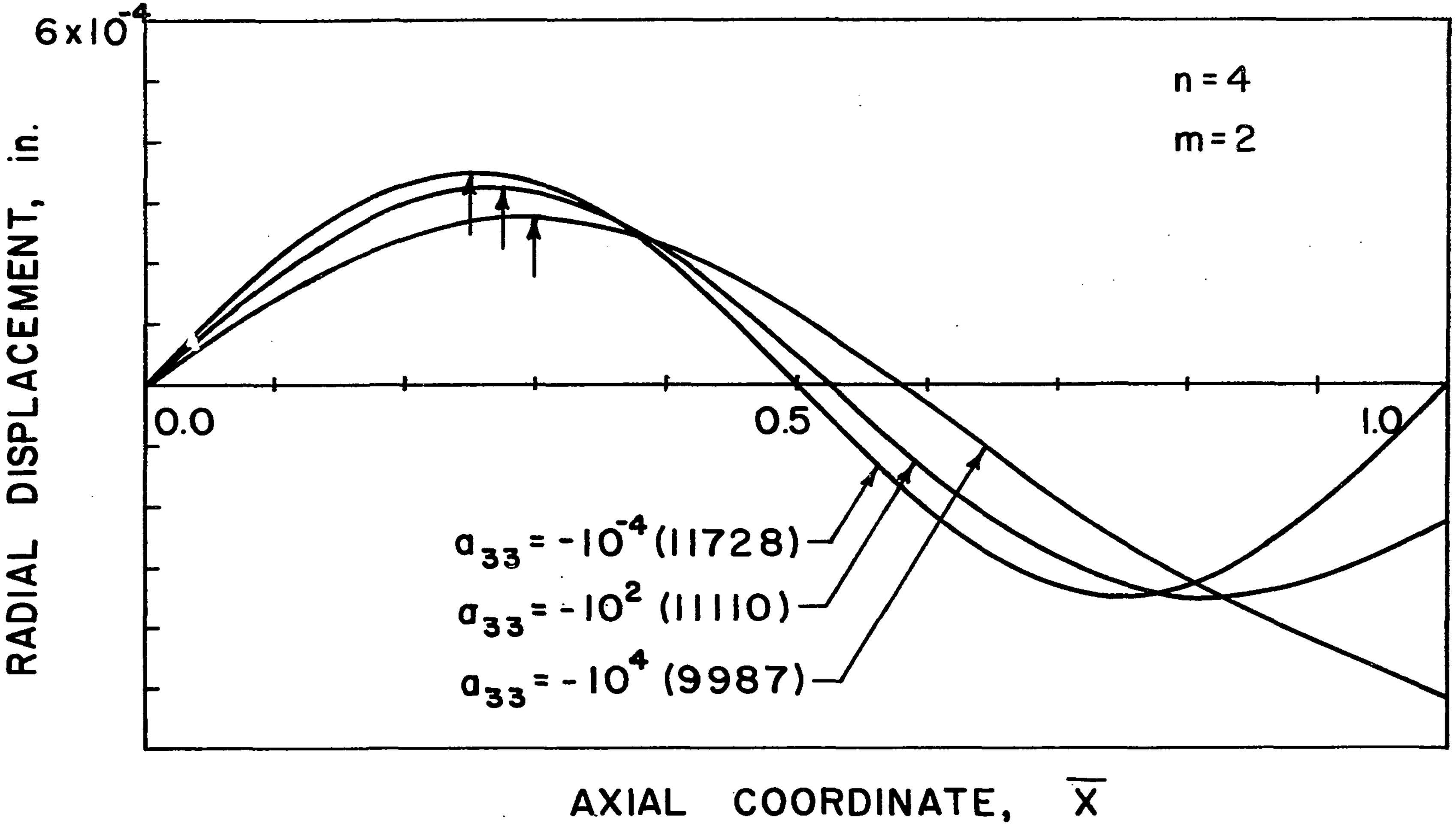

Figure 4.17 Radial Displacement Distribution at Resonance, a_{33}=-10^{-4},-10^{2}, -10^{4}, n=4, m=2.

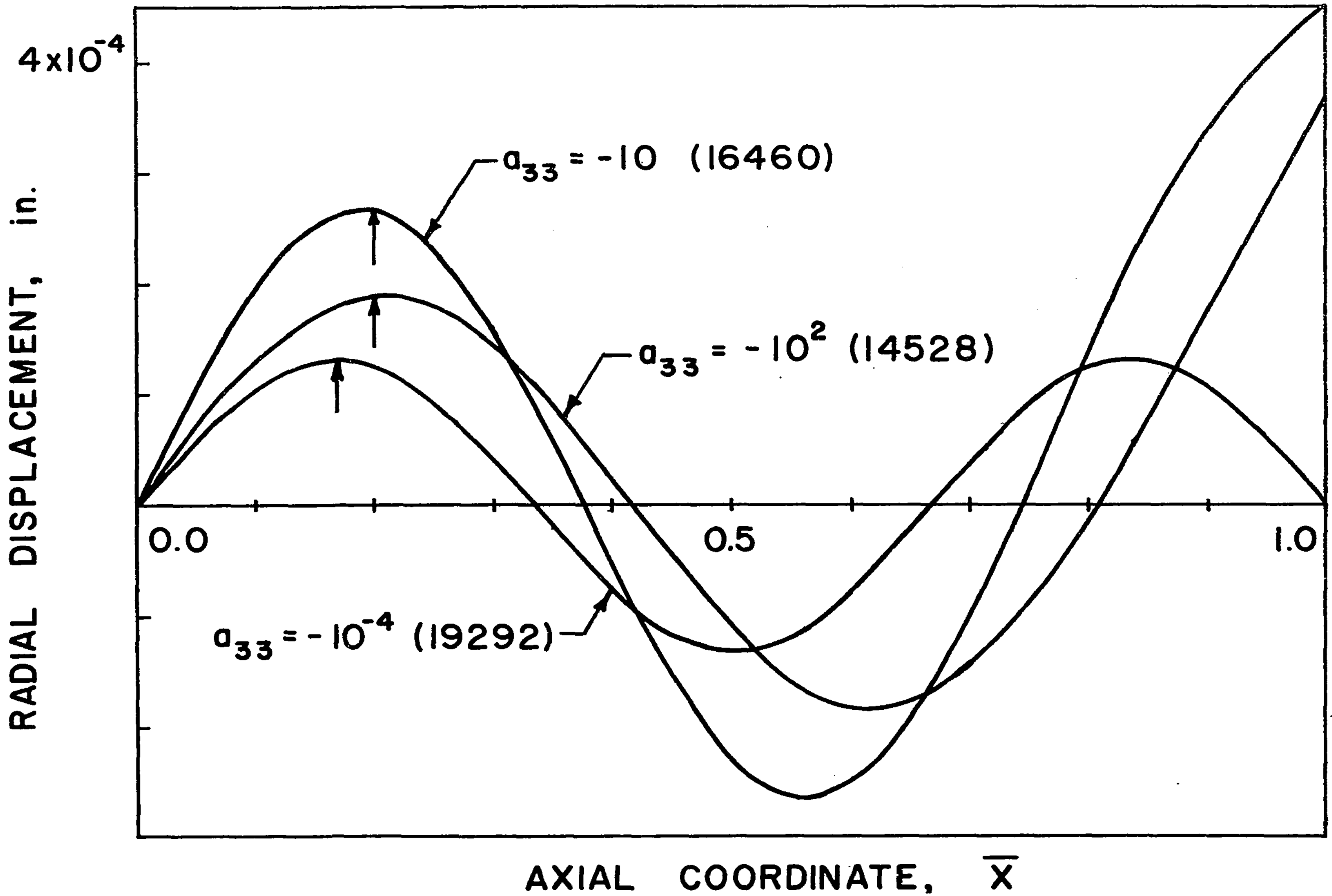

Figure 4.18 Radial Displacement Distribution at

Resonance, $a_{33}=10^{-4}$,,-10, -10^{2}, $n=4$, $m=3$.

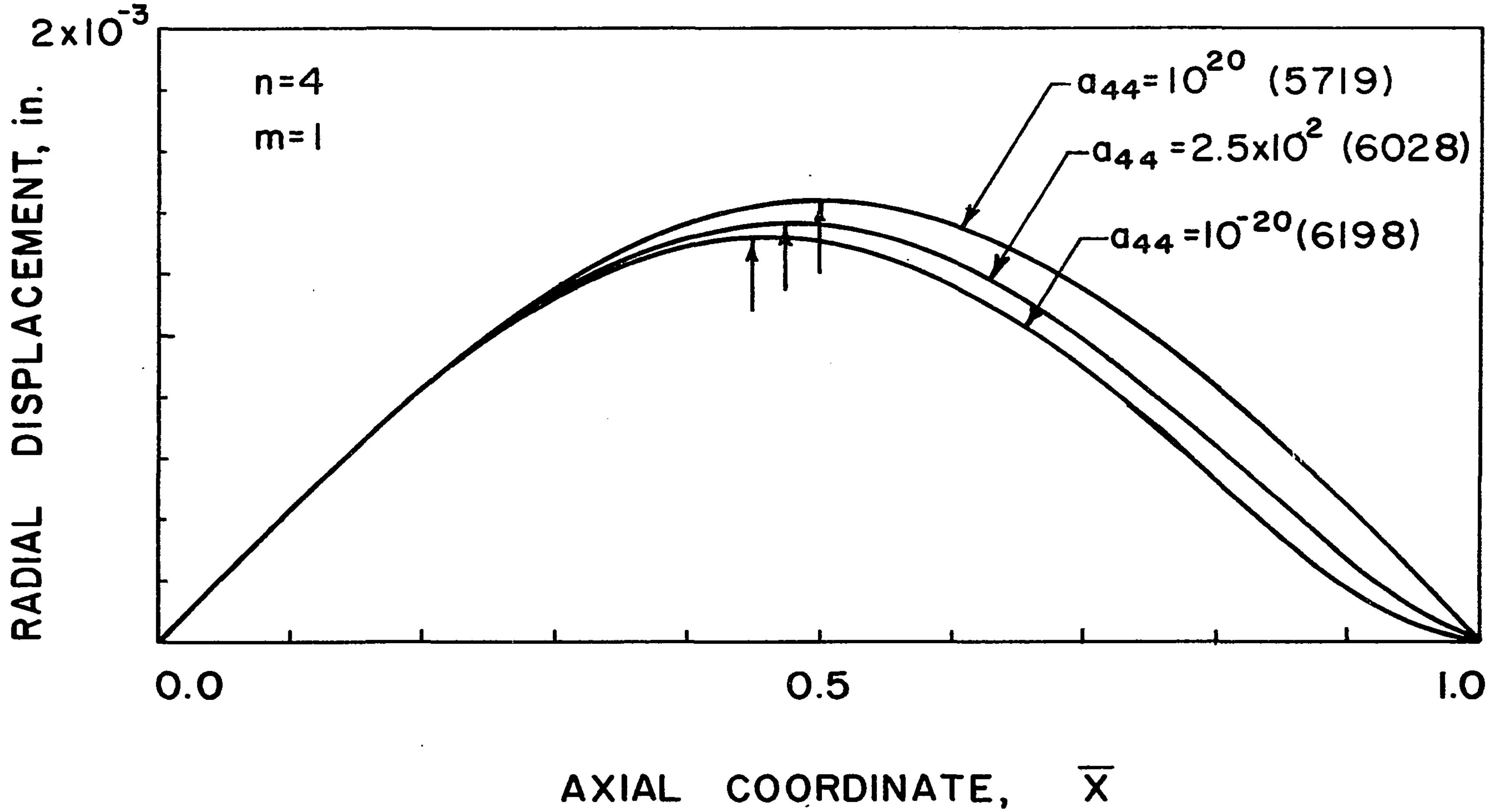

Figure 4.19 Radial Displacement Distribution at Resonance, $a_{44}=10^{-20}$, $2.5x10^{2}$, 10^{20}, n=4, m=1.

