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DEEP POINTS OF CLUSTER VARIETIES

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DEEP POINTS OF CLUSTER VARIETIES

A DISSERTATION APPROVED FOR THE

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BY THE COMMITTEE CONSISTING OF

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Contents

Abstract

A cluster algebra determines an associated cluster variety, and each cluster determines a cluster torus in that variety. The cluster tori sometimes cover the entire cluster variety, but not always; we refer to any points not contained in any cluster torus as deep points. Any geometrically interesting points in the variety (e.g. singularities) must be deep, and many pathologies of the cluster algebra can be localized to those deep points.

We describe the deep points of cluster algebras of unpunctured polygons, unpunctured marked surfaces, punctured polygons, and punctured surfaces with at least two boundary marked points. As a consequence, we classify the deep points of cluster algebras of types A_n and D_n. We also classify the deep points of the Markov cluster algebra —the cluster algebra of the once-punctured torus —and its upper cluster algebra.

We then examine the deep points of skew-symmetrizable cluster algebras that arise from a folding of an acyclic quiver by a finite group of automorphisms. In particular, we study the deep points of cluster algebras of types B_n, C_n, and F_4 via foldings of quivers of types A_{2n-1}, D_{n+1} and E_6, respectively.

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"I cannot think of the deep sea without shuddering at the nameless things that may at this very moment be crawling and floundering on its slimy bed, worshipping their ancient stone idols and carving their own detestable likenesses on submarine obelisks of water-soaked granite. I dream of a day when they may rise above the billows to drag down in their reeking talons the remnants of puny, war-exhausted mankind —of a day when the land shall sink, and the dark ocean floor shall ascend amidst universal pandemonium."

– H. P. Lovecraft, "Dagon"

Chapter 1

Introduction

Cluster algebras are commutative domains with distinguished sets of generators called *cluster variables*, in which the full set of cluster variables can be recovered from *mutation* of overlapping n-element subsets called *clusters*. Cluster algebras were introduced by Fomin and Zelevinsky in a series of papers, the first of which was published in 2002 [FZ02]. They were primarily motivated by the study of dual canonical bases of Lie algebras, but cluster algebras have since been found to have connections to many areas of mathematics and physics. In particular, cluster algebras have been found in the coordinate rings of of many interesting varieties, such as Grassmannians [FZ02, Sco06], Teichmüller spaces [GSV05, FST08, FG06], and affine log Calabi-Yau varieties [GHK15, GHKK18].

For each cluster algebra $\mathcal{A}$ and field $\Bbbk$, there is an associated **cluster variety (of $\Bbbk$-points of $\mathcal{A}$)**, defined as

$$V(\mathcal{A}, \Bbbk) := \mathrm{Hom}_{\mathrm{Ring}}(\mathcal{A}, \Bbbk) = \{\text{ring homomorphisms } p\colon \mathcal{A} \to \Bbbk\}.$$

The above examples can be recovered in this way, as well as many other varieties.

One of the foundational properties of cluster algebras is the *Laurent phenomenon*, which states that every cluster variable of a cluster algebra $\mathcal{A}$ can be written as a Laurent polynomial in the cluster variables of any given cluster; that is, given a cluster $\{x_1, \ldots, x_n\}$ of $\mathcal{A}$, there is an inclusion

$$\mathcal{A} \hookrightarrow \mathbb{Z}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right].$$

Applying $\mathrm{Hom}_{\mathsf{Ring}}(-, \Bbbk)$ to this inclusion gives an open inclusion of varieties

$$(\Bbbk^\times)^n \cong \mathrm{Hom}_{\mathsf{Ring}}\left(\mathbb{Z}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right], \Bbbk\right) \hookrightarrow \mathrm{Hom}_{\mathsf{Ring}}(\mathcal{A}, \Bbbk) = V(\mathcal{A}, \Bbbk).$$

The image of $(\Bbbk^\times)^n$ under this inclusion is called the **cluster torus** of the cluster $\{x_1, \ldots, x_n\}$; this is the subset of $V(\mathcal{A}, \Bbbk)$ on which none of the variables $x_1, \ldots, x_n$ are killed (i.e. sent to zero).

In some cases, the cluster variety $V(\mathcal{A}, \Bbbk)$ is covered by these cluster tori, but not always. We refer to any points that are not contained in any cluster torus as **deep points**. Explicitly, a deep point is a ring homomorphism $\mathcal{A} \to \Bbbk$ that kills at least one cluster variable in every cluster. The set of all such points, called the **deep locus**, forms a closed (possibly empty) subvariety of $V(\mathcal{A}, \Bbbk)$.

1.1 Deep points and defects

While many of the cluster algebras we will discuss are well-behaved, this is certainly not always the case. In [BM25], the following maxim was proposed:

Any bad property of a cluster algebra can be blamed on its deep points.

For example, Proposition 3.2.2 says that any singular point of a cluster variety is necessarily deep.

As a contrapositive to the maxim, if a cluster algebra has no deep points, then it should be well-behaved. For instance, the following theorem says that such a cluster algebra has many desirable properties.

Theorem 3.2.5. *Let $\mathcal{A}$ be a cluster algebra with no deep points for all $\Bbbk$. Then*

1. *$\mathcal{A}$ is non-singular.*
2. *$\mathcal{A}$ is finitely generated.*
3. *$\mathcal{A}$ is equal to its upper cluster algebra.*
4. *$\mathcal{A}$ is a locally acyclic cluster algebra.*

The converse is generally not true, however —a cluster algebra can have deep points and still be well-behaved. For example, all four properties from Theorem 3.2.5 hold for the cluster algebra of the hexagon with boundary coefficients, which has deep points by Theorem 4.3.1.

Despite this, in classifying the deep points of cluster algebras, we are identifying the points where bad behavior *could* arise. As cluster algebras continue to find applications in diverse areas of mathematics and physics, locating potentially problematic points could prove to be very useful.

1.2 A first example - types A_2 and A_3

As a first example, the cluster variables of a type A_2 cluster algebra can be represented by the diagonals of a pentagon, as depicted in Figure 1.1. This representation also encodes the relations between the cluster variables via the *Ptolemy relation* depicted in Figure 1.2, with the boundary edges taking the value of 1. Two cluster variables appear in the same cluster if the corresponding diagonals appear in the same triangulation; together, we refer to the cluster and the corresponding triangulation of the pentagon as a *seed*.

Two seeds are related by a single mutation if the triangulations differ by a *flip* of a single diagonal. As implied by this figure, the type A_2 cluster algebra has a total of five cluster variables, with each cluster variable appearing in exactly two

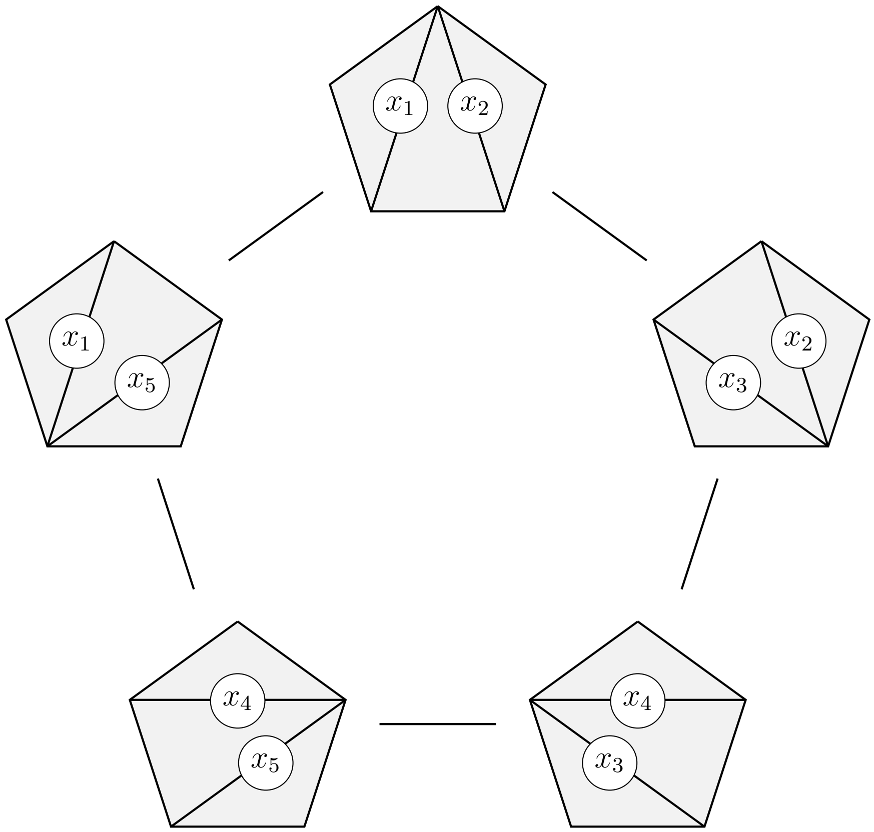


Figure 1.1: A seed pattern for a type A_2 cluster algebra.

of the five clusters. The cluster algebra is generated by these five cluster variables, but the full set of cluster variables can be recovered from any one seed, and is independent of which initial seed is chosen.

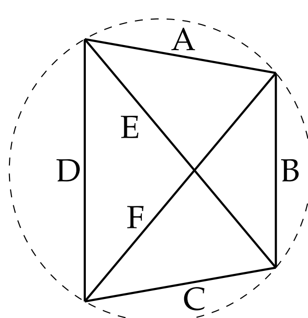


Figure 1.2: A quadrilateral with Ptolemy relation $EF = AC + BD$

If a deep point were to exist in this cluster algebra, it would have to kill at least one diagonal in every triangulation. Suppose we send x_1 to 0; the Ptolemy relation $x_1x_3 = x_2 + 1$ then forces x_2 to be sent to -1. Since a deep point must kill at least one diagonal in each triangulation, x_3 must also be sent to 0, which similarly forces x_4 to be sent to -1. By the same logic, x_5 must be sent to 0; this, however, gives us a contradiction as the Ptolemy relation $x_4x_1 = x_5 + 1$ is not satisfied when both x_1 and x_5 are killed. The choice to kill x_1 was arbitrary, so the type A_2 cluster algebra has no deep points.

The type A_3 cluster algebra does have a deep point, however. This deep point is realized by sending the diagonals of a hexagon to the values indicated in Figure 1.3. One can easily see that any triangulation of the hexagon must include at least one of the arcs that are sent to zero. In fact, Theorem 4.3.5 says that type A_n cluster algebras have a unique deep point if $n \equiv 3 \bmod 4$, or if n is odd and $\mathrm{char}(\mathbb{k}) = 2$; otherwise, the deep locus is empty.

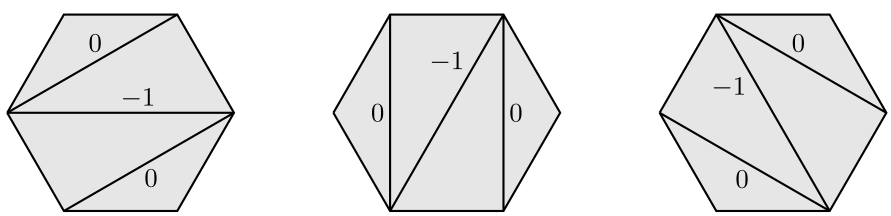
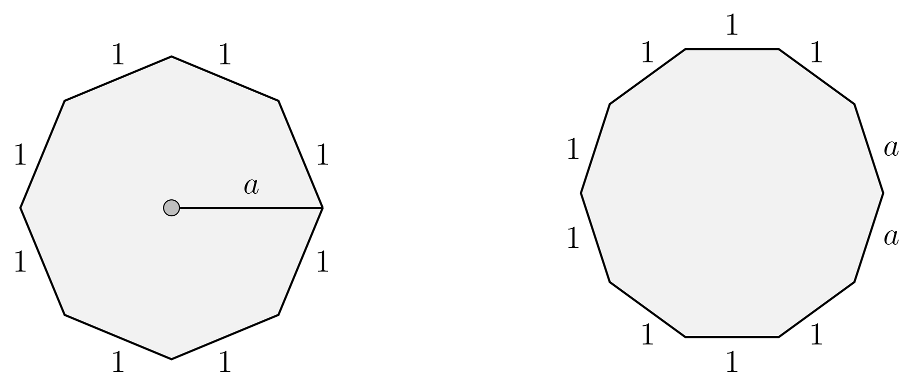


Figure 1.3: The unique deep point in the coefficient-free A_3 cluster algebra

1.3 A second example - type D_8

In another example, the cluster variables of a type D_8 cluster algebra can be represented by the arcs of a once-punctured octagon. In contrast to the unpunctured case, there are triangulations of punctured polygons where an arc cannot be flipped, so we use *tagged triangulations* instead. The Ptolemy relations still hold, but there are now more relations to consider as well (see Section 5.1).

Two cluster variables appear in the same cluster if the corresponding tagged arcs appear in the same tagged triangulation; together, a cluster and its associated tagged triangulation form a seed. Two seeds are related by a single mutation if the tagged triangulations differ by a flip of a single tagged arc. As before, the full set of cluster variables can be recovered from any one seed, and is independent of which initial seed is chosen.



(a) once-punctured octagon

(b) Unpunctured Decagon

Figure 1.4: Cutting of a punctured octagon along a radius.

If $p \in V(\mathcal{A}(D_8), \Bbbk)$ is deep and sends a *plain* radius to a non-zero value $a \in \Bbbk^\times$ as pictured in Figure 1.4a, we can cut along this radius, resulting in the unpunctured decagon in Figure 1.4b. Thus, p corresponds to a deep point of $V(\mathcal{A}(\Delta_{10}), \Bbbk)$ which takes appropriate values on the boundary variables. From Theorem 4.3.1, this gives us a unique deep point that is completely determined by the value of a. Since a can be any non-zero value, this gives us a deep component isomorphic to $\Bbbk^\times$. The same process applied to a $45°$ rotation of the octagon gives another deep copy of $\Bbbk^\times$, as does a change of tagging of the radius at the puncture. As we will show in Chapter 7, every deep point in which a radius is spared is contained in one of these four distinct copies of $\Bbbk^\times$.

Not all deep points of $V(\mathcal{A}(D_8), \Bbbk)$ correspond to deep points of an unpunctured decagon, however. There are also what we refer to as **(totally) radial points**, which kill all radii, regardless of tagging. Every triangulation of the punctured octagon contains at least two radii, so totally radial points are necessarily deep. As noted in Remark 7.4.2, the totally radial points comprise a 5-dimensional component of the deep locus of $V(\mathcal{A}(D_8), \Bbbk)$.

Despite the fact that this five-dimensional component does not intersect the four copies of $\Bbbk^\times$, the union of the five sets is connected. In each of the copies of $\Bbbk^\times$, as the value of the radius approaches zero, they approach a common totally radial point. From work of Benito, Faber, Mourtada, and Schober, we know that the deep locus is singular and isomorphic to the union of the 5 dimensional component and four coordinate axes, with the five sets intersecting at a single point [BFMS23, Theorem A.(4)(c)].

1.4 Outline

The structure of this dissertation is as follows:

- In Chapter 2, we formally define cluster algebras and describe some properties of cluster algebras that will be of use later in the dissertation.
- In Chapter 3, we define cluster varieties and deep points, and discuss some algebraic consequences of the existence or non-existence of deep points.
- In Chapter 4, we define the cluster algebra of a polygon and give a classification of the corresponding deep points. In particular, deep points can only exist when the number of sides n of the polygon Δ_n is even.

  Theorem 4.3.1. *Let* $n \geq 3$.

1. *If n is odd, then the cluster variety $V(\mathcal{A}(\Delta_n), \Bbbk)$ has no deep points.*
2. *If n is even, then the deep locus of $V(\mathcal{A}(\Delta_n), \Bbbk)$ is isomorphic to $(\Bbbk^\times)^{n-1}$.*

By specializing the boundary variables to 1, we then obtain a classification of deep points for cluster algebras of type A_n.

Theorem 4.3.5. *If $n \equiv 3 \bmod 4$ (or $n \equiv 1 \bmod 4$ and $\mathrm{char}(\Bbbk) = 2$), then the cluster variety $V(\mathcal{A}(A_n), \Bbbk)$ has a unique deep point. For all other choices of n and $\Bbbk$, the cluster variety $V(\mathcal{A}(A_n), \Bbbk)$ has no deep points.*

- In Chapter 5, we define the cluster algebra of a surface and describe its relationship to the commutative Kauffman bracket skein algebra of the surface, and discuss cutting and gluing of surfaces.
- In Chapter 6, we extend the results of Chapter 4 to more general unpunctured marked surfaces with boundary. In this case, deep points can only exist if the total number of marked points on the boundary m is even.

Theorem 6.1.15. *Let Σ be a connected, unpunctured, triangulable marked surface with genus g, b-many boundary components, and m-many marked points.*

1. *If m is odd, then $V(\mathcal{A}(\Sigma), \Bbbk)$ has no deep points.*
2. *If m is even, then the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ consists of 2^{2g+b-1}-many disjoint copies of $(\Bbbk^\times)^{2g+b+m-2}$.*

We also show that there are several equivalent combinatorial characterizations of these deep points.

Theorem 6.2.1. *Let Σ be a connected, unpunctured, triangulable marked surface with an even number of marked points. Then the following sets are in bijection.*

1. *Components of the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$, for a fixed field $\Bbbk$.*
2. *Polygonal dissections of Σ, up to congruence.*

3. *Relative cohomology classes in $H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$ with value 1 on the boundary arcs.*
4. *Vanishing classes of Σ; that is, sets of simple, non-boundary marked arcs in Σ which contain an odd number of arcs in every triangle.*
5. *Deep points in $V(\mathcal{A}(\Sigma), \mathbb{Z}_2)$.*
6. *Subsets of D, for a fixed polygonal dissection D of Σ.*
7. *Alternating double covers of Σ; that is, double covers of Σ with an alternating coloring of their marked points, such that the non-trivial deck transformation swaps colors.*

- In Chapter 7, we classify the deep points of once-punctured disks (or polygons). After examining once-punctured bigons, trigons, and quadrilaterals individually, we obtain the following classification for once-punctured disks with $n \geq 5$ boundary marked points.

  Theorem 7.4.1. *Let Σ be a once-punctured disk with $n \geq 5$ marked points on the boundary.*

  1. *If n is odd, then the deep locus is a single component of dimension $2n - 3$.*
  2. *If n is even, then the deep locus contains one component of dimension $2n - 3$ and four distinct subsets of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $(\Bbbk^\times)^n$. The intersection of the closures of these five sets is an $(n-1)$-dimensional subvariety.*

  Specializing the boundary variables to 1, we then obtain a classification of deep points for cluster algebras of type D_n, which is stated in Remark 7.4.2. We also introduce a forgetful map, which gives us the following inclusions.

  Theorem 7.5.2. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary. The forgetful map $\mathcal{F}\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Delta_n)$ induces inclusions of:*

- *the cluster variety $V(\mathcal{A}(\Delta_n), \Bbbk)$ into the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$, and*
- *the cluster variety $V(\mathcal{A}(A_{n-3}), \Bbbk)$ into the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$.*

- In Chapter 8, we extend the results of Chapter 7 to more general punctured marked surfaces with boundary, classifying the deep points of cluster algebras of punctured disks with at least one boundary marked point and punctured surfaces with at least two boundary marked points.

  We again introduce a forgetful map, giving us the following inclusions when there are at least two marked points on the same boundary component.

  Theorem 8.3.1. *Let Σ_q be a connected surface with genus $g \geq 0$, $b \geq 1$ boundary components, q punctures, and $m \geq 2$ marked points on the boundary, with at least 2 of the marked points on the same boundary component. Repeated applications of the forgetful map*

  $$\mathcal{F} \colon \mathcal{A}(\Sigma_q) \to \mathcal{A}(\Sigma_{q-1})$$

  induce inclusions of the cluster varieties

  $$V(\mathcal{A}(\Sigma_j), \Bbbk) \text{ into the deep locus of } V(\mathcal{A}(\Sigma_q), \Bbbk) \text{ for all } 0 \leq j < q.$$

  In particular, there is an inclusion of the cluster variety of the unpunctured surface, $V(\mathcal{A}(\Sigma_0), \Bbbk)$, into the deep locus of the cluster variety of the punctured surface, $V(\mathcal{A}(\Sigma_q), \Bbbk)$.

  Finally, we classify the deep points of the cluster algebra of the once-punctured torus (without boundary), more commonly known as the Markov cluster algebra, as well as the deep points of its upper cluster algebra.

- In Chapter 9, we consider cluster algebras that arise from a folding of an acyclic quiver by a finite group of automorphisms. We show that the deep locus of the type A_{2n-1} cluster algebra embeds in the deep locus of the type

B_n cluster algebra. We then show that there is a unique deep point of the type D_4 cluster algebra that embeds into the deep locus of the type G_2 cluster algebra, and that the totally radial deep points of the type D_{n+1} cluster algebra embed into the deep locus of the type C_n cluster algebra.

Chapter 2

Background

A defining characteristic of a *cluster algebra* is that the algebra is generated by overlapping n-element sets of *cluster variables* called *clusters*, and these clusters are related by a combinatorial rule called *mutation*. This mutation rule is generally encoded in a matrix, but in many cases, it is easier to think of a cluster algebra in terms of a combinatorial object that encodes the relationships between clusters. We begin our discussion with what is probably the simplest such object —the *quiver*.

2.1 Quiver mutation

A **quiver** is a tuple $Q \coloneqq (Q_0, Q_1, s, t)$ consisting of a set of *vertices* Q_0, a set of *arrows* Q_1, a function $s\colon Q_1 \to Q_0$ that assigns a *source* vertex to each arrow, and a function $t\colon Q_1 \to Q_0$ that assigns a *target* vertex to each arrow. We need not distinguish between arrows with the same source and target, so for any arrow $\alpha \in Q_1$ with $s(\alpha) = i$ and $t(\alpha) = j$, we will typically refer to it simply as an arrow $i \to j$. Additionally, we require that $s(\alpha) \neq t(\alpha)$ for all $\alpha \in Q_1$, and if $t(\alpha_1) = s(\alpha_2)$ for some pair of arrows $\alpha_1, \alpha_2 \in Q_1$, then $s(\alpha_1) \neq t(\alpha_2)$; that is, no arrow can begin

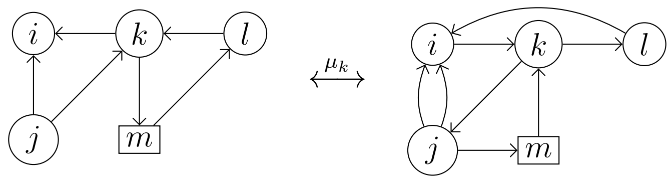


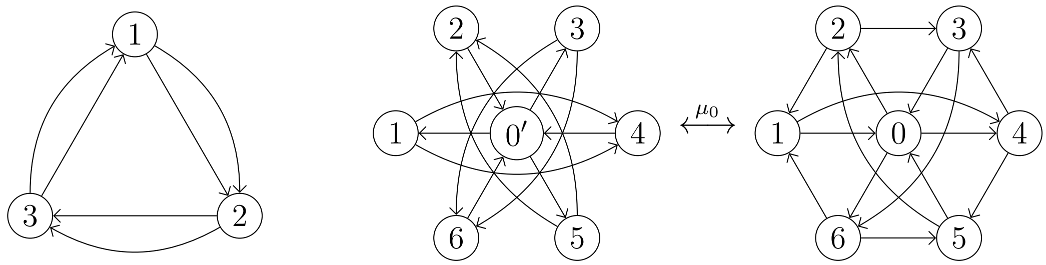
Figure 2.1: Quiver mutation at the vertex k.

and end at the same vertex, and if there is an arrow $i \to j$ then there cannot be an arrow $j \to i$.

Let $E \subseteq Q_0$ denote the set of *mutable* vertices and $F := Q_0 \setminus E$ denote the set of *frozen* vertices. When drawing the quiver as a directed graph, we will denote frozen vertices with a rectangle around the vertex label; for example, the vertex m in Figure 2.1 is frozen. Given any quiver Q, a new quiver $Q' := (Q_0, Q_1', s', t')$ can be produced through **mutation** at any mutable vertex $k \in E$. Quiver mutation at a vertex k, which we denote as μ_k, consists of the following three steps, the results of which are depicted in Figure 2.1:

1. For any sequence of arrows $i \to k \to j$, add an arrow $i \to j$.
2. Remove any 2-cycles created by the addition of arrows in the first step.
3. Reverse the orientation of any arrow incident to k.

Note that $\mu_k(\mu_k(Q)) = Q$, so quiver mutation is an involution. As a result, quiver mutation is an equivalence relation; we say that any two quivers related by a sequence of mutations are mutation-equivalent. Those equivalence classes with finitely many quivers are called *mutation-finite*; the mutation-finite quivers were fully classified by Fekikson, Shapiro, and Tumarkin in [FST12]. Two particularly small mutation classes are the Markov and X_7 quivers, which are depicted in Figure 2.2.



(a) The Markov quiver.   (b) The X_7 quivers.

Figure 2.2: Two classes of mutation-finite quivers.

2.2 Matrix mutation

Given a quiver Q as in Section 2.1, let $m := |Q_0|$ denote the total number of vertices of Q and $n := |E|$ denote the number of mutable vertices. Now, we will construct an **extended exchange matrix** $\widetilde{B}(Q) := (b_{ij})$ with m rows and n columns as follows:

1. Rows $1, \ldots, n$ correspond to the mutable vertices.
2. Rows $n+1, \ldots, m$ correspond to the frozen vertices.
3. The entries b_{ij} are equal to the number of arrows $i \to j$ minus the number of arrows $j \to i$.

The first n rows of $\widetilde{B}(Q)$ represent the mutable part of the quiver and are *skew-symmetric* —that is, $b_{ij} = -b_{ji}$ and all entries on the diagonal are zero. The remaining $m - n$ rows correspond to the frozen variables. By convention, we will omit the arrows between frozen variables, so we can encode all of the data of the quiver without adding extra columns to make the matrix square.

Mutation of the matrix $\widetilde{B}(Q)$ at mutable index $1 \leq k \leq n$ (corresponding to a mutable vertex k) produces a new matrix $\mu_k(\widetilde{B}(Q))$ with the following entries:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0; \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0; \\ b_{ij} & \text{otherwise.} \end{cases} \tag{2.1}$$

Note that $\mu_k(\widetilde{B}(Q)) = \widetilde{B}(\mu_k(Q))$; that is, mutation of an extended exchange matrix associated to a quiver at the index k produces the extended exchange matrix associated to the quiver mutated at vertex k. As in the quiver case, matrix mutation is an involution, and therefore also an equivalence relation.

$$\begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xleftrightarrow{\mu_k} \begin{bmatrix} 0 & -2 & 1 & -1 \\ 2 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

Figure 2.3: Matrix mutation at index k of the extended exchange matrices associated to the quivers in Figure 2.1, with rows indexed i, j, k, l, m in alphabetical order.

Given an $m \times n$ matrix whose top $n \times n$ submatrix is skew-symmetric, one can also convert the matrix to a quiver by reverse-engineering the construction of $\widetilde{B}(Q)$ above. However, it is often necessary to work in the more general class of *skew-symmetrizable* matrices. An $n \times n$ matrix $B = (b_{ij})$ is **skew-symmetrizable** if there exists a diagonal matrix D with positive integer diagonal entries d_i such that DB is skew-symmetric —that is, $d_i b_{ij} = -d_j b_{ji}$ for all i and j. An $m \times n$ matrix with $m \geq n$ whose top $n \times n$ submatrix is skew-symmetrizable is called an *extended skew-symmetrizable* matrix. Mutation of an extended skew-symmetrizable matrix B at an index $1 \leq k \leq n$ can be performed exactly as in the skew-symmetric case.

2.3 Cluster algebras

Given an extended exchange matrix $\widetilde{B}$, attach a *cluster variable* x_i to each row, and let $\mathcal{F} \coloneqq \mathbb{Q}(x_1, \ldots, x_m)$ be the field of rational functions on the initial cluster variables $x_1, \ldots, x_m$. Collectively, we refer to the cluster variables associated to an extended exchange matrix as a *cluster* $\mathbf{x} = \{x_1, \ldots, x_m\}$, and the tuple $(\widetilde{B}, \mathbf{x})$ is called a *seed*. Mutation of a seed at a mutable index k produces a new seed $(\mu_k(\widetilde{B}), \mathbf{x}')$, where $\mathbf{x}' = \mathbf{x} \setminus \{x_k\} \cup \{x'_k\}$, according to the exchange relation:

$$x_k\, x'_k = \prod_{1 \le j \le m} x_j^{\max\left(b_{jk},\, 0\right)} + \prod_{1 \le j \le m} x_j^{\max\left(-b_{jk},\, 0\right)}. \tag{2.2}$$

Let $\mathcal{X}$ be the set of all cluster variables produced through repeated mutation at all mutable indices. The **cluster algebra** $\mathcal{A} = \mathcal{A}(\widetilde{B}, \mathbf{x})$ is the $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}$ and the inverses of any frozen variables.

One remarkable property of cluster algebras is the *Laurent phenomenon* [FZ02, Theorem 3.1]: given any initial seed containing a cluster $\mathbf{x} = \{x_1, \ldots, x_m\}$, the cluster variables of all other clusters can be written as Laurent polynomials in this initial cluster; that is, $\mathcal{A} \hookrightarrow \mathbb{Z}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$. As a result, the algebra depends only on the mutation class of $\widetilde{B}$, so we will often omit the details of the seed and simply write $\mathcal{A}$ or $\mathcal{A}(\widetilde{B})$, or $\mathcal{A}(Q)$ if defined by a quiver.

If $\mathcal{A}$ has finitely many seeds —or equivalently, if the set $\mathcal{X}$ is finite —we say the cluster algebra is of *finite-type*. A full classification of finite-type cluster algebras was given by Fomin and Zelevinsky in [FZ03]. Given a quiver Q, $\mathcal{A}(Q)$ is a finite-type cluster algebra if and only if Q is mutation equivalent to a disjoint union of oriented simply-laced Coxeter-Dynkin diagrams [FST08, Theorem 6.5], pictured in Figure 2.4. Further, different orientations of a simply-laced Coxeter-Dynkin diagram produce isomorphic cluster algebras, so we can refer to the cluster algebra simply by the type of the underlying diagram. Somewhat confusingly, mutation-finite quivers do not necessarily produce finite-type cluster algebras. As an example, the Markov quiver in Figure 2.2a gives rise to a cluster algebra that is not finitely generated [BFZ05, Theorem 1.24].

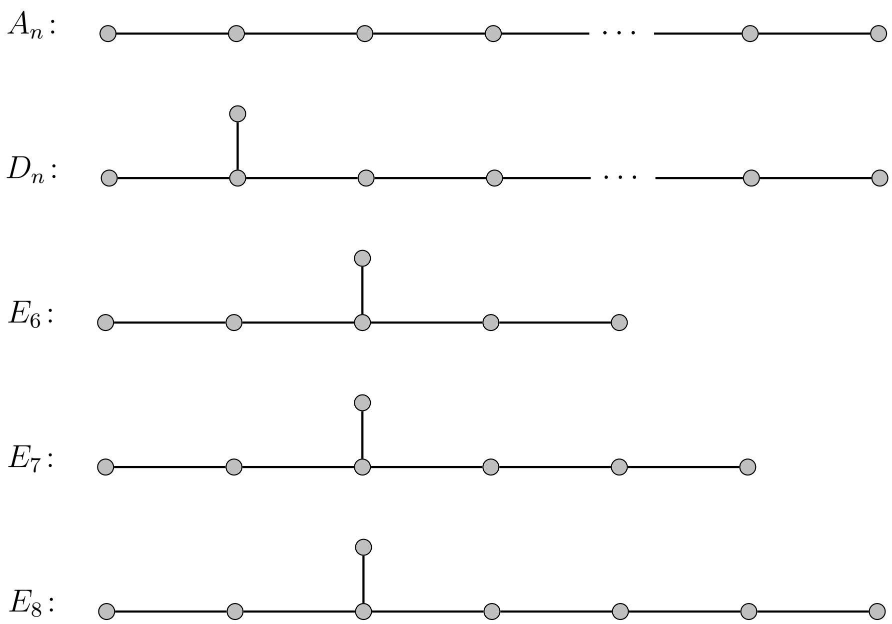


Figure 2.4: Simply-laced Coxeter-Dynkin diagrams.

A related construction is that of the upper cluster algebra, first introduced in [BFZ05]. For each cluster $\mathbf{x}$, we can consider the Laurent algebra generated by those cluster variables $\mathcal{U}_{\mathbf{x}} = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$; the **upper cluster algebra** $\mathcal{U}$ is defined as the intersection of the $\mathcal{U}_{\mathbf{x}}$ over all clusters $\mathbf{x}$; that is,

$$\mathcal{U} = \bigcap_{\text{clusters } \mathbf{x}} \mathcal{U}_{\mathbf{x}} = \bigcap_{\text{clusters } \{x_1,\ldots,x_n\}} \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

For particularly nice cluster algebras, such as *locally acyclic* cluster algebras, $\mathcal{A} = \mathcal{U}$ [Mul14]; for others, such as the Markov cluster algebra associated to the quiver in Figure 2.2a, $\mathcal{A} \subsetneq \mathcal{U}$ [BFZ05, Proposition 1.26].

Coefficients

Depending on the application, it may be necessary to extend the definition of a cluster algebra to include coefficients, as introduced by Fomin and Zelevinsky in [FZ07]. The details of the coefficients can vary, so we will give only a brief introduction here.

In general, given a matrix $\widetilde{B}$ with n columns and $m \geq n$ rows, we treat the top n rows as mutable and the remaining rows as coefficients. Attach a *cluster variable* x_i to each mutable row and a coefficient y_i to all other rows. Collectively, $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a cluster and $\mathbf{y} = \{y_1, \ldots, y_{m-n}\}$ are the coefficients. Together, the tuple $(\widetilde{B}, \mathbf{x}, \mathbf{y})$ is a seed with coefficients.

Let $\mathbb{P}$ be the free abelian group generated by $y_1, \ldots, y_{m-n}$; let $\mathbb{ZP}$ be its integral group ring, the ring of Laurent polynomials in $y_1, \ldots, y_{m-n}$, and let $\mathbb{QP}$ be its field of fractions. Let $\mathcal{F} \coloneqq \mathbb{QP}(x_1, \ldots, x_n)$ be the field of rational functions on the initial cluster variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{QP}$. Additionally, define an auxiliary addition $\oplus$ on $\mathbb{P}$ that is commutative, associative, and distributive over multiplication. The **cluster algebra** $\mathcal{A} = \mathcal{A}(\widetilde{B}, \mathbf{x}, \mathbf{y})$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}$ and the inverses of any frozen variables.

While we have introduced coefficients and frozen variables as distinct, the terms are often used interchangeably in the literature. As a result of the Laurent phenomenon, frozen variables do not appear in the denominators of the Laurent expansions of cluster variables in terms of an initial cluster [FWZ24, Theorem 3.3.6]. Hence, given frozen variables $x_{n+1}, \ldots, x_m$ in a cluster algebra with trivial coefficients, we can instead treat them as coefficients in $\mathbb{ZP} = \mathbb{Z}[x_{n+1}, \ldots, x_m]$. More generally, coefficients should be thought of as a generalization of frozen variables, where the addition and multiplication of $\mathbb{P}$ may differ from those of a traditional polynomial ring.

2.4 Acyclic and locally acyclic cluster algebras

In this section, we will briefly recall some results concerning so-called *acyclic* and *locally acyclic* cluster algebras.

We say that a seed $(Q, \mathbf{x})$ associated to a quiver Q is **acyclic** if Q does not contain any directed cycles. This definition can be extended to a matrix $\widetilde{B} = (b_{ij})$ in the analogous way —a seed $(\widetilde{B}, \mathbf{x})$ is acyclic if there is no sequence of $k > 2$ indices $j_1, j_2, \ldots, j_k = j_1$ such that $b_{j_i\, j_{i+1}} > 0$ for all $1 \leq i < k$. Acyclic cluster algebras have the following nice properties:

Theorem 2.4.1. *[BFZ05, Corollary 1.21] Let $\mathcal{A}$ be the cluster algebra associated with a totally mutable acyclic seed $(\widetilde{B}, \mathbf{x} = \{x_1, \ldots, x_n\})$. Then*

- *$\mathcal{A}$ is generated by $x_1, x_1', \ldots, x_n, x_n'$.*
- *The standard monomials in $x_1, x_1', \ldots, x_n, x_n'$ form a $\mathbb{ZP}$-basis of $\mathcal{A}$.*
- *The mutation relations $x_j x_j' - P_j(\mathbf{x})$ for $1 \leq j \leq n$ generate the ideal $\mathcal{I}$ of relations among the generators $x_1, x_1', \ldots, x_n, x_n'$. (With trivial coefficients, $P_j(\mathbf{x})$ is the right hand side of Equation 2.2.)*
- *These polynomials form a Gröbner basis for $\mathcal{I}$ with respect to any term order in which $x_1', \ldots, x_n'$ are much more expensive than $x_1, \ldots, x_n$.*

In particular, given a totally-mutable acyclic quiver Q with n vertices, $\mathcal{A}(Q)$ is generated by $2n$ variables and n relations. Also, while the above result was initially stated for totally mutable seeds, analogous results hold for quivers with frozen variables, as stated in [Mul13, Lemma 2.3] and [LS23, Theorem 2.4].

Now for a seed $(Q, \mathbf{x})$, let $\{x_1, x_2, \ldots, x_i\} \subset \mathbf{x}$ be a collection of mutable variables; define the **freezing** of $(Q, \mathbf{x})$ at $\{x_1, x_2, \ldots, x_i\}$ to be the same seed, but with the vertices corresponding to the set $\{x_1, x_2, \ldots, x_i\}$ now frozen.