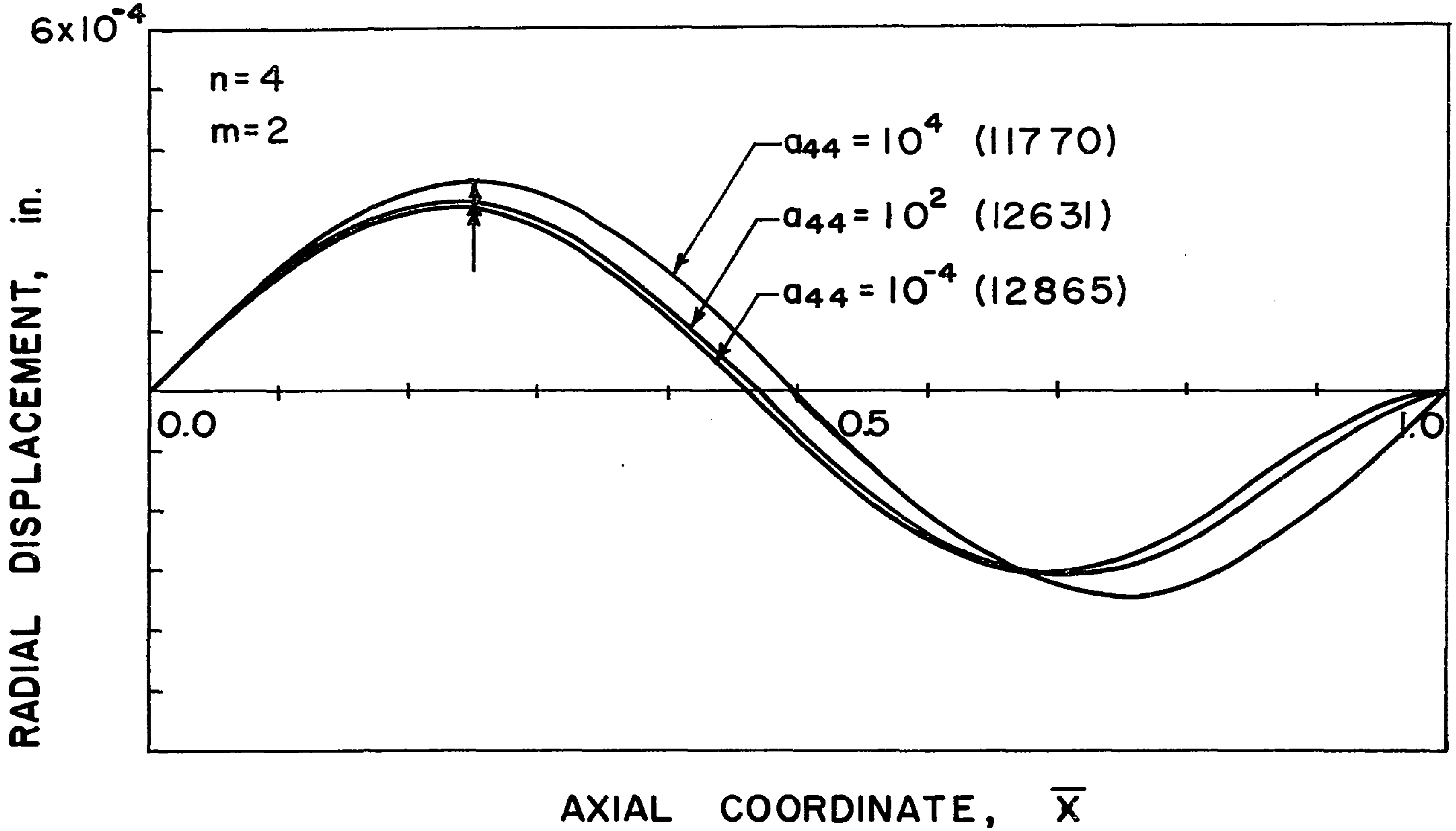

Figure 4.20 Radial Displacement Distribution at Resonance, $a_{44}=10^{-4}$, 10^2, 10^4, n=4, m=2.

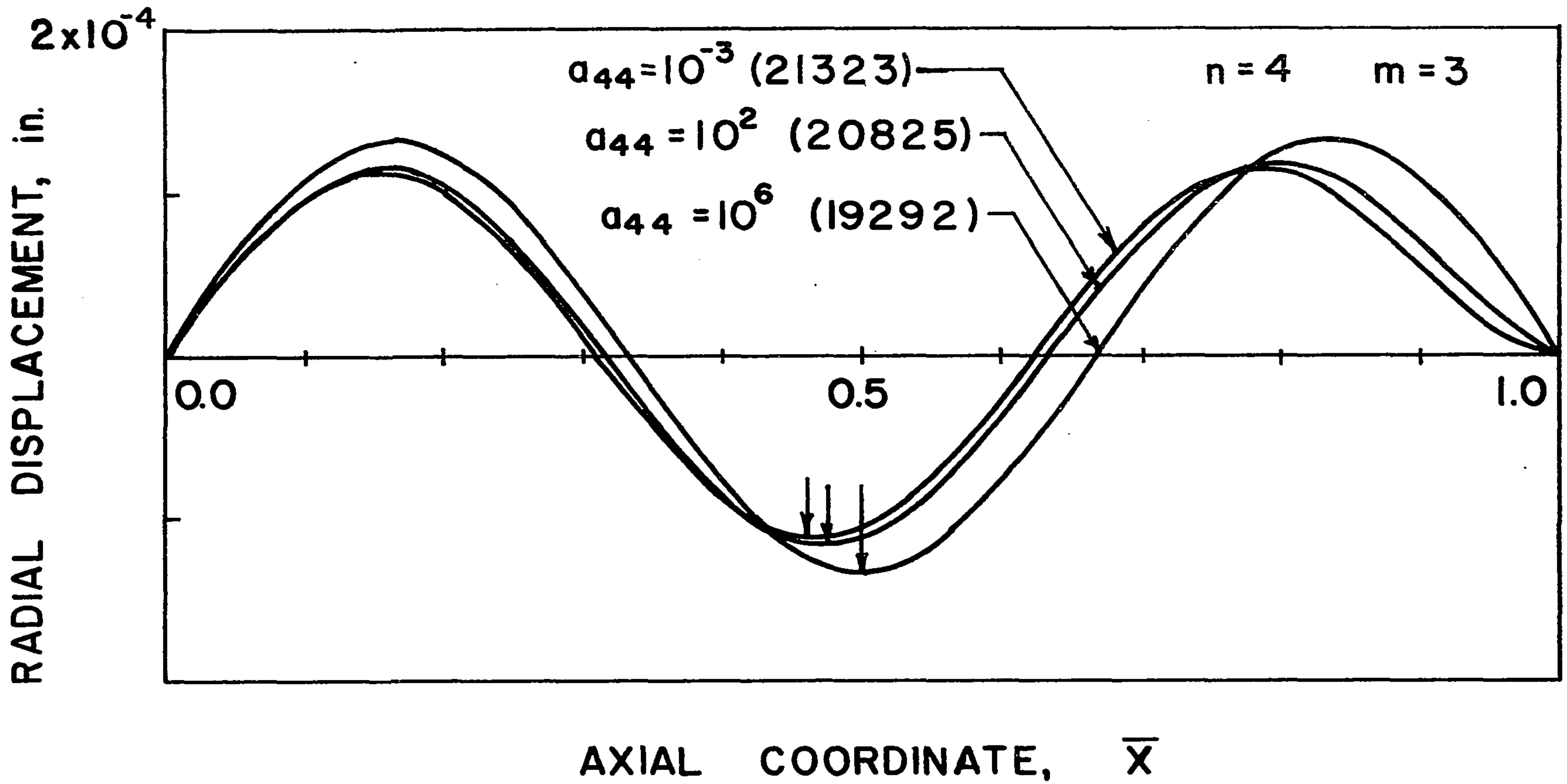

Figure 4.21 Radial Displacement Distribution at Resonance, $a_{44}=10^{-3}$, 10^{2}, 10^{6}, n=4, m=3.

Table 4.1 Effect of Axial Flexibility on Resonant Frequency and Amplitude of Resonance

Axial Flexibility, Dimensionless a_{11}	Circumferential Number n	Axial Half-Wave Number m	Resonant Frequency, Hertz Ω_e	Maximum Radial Displacement, Inches W_n	Maximum Radial Displacement, Dimensionless W^*
-10^{-4}	2	1	7679	--	--
-1	2	1	7676	--	--
-10^{4}	2	1	7671	--	--
-10^{-4}	6	1	7829	--	--
-1	6	1	7733	--	--
-10^{4}	6	1	7699	--	--
-10^{-4}	4	1	5895	$.1345 \times 10^{-2}$	11.48
-1	4	1	5777	$.1399 \times 10^{-2}$	11.48
-10^{4}	4	1	5719	$.1427 \times 10^{-2}$	11.42
-10^{-4}	4	2	11731	$.3493 \times 10^{-3}$	11.78
-1	4	2	11730	$.3482 \times 10^{-3}$	11.72
-10^{4}	4	2	11729	$.3476 \times 10^{-3}$	11.72
-10^{-4}	4	3	19292	$.1326 \times 10^{-3}$	12.10
-1	4	3	19292	$.1325 \times 10^{-3}$	12.10
-10^{4}	4	3	19292	$.1325 \times 10^{-3}$	12.10

Table 4.2 Effect of Tangential Flexibility on Resonant Frequency and Amplitude of Resonance.

Axial Flexibility, Dimensionless a_{22}	Circumferential Number n	Axial Half-Wave Number m	Resonant Frequency, Hertz Ω_e	Maximum Radial Displacement, Inches W_n	Maximum Radial Displacement Dimensionless W^*
-10^{-3}	2	1	7669	--	--
-1	2	1	6606	--	--
-10^{3}	2	1	4706	--	--
-10^{-3}	6	1	7700	--	--
-1	6	1	7673	--	--
-10^{3}	6	1	7661	--	--
-10^{-3}	4	1	5718	$.1427 \times 10^{-2}$	11.42
-1	4	1	5473	$.1521 \times 10^{-2}$	11.20
-10^{3}	4	1	5304	$.1601 \times 10^{-2}$	11.02
-10^{-3}	4	2	11728	$.3476 \times 10^{-3}$	11.70
-1	4	2	11346	$.3655 \times 10^{-3}$	11.50
-10^{3}	4	2	11110	$.3745 \times 10^{-3}$	11.32
-10^{-3}	4	3	19290	$.1330 \times 10^{-3}$	12.1
-1	4	3	19044	$.1406 \times 10^{-3}$	12.5
-10^{3}	4	3	18864	$.1469 \times 10^{-3}$	12.8

Table 4.3 Effect of Transverse Flexibility on Resonant Frequency and Amplitude of Resonance

Axial Flexibility, Dimensionless a_{33}	Circumferential Number n	Axial Half-Wave Number m	Resonant Frequency, Hertz Ω_e	Maximum Radial Displacement, Inches W_n	Maximum Radial Displacement, Dimensionless W^*
-10^{-4}	2	1	7671	--	--
-10	2	1	7659	--	--
-10^{4}	2	1	7631	--	--
-10^{-4}	6	1	7699	--	--
-10	6	1	7497	--	--
-10^{4}	6	1	7238	--	--
-10^{-4}	4	1	5719	$.1427\times10^{-2}$	11.42
-10	4	1	5646	$.1418\times10^{-2}$	11.06
-10^{4}	4	1	5540	$.1391\times10^{-2}$	10.46
-10^{-4}	4	2	11729	$.3476\times10^{-3}$	11.72
-10	4	2	11110	$.3479\times10^{-3}$	10.5
-10^{4}	4	2	9987	$.2751\times10^{-3}$	6.7
				$.5181\times10^{-3}\leftarrow$	
-10^{-4}	4	3	19292	$.1329\times10^{-3}$	12.1
-10	4	3	16460	$.2660\times10^{-3}$	17.62
				$.4528\times10^{-3}\leftarrow$	
-10^{2}	4	3	14528	$.1908\times10^{-3}$	9.85
				$.3680\times10^{-3}\leftarrow$	

$\rightarrow$occurred at $x=x_{max}$

Table 4.4 Effect of Rotational Flexibility on Resonant Frequency and Amplitude of Resonance

Axial Flexibility, Dimensionless a_{44}	Circumferential Number n	Axial Half-Wave Number m	Resonant Frequency, Hertz Ω_e	Maximum Radial Displacement, Inches W_n	Maximum Radial Displacement, Dimensionless W*
10^{-20}	2	1	7982	--	--
1364	2	1	7752	--	--
10^{20}	2	1	7671	--	--
10^{-20}	6	1	8112	--	--
1364	6	1	7791	--	--
10^{20}	6	1	7699	--	--
10^{-20}	4	1	6198	$.1322 \times 10^{-2}$	12.42
10^{3}	4	1	6023	$.1353 \times 10^{-2}$	12.04
10^{6}	4	1	5719	$.1427 \times 10^{-2}$	11.42
10^{-4}	4	2	12865	$.3077 \times 10^{-3}$	12.46
10^{2}	4	2	12631	$.3140 \times 10^{-3}$	12.25
10^{4}	4	2	11770	$.3457 \times 10^{-3}$	11.72
10^{-3}	4	3	21823	$.1128 \times 10^{-3}$	13.14
10^{2}	4	3	20825	$.1182 \times 10^{-3}$	12.54
10^{6}	4	3	19292	$.1330 \times 10^{-3}$	12.10

cient, a_{33}. This same phenomena was reported by Egle [22] in the response of a plate with one edge elastically supported. The physical reason for this behavior is not known.

CHAPTER V

CLOSURE

The present approach provides a powerful tool for the examination of the influence of boundary conditions on the dynamic response of cylindrical shells. The solution which is developed here utilized a simple support without axial constraint at one end, allowing any combination of flexibility at the other, although the solution could be reformulated to account for any combination of natural boundary conditions at one end and elastic boundary conditions at the other. The solution is valid for all values of $n \geq 1$. For $n=0$, the governing equations were developed and the solution can be carried out in the same manner as for $n \geq 1$.

Conclusions to be drawn from the data presented in Chapter IV are that the transverse restraint is more important than are restrictions on slope. Least important are restrictions on axial displacement. These conclusions appear to contradict those drawn by Forsberg [5] but his general conclusions were made with reference to much longer shells. A detailed examination of his results for ratio $\ell/r = 1$ supports these conclusions.

More investigation is needed with respect to the following points:

(1) The range of the solution needs to be extended either by use of multiple precision in the computer evaluation or by completely changing the form of the solution so that it can be properly evaluated for longer shells.

(2) For a fuller understanding of the influence of these various boundary conditions, a knowledge of the stress resultants that occur during vibration is needed.

(3) A more general form of the flexibility influence functions needs to be considered, as well as the effects of boundary dissipation.

(4) The inverse problem, in which the flexibility influence functions could be determined from knowledge of the dynamic response might be practically significant.

(5) Experimental verification is needed to reinforce some of the results.

(6) A transient solution and response to random excitation will ultimately be required.

(7) The more practical, laminated, composite, or stiffened shells warrant further study.

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APPENDIX A

DERIVATION OF THE EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

The procedure which was used in the derivation of the equations of motion is Hamilton's principle. This choice was made because of its simplicity and elegance and because it also gives the form of the boundary conditions. Hamilton's principle states that the actual path followed by a dynamical process is the one which makes

$$\delta \int_{t_0}^{t_1} (T-V+W)dt = 0 \; . \tag{A-1}$$

In other words the line integral of T-V+W takes an extreme value which is a minimum.

The kinetic energy can be readily expressed as

$$T = \frac{1}{2} \rho h \iint_A (\dot{u}^2+\dot{v}^2+\dot{w}^2) r d\theta dx \; , \tag{A-2}$$

where the dot ($\dot{}$) denotes differentiation with respect to time. The quantity W represents the work done by the surface force q and the boundary forces F_1, F_2, F_3, M_1 whose directions are given in Figure 2.2.

$$W = \iint_A -qw \; r d\theta dx + \oint (F_1 u+F_2 v+F_3 w-M_1 \beta_1) r d\theta \tag{A-3}$$

where $\beta_1 = w,_x$ and the line integral is over the boundary at $x=0$ and at $x=x_{max}$. The strain energy is defined as

$$V = \frac{1}{2} \int_V \sigma_{ij} \varepsilon_{ij} dV \tag{A-4}$$

The strain-displacement and the constituitive relations, after applying the assumptions of linear shell theory and of Donnell, are

$$\varepsilon_{xx} = u,_x - z\, w,_{xx}$$

$$\varepsilon_{\theta\theta} = \left(\frac{1}{r} v,_\theta + \frac{w}{r}\right) - \frac{z}{r^2} w,_{\theta\theta}$$

$$\varepsilon_{x\theta} = \left(\frac{1}{r} u,_\theta + v,_x\right) - \frac{2z}{r} w,_{x\theta}$$

$$\sigma_{xx} = \frac{E}{1-\mu^2} (\varepsilon_{xx} + \mu\varepsilon_{\theta\theta})$$

$$\sigma_{\theta\theta} = \frac{E}{1-\mu^2} (\varepsilon_{\theta\theta} + \mu\varepsilon_{xx})$$

$$\sigma_{x\theta} = G\, \varepsilon_{x\theta} \, . \tag{A-5}$$

Here the comma denotes partial differentiation with respect to the subscript which follows (i.e., $u,_x = \frac{\partial u}{\partial x}$). After equations (A-5) are substituted into equation (A-4) and the result has been integrated over the thickness of the shell, the strain energy becomes

$$V = \frac{Eh}{2(1-\mu^2)} \iint_A \left(u^2,_x + \{\frac{1}{r} v,_\theta + \frac{w}{r}\}^2 + 2\mu u,_x\{\frac{1}{r} v,_\theta + \frac{w}{r}\} + \{\frac{1-\mu}{2}\}\{\frac{1}{r} u,_\theta + v,_x\}^2\right) r d\theta dx + \frac{D}{2} \iint_A \left(w^2,_{xx} + \frac{1}{r^4} w^2,_{\theta\theta} + \frac{2\mu}{r^2} w,_{xx} w,_{\theta\theta} + \frac{2}{r^2} (1-\mu) w^2,_{x\theta}\right) r d\theta dx \, . \tag{A-6}$$

After the operation of variation as indicated in equation (A-1) has been carried out

$$\delta T = \rho h \iint_A (\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w}) r d\theta dx \ , \tag{A-7}$$

$$\delta W = \iint_A -q \ \delta w \ r d\theta dx + \oint (F_1 \delta u + F_2 \delta v + F_3 \delta w - M_1 \delta\beta_1) r d\theta , \tag{A-8}$$

$$\delta V = \frac{Eh}{1-\mu^2} \iint_A \Big\{ u,_x \ \delta u,_x + \left(\frac{1}{r} v,_\theta + \frac{w}{r}\right)\left(\frac{1}{r} \delta v,_\theta + \frac{1}{r} \delta w\right)$$

$$+ \mu\left(\frac{1}{r} v,_\theta + \frac{w}{r}\right)\delta u,_x + \mu u,_x\left(\frac{1}{r} \delta v,_\theta + \frac{1}{r} \delta w\right) + \left(\frac{1-\mu}{2}\right)\left(\frac{1}{r} u,_\theta\right.$$

$$\left. + v,_x\right)\left(\frac{1}{r} \delta u,_\theta + \delta v,_x\right)\Big\} r d\theta dx$$

$$+ D\iint \Big\{ w,_{xx} \ \delta w,_{xx} + \frac{1}{r^4} w,_{\theta\theta} \ \delta w,_{\theta\theta} + \frac{\mu}{r^2}\left(w,_{xx} \ \delta w,_{\theta\theta}\right.$$

$$\left. + w,_{\theta\theta}\delta w,_{xx}\right) + \frac{2(1-\mu)}{r} w,_{x\theta} \ \delta w,_{x\theta}\Big\} r d\theta dx \ . \tag{A-9}$$

The time derivitives of the variations in the kinetic energy expression must be removed. Since the operations of variation and differentiation commute, $\delta\dot{u} = \frac{d}{dt}(\delta u)$, and the time integral of the first term of equation (A-7) can be written

$$\int_{t_0}^{t_1} \iint_A \dot{u}\delta\dot{u} \ r d\theta dx \ dt = \iint_A \int_{t_0}^{t_1} \dot{u} \frac{d}{dt} (\delta u) dt \ r d\theta dx \ ,$$

which can be integrated by parts to

$$\iint_A \left[\dot{u}\delta u \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \ddot{u}\delta u dt \right] r d\theta dx \ .$$

The virtual displacements must vanish at the end points of the arbitrary interval $t_0 \le t \le t_1$ and this integral can be written

$$- \int_{t_0}^{t_1} \iint_A \ddot{u}\delta u \ r d\theta dx \ dt \ . \tag{A-10}$$

The derivatives of the virtual displacements in the expression for δV must be removed similarly. For example

$$\iint_A u,_x \, \delta u,_x \, r d\theta dx = \iint_A u,_x \frac{\partial}{\partial x} (\delta u) \, r d\theta dx$$

$$= \int \left[u,_x \, \delta u \Big|_0^\ell - \int u,_{xx} \, \delta u dx \right] r d\theta$$

$$= \oint u,_x \, \delta u \Big|_0^\ell \, r d\theta - \iint_A u,_{xx} \, \delta u \, r d\theta dx \; , \qquad \text{(A-11)}$$

where the operations of variation and differentiation have been interchanged and the result integrated by parts. This step is repeated for every term in equation (A-9) which contains the variation of a derivative. These terms are then substituted back into equation (A-1) which becomes, after regrouping the terms according to virtual displacement

$$\int_{t_0}^{t_1} \iint_A \Bigg[\Big(-\rho h \ddot{u} + \frac{Eh}{1-\mu^2} \{ u,_{xx} + \frac{\mu}{r} v,_{x\theta} + \frac{\mu}{r} w,_x$$

$$+ \frac{1-\mu}{2r^2} u,_{\theta\theta} + \frac{1-\mu}{2r} v,_{x\theta} \} \Big) \delta u + \Big(-\rho h \ddot{v} + \frac{Eh}{1-\mu^2} \{ \frac{1}{r^2} v,_{\theta\theta} + \frac{1}{r^2}$$

$$w,_\theta + \frac{\mu}{r} u,_{x\theta} + \frac{1-\mu}{2r} u,_{x\theta} + \frac{1-\mu}{2} v,_{xx} \} \Big) \delta v + \Big(-\rho h \ddot{w} - q$$

$$- \frac{Eh}{1-\mu^2} \{ \frac{1}{r^2} v,_\theta + \frac{w}{r^2} + \frac{\mu}{r} u,_x \} - D \{ w,_{xxxx} + \frac{2}{r^2} w,_{xx\theta\theta}$$

$$+ \frac{1}{r^4} w,_{\theta\theta\theta\theta} \} \Big) \delta w \Bigg] r d\theta dx dt + \int_{t_0}^{t_1} \oint \Bigg[\Big(F_1 - \frac{Eh}{1-\mu^2} \{ u,_x + \frac{\mu}{r} (v,_\theta$$

$$+ w) \} \Big) \delta u + \Big(F_2 - \frac{Eh}{1-\mu^2} \frac{1-\mu}{2} \{ \frac{1}{r} u,_\theta + v,_x \} \Big) \delta v + \Big(F_3 + D \{ w,_{xxx}$$

$$+ \frac{2-\mu}{r^2} w,_{\theta\theta x} \} \Big) \delta w + \Big(-M_1 - D \{ w,_{xx} + \frac{\mu}{r^2} w,_{\theta\theta} \} \Big) \delta \beta_1 \Bigg] \Bigg|_{x=0}^{x=\ell} r d\theta dt = 0. \qquad \text{(A-12)}$$