Lemma 2.4.2. *[Mul13, Lemma 3.2] Let $\mathcal{A}$ be a cluster algebra, and $\mathcal{A}'$ the freezing of $\mathcal{A}$ at the set $\{x_1, x_2, \ldots, x_i\}$. The following are equivalent:*

- *$\mathcal{A} \subseteq \mathcal{A}'$ as subalgebras of $\mathbb{Q}(x_1, \ldots, x_n)$.*
- *$\mathcal{A}' = \mathcal{A}[x_1^{-1}, \ldots, x_i^{-1}]$ as subalgebras of $\mathbb{Q}(x_1, \ldots, x_n)$.*
- *Every cluster variable in $\mathcal{A}$ can be written as a polynomial in the cluster variables of $\mathcal{A}'$, divided by a monomial in the frozen variables of $\mathcal{A}'$.*

*If any of the above conditions hold, we refer to $\mathcal{A}'$ as a **cluster localization**.*

Denoting $V(\mathcal{A}) := \operatorname{Spec} \mathcal{A}$, a cluster localization gives an open inclusion of schemes $V(\mathcal{A}') \subseteq V(\mathcal{A})$, which we refer to as a **cluster chart**. A cluster algebra $\mathcal{A}$ is called **locally acyclic** if $V(\mathcal{A})$ has a finite cover by acyclic cluster charts. We recall some nice properties of locally acyclic cluster algebras below.

Theorem 2.4.3. *[Mul14, Theorem 2] If $\mathcal{A}$ is an acyclic or locally acyclic cluster algebra, then $\mathcal{A} = \mathcal{U}$.*

Theorem 2.4.4. *[Mul13, Theorem 4.2] If $\mathcal{A}$ is locally acyclic, then $\mathcal{A}$ is finitely generated, integrally closed, and locally a complete intersection.*

Theorem 2.4.5. *[Mul13, Theorem 7.7, Corollary 7.9] If $\mathcal{A}$ is a locally acyclic cluster algebra of full rank, then $\mathbb{Q} \otimes \mathcal{A}$ is regular, $V(\mathcal{A}, \mathbb{C}) := \operatorname{Hom}(\mathcal{A}, \mathbb{C})$ is a smooth complex manifold, and $V(\mathcal{A}, \mathbb{R}) := \operatorname{Hom}(\mathcal{A}, \mathbb{R})$ is a smooth real manifold.*

Remark 2.4.6. Some references use the descriptor *locally acyclic* when referring to quivers for which the *Louise algorithm* is successful. Louise quivers are locally acyclic [MS16, Proposition 2.6], but as far as the author is aware, it is still not known whether all locally acyclic quivers are Louise.

Chapter 3

Deep points of cluster varieties

We now begin our discussion of *deep points of cluster varieties*, the primary object of interest in this dissertation.

Remark 3.0.1. Portions of this text appear in the article *Deep Points of Cluster Algebras* by James Beyer and Greg Muller [BM25], and are included here with permission of all authors.

3.1 Cluster varieties

Given a cluster algebra $\mathcal{A}$ and a field $\Bbbk$, the **cluster variety** is the set

$$V(\mathcal{A},\Bbbk) \coloneqq \mathrm{Hom}_{\mathrm{Ring}}(\mathcal{A},\Bbbk) \coloneqq \{\text{ring homomorphisms } p : \mathcal{A} \to \Bbbk\}$$

This has the *Zariski topology* in which a closed set consists of homomorphisms which send a fixed ideal to 0. Each element $a \in \mathcal{A}$ determines a function $f_a : V(\mathcal{A},\Bbbk) \to \Bbbk$ by the rule that $f_a(p) \coloneqq p(a)$. The notation $f_a(p)$ is more useful when regarding $V(\mathcal{A},\Bbbk)$ as a geometric object with points $\{p\}$, and the notation $p(a)$ is more useful when regarding it as a set of homomorphisms $\{p\}$.

Remark 3.1.1. If $\mathcal{A}$ is finitely generated, then $V(\mathcal{A}, \Bbbk)$ may be identified with an affine $\Bbbk$-variety. Explicitly, a finite presentation $\mathcal{A} \simeq \mathbb{Z}[g_1, g_2, \ldots, g_m]/\langle p_1, p_2, \ldots, p_n\rangle$ induces a homeomorphism between $V(\mathcal{A}, \Bbbk)$ and the solution set of the system $\{p_1 = 0, p_2 = 0, \ldots, p_n = 0\}$ inside $\Bbbk^m$.

In general, $V(\mathcal{A}, \Bbbk)$ is the set of $\Bbbk$-points of the affine scheme $\mathrm{Spec}(\mathcal{A})$. While this scheme need not be finite-type, we still refer to $V(\mathcal{A}, \Bbbk)$ as a 'variety' for simplicity. Many of the results of this paper can be reformulated for schemes, but we use varieties to appeal to a wider audience.

By the Laurent phenomenon, each cluster $\{x_1, x_2, \ldots, x_n\} \subset \mathcal{A}$ gives an inclusion of rings

$$\mathcal{A} \hookrightarrow \mathcal{A}[x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}] = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$$

A homomorphism $p : \mathcal{A} \to \Bbbk$ factors through this localization map if and only if $p(x_1), p(x_2), \ldots, p(x_n)$ are non-zero. Therefore, we have an open inclusion

$$(\Bbbk^\times)^n \simeq \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}], \Bbbk) \hookrightarrow \mathrm{Hom}_{\mathrm{Ring}}(\mathcal{A}, \Bbbk) =: V(\mathcal{A}, \Bbbk)$$

whose image is the set of homomorphisms $p : \mathcal{A} \to \Bbbk$ such that $p(x_1), p(x_2), \ldots,$ $p(x_n)$ are non-zero; or equivalently, the open subset of $V(\mathcal{A}, \Bbbk)$ where none of the functions $f_{x_1}, f_{x_2}, \ldots, f_{x_n}$ vanish. This image is called the **cluster torus** associated to the cluster $\{x_1, x_2, \ldots, x_n\}$.[1]

Warning 3.1.2. The phrase 'cluster variety' has been used in at least two other ways in the literature:

- The union of the cluster tori (e.g. [FG09]).
- A certain *blowup at infinity* of an algebraic torus (e.g. [GHK15]).

[1] The reader is cautioned that a cluster torus is not topological torus; rather, it is a *split algebraic torus*.

These references generally only consider varieties *up to birational equivalence* or *up to codimension 2* (e.g. [GHK15, Section 3.2]), in which case they are equivalent to our definition. By contrast, we are interested in the special points and fine geometry of $V(\mathcal{A}, \Bbbk)$, so we need to be more explicit and careful about which definition we are using.

3.2 Deep points

Definition 3.2.1. A point $p \in V(\mathcal{A}, \Bbbk)$ is a **deep point** if, for all clusters

$$\{x_1, x_2, \ldots, x_n\} \subset \mathcal{A},$$

there is a cluster variable $x_i \in \{x_1, \ldots, x_n\}$ such that $p(x_i) = 0$.

The set of all deep points in $V(\mathcal{A}, \Bbbk)$ is called the **deep locus** of $V(\mathcal{A}, \Bbbk)$.

Algebraic properties and consequences

One of the driving factors behind our classification of deep points is that many undesirable properties of algebras can be blamed on the presence of deep points. Perhaps the most glaring example is that of singularities.

Proposition 3.2.2. *A singular point in a cluster variety must be deep.*

Proof. If $p \in V(\mathcal{A}, \Bbbk)$ is not deep, then it is contained in a cluster torus, and so there is an open neighborhood of p which is non-singular. □

Deep points can also be translated to the algebraic side. Define the **deep ideal** $\mathsf{D}(\mathcal{A})$ of a cluster algebra $\mathcal{A}$ to be the ideal generated by the products of the elements in each cluster; that is,

$$\mathsf{D}(\mathcal{A}) \coloneqq \sum_{\text{clusters } \{x_1, x_2, \ldots, x_n\}} \mathcal{A} x_1 x_2 \cdots x_n$$

As the following proposition shows, the deep locus is defined by the vanishing of this ideal.

Proposition 3.2.3. *A point $p \in V(\mathcal{A}, \Bbbk)$ is deep if and only if $p(\mathsf{D}(\mathcal{A})) = 0$. Therefore, the deep locus of $V(\mathcal{A}, \Bbbk)$ is the Zariski closed subset defined by the vanishing of the deep ideal $\mathsf{D}(\mathcal{A})$.*

Proof. The first claim follows from equivalences between the the following statements.

1. p is deep; that is, for each cluster $\{x_1, x_2, \ldots, x_n\}$, there exists x_i such that $p(x_i) = 0$.
2. For each cluster $\{x_1, x_2, \ldots, x_n\}$, $p(x_1 x_2 \cdots x_n) = 0$.
3. $p(\mathsf{D}(\mathcal{A})) = 0$.

These all follow from basic properties of homomorphisms; note that $(2) \Rightarrow (1)$ uses that $\Bbbk$ is a field. The second claim follows from the first claim by the definition of the Zariski topology. □

A cluster algebra may have deep points over some fields but not others (e.g. Theorem 4.3.5), so a lack of deep points over a specific field generally doesn't say much about the algebra. Rather, it is the absence of deep points over all fields which will have the strongest algebraic consequences.

Proposition 3.2.4. *Given a cluster algebra $\mathcal{A}$, the following are equivalent.*

1. *The deep ideal $\mathsf{D}(\mathcal{A})$ of $\mathcal{A}$ is trivial; that is, $1 \in \mathsf{D}(\mathcal{A})$.*
2. *$V(\mathcal{A}, \Bbbk)$ has no deep points, for all fields $\Bbbk$.*

3. *$V(\mathcal{A}, \Bbbk)$ is covered by finitely many cluster tori, for all fields $\Bbbk$.*

Proof. If $\mathsf{D}(\mathcal{A})$ is non-trivial, then it is contained is some maximal ideal $\mathfrak{m}$. The quotient map $\mathcal{A} \to \mathcal{A}/\mathfrak{m}$ is a deep point in $V(\mathcal{A}, \mathcal{A}/\mathfrak{m})$, and so $(2) \Rightarrow (1)$. Conversely, if the deep ideal $\mathsf{D}(\mathcal{A})$ is trivial, then we may write 1 as

$$1 = \sum_{\{x_1,x_2,\ldots,x_n\}\in S} a_{\{x_1,x_2,\ldots,x_n\}} x_1 x_2 \cdots x_n$$

where each $a_{\{x_1,x_2,\ldots,x_n\}} \in \mathcal{A}$ and S is a finite set of clusters. Applying $p \in V(\mathcal{A}, \Bbbk)$ to this equality,

$$1 = \sum_{\{x_1,x_2,\ldots,x_n\}\in S} p(a_{\{x_1,x_2,\ldots,x_n\}}) p(x_1) p(x_2) \cdots p(x_n)$$

Since the right-hand side cannot vanish, there must be some cluster in

$$\{x_1, x_2, \ldots, x_n\} \in S$$

such that $p(x_1)p(x_2)\cdots p(x_n) \neq 0$, and so the point p is in the cluster torus corresponding to $\{x_1, x_2, \ldots, x_n\}$. Therefore, the cluster tori indexed by S cover $V(\mathcal{A}, \Bbbk)$, so $(1) \Rightarrow (3)$. Finally, $(3) \Rightarrow (2)$ follows from the definition of a deep point. $\square$

Theorem 3.2.5. *Let $\mathcal{A}$ be a cluster algebra with no deep points for all $\Bbbk$ (or equivalently, the deep ideal $\mathsf{D}(\mathcal{A})$ is trivial). Then*

1. *$\mathcal{A}$ is non-singular.*
2. *$\mathcal{A}$ is finitely generated.*
3. *$\mathcal{A}$ is equal to its upper cluster algebra.*
4. *$\mathcal{A}$ is a locally acyclic cluster algebra.*

Proof. The non-singularity of $\mathcal{A}$ (or equivalently, of $V(\mathcal{A}, \Bbbk)$ for all $\Bbbk$) follows from Proposition 3.2.2. In the terminology of [Mul13], the localization

$$\mathcal{A} \to \mathcal{A}[x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}]$$

is the *cluster localization* corresponding to freezing an entire cluster; the corresponding cluster algebra has no mutable variables and is therefore vacuously acyclic. If the deep ideal of $\mathcal{A}$ is trivial, then finitely many of these cluster localizations cover the cluster variety (Proposition 3.2.4) and so $\mathcal{A}$ is locally acyclic. Local acyclicity implies finite generation and $\mathcal{A} = \mathcal{U}$ by [Mul14, Theorem 2]. $\square$

Deep points and upper cluster algebras

The story up to this point can be repeated for the upper cluster algebra $\mathcal{U}$ of a cluster algebra $\mathcal{A}$. The **upper cluster variety** is the set

$$V(\mathcal{U}, \Bbbk) := \mathrm{Hom}_{\mathrm{Ring}}(\mathcal{U}, \Bbbk) := \{\text{ring homomorphisms } p : \mathcal{U} \to \Bbbk\}$$

endowed with the Zariski topology. Each cluster $\{x_1, x_2, \ldots, x_n\}$ determines an open inclusion

$$(\Bbbk^\times)^n \simeq \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}], \Bbbk) \hookrightarrow \mathrm{Hom}_{\mathrm{Ring}}(\mathcal{U}, \Bbbk) =: V(\mathcal{U}, \Bbbk)$$

whose image we call a **cluster torus** in $V(\mathcal{U}, \Bbbk)$. The points in $V(\mathcal{U}, \Bbbk)$ which are not in any cluster torus are the **deep points**, which collectively form the **deep locus**, which is defined by the vanishing of a **deep ideal** $\mathsf{D}(\mathcal{U})$ in $\mathcal{U}$.

We can relate this back to the cluster algebra $\mathcal{A}$ using the inclusion $\mathcal{A} \hookrightarrow \mathcal{U}$. Applying the functor $\mathrm{Hom}_{\mathrm{Ring}}(-, \Bbbk)$ gives a morphism of varieties $V(\mathcal{U}, \Bbbk) \to V(\mathcal{A}, \Bbbk)$.

Proposition 3.2.6. *The map $V(\mathcal{U}, \Bbbk) \to V(\mathcal{A}, \Bbbk)$ restricts to an isomorphism on each cluster torus, and therefore restricts to an isomorphism between the complements of the deep loci.*

Proof. The localizations of $\mathcal{A}$ and $\mathcal{U}$ at a cluster equal the Laurent ring, so they are equal.

$$\mathcal{U}[x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}] = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] = \mathcal{A}[x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}]$$

Therefore, the map $V(\mathcal{U}, \Bbbk) \to V(\mathcal{A}, \Bbbk)$ restricts to an isomorphism between the cluster tori

$$V(\mathcal{U}[x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}], \Bbbk) \xrightarrow{\sim} V(\mathcal{A}[x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}], \Bbbk)$$

Since this holds for each cluster, the morphism restricts to an isomorphism between the union of the cluster tori, which are the complements of the deep loci. $\square$

The proposition tells us that any difference between a cluster algebra $\mathcal{A}$ and its upper cluster algebra $\mathcal{U}$ corresponds to a difference between the deep loci. An example of this is the Markov cluster algebra (discussed in Section 8.4), where the map $V(\mathcal{U}, \Bbbk) \to V(\mathcal{A}, \Bbbk)$ collapses a line in the deep locus of $V(\mathcal{U}, \Bbbk)$ to a single deep point in $V(\mathcal{A}, \Bbbk)$, and is otherwise an isomorphism (see Remark 8.4.9).

Remark 3.2.7. More generally, one can consider any *intermediate cluster algebra* $\mathcal{A} \subseteq R \subseteq \mathcal{U}$; examples include marked skein algebras [Mul16] and the span of the theta functions [GHKK18]. We may define deep points in the intermediate cluster variety $V(R, \Bbbk)$, and we have morphisms

$$V(\mathcal{U}, \Bbbk) \to V(R, \Bbbk) \to V(\mathcal{A}, \Bbbk)$$

which are isomorphisms away from the respective deep loci.

Chapter 4

Deep points of cluster algebras of polygons

We now turn to describing the deep locus of a number of specific cluster algebras, starting with cluster algebras of polygons.

Remark 4.0.1. Portions of this text appear in the article *Deep Points of Cluster Algebras* by James Beyer and Greg Muller [BM25], and are included here with permission of all authors.

4.1 The cluster algebra(s) of a polygon

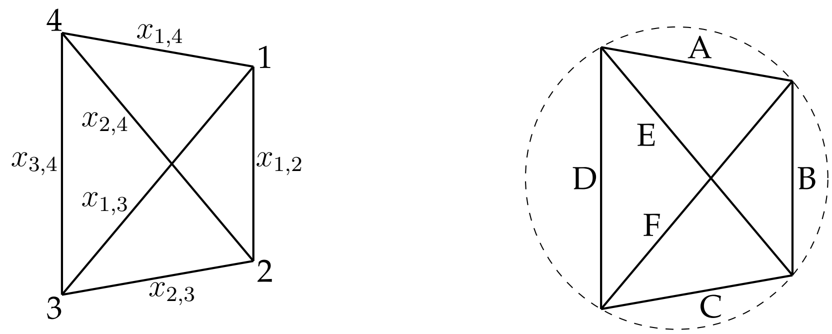
We start with an algebra which is not quite a cluster algebra (by our definition), but can be made into a cluster algebra in two closely related ways.

For any integer $n \geq 3$, define the $(2, n)$**-th Plücker algebra** as follows:

$$P_{2,n} := \mathbb{Z}[x_{i,j}, \forall 1 \leq i < j \leq n] / \langle x_{i,k}x_{j,l} - (x_{i,j}x_{k,l} + x_{i,l}x_{j,k}), \forall 1 \leq i < j < k < l \leq n \rangle$$

The generators and relations of this algebra may be visualized as follows. Let Δ_n

be a convex n-gon with the vertices indexed clockwise from 1 to n. For each pair $1 \leq i < j \leq n$, we identify the generator $x_{i,j}$ with the line segment between vertices i and j. Inspired by this realization, we call $x_{i,j}$ an **edge** if $j = i+1$ or $(i,j) = (1,n)$; otherwise, $x_{i,j}$ is a **diagonal**. The relations in the algebra are parameterized by pairs of crossing diagonals, and can be visualized as in Figure 4.1a.[1]



(a) A Ptolemy relation among variables in a Plücker algebra
$x_{1,3}x_{2,4} = x_{1,2}x_{3,4} + x_{1,4}x_{2,3}$

(b) A Ptolemy relation among lengths between four cocircular points
$EF = AC + BD$

Figure 4.1: The Ptolemy relations

The algebra $P_{2,n}$ is *almost* a cluster algebra of type A_{n-3}. The issue lies in the edge elements; they behave like frozen variables, but our definition of cluster algebra requires that frozen variables have inverses. There are two natural ways to resolve this, both of which will be of interest.

- The **cluster algebra of the n-gon (with boundary coefficients)** is the localization

$$\mathcal{A}(\Delta_n) \coloneqq P_{2,n}[x_{i,j}^{-1},\ \forall \text{ edges } x_{i,j}]$$

- The **cluster algebra of type A_{n-3} (with no frozens)** is the quotient

$$\mathcal{A}(A_{n-3}) \coloneqq P_{2,n}/\langle x_{i,j} - 1,\ \forall \text{ edges } x_{i,j}\rangle$$

[1] These relations are a special case of the *Plücker relations* of a Grassmannian, the *Ptolemy relations* in convex geometry, and the *Kauffman skein relations* in knot theory.

These two algebras are related by the quotient isomorphism

$$\mathcal{A}(\Delta_n)/\langle x_{i,j}-1,\ \forall\text{ edges } x_{i,j}\rangle \simeq \mathcal{A}(A_{n-3})$$

As the names imply, these algebras are both cluster algebras.

Theorem 4.1.1 ([FZ03, Section 12])**.** *The algebras $\mathcal{A}(\Delta_n)$ and $\mathcal{A}(A_{n-3})$ are both (isomorphic to) cluster algebras of type A_{n-3}. In both, the mutable cluster variables are the diagonals in Δ_n and the clusters are the triangulations of Δ_n. The frozen variables in $\mathcal{A}(\Delta_n)$ are the edges, while $\mathcal{A}(A_{n-3})$ has no frozens.*

Since these cluster algebras are defined in terms of generators and relations, the associated cluster variety may be characterized in terms of these presentations.

Corollary 4.1.2. *A point $p\in V(\mathcal{A}(\Delta_n),\mathbb{k})$ (respectively, $V(\mathcal{A}(A_{n-3}),\mathbb{k})$) is equivalent to a map*

$$p:\{\textit{diagonals and edges in } \Delta_n\}\to\mathbb{k}$$

such that the Ptolemy relations hold and edges are not sent to 0 (respectively, edges are sent to 1). Such a point is deep if and only if every triangulation of Δ_n contains a diagonal which is sent to 0.

Remark 4.1.3. If we choose a specific polygon Δ_n in the plane whose vertices lie on a common circle as in Figure 4.1b (called a *cocircular n-gon*), then sending each arc in Δ_n to its length satisfies the Ptolemy relations by *Ptolemy's Theorem*. In this way, we may identify the set of cocircular n-gons up to congruence with a proper subset of the cluster variety $V(\mathcal{A}(\Delta_n),\mathbb{k})$.

Remark 4.1.4. If we choose a $2\times n$ matrix M such that cyclically adjacent columns are linearly independent, then sending each $x_{i,j}$ to the determinant of the ith and jth columns of M satisfies the Ptolemy relations by the *Plücker embedding*. In this way, we may identify the set of such matrices (up to left multiplication by $\mathrm{SL}(2, \Bbbk)$) with the cluster variety $V(\mathcal{A}(\Delta_n), \Bbbk)$.

4.2 Vanishing lemmas

By a **triangle** in Δ_n, we mean a triple of the form $\{x_{i,j}, x_{i,k}, x_{j,k}\}$; that is, the line segments bounding a triangle of vertices in Δ_n. The following lemma demonstrates that a deep point of $\mathcal{A}(\Delta_n)$ must kill an odd number of diagonals in each triangle in Δ_n.

Lemma 4.2.1. *Let $p \in V(\mathcal{A}(\Delta_n), \Bbbk)$ and let $\{x_{i,j}, x_{i,k}, x_{j,k}\}$ be a triangle in Δ_n.*

1. *If p kills any two of $\{x_{i,j}, x_{i,k}, x_{j,k}\}$, then p also kills the third.*
2. *If p is deep, then p kills at least one of $\{x_{i,j}, x_{i,k}, x_{j,k}\}$.*

Proof. (1) Assume that $p(x_{i,j}) = p(x_{i,k}) = 0$. Since p cannot vanish on an edge, $x_{i,j}$ cannot be an edge and so $j \geq i + 2$. Applying p to the Ptolemy relation of the quadrilateral with vertices $i, i+1, j, k$ and using the fact that $p(x_{i,i+1}) \neq 0$,

$$p(x_{j,k}) = \frac{p(x_{i,j})p(x_{i+1,k}) - p(x_{i,k})p(x_{i+1,k})}{p(x_{i,i+1})} = 0$$

An identical argument applies if p kills any other pair in $\{x_{i,j}, x_{i,k}, x_{j,k}\}$.

(2) Assume, for contradiction, that p is deep and there is some triangle in Δ_n on which it does not vanish. Therefore, the set of triangulations of Δ_n which contain a triangle on which p does not vanish is non-empty. Among all such triangulations, choose a triangulation S which contains the minimal possible number of diagonals killed by p.

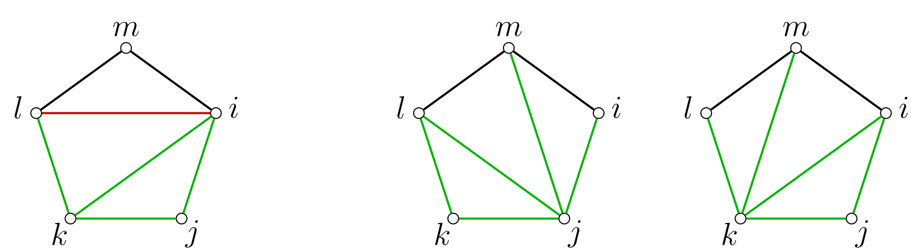
Since p kills diagonals in some triangles in S but not others, there must be an adjacent pair of triangles in S such that p kills a diagonal in one triangle and no diagonals in the other. We focus on the pentagon consisting of these two triangles and the triangle on the other side of the diagonal killed by p. Flipping and rotating indices in Δ_n as necessary, we may choose indices $i < j < k < l < m$ for the pentagon so that $\overline{ijk}$ is the triangle on which p is does not vanish, $\overline{ikl}$ is the adjacent triangle in S containing a diagonal $x_{i,l}$ killed by p, and $\overline{ilm}$ is the other triangle in S containing $x_{i,l}$ (Figure 4.2a).

By assumption, $p(x_{i,j}), p(x_{i,k})$, and $p(x_{j,k})$ are non-zero and $p(x_{i,l}) = 0$. We repeatedly apply part (1) to triangles in this pentagon.

- Since $p(x_{i,j}) \neq 0$ and $p(x_{i,l}) = 0$, $p(x_{j,l}) \neq 0$.
- Since $p(x_{i,k}) \neq 0$ and $p(x_{i,l}) = 0$, $p(x_{k,l}) \neq 0$.
- Since $p(x_{j,k}) \neq 0$, at most one of $p(x_{j,m})$ and $p(x_{k,m})$ can be 0.

Therefore, either $p(x_{j,l})$ and $p(x_{j,m})$ are non-zero, or $p(x_{i,k})$ and $p(x_{k,m})$ are non-zero (Figure 4.2b).

As a consequence, replacing $\{x_{i,k}, x_{i,l}\}$ in S with $\{x_{j,l}, x_{j,m}\}$ or $\{x_{i,k}, x_{k,m}\}$ produces a triangulation of Δ_n with strictly fewer arcs killed by p and at least one triangle on which p does not vanish. This contradicts the minimality of S. $\square$



(a) Part of a triangulation in which $x_{i,l}$ vanishes

(b) Two possible modifications, one of which must have fewer vanishing arcs.

Figure 4.2: Modifying a triangulation to reduce the number of vanishing arcs.

As a consequence, deep points of $\mathcal{A}(\Delta_n)$ must kill a specific set of diagonals.

Lemma 4.2.2. *Let $p \in V(\mathcal{A}(\Delta_n), \Bbbk)$. Then p is deep if and only if, for all i, j,*

$$p(x_{i,j}) = 0 \text{ if and only if } j - i \text{ is even} \tag{4.1}$$

Proof. Since every triangulation of Δ_n contains a diagonal $x_{i,j}$ with $j - i$ even, a point satisfying (4.1) is deep. Next, we prove by induction on $j - i$ that a deep point p must satisfy (4.1). If $j - i = 1$, then $x_{i,j}$ is an edge of Δ_n and so $p(x_{i,j}) \neq 0$. If $j - i > 1$, apply Lemma 4.2.1 to the triangle $\{x_{i,j-1}, x_{i,j}, x_{j-1,j}\}$. Since $x_{j-1,j}$ is an edge, $p(x_{j-1,j}) \neq 0$ and so exactly one of $p(x_{i,j-1})$ and $p(x_{i,j})$ is 0. If (4.1) holds for $p(x_{i,j-1})$, then it holds for $p(x_{i,j})$, completing the induction. □

4.3 Deep points

We can now classify the deep points of both $\mathcal{A}(\Delta_n)$ and $\mathcal{A}(A_n)$.

Theorem 4.3.1. *Let $n \geq 3$.*

1. *If n is odd, then the cluster variety $V(\mathcal{A}(\Delta_n), \Bbbk)$ has no deep points.*
2. *If n is even, then a choice of values on the edges of Δ_n*

$$p : \{\text{edges in } \Delta_n\} \to \Bbbk^\times$$

extends to a deep point of $V(\mathcal{A}(\Delta_n), \Bbbk)$ if and only if

$$\frac{p(x_{1,2})p(x_{3,4}) \cdots p(x_{n-1,n})}{p(x_{1,n})p(x_{2,3}) \cdots p(x_{n-2,n-1})} = (-1)^{\frac{n+2}{2}} \tag{4.2}$$

and this extension is unique, with values on an arbitrary diagonal $x_{i,j}$ given by

$$p(x_{i,j}) = \left\{ \begin{array}{cc} (-1)^{\frac{j-i-1}{2}} \dfrac{p(x_{i,i+1})p(x_{i+2,i+3}) \cdots p(x_{j-1,j})}{p(x_{i+1,i+2}) \cdots p(x_{j-2,j-1})} & \text{if } i \not\equiv j \bmod 2 \\ 0 & \text{if } i \equiv j \bmod 2 \end{array} \right\} \tag{4.3}$$

As a consequence, the deep locus of $V(\mathcal{A}(\Delta_n), \Bbbk)$ freely determined by its values on all but one edge, and is therefore isomorphic to $(\Bbbk^\times)^{n-1}$.

Proof. Fix a deep point $p \in V(\mathcal{A}(\Delta_n), \Bbbk)$ and consider a number of cases.

1. If n is odd, then $n-1$ is even, and so Lemma 4.2.2 forces $p(x_{1,n}) = 0$. However, $x_{1,n}$ is an edge in Δ_n, and so this is impossible. Therefore, $V(\mathcal{A}(\Delta_n), \Bbbk)$ has no deep points when n is odd.

2. a) If $n = 4$, the two triangulations of Δ_4 each contain a unique diagonal, and so p is deep if and only if $p(x_{1,3}) = p(x_{2,4}) = 0$. A deep point of $\mathcal{A}(\Delta_4)$ is then determined by its non-zero values on the edges, subject to the Ptolemy relation $0 = p(x_{1,2})p(x_{3,4}) + p(x_{2,3})p(x_{1,4})$. Rewriting yields (4.2):

$$\frac{p(x_{1,2})p(x_{3,4})}{p(x_{2,3})p(x_{1,4})} = -1 = (-1)^{\frac{4+2}{2}}.$$

   b) If $n > 4$ is even, assume (for induction) that the theorem holds for all $\Delta_{n'}$ with $n' < n$, and consider a deep point $p \in V(\mathcal{A}(\Delta_n), \Bbbk)$. Lemma 4.2.2 forces $p(x_{i,j}) = 0$ if and only if $j - i$ is even. Consider $x_{i,j}$ with $j - i$ odd, and let $\Delta_{i,j}$ be the convex hull of the vertices $i, i+1, \ldots, j-1, j$. The restriction of p to edges and diagonals in $\Delta_{i,j}$ defines a point in $V(\mathcal{A}(\Delta_{i,j}), \Bbbk)$; furthermore, since p kills at least one diagonal in every triangle (Lemma 4.2.1), this restriction is deep. By the inductive hypothesis on the $(j-i-1)$-gon $\Delta_{i,j}$, we know that p satisfies Condition (4.2) around the edges of $\Delta_{i,j}$:

$$\frac{p(x_{i,i+1})p(x_{i+2,i+3})\cdots p(x_{j-1,j})}{p(x_{i,j})p(x_{i+1,i+2})\cdots p(x_{j-2,j-1})} = (-1)^{\frac{j-i+1}{2}}$$

   Multiplying both sides by $(-1)^{\frac{j-i-1}{2}} p(x_{i,j})$ gives Formula (4.3) for $p(x_{i,j})$ when $j - i$ is odd. Since the values on the edges determines the value on each diagonal, p is uniquely determined by its values on the edges.

   To verify Condition (4.2) for Δ_n, consider the convex hull $\Delta_{j,i}$ of the vertices $j, j+1, \ldots, n, 1, \ldots, i$. By the same argument as before, p restricts to a deep point in $V(\mathcal{A}(\Delta_{j,i}), \Bbbk)$. By another application of the inductive hypothesis, p satisfies Condition (4.2) around the edges of $\Delta_{j,i}$:

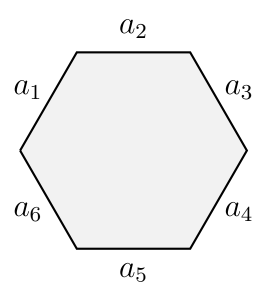
$$\frac{p(x_{j,j+1})p(x_{j+2,j+3})\cdots p(x_{i-1,i})}{p(x_{i,j})p(x_{j+1,j+2})\cdots p(x_{i-2,i-1})} = (-1)^{\frac{n-i+j+1}{2}}$$

Multiplying both sides by $(-1)^{\frac{n-i+j+1}{2}} p(x_{i,j})$ gives another formula for $p(x_{i,j})$. Since the two formulas must be equal,

$$\begin{aligned}(-1)^{\frac{j-i+1}{2}} &\frac{p(x_{i,i+1})p(x_{i+2,i+3})\cdots p(x_{j-1,j})}{p(x_{i+1,i+2})\cdots p(x_{j-2,j-1})} \\ &= (-1)^{\frac{n-i+j+1}{2}} \frac{p(x_{j,j+1})p(x_{j+2,j+3})\cdots p(x_{i-1,i})}{p(x_{j+1,j+2})\cdots p(x_{i-2,i-1})}\end{aligned}$$

Moving the values of p to the left and the signs to the right proves that p satisfies Condition (4.2) for the edges in the boundary of Δ_n, completing the induction. $\square$

Example 4.3.2. Consider the hexagon with values $a_1, a_2, \ldots, a_6$ on the six edges.



By Theorem 4.3.1, these values extend to a (necessarily unique) deep point if and only if

$$\frac{a_1 a_3 a_5}{a_2 a_4 a_6} = 1 \tag{4.4}$$

Assuming this holds, the values at this deep point on the diagonals are depicted in Figure 4.3.

Note that Condition (4.4) implies that $-\frac{a_1 a_3}{a_2} = -\frac{a_4 a_6}{a_5}$. More generally, every diagonal is defined by two equivalent formulas corresponding to the edges on

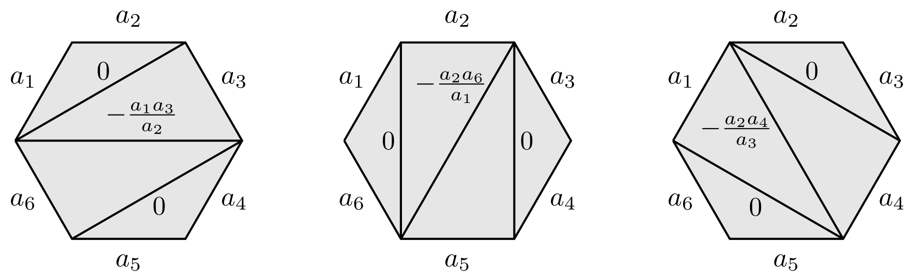


Figure 4.3: A general deep point in the cluster algebra of a hexagon

either side. Setting all boundary values $a_i = 1$ gives the unique deep point in the coefficient-free cluster algebra of type A_3 (Figure 1.3).

A key tool of the proof and key takeaway from the theorem is that a point p of $\mathcal{A}(\Delta_n)$ is deep if and only if, for any diagonal which cuts Δ_n into two even-sided polygons, the restrictions of p to each of those polygons are deep points in the respective cluster varieties. Stated precisely:

Corollary 4.3.3. *Let $x_{i,j}$ be a diagonal in Δ_n with $j-i$ odd, and let $\Delta_{i,j}$ and $\Delta_{j,i}$ be the convex hulls of $\{i, i+1, \ldots, j-1, j\}$ and $\{j, j+1, \ldots, n, 1, \ldots, i-1, i\}$, respectively. Then $p \in V(\mathcal{A}(\Delta_n), \Bbbk)$ is deep if and only if the restrictions of p to $\Delta_{i,j}$ and $\Delta_{j,i}$ define deep points in $V(\mathcal{A}(\Delta_{i,j}), \Bbbk)$ and $V(\mathcal{A}(\Delta_{j,i}), \Bbbk)$.*

Remark 4.3.4. As an application, the non-zero values of Formula (4.3) follow from Condition (4.2) for $\Delta_{i,j}$, and Condition (4.2) for Δ_n follows from Condition (4.2) for $\Delta_{i,j}$ and $\Delta_{j,i}$.

Theorem 4.3.1 gives a classification of deep points in coefficient-free cluster algebras of type A, by specializing the values of the edges to 1 and shifting the indexing by 3.

Theorem 4.3.5. *If $n \equiv 3 \bmod 4$ (or $n \equiv 1 \bmod 4$ and $\operatorname{char}(\Bbbk) = 2$), then the cluster variety $V(\mathcal{A}(A_n), \Bbbk)$ has a unique deep point whose values on the diagonals in Δ_{n+3} are* *given by*

$$p(x_{i,j}) = \left\{ \begin{array}{cc} (-1)^{\frac{j-i-1}{2}} & \textit{if } i \not\equiv j \bmod 2 \\ 0 & \textit{if } i \equiv j \bmod 2 \end{array} \right\} \tag{4.5}$$

For all other choices of n and $\Bbbk$, the cluster variety $V(\mathcal{A}(A_n), \Bbbk)$ has no deep points.

Proof. Since the value of p on each edge must be 1, Condition (4.2) becomes $1 = (-1)^{\frac{(n+3)+2}{2}}$, which holds if and only if $n \equiv 3 \bmod 4$ (or $n \equiv 1 \bmod 4$ and $\mathrm{char}(\Bbbk) = 2$). Formula (4.3) reduces to (4.5). $\square$

Remark 4.3.6. Comparing with the classification of singularities in [BFMS23, Theorem A], we see that the deep points of $\mathcal{A}(A_n)$ are precisely the singularities. By contrast, $V(\mathcal{A}(\Delta_n), \mathbb{C})$ is non-singular ([Mul13, Corollary 7.9]), and so the deep points in Theorem 4.3.1 are not singularities.