Since the time interval $t_0 \rightarrow t_1$ in equation (A-12) is arbitrary, the space integrals must vanish. Also the variations

δu, δv, δw, $\delta\beta$ are arbitrary everywhere except at the boundaries, therefore the integrands must vanish. Therefore, the equations of motion are as follows:

$$u,_{xx} + \frac{\mu}{r} v,_{x\theta} + \frac{\mu}{r} w,_{x} + \left(\frac{1-\mu}{2r}\right) u,_{\theta\theta} + \left(\frac{1-\mu}{2r}\right) v,_{x\theta} - \rho \frac{(1-\mu^2)}{E} \ddot{u} = 0 ,$$

$$\frac{1}{r^2} v,_{\theta\theta} + \frac{1}{r^2} w,_{\theta} + \frac{\mu}{r} u,_{x\theta} + \left(\frac{1-\mu}{2r}\right) u,_{x\theta} + \left(\frac{1-\mu}{2}\right) v,_{xx} - \rho \frac{(1-\mu^2)}{E} \ddot{v} = 0 ,$$

$$\frac{1}{r^2} v,_{\theta} + \frac{w}{r^2} + \frac{\mu}{r} u,_{x} + \frac{h^2}{12} \nabla^4 w + q \frac{(1-\mu^2)}{Eh} + \rho \frac{(1-\mu^2)}{E} \ddot{w} = 0 . \tag{A-13}$$

The boundary conditions also come from equation (A-12). At both $x=0$ and $x=\ell$ we must have

$$\delta u = 0 \text{ or } F_1 = \frac{Eh}{1-\mu^2}\left(u,_{x} + \frac{\mu}{r}(v,_{\theta} + w)\right) ,$$

$$\delta v = 0 \text{ or } F_2 = \frac{Eh}{1-\mu^2}\left(\frac{1-\mu}{2}\right)\left(\frac{1}{r} u,_{\theta} + v,_{x}\right) ,$$

$$\delta w = 0 \text{ or } F_3 = -D(w,_{xxx} + \frac{2-\mu}{r^2} w,_{\theta\theta x}) ,$$

$$\delta\beta_1 = 0 \text{ or } M_1 = -D(w,_{xx} + \frac{\mu}{r^2} w,_{\theta\theta}) . \tag{A-14}$$

These boundary conditions can be expressed in terms of the dimensionless axial coordinate, as was done in Chapter II, by introducing the transformations

$$\bar{x} = \frac{x}{r} ,$$

$$T = \omega t ,$$

$$w(x,\theta,t) = \bar{w}(\bar{x},\theta,T), \text{ etc.}$$

In terms of these quantities, dimensionless boundary forces are derived from equations (A-14) as follows:

$$\bar{F}_1 = \frac{F_1}{K} = \frac{1}{r}\left(\bar{u},_{\bar{x}} + \mu(\bar{v},_{\theta} + \bar{w})\right) ,$$

$$\bar{F}_2 = \frac{F_2}{K} = \frac{1}{r}\left(\frac{1-\mu}{2}\right)(\bar{u},_{\theta} + \bar{v},_{\bar{x}}) \ ,$$

$$\bar{F}_3 = \frac{F_3}{K} = -\frac{1}{r} R^2\left(\bar{w},_{\bar{x}\bar{x}\bar{x}} + (2-\mu)\bar{w},_{\theta\theta\bar{x}}\right) \ ,$$

$$\bar{M}_1 = \frac{M_1}{Kr} = -\frac{1}{r} R^2(\bar{w},_{\bar{x}\bar{x}} + \mu\bar{w},_{\theta\theta}) \ , \qquad \text{(A-15)}$$

where $K = \frac{Eh}{1-\mu^2}$.

Use is made of these quantities in the force deflection relationships of the parent structure, equations (2-26). The first of equations (2-26) is, for example

$$du = (a_{11}F_1 + a_{13}F_3 + a_{14}M_1)rd\xi \ .$$

By use of equations (A-15), this can also be written

$$du = (a_{11}K\bar{F}_1 + a_{13}K\bar{F}_3 + a_{14}Kr\bar{M}_1)rd\xi \ .$$

The groups $a_{11}K$, $a_{13}K$ and $a_{14}Kr$ are dimensionless and will be referred to as $\bar{a}_{11}$, $\bar{a}_{13}$, $\bar{a}_{14}$. The other dimensionless flexibility functions are defined similarly from consideration of the remaining three equations of the set of equations (2-26). These are summarized as follows

$$\bar{a}_{11} = Ka_{11} \ ,$$
$$\bar{a}_{13} = Ka_{13} \ ,$$
$$\bar{a}_{14} = Kra_{14} \ ,$$
$$\bar{a}_{22} = Ka_{22} \ ,$$
$$\bar{a}_{31} = Ka_{31} \ ,$$
$$\bar{a}_{33} = Ka_{33} \ ,$$
$$\bar{a}_{34} = Kra_{34} \ ,$$
$$\bar{a}_{41} = Kra_{41} \ ,$$
$$\bar{a}_{43} = Kra_{43} \ ,$$

$$\bar{a}_{44} = Kr^2 a_{44} \ .$$

It should be noted that the $\bar{a}_{ij}$ all contain the parameter K which contains the modulus of elasticity of the cylindrical shell which is a complex quantity. In all of the results which have been presented, the $\bar{a}_{ij}$ were taken as real numbers. This is physically equivalent to the assumption that the damping parameter of the parent structure is the same as that of the cylindrical shell.

APPENDIX B

ORIGIN OF NON DIMENSIONAL DISPLACEMENT

The non-dimensional displacement w*, which was employed in the plotting of the results in Chapter IV was motivated by the normal mode expansion of the dynamic response which follows. In the normal mode expansion, each normal mode (assumed to be a known function) is multiplied by a time dependent coefficient and the solution given by the following summations:

$$u(x,\theta,t) = \sum_n \sum_m w_{mn}(t)\,\phi^u_{mn}(x,\theta) \ ,$$

$$v(x,\theta,t) = \sum_n \sum_m w_{mn}(t)\,\phi^v_{mn}(x,\theta) \ ,$$

$$w(x,\theta,t) = \sum_n \sum_m w_{mn}(t)\,\phi^w_{mn}(x,\theta) \ . \qquad \text{(B-1)}$$

These assumed solutions are substituted into the governing equations of motion (A-13), the result of which is

$$\sum_n \sum_m \left[w_{mn}\left(\frac{\partial^2\phi^u_{mn}}{\partial x^2} + \frac{1-\mu}{2r^2}\frac{\partial^2\phi^v_{mn}}{\partial\theta^2} + \frac{1+\mu}{2r}\frac{\partial^2\phi^v_{mn}}{\partial x\partial\theta} + \frac{\mu}{r}\frac{\partial\phi^w_{mn}}{\partial x}\right) - \frac{1-\mu^2}{E^*}\rho\ddot{w}_{mn}\phi^u_{mn}\right] = 0 \ ,$$

$$\sum_n \sum_m \left[w_{mn}\left(\frac{1+\mu}{2r}\frac{\partial^2\phi^u_{mn}}{\partial x\partial\theta} + \frac{1-\mu}{2}\frac{\partial^2\phi^v_{mn}}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2\phi^v_{mn}}{\partial\theta^2} + \frac{1}{r^2}\frac{\partial\phi^w_{mn}}{\partial\theta}\right)\right.$$

$$- \frac{1-\mu^2}{E^*} \rho \ddot{w}_{mn} \phi^v_{mn} \Bigg] = 0 \ ,$$

$$\sum_n \sum_m \Bigg[w_{mn} \left(\frac{\mu}{r} \frac{\partial \phi^u_{mn}}{\partial x} + \frac{1}{r^2} \frac{\partial \phi^v_{mn}}{\partial \theta} + \frac{1}{r^2} \phi^w_{mn} + \frac{h^2}{12} \nabla^4 \phi^w_{mn} \right) + \frac{1-\mu^2}{E^*}$$

$$\rho \ddot{w}_{mn} \phi^w_{mn} \Bigg] = \frac{-q(1-\mu^2)}{E^* h} \quad . \tag{B-2}$$

The normal modes, however, each satisfy the equations of motion for free undamped vibration. Thus

$$\frac{\partial^2 \phi^u_{mn}}{\partial x^2} + \frac{1-\mu}{2r^2} \frac{\partial^2 \phi^v_{mn}}{\partial \theta^2} + \frac{1+\mu}{2r} \frac{\partial^2 \phi^v_{mn}}{\partial x \partial \theta} + \frac{\mu}{r} \frac{\partial \phi^w_{mn}}{\partial x} = - \frac{1-\mu^2}{E} \rho \omega_n^2 \phi^u_{mn} \ ,$$

$$\frac{1+\mu}{2r} \frac{\partial^2 \phi^u_{mn}}{\partial x \partial \theta} + \frac{1-\mu}{2} \frac{\partial^2 \phi^v_{mn}}{\partial x^2} + \frac{1}{r^2} \frac{\partial^2 \phi^v_{mn}}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial \phi^w_{mn}}{\partial \theta} = - \frac{1-\mu^2}{E} \rho \omega_n^2 \phi^v_{mn} ,$$

$$\frac{\mu}{r} \frac{\partial \phi^u_{mn}}{\partial x} + \frac{1}{r^2} \frac{\partial \phi^v_{mn}}{\partial \theta} + \frac{1}{r^2} \phi^w_{mn} + \frac{h^2}{12} \nabla^4 \phi^w_{mn}$$

$$= + \frac{1-\mu^2}{E} \rho \omega_n^2 \phi^w_{mn} \ . \tag{B-3}$$

Combining equations (B-2) and (B-3) leads to the set of equations

$$\sum_n \sum_m \left(\ddot{w}_{mn} + \omega_n^2 (1+i\gamma) w_{mn} \right) \phi^u_{mn} = 0 \ ,$$

$$\sum_n \sum_m \left(\ddot{w}_{mn} + \omega_n^2 (1+i\gamma) w_{mn} \right) \phi^v_{mn} = 0 \ ,$$

$$\sum_n \sum_m \left(\ddot{w}_{mn} + \omega_n^2 (1+i\gamma) w_{mn} \right) \phi^w_{mn} = - \frac{q}{\rho h} \ . \tag{B-4}$$