Chapter 5

Surface type cluster algebras

In [GSV05, FG06, FST08], it was shown how a triangulable marked surface Σ determines a cluster algebra $\mathcal{A}(\Sigma)$ with boundary coefficients. Cluster algebras of punctured surfaces were examined more closely in [FT18] using tools from Penner's *Decorated Teichmüller Theory* [Pen12]. We will review the necessary parts of these constructions with the aim of classifying the deep loci of these algebras.

Remark 5.0.1. Portions of this text appear in the article *Deep Points of Cluster Algebras* by James Beyer and Greg Muller, published in *International Mathematics Research Notices* [BM25].

5.1 Cluster algebras of marked surfaces

A **marked surface** consists a smooth, oriented, compact surface-with-boundary Σ, together with a finite set of **marked points** $\mathcal{M}$. Marked points in the interior of Σ are called **punctures**, and so a marked surface is **unpunctured** if $\mathcal{M} \subset \partial\Sigma$. A connected marked surface is determined (up to diffeomorphisms that preserve the marked points and orientation) by a few numbers:

- The genus g of the underlying surface.
- The number b of boundary components.
- The number q of punctures.
- The unordered numbers $m_1, m_2, \ldots, m_b$ of marked points on each boundary component.

A **marked curve** in a marked surface Σ is an immersion of a compact, connected curve-with-boundary[1] into Σ such that

- any endpoints map to $\mathcal{M}$,
- the interior maps to $\Sigma \setminus \mathcal{M}$,
- any *crossings* (points in $\Sigma \setminus \mathcal{M}$ with more than one preimage) must be simple (exactly two preimages) and transverse (the two preimages have distinct derivatives), and
- the set of crossings is finite.

Intuitively, a marked curve is a curve drawn on the surface Σ which may cross itself finitely many times (but only transversely) and either ends at marked points or forms a closed loop. A **marked multicurve** is an immersion of finitely many compact curves-with-boundary satisfying these properties. We consider two marked curves (resp. multicurves) **homotopic** if they are related by orientation-reversal and homotopies through the class of marked curves (resp. multicurves).[2]

Some additional terminology for marked curves:

- A **marked arc** is a marked curve with two endpoints (i.e. an immersed interval).
- A **loop** is a marked curve with no end points (i.e. an immersed circle).

[1]Note that a compact, connected curve-with-boundary must be diffeomorphic to either an interval or a circle.

[2]That is, homotopies which are marked curves (resp. multicurves) at all times. In particular, the immersion condition prohibits homotopies which change the *framing* in the sense of knot theory.

- A marked curve is **simple** if it has no crossings and is not contractible.
- A multicurve (i.e. a finite set of marked curves) is **compatible** if there are no crossings and every curve is simple.
- A **boundary curve** is a marked curve homotopic to a marked curve contained in $\partial\Sigma$.[3]

Example 5.1.1. A simple but important class of marked surfaces are the **(topological) polygons**. These are discs with no punctures and marked points along the boundary. Such a marked surface with m-many marked points is homeomorphic to any convex m-gon in the plane; under such a homeomorphism, each edge and diagonal determines a marked arc.

Example 5.1.2. A less trivial example is the $(2,2)$**-annulus**, which is what we call the annulus with 2 marked points on each boundary component. It will be convenient to depict the annulus as a rectangle with two opposite sides identified (drawn dashed in Figure 5.1). Up to homotopy, this marked surface has a unique simple loop and four boundary arcs.

(a) Some simple marked arcs in the (2,2)-annulus.

(b) The unique simple loop in the (2,2)-annulus.

Figure 5.1: Marked curves in the (2,2)-annulus.

A **triangulation** of a marked surface Σ is a simple marked multicurve which divides the surface into a union of topological triangles, together with the set of

[3]Since the interior of a marked curve cannot pass through a marked point, a boundary marked curve must either be a simple marked arc connecting adjacent marked points along the boundary, or a simple loop homotopic to an unmarked boundary component.

boundary arcs.[4] If a marked surface admits a triangulation, we say it is **triangulable**.

Proposition 5.1.3 ([FST08, Prop. 2.10])**.** *Let Σ be a connected marked surface with genus g, b-many marked points, q-many punctures, and m-many marked points on the boundary. Then Σ is triangulable if and only if the set of marked points M is not empty, every component of the boundary $\partial\Sigma$ contains at least one marked point, and the quantity*

$$n(\Sigma) \coloneqq 6g + 3b + 3q + 2m - 6$$

is positive. Every triangulation of Σ has $n(\Sigma)$-many marked arcs (counting boundary arcs).

Remark 5.1.4. Every boundary marked arc is in every triangulation, and there are m-many of them. Therefore, every triangulation contains $6g+3b+3q+m-6$ many non-boundary marked arcs; this matches the formula in [FST08, Prop. 2.10] and is the number of mutable cluster variables in the corresponding cluster algebra.

Example 5.1.5. Two triangulations of the $(2, 2)$-annulus are given in Figure 5.2. Note that these triangulations each have 4 non-boundary arcs; together with the 4 boundary arcs, this matches

$$n(\Sigma) = 6g + 3b + 3q + 2m - 6 = 6 \cdot 0 + 3 \cdot 2 + 3 \cdot 0 + 2 \cdot 4 - 6 = 8$$

Figure 5.2: Two marked triangulations of the $(2, 2)$-annulus.

[4] Including the boundary arcs in a triangulation is non-standard, but will simplify the connection to cluster algebras.

Cluster algebras of unpunctured surfaces

Given a triangulation of an unpunctured surface, we can construct a quiver in the following steps:

1. Assign a mutable vertex to each interior arc.
2. Assign a frozen vertex to each boundary arc.
3. Add arrows between the sides of each triangle in the clockwise direction, omitting any arrows between frozen vertices.

Quiver mutation is then related by a *flip* of a diagonal in any quadrilateral, as pictured in Figure 5.3.

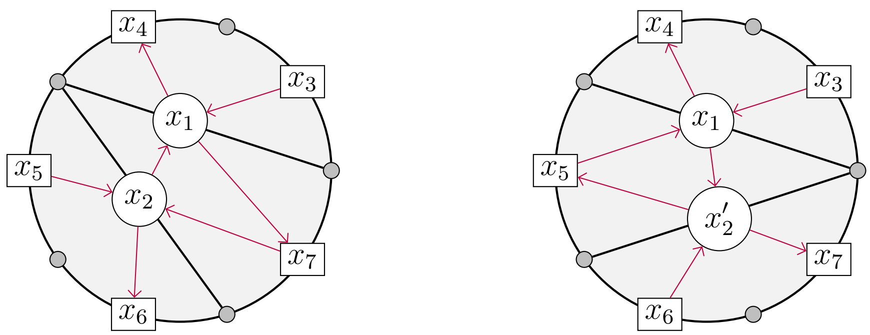
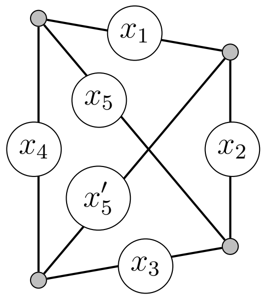


Figure 5.3: Quivers (in purple) associated to two triangulations of the unpunctured disk with five marked points on the boundary, related by a flip of the diagonal marked x_2.

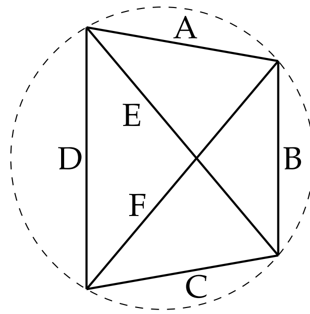
Seed mutation follows from the previous description given in Section 2.3, but also has a geometric description. As depicted in Figure 5.4, the relation between the two diagonals of a given quadrilateral is given by the Ptolemy relation: the product of the diagonals equals the sum of products of opposite sides.

Cluster algebras of punctured surfaces

While all triangulations of unpunctured surfaces are related by flips of diagonals in quadrilaterals [Hat91, Har86], this is no longer true in the presence of punc-



(a) A Ptolemy relation among variables in a cluster algebra
$x_5x_5' = x_1x_3 + x_2x_4$



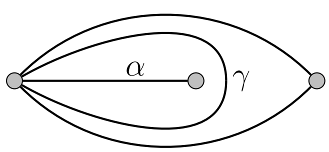
(b) A Ptolemy relation among lengths between four cocircular points
$EF = AC + BD$

Figure 5.4: The Ptolemy relations

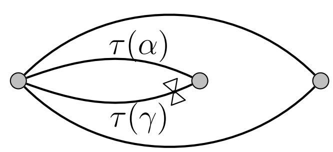
tures. Consider the bigon in Figure 5.5a; there is no triangulation with the arc α removed and replaced by another arc. To fix this issue, we utilize the notion of *tagged triangulations*, as defined in [FST08].

Definition 5.1.6. [FST08, Definition 7.1] An arc γ in Σ has two *ends* obtained by arbitrarily cutting the arc into three connected pieces and discarding the middle one. A **tagged arc** is an arc in which each of its ends has been tagged in one of two ways —*plain* or *notched* —so that the following conditions are met:

- the arc does not cut out a once-punctured monogon,
- an endpoint at the boundary is tagged plain, and
- if both ends of the arc are at the same marked point (or puncture), then they are tagged in the same way.



(a) A self-folded triangle.



(b) A tagged triangulation.

Figure 5.5: A punctured bigon. Plain arcs are drawn without decoration and notched arcs are decorated with a bowtie (i.e. the $\bowtie$ symbol).

Each ordinary arc γ can be replaced with a tagged arc $\tau(\gamma)$ as follows:

- If γ does not cut out a punctured monogon, then $\tau(\gamma)$ is γ with both ends tagged plain.
- If γ cuts out a punctured monogon, then both ends of γ are adjacent to the point a. Labeling the puncture b, there is a unique ordinary arc α connecting a to b that is compatible with γ. Then $\tau(\gamma)$ is obtained by tagging α plain at a and notched at b.

Remark 5.1.7. Note that since Σ is triangulable, it is not a sphere with three punctures. As a result, an arc γ cannot cut out punctured monogons on both sides, so the punctured monogon cut out by γ is unique.

Referring again to Figure 5.5, we can see that the arc γ in Figure 5.5a is replaced with the arc $\tau(\gamma)$ in Figure 5.5b, a copy of the ordinary arc α which is notched at the puncture. A **tagged triangulation** is then a maximal collection of compatible tagged arcs, according to the compatibility relation defined below.

Definition 5.1.8. [FST08, Definition 7.4] Two tagged arcs $\alpha, \beta \in \Sigma$ are called **compatible** if and only if the following conditions are satisfied:

- the untagged versions of α and β are compatible;
- if the untagged versions of α and β are different, and α and β share an endpoint at a, then the ends of α and β connected to a must have the same tagging;
- if the untagged versions of α and β coincide, then at least one end of α must be tagged in the same way as the corresponding end of β.

Remark 5.1.9. [FST08, Remark 7.6] If two plain arcs α and β are compatible, then the tagged arcs $\tau(\alpha)$ and $\tau(\beta)$ are compatible. The converse is false: in a once-punctured bigon, the two curves that cut out once-punctured monogons are not compatible even though the tagged arcs representing them are compatible.

Given this definition of compatibility, every non-boundary arc in a triangulation of Σ can be removed and replaced with another arc to produce a different triangulation, resulting in the following theorem:

Theorem 5.1.10. *[FST08, Theorem 7.10] If Σ is not a closed surface with exactly one puncture, then any two tagged triangulations are connected by a sequence of flips.*

Flips of tagged arcs inside of a punctured bigon are demonstrated in Figure 5.6.

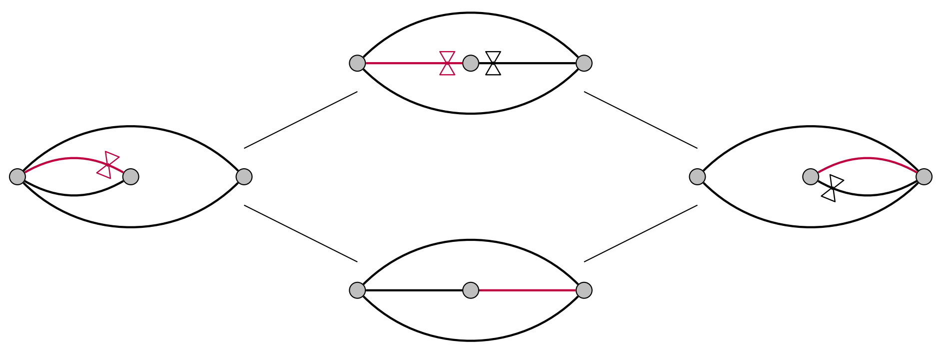
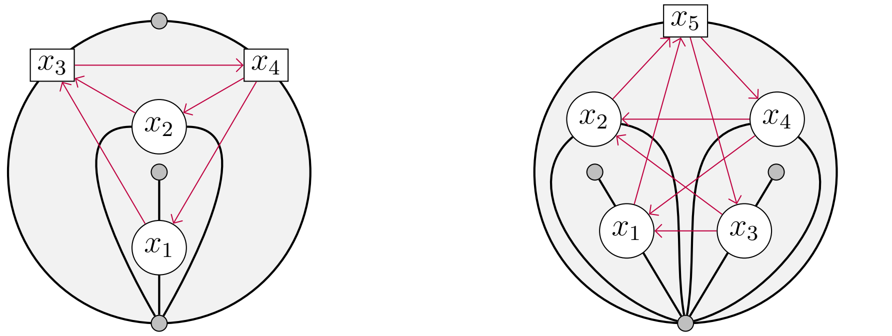


Figure 5.6: Tagged triangulations of a punctured bigon. Edges connect tagged triangulations related by a flip of a single tagged arc.

Given a triangulation of Σ without self-folded triangles, the construction of a quiver follows in exactly the same way as in the unpunctured case. In the presence of self-folded triangles, the quiver can be generated by treating both arcs of the self-folded triangle as the third side of the other triangle containing the arc cutting out a punctured monogon, as in Figure 5.7.

As noted in [FT18, Chapter 5], it follows from [FST08, Proposition 12.3] that replacing ideal triangulations with tagged triangulations does not extend the class of associated exchange matrices: each matrix corresponding to a tagged triangulation is identical, up to simultaneous permutations of rows and columns, to a matrix corresponding to an ordinary ideal triangulation. Since the matrices are extended skew-symmetric, we can conclude that the same is true of the associated quivers.

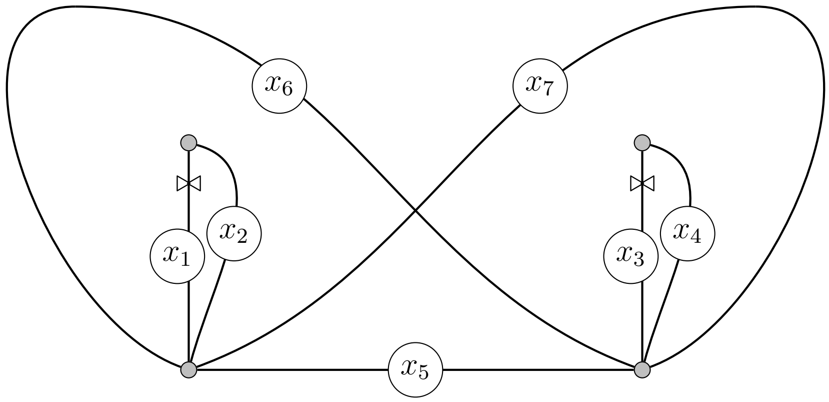


(a) A once-punctured bigon. (b) A twice-punctured monogon.

Figure 5.7: Quivers (in purple) associated to triangulations of a once-punctured bigon and a twice-punctured monogon.

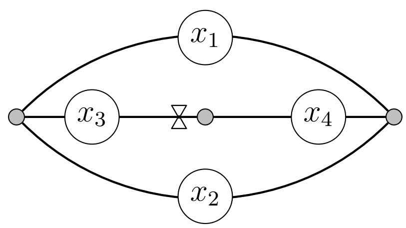
While the Ptolemy relations from triangulations of unpunctured surfaces still hold for tagged triangulations of punctured surfaces, they can become more complicated. As noted in [FT18, Definition 8.4], some of the arcs bounding the quadrilateral may bound a once-punctured monogon, and thus, may not be present in all triangulations involving the other arcs. In such a case, we should replace every such arc by the product of the two tagged arcs that it encloses. See Figure 5.8 (reproduced from Figure 19 of [FT18, Chapter 8]) for an example of a Ptolemy relation in a tagged triangulation of a four-punctured sphere.

In addition to the necessary adaptation of the Ptolemy relations in the case where one of the bounding arcs also bounds a punctured monogon, we also must adapt the Ptolemy relation in the case of a punctured bigon. As depicted in Figure 5.9, the product of two incompatible arcs inside a punctured bigon is equal to the sum of the two bounding arcs [FT18, Definition 8.5].



$$x_6x_7 = x_1x_2x_3x_4 + x_5^2 \tag{5.1}$$

Figure 5.8: Ptolemy relation for a tagged flip in a 4-punctured sphere.



$$x_3x_4 = x_1 + x_2 \tag{5.2}$$

Figure 5.9: Ptolemy relation for a punctured bigon.

5.2 Kauffman skein algebras of marked surfaces

The skein algebra of an unpunctured marked surface

The connection between cluster algebras of marked surfaces with boundary and *Kauffman skein algebras* was explored in [MW13, Mul16]. In many cases it is more practical to work with the skein algebra of the marked surface, which is generated by marked curves with topologically-defined relations.

Given a marked surface Σ, let $\mathbb{Z}\mathrm{Multi}(\Sigma)$ denote the abelian group of $\mathbb{Z}$-linear combinations of marked multicurves in Σ up to homotopy. Define the **skein al-**

gebra $\mathrm{Sk}_1(\Sigma)$ to be the quotient of $\mathbb{Z}\mathrm{Multi}(\Sigma)$ by the subgroup generated by the families of relations depicted in Figure 5.10.[5] In each figure, the dashed circle represents a small contractible neighborhood in Σ, and all elements considered agree outside this circle. We let $\langle\gamma\rangle$ denote the element of $\mathrm{Sk}_1(\Sigma)$ corresponding to a marked multicurve γ.

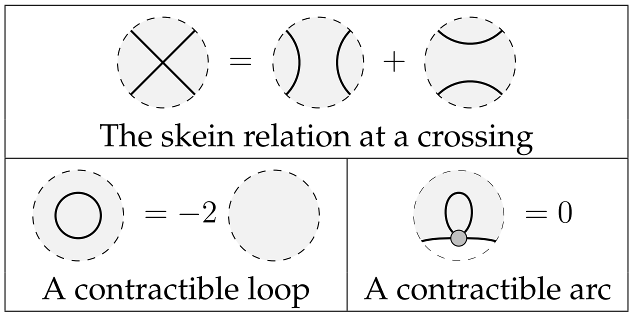


Figure 5.10: The $q = 1$ marked Kauffman skein relations

Given two marked multicurves γ and λ, their **superposition** $\gamma \cup \lambda$ is the union of the corresponding immersed curves. This may not be a marked multicurve, as the crossings may no longer be transverse or have more than two preimages. This can be resolved by replacing γ and λ with equivalent marked multicurves γ' and λ' whose superposition is a marked multicurve. While the resulting multicurve $\gamma' \cup \lambda'$ depends on the choice of γ' and λ', the corresponding element $\langle\gamma' \cup \lambda'\rangle$ in the skein algebra does not [Mul16, Proposition 3.5]. This gives a well-defined **superposition product** on the skein algebra $\mathrm{Sk}_1(\Sigma)$.

Theorem 5.2.1. *[Mul16, Corollary 6.16] Let Σ be an unpunctured, triangulable marked surface. Then the superposition product makes $\mathrm{Sk}_1(\Sigma)$ into a commutative domain.*

Example 5.2.2. If Σ is a polygon with m-many marked points (with marked points implicitly serving as the vertices of the polygon), we may index the marked points

[5] While we use the term 'skein algebra' here for brevity, $\mathrm{Sk}_1(\Sigma)$ would more properly be called the *marked Kauffman skein algebra of* $(\Sigma, \mathcal{M})$ *at* $q = 1$ to distinguish it from other variations.

$1, 2, \ldots, m$. This identifies the simple marked arcs with edges and diagonals in Δ_m, and induces an isomorphism

$$\mathrm{Sk}_1(\Delta_n) \simeq P_{2,n}$$

As a consequence, the localization $\mathrm{Sk}_1(\Delta_n)[\partial\Sigma^{-1}]$ is a cluster algebra of type A, and its deep points are characterized by Theorem 4.3.1.

The tagged skein algebra of a punctured surface

Extension of the Kauffman skein relations to punctured surfaces has been explored in [MSW13, Thu14, FT18], and more recently in [Wil20, GLFW24, MQ23, BKK24]. In addition to the relations of Figure 5.10, the tagged skein algebra requires two additional relations pictured in Figure 5.11.

Similar to the unpunctured case, let $\mathbb{Z}\mathrm{Multi}^{\bowtie}(\Sigma)$ denote the abelian group of $\mathbb{Z}$-linear combinations of tagged marked multicurves in Σ up to homotopy. Define the **tagged skein algebra** $\overline{\mathrm{Sk}}^{\bowtie}(\Sigma)$ to be the quotient of $\mathbb{Z}\mathrm{Multi}^{\bowtie}(\Sigma)$ by the subgroup generated by the families of relations depicted in Figures 5.10 and 5.11. Further, define a version of the tagged skein algebra localized at the boundary, $\mathrm{Sk}^{\bowtie}(\Sigma) := \overline{\mathrm{Sk}}^{\bowtie}(\Sigma)[\partial\Sigma^{\pm 1}]$.

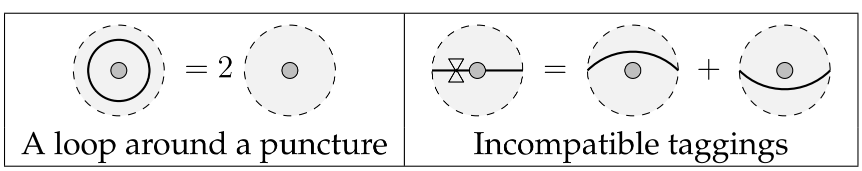


Figure 5.11: The $q = 1$ tagged Kauffman skein relations

The reader should refer to [MQ23] for more details on tagged skein algebras. We will introduce details only as needed, beginning with the following result.

Lemma 5.2.3. *[MQ23, Lemma 3.10] Simultaneously changing all tags at a puncture p induces an involutive automorphism i_p of $\mathrm{Sk}^{\bowtie}(\Sigma)$.*

Relationship between cluster algebras and skein algebras

Generalizing how the cluster algebra of a polygon was defined as a localization of the Plücker algebra in Section 4, the cluster algebra of an unpunctured marked surface can be defined as a localization of the skein algebra.

Theorem 5.2.4. *Let Σ be an unpunctured, triangulable marked surface. Then the localization*

$$\mathrm{Sk}_1(\Sigma)[\partial\Sigma^{-1}]$$

of the skein algebra $\mathrm{Sk}_1(\Sigma)$ at the set of boundary arcs $\partial\Sigma$ is a cluster algebra. The frozen cluster variables are the boundary arcs, the mutable cluster variables are the simple non-boundary arcs, and the clusters are the triangulations of Σ.

Proof. Let $\mathcal{A}(\Sigma)$ be the Fomin-Shapiro-Thurston cluster algebra of Σ and let $\mathcal{U}(\Sigma)$ be its upper cluster algebra. When $|\mathcal{M}| \geq 2$, $\mathcal{A}(\Sigma) = \mathcal{U}(\Sigma)$ by [Mul13, Theorems 4.1 and 10.6]. When $|\mathcal{M}| = 1$, $\mathcal{A}(\Sigma) = \mathcal{U}(\Sigma)$ by [CLS15]. By [Mul16, Theorem 7.15], there are containments

$$\mathcal{A}(\Sigma) \subseteq \mathrm{Sk}_1(\Sigma)[\partial\Sigma^{-1}] \subseteq \mathcal{U}(\Sigma)$$

Therefore, all three algebras are equal. $\square$

When the surface is punctured, there may not necessarily be an isomorphism between the cluster algebra and the localized commutative tagged skein algebra $\mathrm{Sk}_1^{\bowtie}(\Sigma)$. We can show that this is an isomorphism in many cases, however.

Theorem 5.2.5. *Let Σ be a triangulable marked surface (possibly with punctures) such that at least one of the following is true:*

1. *Σ is a disk.*
2. *Σ embeds into a disk.*

3. Σ *has at least two boundary marked points in each connected component.*

. *Then the localized tagged skein algebra* $\mathrm{Sk}^{\bowtie}(\Sigma)$ *is a cluster algebra. The frozen cluster variables are the boundary arcs, the mutable cluster variables are the simple (tagged) non-boundary arcs, and the clusters are the triangulations of* Σ.

Proof. Let $\mathcal{A}(\Sigma)$ be the Fomin-Shapiro-Thurston cluster algebra of Σ and let $\mathcal{U}(\Sigma)$ be its upper cluster algebra. Then $\mathcal{A}(\Sigma)$ is locally acyclic by Lemma 10.4, Theorem 10.5, and Theorem 10.6 of [Mul13]. Consequently, $\mathcal{A}(\Sigma) = \mathcal{U}(\Sigma)$ by [Mul14, Theorem 2].

By [MQ23, Proposition 3.18], there are containments

$$\mathcal{A}(\Sigma) \subseteq \mathrm{Sk}_1^{\bowtie}(\Sigma) \subseteq \mathcal{U}(\Sigma).$$

Therefore, all three algebras are equal. □

Since the skein algebra is generated by the simple marked curves and the relations are generated by the $q = 1$ marked Kauffman skein relations, the cluster variety may be characterized as follows.

Corollary 5.2.6. *Let* Σ *be an unpunctured, triangulable marked surface. A point* $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ *is equivalent to a map*

$$p : \{\textit{simple marked curves in } \Sigma \textit{ up to homotopy}\} \to \Bbbk$$

such that the $q = 1$ *marked Kauffman skein relations hold (Figure 5.10) and boundary arcs are not sent to* 0. *Such a point is deep if and only if every triangulation contains an arc which is sent to* 0.

Corollary 5.2.7. *Let* Σ *be a triangulable marked surface (possibly with punctures) such that at least one of the following is true:*

1. Σ *is a disk.*

2. Σ *embeds into a disk.*

3. Σ *has at least two boundary marked points in each connected component.*

. *A point* $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ *is equivalent to a map*

$$p : \{\textit{simple marked curves in } \Sigma \textit{ up to homotopy}\} \to \Bbbk$$

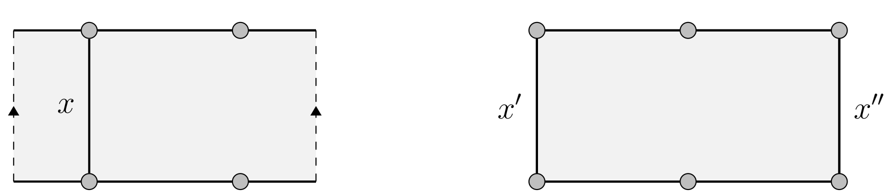
such that the tagged skein relations hold (Figures 5.10 and 5.11) and boundary arcs are not sent to 0. *Such a point is deep if and only if every triangulation contains an arc which is sent to* 0.

From this point on, we identify points of $V(\mathcal{A}(\Sigma), \Bbbk)$ with such functions without comment.

5.3 Cutting and gluing

Cutting and gluing unpunctured surfaces

Given a simple marked arc s in a marked surface Σ, we may construct a new marked surface $\Sigma \smallsetminus s$ by **cutting** along s; that is, removing s and adding two copies of s (one to each side of the original s) (see Figure 5.12). More generally, given a compatible collection of non-boundary marked arcs S, we construct $\Sigma \smallsetminus S$ by cutting along the arcs in S in any order.



(a) The $(2,2)$-annulus with marked arc x

(b) A hexagon formed by cutting along the marked arc x.

Figure 5.12: A cutting of the (2,2)-annulus.

For unpunctured surfaces, the cluster algebras of Σ and $\Sigma \smallsetminus S$ are related as follows.

Proposition 5.3.1. *Let Σ be an unpunctured, triangulable marked surface, and let S be a compatible collection of non-boundary marked arcs in Σ. Then the map $\Sigma \smallsetminus S \to \Sigma$ induces an isomorphism*

$$\mathcal{A}(\Sigma \smallsetminus S)/\langle s' - s'', \forall s \in S\rangle \xrightarrow{\sim} \mathcal{A}(\Sigma)[S^{-1}]$$

Here, s' and s'' denote the two preimages of s in $\Sigma \smallsetminus S$.

Proof. The topological map $\Sigma \smallsetminus S \to \Sigma$ is an immersion, and so it sends marked curves in $\Sigma \smallsetminus S$ to marked curves in Σ. Since the skein relations are local, this induces a morphism

$$\mathrm{Sk}_1(\Sigma \smallsetminus S) \to \mathrm{Sk}_1(\Sigma)$$

For each $s \in S$, the two boundary arcs s' and s'' in $\Sigma \smallsetminus S$ map to s. It follows that the above morphism factors through the quotient

$$\mathrm{Sk}_1(\Sigma \smallsetminus S)/\langle s' - s'', \forall s \in S\rangle \to \mathrm{Sk}_1(\Sigma)$$

By [Mul16, Corollary 4.13], for each simple marked arc x in Σ, there is some $k > 0$ such that for any $\langle y\rangle \in \mathrm{Sk}_1(\Sigma)$, $\langle x^k\rangle\langle y\rangle$ does not intersect x. These exponents are multiplicative, so for any marked multiarc γ in Σ, there is a monomial S^m in S such that $\langle S^m\rangle\langle\gamma\rangle$ is in the image of $\mathrm{Sk}_1(\Sigma \smallsetminus S)$. Therefore, the map

$$\mathrm{Sk}_1(\Sigma \smallsetminus S)[\partial^{-1}]/\langle s' - s'', \forall s \in S\rangle \to \mathrm{Sk}_1(\Sigma)[\partial^{-1}, S^{-1}]$$

is an isomorphism. This translates to cluster algebras as the desired isomorphism

$$\mathcal{A}(\Sigma \smallsetminus S)/\langle s' - s'', \forall s \in S\rangle \xrightarrow{\sim} \mathcal{A}(\Sigma)[S^{-1}] \qquad \square$$

Example 5.3.2. Let Σ be the $(2,2)$-annulus and let x be a marked arc connecting opposite boundary components (Figure 5.12a). Cutting along this arc gives a hexagon $\Sigma \setminus x$ with a pair of opposite boundary edges labeled x' and x'' (Figure 5.12b). Then Proposition 5.3.1 states there is an isomorphism

$$\mathcal{A}(\Sigma \setminus x)/\langle x' - x'' \rangle \xrightarrow{\sim} \mathcal{A}(\Sigma)[x^{-1}]$$

which sends the cluster variable in $\mathcal{A}(\Sigma \setminus x)$ corresponding to the simple marked arc γ in $\Sigma \setminus x$ to the cluster variable in $\mathcal{A}(\Sigma)$ corresponding to the image of γ in Σ. Note that $\mathcal{A}(\Sigma \setminus x) \simeq \mathcal{A}(\Delta_6)$ is a finite-type cluster algebra of type A_3, whereas $\mathcal{A}(\Sigma)$ is an infinite-type cluster algebra.

The isomorphism from Proposition 5.3.1 fits into the following diagram, in which the map on the left is the localization map and the map on the right is the quotient map.

$$\begin{tikzcd}
\mathcal{A}(\Sigma) \arrow[d, hook] & \mathcal{A}(\Sigma \setminus S) \arrow[d, two heads] \\
\mathcal{A}(\Sigma)[S^{-1}] & \mathcal{A}(\Sigma \setminus S)/\langle s' - s'', \forall s \in S \rangle \arrow[l, "\sim"']
\end{tikzcd} \tag{5.3}$$

The isomorphism can be translated to cluster varieties via the following terminology.

- A point in $V(\mathcal{A}(\Sigma), \Bbbk)$ is **non-zero on** S if it is non-zero on each arc in S.
- A point in $V(\mathcal{A}(\Sigma \setminus S), \Bbbk)$ is **gluable along** S if it has the same value on both preimages of each edge in S.

The first type of points are in bijection with homomorphisms $\mathcal{A}(\Sigma)[S^{-1}] \to \Bbbk$ and the second type of points are in bijection with homomorphisms

$$\mathcal{A}(\Sigma \setminus S)/\langle s' - s'', \forall s \in S \rangle \to \Bbbk.$$

Therefore, Proposition 5.3.1 implies the following.

Proposition 5.3.3. *Let Σ be an unpunctured, triangulable marked surface, and let S be a compatible collection of non-boundary marked arcs in Σ. Then restricting values from marked curves in Σ to marked curves in $\Sigma \smallsetminus S$ gives an isomorphism between:*

1. *The open subvariety of $V(\mathcal{A}(\Sigma), \Bbbk)$ consisting of points which are non-zero on S.*
2. *The closed subvariety of $V(\mathcal{A}(\Sigma \smallsetminus S), \Bbbk)$ consisting of points which are gluable along S.*

We refer to this as the **cutting isomorphism** between the two subvarieties:

$$\left\{ \begin{matrix} \text{points in } V(\mathcal{A}(\Sigma), \Bbbk) \\ \text{non-zero on } S \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{points in } V(\mathcal{A}(\Sigma \smallsetminus S), \Bbbk) \\ \text{gluable along } S \end{matrix} \right\} \tag{5.4}$$

Additionally, we refer to the action of the inverse map as **gluing** points.

Example 5.3.4. Let Σ, x, and $\Sigma \smallsetminus x$ be as in Figure 5.12 and Example 5.3.2. The cutting isomorphism is

$$\left\{ \begin{matrix} \text{points in } V(\mathcal{A}(\Sigma), \Bbbk) \\ \text{non-zero on } x \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{points in } V(\mathcal{A}(\Sigma \smallsetminus x), \Bbbk) \\ \text{with the same value on } x' \text{ and } x'' \end{matrix} \right\}$$

Using Corollary 5.2.6, we can reformulate this as a bijection between the following two sets:

- Functions $\{$simple marked curves in Σ up to homotopy$\} \to \Bbbk$ which satisfy the $q = 1$ Kauffman skein relations, are non-zero on boundary arcs, and are non-zero on x.
- Functions $\{$simple marked curves in $\Sigma \smallsetminus x$ up to homotopy$\} \to \Bbbk$ which satisfy the $q = 1$ Kauffman skein relations, are non-zero on boundary arcs, and have the same value on x' and x''.

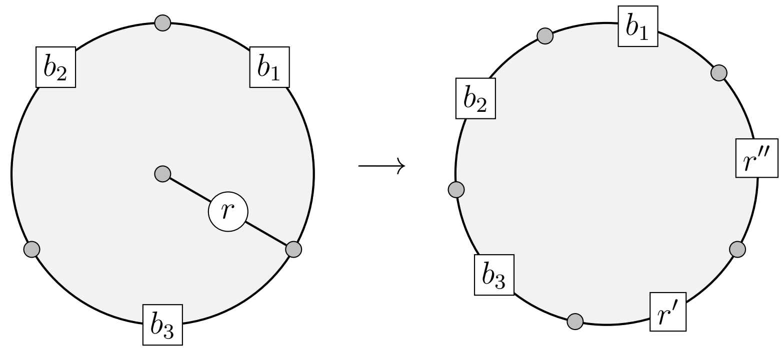


Figure 5.13: Cutting along a radius

Cutting and gluing punctured surfaces

We now wish to extend these cutting and gluing results to punctured surfaces. In order to do so, we must extend [Mul16, Lemma 4.11] to tagged multicurves in punctured surfaces.

We define a **simple (tagged) multicurve** X in Σ as a transverse (tagged) multicurve with with no crossings, no contractible loops, and no contractible arcs; let $[X]$ denote the corresponding element in $\mathrm{Sk}^{\bowtie}(\Sigma)$. Additionally, for each element $x \in \mathrm{Sk}^{\bowtie}(\Sigma)$, there is a unique subset $\mathrm{Supp}(x)$ of simple multicurves (called the *support* of x) and unique $\lambda_Y \in \mathbb{Z}$ for each $Y \in \mathrm{Supp}(x)$ such that

$$x = \sum_{Y \in \mathrm{Supp}(x)} \lambda_Y [Y].$$

Definition 5.3.5. Given simple multicurves X and Y, define the **intersection pairing** of X and Y as $\mu(X, Y) := A + D$, where

- A is the minimum number of crossings between all X' and Y', over all transverse pairs (X', Y') homotopic to (X, Y); note that intersections at marked points are not counted, and
- $D := D(X, Y) \cdot D(Y, X)$, where $D(\alpha, \beta)$ is the number of ends of β that are incident to an end of α and are tagged differently than α at that puncture.

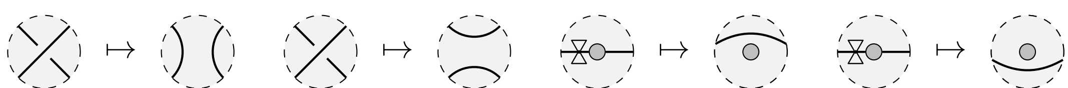
Remark 5.3.6. A similar definition of intersection pairing appears in [FST08, Definition 8.4] with two additional summands. These additional terms are always non-positive, so our definition of intersection pairing will always produce a number at least as large. We also change D to the product of $D(X,Y)$ and $D(Y,X)$ (instead of just $D(X,Y)$), so again, our intersection pairing will always be at least as large as the one defined in [FST08].

Lemma 5.3.7. *If x is a simple tagged arc in Σ, then for all $y \in \mathrm{Sk}^{\bowtie}(\Sigma)$ such that $\mu([x],y) > 0$,*

$$\mu([x],[x]y) \leq \mu([x],y) - 1.$$

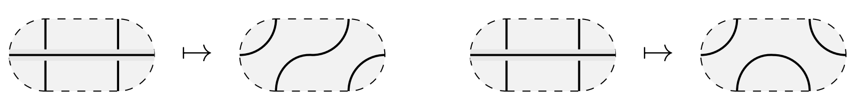
We adapt the proof of [Mul16, Lemma 4.11] to the setting of tagged arcs.

Proof. First consider the case when y is a simple multicurve Y so that $x \cdot Y$ is transverse, and $x \cdot Y$ has $\mu(x,Y)$ crossings (the minimal number, up to homotopy). Consider the set I of multicurves which can be obtained by applying some combination of the following local relations to each crossing in $x \cdot Y$:



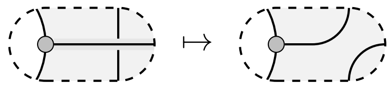
Since the simple multicurves in the support $\mathrm{Supp}([x][Y])$ come from applying the above relations to the crossings in $x \cdot Y$, we have $\mathrm{Supp}([x][Y]) \subset I$.

Consider a simple multicurve $Z \in I$. For two adjacent crossings in $x \cdot Y$ along x, there are two local possibilities for Z, up to reflection across x.



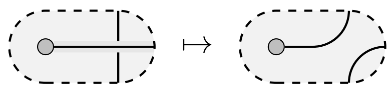
In this local picture, the first case is homotopic to a multicurve with crossing x once, and the second is homotopic to a multicurve which does not cross x.

Similarly, between a crossing in $x \cdot Y$ and an end of x at a boundary, there is one local possibility for Z, up to reflection across x.



This local picture is homotopic to one which does not cross x.

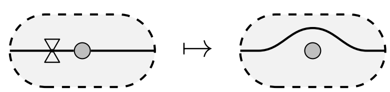
Between a crossing in $x \cdot Y$ and an end of x at a puncture, there is again one local possibility for Z, up to reflection across x.



This local picture is homotopic to one which does not cross x.

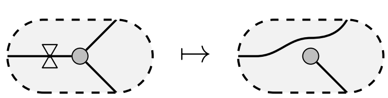
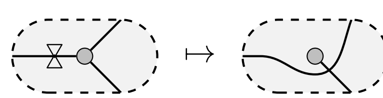
Finally, we consider the case where $x \cdot Y$ has incompatible taggings at an end of x at a puncture, so $D > 0$, which we will break into multiple cases.

1. Suppose that x and Y both have a single end incident to the puncture and they are tagged differently. Again, there is one local possibility for Z, up to reflection across x.

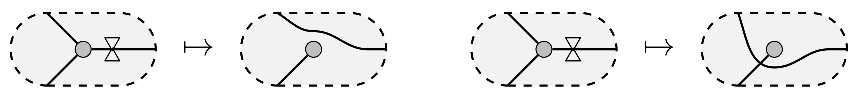
   This local picture is homotopic to one which does not cross x.

2. Suppose that one end of x is at the puncture, but Y has at least 2 ends at the puncture tagged differently than x. Here, the local contribution to $\mu(x, Y)$ is the value of $D \geq 2$. When $D = 2$, there are two local possibilities, up to reflection across x.

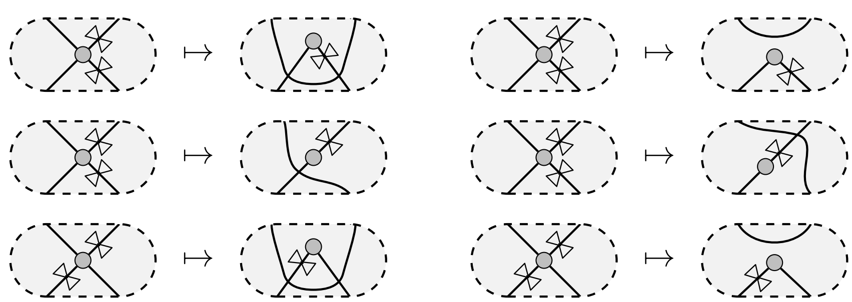
   In this local picture, both cases are homotopic to a multicurve intersecting x once. For $D > 2$, the same patterns (with additional arcs at the puncture tagged differently than x) hold, giving us multicurves that have strictly fewer intersections with x.

3. Suppose that x has both ends at the puncture, and Y has a single end at the puncture which is tagged differently than x. Here, the local contribution to $\mu(x, Y)$ is the value of $D = 2$.



In this local picture, the first case is homotopic to a multicurve that does not intersect x, and the second case is homotopic to a multicurve that intersects x once.

4. Suppose that x has both ends at the puncture, and Y has at least 2 ends at the puncture tagged differently than x. Here, the local contribution to $\mu(x, Y)$ is the value of $D \geq 4$. When $D = 4$, there are six possibilities, up to reflection and rotation.



In this local picture, all cases are homotopic to a multicurve that has strictly fewer intersections with x. In the left column, we have $A + D = 1 + 2 = 3 < 4$; in the right column, we have $A + D = 0 + 2 = 2 < 4$. For $D > 4$, the same patterns (with additional arcs at the puncture tagged differently than x) hold, giving us multicurves that have strictly fewer intersections with x. In particular, the value of D will be cut in half, while A will increase by a number less than the number of ends of Y incident to the puncture, so the total number of intersections has decreased.

Hence, Z is homotopic to a simple multicurve Z', such that $x \cdot Z'$ is transverse and $x \cdot Z'$ has strictly fewer crossings than $x \cdot Y$. Since the latter already has $\mu(x, Y)$ crossings,

$$\mu([x], [Z']) \leq \mu(x, Y) - 1.$$

Because $\mathrm{Supp}([x][Y]) \subset I$,

$$\mu([x], [x][Y]) \leq \mu([x], [Y]) - 1.$$

The general form of the lemma follows from this case. $\square$

Corollary 5.3.8. *If x is a simple tagged arc in Σ, then for all $y \in \mathrm{Sk}^{\bowtie}(\Sigma)$,*

$$\mu([x], [x]^{\mu([x],y)} y) = 0.$$

The proof is identical to that of [Mul16, Corollary 4.13].

Proof. By iterating Lemma 5.3.7, if $i \leq \mu([x], [x]^i y)$, then $\mu([x], [x]^i y) \leq \mu([x], y) - i$. In particular, $\mu([x], [x]^{\mu([x],y)} y) \leq 0$, so it is zero. $\square$

By replacing the reference to [Mul16, Corollary 4.13] in the proof of Proposition 5.3.1 with a reference to Proposition 5.3.8, and replacing $\mathrm{Sk}_1(\Sigma)$ with $\mathrm{Sk}^{\bowtie}(\Sigma)$, we now have the following results for punctured surfaces.

Proposition 5.3.9. *Let Σ be a triangulable marked surface, and let S be a compatible collection of non-boundary (tagged) marked arcs in Σ. Then the map $\Sigma \smallsetminus S \to S$ induces an isomorphism*

$$\mathcal{A}(\Sigma \smallsetminus S)/\langle s' - s'', \forall s \in S \rangle \xrightarrow{\sim} \mathcal{A}(\Sigma)[S^{-1}]$$

Here, s' and s'' denote the two preimages of s in $\Sigma \smallsetminus S$.

Proposition 5.3.10. *Let Σ be a triangulable marked surface, and let S be a compatible collection of non-boundary (tagged) marked arcs in Σ. Then restricting values from (tagged) marked curves in Σ to (tagged) marked curves in $\Sigma \smallsetminus S$ gives an isomorphism between:*

1. *The open subvariety of $V(\mathcal{A}(\Sigma), \Bbbk)$ consisting of points which are non-zero on S.*
2. *The closed subvariety of $V(\mathcal{A}(\Sigma \smallsetminus S), \Bbbk)$ consisting of points which are gluable along S.*

Chapter 6

Deep points of unpunctured marked surfaces

In this chapter, we classify and parameterize the deep points of unpunctured and triangulable marked surfaces.

Remark 6.0.1. Portions of this text appear in the article *Deep Points of Cluster Algebras* by James Beyer and Greg Muller [BM25], and are included here with permission of all authors.

6.1 Characterization of deep points in unpunctured surfaces

In this section, we characterize the deep points of these surface cluster algebras. Our strategy is to reduce to the polygonal case by cutting along collections of arcs called *polygonal dissections*.

Vanishing lemma

The first step in characterizing deep points of $\mathcal{A}(\Sigma)$ is to characterize which marked arcs can simultaneously vanish. A **triangle of arcs** in Σ is a compatible triple of marked arcs $\{x, y, z\}$ which bound a disk in Σ. The following lemma shows that, for connected surfaces, a deep point must kill an odd number of arcs in each triangle, generalizing Lemma 4.2.1.

Lemma 6.1.1. *Let Σ be a connected, unpunctured, triangulable marked surface, let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$, and let $\{x, y, z\}$ be a triangle of arcs in Σ.*

1. *If p kills any two of $\{x, y, z\}$, then p also kills the third.*
2. *If p is deep, then p kills at least one of $\{x, y, z\}$.*

Proof. (1) Assume that $p(x) = p(y) = 0$; note that neither x nor y can be boundary arcs. Let a be a boundary arc with an endpoint at the shared vertex of x and y. Then a, x, z form three of the four edges of a quadrilateral (Figure 6.1); let b be the other edge, and let c be the other diagonal.[1]

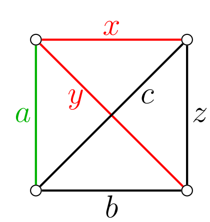


Figure 6.1: Extending a triangle to a quadrilateral in Σ

These arcs satisfy the skein relation $cy = az + bx$. Applying p to this relation gives

$$p(c)p(y) = p(a)p(z) + p(b)p(x)$$

Since $p(x) = p(y) = 0$ and $p(a) \neq 0$ (as a is a boundary arc), this forces $p(z) = 0$.

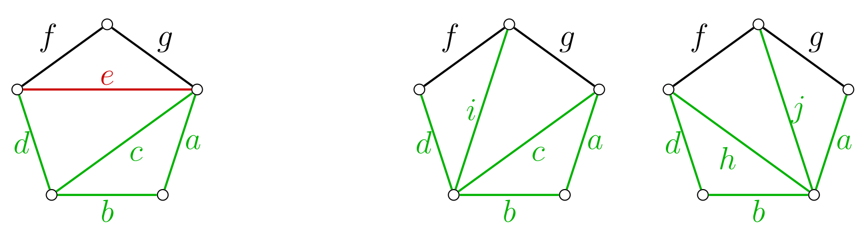
[1]Despite the figure, the arcs a, b, x, z need not be distinct.

(2) Consider $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ and assume that there is a triangle in Σ with no vanishing arcs. We claim there is a triangulation of T on which p doesn't vanish. To show this, choose a triangulation T of Σ containing the non-vanishing triangle. Either p does not vanish on all of T (and we are done), or p kills at least one arc in T.

Assume p kills at least one arc in T. Since Σ is connected, we may find adjacent triangles $\{a, b, c\}$ and $\{c, d, e\}$ in T such that p does not kill any of $\{a, b, c\}$ and kills at least one of $\{c, d, e\}$. Since $p(c) \neq 0$, part (1) implies that p kills exactly one of d and e; relabeling as needed, we assume that $p(d) \neq 0$ and $p(e) = 0$. Let $\{e, f, g\}$ be the other triangle in T containing the arc e, and consider the pentagon in Σ formed by these three triangles (Figure 6.2a); let h, i, j be the other arcs in the figure (as in Figure 6.2b). We apply part (1) to two triangles in this pentagon.

- Consider the triangle $\{a, e, h\}$. Since $p(a) \neq 0$ and $p(e) = 0$, $p(h) \neq 0$.
- Consider the triangle $\{b, i, j\}$. Since $p(b) \neq 0$, at least one of $p(i), p(j)$ is not 0.

Therefore, either $p(c)p(i) \neq 0$ or $p(h)p(j) \neq 0$. If we replace the arcs $\{c, e\}$ in T with either $\{c, i\}$ or $\{h, j\}$, we obtain a triangulation of Σ in which p kills one fewer arc than T. Iterating this argument produces a triangulation of Σ on which p does not vanish, and so p cannot be deep. $\square$



(a) Part of a triangulation in which e vanishes

(b) Two possible modifications, one of which must have fewer vanishing arcs.

Figure 6.2: Modifying a triangulation to reduce the number of vanishing arcs.

Warning 6.1.2. Unlike the polygonal case, this lemma is not enough to show that every deep point kills the same set of arcs (as in Lemma 4.2.2); in fact, different deep points of $\mathcal{A}(\Sigma)$ can kill different sets of arcs. This results in the deep locus of $\mathcal{A}(\Sigma)$ having multiple components (Theorem 6.1.15).

Polygonal dissections

To reduce to the polygonal case, we want to find collections of arcs in Σ whose cutting is a polygon.

Proposition 6.1.3. *Let Σ be a connected, triangulable marked surface. Then there exists a compatible collection of marked arcs D such that the cutting $\Sigma \setminus D$ (as defined in Section 5.3) is a polygon.*

Let us call such a set D of arcs a **polygonal dissection** of Σ.

Proof. Choose a triangulation of Σ. The dual graph of such a triangulation is connected, and therefore it admits a spanning tree. Choose a spanning tree, and let D denote the set of arcs in the triangulation which are not in the spanning tree. Then the triangulation descends to a triangulation of $\Sigma \setminus D$ whose dual graph is a tree; therefore, $\Sigma \setminus D$ is a polygon. $\square$

Proposition 6.1.4. *Let Σ be a connected, triangulable marked surface with genus g, b-many boundary components, q-many punctures, and m-many boundary marked points. Then every polygonal dissection of Σ cuts along*

$$d(\Sigma) := 2g + b + q - 1$$

many arcs and produces a

$$\delta(\Sigma) := 4g + 2b + 2q + m - 2$$

sided polygon.

Proof. Let D be a polygonal dissection of Σ and choose a triangulation T of Σ such that $D \subset T$. We can compute the Euler characteristic of Σ as $\chi(\Sigma) = V - E + F = (m+q) - |T| + F$. From Proposition 5.1.3, a triangulation of Σ consists of $|T| = n(\Sigma) = 6g+3b+3q+2m-6$ arcs, so $F = \chi(\Sigma)+n(\Sigma)-m-q = \chi(\Sigma)+(6g+3b+2q+m-6)$. Using the Euler characteristic formula $\chi(\Sigma) = 2-2g-b$, we have $F = (2-2g-b)+(6g+3b+2q+m-6) = 4g+2b+2q+m-4$.

Observe that a triangulation of the polygon $\Sigma \smallsetminus D$ has $n(\Sigma)+d(\Sigma)$ arcs which divide $\Sigma \smallsetminus D$ into F triangles. Hence, the polygon has $\delta(\Sigma) = F+2 = 4g+2b+2q+m-2$ vertices and sides. A triangulation of a polygon of $F+2$ sides will have $2(F+2)-3$ arcs, so solving the equation $n(\Sigma)+d(\Sigma) = 2(F+2)-3$, we find that the number of arcs in a polygonal dissection is $d(\Sigma) = 2g+b+q-1$. $\square$

In what follows, we will need more than the existence of polygonal dissections; we need polygonal dissections on which any given point of $\mathcal{A}(\Sigma)$ is nonzero. Since a point in $V(\mathcal{A}(\Sigma), \Bbbk)$ can never kill exactly two of the three arcs in a triangle (Lemma 6.1.1), it will suffice to prove the following.

Lemma 6.1.5. *Let Σ be a connected, unpunctured, triangulable marked surface, and let $\mathcal{V}$ be a set of simple, non-boundary marked arcs which does not contain exactly two of the three arcs in any triangle in Σ. Then there exists a polygonal dissection D of Σ consisting of arcs not in $\mathcal{V}$.*

Proof. Choose a polygonal dissection D_0 of Σ, and suppose there is some arc $p \in D_0 \cap \mathcal{V}$. Because $\mathcal{V}$ does not contain boundary arcs in Σ, some but not all edges in the boundary of $\Sigma \smallsetminus D_0$ come from arcs in $\mathcal{V}$. Choose $\alpha \in \mathcal{V}$ and $\beta \notin \mathcal{V}$ to be arcs in Σ which become adjacent edges in the boundary of $\Sigma \smallsetminus D_0$, and let γ be the arc in Σ

that completes a triangle with α and β in $\Sigma \smallsetminus D_0$. By the assumption on triangles, we know that $\gamma \notin \mathcal{V}$. Then $D_1 := (D_0 \cup \{\gamma\}) \smallsetminus \{\alpha\}$ is a polygonal dissection of Σ containing strictly fewer arcs in $\mathcal{V}$ than D_0. Repeated applications of this process will eventually produce a polygonal dissection D disjoint from $\mathcal{V}$. $\square$

The special case of $\mathcal{V} := \{a \text{ such that } p(a) = 0\}$ is the following.

Corollary 6.1.6. *Let Σ be a connected, unpunctured, triangulable marked surface. For every point $p \in V(\mathcal{A}(\Sigma), \Bbbk)$, there is a polygonal dissection D of Σ on which p is non-zero.*

This can be interpreted geometrically as follows. By Proposition 5.3.3, each polygonal dissection D determines an open embedding

$$\{\text{points in } V(\mathcal{A}(\Sigma \smallsetminus D), \Bbbk) \text{ gluable along } D\} \hookrightarrow V(\mathcal{A}(\Sigma), \Bbbk) \tag{6.1}$$

The corollary is equivalent to the fact that these open charts collectively cover the cluster variety.

Cutting and gluing deep points

We now know that every point in the cluster variety $V(\mathcal{A}(\Sigma), \Bbbk)$ is non-zero on some polygonal dissection, and is therefore contained in an open neighborhood given by gluing the cluster variety of a polygon (Equation (6.1)). Since Theorem 4.3.1 gives a characterization of the deep points in the latter, we need to understand how cutting and gluing affect deepness. The immediate result is the following.

Proposition 6.1.7. *The cutting isomorphism restricts to an inclusion on deep points; that is,*

$$\left\{ \begin{matrix} \textit{deep points in } V(\mathcal{A}(\Sigma), \Bbbk) \\ \textit{non-zero on } S \end{matrix} \right\} \hookrightarrow \left\{ \begin{matrix} \textit{deep points in } V(\mathcal{A}(\Sigma \smallsetminus S), \Bbbk) \\ \textit{gluable along } S \end{matrix} \right\}$$

Proof. Let p be a deep point of $V(\mathcal{A}(\Sigma), \Bbbk)$ which is non-zero on S, and let p' be its image under the cutting isomorphism. Any triangulation of $\Sigma \setminus S$ is the image of a triangulation of Σ which contains S. Since p is deep, it must vanish on at least one arc in the triangulation of $\Sigma \setminus S$, and so p' must vanish on at least one arc in the triangulation of Σ. $\square$

Unfortunately, this restriction is not always a bijection! As the following example shows, the cutting of a non-deep point may become deep.

Example 6.1.8. For any non-zero $a \in \Bbbk$, the values depicted in Figure 6.3 uniquely determine a point in $V(\mathcal{A}(\Delta_6), \Bbbk)$ which is not deep (since no arc in the given triangulation vanishes), but which becomes deep when cut along a certain diagonal (since any triangulation of the disconnected surface must contain one of the two vanishing diagonals in the right-hand component).

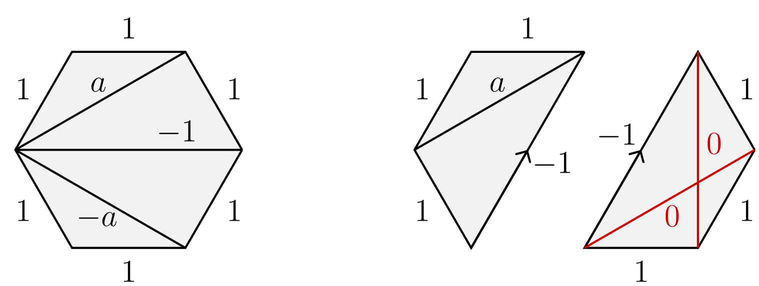


Figure 6.3: A non-deep point (left) which becomes deep when cut (right)

The problem is that deep points of a disconnected surface behave counterintuitively. If Σ is a disjoint union of two components Σ_1 and Σ_2, then a point p of $V(\mathcal{A}(\Sigma), \Bbbk)$ is equivalent to a pair of points p_1 and p_2 of $V(\mathcal{A}(\Sigma_1), \Bbbk)$ and $V(\mathcal{A}(\Sigma_2), \Bbbk)$, respectively; that is,

$$V(\mathcal{A}(\Sigma), \Bbbk) \simeq V(\mathcal{A}(\Sigma_1), \Bbbk) \times V(\mathcal{A}(\Sigma_2), \Bbbk)$$

Since a triangulation of Σ is a union of triangulations of Σ_1 and Σ_2, we see that a point p of $V(\mathcal{A}(\Sigma), \Bbbk)$ is deep if and only if *either* of the points p_1 or p_2 are deep.

We introduce a definition which behaves better for disconnected surfaces and which won't be used beyond the following lemma. Let us say a point p in $V(\mathcal{A}(\Sigma), \Bbbk)$ is **deep-on-components** if the restriction of p to each connected component of Σ is deep; that is, it kills a marked arc in each triangulation of each connected component. Every deep-on-components point is deep, but the converse fails (for example, the deep point on the right side of Figure 6.3 is not deep-on-components).

Many prior results extend to disconnected surfaces by replacing *deep* with *deep-on-components*. For example, applying Lemma 6.1.1 to each connected component of Σ, we see that a point p of $V(\mathcal{A}(\Sigma), \Bbbk)$ is deep-on-components if and only if it kills an odd number of arcs in any triangle in Σ.

Lemma 6.1.9. *Let Σ be an unpunctured, triangulable marked surface. For any compatible collection of non-boundary arcs S in Σ, the cutting isomorphism* (5.4) *restricts to a bijection between:*

1. *The deep-on-components points of $V(\mathcal{A}(\Sigma), \Bbbk)$ which are non-zero on S.*
2. *The deep-on-components points of $V(\mathcal{A}(\Sigma \smallsetminus S), \Bbbk)$ which are gluable along S.*

Proof. Fix a point $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ which is non-zero on S, and let $p' \in V(\mathcal{A}(\Sigma \smallsetminus S), \Bbbk)$ be its image under the cutting isomorphism.

$(1) \Rightarrow (2)$. If p is deep-on-components, then it kills at least one marked arc in every triangle of Σ (by Lemma 6.1.1 for each connected component). If Σ' is a connected component of $\Sigma \smallsetminus S$, then p' kills at least one marked arc in every triangle in Σ'; in particular, it kills at least one marked arc in every triangulation of Σ'. Therefore, p' is deep-on-components.

$(2) \Rightarrow (1)$. Assume that p' is deep-on-components, and consider various cases.

First, assume Σ is a polygon; note that this forces $\Sigma \smallsetminus S$ to be a disjoint union of polygons. Index the elements of S as $\alpha_1, \alpha_2, \ldots, \alpha_n$. For any i between 0 and n, let V_i denote the set of points in $V(\mathcal{A}(\Sigma \smallsetminus \{\alpha_1, \alpha_2, \ldots, \alpha_i\}), \Bbbk)$ which are gluable along $\{\alpha_1, \alpha_2, \ldots, \alpha_i\}$ and non-zero on $\{\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n\}$. Then the cutting isomorphism (5.4) factors as a composition of isomorphisms

$$V_0 \xrightarrow{\sim} V_1 \xrightarrow{\sim} \cdots \xrightarrow{\sim} V_{n-1} \xrightarrow{\sim} V_n$$

where each $V_i \to V_{i+1}$ is (a restriction of) the cutting isomorphism along α_{i+1}. Each of these cuts divides a polygon into two smaller polygons; by Corollary 4.3.3, a point which is deep on both smaller polygons corresponds to a point which is deep on their gluing. As a consequence, a deep-on-components point in V_{i+1} glues to a deep-on-components point in V_i. Iterating along the composition above, the deep-on-components point p' glues to a deep-on-components point p.

Next, assume that Σ is connected and $\Sigma \smallsetminus S$ is a disjoint union of polygons. Then the universal cover $\widehat{\Sigma}$ of Σ can be covered by the lifts of these polygons. Consider three marked arcs $\{\alpha, \beta, \gamma\}$ which bound a triangle in Σ, and choose lifts $\{\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}\}$ which bound a triangle in $\widehat{\Sigma}$. Since this triangle is compact, it is contained in the union of finitely many lifts of components in $\Sigma \smallsetminus S$. Let Σ' denote this union; since it is a finite union of polygons in a simply connected space, it is also a polygon. Since the point p' is deep-on-components, it gives a deep point on each of the lifts of these components. By the previous case, these deep points glue together to a deep point $\widehat{p}$ on the polygon Σ'. By Lemma 4.2.1, an odd number of $\widehat{p}(\widehat{\alpha}), \widehat{p}(\widehat{\beta}), \widehat{p}(\widehat{\gamma})$ must vanish, and so an odd number of $p(\alpha), p(\beta), p(\gamma)$ must vanish. Therefore, p kills an odd number of marked arcs in every triangle, and so it is deep-on-components.

Next, assume that Σ is connected. By Corollary 6.1.6, there are polygonal dissections of each connected component of $\Sigma \setminus S$ on which p' is non-zero.

By redefining S to be the union of the original S together with the arcs in these polygonal dissections, we reduce to the previous case.

Finally, if Σ is disconnected, then we may apply the previous case to each component of Σ and deduce that p is deep-on-components. $\square$

In what follows, we only need the specialization of Lemma 6.1.9 in the case of polygonal dissections.

Corollary 6.1.10. *Let Σ be a connected, unpunctured, triangulable marked surface. For any polygonal dissection D in Σ, the cutting isomorphism* (5.4) *restricts to a bijection between:*

1. *The deep points in $V(\mathcal{A}(\Sigma), \Bbbk)$ which are non-zero on D.*
2. *The deep points in $V(\mathcal{A}(\Sigma \setminus D), \Bbbk)$ which are gluable along D.*

Comparing polygonal dissections

While Corollary 6.1.10 allows us to construct deep points of $\mathcal{A}(\Sigma)$ by gluing deep points of polygons, a single deep point of $\mathcal{A}(\Sigma)$ may be non-zero on many polygonal dissections, and can therefore be constructed via many different gluings.

In this section, we use relative cohomology classes in $H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$ to characterize when a deep point is non-zero on a polygonal dissection, and when two polygonal dissections determine the same set of deep points. Note that every marked arc in Σ is a 1-cycle relative to the marked points $\mathcal{M}$, and so elements of $H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$ have a well-defined value on each marked arc in Σ.

Proposition 6.1.11. *Let Σ be a connected, unpunctured, triangulable marked surface with an even number of marked points.*

1. *Given a deep point p of $V(\mathcal{A}(\Sigma), \Bbbk)$, there is a unique cohomology class $\chi_p \in H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$ such that the value of χ_p on each simple marked arc a in Σ is*

$$\chi_p(a) \coloneqq \left\{ \begin{array}{ll} 0 & \textit{if } p(a) = 0 \\ 1 & \textit{if } p(a) \neq 0 \end{array} \right\} \tag{6.2}$$

2. *Given a polygonal dissection D of Σ, there is a unique cohomology class $\chi_D \in H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$ such that χ_D is non-zero on each marked arc in D and each boundary arc of Σ.*
3. *A deep point p is non-zero on a polygonal dissection D if and only if $\chi_p = \chi_D$.*

Proof. A (marked) triangulation T of Σ determines a simplicial decomposition of Σ whose 0-cells are the marked points $\mathcal{M}$, whose 1-cells are the marked arcs in T, and whose 2-cells are the triangles in $\Sigma \smallsetminus T$. Since singular and simplicial cohomology are isomorphic in this case, we may compute the cohomology groups $H^\bullet(\Sigma, \mathcal{M}; \mathbb{Z}_2)$ as the cohomology of the reduced simplicial cochain complex

$$\underbrace{\mathbb{Z}_2^{\mathcal{M}} / \mathbb{Z}_2^{\mathcal{M}}}_{\simeq 0} \to \mathbb{Z}_2^T \to \mathbb{Z}_2^{\Sigma \smallsetminus T}$$

As a consequence, elements of $H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$ may be realized as functions $T \to \mathbb{Z}_2$ such that the sum of the values around each triangle is 0 (the 1-cocycle condition). In particular:

- Given a deep point p of $V(\mathcal{A}(\Sigma), \Bbbk)$, choose a triangulation T and define a function $\chi_p : T \to \mathbb{Z}_2$ by applying Formula (6.2) to each marked arc in T. The sum around each triangle in T is 0 by Lemma 6.1.1, so this defines an element $\chi_p \in H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$.

  Given a non-boundary arc a in T, let a' be the *flip* of a; that is, the unique-up-to-homotopy marked arc such that $T \smallsetminus \{a\} \cup \{a'\}$ is a triangulation and a is not homotopic to a'. Choose arcs $b, c \in T$ such that a', b, c bound a triangle.

By Lemma 6.1.1, $p(a') = 0$ if and only if an even number of $p(b)$ and $p(c)$ are 0; similarly, by the cocycle condition, $\chi_p(a') = 0$ if and only if an even number of $\chi_p(b)$ and $\chi_p(c)$ are 0. Therefore, $\chi_p(a') = 0$ if and only if $p(a) = 0$, so Formula (6.2) holds for a'. Since any simple marked arc in Σ can be realized by a sequence of flips, Formula (6.2) holds for all simple marked arcs in Σ.

- Consider a polygonal dissection D of Σ. For any triangulation T of Σ which contains D, define a function $\chi_D : T \to \mathbb{Z}_2$ by the rule that, for each a in T,

$$\chi_D(a) \coloneqq \left\{ \begin{array}{ll} 0 & \text{if } \Sigma \smallsetminus (D \cup \{a\}) \text{ consists of two odd-sided polygons} \\ 1 & \text{if } \Sigma \smallsetminus (D \cup \{a\}) \text{ consists of two even-sided polygons} \\ 1 & \text{if } a \text{ is in } D \text{ or } \partial\Sigma \end{array} \right\}$$

  It is straight-forward to show this satisfies the 1-cocycle condition (for example, by induction on the size of the smaller polygon in $\Sigma \smallsetminus (D \cup \{a\})$), so it defines an element in $H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$.

Since each χ_p is non-zero on every boundary arc in Σ, the uniqueness condition in (2) implies that $\chi_p = \chi_D$ if and only if χ_p is non-zero on the arcs in D; equivalently, if p is non-zero on D. $\square$

As a consequence, two polygonal dissections can be used to construct the same deep points if and only if they have the same cohomology class. We use this to define an equivalence relation.

Definition 6.1.12. Let Σ be as in Proposition 6.1.11. Then two polygonal dissections of Σ are **congruent** if their relative cohomology classes are equal.

Proposition 6.1.13. *Let Σ be a connected, unpunctured, triangulable marked surface with an even number of marked points. Then the function $D \mapsto \chi_D$ defines a bijection between*

1. *polygonal dissections D of Σ up to congruence, and*

2. *elements* χ *in* $H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$ *with value* 1 *on every boundary marked arc in* Σ.

The size of each set is 2^{2g+b-1}, *where* g *is the genus and* b *is the number of boundary components.*

Proof. The function $D \mapsto \chi_D$ is injective by the definition of congruence. Given χ satisfying (2), the set of simple marked arcs killed by χ contains an odd number of arcs in each triangle. By Lemma 6.1.5, there exists a polygonal dissection D on which χ is non-zero; by the uniqueness in Proposition 6.1.11.2, $\chi_D = \chi$. Therefore, the function $D \mapsto \chi_D$ is surjective.

For cardinality, consider (part of) the long exact sequence induced by the triple $(\Sigma, \partial\Sigma, \mathcal{M})$.

$$\cdots \to H^0(\partial\Sigma, \mathcal{M}; \mathbb{Z}_2) \to H^1(\Sigma, \partial\Sigma; \mathbb{Z}_2) \xrightarrow{f} H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2) \xrightarrow{g} H^1(\partial\Sigma, \mathcal{M}; \mathbb{Z}_2) \to \cdots$$

Since every component of $\partial\Sigma$ contains a marked point, $H^0(\partial\Sigma, \mathcal{M}; \mathbb{Z}_2) = 0$ and so the map labeled f above is injective. The map labeled g above sends χ to its values on the boundary arcs, and so the set of relative cohomology classes coming from polygonal dissections is precisely the preimage of the element in $H^1(\partial\Sigma, \mathcal{M}; \mathbb{Z}_2)$ which sends each boundary arc to 1. Since this preimage is non-empty, it has the same cardinality of as the preimage of 0; that is, the kernel of g. Since f is injective and the sequence is exact, the kernel of g has the same cardinality as $H^1(\Sigma, \partial\Sigma; \mathbb{Z}_2)$.

The dimension of this relative cohomology group can be computed via Euler characteristics.

$$\begin{aligned}
&\sum_{i=0}^{2} (-1)^i \dim_{\mathbb{Z}_2}(H^i(\Sigma, \partial\Sigma; \mathbb{Z}_2)) \\
&\qquad = \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{Z}_2}(H^i(\Sigma; \mathbb{Z}_2)) - \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{Z}_2}(H^i(\partial\Sigma; \mathbb{Z}_2)) \\
&\qquad = (2 - 2g - b) + 0
\end{aligned}$$

Since Σ is connected, $\dim(H^0(\Sigma, \partial\Sigma; \mathbb{Z}_2)) = 0$ and $\dim(H^2(\Sigma, \partial\Sigma; \mathbb{Z}_2)) = 1$. Therefore,

$$\dim_{\mathbb{Z}_2}(H^1(\Sigma, \partial\Sigma; \mathbb{Z}_2)) = 2g + b - 1$$

and so the cardinality of $H^1(\Sigma, \partial\Sigma; \mathbb{Z}_2)$ is 2^{2g+b-1}. $\square$

Remark 6.1.14. Congruence classes of polygonal dissections are in bijection with several other sets of potential interest; we explore these different interpretations in Section 6.2.

Deep points of marked surfaces

We can now combine our results to characterize and parameterize the deep locus of $V(\mathcal{A}(\Sigma), \mathbb{k})$.

Theorem 6.1.15. *Let Σ be a connected, unpunctured, triangulable marked surface with genus g, b-many boundary components, and m-many marked points.*

1. *If m is odd, then $V(\mathcal{A}(\Sigma), \mathbb{k})$ has no deep points.*
2. *If m is even, then each deep point of $V(\mathcal{A}(\Sigma), \mathbb{k})$ is non-zero on some polygonal dissection.*
   a) *Given a polygonal dissection D of Σ, a function*

   $$p : \{\textit{boundary arcs in } \Sigma\} \cup D \to \mathbb{k}^\times$$

   extends to a (necessarily unique) deep point of $V(\mathcal{A}(\Sigma), \mathbb{k})$ which is non-zero on D if and only if the alternating product of the values of p around the boundary of $\Sigma \smallsetminus D$ is $(-1)^{b+\frac{m}{2}}$.

   b) *Two polygonal dissections of Σ parameterize the same deep points if and only if they are congruent (see Definition 6.1.12); otherwise they parameterize disjoint sets of deep points.*

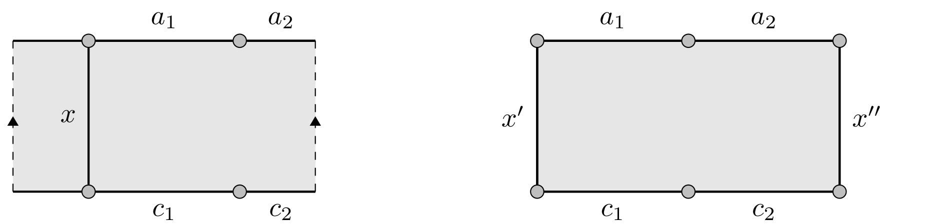
Therefore, the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ consists of 2^{2g+b-1}-many disjoint copies of $(\Bbbk^\times)^{2g+b+m-2}$.

Proof. Let D be a polygonal dissection of Σ. By Corollary 6.1.10, deep points of $V(\mathcal{A}(\Sigma), \Bbbk)$ which are non-zero on D are in bijection with deep points of $V(\mathcal{A}(\Sigma \smallsetminus D), \Bbbk)$ which are gluable along D. By Proposition 6.1.4, D contains $(2g + b - 1)$-many arcs, and $\Sigma \smallsetminus D$ is a polygon with $(4g + 2b + m - 2)$-many sides. By Theorem 4.3.1, $V(\mathcal{A}(\Sigma \smallsetminus D), \Bbbk)$ has no deep points if $(4g + 2b + m - 2)$ is odd, or equivalently, if m is odd. This verifies Part (1) of the theorem.

If m is even, then $V(\mathcal{A}(\Sigma \smallsetminus D), \Bbbk)$ has a unique deep point for each choice of values on the boundary of $\Sigma \smallsetminus D$, subject to the product in Condition (4.2). If we fix a boundary arc α in Σ, then we may freely choose the values of p on the $(m-1)$-many other boundary arcs and the $(2g + b - 1)$-many arcs in D to be any non-zero numbers in $\Bbbk$. Since $p(\alpha)$ will appear once in the alternating product, there is a unique non-zero value which makes Condition (4.2) true. Therefore, a deep point in $V(\mathcal{A}(\Sigma), \Bbbk)$ which is non-zero on D is freely determined by the choice of $(2g + b + m - 2)$-many non-zero elements in $\Bbbk$. This verifies part (2a) and the shape of each deep component.