Equations (B-4) are multiplied by ϕ^u_{pq}, ϕ^v_{pq}, ϕ^w_{pq} respectively and then added together so that the orthogonality condition can be used, the orthogonality equation being [3]

$$\iint_A \left(\phi^u_{mn} \phi^u_{pq} + \phi^v_{mn} \phi^v_{pq} + \phi^w_{mn} \phi^w_{pq} \right) dA$$

$$= \delta_{mp} \delta_{nq} \iint_A \left(\phi^{u2}_{mn} + \phi^{v2}_{mn} + \phi^{w2}_{mn} \right) dA$$

$$= \delta_{mp}\delta_{nq}N_{pq} \quad . \tag{B-5}$$

After these operations have been carried out, equations (B-4) reduce to

$$\ddot{w}_{pq} + \omega_n^2(1+i\gamma)w_{pq} = -\frac{1}{N_{pq}}\frac{1}{\rho h}\iint_A q\ \Phi^w_{pq}\ dA \ . \tag{B-6}$$

Now since $q = -\frac{Q_0}{r}\delta(x-x_0)\delta(\theta)e^{i\omega t}$ equation (B-6) becomes

$$\ddot{w}_{pq} + \omega_n^2(1+i\gamma)w_{pq} = \frac{Q_0}{\rho h}\frac{1}{N_{pq}}\Phi^w_{pq}(x_0,0)e^{i\omega t} \ . \tag{B-7}$$

The solution to this equation is well known

$$w_{pq} = \frac{Q_0 M_{pq}\ e^{i(\omega t-\alpha)}}{2\pi r\ell\rho h\omega_n^2\left[\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \gamma^2\right]^{\frac{1}{2}}} \tag{B-8}$$

where

$$M_{pq} = \frac{1}{N_{pq}}\Phi^w_{pq}(x_0,0)2\pi r\ell \ .$$

The factor $2\pi r\ell$ is included so that the number M_{pq} is dimensionless. Therefore, at $\omega=\omega_n$

$$2\pi r\ell\left|w_{pq}\right|\rho h\omega_n^2\frac{\gamma}{Q_0} = w^* = M_{pq} \ .$$

APPENDIX C

COMPUTER PROGRAM

The computer program which was used to evaluate the solution is listed at the end of this section. It is written in Fortran IV. Several versions of the program were used. The basic one, called AIJSUM, is described here. This program "cycles" in frequency, computing at each frequency point the desired quantities. At each frequency, the displacement response at specified points are computed for those circumferential numbers specified and these results are summed. This is done for each of the various values of the damping parameter which the programer specifies.

To use this program, one must specify the following parameters:

EL = length in inches,

RAD = radius in inches,

H = thickness in inches,

E = modulus of elasticity, psi,

MU = Poisson's ratio,

RHO = mass density, lb.-sec^2/in^4,

GAMMA1 = increment in damping parameter,

QL = magnitude of load in pounds,

X0 = point of application of load, inches,

FREQ = starting frequency in Hertz,

FREQ2 = frequency increment in Hertz,

N3 = number of terms to be included in summation,

CNF = first circumferential number to be included in summation,

N5 = number of increments desired in axial coordinate,

N7 = number of different values of damping to be used,

N9 = number of frequency points to be used.

The system will print out at each frequency point the following information: H, EL, RAD, QL, X0, GAMMA, FREQ, RN(real frequency parameter), RSQ (geometrical parameter), FLXCF(I,J) (4x4 array of flexibility coefficients). This will be followed by a tabulation of the quantities XBAR (axial coordinate/radius), WXM (magnitude of radial displacement in inches), PHASE1 (phase angle between QL and WXM), WSTAR (dimensionless radial displacement).

The quantity WXM is the magnitude of the complex sum over the specified number of circumferential numbers. It might be desirable to have the results printed for each term in the summation individually as well as the sum itself. If so, the C should be removed from the first column of those cards which are labelled with a 2 in column 73. When this is done, the values of the roots are printed with the results for each term in the summation. Two sets of roots are printed - those for real modulus and those for a complex modulus of elasticity.

The listing for one other version of the program will be given here. This one, called A_N_M_, searches for the resonant frequency. With zero damping, it cycles using large frequency increments until a phase shift of 180° is encountered, then backs up to the previous frequency point and proceeds to cycle with a frequency increment only half as large as before. Each time the phase shift is encountered, the frequency increment is halved until the new increment is less than one Hertz. This process is repeated again with a new set of flexibility coefficients, as many times as desired. To use this program one must specify the following parameters.

EL = length,

RAD = radius,

H = thickness,

QL = magnitude of load,

XO = point of application of load,

MU = Poisson's ratio,

E = modulus of elasticity,

RHO = mass density,

CN = circumferential number,

N5 = number of increments desired in the axial coordinate

FREQ = starting frequency,

FREQ2 = frequency increment

N9 = number of frequencies used before cutoff if no phase shift is encountered,

N11 = number of different sets of flexibility coefficients to be used.