If two polygonal dissections have any deep points in common, they must have the same cohomology class by Proposition 6.1.11, in which case all of their deep points are the same. This verifies Part (2b). Since there are 2^{2g+b-1}-many equivalence classes of polygonal dissections (Proposition 6.1.13), this will be the number of components of the deep locus. $\square$

Example 6.1.16. Let Σ be the $(2, 2)$-annulus, the genus $g = 0$ surface with $b = 2$ boundary components and 2 marked points on each boundary, for a total of $m = 4$ marked points. Figure 6.4a depicts Σ as a rectangle with opposite sides identified (drawn dashed). The marked points divide each boundary component into two boundary arcs, which we label a_1, a_2 and c_1, c_2 as in Figure 6.4.



(a) The $(2,2)$-annulus Σ with marked arc x

(b) A hexagon Δ formed by cutting Σ along the polygonal dissection $D = \{x\}$.

Figure 6.4: A polygonal dissection of the (2,2)-annulus.

The theorem predicts that the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ consists of 2 copies of $(\Bbbk^\times)^4$; we show this now. Consider the polygonal dissection of Σ consisting of the marked arc x in Figure 6.4a, whose cutting is the hexagon Δ in Figure 6.4b (note that the arc x becomes boundary arcs x' and x'' in Δ). By Theorem 4.3.1, a function

$$p : \{a_1, a_2, c_1, c_2, x', x''\} \to \Bbbk^\times$$

extends to a (necessarily unique) deep point in $V(\mathcal{A}(\Delta), \Bbbk)$ if and only if Condition (4.2) holds; that is,

$$\frac{p(a_1)p(x'')p(c_1)}{p(a_2)p(c_2)p(x')} = 1 = (-1)^{\frac{6+2}{2}}$$

Such a deep point is gluable along x if and only if $p(x') = p(x'')$; therefore, a function

$$p : \{a_1, a_2, c_1, c_2, x\} \to \Bbbk^\times$$

extends to a (necessarily unique) deep point in $V(\mathcal{A}(\Sigma), \Bbbk)$ if and only if

$$\frac{p(a_1)p(x)p(c_1)}{p(a_2)p(c_2)p(x)} = \frac{p(a_1)p(c_1)}{p(a_2)p(c_2)} = 1 = (-1)^{2+\frac{4}{2}}$$

Note that this condition can be used to determine one value in terms of the other three. Therefore, a deep point in $V(\mathcal{A}(\Sigma), \Bbbk)$ which is non-zero on $\{x\}$ is freely determined by a function

$$p : \{a_1, a_2, c_1, x\} \to \Bbbk^\times$$

So, the set of such deep points forms a copy of $(\Bbbk^\times)^4$ inside $V(\mathcal{A}(\Sigma), \Bbbk)$.

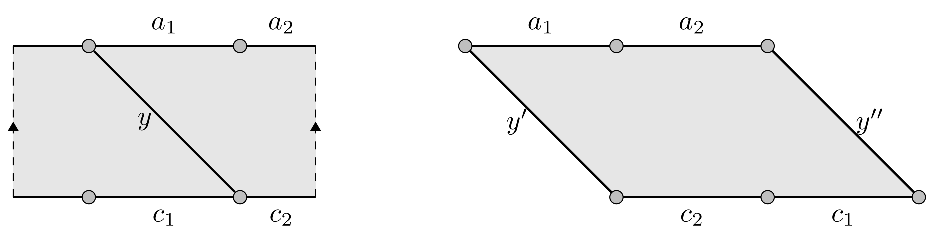
The other component of the deep locus can be parametrized by finding a polygonal dissection which is not congruent to the first; for example, the marked arc y in Figure 6.5a. By the same logic as before, a function

$$p : \{a_1, a_2, c_1, c_2, y\} \to \Bbbk^\times$$

extends to a (necessarily unique) deep point of $\mathcal{A}(\Sigma)$ if and only if

$$\frac{p(a_1)p(c_2)}{p(a_2)p(c_1)} = 1$$

The set of such deep points forms another copy of $(\Bbbk^\times)^4$ inside $V(\mathcal{A}(\Sigma), \Bbbk)$.



(a) The $(2, 2)$-annulus Σ, with marked arc y

(b) A hexagon Δ formed by cutting Σ along the polygonal dissection $D = \{y\}$.

Figure 6.5: Another polygonal dissection of the (2,2)-annulus.

6.2 The deep components of unpunctured marked surfaces

By Theorem 6.1.15, the deep locus of an unpunctured marked surface with an even number of marked points consists of 2^{2g+b-1}-many disjoint algebraic tori. Proposition 6.1.13 showed these tori may be indexed by either (a) polygonal dissections of

Σ up to congruence, or (b) relative cohomology classes in $H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$ with value 1 on the boundary arcs.

In this section, we construct several other sets which index the components of the deep locus. The results of this section are summarized by the following theorem.

Theorem 6.2.1. *Let Σ be a connected, unpunctured, triangulable marked surface with an even number of marked points. Then the following sets are in bijection.*

1. *Components of the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$, for a fixed field $\Bbbk$.*
2. *Polygonal dissections of Σ, up to congruence.*
3. *Relative cohomology classes in $H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$ with value 1 on the boundary arcs.*
4. Vanishing classes *of Σ; that is, sets of simple, non-boundary marked arcs in Σ which contain an odd number of arcs in every triangle.*
5. *Deep points in $V(\mathcal{A}(\Sigma), \mathbb{Z}_2)$.*
6. *Subsets of D, for a fixed polygonal dissection D of Σ.*
7. Alternating double covers *of Σ; that is, double covers of Σ with an alternating coloring of their marked points, such that the non-trivial deck transformation swaps colors.*

Remark 6.2.2. We construct bijections between these sets which are 'canonical', in that they commute with diffeomorphisms that preserve the marked points and orientation. This is important, because the set of spin structures on Σ will always have the same cardinality as each of the sets in Theorem 6.2.1; however, it is impossible to choose a bijection which commutes with such diffeomorphisms (see the subsection on Spin structures below).

Convention 6.2.3. In this section, Σ will always denote a connected, unpunctured, triangulable marked surface with genus g, b-many marked points, and an even number m of marked points.

Vanishing classes

The elementary tool in Section 6 is Lemma 6.1.1, which states that a deep point of $V(\mathcal{A}(\Sigma), \Bbbk)$ must vanish on an odd number of arcs in each triangle in Σ. In contrast with the polygonal case (Lemma 4.2.1), this does not uniquely determine the set of simple arcs (i.e. cluster variables) killed by a given deep point. We give such a set of arcs a name.

Definition 6.2.4. A **vanishing class** of Σ is a subset $\mathcal{V}$ of the set of simple, non-boundary marked arcs in Σ which contains an odd number of arcs in each triangle in Σ.

By Lemma 6.1.1, each deep point p of $V(\mathcal{A}(\Sigma), \Bbbk)$ determines a vanishing class

$$\mathcal{V}(p) := \{\text{simple marked arcs } a \text{ in } \Sigma \text{ such that } p(a) = 0\} \tag{6.3}$$

Note that a vanishing class is usually infinite.

Example 6.2.5. A (unpunctured) polygon has a unique vanishing class, consisting of the set of arcs whose two indices differ by an even integer (c.f. Lemma 4.2.2).

Example 6.2.6. The $(2, 2)$-annulus has 2 vanishing classes. One way to describe these classes is via the two colorings of the marked points in Figure 6.6. For each, the set of simple marked arcs whose endpoints have the same colors is a vanishing class, and these are the only two vanishing classes.

Warning 6.2.7. It is not always possible to describe the vanishing classes by coloring the vertices of Σ, as in Figure 6.6! More generally, one needs to pass to a double cover (see below).

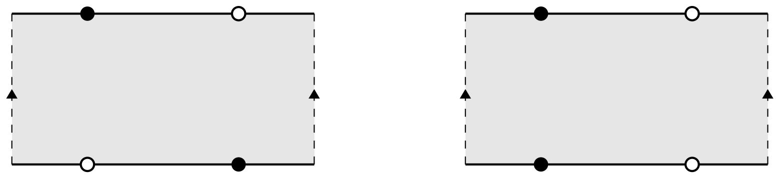


Figure 6.6: Coloring vertices to describe vanishing classes of the $(2,2)$-annulus

Proposition 6.2.8. *The vanishing class of a deep point (Equation* (6.3)*) is constant on components of the deep locus of* $V(\mathcal{A}(\Sigma), \Bbbk)$*, and induces a bijection between components of the deep locus of* $V(\mathcal{A}(\Sigma), \Bbbk)$ *and vanishing classes of* Σ*.*

Proof. By Proposition 6.1.11.1, the cohomology class χ_p of a deep point p is uniquely determined by the set of simple non-boundary arcs it sends to 0; that is, its vanishing class $\mathcal{V}(p)$. Therefore, $\mathcal{V}(p) = \mathcal{V}(p')$ if and only if $\chi_p = \chi_{p'}$ if and only if p and p' are in the same component of the deep locus.

To show surjectivity, consider a vanishing class $\mathcal{V}$. By Lemma 6.1.5, we can find a polygonal dissection D disjoint from $\mathcal{V}$. By Theorem 6.1.15, we can find a deep point p of $V(\mathcal{A}(\Sigma), \Bbbk)$ which is non-zero on D. By Proposition 6.1.11, $\chi_p = \chi_D$; therefore, $\mathcal{V}(p) = \mathcal{V}$. $\square$

Corollary 6.2.9. *Every polygonal dissection of* Σ *is disjoint from a unique vanishing class of* Σ*.*

Proof. Existence follows from Lemma 6.1.5. Let $\mathcal{V}$ and $\mathcal{V}'$ be vanishing classes disjoint from a polygonal dissection D. By Proposition 6.2.8, there are deep points p, p' such that $\mathcal{V} = \mathcal{V}(p)$ and $\mathcal{V}' = \mathcal{V}(p')$. Since p and p' are non-zero on D, $\chi_p = \chi_D = \chi_{p'}$; therefore, $\mathcal{V} = \mathcal{V}'$. $\square$

Deep points over $\mathbb{Z}_2$

When $\Bbbk = \mathbb{Z}_2$ is the field with two elements, the algebraic torus $(\Bbbk^\times)^n$ consists of a single point for any n. Therefore, the deep points of $V(\mathcal{A}(\Sigma), \mathbb{Z}_2)$ are in bijection with polygonal dissections of D up to congruence by Theorem 6.1.15, and so they may be used to index the deep components of $V(\mathcal{A}(\Sigma), \Bbbk)$ for any field $\Bbbk$.

This correspondence may be realized more directly as follows.

Proposition 6.2.10. *Let $\mathcal{V}$ be a vanishing class of Σ. Then the function from the set of simple marked arcs in Σ to $\mathbb{Z}_2$ defined by*

$$a \mapsto \left\{ \begin{array}{ll} 0 & \text{if } a \in \mathcal{V} \\ 1 & \text{if } a \notin \mathcal{V} \end{array} \right\} \tag{6.4}$$

1. *extends uniquely to a relative cohomology class $\chi_{\mathcal{V}} \in H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$ with value 1 on the boundary arcs, and*
2. *extends uniquely to a deep point $p_{\mathcal{V}} \in V(\mathcal{A}(\Sigma), \mathbb{Z}_2)$; that is, a ring homomorphism $\mathcal{A}(\Sigma) \to \mathbb{Z}_2$ which kills at least one marked arc in each triangulation.*

Furthermore, every such relative cohomology class and deep point arises in this way.

Proof. Part (1) is a restatement of Proposition 6.1.11.1. To show Part (2), let $\mathcal{V}$ be a vanishing class of Σ. By Proposition 6.2.8 and the fact that $|\mathbb{Z}_2^\times| = 1$, there is a unique deep point $p \in V(\mathcal{A}(\Sigma), \mathbb{Z}_2)$ such that $\mathcal{V}(p) = \mathcal{V}$. For any simple marked arc a, $p(a) = 0$ if and only if $a \in \mathcal{V}(p)$ by Equation (6.3). Since there is only one other element in $\mathbb{Z}_2$, $p(a) = 1$ if and only if $a \notin \mathcal{V}(p)$. Therefore, p extends the function given in Equation (6.4).

Therefore, these constructions define injections from the set of vanishing classes to (1) the sets of relative cohomology classes with value 1 on the boundary and (2) deep points in $V(\mathcal{A}(\Sigma), \mathbb{Z}_2)$. Since all three sets have cardinality 2^{2g+b-1}, these constructions are bijections. $\square$

Subsets of a polygonal dissection

One of the simpler characterizations of the deep components is by the subset of arcs they kill in a fixed polygonal dissection.

Proposition 6.2.11. *Let D be a polygonal dissection of Σ. For every subset $S \subset D$, there is a unique vanishing class $\mathcal{V}$ such that $D \cap \mathcal{V} = S$.*

Proof. By Corollary 6.2.9, there exists a unique vanishing class of Σ disjoint from D.

For our inductive hypothesis, let $S \subseteq D$ be nonempty and assume that for each proper subset $S' \subsetneq S$ and every polygonal dissection D' containing S', there exists a unique vanishing class $\mathcal{V}'$ such that $D' \cap \mathcal{V}' = S'$. Because Σ has nonempty boundary, we can choose arcs $\alpha \in S$ and $\beta \notin S$ which are adjacent edges in the boundary of $\Sigma \smallsetminus D$. Let γ be the arc in Σ that completes a triangle with α and β in $\Sigma \smallsetminus D$. Then $D_0 \coloneqq (D \cup \{\gamma\}) \smallsetminus \{\alpha\}$ is a polygonal dissection, and $S_0 \coloneqq S \smallsetminus \{a\}$ is a proper subset of S that is contained in D_0. By the inductive hypothesis, there exists a unique vanishing class $\mathcal{V}_0$ such that $D_0 \cap \mathcal{V}_0 = S_0$. Since $\{\beta, \gamma\} \subset (D_0 \cup \partial\Sigma)$ and $\{\beta, \gamma\} \cap S_0 = \emptyset$, Lemma 6.1.1 implies that $\alpha \in \mathcal{V}_0$. Hence, $\mathcal{V}_0$ is the unique vanishing class such that $D \cap \mathcal{V}_0 = S$. $\square$

Note that the number of arcs in D is $2g + b - 1$ by Proposition 6.1.4, so the number of subsets of D is 2^{2g+b-1}. By Theorem 6.1.15 and Proposition 6.2.8, the number of vanishing classes is also 2^{2g+b-1}. Since the sets have the same cardinality, Proposition 6.2.11 implies that these sets are in bijection.

Alternating double covers

It is well-known that elements of $H^1(\Sigma; \mathbb{Z}_2)$ may be identified with double covers of Σ up to equivalence. Over the next several sections, we relate elements of $H^1(\Sigma, \mathcal{M}; \mathbb{Z}_2)$ with value 1 on the boundary arcs to double coverings of Σ endowed with an *alternating coloring* of their marked points.

Definition 6.2.12. An **alternating double cover** of Σ is a (not necessarily connected) 2-fold covering space[2] $\Sigma' \to \Sigma$, with a black and white coloring of the marked points of Σ', such that

- every pair of adjacent marked points in Σ' has opposite colors, and
- the preimages of each marked point in Σ consists of one point of each color.

An **equivalence** of alternating double covers of Σ is a diffeomorphism which commutes with the covering map to Σ and either sends every marked point to a marked point of the same color, or sends every marked point to a marked point of the opposite color.

Swapping the colors in an alternating double cover yields an equivalent double cover; for this reason, the specific colors do not matter as much as the underlying partition of the marked points. We say an alternating double cover is **trivial** if the underlying covering space Σ' is a trivial double cover of Σ (that is, two copies of Σ).

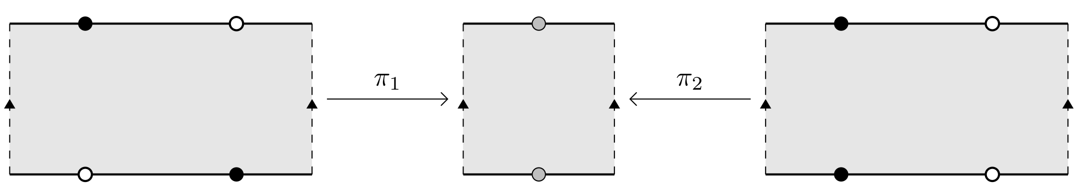


Figure 6.7: Alternating double covers of the $(1, 1)$-annulus (up to equivalence)

[2]A **covering space** of a marked surface is a covering space of the underlying surface whose marked points are the points which map to marked points in the original surface.

Example 6.2.13. The $(1, 1)$-annulus (that is, the annulus with 1 marked point on each boundary component) has two alternating double covers up to equivalence (Figure 6.7). Note that a trivial double cover consisting of two copies of Σ would have been allowed, except there is no way to color the marked points in such a double cover so as to satisfy the alternating condition.

Constructing alternating double covers

The natural way to construct an alternating double cover is to first choose a (not necessarily connected) double cover of Σ, and then attempt to color the marked points in Σ'. The following proposition summarizes when and how this works.

Proposition 6.2.14. *A double cover $\Sigma' \to \Sigma$ admits an alternating coloring if and only if*

- *each component in $\partial\Sigma$ with an odd number of marked points has connected preimage, and*
- *each component in $\partial\Sigma$ with an even number of marked points has disconnected preimage.*

Such a double cover exists if and only if the total number of marked points in Σ is even, in which case there are 2^{2g}-many such double covers and each admit 2^{b-1}-many alternating colorings up to equivalence.

Remark 6.2.15. So, Σ admits a trivial alternating double cover if and only if each component of $\partial\Sigma$ has an even number of marked points, in which case there are 2^{b-1}-many trivial alternating double covers.

Proof. Consider a double cover $\Sigma' \to \Sigma$, and let S_i be a boundary component of Σ with m_i-many marked points.

1. If the preimage of S_i in Σ' is connected, then this preimage is a single circle with $2m_i$-many marked points. There are always exactly two ways to color

these points so that adjacent points have opposite colors. However, the two preimages of a marked point in S_i are m_i-many steps apart, so they will receive opposite colors if and only if m_i is odd.

2. If the preimage of S_i in Σ' is disconnected, then this preimage consists of two circles, each with m_i-many marked points. Each of these circles may be given an alternating coloring if and only if m_i is even. Once such a coloring is chosen, the alternating coloring of the other circle is determined by the condition that preimages have opposite colors.

This confirms that the parity conditions on boundary components of Σ are a necessary and sufficient for the existence of alternating colorings. Furthermore, when they exist, we see that there are always exactly 2 choices of alternating coloring for each boundary component in Σ. Since swapping colors is an equivalence, this gives 2^{b-1}-many alternating colorings of Σ' up to equivalence.

To count double covers of Σ satisfying the parity conditions, recall that double covers of a manifold Σ are in bijection with $H^1(\Sigma, \mathbb{Z}_2)$, where a closed loop in Σ lifts along the double cover if and only if it is killed by the corresponding cohomology class.

Define $\chi_0 \in H^1(\partial\Sigma; \mathbb{Z}_2)$ to be the class which sends each boundary component in Σ to the parity of the number of marked points on that boundary component. Then double covers of Σ which satisfy the parity conditions are in bijection the preimage of χ_0 under the map

$$H^1(\Sigma; \mathbb{Z}_2) \to H^1(\partial\Sigma; \mathbb{Z}_2)$$

Consider the long exact sequence (with dimensions computed in the proof of Proposition 6.1.13)

$$
\begin{aligned}
0 \longrightarrow \underbrace{H^0(\Sigma;\mathbb{Z}_2)}_{\dim=1} \longrightarrow \underbrace{H^0(\partial\Sigma;\mathbb{Z}_2)}_{\dim=b} \longrightarrow \underbrace{H^1(\Sigma,\partial\Sigma;\mathbb{Z}_2)}_{\dim=2g+b-1} & \xrightarrow{f} \\
\underbrace{H^1(\Sigma;\mathbb{Z}_2)}_{\dim=2g+b-1} \xrightarrow{g} \underbrace{H^1(\partial\Sigma;\mathbb{Z}_2)}_{\dim=b} \xrightarrow{h} \underbrace{H^2(\Sigma,\partial\Sigma;\mathbb{Z}_2)}_{\dim=1} & \longrightarrow 0
\end{aligned}
$$

Under the identification $H^2(\Sigma,\partial\Sigma;\mathbb{Z}_2) \simeq \mathbb{Z}_2$, h sends a class in $H^1(\partial\Sigma;\mathbb{Z}_2)$ to the sum of its values on the boundary components. In particular, Σ admits an alternating double cover if and only if it admits a double cover satisfying the parity conditions if and only if the preimage of χ_0 is non-empty if and only if $h(\chi_0) = 0$ if and only if the total number of marked points m is even.

When $h(\chi_0) = 0$, the cardinality of the preimage $g^{-1}(\chi_0)$ is equal to the cardinality of the preimage $g^{-1}(0)$ of 0; that is, the kernel of g. Using the above dimensions, we compute that

$$
\begin{aligned}
\dim(\ker(g)) &= \dim(\operatorname{im}(f)) \\
&= \dim(H^1(\Sigma,\partial\Sigma;\mathbb{Z}_2)) - \dim(H^0(\partial\Sigma;\mathbb{Z}_2)) + \dim(H^0(\Sigma;\mathbb{Z}_2)) \\
&= 2g
\end{aligned}
$$

Therefore, the number of preimages of χ_0 is 2^{2g} when m is even and 0 otherwise.

$\square$

Alternating double covers and deep components

Assuming that the total number of marked points m is even, each polygonal dissection of Σ can be used to construct an alternating double cover of Σ, as follows.

Construction 6.2.16. Let D be a polygonal dissection of Σ. By Proposition 6.1.4, $\Sigma \smallsetminus D$ has an even number of sides. As a result, there are two ways to color the marked points in $\Sigma \setminus D$ alternating between black and white (differing by swapping the colors). Take two copies of the polygon $\Sigma \setminus D$, one with each of the two possible alternating colorings. For each arc in D, glue the two pairs of preimages of that arc back together in such a way that the colorings at the endpoints match. The resulting surface Σ' is an alternating double cover of Σ (e.g. Figure 6.8).

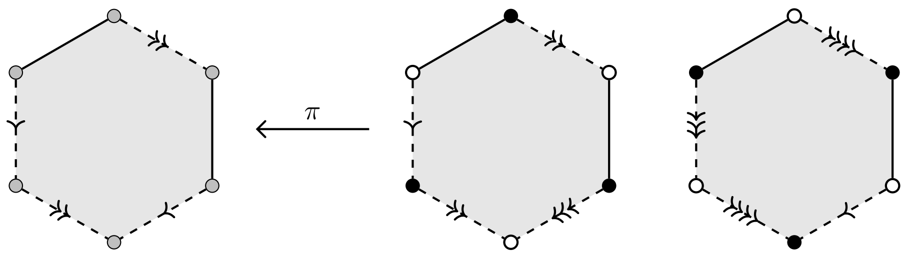


Figure 6.8: Constructing the alternating double covering of a polygonal dissection

Conversely, alternating double covers determine vanishing classes via the following observation. Given a simple marked arc a in Σ, there are two lifts of a along an alternating double cover $\Sigma' \to \Sigma$. Since these two lifts are related by the unique non-trivial deck transformation, either (a) both lifts of a have same colored endpoints, or (b) both lifts of a have opposite colored endpoints.

Proposition 6.2.17. *Given an alternating double cover $\Sigma' \to \Sigma$, the set*

$$\mathcal{V}(\Sigma') := \{\textit{simple marked arcs } a \textit{ in } \Sigma \textit{ such that either lift of } a \textit{ to } \Sigma \textit{ has same colored ends}\}$$

is a vanishing class of Σ. If D is a polygonal dissection of Σ with associated alternating double cover Σ' (Construction 6.2.16), then $\chi_{\mathcal{V}(\Sigma')} = \chi_D$.

Proof. If a is a boundary arc in Σ, then both lifts of a to Σ' are boundary arcs, so the endpoints have opposite colors and $a \notin \mathcal{V}(\Sigma')$. Consider a triangle $\{a, b, c\}$ in Σ, and let $\{a', b', c'\}$ be a lift to Σ'. If all three vertices of the triangle $\{a', b', c'\}$ have the same color, then all three of $\{a, b, c\}$ are in $\mathcal{V}(\Sigma')$. If one vertex of the triangle

$\{a', b', c'\}$ has a different color than the other two, then exactly one of $\{a, b, c\}$ is in $\mathcal{V}(\Sigma')$. Therefore, $\mathcal{V}(\Sigma')$ is a vanishing class of Σ.

Let D be a polygonal dissection of Σ with associated alternating double cover Σ'. Since the coloring of marked points in Σ' alternates around the boundary of each copy of $\Sigma \smallsetminus D$ in Construction 6.2.16, both lifts of an arc in D to an arc in Σ' have opposite colored ends. As a consequence, D is disjoint from $\mathcal{V}(\Sigma')$.

By Proposition 6.1.11.2, $\chi_{\mathcal{V}(\Sigma')}(a) = 1$ for each arc $a \in D$ and each boundary arc a. By the uniqueness in Proposition 6.1.11.3, $\chi_{\mathcal{V}(\Sigma')} = \chi_D$. □

Corollary 6.2.18. *Construction 6.2.16 and Proposition 6.2.17 define bijections between*

- *polygonal dissections of* Σ *up to congruence,*
- *alternating double covers of* Σ *up to equivalence, and*
- *vanishing classes of* Σ*.*

This implies that alternating double covers of Σ are in bijection with the other sets in Theorem 6.2.1. Many of these bijections can be formulated directly; for example, if p is a deep point of $V(\mathcal{A}(\Sigma), \Bbbk)$ with vanishing class $\mathcal{V}(p)$, then the alternating double cover $\Sigma' \to \Sigma$ associated to $\mathcal{V}(p)$ is the unique alternating double cover of Σ such that the pullback of p to $V(\mathcal{A}(\Sigma'), \Bbbk)$ kills precisely the marked arcs in Σ' whose ends have the same color.

Spin structures

Finally, we dispel a tempting misconception: that components of the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ may be identified with spin structures on Σ.

There are many equivalent ways to define spin structures on a surface-with-boundary; here is one. Define the **direction bundle** $\mathcal{D}\Sigma$ to Σ to be the quotient of the punctured tangent bundle $\mathcal{T}^{\times}\Sigma$ by scaling by $\mathbb{R}_+$.[3] Elements of $\mathcal{D}\Sigma$ are non-zero tangent vectors to Σ up to a positive scalar. The direction bundle is a bundle of circles over the base Σ.

A **spin structure** on Σ is a choice of double cover $\mathcal{S}\Sigma \to \mathcal{D}\Sigma$ such that the preimage of any circular fiber in $\mathcal{D}\Sigma$ is connected.

The set of spin structures on Σ is a torsor over $H^1(\Sigma; \mathbb{Z}_2)$ [Joh80, Theorem 3A], and so there are 2^{2g+b-1}-many spin structures on Σ, which coincides with the number of deep components of $V(\mathcal{A}(\Sigma), \Bbbk)$. Therefore, bijections between the two sets exist; however, the following example shows that no such bijection can be 'natural'.

Proposition 6.2.19. *Let Σ be the $(1,1)$-annulus (so $2^{2g+b-1} = 2$), and let $\gamma : \Sigma \to \Sigma$ be the Dehn twist around the unique simple loop in Σ. Then*

1. *the induced action of γ on the set of spin structures of Σ is trivial, but*
2. *the induced action of γ on the set of components of the deep locus of $\mathcal{A}(\Sigma)$ is non-trivial.*

As a consequence, there can be no bijection between the two sets which commutes with the action of the mapping class group of Σ.

Proof. Let ℓ be the (homotopy class of) the simple loop in Σ (Figure 5.1b). The derivative of a parametrization of ℓ lifts to a loop $d\ell$ in $\mathcal{D}\Sigma$, and the homotopy class of $d\ell$ is independent of the choice of parametrization. The two spin structures on Σ differ in whether $d\ell$ has connected or disconnected preimage along the double cover $\mathcal{S}\Sigma \to \mathcal{D}\Sigma$. Since the Dehn twist around ℓ fixes $d\ell$, the Dehn twist does not exchange these two spin structures.

[3] Given a Riemannian metric of Σ, the direction bundle may be identified with the unit tangent bundle.

Conversely, the Dehn twist around ℓ exchanges the two alternating double covers of Σ (Figure 6.7). Therefore, the action of the Dehn twist on the other sets in Theorem 6.2.1 is also non-trivial. $\square$

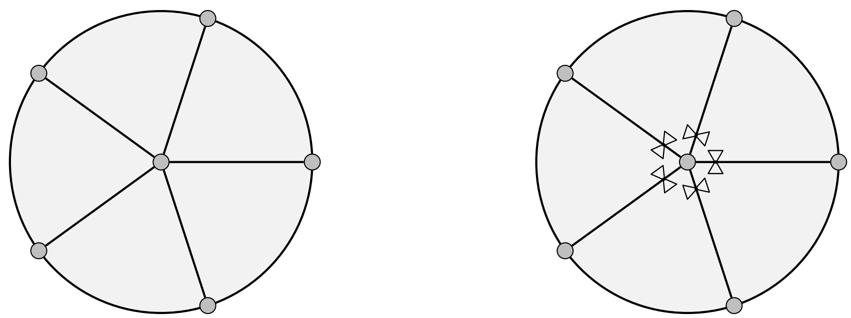
Chapter 7

Deep points of once-punctured marked disks

Convention 7.0.1. In Chapter 4, we considered cluster algebras of polygons. These topological n-gons are unpunctured disks with n marked points on the boundary, with the marked points implicitly located at the vertices of the polygon. In this chapter, we will generally refer to the surfaces as disks, but the reader may follow the previous convention and understand these disks as polygons (with marked points implicitly at the vertices) when it is helpful.

Recall that a **puncture** in a surface Σ is a distinguished (or marked) point in the interior of Σ. A (tagged) **radius** is a simple arc with one end at the boundary and the other end at a puncture (tagged plain or notched). In a disk Σ with one puncture, there is a triangulation consisting entirely of boundary arcs and plain radii, so any deep point $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ must kill at least one plain radius; likewise, there is a triangulation with all notched radii, so p must kill at least one notched radius. We will refer to these triangulations, pictured in Figure 7.1, as *radial triangulations*.

There are also deep points that kill all radii of one or both taggings. We will



(a) A plain radial triangulation. (b) A notched radial triangulation.

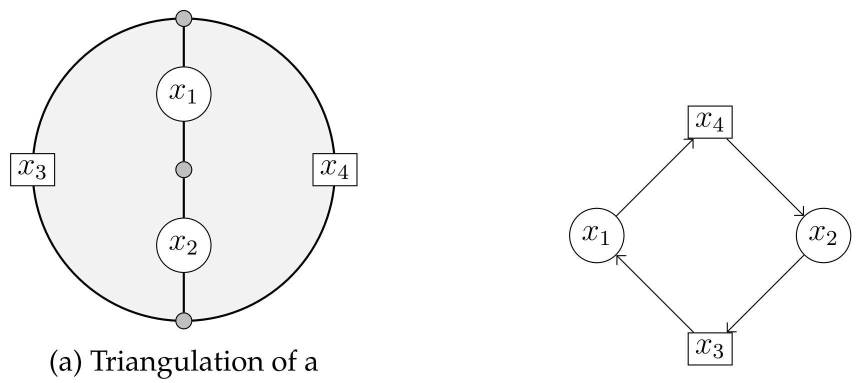
Figure 7.1: Radial triangulations of a punctured disk.

refer to a deep point that kills all radii of both taggings as a **(totally) radial deep point**. If a deep point kills all plain radii (but not all notched radii), we will refer to it as a **plain deep point**; similarly, if a deep point kills all notched radii (but not all plain radii), we will refer to it as a **notched deep point**.

Unfortunately, many of the tools used in the classification of deep points of cluster algebras of unpunctured surfaces will not directly apply to cluster algebras of punctured surfaces. In this chapter, we will consider disks with a single puncture with the goal of later generalizing results where possible. We begin with some low-complexity examples.

7.1 Minimal examples

Notation 7.1.1. Throughout this section, we will denote the cluster variable associated to the plain tagging of a radius as x_j and the variable associated to the notched tagging of that radius as $x_j^\vee$.



(a) Triangulation of a once-punctured bigon.

(b) The associated quiver.

Figure 7.2: Initial triangulations of a once-punctured bigon.

Once-punctured bigon

The once-punctured bigon, a punctured disk with 2 boundary marked points as depicted in Figure 7.2, has 4 triangulations with a total of 4 mutable arcs, as demonstrated in Figure 5.6. There are no arrows between the radii in any of the four quivers, but there are arrows to the frozen variables which give us the relations

$$x_1 x_2^\vee = x_3 + x_4 \quad \text{and} \quad x_2 x_1^\vee = x_3 + x_4.$$

Any deep point $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ would need to kill at least one of the plain radii and at least one of the notched radii, and consequently satisfy $0 = p(x_3) + p(x_4)$. Further, since p also needs to kill at least one radius in every triangulation (see Figure 5.6), p must kill mismatched taggings —that is, if $p(x_1) = 0$ then $p(x_2^\vee) = 0$ as well, and if $p(x_2) = 0$ then $p(x_1^\vee) = 0$. Finally, both $p(x_1)p(x_2^\vee) = 0$ and $p(x_1^\vee)p(x_2) = 0$, so combined with the above, at most of one of the four radii can take a non-zero value.

This gives us four subsets of deep points corresponding to the following values of $p(x_1, x_1^\vee, x_2, x_2^\vee, x_3, x_4)$ for $a \in \Bbbk$ and $c \in \Bbbk^\times$:

$$(a, 0, 0, 0, c, -c), \quad (0, a, 0, 0, c, -c), \quad (0, 0, a, 0, c, -c), \quad \text{and} \quad (0, 0, 0, a, c, -c).$$

Each of these subsets of deep points is two-dimensional, and they intersect along the one-dimensional algebraic torus $(0, 0, 0, 0, c, -c)$.

Remark 7.1.2. Observe that the mutable part of the quiver is the disjoint union of two A_1 quivers, so $\mathcal{A}(\Sigma)$ is a cluster algebra of type $A_1 \times A_1$. If we specialize the two frozen boundary variables to 1, we get relations of the form $x_i x_j^\vee = 2$, so the cluster variety of the type $A_1 \times A_1$ cluster algebra with trivial coefficients has no deep points if $\mathrm{char}(\Bbbk) \neq 2$. If $\mathrm{char}(\Bbbk) = 2$, there are four sets of deep points as before:

$$(a, 0, 0, 0, 1, 1), \quad (0, a, 0, 0, 1, 1), \quad (0, 0, a, 0, 1, 1), \quad \text{and} \quad (0, 0, 0, a, 1, 1).$$

These sets are one-dimensional and intersect at the point $(0, 0, 0, 0, 1, 1)$.

Once-punctured trigon

The once-punctured trigon, a punctured disk with three boundary marked points, has 9 mutable arcs —3 plain radii, 3 notched radii, and 3 arcs which bound punctured bigons, as pictured in Figure 7.3 —and a total of 14 triangulations. Examining the figure, we see that the three mutable vertices form a quiver in the mutation class of the A_3 quiver.

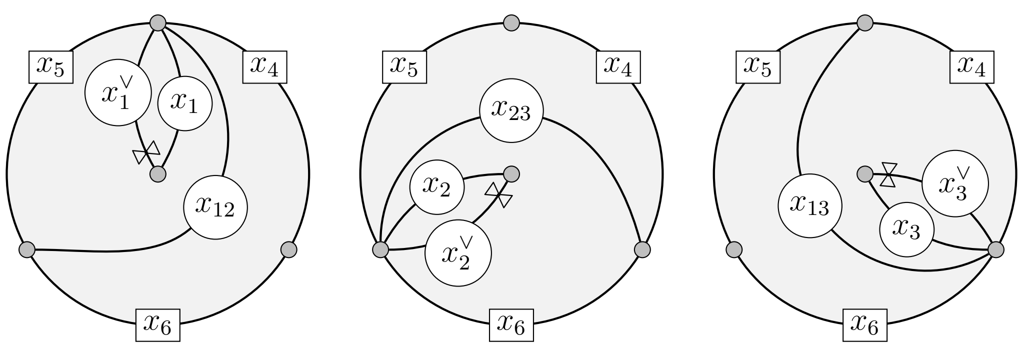


Figure 7.3: Arcs in the punctured trigon.

Remark 7.1.3. From Theorem 4.3.5, if we specialize the frozen boundary variables to 1, there is a unique deep point which is realized here by sending $x_{12}, x_{13},$ and x_{23} to a value of -1.

Let us choose the triangulation on the left of Figure 7.3 as our initial seed, giving us an initial quiver

$$\begin{tikzcd}
 & \boxed{x_4} & & \boxed{x_6} \arrow[dl] & \\
x_1^\vee \arrow[rr] & & x_{12} \arrow[ul] \arrow[d] & & x_1 \arrow[ll] \\
 & & \boxed{x_5} \arrow[ull] \arrow[urr] & &
\end{tikzcd}.$$

The mutable part of this quiver is acyclic, so we know from [BFZ05, Corollary 1.21] (or stated more directly for our application, [LS23, Theorem 2.4]) that $\mathcal{A}(\Sigma)$ is generated by an initial seed, along with one mutation at each interior arc. The cluster algebra is then generated by $x_1, x_2^\vee, x_1^\vee, x_2, x_{12}$ and x_{13}, along with the frozen variables and their inverses; the ideal of relations is generated by:

$$x_1 x_2^\vee = x_1^\vee x_2 = x_{12} + x_5 \quad \text{and} \quad x_{12}x_{13} = x_1 x_1^\vee x_6 + x_4 x_5.$$

Other relations include, but are not limited to:

$$\begin{aligned}
x_2 x_3^\vee &= x_2^\vee x_3 = x_{23} + x_6 & x_{23}x_{12} &= x_2 x_2^\vee x_4 + x_5 x_6 \\
x_3 x_1^\vee &= x_3^\vee x_1 = x_{13} + x_4 & x_{13}x_{23} &= x_3 x_3^\vee x_5 + x_4 x_6 \\
x_1 x_{23} &= x_2 x_4 + x_3 x_5 & x_1^\vee x_{23} &= x_2^\vee x_4 + x_3^\vee x_5 \\
x_2 x_{13} &= x_1 x_6 + x_3 x_5 & x_2^\vee x_{13} &= x_1^\vee x_6 + x_3^\vee x_5 \\
x_3 x_{12} &= x_1 x_6 + x_2 x_4 & x_3^\vee x_{12} &= x_1^\vee x_6 + x_2^\vee x_4
\end{aligned}$$

We know that any deep point must kill at least one plain radius and at least one notched radius. Suppose $p(x_3) = 0$, but $p(x_2) \neq 0$ and $p(x_1) \neq 0$. Then in order to kill at least one arc in each triangulation, $p(x_{12}) = 0$. The relations $x_1 x_2^\vee = x_2 x_1^\vee = x_{12} + x_5$ then imply that $p(x_2^\vee) \neq 0$ and $p(x_1^\vee) \neq 0$ as well, so $p(x_3^\vee) = 0$. Then from the triangulations pictured in Figure 7.3, we now see that p must kill x_{23} and x_{13}; the relation $x_{13}x_{23} = x_3 x_3^\vee x_5 + x_4 x_6$ then implies that $p(x_4)p(x_6) = 0$, which is not possible. Hence, p cannot kill a single plain radius while sparing the other two.

Next, suppose p kills both x_1 and x_2. From the relation $x_1 x_{23} = x_2 x_4 + x_3 x_5$, we have $p(x_3)p(x_5) = 0$, which implies that $p(x_3) = 0$ as well, so p must kill all plain radii. Repeating these arguments with notched radii, we see that a deep point p must kill all six radii; that is, p is totally radial.

From the above relations, we see that sending the 6 radii to 0 gives us the following relations:

$$0 = p(x_{12}) + p(x_5), \qquad 0 = p(x_{23}) + p(x_6), \qquad 0 = p(x_{13}) + p(x_4).$$

Hence, any deep point must kill all radii and send each pair of arcs bounding punctured bigons to additive inverses. While six variables are sent to non-zero values, there are only three possible choices for their values. Therefore, the deep locus is isomorphic to the algebraic torus $(\mathbb{k}^\times)^3$.

7.2 Characterization of deep points

As previously noted, many of the tools used in the classification of deep points in the unpunctured case will not directly apply to punctured surfaces. In this section, we begin to characterize some local results for once-punctured disks with $n \geq 4$ marked points on the boundary.

Vanishing Lemmas

Consider Figure 7.4 below. To make the following arguments easier to follow, we will conflate the cluster variables with their image (i.e. we will write a instead of $p(x_i) = a$). We will use the convention that if r_j is the value of a plain radius, $r_j^\vee$ will be the value of the notched version of the same radius. The dashed arc labeled a is not a boundary arc when $n \geq 5$, but is a boundary arc when $n = 4$. Arcs labeled c_j cut out a punctured bigon along with the boundary arc b_j; the arc c_a cuts out a punctured bigon with the arc labeled a. Arcs labeled d_{ij} cut out punctured trigons along with boundary arcs b_i and b_j; arcs labeled d_{ia} cut out punctured trigons with the boundary arc b_i and the arc labeled a.

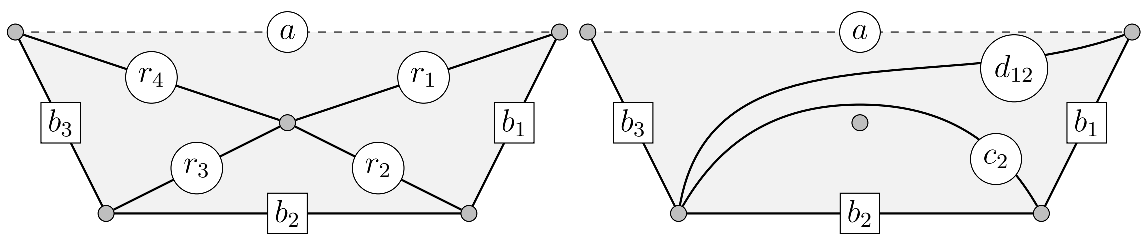


Figure 7.4: Arcs in a punctured quadrilateral.

For reference, extrapolating from Figure 7.4, we have the following Ptolemy relations:

$$c_2 r_1 = d_{12} r_2 + r_3 b_1 \qquad c_2 r_4 = d_{23} r_3 + r_2 b_3$$

$$c_3 r_2 = d_{23} r_3 + r_4 b_2 \qquad c_3 r_1 = d_{3a} r_4 + r_3 a$$

$$c_a r_3 = d_{3a} r_4 + r_1 b_3 \qquad c_a r_2 = d_{1a} r_1 + r_4 b_1$$

$$c_1 r_4 = d_{1a} r_1 + r_2 a \qquad c_1 r_3 = d_{12} r_2 + r_1 b_2$$

$$d_{23} d_{12} = b_1 b_3 + a c_2 \qquad d_{3a} d_{23} = a b_2 + b_1 c_3$$

$$d_{1a} d_{3a} = b_1 b_3 + b_2 c_a \qquad d_{12} d_{1a} = a b_2 + b_3 c_1$$

Lemma 7.2.1. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ have image as above. If $r_2 = 0$ and $r_3 = 0$, then $r_1 = 0$ and $r_4 = 0$.*

Proof. First consider the mutation relation $r_3 c_1 = r_1 b_2 + r_2 d_{12}$, which implies that $r_1 b_2 = 0$. Because it corresponds to a boundary arc, $b_2 \neq 0$, so $r_1 = 0$.

Next, consider the mutation relation $r_4 c_1 = r_2 a + r_1 d_{1a}$, which implies that $r_4 c_1 = 0$, and the relation $r_1 r_2^{\vee} = c_1 + b_1$, which implies $c_1 + b_1 = 0$. Since b_1 corresponds to a boundary arc, $b_1 \neq 0$, so $c_1 \neq 0$ as well. Therefore, $r_4 = 0$. □

That is, if a point p kills two radially adjacent radii, then it must also kill the adjacent radius on either side. To obtain the result for notched radii, simply replace all r_i in Lemma 7.2.1 with $r_i^{\vee}$.

Corollary 7.2.2. *Let Σ be a once-punctured disk with $n \geq 2$ marked points on the boundary, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$. If p kills two radially adjacent plain (resp. notched) radii, then p kills all plain (resp. notched) radii.*

Proof. If $n = 2$, the result is clear as there are only two plain (resp. notched) radii. If $n \geq 4$, the corollary follows from iteratively applying Lemma 7.2.1 around the puncture.

From the previous discussion of the $n = 3$ case, the relation $x_1 x_{23} = x_2 x_4 + x_3 x_5$ implies that if x_1 and x_2 are killed, then x_3 is killed as well. By symmetry, if any two plain (resp. notched) radii are killed, the third must be killed as well. □

Remark 7.2.3. Note that while we have deep points in mind, Lemma 7.2.1 and Corollary 7.2.2 do not require that p be a deep point a priori, nor have we proven that they imply deepness.

We also have the following porism from the proof of Lemma 7.2.1, which emphasizes an idea we will use later. The reader should note, however, that we have not shown that this porism applies to c_a when $n \geq 5$.

Porism 7.2.4. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ have image as above. If p kills two radially adjacent radii, either both plain or both notched, then for all c_i which cut out a once-punctured bigon along with a boundary arc b_i, $c_i = -b_i$, so $c_i \neq 0$.*

Next, we want to examine what happens to the plain (resp. notched) radii when all notched (resp. plain) radii are killed by a deep point. Note that while each deep point must kill at least one radius of each tagging, if $n \geq 4$, we have not shown that at least one radius of each type must be killed in each quadrilateral with 3 boundary sides as shown.

Lemma 7.2.5. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a deep point with image as above, such that p kills all notched radii; that is, $r_i^\vee = 0$ for all i.*

1. *$b_1b_3 - ab_2 = 0$ if and only if $d_{23} = d_{1a} = 0$ or $d_{12} = d_{3a} = 0$.*
   a) *If $d_{23} = d_{1a} = 0$, then $r_2 = 0$ if and only if $r_4 = 0$.*
   b) *If $d_{12} = d_{3a} = 0$, then $r_1 = 0$ if and only if $r_3 = 0$.*
2. *if $b_1b_3 - ab_2 \neq 0$ and $r_2 = 0$, then p does not kill any d_{ij}, and p kills all plain radii.*

Proof. By assumption, p kills all notched radii, so Porism 7.2.4 gives us $c_i = -b_i$. From the relations $r_1r_3^\vee = d_{12} + d_{3a} = 0$ and $r_2r_4^\vee = d_{23} + d_{1a} = 0$, we also have $d_{12} = -d_{3a}$ and $d_{23} = -d_{1a}$. Substituting, we get the relation

$$d_{23}d_{12} = b_1b_3 + ac_2 = b_1b_3 - ab_2.$$

1. If $b_1b_3 - ab_2 = 0$, then either $d_{23} = 0$ or $d_{12} = 0$. Thus, if $b_1b_3 - ab_2 = 0$, then either $d_{23} = d_{1a} = 0$ or $d_{12} = d_{3a} = 0$. Conversely, if either $d_{23} = 0$ or $d_{3a} = 0$, then $b_1b_3 - ab_2 = 0$.
   a) Suppose $d_{23} = d_{1a} = 0$. We have the simplification $-b_2r_4 = b_3r_2$; since $b_i \neq 0$ for all i, $r_4 = 0$ if and only if $r_2 = 0$.
   b) Next, suppose $d_{12} = d_{3a} = 0$. We have the simplification $-b_2r_1 = b_1r_3$; since $b_i \neq 0$ for all i, $r_3 = 0$ if and only if $r_1 = 0$.
2. Now suppose that $b_1b_3 - ab_2 \neq 0$ and $r_2 = 0$. Since $d_{23}d_{12} = b_1b_3 - ab_2$, none of the d_{ij} are killed by p. Substituting for r_2 and simplifying, we have:

$$-b_2r_1 = r_3b_1 \qquad\qquad -b_2r_4 = d_{23}r_3$$

$$0 = d_{23}r_3 + r_4b_2 \qquad\qquad 0 = d_{1a}r_1 + r_4b_1$$

Hence, $r_3 = \dfrac{-b_2}{b_1}r_1 = \dfrac{-b_2}{d_{23}}r_4$. Substituting into the equation $c_3r_1 = d_{3a}r_4 + r_3a$, we get

$$c_3r_1 = \frac{d_{23}d_{3a}r_1}{b_1} - \frac{ab_2r_1}{b_1},$$

so $b_1b_3r_1 = (ab_2 - d_{23}d_{3a})r_1$, and $b_1b_3r_1 = (ab_2 - b_1b_3 + ab_2)r_1$. Simplifying, we get $2(b_1b_3 - ab_2)r_1 = 0$. Since $b_1b_3 - ab_2 \neq 0$, it must be that $r_1 = 0$, so all other plain radii are killed by p as well as a result of Corollary 7.2.2. □

By instead killing all plain radii, we immediately get the analogous result with the taggings swapped.

Corollary 7.2.6. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a deep point with image as above, such that p kills all plain radii; that is, $r_i = 0$ for all i.*

1. *$b_1b_3 - ab_2 = 0$ if and only if $d_{23} = d_{1a} = 0$ or $d_{12} = d_{3a} = 0$.*
   a) *If $d_{23} = d_{1a} = 0$, then $r_2^\vee = 0$ if and only if $r_4^\vee = 0$.*

b) If $d_{12} = d_{3a} = 0$, then $r_1^\vee = 0$ if and only if $r_3^\vee = 0$.

2. *if $b_1b_3 - ab_2 \neq 0$ and $r_2^\vee = 0$, then p does not kill any d_{ij}, and p kills all notched radii.*

Corollary 7.2.7. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a deep point with image as above, such that p kills all notched (resp. plain) radii. If $b_1b_3 - ab_2 \neq 0$, then p kills all plain (resp. notched) radii as well.*

Proof. Since p is a deep point, it must kill at least one plain radius in Σ. Choose a punctured quadrilateral subsurface matching Figure 7.4 such that the plain radius being killed is in the r_2 position. Since $b_1b_3 - ab_2 \neq 0$, Lemma 7.2.5 implies that all plain radii in Σ are killed by p. To prove for notched radii, replace Lemma 7.2.5 with Corollary 7.2.6. □

Next, we will consider the deep points that do not kill all radii of a given tagging.

Lemma 7.2.8. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a deep point such that $r_2 = 0$ but $r_3 \neq 0$. Then*

1. *$r_2^\vee = 0$.*
2. *$r_1 \neq 0$.*
3. *$r_4 = 0$ if and only if $d_{1a} = d_{23} = 0$.*
4. *if $r_4 = 0$, then $b_1b_3 - ab_2 = 0$.*

Proof. From the relation $r_2r_3^\vee = b_2 + c_2 = 0$, we have $r_3r_2^\vee = 0$ as well, so $r_2^\vee = 0$. From the contrapositive of Corollary 7.2.2, since $r_3 \neq 0$, p cannot kill both r_1 and r_2; consequently, $r_1 \neq 0$.

Next, consider the relation $r_2b_3 + r_4b_2 = r_3d_{1a}$, which simplifies to $r_4b_2 = r_3d_{1a}$. Since $r_3 \neq 0$, we have $r_4 = 0$ if and only if $d_{1a} = 0$. From the relation $r_4r_2^\vee = 0 = d_{1a} + d_{23}$, we likewise have that $d_{1a} = 0$ if and only if $d_{23} = 0$. Combining with the relation $d_{1a}d_{12} = ab_2 + c_1b_3 = ab_2 - b_1b_3$, if $r_4 = 0$ then $b_1b_3 - ab_2 = 0$. □

Cutting and gluing

To further classify the deep points of once-punctured disks, we will want to extend and utilize the idea of cutting and gluing used in previous chapters.

Lemma 7.2.9. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be deep. There is an index i such that $c_i \neq 0$, $r_i = 0$, and $r_i^\vee = 0$.*

Proof. Since p is deep, it must kill at least one plain radius, so choose an indexing such that $r_2 = 0$. If $r_3 = 0$, then p must kill all plain radii by Lemma 7.2.1. Then for all i, we have $r_ir_{i+1}^\vee = c_i + b_i = 0$, so $c_i \neq 0$ for $1 \leq i \leq n$. Since p must also kill at least one notched radius, there is an index i such that $r_i^\vee = 0$, $r_i = 0$, and $c_i \neq 0$.

If $r_3 \neq 0$, then by Lemma 7.2.8, we have $r_2^\vee = 0$. Then $r_2r_1^\vee = c_2 + b_2 = 0$, and we have $c_2 \neq 0$. □

Cutting Σ along c_i produces a disconnected surface $\Sigma \smallsetminus c_i = \Sigma_1 \sqcup \Sigma_n$, where Σ_1 is a once-punctured bigon and Σ_n is homeomorphic to the unpunctured n-gon Δ_n. Recall that a deep point $p \in V(\mathcal{A}(\Sigma \smallsetminus c_i), \Bbbk)$ must kill at least one arc in every triangulation of $\Sigma \smallsetminus c_i$, so it must kill at least one arc in every triangulation of either Σ_1 or Σ_n, but not necessarily both. For convenience, let us denote the separate restrictions as $p_1 \coloneqq p|_{\Sigma_1}$ and $p_n \coloneqq p|_{\Sigma_n}$.

Recall from Proposition 6.1.7 that

$$\left\{\begin{matrix}\text{deep points in } V(\mathcal{A}(\Sigma),\Bbbk) \\ \text{non-zero on } S\end{matrix}\right\} \hookrightarrow \left\{\begin{matrix}\text{deep points in } V(\mathcal{A}(\Sigma \smallsetminus S),\Bbbk) \\ \text{gluable along } S\end{matrix}\right\}$$

As noted in the discussion following that proposition, this is not always a bijection. Let us now examine whether this is a bijection for $S = \{c_i\}$.

Proposition 7.2.10. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary. There is a bijection*

$$\left\{\begin{matrix}\textit{totally radial deep} \\ \textit{points in } V(\mathcal{A}(\Sigma),\Bbbk) \\ \textit{non-zero on } c_i\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{totally radial deep} \\ \textit{points in } V(\mathcal{A}(\Sigma \smallsetminus c_i),\Bbbk) \\ \textit{gluable along } c_i\end{matrix}\right\}$$

Further, the subvariety of totally radial deep points of $V(\mathcal{A}(\Sigma),\Bbbk)$ is isomorphic to the cluster variety $V(\mathcal{A}(\Delta_n),\Bbbk)$.

Proof. Suppose that $p \in V(\mathcal{A}(\Sigma),\Bbbk)$ is a totally radial deep point, so it kills all radii. Then its image under the gluing isomorphism $\bar{p}$ must also kill all radii in $\Sigma \smallsetminus c_i$. Since $\Sigma \setminus c_i$ is disconnected, $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus c_i),\Bbbk)$ is deep if either the restriction p_1 to the punctured bigon is deep, the restriction p_n to the unpunctured n-gon is deep, or both. Since p_1 kills all radii in the punctured bigon, it is deep, and $\bar{p}$ is deep.

Now, suppose that $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus c_i),\Bbbk)$ is a totally radial deep point, so it kills all radii. Since there are no radii in Σ_n, there are no conditions on p_n, but p_1 kills all radii in Σ_1. Lifting to $p \in V(\mathcal{A}(\Sigma),\Bbbk)$, we see that p kills r_i, $r_i^\vee$, r_{i+1}, and $r_{i+1}^\vee$. By Corollary 7.2.2, p must kill all radii of both taggings, so p is a totally radial deep point.

Since the boundary values of p_1 on Σ_1 are determined by the boundary values of p_n on Σ_n, every totally radial point p is determined by a point $p_n \in V(\mathcal{A}(\Sigma_n),\Bbbk) \cong$

$V(\mathcal{A}(\Delta_n), \Bbbk)$. On the other hand, totally radial deep points on Σ have no restriction on boundary values, so every point of $V(\mathcal{A}(\Sigma_n), \Bbbk) \cong V(\mathcal{A}(\Delta_n), \Bbbk)$ can be realized as the restriction p_n of a totally radial deep point $\bar{p}$. $\square$

Remark 7.2.11. The reader should note that Proposition 7.2.10 says that the subvariety of totally radially deep points is isomorphic to the entire cluster variety of the unpunctured disk with n boundary marked points (or equivalently, the n-gon), and not just the deep locus of that cluster variety as one might expect.

Next, let us consider a deep point that does not kill all radii. Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a deep point such that $r_2 = 0$ and $r_3 \neq 0$. First, since $r_2 r_3^\vee = 0 = c_2 + b_2$, we have $c_2 = -b_2$ and $c_2 \neq 0$. From Lemma 7.2.8, we have $r_2^\vee = 0$ as well.

Recall from Proposition 6.1.7 that, since $c_2 \neq 0$, the image of p under the cutting isomorphism is a deep point $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus c_2), \Bbbk)$ that is gluable along c_2. Using the same notation as above, since $\bar{p}$ is deep, we know that either p_1 is deep on the once-punctured bigon, p_n is deep on the unpunctured n-gon, or both.

1. If p_1 is deep on the once-punctured bigon, then at least one arc in the triangulation $\{r_3, r_3^\vee\}$ must be killed, so we know that $r_3^\vee = 0$. By Corollary 7.2.2, this means that p must kill all notched radii. Since $r_3 \neq 0$, the contrapositive of Lemma 7.2.5 part 2 gives us $b_1 b_3 - a b_2 = 0$, so $b_1 b_3 + a c_2 = 0$.
   - **Claim:** $r_4 = 0$.

     Proof of Claim: Seeking a contradiction, suppose $r_4 \neq 0$. Since all notched radii are killed by p, we have $r_3 r_4^\vee = c_3 + b_3 = 0$, so $c_3 \neq 0$. If we cut along c_3 instead of c_2, $\widehat{p} \in V(\mathcal{A}(\Sigma \smallsetminus c_3), \Bbbk)$ must be deep. Since both $r_3 \neq 0$ and $r_4 \neq 0$, the restriction of $\widehat{p}$ to the bigon is not deep, so it must instead be deep on the unpunctured n-gon. Since $\widehat{p}$ is deep on the unpunctured

n-gon, p must kill both d_{23} and d_{3a}; since all notched radii are killed, we have $d_{23} = d_{1a} = 0$ and $d_{12} = d_{3a} = 0$. By Lemma 7.2.8, $r_4 = 0$. $\square$

By Corollary 7.2.2, we also know that $r_5 \neq 0$ since p does not kill all plain radii. Repeatedly applying the proof of the claim around the puncture, we see that p must kill all even-indexed plain radii and spare all odd-indexed radii. Note that if n is odd, all radii are even-indexed, so all deep points must be totally radial. Hence, under our assumptions, n must be even.

If $n = 4$, we observe that $r_1 b_2 + r_3 b_1 = 0$ and $r_3 b_4 + r_1 b_3 = 0$, so $r_1 \neq 0$ and we have

$$r_1 = \frac{-b_1}{b_2} r_3 = \frac{b_1 b_3}{b_2 b_4} r_1.$$

More generally, all even-indexed radii are killed by p, so

$$r_{j+2} b_{j+3} + r_j b_{j+2} = 0.$$

As a result, we have

$$r_1 = (-1) \frac{b_1}{b_2} r_3 = (-1)^2 \frac{b_1 b_3}{b_2 b_4} r_5 = \cdots = (-1)^{n/2} \frac{b_1 b_3 \cdots b_{n-1}}{b_2 b_4 \cdots b_n} r_1.$$

Since $r_1 \neq 0$, the values on the boundary edges satisfy the equation

$$\frac{b_1 b_3 \cdots b_{n-1}}{b_2 b_4 \cdots b_n} = (-1)^{n/2}. \tag{7.1}$$

As a result, the boundary values on the unpunctured n-gon satisfy the equation

$$\frac{b_1 b_3 \cdots b_{n-1}}{c_2 b_4 \cdots b_n} = (-1)^{\frac{n+2}{2}}, \tag{7.2}$$

and p_n is necessarily deep on the unpunctured n-gon by Theorem 4.3.1. That is, if n is even and p_1 is deep, then p_n is deep as well.

2. If n is odd, then Theorem 4.3.1 says that p_n cannot be deep.

3. If n is even and p_n is deep, Theorem 4.3.1 gives us the boundary relation

$$\frac{b_1 b_3 \cdots b_{n-1}}{c_2 b_4 \cdots b_n} = (-1)^{\frac{n+2}{2}}, \quad \text{and} \quad a = \frac{b_1 b_3}{b_2},$$

so $b_1 b_3 - a b_2 = 0$. The same theorem gives us $d_{12} = d_{23} = 0$, and since $r_2 = 0$, we have $d_{23} = -d_{1a}$, so $d_{1a} = 0$ as well. By Lemma 7.2.8, we also now have $r_4 = 0$. Since $r_2 = 0$ and $r_3 \neq 0$, Corollary 7.2.2 says that $r_1 \neq 0$ as well.

By symmetry (repeating with a cut of Σ along c_{2k} for all k), we see that all even-indexed plain radii are killed and all odd-indexed plain radii are spared. By the same argument, all non-radial arcs of even length must be killed, so all $d_{ij} = 0$. Since all non-radial arcs of even length are killed, the relations $r_{2k+1} r^{\vee}_{2k+3} = 0 + 0$ force all odd-indexed notched radii to be killed. Since $r^{\vee}_2 = 0$, we have adjacent notched radii being killed, so all notched radii are killed by Corollary 7.2.2.

Note that since all notched radii are killed and $r_2 = 0$, the restriction to p_1 is deep as well. That is, if n is even and p_n is deep, then p_1 is deep as well.

Recall that a point p in $V(\mathcal{A}(\Sigma \smallsetminus c_i), \Bbbk)$ is called *deep-on-components* if its restriction to each component is deep. As a result, we can characterize the non-totally radial points as follows:

Proposition 7.2.12. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary. There is a bijection*

$$\left\{ \begin{array}{c} \textit{non-totally radial deep} \\ \textit{points in } V(\mathcal{A}(\Sigma), \Bbbk) \\ \textit{non-zero on } c_i \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{non-totally radial} \\ \textit{deep-on-components} \\ \textit{points in } V(\mathcal{A}(\Sigma \smallsetminus c_i), \Bbbk) \\ \textit{gluable along } c_i \end{array} \right\}$$

Classifying deep points

As a result of the above exposition, we have the following lemmas.

Lemma 7.2.13. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary.*

1. *Let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a deep point such that $r_2 = 0$ and $r_3 \neq 0$. Then p kills all notched radii.*
2. *Let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a deep point such that $r_2^\vee = 0$ and $r_3^\vee \neq 0$. Then p kills all plain radii.*

Lemma 7.2.14. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary.*

1. *Let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a deep point with image as above, such that p kills all notched radii. Then p either kills all radii, or p kills alternating plain radii (that is, all even-indexed or all odd-indexed plain radii).*
2. *Let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a deep point with image as above, such that p kills all plain radii. Then p either kills all radii, or p kills alternating notched radii (that is, all even-indexed or all odd-indexed notched radii).*

Corollary 7.2.15. *Let Σ be a once-punctured disk with $n \geq 5$ marked points on the boundary, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a deep point. If n is odd, then p kills all radii of both taggings; that is, p is a totally radial deep point.*

Lemma 7.2.16. *Let Σ be a once-punctured disk with $n = 2k \geq 4$ marked points on the boundary, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a deep point such that not all radii are killed; that is, p is not a totally radial deep point. Then the values on the boundary edges satisfy the equation*

$$\frac{b_1 b_3 \cdots b_{n-1}}{b_2 b_4 \cdots b_n} = (-1)^{n/2} \tag{7.3}$$

Finally, we can combine the above results to complete the classification of deep points of once-punctured disks.

Theorem 7.2.17. *Let Σ be a once-punctured disk with $n \geq 4$ marked points on the boundary, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a deep point.*

1. *If n is odd, then p kills all radii of both taggings; that is, p is a totally radial deep point.*
2. *If n is even, then p is one of the following three types:*
   a) *p is a totally radial deep point; that is, it kills all radii of both taggings.*
   b) *p is a notched deep point; that is, it kills all notched radii, but not all plain radii. In this case, p kills alternating plain radii —either all odd-indexed plain radii or all even-indexed plain radii —and the values of the boundary edges satisfy Equation 7.3.*
   c) *p is a plain deep point; that is, it kills all plain radii, but not all notched radii. In this case, p kills alternating notched radii —either all odd-indexed notched radii or all even-indexed notched radii —and the values of the boundary edges satisfy Equation 7.3.*

7.3 Once-punctured quadrilateral

We will now use the results of the previous section to explicitly compute the deep points corresponding to the once-punctured disk with $n = 4$ marked points on the boundary; let Σ denote this surface. As seen in Figure 7.5, the mutable part of the quiver is an orientation of a type D_4 Coxeter-Dynkin diagram. The mutable part of this quiver is acyclic, so we know from [BFZ05, Corollary 1.21] (or stated more directly for our application, [LS23, Theorem 2.4]) that $\mathcal{A}(\Sigma)$ is generated by an initial seed, along with one mutation at each interior arc. Using this quiver as our initial seed, the cluster algebra is generated by $x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4$ and x_4', along with the frozen variables and their inverses, where x_2 denotes the plain radius realized by a flip of $x_1^\vee$ in the given triangulation. The ideal of relations is generated by:

$$x_1 x_2^\vee = x_3 + x_5 \qquad\qquad x_1^\vee x_2 = x_3 + x_5$$

$$x_3 x_3' = x_1 x_1^\vee x_6 + x_4 x_5 \qquad\qquad x_4 x_4' = x_3 x_7 + x_6 x_8.$$

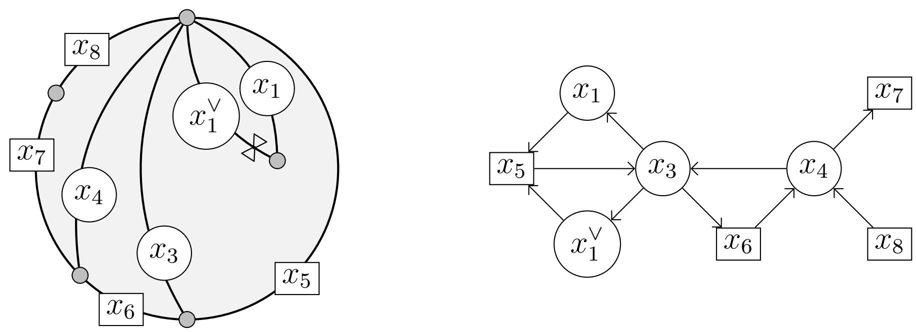


Figure 7.5: A triangulation of a once-punctured quadrilateral and its associated quiver.

From Theorem 7.2.17, we know that every deep point is one of three types, which we will consider separately. Throughout the following, let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be deep. Recall that each triangulation will contain at least one arc killed by p, so p must kill at least one plain radius and at least one notched radius; also, p does not kill any boundary frozen variables. In relating previous results, the arcs in Figure 7.4 correspond to the variables in Figure 7.5 via

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (r_1, r_2^\vee, r_1^\vee, r_2, c_1, d_{3a}, d_{12}, d_{1a})$$

$$p(x_5, x_6, x_7, x_8) = (b_1, b_2, b_3, a)$$

1. First, suppose that p kills all radii of both taggings. Because p kills all radii, we have $0 = p(x_3') + p(x_4)$; likewise, $p(x_3) + p(x_5) = 0$. We also have the relation

$$p(x_4)p(x_4') = p(x_3)p(x_7) + p(x_6)p(x_8) = p(x_6)p(x_8) - p(x_5)p(x_7).$$

a) If $p(x_4) \neq 0$, then $p(x_4)$ and the values of the boundary variables determine the values of both x_4' and x_3' (with both non-zero). Thus, such a deep point is of the form

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (0, 0, 0, 0, -a, -u, u, \frac{bd - ac}{u})$$
$$p(x_5, x_6, x_7, x_8) = (a, b, c, d)$$

for some $u, a, b, c, d \in \Bbbk^\times$. These points determine a deep subset of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $(\Bbbk^\times)^5$.

b) On the other hand, if $p(x_4) = 0$, then $p(x_3') = 0$ and

$$p(x_6)p(x_8) - p(x_5)p(x_7) = 0,$$

in which case $p(x_4')$ can take any value.

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (0, 0, 0, 0, -a, 0, 0, u)$$
$$p(x_5, x_6, x_7, x_8) = (a, b, c, \frac{ac}{b})$$

for some $a, b, c \in \Bbbk^\times$ and $u \in \Bbbk$. These points determine a deep subset of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $\Bbbk \times (\Bbbk^\times)^3$.

Further, as a result of Proposition 7.2.10, we know that the union of these deep subsets is isomorphic to the cluster variety $V(\mathcal{A}(\Delta_4), \Bbbk)$.

2. Second, suppose that p kills all notched radii, but not all plain radii. These points can be divided into two types: those that kill only the odd-indexed plain radii and those that kill only the even-indexed plain radii.

   By Lemma 7.2.16, we have $p(x_6)p(x_8) = p(x_5)p(x_7)$. Further, because p kills all notched radii, we have $p(x_3) + p(x_5) = 0$. Finally, by Proposition 7.2.12, p restricts to a deep point on any unpunctured quadrilateral $\Sigma \setminus c_i$. By Theorem 4.3.1, p must kill all non-radial arcs of even length; as a result, p kills all of x_3', x_4, and x_4'.

a) Suppose that p kills only the odd-indexed plain radii. Such a deep point is of the form

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (0, 0, 0, u, -a, 0, 0, 0)$$
$$p(x_5, x_6, x_7, x_8) = (a, b, c, \frac{ac}{b})$$

for some $u, a, b, c \in \Bbbk^\times$. These points determine a subset of the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $(\Bbbk^\times)^4$.

b) Suppose that p kills only the even-indexed plain radii. Such a deep point is of the form

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (u, 0, 0, 0, -a, 0, 0, 0)$$
$$p(x_5, x_6, x_7, x_8) = (a, b, c, \frac{ac}{b})$$

for some $u, a, b, c \in \Bbbk^\times$. These points determine a subset of the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $(\Bbbk^\times)^4$.

3. Third, suppose that p kills all plain radii, but not all notched radii. These points can be divided into two types: those that kill only the odd-indexed notched radii and those that kill only the even-indexed notched radii.

As in the previous case, Lemma 7.2.16 implies that $p(x_6)p(x_8) - p(x_5)p(x_7) = 0$. Further, because p kills all plain radii, we have $p(x_3) + p(x_5) = 0$. Finally, by Proposition 7.2.12, p restricts to a deep point on any unpunctured quadrilateral $\Sigma \setminus c_i$. By Theorem 4.3.1, p must kill all non-radial arcs of even length; as a result, p kills all of x_3', x_4, and x_4'.

a) Suppose that p kills only the odd-indexed notched radii. Such a deep point is of the form

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (0, u, 0, 0, -a, 0, 0, 0)$$
$$p(x_5, x_6, x_7, x_8) = (a, b, c, \frac{ac}{b})$$

for some $u, a, b, c \in \Bbbk^\times$. These points determine a subset of the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $(\Bbbk^\times)^4$.

b) Suppose that p kills only the even-indexed notched radii. Such a deep point is of the form

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (0, 0, u, 0, -a, 0, 0, 0)$$
$$p(x_5, x_6, x_7, x_8) = (a, b, c, \frac{ac}{b})$$

for some $u, a, b, c \in \Bbbk^\times$. These points determine a subset of the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $(\Bbbk^\times)^4$.

Combining the above, we have the following result:

Proposition 7.3.1. *Let Σ be a once-punctured disk with 4 marked points on the boundary. Then the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ is comprised of four subsets isomorphic to $(\Bbbk^\times)^4$, one subset isomorphic to $\Bbbk \times (\Bbbk^\times)^3$, and one subset isomorphic to $(\Bbbk^\times)^5$. The intersection of the closures of these six sets is a three-dimensional algebraic torus consisting of deep points of the form*

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (0, 0, 0, 0, -a, 0, 0, 0)$$
$$p(x_5, x_6, x_7, x_8) = (a, b, c, \frac{ac}{b})$$

for all $a, b, c \in \Bbbk^\times$.

Remark 7.3.2. If we specialize the frozen boundary variables to 1, the deep locus is comprised of six lines which intersect at a single point. We have the constraint

$$p(x_6)p(x_8) - p(x_5)p(x_7) = 1 - 1 = 0,$$

so away from the point of intersection, each deep point is of one of the forms

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (0,0,0,0,-1,-u,u,0),$$

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (0,0,0,0,-1,0,0,u),$$

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (0,0,0,u,-1,0,0,0),$$

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (u,0,0,0,-1,0,0,0),$$

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (0,u,0,0,-1,0,0,0), \text{ or}$$

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4') = (0,0,u,0,-1,0,0,0),$$

and the point of intersection is the point

$$(0,0,0,0,-1,0,0,0,1,1,1,1) \in \Bbbk^8 \times (\Bbbk^\times)^4.$$

An abbreviated version of this specialized result for the D_4 cluster variety previously appeared as Remark 6.11 in [CGSS24], calculated using a different initial seed. We should also remark that this specialized deep locus coincides exactly with the singular locus of $\mathcal{A}(D_4)$ in part (4)(a) of [BFMS23, Theorem A].

Note that we can also characterize the deep points of $V(\mathcal{A}(D_4), \Bbbk)$ topologically as belonging to one of seven types:

1. those that spare the even-indexed plain radii,
2. those that spare the odd-indexed plain radii,
3. those that spare the even-indexed notched radii,
4. those that spare the odd-indexed notched radii,
5. those that spare the diagonals connecting even-indexed vertices,
6. those that spare the diagonals connecting odd-indexed vertices, and
7. the point that does not spare any diagonals or radii.

Note that all deep points spare the c_i, the arcs that cut out punctured bigons along with boundary arcs.

Warning 7.3.3. Although Proposition 7.2.10 implies that the subset corresponding to totally radial deep points is isomorphic to $V(\mathcal{A}(\Delta_4), \mathbb{k})$, after specializing the boundary variables to 1, the subvariety of totally-radial points is not exactly $V(\mathcal{A}(A_1), \mathbb{k})$. We are specializing the boundary values of the punctured disk, so the value of c_i is -1, and the unpunctured polygon we get after cutting does not have boundary values all equal to 1.