The program as now set up changes only one flexibility coefficient at a time. This is now done as in statement 240 but any other rule could be easily substituted.

```
C      AIJSUM
C      N3=NUMBER OF TERMS TO BE INCLUDED IN SUMMATION
C      CNF=FIRST CIRCUMFERENTIAL NUMBER
C      N5=NUMBER OF INCREMENTS DESIRED IN XBAR
C      N7=NUMBER OF VALUES OF DAMPING TO BE USED
C      N9=NUMBER OF FREQUENCY INCREMENTS DESIRED
C      N2,N4,N6,N8=COUNTERS
       COMPLEX*16 P(8),PP(8),A(4),B(4),C(4),AP(4),BP(4),CP(4),
      2CPP(4),CPPP(4),CF(4,4),RHS(4),UXQP,WXQP,WXQPP,WXQPPP,VXQP,
      3FSW(8,4),FSU(8,4),FSV(8,4),QSW(8),QSU(8),QSV(8),
      4SE(8),SP(8),SSTAR(8),FLXCF(4,4),DENOM,FN,FN2,S1,S2,S3,S4,
      5S5,S6,S7,S8,WXQ,UXQ,VXQ,Z,SAVE,BIGA
       COMPLEX*16 SPP(8),SP2
       COMPLEX*16 T11,T12,T13,T21,T22,T23,T31,T32,T33
       COMPLEX*16 M11,M12,M13,M21,M22,M23,M31,M32,M33
       COMPLEX WNX,UNX,VNX
       COMPLEX WX(10,50,5),SWX
       REAL*8 SCOF(9),COF(9),ROOTR(8),ROOTI(8)
       REAL MU,MU1,MU2,MESS
  500 FORMAT(1H1,10X,'THICKNESS=',F8.2,'  INCHES',/,
     211X,'LENGTH    =',F8.2,'  INCHES',/,
     311X,'RADIUS    =',F8.2,'  INCHES',/,
     411X,'LOAD      =',F8.2,'  POUNDS',/,
     511X,'X0        =',F8.2,'  INCHES',/,
     611X,'GAMMA     =',F8.2)
  501 FORMAT(11X,'FREQ   =',F8.2,'  CPS',/,11X,'RN      =',E10.2,/,
     211X,'RSQ    =',E10.2,/,11X,'Q      =',E10.2)
  502 FORMAT(1H1,10X,'FREQ = ',F8.2,'  CPS',/,11X,
     2'CN    =',F8.2,/,11X,'GAMMA= ',F10.4)
  504 FORMAT(2E12.4,2X,2E12.4,2X,2E12.4,2X,2E12.4)
  506 FORMAT(10X,'UNO     =',2D24.16,/,10X,'VNPO    =',2D24.16,/,
     210X,'WNPO    =',2D24.16,/,10X,'WNPPPO =',2D24.16)
  507 FORMAT(10X,'KS = ',F8.2)
  508 FORMAT(3X,'XBAR=',F8.2,3X,'WNXM=',E12.4,3X,'PHASE1=',
     2F8.2,3X,'WNX=',2E12.4)
  509 FORMAT(5X,'EIR=',F8.2)
  510 FORMAT(4E12.4)
```

```
514 FORMAT(3X,'XBAR=',F9.3,3X,'WXM=',E12.4,3X,'PHASE1=',F8.2,
   23X,'WSTAR=',E12.4)
515 FORMAT(D18.4,D12.4)
    FREQ=10000.0
    FREQ2=100.0
    CNF=4.0
    N3=1
    N5=20
    N7=3
    N9=40
    NL=N5+1
    GAMMA1=0.01
    H=0.15
    RAD=3.0
    EL=3.0
    X0=0.75
    QL=10.0
    MU=0.3
    E=30.0E+06
    RHO=0.731E-03
    Z=DCMPLX(0.0D0,1.0D0)
    XBARM=EL/RAD
    X0=X0/RAD
    MU1=(1.0-MU)/2.0
    MU2=(1.0+MU)/2.0
    Q=-0.29*RAD*QL/(E*H)
    RSQ=(H**2)/(12.0*(RAD**2))
    FREQ=FREQ-FREQ2
    MESS=248.0*RAD*EL*RHO*H/QL
    READ(05,515)((FLXCF(I,J),I=1,4),J=1,4)
    N8=0
 03 FREQ=FREQ+FREQ2
    CN=CNF-1.0
    N2=0
    N8=N8+1
    RN=RHO*35.8*(FREQ**2)*(RAD**2)/E
```

```
      RN2=RN**2
      RN3=RN**3
02    CN=CN+1.0
      N2=N2+1
      GAMMA=-GAMMA1
      N6=0
      CN2=CN**2
      CN3=CN**3
      CN4=CN**4
      CN5=CN**5
      CN6=CN**6
      CN7=CN**7
      CN8=CN**8
      SCOF(1)=MU1*CN8*RSQ-(1.0+MU1)*CN5*RN*RSQ+CN4*RN2*RSQ
     2-MU1*CN4*RN+(1.0+MU1)*CN2*RN2-MU1*CN2*RN+RN2-RN3
      SCOF(2)=0.0
      SCOF(3)=-4.0*MU1*CN6*RSQ+3.0*(1.0+MU1)*CN4*RN*RSQ
     2+2.0*MU1*CN2*RN-2.0*CN2*RN2*RSQ+(1.0+MU1-MU**2)*RN
     3-(1.0+MU1)*RN2
      SCOF(4)=0.0
      SCOF(5)=6.0*MU1*CN4*RSQ-3.0*(1.0+MU1)*CN2*RN*RSQ
     2+RN2*RSQ-MU1*RN+MU1*(1.0-MU**2)
      SCOF(6)=0.0
      SCOF(7)=((1.0+MU1)*RN-4.0*MU1*CN2)*RSQ
      SCOF(8)=0.0
      SCOF(9)=MU1*RSQ
      CALL POLRT(SCOF,COF,8,ROOTR,ROOTI,EIR)
      IF(EIR)11,11,10
10    WRITE(06,509)EIR
11    AA=SCOF(9)
      BB=SCOF(7)
      CC=SCOF(5)
      DD=SCOF(3)
      EE=SCOF(1)
      BBP=-(1.0+MU1)*RN*RSQ
      CCP=3.0*(1.0+MU1)*CN2*RN*RSQ-2.0*RN2*RSQ+MU1*RN
```

```
      DDP=-3.0*(1.0+MU1)*CN4*RN*RSQ-2.0*MU1*CN2*RN+4.0*CN2*RN2*RSQ
     2-(1.0+MU1-MU**2)*RN+2.0*(1.0+MU1)*RN2
      EEP=(1.0+MU1)*CN6*RN*RSQ+MU1*CN4*RN-2.0*CN4*RN2*RSQ
     2-2.0*(1.0+MU1)*CN2*RN2+MU1*CN2*RN-2.0*RN2+3.0*RN3
      BBPP=2.0*(1.0+MU1)*RN*RSQ
      CCPP=-5.0*(1.0+MU1)*CN2*RN*RSQ+6.0*RN2*RSQ-2.0*MU1*RN
      DDPP=6.0*(1.0+MU1)*CN4*RN*RSQ+4.0*MU1*CN2*RN-12.0*CN2*RN2*RSQ
     2+2.0*(1.0+MU1-MU**2)*RN-6.0*(1.0+MU1)*RN2
      EEPP=-2.0*(1.0+MU1)*CN6*RN*RSQ-2.0*MU1*CN4*RN+6.0*CN4*RN2*RSQ
     2+6.0*(1.0+MU1)*CN2*RN2-2.0*MU1*CN2*RN+6.0*RN2-12.0*RN3
      DO 20 J=1,8
      SE(J)=DCMPLX(ROOTR(J),ROOTI(J))
      S1=SE(J)
      S2=S1**2
      S3=S1**3
      S4=S1**4
      S5=S1**5
      S6=S1**6
      S7=S1**7
      SP(J)=(BBP*S6+CCP*S4+DDR*S2+EEP)/(8.0*AA*S7+6.0*BB*S5
     2+4.0*CC*S3+2.0*DD*S1)*(-1.0)
      SP2=SP(J)**2
   20 SPP(J)=((EEPP+DDPP*S2+CCPP*S4+BBPP*S6)+(4.0*DDP*S1+8.0*CCP*S3
     2+12.0*BBP*S5)*SP(J)+(2.0*DD+12.0*CC*S2+30.0*BB*S4+56.0*AA*S6)*SP2)
     3/(2.0*DD*S1+4.0*CC*S3+6.0*BB*S5+8.0*AA*S7)*(-1.0)
   01 GAMMA=GAMMA+GAMMA1
      N6=N6+1
      N4=0
C     WRITE(06,502)FREQ,CN,GAMMA                                      2
      FN=RN/(1.0+Z*GAMMA)
      FN2=FN**2
      DO 24 J2=1,8
      SSTAR(J2)=SE(J2)+Z*GAMMA*SP(J2)-0.50*(GAMMA**2)*SPP(J2)
C     WRITE(06,510)SE(J2),SSTAR(J2)                                   2
   24 CONTINUE
      DO 30 K=1,8
```

```
      S1=SSTAR(K)
      S2=S1**2
      DENOM=DCMPLX(1.0D0,0.0D0)
      DO 14 L=1,8
      IF(L-K)12,14,12
   12 DENOM=(SSTAR(K)-SSTAR(L))*DENOM
   14 CONTINUE
      DENOM=AA*DENOM
      T11=S2+FN-MU1*CN2
      T12=MU2*CN*S1
      T13=MU*S1
      T21=-T12
      T22=MU1*S2+FN-CN2
      T23=-CN
      T31=T13
      T32=CN
      T33=RSQ*((S2-CN2)**2)+1.0-FN
      M11=T22*T33-T32*T23
      M12=T31*T23-T21*T33
      M13=T21*T32-T31*T22
      M21=T32*T13-T12*T33
      M22=T11*T33-T31*T13
      M23=T31*T12-T11*T32
      M31=T12*T23-T22*T13
      M32=T21*T13-T11*T23
      M33=T11*T22-T21*T12
      FSW(K,1)=(S1*M13-MU2*CN*M23+MU*M33)/DENOM
      FSW(K,2)=MU1*M23/DENOM
      FSW(K,3)=(S2-2.0*CN2)*RSQ*M33/DENOM
      FSW(K,4)=RSQ*M33/DENOM
      QSW(K)=-Q*M33/DENOM
      FSV(K,1)=(S1*M12-MU2*CN*M22+MU*M32)/DENOM
      FSV(K,2)=MU1*M22/DENOM
      FSV(K,3)=(S2-2.0*CN2)*RSQ*M32/DENOM
      FSV(K,4)=RSQ*M32/DENOM
      QSV(K)=-Q*M32/DENOM
```

```
      FSU(K,1)=(S1*M11-MU2*CN*M21+MU*M31)/DENOM
      FSU(K,2)=MU1*M21/DENOM
      FSU(K,3)=(S2-2.0*CN2)*RSQ*M31/DENOM
      FSU(K,4)=RSQ*M31/DENOM
   30 QSU(K)=-Q*M31/DENOM
   36 XBAR=XBARM*(1.0-(1.0/N5)*N4)
      N4=N4+1
      DO 40 M=1,8
   40 P(M)=CDEXP(SSTAR(M)*XBAR)
      DO 50 N=1,4
      C(N)=DCMPLX(0.0D0,0.0D0)
      A(N)=DCMPLX(0.0D0,0.0D0)
      B(N)=DCMPLX(0.0D0,0.0D0)
      DO 50 II=1,8
      C(N)=C(N)+FSW(II,N)*P(II)
      A(N)=A(N)+FSU(II,N)*P(II)
   50 B(N)=B(N)+FSV(II,N)*P(II)
      IF(XBAR-XO)170,60,60
   60 DO 70 JJ=1,8
   70 PP(JJ)=CDEXP(SSTAR(JJ)*(XBAR-XO))
      WXQ=DCMPLX(0.0D0,0.0D0)
      UXQ=DCMPLX(0.0D0,0.0D0)
      VXQ=DCMPLX(0.0D0,0.0D0)
      DO 80 KK=1,8
      WXQ=WXQ+QSW(KK)*PP(KK)
      UXQ=UXQ+QSU(KK)*PP(KK)
   80 VXQ=VXQ+QSV(KK)*PP(KK)
      IF(XBAR-XBARM)170,90,90
   90 DO 100 LL=1,4
      AP(LL)=DCMPLX(0.0D0,0.0D0)
      BP(LL)=DCMPLX(0.0D0,0.0D0)
      CP(LL)=DCMPLX(0.0D0,0.0D0)
      CPP(LL)=DCMPLX(0.0D0,0.0D0)
      CPPP(LL)=DCMPLX(0.0D0,0.0D0)
      DO 100 MM=1,8
      AP(LL)=AP(LL)+SSTAR(MM)*FSU(MM,LL)*P(MM)
```

```
      BP(LL)=BP(LL)+SSTAR(MM)*FSV(MM,LL)*P(MM)
      CP(LL)=CP(LL)+SSTAR(MM)*FSW(MM,LL)*P(MM)
      CPP(LL)=CPP(LL)+(SSTAR(MM)**2)*FSW(MM,LL)*P(MM)
100   CPPP(LL)=CPPP(LL)+(SSTAR(MM)**3)*FSW(MM,LL)*P(MM)
      UXQP=DCMPLX(0.0D0,0.0D0)
      VXQP=DCMPLX(0.0D0,0.0D0)
      WXQP=DCMPLX(0.0D0,0.0D0)
      WXQPP=DCMPLX(0.0D0,0.0D0)
      WXQPPP=DCMPLX(0.0D0,0.0D0)
      DO 110 NN=1,8
      UXQP=UXQP+SSTAR(NN)*QSU(NN)*PP(NN)
      VXQP=VXQP+SSTAR(NN)*QSV(NN)*PP(NN)
      WXQP=WXQP+SSTAR(NN)*QSW(NN)*PP(NN)
      WXQPP=WXQPP+(SSTAR(NN)**2)*QSW(NN)*PP(NN)
110   WXQPPP=WXQPPP+(SSTAR(NN)**3)*QSW(NN)*PP(NN)
      DO 150 M=1,4
      CF(1,M)=-A(M)+FLXCF(1,1)*AP(M)+MU*CN*FLXCF(1,1)*B(M)
     2+(MU*FLXCF(1,1)+MU*CN2*RSQ*FLXCF(1,4))*C(M)
     3+(2.0-MU)*CN2*RSQ*FLXCF(1,3)*CP(M)
     4-RSQ*FLXCF(1,4)*CPP(M)-RSQ*FLXCF(1,3)*CPPP(M)
      CF(2,M)=-B(M)-MU1*CN*FLXCF(2,2)*A(M)+MU1*FLXCF(2,2)*BP(M)
      CF(3,M)=-C(M)+FLXCF(3,1)*AP(M)+MU*CN*FLXCF(3,1)*B(M)
     2+(MU*FLXCF(3,1)+MU*CN2*RSQ*FLXCF(3,4))*C(M)
     3+(2.0-MU)*CN2*RSQ*FLXCF(3,3)*CP(M)
     4-RSQ*FLXCF(3,4)*CPP(M)-RSQ*FLXCF(3,3)*CPPP(M)
150   CF(4,M)=-CP(M)+FLXCF(4,1)*AP(M)+MU*CN*FLXCF(4,1)*B(M)
     2+(MU*FLXCF(4,1)+MU*CN2*RSQ*FLXCF(4,4))*C(M)
     3+(2.0-MU)*CN2*RSQ*FLXCF(4,3)*CP(M)
     4-RSQ*FLXCF(4,4)*CPP(M)-RSQ*FLXCF(4,3)*CPPP(M)
      RHS(1)=UXQ-FLXCF(1,1)*UXQP-MU*CN*FLXCF(1,1)*VXQ
     2-(MU*FLXCF(1,1)+MU*CN2*RSQ*FLXCF(1,4))*WXQ
     3-(2.0-MU)*CN2*RSQ*FLXCF(1,3)*WXQP
     4+RSQ*FLXCF(1,4)*WXQPP+RSQ*FLXCF(1,3)*WXQPPP
      RHS(2)=VXQ+MU1*CN*FLXCF(2,2)*UXQ-MU1*FLXCF(2,2)*VXQP