7.4 Once-punctured disks with at least five boundary marked points

Continuing in the application of results of Section 7.2, we now proceed with the explicit computation of deep points for once-punctured disks with $n \geq 5$ marked points on the boundary. As seen in Figure 7.6, the mutable part of the quiver is an orientation of a type D_n Coxeter-Dynkin diagram. The mutable part of this quiver is acyclic, so we know from [BFZ05, Corollary 1.21] (or stated more directly for our application, [LS23, Theorem 2.4]) that $\mathcal{A}(\Sigma)$ is generated by an initial seed, along with one mutation at each interior arc. Using this quiver as our initial seed, the cluster algebra is generated by $x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', \ldots, x_n, x_n'$, along with the frozen variables and their inverses, where x_2 denotes the plain radius realized by a flip of $x_1^\vee$ in the given triangulation. The ideal of relations is generated by

$$
\begin{aligned}
x_1 x_2^\vee &= x_3 + x_{n+1} & \qquad x_3 x_3' &= x_1 x_1^\vee x_{n+2} + x_4 x_{n+1} \\
x_1^\vee x_2 &= x_3 + x_{n+1} & \qquad x_4 x_4' &= x_3 x_{n+3} + x_5 x_{n+2} \\
&\vdots & &\vdots \\
x_{n-1} x_{n-1}' &= x_{n-2} x_{2n-2} + x_n x_{2n-3} & \qquad x_n x_n' &= x_{n-1} x_{2n-1} + x_{2n} x_{2n-2}
\end{aligned}
$$

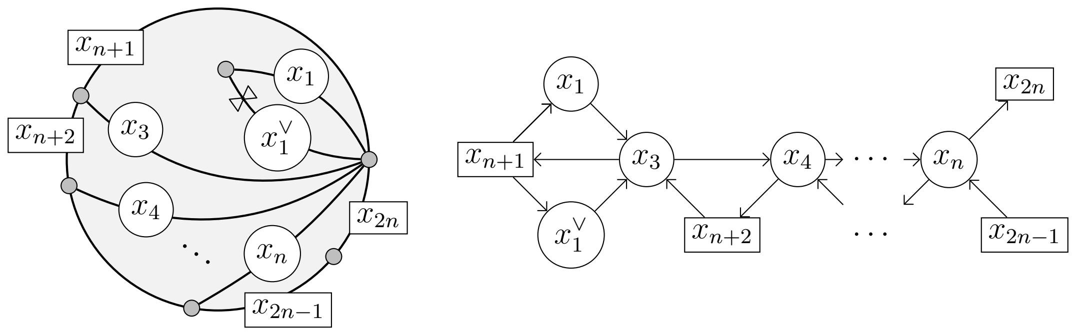


Figure 7.6: A triangulation of a once-punctured n-gon and its associated quiver.

As stated in Theorem 7.2.17, if n is odd, then the only deep points are those that kill all radii of both taggings —the totally radial deep points. If n is even, then we will have all three types of deep points, which we will consider separately. Throughout the following, let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be deep. Recall that each triangulation will contain at least one arc killed by p, so p must kill at least one plain radius and at least one notched radius; also, p does not kill any boundary frozen variables.

1. First, suppose that p is a totally radial deep point; that is, it kills all radii of both taggings. Since all radii are killed, we have $p(x_3) + p(x_{n+1}) = 0$ and $p(x_3') + p(x_4) = 0$.

   Proposition 7.2.10 implies that the totally radial points comprise a subvariety isomorphic to $V(\mathcal{A}(\Delta_n), \Bbbk)$. Cutting along c_i disconnects the disk into a once-punctured bigon and an unpunctured n-gon. As the totally radial deep points do not have a requirement on the boundary values, c_i can take on any non-zero value, and the value of b_i is determined by the value of c_i. The restriction to the unpunctured n-gon need not be deep, so the remaining arcs can take on any values satisfying the Ptolemy relations. The ideal of relations for $\mathcal{A}(\Delta_n)$ is generated by $n-3$ equations corresponding to mutation relations, which form a Gröbner basis [BFZ05, Corollary 1.17]. Choosing the values of the initial $n-3$ diagonal arcs of the unpunctured n-gon $\Sigma \setminus c_i$, in addition to the n boundary arcs, determines the values on the other generators via these defining equations. Thus, the subvariety of $V(\mathcal{A}(\Sigma), \Bbbk)$ corresponding to totally radial points is of dimension $2n - 3$.

2. Second, suppose that n is even and p is a notched deep point; that is, it kills all notched radii, but not all plain radii. In this case, p kills alternating plain radii —either all odd-indexed plain radii or all even-indexed plain radii —and the values of the boundary edges are subject to Equation 7.3.
   a) Suppose that p kills only the odd-indexed plain radii. Then $p(x_2) \neq 0$, and $p(x_3) = -p(x_{n+1}) \neq 0$, so we can cut along the arc labeled x_3. From Proposition 7.2.12, p restricts to a deep point on the unpunctured n-gon which is determined by its values on the boundary arcs. Since the boundary arcs must satisfy Equation 7.3, we can choose non-zero values for $n - 1$ of the boundary arcs and the last value is determined. Along with the choice of non-zero value for x_2, this gives us a total of n choices. Thus, these deep points constitute a subset of the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $(\Bbbk^\times)^n$.
   b) Suppose that p kills only the even-indexed plain radii. Then $p(x_1) \neq 0$, and $p(x_3) = -p(x_{n+1}) \neq 0$, so we can cut along the arc labeled x_3. From Proposition 7.2.12, p restricts to a deep point on the unpunctured n-gon which is determined by its values on the boundary arcs. Since the boundary arcs must satisfy Equation 7.3, we can choose non-zero values for $n - 1$ of the boundary arcs and the last value is determined. Along with the choice of non-zero value for x_1, this gives us a total of n choices. Thus, these deep points constitute a subset of the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $(\Bbbk^\times)^n$.

3. Third, suppose that n is even and p is a plain deep point; that is, it kills all

plain radii, but not all notched radii. In this case, p kills alternating notched radii —either all odd-indexed notched radii or all even-indexed notched radii —and the values of the boundary edges are subject to Equation 7.3.

a) Suppose that p kills only the odd-indexed notched radii. Then $p(x_2^\vee) \neq 0$, and $p(x_3) = -p(x_{n+1}) \neq 0$, so we can cut along the arc labeled x_3. From Proposition 7.2.12, p restricts to a deep point on the unpunctured n-gon which is determined by its values on the boundary arcs. Since the boundary arcs must satisfy Equation 7.3, we can choose non-zero values for $n-1$ of the boundary arcs and the last value is determined. Along with the choice of non-zero value for $x_2^\vee$, this gives us a total of n choices. Thus, these deep points constitute a subset of the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $(\Bbbk^\times)^n$.

b) Suppose that p kills only the even-indexed notched radii. Then $p(x_1^\vee) \neq 0$, and $p(x_3) = -p(x_{n+1}) \neq 0$, so we can cut along the arc labeled x_3. From Proposition 7.2.12, p restricts to a deep point on the unpunctured n-gon which is determined by its values on the boundary arcs. Since the boundary arcs must satisfy Equation 7.3, we can choose non-zero values for $n-1$ of the boundary arcs and the last value is determined. Along with the choice of non-zero value for $x_1^\vee$, this gives us a total of n choices. Thus, these deep points constitute a subset of the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $(\Bbbk^\times)^n$.

Again, cutting along c_i disconnects the disk into a once-punctured bigon and an unpunctured n-gon. As the plain and notched radial deep points require the boundary values adhere to Equation 7.3, n must be even, and

$$-c_i = b_i = (-1)^{n/2} \frac{b_{i+1} \cdots b_{i+n-1}}{b_{i+2} \cdots b_{i+n-2}}.$$

As a result, we have the the relation

$$\frac{b_{i+1}\cdots b_{i+n-1}}{c_i b_{i+2}\cdots b_{i+n-2}} = (-1)^{\frac{n+2}{2}}.$$

Thus, for each fixed choice of non-zero value on the spared radius, we have a subvariety of $V(\mathcal{A}(\Sigma \setminus c_i))$ isomorphic to the deep locus of $V(\mathcal{A}(\Delta_n), \Bbbk)$, which is of dimension $n-1$. For $n \geq 4$, any deep point p will kill three of the four radii in the punctured bigon, while the fourth can take on any non-zero value. Thus, we have four deep subsets of $V(\mathcal{A}(\Sigma), \Bbbk)$, each isomorphic to $(\Bbbk^\times)^n$.

Theorem 7.4.1. *Let Σ be a once-punctured disk with $n \geq 5$ marked points on the boundary.*

1. *If n is odd, then the deep locus is a single component of dimension $2n-3$.*
2. *If n is even, then the deep locus contains one component of dimension $2n-3$ and four distinct subsets of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $(\Bbbk^\times)^n$. The intersection of the closures of these five sets is the $(n-1)$-dimensional subvariety consisting of deep points of the form*

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4', \ldots, x_n, x_n') = (0,0,0,0,-b_1,0,0,0,\ldots,0,0)$$
$$p(x_{n+1}, x_{n+2}, \ldots, x_{2n-1}, x_{2n}) = \left(b_1, b_2, \ldots, b_{n-1}, (-1)^{n/2}\,\frac{b_1 b_3 \cdots b_{n-1}}{b_2 b_4 \cdots b_{n-2}}\right)$$

for all $b_1, \ldots, b_{n-1} \in \Bbbk^\times$.

Remark 7.4.2. If we specialize the frozen boundary variables to 1, then Equation 7.3 is satisfied only when $n \equiv 0 \bmod 4$ or $\operatorname{char}(\Bbbk) = 2$ and $n \equiv 0 \bmod 2$. Thus, we have the following description of the deep locus:

1. If $n \equiv 0 \bmod 4$ or $\operatorname{char}(\Bbbk) = 2$ and $n \equiv 0 \bmod 2$, then the deep locus is comprised of one $(n-3)$-dimensional component and four copies of $\Bbbk^\times$. The

intersection of the closures of these sets is the point

$$p(x_1, x_2^\vee, x_1^\vee, x_2, x_3, x_3', x_4, x_4', \dots, x_n, x_n') = (0, 0, 0, 0, -1, 0, 0, 0, \dots, 0, 0)$$

$$p(x_{n+1}, x_{n+2}, \dots, x_{2n-1}, x_{2n}) = (1, 1, \dots, 1, 1)$$

in $\Bbbk^{2n} \times (\Bbbk^\times)^n$.

2. Otherwise, the deep locus is comprised of a single $(n-3)$-dimensional component.

This coincides exactly with the singular locus of $\mathcal{A}(D_n)$ in parts (4)(b) and (4)(c) of [BFMS23, Theorem A].

7.5 A forgetful map

From Proposition 7.2.10, the subvariety of totally radial deep points is isomorphic to $V(\mathcal{A}(\Delta_n), \Bbbk)$. We will examine that statement a little more closely in just a moment. But first, an aside.

The forgetful map and the Birman exact sequence

Let Σ be a surface and let (Σ, x) denote the surface with a distinguished point in the interior. There is a natural homomorphism of mapping class groups

$$\mathcal{F}orget \colon \operatorname{Mod}(\Sigma, x) \to \operatorname{Mod}(\Sigma)$$

called the *forgetful map*. As noted in [FM12, Section 4.2], $\mathcal{F}orget$ is the map induced by the inclusion $\Sigma \smallsetminus x \hookrightarrow \Sigma$, and is realized by forgetting about the distinguished point x.

There is a related map

$$\mathcal{P}ush \colon \pi_1(\Sigma, x) \to \operatorname{Mod}(\Sigma, x)$$

that can be informally thought of as putting a finger on x and dragging it along a path, while also dragging the rest of the surface along with it. These maps famously fit together in the *Birman exact sequence*.

Theorem 7.5.1 (Birman Exact Sequence). *[FM12, Theorem 4.6] Let Σ be a surface with $\chi(\Sigma) < 0$. Let (Σ, x) be the surface obtained from Σ by marking a point x in the interior of Σ. Then the following sequence is exact:*

$$1 \longrightarrow \pi_1(\Sigma, x) \xrightarrow{\mathcal{P}ush} \mathrm{Mod}(\Sigma, x) \xrightarrow{\mathcal{F}orget} \mathrm{Mod}(\Sigma) \longrightarrow 1$$

The image of the $\mathcal{P}ush$ map along a given path is a homeomorphism of (Σ, x), so it sends a triangulation to a triangulation with an isomorphic quiver [BS15, Proposition 8.5][1]. As such, in terms of cluster algebras, the image of the $\mathcal{P}ush$ map along a given path is an element of the *cluster modular group* —informally, an automorphism of the cluster algebra that preserves the quiver. Allowing for changing the taggings of all arcs incident to a puncture, the cluster modular group of a cluster algebra associated to a q-punctured surface with boundary is isomorphic to $\mathrm{Mod}(\Sigma, x) \ltimes \mathbb{Z}_2^q$ [BS15, Equation 8.4]. The $\mathcal{F}orget$ map forgets taggings, as all arcs ending at boundary marked points are tagged plain; as a result, the automorphisms that switch taggings must be in the kernel of $\mathcal{F}orget$. However, the $\mathcal{P}ush$ map does not change taggings of arcs, so the automorphisms that change taggings are not in the image of $\mathcal{P}ush$. Therefore, the induced sequence on cluster modular groups is not exact.

Now, back to our discussion of deep points.

[1]Assuming Σ is not a once-punctured disk with 2 or 4 marked points on the boundary or a twice-punctured disk with 2 marked points on the boundary.

The forgetful map and deep points

Let Δ_n be an unpunctured disk with $n \geq 4$ marked points on the boundary, and let Σ denote the same disk with a distinguished marked point (puncture) in the interior. As described in the cutting and gluing subsection of Section 7.2, if we cut Σ along c_i, we get a disconnected surface with two components —a once-punctured bigon Σ_1 and an unpunctured n-gon Σ_n. The unpunctured disks Δ_n and Σ_n are homeomorphic, so there is an induced map

$$\mathcal{F}\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Delta_n)$$

realized by forgetting the once-punctured bigon Σ_1, which in turn gives us a dual map on varieties

$$V(\mathcal{A}(\Delta_n), \Bbbk) \to V(\mathcal{A}(\Sigma), \Bbbk).$$

Proposition 7.2.10 tells us that the points of $V(\mathcal{A}(\Delta_n), \Bbbk)$ are in bijection with the totally radial deep points of $V(\mathcal{A}(\Sigma_n), \Bbbk)$, so this map induces an inclusion of the cluster variety of the unpunctured disk (or polygon) $V(\mathcal{A}(\Delta_n), \Bbbk)$ into the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$. Restricting $\mathcal{A}(\Delta_n)$ to $\mathcal{A}(A_{n-3})$ by specializing the frozen boundary variables to 1, we also get an inclusion of the cluster variety $V(\mathcal{A}(A_{n-3}), \Bbbk)$ into the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$.

Theorem 7.5.2. *The forgetful map $\mathcal{F}\colon \mathcal{A}(\Sigma) \to \mathcal{A}(\Delta_n)$ induces inclusions of:*

- *the cluster variety $V(\mathcal{A}(\Delta_n), \Bbbk)$ into the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$, and*
- *the cluster variety $V(\mathcal{A}(A_{n-3}), \Bbbk)$ into the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$.*

Recall, however, that specializing the frozen boundary variables of Σ to 1 forces one of the frozen boundary variables of Σ_n to be -1. Thus, $\mathcal{F}$ does not induce an inclusion of $V(\mathcal{A}(A_{n-3}), \Bbbk)$ into the deep locus of $V(\mathcal{A}(D_n), \Bbbk)$, but rather a deep subvariety of $V(\mathcal{A}(\Sigma), \Bbbk)$ corresponding to points assigning boundary values $(-1, 1, 1, \ldots, 1)$.

Chapter 8

Deep points of punctured marked surfaces

In this chapter, we will classify and parameterize the deep points of several classes of punctured marked surfaces. We will assume throughout the chapter that our surface is connected, orientable, and triangulable, and will take care to specify the genus, number of boundary components, and number and configuration of marked points at each step.

Recall from Theorem 5.2.5 that $\mathcal{A}(\Sigma) \cong \mathrm{Sk}^{\bowtie}(\Sigma)$ if Σ is a triangulable marked surface such that at least one of the following is true:

1. Σ is a disk.
2. Σ embeds into a disk.
3. Σ has at least two boundary marked points in each connected component.

. As such, we will focus on classifying deep points of these surfaces first. Since we assume that the surface is triangulable, the second type —those that embed in a disk —will be covered by the first and third types, so we will not address it directly. We will initiate a discussion of other surfaces, such as the once-punctured torus (without boundary) at the end of the chapter.

Recall that a (tagged) radius is a simple arc with one end at the boundary and the other end at a puncture (tagged plain or notched). In contrast to the once-punctured disks we discussed in Chapter 7, in a surface with multiple punctures, not every simple arc emanating from a puncture is a radius. We refer to arcs with both ends at punctures as **tunnels**. While every triangulation includes at least two arcs emanating from each puncture, a triangulation may not include radii emanating from each puncture, as demonstrated in Figure 8.1;

If a deep point p kills all arcs emanating from a puncture ξ, we will say that ξ is **quashed** by p. If all punctures are quashed by p, we will refer to p as a **malevolent** deep point; on the other hand, if p does not quash any punctures, we will refer to it as **benevolent**. Note that because every triangulation will include multiple arcs emanating from each puncture, if any one puncture is quashed by p, that is enough to guarantee the deepness of p and we should not place any a priori expectations on what happens at the other punctures.

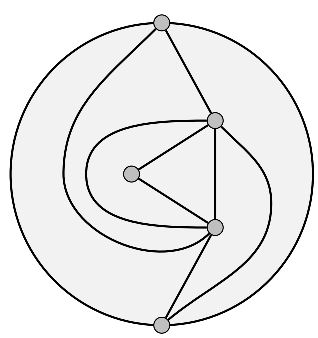


Figure 8.1: A triangulation of a thrice-punctured bigon where not every puncture is connected to the boundary by a radius.

8.1 Punctured disks

Another difference from the case of once-punctured disks is that a disk with $q \geq 2$ punctures and a single marked point on the boundary is triangulable, as demonstrated in Figure 8.2. We will divide our discussion of punctured disks into two parts —those disks with $n \geq 2$ marked points on the boundary and those with only $n = 1$ marked point on the boundary.

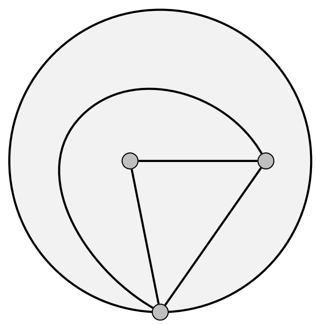


Figure 8.2: A triangulation of a twice-punctured monogon.

Before moving on to the separate cases, let us prove some results that will apply in both settings.

Cutting along radii

First, we want to examine what happens when we cut along a radius. Suppose that a deep point p assigns a non-zero value to a radius r. Then we can undo the tagging on $r^\vee$ as demonstrated in Figure 8.3, and then cut along r from the boundary up to the puncture. Note that this creates not just two copies of the radius r, but also two copies of the boundary marked point.

Observe that if the notched version of that radius $r^\vee$ is instead non-zero, we can apply an isomorphism to the cluster algebra that switches the taggings at a given puncture before performing the cut. We will ignore this step in most the following

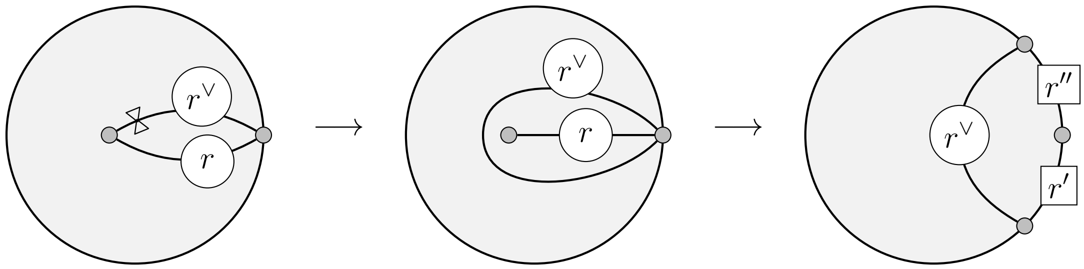


Figure 8.3: Cutting along a radius

arguments and simply assume that the radius being cut has a plain tagging, but we will need to remember to consider the taggings when counting the number of deep components.

First, a proposition that we could have proven in the previous chapter:

Proposition 8.1.1. *Let Σ be a disk with $n \geq 2$ marked points on the boundary and $q = 1$ punctures. There is a bijection*

$$\left\{\begin{array}{c}\textit{non-totally radial deep}\\ \textit{points in } V(\mathcal{A}(\Sigma), \Bbbk)\\ \textit{non-zero on a radius } r_i\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{deep points in}\\ V(\mathcal{A}(\Sigma \smallsetminus r_i), \Bbbk)\\ \textit{gluable along } r_i\end{array}\right\}$$

Proof. Suppose that $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ is a deep point such that p spares some radius r_i, which connects the puncture ξ_i to the boundary marked point m_i. From Proposition 6.1.7, we know that its image under the cutting isomorphism $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus r_i), \Bbbk)$ must be deep as well.

Now, suppose that $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus r_i), \Bbbk)$ is a deep point that is gluable along r_i. That is, two adjacent boundary arcs are assigned the same value $p(r_i)$; denote the marked point between these two boundary arcs as ξ_i'.

Since $\Sigma \smallsetminus r_i$ is an unpunctured $(n+2)$-gon, Theorem 4.3.1 says that n must be even and the values on the boundary arcs must satisfy

$$\frac{b_1 \cdots b_{n-1}\, p(r_i)}{b_2 \cdots b_n\, p(r_i)} = \frac{b_1 \cdots b_{n-1}}{b_2 \cdots b_n} = (-1)^{\frac{(n+2)+2}{2}} = (-1)^{n/2}.$$

By the same theorem, we see that the diagonals from boundary marked points to ξ_i' alternate between zero and non-zero values, while all length 2 arcs that form triangles with two boundary arcs are killed. In particular, the image of $r_i^\vee$ is killed.

Note that every triangulation of Σ will contain at least two radii. Suppose that T is a triangulation of Σ containing two radii spared by p. These radii are arcs in $\Sigma \smallsetminus r_i$ and are both spared by $\bar{p}$. By Lemma 4.2.1, the arc completing a triangle with these arcs must be killed by $\bar{p}$, and its gluing in Σ is killed by p. Thus, p kills at least one arc in every triangulation of Σ. Hence, $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ is a non-totally radial deep point. □

We already know from Theorem 7.2.17 that if p is not a totally radial deep point, there is a radius that is spared by p, so the above holds for all non-totally radial deep points. Before extending this Proposition to allow for multiple punctures, we want to show that a benevolent deep point always spares at least one radius.

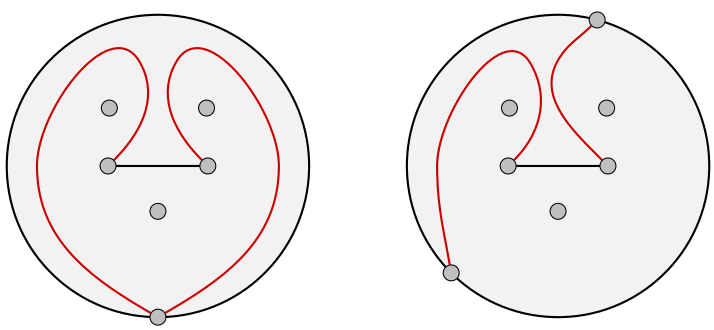


Figure 8.4: Unpunctured quadrilaterals with a tunnel opposite a boundary component.

Lemma 8.1.2. *Let Σ be a disk with $n \geq 1$ marked points on the boundary and $q \geq 2$ punctures. For any given tunnel τ, there is an unpunctured quadrilateral in Σ bounded by one boundary arc, two radii, and the tunnel τ.*

Proof. As in Figure 8.4, take two radii emanating from opposite ends of a tunnel and have them travel together in parallel between any other punctures toward the boundary. Before reaching the boundary, have them travel in opposite directions along the boundary until they each reach a marked point. □

Lemma 8.1.3. *Let Σ be a disk with $n \geq 1$ marked points on the boundary and $q \geq 2$ punctures, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a benevolent deep point. Then p must spare at least one radius.*

Proof. Seeking a contradiction, suppose that p kills all radii. Since p is benevolent by assumption, p must spare a tunnel τ. By Lemma 8.1.2, there exists a quadrilateral consisting of two radii, a boundary component b, and τ, with τ opposite b as in Figure 8.4. The Ptolemy relation on such a quadrilateral is then $0 = p(\tau)p(b) + 0$, but both $p(b)$ and $p(\tau)$ are non-zero by assumption. This is a contradiction, so p must spare at least one radius. □

And now, we extend Proposition 8.1.1 to the current setting.

Proposition 8.1.4. *Let Σ be a disk with $n \geq 1$ marked points on the boundary and $q \geq 2$ punctures. There is a bijection*

$$\left\{\begin{array}{c}\textit{benevolent deep points}\\ \textit{in } V(\mathcal{A}(\Sigma), \Bbbk)\\ \textit{non-zero on } r_i\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{benevolent deep points}\\ \textit{in } V(\mathcal{A}(\Sigma \smallsetminus r_i), \Bbbk)\\ \textit{gluable along } r_i\end{array}\right\}$$

Proof. Suppose that $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ is a benevolent deep point such that p spares the radius r_i, with ends at the puncture ξ_i and marked point m_i. From Proposition 6.1.7, we know that its image under the cutting isomorphism $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus r_i), \Bbbk)$ must be deep as well. Since p does not quash any punctures, $\bar{p}$ will not quash any remaining puncture after cutting, so $\bar{p}$ is a benevolent deep point on $\Sigma \smallsetminus r_i$.

To prove the opposite direction, we use induction on the number of punctures. For the following, assume that $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus r_i), \Bbbk)$ is a benevolent deep point that is gluable along r_i.

- First, suppose that $q = 2$. Since $\Sigma \smallsetminus r_i$ is a once-punctured $(n+2)$-gon, Theorem 7.2.17 says that since p is non-zero on r_i, it must kill alternating plain radii and all notched radii, and the boundary values satisfy

$$\frac{b_1 \cdots b_{n-1}\, r_i}{b_2 \cdots b_n\, r_i} = \frac{b_1 \cdots b_{n-1}}{b_2 \cdots b_n} = (-1)^{\frac{n+2}{2}}.$$

  Also, n must be even. Since $\bar{p}$ is gluable along r_i, p will have a non-zero value on r_i, so p does not quash either puncture.

  Also, observe that any triangulation of Σ must either include radially adjacent radii or a length 2 diagonal. Thus, every triangulation must include an arc that is killed by p as a gluing of $\bar{p}$, and p is therefore a benevolent deep point that spares r_i.

- For our inductive hypothesis, now suppose that $q > 2$ and the bijection holds for all $q_0 < q$. Since $\Sigma \smallsetminus r_i$ is a $(q-1)$-punctured $(n+2)$-gon, our inductive hypothesis says that there exists an ordered sequence of non-boundary arcs $r_1, \ldots, r_{q-1}$ in $\Sigma \smallsetminus r_i$ that are spared in $\bar{p}$ such that the sets

$$R_k \coloneqq \{r_1, \ldots, r_k\}, \text{ and}$$

$$S_k \coloneqq \{\text{benevolent deep points } p \in V(\mathcal{A}(\Sigma \smallsetminus R_{k-1}), \Bbbk) \text{ non-zero on } R_k\}$$

  give us bijections

$$S_2 \leftrightarrow S_3 \leftrightarrow \cdots \leftrightarrow S_{q-2} \leftrightarrow S_{q-1}.$$

  There is then a benevolent deep point $\widehat{p} \in V(\mathcal{A}(\Sigma \smallsetminus R_{q-1}), \Bbbk)$ that glues to a benevolent deep point on $\Sigma \smallsetminus r_i$. Hence, $\widehat{p}$ kills alternating plain radii and all notched radii in $\Sigma \setminus R_{q-1}$, and satisfies the boundary relation

$$\frac{b_1 \cdots b_{n-1}\, r_1 \cdots r_{q-1}\, r_i}{b_2 \cdots b_n\, r_1 \cdots r_{q-1}\, r_i} \;=\; \frac{b_1 \cdots b_{n-1}}{b_2 \cdots b_n} \;=\; (-1)^{\frac{n+2q+2}{2}}.$$

As a result, n must be even, and alternating radii around each puncture must be killed by p.

Again, observe that any triangulation of Σ must either include radially adjacent radii or a length 2 diagonal. Thus, every triangulation must include an arc that is killed by p as a gluing of $\bar{p}$, and p is therefore a benevolent deep point that spares r_i.

□

Remark 8.1.5. When we say *alternating radii around each puncture*, we mean the radii connecting that puncture to either the even-indexed marked points or the odd-indexed marked points. It is possible that a triangulation will contain multiple radii with the same endpoints (e.g. Figure 8.2), with both radii killed or both spared. This will be further clarified in Lemma 8.1.7.

Let us now emphasize a few takeaways from the above proof in the form a porism.

Porism 8.1.6. *Let Σ be a disk with $n \geq 1$ marked points on the boundary and $q \geq 2$ punctures, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be deep.*

1. *If n is odd, then p must quash at least one puncture.*
2. *If p is benevolent, then p must satisfy the boundary relation*

$$\frac{b_1 \cdots b_{n-1}}{b_2 \cdots b_n} \;=\; (-1)^{\frac{n+2q+2}{2}}.$$

3. *If p is benevolent, then around each puncture, p must kill alternating radii of one tagging and all radii of the opposite tagging.*

Disks with at least 2 marked points on the boundary

We examine the deep points in three groups: the malevolent points, those that quash a proper subset of the punctures, and those that quash no punctures. Let Σ be a disk with $n \geq 2$ marked points on the boundary and $q \geq 2$ punctures, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be deep.

- Suppose p is a malevolent deep point, so every puncture is quashed by p. Let $c_{i,1}$ be an arc that cuts out a once-punctured bigon opposite the boundary arc b_i. For $2 \leq j \leq p$, let $c_{i,j}$ be an arc that cuts out a once-punctured bigon opposite the arc $c_{i,j-1}$. Since every arc emanating from every puncture is killed by p, we know that $c_{i,1} = -b_i \neq 0$. Cutting along $c_{i,1}$, $\Sigma \setminus c_i$ has two components —a once-punctured bigon and a $(q-1)$-punctured n-gon. Repeating this process, we see that $c_{i,j} = (-1)^j\, b_i$, so $c_{i,j} \neq 0$ and we can cut along each $c_{i,j}$. The cut surface $\Sigma \smallsetminus \{c_{i,1}, \ldots, c_{i,q}\}$ then has $q+1$ components —an unpunctured n-gon and q once-punctured bigons.

  The deep point $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus \{c_{i,1}, \ldots, c_{i,q}\}), \Bbbk)$ restricts to a deep point on each of once-punctured bigons, so it need not be deep on the unpunctured n-gon. Also, the value of the boundary arc b_i is determined by the value of the corresponding boundary arc $c_{i,q}$ in the unpunctured n-gon. As such, the set of malevolent deep points is isomorphic to the cluster variety $V(\mathcal{A}(\Delta_n), \Bbbk)$, which is of dimension $2n-3$.

- Suppose p is not a malevolent deep point, but at least one puncture ξ_i is quashed by p. Let $c_{i,1}$ be an arc that cuts out a once-punctured bigon opposite the boundary arc b_i, such that the bigon contains the puncture ξ_i. Since all arcs emanating from ξ_i are killed by p, we have $c_{i,1} = -b_i \neq 0$. Cutting along $c_{i,1}$, $\Sigma \setminus c_{i,1}$ has two components —a once-punctured bigon Σ_1 and a $(q-1)$-

punctured n-gon Σ°. The deep point $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus c_{i,1}), \Bbbk)$ restricts to a deep point on Σ_1, so it need not be deep on Σ°. Also, the value of the boundary arc b_i is determined by the value of the corresponding boundary arc $c_{i,1}$ in Σ°. As such, the set of deep points that quash a given puncture is isomorphic to $V(\mathcal{A}(\Sigma^\circ), \Bbbk)$, the cluster variety corresponding to the $(q-1)$-punctured n-gon. Since there are q punctures, there are q such deep subsets, with intersection equal to the malevolent points described in the previous bullet.

- Suppose no punctures are quashed by p. From Lemma 8.1.3, we know that at least one radius is spared. From Porism 8.1.6, we know three things:
  - If n is odd, then p must quash at least one puncture.
  - If p is benevolent, then p must satisfy the boundary relation
  $$\frac{b_1 \cdots b_{n-1}}{b_2 \cdots b_n} \;=\; (-1)^{\frac{n+2q+2}{2}}.$$
  - If p is benevolent, then around each puncture, p kills alternating radii of one tagging and all radii of the opposite tagging.

  Thus, if n is odd, there are no benevolent deep points. If n is even, then the boundary values must satisfy the given relation, so so choosing $n-1$ of them determines the last one. Also, p kills alternating radii of one tagging at each puncture, so once the boundary values are chosen, a choice of non-zero value for one radius at each puncture determines the rest. Recall that we have 2 possible taggings at each puncture, and a choice as to whether the odd-indexed or even-indexed radii are killed, giving us 4^q sets of benevolent deep points, each isomorphic to $(\Bbbk^\times)^{q+n-1}$.

Monogons

We again examine the deep points in three groups: the malevolent points, those that quash a proper subset of the punctures, and those that quash no punctures. Let Σ be a disk with $n = 1$ marked points on the boundary and $q \geq 2$ punctures, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be deep.

- Suppose p is a malevolent deep point, so every puncture is quashed by p.
  1. If $q = 2$, then there is only one arc with a non-zero value —the boundary arc —and it will appear in Ptolemy relations only as a product with some other arc, all Ptolemy relations are simplified to $0 = 0$. Thus, the value on the boundary arc can be any non-zero value, giving us a deep subset of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $\Bbbk^{\times}$.
  2. If $q \geq 3$, there are arcs which cut out k-punctured monogons for $2 \leq k \leq q-1$, which we denote c_k. Observe that the arc c_{q-1} forms a once-punctured bigon with the boundary arc b. Since all radii are killed, we have the relation $c_{q-1} = -b_1 \neq 0$. Likewise, $c_{j-1} = -c_j$ for all $2 < j \leq q-1$. Cutting along $\{c_2, \ldots, c_{q-1}\}$ gives us $q-1$ components: a twice-punctured monogon and $q-2$ once-punctured bigons. Since the value of the boundary arc b_1 determines the values of the boundary arcs of the other components, there is a single choice of non-zero value, giving us a single deep subset of $V(\mathcal{A}(\Sigma), \Bbbk)$ isomorphic to $\Bbbk^{\times}$.
- Suppose p is not a malevolent deep point, but at least one puncture ξ_i is quashed by p.
  - First, suppose that $q = 2$. Since p is not malevolent, there is some other puncture ξ_j that is not quashed by p. Thus, we know there is at least one arc α emanating from ξ_j that is spared by p. Likewise, there is an

arc β that cuts out a bigon with α containing the quashed puncture ξ_i. Since $p(\alpha) \neq 0$, and all arcs emanating from ξ_i are killed by p, the relation $p(\alpha) + p(\beta) = 0$ implies that $p(\beta) \neq 0$ as well.

Cutting along the arcs $\{\alpha, \beta\}$ produces one of two options, depending on whether the arcs are tunnels or one end is at the boundary. If both ends are at punctures, then we have a disconnected surface with one component a once-punctured bigon and the other a $(q-3)$-punctured $(2,1)$-annulus. If one end is at the boundary, then we have a disconnected surface with one component a once-punctured bigon and the other a $(q-2)$-punctured trigon. Denote the once-punctured bigon as Σ_1 and the other component as Σ°. Since the restriction of p to Σ_1 is deep, the restriction to Σ° need not be deep. Thus, the set of deep points that quash a given puncture is isomorphic to $V(\mathcal{A}(\Sigma^\circ), \Bbbk)$, the cluster variety corresponding to the surface with a punctured bigon cut out.

– Now, suppose that $q \geq 3$, and let ξ_1 be a puncture that is quashed by p. Denote by c_1 the arc that cuts out a $(q-1)$-punctured monogon not containing ξ_1. Along with the boundary arc b_1, the arc c_1 cuts out a once-punctured bigon containing ξ_1. Since all radii from ξ_1 are killed by p, we have the relation $c_1 = -b_1 \neq 0$; cutting along c_1 gives us two connected components —a once-punctured bigon Σ_1 and a $(q-1)$-punctured monogon Σ°. Because the restriction of p to Σ_1 is deep, the restriction to Σ° need not be deep. Further, the values on the boundary edges of Σ_1 are determined by the boundary value on Σ°, so the set of deep points that quash a given puncture is isomorphic to $V(\mathcal{A}(\Sigma^\circ), \Bbbk)$, the cluster variety corresponding to a $(q-1)$-punctured monogon.

Finally, since there are q punctures, there are q such deep subsets, with intersection equal to the set of malevolent points.

- From Porism 8.1.6, we know that if n is odd, then p must quash at least one puncture. Thus, there are no benevolent deep points in a punctured monogon.

Benevolent deep points and radii

Before moving on to more general surfaces, we want to highlight how a benevolent deep point handles radii. In particular, in a disk with at least 2 punctures, there are radii with the same endpoints that are not homotopic, so it is natural to ask how the values on these radii are related.

Lemma 8.1.7 (The Unicorn Lemma). *Let Σ be a disk with $n \geq 1$ marked points on the boundary and $q \geq 2$ punctures, and let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a benevolent deep point. If r_1 and r_2 are radii with the same endpoints, then*

$$p(r_1) = (-1)^k \, p(r_2)$$

for some $k \in \mathbb{N}$.

Proof. If r_1 and r_2 are compatible —that is, their homotopy classes contain representative arcs that do not intersect —then they bound a k-punctured bigon for some $k \geq 1$. Since p kills all radii of one of the taggings, if $k = 1$, we have the relation $p(r_1) + p(r_2) = 0$. If $k > 1$, there are radii $c_1, \ldots, c_{k-1}$ that cut the k-punctured bigon into k once-punctured bigons. The relations $p(r_1) + p(c_1) = 0$, $p(c_{k-1}) + p(r_2) = 0$, and $p(c_i) + p(c_{i+1}) = 0$ for all $1 \leq i \leq k-2$ combine to give us the desired relation.

If r_1 and r_2 are not compatible, we can construct a sequence of radii with the same endpoints $r_1 = c_0, c_1, \ldots, c_\ell, c_{\ell+1} = r_2$ such that c_j is compatible with both

c_{j-1} and c_{j+1} for all $1 \leq j < \ell$. To begin, we choose minimally intersecting representatives of r_1 and r_2 that intersect transversely at each crossing. Next, assign an orientation to each radius so that both radii "flow" from the boundary marked point m to the puncture ξ.