      RHS(3)=WXQ-FLXCF(3,1)*UXQP-MU*CN*FLXCF(3,1)*VXQ
     2-(MU*FLXCF(3,1)+MU*CN2*RSQ*FLXCF(3,4))*WXQ
```

```
     3-(2.0-MU)*CN2*RSQ*FLXCF(3,3)*WXQP
     4+RSQ*FLXCF(3,4)*WXQPP+RSQ*FLXCF(3,3)*WXQPPP
      RHS(4)=WXQP-FLXCF(4,1)*UXQP-MU*CN*FLXCF(4,1)*VXQ
     2-(MU*FLXCF(4,1)+MU*CN2*RSQ*FLXCF(4,4))*WXQ
     3-(2.0-MU)*CN2*RSQ*FLXCF(4,3)*WXQP
     4+RSQ*FLXCF(4,4)*WXQPP+RSQ*FLXCF(4,3)*WXQPPP
      KKK=4
      CALL CSIMQ(CF,RHS,KKK,KS)
      IF(KS)162,162,160
  160 WRITE(06,507)KS
C     WRITE(06,506)(RHS(NW),NW=1,4)                                 2
  162 CONTINUE
  170 IF(XBAR-XO)190,180,180
  180 WNX=WXQ
      DO 182 I=1,4
  182 WNX=WNX+RHS(I)*C(I)
      GO TO 200
  190 WNX=CMPLX(0.0,0.0)
      DO 192 I=1,4
  192 WNX=WNX+RHS(I)*C(I)
  200 WX(N2,N4,N6)=WNX
      WNXM=CABS(WNX)
      PHASE1=57.29578*ATAN2(AIMAG(WNX),REAL(WNX))
C 210 WRITE(06,508)XBAR,WNXM,PHASE1,WNX                             2
      IF(N4-NL)36,220,220
  220 IF(N6-N7)01,230,230
  230 IF(N2-N3)02,240,240
  240 DO 246 I=1,N7
      GAMMA=GAMMA1*(I-1)
      WRITE(06,500)H,EL,RAD,QL,XO,GAMMA
      WRITE(06,501)FREQ,RN,RSQ,Q
      WRITE(06,504)((FLXCF(L,M),L=1,4),M=1,4)
      DO 246 J=1,NL
      XBAR=XBARM*(1.0-(1.0/N5)*(J-1))
      SWX=CMPLX(0.0,0.0)
      DO 242 K=1,N3
```

```
242 SWX=SWX+WX(K,J,I)
    WXM=CABS(SWX)
    PHASE1=57.29578*ATAN2(AIMAG(SWX),REAL(SWX))
    WSTAR=MESS*GAMMA*(FREQ**2)*WXM
246 WRITE(06,514)XBAR,WXM,PHASE1,WSTAR
    IF(N8-N9)03,250,250
250 CALL EXIT
    END
```

```
      SUBROUTINE POLRT(XCOF,COF,M,ROOTR,ROOTI,IER)
      DIMENSION XCOF(1),COF(1),ROOTR(1),ROOTI(1)                        PLRT 460
      DOUBLE PRECISION XO,YO,X,Y,XPR,YPR,UX,UY,V,YT,XT,U,XT2,YT2,SUMSQ, PLRT 470
     1 DX,DY,TEMP,ALPHA                                                 PLRT 480
      DOUBLE PRECISION XCOF,COF,ROOTR,ROOTI
      IFIT=0                                                            PLRT 680
      N=M                                                               PLRT 690
      IER=0                                                             PLRT 700
      IF(XCOF(N+1))10,25,10                                             PLRT 710
   10 IF(N) 15,15,32                                                    PLRT 720
   15 IER=1                                                             PLRT 760
   20 RETURN                                                            PLRT 770
   25 IER=4                                                             PLRT 810
      GO TO 20                                                          PLRT 820
   30 IER=2                                                             PLRT 860
      GO TO 20                                                          PLRT 870
   32 IF(N-36) 35,35,30                                                 PLRT 880
   35 NX=N                                                              PLRT 890
      NXX=N+1                                                           PLRT 900
      N2=1                                                              PLRT 910
      KJ1 = N+1                                                         PLRT 920
      DO 40 L=1,KJ1                                                     PLRT 930
      MT=KJ1-L+1                                                        PLRT 940
   40 COF(MT)=XCOF(L)                                                   PLRT 950
   45 XO=.00500101                                                      PLRT 990
      YO=0.01000101                                                     PLRT1000
      IN=0                                                              PLRT1040
   50 X=XO                                                              PLRT1050
      XO=-10.0*YO                                                       PLRT1090
      YO=-10.0*X                                                        PLRT1100
      X=XO                                                              PLRT1140
      Y=YO                                                              PLRT1150
      IN=IN+1                                                           PLRT1160
      GO TO 59                                                          PLRT1170
   55 IFIT=1                                                            PLRT1180
      XPR=X                                                             PLRT1190
```

```
      YPR=Y                                                             PLRT1200
   59 ICT=0                                                             PLRT1240
   50 UX=0.0                                                            PLRT1250
      UY=0.0                                                            PLRT1260
      V =0.0                                                            PLRT1270
      YT=0.0                                                            PLRT1280
      XT=1.0                                                            PLRT1290
      U=COF(N+1)                                                        PLRT1300
      IF(U) 65,130,65                                                   PLRT1310
   65 DO 70 I=1,N                                                       PLRT1320
      L =N-I+1                                                          PLRT1330
      TEMP=COF(L)                                                       PLRT1340
      XT2=X*XT-Y*YT                                                     PLRT1350
      YT2=X*YT+Y*XT                                                     PLRT1360
      U=U+TEMP*XT2                                                      PLRT1370
      V=V+TEMP*YT2                                                      PLRT1380
      FI=I                                                              PLRT1390
      UX=UX+FI*XT*TEMP                                                  PLRT1400
      UY=UY-FI*YT*TEMP                                                  PLRT1410
      XT=XT2                                                            PLRT1420
   70 YT=YT2                                                            PLRT1430
      SUMSQ=UX*UX+UY*UY                                                 PLRT1440
      IF(SUMSQ) 75,110,75                                               PLRT1450
   75 DX=(V*UY-U*UX)/SUMSQ                                              PLRT1460
      X=X+DX                                                            PLRT1470
      DY=-(U*UY+V*UX)/SUMSQ                                             PLRT1480
      Y=Y+DY                                                            PLRT1490
   78 IF(DABS(DY)+DABS(DX)-1.0D-05) 100,80,80                           PLRT1500
   80 ICT=ICT+1                                                         PLRT1540
      IF(ICT-500) 60,85,85                                              PLRT1550
   85 IF(IFIT)100,90,100                                                PLRT1560
   90 IF(IN-5) 50,95,95                                                 PLRT1570
   95 IER=3                                                             PLRT1610
      GO TO 20                                                          PLRT1620
  100 DO 105 L=1,NXX                                                    PLRT1630
      MT=KJ1-L+1                                                        PLRT1640
```

```
      TEMP=XCOF(MT)
      XCOF(MT)=COF(L)
  105 COF(L)=TEMP
      ITEMP=N
      N=NX
      NX=ITEMP
      IF(IFIT) 120,55,120
  110 IF(IFIT) 115,50,115
  115 X=XPR
      Y=YPR
  120 IFIT=0
  122 IF(DABS(Y)-1.0D-4*DABS(X)) 135,125,125
  125 ALPHA=X+X
      SUMSQ=X*X+Y*Y
      N=N-2
      GO TO 140
  130 X=0.0
      NX=NX-1
      NXX=NXX-1
  135 Y=0.0
      SUMSQ=0.0
      ALPHA=X
      N=N-1
  140 COF(2)=COF(2)+ALPHA*COF(1)
  145 DO 150 L=2,N
  150 COF(L+1)=COF(L+1)+ALPHA*COF(L)-SUMSQ*COF(L-1)
  155 ROOTI(N2)=Y
      ROOTR(N2)=X
      N2=N2+1
      IF(SUMSQ) 160,165,160
  160 Y=-Y
      SUMSQ=0.0
      GO TO 155
  165 IF(N) 20,20,45
      END
```

```
      SUBROUTINE CSIMQ(CF,RHS,N,KS)
      COMPLEX*16 CF(16),RHS(4),BIGA,SAVE
      TOL=0.0
      KS=0
      JJ=-4
      DO 565 J=1,4
      JY=J+1
      JJ=JJ+5
      BIGA=DCMPLX(0.0D0,0.0D0)
      IT=JJ-J
      DO 530 I=J,4
      IJ=IT+I
      IF(CDABS(BIGA)-CDABS(CF(IJ)))520,530,530
  520 BIGA=CF(IJ)
      IMAX=I
  530 CONTINUE
      IF(CDABS(BIGA)-TOL)535,535,540
  535 KS=1
      RETURN
  540 II=J+4*(J-2)
      IT=IMAX-J
      DO 550 K=J,4
      I1=I1+4
      I2=I1+IT
      SAVE=CF(I1)
      CF(I1)=CF(I2)
      CF(I2)=SAVE
  550 CF(I1)=CF(I1)/BIGA
      SAVE=RHS(IMAX)
      RHS(IMAX)=RHS(J)
      RHS(J)=SAVE/BIGA
      IF(J-4)555,570,555
  555 IQS=4*(J-1)
      DO 565 IX=JY,4
      IXJ=IQS+IX
      IT=J-IX
```

```
      DO 560 JX=JY,4
      IXJX=4*(JX-1)+IX
      JJX=IXJX+IT
560   CF(IXJX)=CF(IXJX)-(CF(IXJ)*CF(JJX))
565   RHS(IX)=RHS(IX)-(RHS(J)*CF(IXJ))
570   NY=3
      IT=16
      DO 580 J=1,NY
      IA=IT-J
      IB=4-J
      IC=4
      DO 580 K=1,J
      RHS(IB)=RHS(IB)-CF(IA)*RHS(IC)
      IA=IA-4
580   IC=IC-1
      RETURN
      END
```

```
C
C  A__N_M_
C
C  THIS PROGRAM SEARCHS FOR THE RESONANT FREQUENCY,BASED ON
C  SHIFT IN PHASE ANGLE
C     N5=NUMBER OF INCREMENTS DESIRED IN AXIAL COORDINATE
C     N9=NUMBER OF FREQUENCIES USED BEFORE CUTOFF IF NO
C        PHASE SHIFT IS ENCOUNTERED
C     N11=NUMBER OF SETS OF FLEXIBILITY COEFFICIENTS TO BE USED
      COMPLEX*16 P(8),PP(8),A(4),B(4),C(4),AP(4),BP(4),CP(4),
     2CPP(4),CPPP(4),CF(4,4),RHS(4),UXQP,WXQP,WXQPP,WXQPPP,VXQP,
     3FSW(8,4),FSU(8,4),FSV(8,4),QSW(8),QSU(8),QSV(8),
     4SSTAR(8),FLXCF(4,4),DENOM,S1,S2,WXQ,UXQ,VXQ,SAVE,BIGA
      COMPLEX*16 T11,T12,T13,T21,T22,T23,T31,T32,T33
      COMPLEX*16 M11,M12,M13,M21,M22,M23,M31,M32,M33
      COMPLEX WNX,UNX,VNX
      REAL*8 SCOF(9),COF(9),ROOTR(8),ROOTI(8)
      REAL MU,MU1,MU2
      DIMENSION PHA(100)
  500 FORMAT(1H1,10X,'THICKNESS=',F8.2,'  INCHES',/,
     211X,'LENGTH   =',F8.2,'  INCHES',/,
     311X,'RADIUS   =',F8.2,'  INCHES',/,
     411X,'LOAD     =',F8.2,'  POUNDS',/,
     511X,'XO       =',F8.2,'  INCHES',/,11X,'CN       =',F8.2)
  501 FORMAT(11X,'FREQ  =',F8.2,'  CPS',/,11X,'RN    =',E10.2,/,
     211X,'RSQ   =',E10.2,/,11X,'Q     =',E10.2)
  506 FORMAT(10X,'UNO    =',2E12.4,/,10X,'VNPO   =',2E12.4,/,
     210X,'WNPO   =',2E12.4,/,10X,'WNPPPO =',2E12.4)
  507 FORMAT(10X,'KS = ',F8.2)
  508 FORMAT(3X,'XBAR=',F9.3,3X,'WNXM=',E12.4,3X,'PHASE1=',
     2F8.2,3X,'WNX=',2E12.4)
  509 FORMAT(5X,'EIR=',F8.2)
  510 FORMAT(2E12.4)
  515 FORMAT(D18.4,D12.4)
  516 FORMAT(2E10.2,2X,2E10.2,2X,2E10.2,2X,2E10.2)
      EL=3.0
```

```
      RAD=3.0
      H=0.15
      N5=20
      NL=N5+1
      N9=40.0
      N10=0
      N11=10
      X0=1.5
      CN=3.0
      QL=10.0
      MU=0.3
      E=30.0E+C6
      RHO=0.731E-C3
      XBARM=EL/RAD
      X0=X0/RAD
      MU1=(1.0-MU)/2.0
      MU2=(1.0+MU)/2.0
      Q=-0.29*RAD*QL/(E*H)
      RSQ=(H**2)/(12.0*(RAD**2))
      CN2=CN**2
      CN3=CN**3
      CN4=CN**4
      CN5=CN**5
      CN6=CN**6
      CN7=CN**7
      CN8=CN**8
      READ(C5,515)((FLXCF(I,J),I=1,4),J=1,4)
   01 FREQ=2000.0
      FREQ2=1CCC.C
      N8=0
      NT=0
   03 IF(NT)04,C4,05
   05 FREQ2=FREQ2/2.0
      IF(FREQ2-1.C)240,240,04
   04 FREQ=FREQ+FREQ2
      N4=0
```

```
      N8=N8+1
      RN=RHC*35.8*(FREQ**2)*(RAD**2)/E
      RN2=RN**2
      RN3=RN**3
      SCOF(1)=MU1*CN8*RSQ-(1.0+MU1)*CN6*RN*RSQ+CN4*RN2*RSQ
     2-MU1*CN4*RN+(1.0+MU1)*CN2*RN2-MU1*CN2*RN+RN2-RN3
      SCOF(2)=0.0
      SCOF(3)=-4.0*MU1*CN6*RSQ+3.0*(1.0+MU1)*CN4*RN*RSQ
     2+2.0*MU1*CN2*RN-2.0*CN2*RN2*RSQ+(1.0+MU1-MU**2)*RN
     3-(1.0+MU1)*RN2
      SCOF(4)=0.0
      SCOF(5)=6.0*MU1*CN4*RSQ-3.0*(1.0+MU1)*CN2*RN*RSQ
     2+RN2*RSQ-MU1*RN+MU1*(1.0-MU**2)
      SCOF(6)=0.0
      SCOF(7)=((1.0+MU1)*RN-4.0*MU1*CN2)*RSQ
      SCOF(8)=0.0
      SCOF(9)=MU1*RSQ
      CALL POLRT(SCOF,COF,8,ROOTR,ROOTI,EIR)
      IF(EIR)11,11,10
   10 WRITE(06,509)EIR
   11 WRITE(06,500)H,EL,RAD,QL,XO,CN
      WRITE(06,501)FREQ,RN,RSQ,Q
      WRITE(06,516)((FLXCF(I,J),I=1,4),J=1,4)
      DO 24 I=1,8
      SSTAR(I)=DCMPLX(ROOTR(I),ROOTI(I))
   24 WRITE(06,510)SSTAR(I)
      DO 30 K=1,8
      S1=SSTAR(K)
      S2=S1**2
      DENOM=DCMPLX(1.0D0,0.0D0)
      DO 14 L=1,8
      IF(L-K)12,14,12
   12 DENOM=(SSTAR(K)-SSTAR(L))*DENOM
   14 CONTINUE
      DENOM=DENOM*SCOF(9)
      T11=S2+RN-MU1*CN2
```

```
      T12=MU2*CN*S1
      T13=MU*S1
      T21=-T12
      T22=MU1*S2+RN-CN2
      T23=-CN
      T31=T13
      T32=CN
      T33=RSQ*((S2-CN2)**2)+1.0-RN
      M11=T22*T33-T32*T23
      M12=T31*T23-T21*T33
      M13=T21*T32-T31*T22
      M21=T32*T13-T12*T33
      M22=T11*T33-T31*T13
      M23=T31*T12-T11*T32
      M31=T12*T23-T22*T13
      M32=T21*T13-T11*T23
      M33=T11*T22-T21*T12
      FSW(K,1)=(S1*M13-MU2*CN*M23+MU*M33)/DENOM
      FSW(K,2)=MU1*M23/DENOM
      FSW(K,3)=(S2-2.0*CN2)*RSQ*M33/DENOM
      FSW(K,4)=FSQ*M33/DENOM
      QSW(K)=-Q*M33/DENOM
      FSV(K,1)=(S1*M12-MU2*CN*M22+MU*M32)/DENOM
      FSV(K,2)=MU1*M22/DENOM
      FSV(K,3)=(S2-2.0*CN2)*RSQ*M32/DENOM
      FSV(K,4)=FSQ*M32/DENOM
      QSV(K)=-Q*M32/DENOM
      FSU(K,1)=(S1*M11-MU2*CN*M21+MU*M31)/DENOM
      FSU(K,2)=MU1*M21/DENOM
      FSU(K,3)=(S2-2.0*CN2)*RSQ*M31/DENOM
      FSU(K,4)=FSQ*M31/DENOM
   30 QSU(K)=-Q*M31/DENOM
   36 XBAR=XBARM*(1.0-(1.0/N5)*N4)
      N4=N4+1
      DO 40 M=1,8
   40 P(M)=CDEXP(SSTAR(M)*XBAR)
```

```
      DO 50 N=1,4
      C(N)=DCMPLX(0.0D0,0.0D0)
      A(N)=DCMPLX(0.0D0,0.0D0)
      B(N)=DCMPLX(0.0D0,0.0D0)
      DO 50 II=1,8
      C(N)=C(N)+FSW(II,N)*P(II)
      A(N)=A(N)+FSU(II,N)*P(II)
   50 B(N)=B(N)+FSV(II,N)*P(II)
      IF(XBAR-XC)170,60,60
   60 DO 70 JJ=1,8
   70 PP(JJ)=CDEXP(SSTAR(JJ)*(XBAR-XO))
      WXQ=DCMPLX(0.0D0,0.0D0)
      UXQ=DCMPL>(C.0D0,0.0D0)
      VXQ=DCMPLX(C.0D0,0.0D0)
      DO 80 KK=1,8
      WXQ=WXQ+QSW(KK)*PP(KK)
      UXQ=UXQ+QSU(KK)*PP(KK)
   80 VXQ=VXQ+QSV(KK)*PP(KK)
      IF(XBAR-XEARM)170,90,90
   90 DO 100 LL=1,4
      AP(LL)=DCMPLX(0.0D0,0.0D0)
      BP(LL)=DCMPLX(0.0D0,0.0D0)
      CP(LL)=DCMPLX(0.0D0,0.0D0)
      CPP(LL)=DCMPLX(0.0D0,0.0D0)
      CPPP(LL)=DCMPLX(0.0D0,0.0D0)
      DO 100 MM=1,8
      AP(LL)=AP(LL)+SSTAR(MM)*FSU(MM,LL)*P(MM)
      BP(LL)=BP(LL)+SSTAR(MM)*FSV(MM,LL)*P(MM)
      CP(LL)=CP(LL)+SSTAR(MM)*FSW(MM,LL)*P(MM)
      CPP(LL)=CPP(LL)+(SSTAR(MM)**2)*FSW(MM,LL)*P(MM)
  100 CPPP(LL)=CPPP(LL)+(SSTAR(MM)**3)*FSW(MM,LL)*P(MM)
      UXQP=DCMPLX(0.0D0,0.0D0)
      VXQP=DCMPLX(0.0DC,0.0D0)
      WXQP=DCMPLX(0.0D0,0.0D0)
      WXQPP=DCMPLX(0.0D0,0.0D0)
      WXQPPP=DCMPLX(0.0D0,0.0D0)
```

```
      DO 110 NN=1,8
      UXQP=UXQP+SSTAR(NN)*QSU(NN)*PP(NN)
      VXQP=VXQP+SSTAR(NN)*QSV(NN)*PP(NN)
      WXQP=WXQP+SSTAR(NN)*QSW(NN)*PP(NN)
      WXQPP=WXQPP+(SSTAR(NN)**2)*QSW(NN)*PP(NN)
  110 WXQPPP=WXQPPP+(SSTAR(NN)**3)*QSW(NN)*PP(NN)
      DO 150 M=1,4
      CF(1,M)=-A(M)+FLXCF(1,1)*AP(M)+MU*CN*FLXCF(1,1)*B(M)
     2+(MU*FLXCF(1,1)+MU*CN2*RSQ*FLXCF(1,4))*C(M)
     3+(2.0-MU)*CN2*RSQ*FLXCF(1,3)*CP(M)
     4-RSQ*FLXCF(1,4)*CPP(M)-RSQ*FLXCF(1,3)*CPPP(M)
      CF(2,M)=-B(M)-MU1*CN*FLXCF(2,2)*A(M)+MU1*FLXCF(2,2)*BP(M)
      CF(3,M)=-C(M)+FLXCF(3,1)*AP(M)+MU*CN*FLXCF(3,1)*B(M)
     2+(MU*FLXCF(3,1)+MU*CN2*RSQ*FLXCF(3,4))*C(M)
     3+(2.0-MU)*CN2*RSQ*FLXCF(3,3)*CP(M)
     4-RSQ*FLXCF(3,4)*CPP(M)-RSQ*FLXCF(3,3)*CPPP(M)
  150 CF(4,M)=-CP(M)+FLXCF(4,1)*AP(M)+MU*CN*FLXCF(4,1)*B(M)
     2+(MU*FLXCF(4,1)+MU*CN2*RSQ*FLXCF(4,4))*C(M)
     3+(2.0-MU)*CN2*RSQ*FLXCF(4,3)*CP(M)
     4-RSQ*FLXCF(4,4)*CPP(M)-RSQ*FLXCF(4,3)*CPPP(M)
      RHS(1)=UXQ-FLXCF(1,1)*UXQP-MU*CN*FLXCF(1,1)*VXQ
     2-(MU*FLXCF(1,1)+MU*CN2*RSQ*FLXCF(1,4))*WXQ
     3-(2.0-MU)*CN2*RSQ*FLXCF(1,3)*WXQP
     4+RSQ*FLXCF(1,4)*WXQPP+RSQ*FLXCF(1,3)*WXQPPP
      RHS(2)=VXQ+MU1*CN*FLXCF(2,2)*UXQ-MU1*FLXCF(2,2)*VXQP
      RHS(3)=WXQ-FLXCF(3,1)*UXQP-MU*CN*FLXCF(3,1)*VXQ
     2-(MU*FLXCF(3,1)+MU*CN2*RSQ*FLXCF(3,4))*WXQ
     3-(2.0-MU)*CN2*RSQ*FLXCF(3,3)*WXQP
     4+RSQ*FLXCF(3,4)*WXQPP+RSQ*FLXCF(3,3)*WXQPPP
      RHS(4)=WXQP-FLXCF(4,1)*UXQP-MU*CN*FLXCF(4,1)*VXQ
     2-(MU*FLXCF(4,1)+MU*CN2*RSQ*FLXCF(4,4))*WXQ
     3-(2.0-MU)*CN2*RSQ*FLXCF(4,3)*WXQP
     4+RSQ*FLXCF(4,4)*WXQPP+RSQ*FLXCF(4,3)*WXQPPP
      KKK=4
      CALL CSIMQ(CF,RHS,KKK,KS)
      IF(KS)162,162,160
```

```
  160 WRITE(06,507)KS
  162 WRITE(06,506)(RHS(NW),NW=1,4)
  170 IF(XBAR-XC)190,180,180
  180 WNX=WXC
      DO 182 I=1,4
  182 WNX=WNX+RHS(I)*C(I)
      GO TO 200
  190 WNX=CMPLX(0.0,0.0)
      DO 192 I=1,4
  192 WNX=WNX+RHS(I)*C(I)
  200 WNXM=CABS(WNX)
      PHASE1=57.29578*ATAN2(AIMAG(WNX),REAL(WNX))
      IF(N4-18)210,204,210
  204 PHA(N8)=ABS(PHASE1)
  210 WRITE(06,508)XBAR,WNXM,PHASE1,WNX
      IF(N4-NL)36,220,220
  220 IF(N8-1)228,228,222
  222 IF(ABS(PHA(N8)-PHA(N8-1))-20.0)228,228,224
  224 FREQ=FREQ-FREQ2
      NT=1
      N8=N8-1
      GO TO 05
  228 IF(N8-N9)03,250,250
  240 FLXCF(2,2)=FLXCF(2,2)*10.0
      N10=N10+1
      IF(N10-N11)01,01,250
  250 CALL EXIT
      END
C
C  SUBROUTINES FCLRT AND CSIMQ AS LISTED IN AIJSUM ARE REQUIRED
C
```