Now, let c_1 be the radius that starts at m and follows r_2 until its first intersection with with r_1, then turns to follow r_1 in the direction of the puncture ξ. This new radius c_1 is such that $\mu(r_1, c_1) = 0$ and $\mu(c_1, r_2) < \mu(r_1, r_2)$.[1] Since r_1 and c_1 have non-intersecting homotopy representatives, $p(r_1) = (-1)^{k_1}\, p(c_1)$ for some $k_1 \in \mathbb{N}$ from the first paragraph of this proof.

If $\mu(c_1, r_2) > 0$, repeat the process, choosing minimally intersecting representatives of c_1 and r_2 that intersect transversely at each crossing, and assigning an orientation to each radius so that both radii flow from m to ξ. Let c_2 be the radius that starts at m and follows r_2 until its first intersection with with c_1, then turns to follow c_1 in the direction of the puncture ξ. This radius c_2 is such that $\mu(c_1, c_2) = 0$ and $\mu(c_2, r_2) < \mu(c_1, r_2)$. Since c_1 and c_2 have non-intersecting homotopy representatives, $p(c_1) = (-1)^{k_2}\, p(c_2)$ for some $k_2 \in \mathbb{N}$. If $\mu(c_2, r_2) > 0$, repeat this process until we have a radius c_ℓ such that $\mu(c_\ell, r_2) = 0$, so that $p(c_\ell) = (-1)^{k_\ell}\, p(r_2)$ for some $k_\ell \in \mathbb{N}$.

Finally, we have a sequence $r_1 = c_0, c_1, \ldots, c_\ell, c_{\ell+1} = r_2$ of pairwise compatible radii, with each pair giving us a relation $p(c_{j-1}) = (-1)^{k_j}\, p(c_j)$. Combining these relations, we have

$$p(r_1) = (-1)^{\sum_{i=1}^{\ell} k_i}\, p(r_2). \qquad \square$$

Remark 8.1.8. The construction used in the proof of Lemma 8.1.7 is inspired by the *unicorn paths* of [HPW15] and *surgery paths* of [Hat91].

[1] This is the intersection pairing of Definition 5.3.5.

Corollary 8.1.9. *Let Σ be a disk with $n \geq 1$ marked points on the boundary and $q \geq 2$ punctures, and let $\mathfrak{R}_{i,j}$ denote the set of all radii connecting a puncture ξ_i to a marked point m_j. If $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ is a benevolent deep point, then either p kills every radius in $\mathfrak{R}_{i,j}$ or p spares every radius in $\mathfrak{R}_{i,j}$.*

8.2 Punctured surfaces with boundary

We now turn our attention to more general punctured surfaces, possibly with positive genus or multiple boundary components. Because we assume that they are triangulable, we require there be at least one marked point on every boundary component. We will also require at least two marked points in each connected component; if there is only one boundary component, it must contain at least two marked points.

Points that quash at least one puncture

In this section, we will examine only the malevolent points and those that quash a proper subset of the punctures, and defer discussion of benevolent points. First, we will examine the case where there are at least two marked points on one of the boundary components. After that, we will examine surfaces with multiple boundary components, each containing a single marked point.

Surfaces with two marked points on the same boundary component

Consider a connected surface Σ with genus $g \geq 0$, $b \geq 1$ boundary components, $q \geq 1$ punctures, and $m \geq 2$ marked points, with at least two of the marked points on the same boundary component. Let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be deep.

- Suppose p is a malevolent deep point, so every puncture is quashed by p. Let $c_{i,1}$ be an arc that cuts out a once-punctured bigon opposite the boundary arc b_i. For $2 \leq j \leq p$, let $c_{i,j}$ be an arc that cuts out a once-punctured bigon opposite the arc $c_{i,j-1}$. Since every arc emanating from every puncture is killed by p, we know that $c_{i,1} = -b_i \neq 0$. The cut surface $\Sigma \setminus c_{i,1}$ has two components —a once-punctured bigon and a $(q-1)$-punctured n-gon. Since $c_{i,j} = (-1)^j\, b_i$, we know $c_{i,j} \neq 0$, so we can cut along each $c_{i,j}$. The cut surface $\Sigma \smallsetminus \{c_{i,1}, \ldots, c_{i,q}\}$ then has $q+1$ components —an unpunctured version of Σ which we will call $\widetilde{\Sigma}$, and q once-punctured bigons.

  The deep point $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus \{c_{i,1}, \ldots, c_{i,q}\}), \Bbbk)$ restricts to a deep point on each of the once-punctured bigons, so it need not be deep on the unpunctured surface $\widetilde{\Sigma}$. Also, the boundary values on the once-punctured bigons are determined by the boundary values on $\widetilde{\Sigma}$, so p is determined by its values on $\widetilde{\Sigma}$. As such, the set of malevolent points is isomorphic to the cluster variety coming from the unpunctured surface, $V(\mathcal{A}(\widetilde{\Sigma}), \Bbbk)$.

- Suppose p is not a malevolent deep point, but at least one puncture ξ_i is quashed by p. Let $c_{i,1}$ be an arc that cuts out a once-punctured bigon opposite the boundary arc b_i, such that the bigon contains the puncture ξ_i. Since all arcs emanating from ξ_i are killed by p, we have $c_{i,1} = -b_i \neq 0$. The cut surface $\Sigma \setminus c_{i,1}$ has two components —a once-punctured bigon Σ_1 and a $(q-1)$-punctured surface Σ°. The deep point $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus c_{i,1}), \Bbbk)$ restricts to a deep point on Σ_1, so it need not be deep on Σ°. As such, the set of deep points which quash a given puncture is isomorphic to $V(\mathcal{A}(\Sigma^\circ), \Bbbk)$, the cluster variety corresponding to the $(q-1)$-punctured surface.

  Since there are q punctures, there are q such sets of deep points, with intersection equal to the malevolent points.

Surfaces with only one marked point on each boundary component

Now, consider a connected surface Σ with genus $g \geq 0$, $b \geq 2$ boundary components, $q \geq 1$ punctures, and $m \geq 2$ marked points, with the assumption that each boundary component contains only a single marked point. Let $p \in V(\mathcal{A}(\Sigma), \mathbb{k})$ be deep.

First, let us prove a short lemma that will be useful in the following discussion.

Lemma 8.2.1. *Let p and Σ be as defined above. Then p spares at least one arc in Σ.*

Proof. Consider the arcs as depicted in Figure 8.5. The arcs a_q, b_1, a_0', and b_2 form an unpunctured quadrilateral with diagonals d_1 and d_2. This gives us a Ptolemy relation $d_1 d_2 = a_q a_0' + b_1 b_2$. Since b_1 and b_2 are boundary arcs, we have $b_1 b_2 \neq 0$, so at least one of the other terms must be non-zero as well. Therefore, either $a_q a_0' \neq 0$ or $d_1 d_2 \neq 0$. □

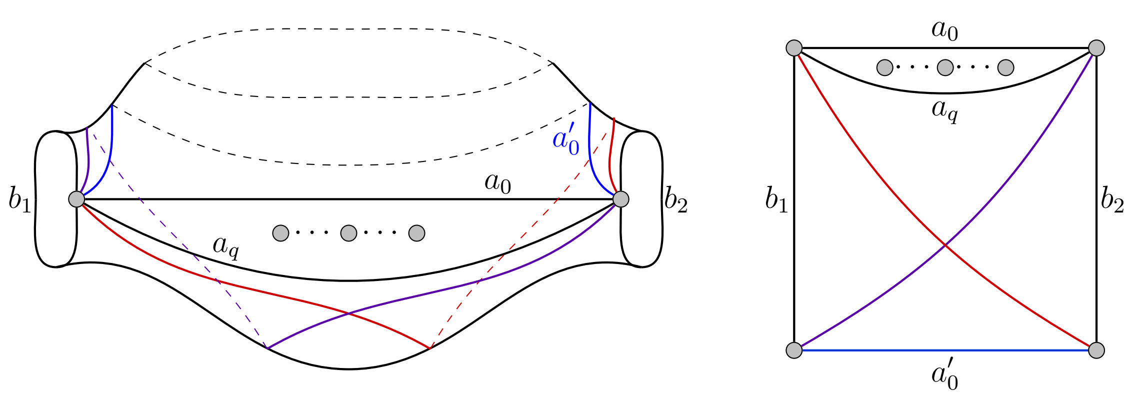


Figure 8.5: Local picture of arcs forming an unpunctured quadrilateral in the surface with one marked point on each boundary

Now, we can proceed with our discussion of the deep points that quash at least one puncture.

- Suppose that p is a malevolent point, so every puncture is quashed by p. As a result of Lemma 8.2.1, we know that p spares at least one arc; let us denote this arc as c_0. Let c_1 be an arc that cuts out a once-punctured bigon opposite c_0. Since all arcs emanating from each puncture are killed by p, we have $p(c_0) + p(c_1) = 0$. Now, we cut along $\{c_0, c_1\}$.

Let c_2 be an arc in $\Sigma \smallsetminus \{c_0, c_1\}$ that cuts out a once-punctured bigon with c_1. Since all arcs emanating from each puncture are killed by p, we have $p(c_1) + p(c_2) = 0$. Now cut along $\{c_2\}$. Repeat this process until we have $q+1$ components, q once-punctured bigons and an unpunctured surface, which we will call Σ°.

The deep point $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus \{c_0, \dots, c_q\}), \Bbbk)$ restricts to a deep point on each of the once-punctured bigons, so it need not be deep on the unpunctured surface Σ°. Also, the boundary values on the once-punctured bigons are determined by the boundary values on Σ°, so p is determined by its values on Σ°. As such, the set of malevolent points is isomorphic to the cluster variety coming from the unpunctured surface, $V(\mathcal{A}(\Sigma^\circ), \Bbbk)$.

- Now, suppose that p is not a malevolent deep point, but at least one puncture ξ_i is quashed by p. As a result of Lemma 8.2.1, we know that p spares at least one arc; let us denote this arc as c_0. Let c_1 be an arc that cuts out a once-punctured bigon opposite c_0, such that ξ_i is contained in this bigon. Since all arcs emanating from ξ_i are killed by p, we have $p(c_0) + p(c_1) = 0$. Now, we cut along $\{c_0, c_1\}$, giving us two components: a once-punctured bigon and a $(q-1)$-punctured surface, which we will call Σ°.

  The deep point $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus \{c_0, c_1\}), \Bbbk)$ restricts to a deep point on the once-punctured bigon, so it need not be deep on the unpunctured surface Σ°. Also, the boundary values on the once-punctured bigon are determined by the boundary values on Σ°, so p is determined by its values on Σ°. As such, the set of deep points that quash ξ_i is isomorphic to the cluster variety coming from the $(q-1)$-punctured surface, $V(\mathcal{A}(\Sigma^\circ), \Bbbk)$.

As there are q punctures, there are q such sets of deep points, with intersection equal to the malevolent points.

Remark 8.2.2. Given that the surface Σ has at least two boundary components, if we assume that a given puncture is quashed, then we can use a boundary component as one side of the cut out bigon as in previous arguments. This gives an equivalent argument with the more informative description of the set of malevolent points being isomorphic to the cluster variety of the unpunctured version of Σ.

Benevolent Points

Before examining the benevolent points, we want to extend previous results regarding cutting along radii.

Cutting along radii

Lemma 8.2.3. *Let Σ be a surface with genus $g \geq 0$, $b \geq 1$ boundary components, $m \geq 2$ marked points on the boundary, and $q \geq 2$ punctures. For any given tunnel τ, there is an unpunctured quadrilateral in Σ bounded by one boundary arc, two radii, and the tunnel τ.*

Proof. The proof is the same as in Lemma 8.1.2. Take two radii emanating from opposite ends of a tunnel and have them travel together in parallel between any other punctures toward a boundary component. Before reaching the boundary, have them travel in opposite directions along the boundary until they each reach a marked point. $\square$

Lemma 8.2.4. *Let Σ be a surface with genus $g \geq 0$, $b \geq 1$ boundary components, $m \geq 2$ marked points on the boundary, and $q \geq 1$ punctures. If $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ does not quash any punctures, then it spares at least one radius.*

Proof. Seeking a contradiction, suppose that p kills all radii. If $q = 1$, then the only arcs emanating from the puncture are radii. If p kills all radii, then p quashes the puncture so we have a contradiction.

Now, suppose that $q \geq 2$. Since p does not quash any punctures, there must exist a tunnel τ that is spared by p. By Lemma 8.2.3, there exists a quadrilateral consisting of two radii, a boundary component b, and τ, with τ opposite b as in Figure 8.4. The Ptolemy relation on such a quadrilateral is then $0 = p(\tau)p(b) + 0$, but both $p(b)$ and $p(\tau)$ are non-zero by assumption. This is a contradiction, so p must spare at least one radius. □

Lemma 8.2.5. *Let Σ be a surface with genus $g \geq 0$, $b \geq 1$ boundary components, $m \geq 2$ marked points on the boundary, and $q \geq 1$ punctures. If $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ does not quash any punctures, then there exist a collection of arcs $r_1, \ldots, r_q$ such that $p(r_i) \neq 0$ for all $1 \leq i \leq q$, and $\widehat{\Sigma} \coloneqq \Sigma \setminus \{r_1, \ldots, r_q\}$ is an unpunctured surface with genus g, b boundary components, and $m + 2q$ marked points on the boundary.*

Proof. By Lemma 8.2.4, there exists a radius r_1 such that $p(r_1) \neq 0$. Cutting along r_1, we get a surface Σ_1 with genus g, b boundary components, $m+2$ marked points, and $q - 1$ punctures.

Repeatedly applying Lemma 8.2.4, after each cut we have a new surface with two additional marked points and one fewer puncture. After q steps, we have an unpunctured surface $\widehat{\Sigma}$ with genus g, b boundary components, and $m + 2q$ marked points on the boundary. □

Now, we wish to extend Corollary 6.1.6 to the setting of punctured surfaces.

Proposition 8.2.6. *Let Σ be a connected surface with genus $g \geq 0$, $b \geq 1$ boundary components, $m \geq 2$ marked points on the boundary, and $q \geq 1$ punctures. If $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ is a benevolent deep point, then there is a polygonal dissection D of Σ on which p is non-zero.*

Proof. First, let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a benevolent deep point. By Lemma 8.2.5, there exist arcs $r_1, \ldots, r_q$ such that $p(r_i) \neq 0$ for $1 \leq i \leq q$, and the cut surface $\widehat{\Sigma} := \Sigma \smallsetminus \{r_1, \ldots, r_q\}$ is an unpunctured surface with genus g, b boundary components, and $m + 2q$ marked points on the boundary. By Proposition 6.1.7, the image of p under the cutting isomorphism $\bar{p} \in V(\mathcal{A}(\widehat{\Sigma}), \Bbbk)$ is deep. From Theorem 6.1.15, there exists a polygonal dissection D' of $\widehat{\Sigma}$ that is non-zero under $\bar{p}$. Lifting, we have that $D := D' \cup \{r_1, \ldots, r_q\}$ is a polygonal dissection of Σ on which p is non-zero. $\square$

Next, we wish to extend Proposition 8.1.4 to the current setting.

Proposition 8.2.7. *Let Σ be a connected surface with genus $g \geq 0$, $b \geq 1$ boundary components, $m \geq 2$ marked points on the boundary, and $q \geq 1$ punctures. Let D be a polygonal dissection of Σ. There is a bijection*

$$\left\{\begin{array}{c}\textit{benevolent deep points}\\ \textit{in } V(\mathcal{A}(\Sigma), \Bbbk)\\ \textit{non-zero on } D\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{deep points in}\\ V(\mathcal{A}(\Sigma \smallsetminus D), \Bbbk)\\ \textit{gluable along } D\end{array}\right\}$$

Proof. First, let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a benevolent deep point that is non-zero on D. By Proposition 6.1.7, its image under the cutting isomorphism $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus D), \Bbbk)$ is deep. Since $\Sigma \smallsetminus D$ is an unpunctured polygon, Theorem 4.3.1 tells us that m must be even. Further, combining with Proposition 6.1.13, the alternating product of the boundary arcs of $\Sigma \smallsetminus D$ is

$$(-1)^{\frac{4g+2b+m+2q}{2}} = (-1)^{b+q+\frac{m}{2}}.$$

Now suppose that $\bar{p} \in V(\mathcal{A}(\Sigma \smallsetminus D), \Bbbk)$ is a deep point that is gluable along D. Since D is a polygonal dissection of Σ, there is a subset of q arcs $R \coloneqq \{r_1, \ldots, r_q\} \subset D$ such that each $r_i \in R$ has at least one end at a puncture, $\Sigma \smallsetminus R$ is an unpunctured surface, and $D' \coloneqq D \smallsetminus R$ is a polygonal dissection of $\Sigma \smallsetminus R$. By Corollary 6.1.10, we have a bijection between

- deep points of $V(\mathcal{A}(\Sigma \smallsetminus R), \Bbbk)$ which are non-zero on D', and
- deep points of $V(\mathcal{A}(\Sigma \smallsetminus D), \Bbbk)$ which are gluable along D'.

Thus, we only need to consider the gluing along the arcs of R.

By our assumption of gluability, there are two boundary edges of $\Sigma \smallsetminus D$ that glue to give a radius r_q from the boundary of Σ to a puncture ξ_q. By Lemma 4.2.1, $\bar{p}$ kills an odd number of arcs in every triangle of $\Sigma \smallsetminus D$. Considering a fan triangulation of $\Sigma \smallsetminus D$ with all diagonals emanating from the marked point (or a choice of one of the marked points) corresponding to ξ_q, Theorem 4.3.1 says that $\bar{p}$ will kill alternating diagonals; as a result, p will kill alternating radii emanating from ξ_q in Σ. Every triangulation of Σ will contain at least two arcs emanating from ξ_q, but does not necessarily kill at least one such arc in every triangulation; suppose that T is a triangulation of Σ containing two arcs emanating from ξ_q that are spared by p, and denote these two spared arcs as r_1 and r_2. If r_1 and r_2 are arcs in $\Sigma \smallsetminus D$, then they are spared by $\bar{p}$ as well. Let γ be the arc in $\Sigma \smallsetminus D$ that completes a triangle with r_1 and r_2, and observe that γ is also in T. Since $\bar{p}$ kills an odd number of arcs in every triangle, the arc γ is necessarily killed by $\bar{p}$. Thus, p must also kill γ, so p kills at least one arc in T. Thus, p must kill at least one arc in every triangulation of Σ, so p is deep.

Now, suppose that r_1, r_2 in T are not arcs in $\Sigma \setminus D$, and let γ denote an arc in T that completes a triangle with r_1 and r_2. Since $\Sigma \setminus R$ is an unpunctured surface, its universal cover $\widehat{\Sigma}$ is covered by lifts of the polygon $\Sigma \setminus D$. Choose lifts $\widehat{r_1}$, $\widehat{r_2}$, and $\widehat{\gamma}$ that bound a triangle in $\widehat{\Sigma}$. Since this triangle is compact, it is contained in the union of finitely many lifts of $\Sigma \setminus D$; let Σ' denote this union. As Σ' is a union of polygons in a simply connected space, it is again a polygon. Since $\bar{p}$ is deep, it gives a deep point on each lift of $\Sigma \setminus D$. As in the proof of Lemma 6.1.9, these deep points glue to a deep point $\widehat{p} \in V(\mathcal{A}(\Sigma'), \Bbbk)$. By Lemma 4.2.1, $\widehat{p}$ must kill an odd number of arcs in the triangle $\{\widehat{r_1}, \widehat{r_2}, \widehat{\gamma}\}$, so $\widehat{p}(\widehat{\gamma}) = 0$. This descends to the deep point $\bar{p}$, which glues along D to give the point p. As a result, p must kill γ as well, so p kills at least one arc of T. Hence, p must kill at least one arc in every triangulation of Σ, so p is deep.

Finally, since $\bar{p}$ is gluable along D, it is gluable along R, so there is at least one arc emanating from each puncture that is spared by p. Therefore, p is a benevolent deep point. $\square$

Characterization of benevolent points

By the bijection in Proposition 8.2.7, we know that the benevolent deep points nonzero on a polygonal dissection D are in correspondence with the deep points of a polygon with $4g + 2b + 2q + m - 2$ sides, with each of the arcs in D corresponding to two sides of the polygon. Since both edges corresponding to an arc in D take the same value, we only have $2g + b + q + m - 1$ choices of values for the edges of the polygon. Following from Theorem 4.3.1, we have the following:

Proposition 8.2.8. *Let Σ be a connected surface with genus $g \geq 0$, $b \geq 1$ boundary components, $m \geq 2$ marked points on the boundary, and $q \geq 1$ punctures. Let D be a* *polygonal dissection of Σ.*

1. *If m is odd, then $V(\mathcal{A}(\Sigma), \Bbbk)$ has no benevolent deep points.*
2. *If m is even, then $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ is deep if and only if the alternating product around the boundary edges of $\Sigma \setminus D$ equals*

$$(-1)^{\frac{4g+2b+2q+m}{2}} = (-1)^{b+q+\frac{m}{2}}.$$

As a consequence, these benevolent deep points are determined by the values on all but one boundary edge of $\Sigma \setminus D$, so they determine 2^q deep subsets isomorphic to $(\Bbbk^\times)^{2g+b+q+m-2}$, one for each choice of tagging at each puncture.

It remains to count the number of distinct polygonal dissections of Σ. By Proposition 6.1.13, each unpunctured surface has 2^{2g+b-1} polygonal dissections, up to congruence. Thus, for each collection R of arcs such that $\Sigma \setminus R$ is an unpunctured surface with the same genus and number of boundary components as Σ, there are 2^{2g+b-1} distinct polygonal dissections of $\Sigma \setminus R$. The arcs of each polygonal dissection D' are disjoint from R except at the boundary, so $\Sigma \setminus D'$ is a q-punctured polygon —a disk with q punctures and $4g + 2b + m - 2$ marked points on the boundary.

From Proposition 8.1.4 and the discussion on punctured disks, we know that a benevolent deep point kills alternating radii around each puncture in $\Sigma \setminus D'$, and choosing a non-zero value for one arc emanating from each puncture determines the values of the other arcs emanating from that puncture. Further clarifying this point, Corollary 8.1.9 tells us that a benevolent deep point either kills all radii with the same endpoints in the punctured polygon $\Sigma \setminus D'$ or spares all of them. That is, at each puncture, either the even-indexed radii are killed or the odd-indexed radii are killed, and a choice of non-zero value on one radius determines all remaining values around a given puncture. As such, there are 2^q ways to choose which radii receive non-zero values; hence, there are 2^q ways to choose the collection R, up to equivalence. This gives us a total of $2^{2g+b+q+1}$ distinct polygonal dissections. Combining with the above proposition, we have the following result:

Theorem 8.2.9. *Let Σ be a connected surface with genus $g \geq 0$, $b \geq 1$ boundary components, $m \geq 2$ marked points on the boundary, and $q \geq 1$ punctures.*

1. *If m is odd, then $V(\mathcal{A}(\Sigma), \Bbbk)$ has no benevolent deep points.*
2. *If m is even, then the deep locus of $V(\mathcal{A}(\Sigma), \Bbbk)$ contains $2^{2g+b+2q+1}$ sets of benevolent points, each isomorphic to $(\Bbbk^\times)^{2g+b+q+m-2}$.*

Benevolent deep points and radii

Unfortunately, Lemma 8.1.7 and Corollary 8.1.9 do not extend to more general surfaces. That is, if a surface Σ has positive genus or multiple boundary components, there may exist radii r_1 and r_2 with the same endpoints such that a benevolent deep point kills r_1 and spares r_2, or vice versa.

Example 8.2.10. Let Σ be a $(1, 1)$-annulus with a single puncture —that is, the once-punctured genus 0 surface with one marked point on each of its two boundary components. Let $p \in V(\mathcal{A}(\Sigma), \Bbbk)$ be a benevolent deep point; by Lemma 8.2.4, at least one radius r_0 must be spared by p. Cutting along r_0 gives us an unpunctured $(1, 3)$-annulus Σ', and the deep points on Σ' with the same value on r_0 descend to deep points on Σ which are non-zero on r_0 by Proposition 8.2.7, so we know that such a p exists.

Suppose that r_0 is as pictured in Figure 8.6, and r_1 is a radius with the same endpoints as in the same figure. After cutting along r_0, we have the image on the right. From Lemma 6.1.1, we know that p kills an odd number of arcs in each triangle of the unpunctured surface. Since $r_0 \neq 0$, we know that exactly one of c_1

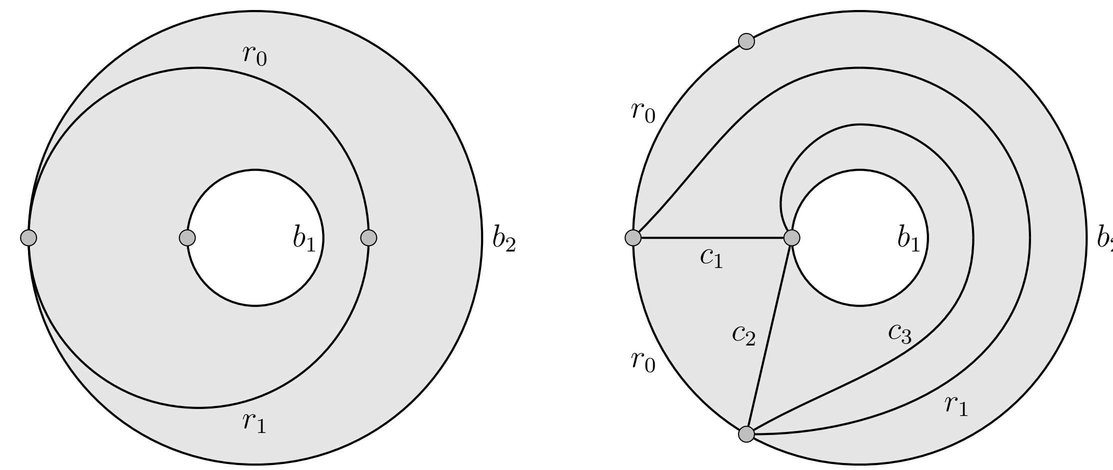


Figure 8.6: A once-punctured $(1, 1)$-annulus (left) and the result of cutting along r_0 (right)

and c_2 is killed by p. If $p(c_2) = 0$ and $p(c_1) \neq 0$, then $p(c_3) \neq 0$, so $p(r_1) = 0$. If instead $p(c_1) = 0$ and $p(c_2) \neq 0$, then $p(c_3) = 0$, so again $p(r_1) = 0$. In particular, r_1 is killed by p. As a result, r_0 and r_1 are an example of radii with the same endpoints where p kills one and spares the other.

8.3 A forgetful map

Let Σ_q be a connected surface with genus $g \geq 0$, $b \geq 1$ boundary components, $q \geq 1$ punctures, and $m \geq 2$ marked points on the boundary, with at least 2 of the marked points on the same boundary component, and let Σ_0 denote the same surface with no punctures.

Choose an arbitrary ordering of the punctures $\xi_1, \xi_2, \ldots, \xi_q$. As discussed in the previous sections, the points of $V(\mathcal{A}(\Sigma_q), \Bbbk)$ that quash ξ_1 correspond to a deep subvariety V_1 isomorphic to $V(\mathcal{A}(\Sigma_{q-1}), \Bbbk)$. Likewise, the points of $V(\mathcal{A}(\Sigma_q), \Bbbk)$ that quash ξ_1 and ξ_2 correspond to a deep subvariety V_2 isomorphic to $V(\mathcal{A}(\Sigma_{q-2}), \Bbbk)$, and $V_2 \subset V_1$. Continuing for any $1 \leq j < q$, the points of $V(\mathcal{A}(\Sigma_q), \Bbbk)$ that quash the first $j+1$ punctures correspond to a deep subvariety V_{j+1} which is isomorphic to $V(\mathcal{A}(\Sigma_{q-j-1}),\Bbbk)$, and $V_{j+1} \subset V_j$. This gives us a nested inclusion of deep subvarieties

$$V_q \coloneqq V(\mathcal{A}(\Sigma_0),\Bbbk) \subset V_{q-1} \subset \cdots \subset V_2 \subset V_1.$$

Recall that such inclusions are achieved by cutting along an arc c_i, where c_i and the boundary arc b_i together cut out a bigon containing a puncture that is being quashed by p. Cutting Σ_q along c_i, we get a disconnected surface with two components —a once-punctured bigon Σ_1 and a $(q-1)$-punctured surface Σ_{q-1}. This gives us an induced map

$$\mathcal{F}\colon \mathcal{A}(\Sigma_q) \to \mathcal{A}(\Sigma_{q-1})$$

realized by forgetting the once-punctured bigon Σ_1, which in turn gives us a dual map on varieties

$$V(\mathcal{A}(\Sigma_{q-1}),\Bbbk) \to V(\mathcal{A}(\Sigma_q),\Bbbk).$$

As noted above, this map is an inclusion, and its image is contained entirely within the deep locus of $V(\mathcal{A}(\Sigma_q),\Bbbk)$.

Theorem 8.3.1. *Let Σ_q be a connected surface with genus $g \geq 0$, $b \geq 1$ boundary components, q punctures, and $m \geq 2$ marked points on the boundary, with at least 2 of the marked points on the same boundary component. Repeated applications of the forgetful map*

$$\mathcal{F}\colon \mathcal{A}(\Sigma_q) \to \mathcal{A}(\Sigma_{q-1})$$

induce inclusions of the cluster varieties

$$V(\mathcal{A}(\Sigma_j),\Bbbk) \text{ into the deep locus of } V(\mathcal{A}(\Sigma_q),\Bbbk) \text{ for all } 0 \leq j < q.$$

In particular, there is an inclusion of the cluster variety of the unpunctured surface, $V(\mathcal{A}(\Sigma_0), \mathbb{k})$*, into the deep locus of the cluster variety of the punctured surface,* $V(\mathcal{A}(\Sigma_q), \mathbb{k})$.

Remark 8.3.2. The results of the theorem may also apply in other cases. As noted in Remark 8.2.2, a similar argument can be made if the surface has $b \geq 2$ boundary components with a single marked point on each.

8.4 Punctured surfaces without boundary

Unfortunately, the cluster algebra $\mathcal{A}$ of a punctured surface without boundary is not locally acyclic [Mul13, Theorem 10.9]. In most cases, if not all, $\mathcal{A} \subsetneq \mathcal{U}$ for these cluster algebras. As such, the deep locus of $\mathcal{A}$ is contained in the deep locus of $\mathcal{U}$, but they may not be equal. One such example arises from the once-punctured torus.

The Markov cluster algebra

Let Σ be the closed surface of genus 1 with one puncture. The cluster algebra $\mathcal{A}(\Sigma)$ is better known as the *Markov cluster algebra*, and its associated quiver is the so-called Markov quiver depicted in Figure 8.7. In addition to being the cluster algebra of the surface Σ, the Markov cluster algebra arises in the study of *Markov triples* [FG07, Appendix B] and is the smallest example of many pathological behaviors in cluster algebras; e.g. it is not equal to its own cluster algebra [BFZ05], it is not finitely generated, and it is non-Noetherian [Mul13].

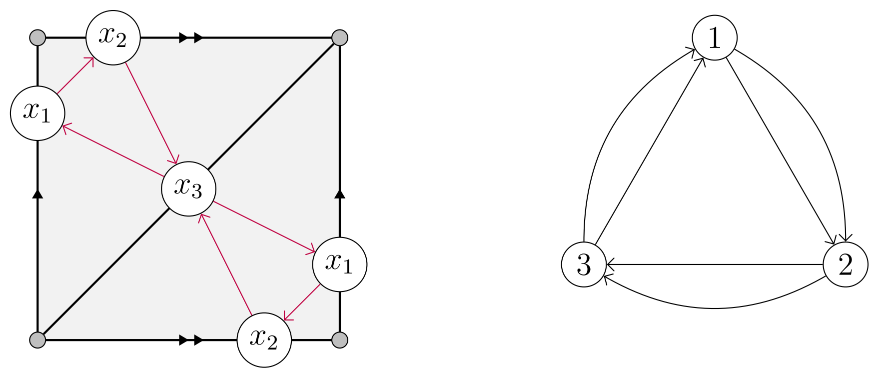


Figure 8.7: A triangulation of a once-punctured torus and its associated quiver.

Mutation-cyclic cluster algebras of rank 3

We start with some general results about cluster algebras of rank 3 which are **mutation-cyclic**; that is, not acyclic.[2] Ironically, the abundance of cycles in the mutation-equivalent quivers implies a lack of cycles in the exchange graph.

Theorem 8.4.1 ([War14, Theorem 10.1])**.** *The exchange graph of a mutation-cyclic skew-symmetric cluster algebra of rank 3 is a 3-regular tree.*

Equivalently, there is a unique reduced[3] sequence of mutations between any two clusters.

The theorem allows us to partition the cluster variables in such a cluster algebra into three types. For the sake of terminology, let us choose an initial cluster in $\mathcal{A}$ and denote the cluster variables $\{x, y, z\}$. We recursively define the x**-type** cluster variables in $\mathcal{A}$ to consist of x itself and any mutation of an x-type cluster variable; y**-type** and z**-type** cluster variables are defined similarly.

Proposition 8.4.2. *Every cluster variable in a mutation-cyclic skew-symmetric cluster algebra of rank 3 is exactly one of the three types, and every cluster consists of one variable of each type.*

[2] A sharp characterization of mutation-cyclic quivers on three vertices appears in [BBH11].

[3] By *reduced*, we mean that the sequence never mutates the same vertex twice in a row.

Proof. By Theorem 8.4.1, every cluster in $\mathcal{A}$ can be obtained from the initial cluster by a unique reduced sequence of mutations. Each mutation replaces a single cluster variable by another variable of the same type, and so each cluster in the sequence consists of one variable of each type. $\square$

This has the following application for the deep points of the cluster algebra of any mutation-cyclic quiver with three vertices.

Lemma 8.4.3. *Let $\mathcal{A}$ be a mutation-cyclic skew-symmetric cluster algebra of rank 3.*

1. *If $p \in V(\mathcal{A}, \Bbbk)$ is deep, then either*
   a) *p kills every cluster variable of one type and no other cluster variables, or*
   b) *p kills every cluster variable.*
2. *Two deep points in $p \in V(\mathcal{A}, \Bbbk)$ with the same values on one cluster must be equal.*

Proof. Let $\{\overline{x}, \overline{y}, \overline{z}\}$ be a cluster in $\mathcal{A}$, in which $\overline{x}$ is the x-type variable, $\overline{y}$ is the y-type variable, and $\overline{z}$ is the z-type variable. Since $\mathcal{A}$ is mutation-cyclic, the quiver associated to $\{\overline{x}, \overline{y}, \overline{z}\}$ is cyclic and the mutation relation for $\overline{y}$ will be of the form $\overline{yy}' = \overline{x}^b + \overline{z}^c$ for some $b, c > 0$.

Since p is deep, it must kill at least one of $\overline{x}, \overline{y}, \overline{z}$. Suppose p kills at least two of $\overline{x}, \overline{y}, \overline{z}$; without loss of generality, assume $p(\overline{x}) = p(\overline{y}) = 0$. Applying p to $\overline{yy}' = \overline{x}^b + \overline{z}^c$ yields $p(\overline{z})^c = 0$, so $p(\overline{z}) = 0$. Therefore, if p kills two cluster variables in a cluster, it must kill the third. Since adjacent clusters have two variables in common with this cluster, p must kill all three cluster variables in those clusters as well. Iteratively, p must then kill all three cluster variables in every cluster.

Next, suppose p kills only one of $\overline{x}, \overline{y}, \overline{z}$; without loss of generality, assume $p(\overline{x}) = 0$ and $p(\overline{y}), p(\overline{z}) \neq 0$. Since p is deep, the mutated variable $\overline{x}'$ in the cluster $\{\overline{x}', \overline{y}, \overline{z}\}$ must also be sent to zero by p. Mutating at $\overline{y}$, we have $\overline{yy}' = \overline{x}^b + \overline{z}^c$, and so $p(\overline{y}') = \dfrac{p(\overline{z})^c}{p(\overline{y})} \neq 0$; the mutation of $\overline{z}$ is similarly not killed. Iteratively, we see that all of the x-type variables are killed by p, and none of the y- or z-type variables are killed by p.

Let p, p' be deep points with the same values on $\{\overline{x}, \overline{y}, \overline{z}\}$. If $p(\overline{x}) = p(\overline{y}) = p(\overline{z}) = 0$, then p and p' kill every cluster variable; since this is a generating set of $\mathcal{A}$, they must coincide. Suppose they both kill cluster variables of one type and no other cluster variables; without loss of generality, assume they both kill all x-type cluster variables. Mutating at $\overline{y}$, we have $\overline{y}\overline{y}' = \overline{x}^b + \overline{z}^c$, and so $p(\overline{y}') = \dfrac{p(\overline{z})^c}{p(\overline{y})} = \dfrac{p'(\overline{z})^c}{p'(\overline{y})} = p'(\overline{y}')$. Therefore, p and p' have the same values on all adjacent clusters as well. Iteratively, we see that p and p' have the same values on all cluster variables; since this is a generating set of $\mathcal{A}$, they must coincide. $\square$

Warning 8.4.4. It is not true that two (not necessarily deep) points in $V(\mathcal{A}, \Bbbk)$ with the same values on a cluster must be equal. For example, the Markov cluster algebra has multiple complex points which send the initial cluster (x, y, z) to $(0, 1, i)$, but only one of these points is deep.

Warning 8.4.5. Lemma 8.4.3 does not guarantee the existence of any deep points! General mutation-cyclic rank 3 cluster algebras are poorly behaved (e.g. they are always non-Noetherian) and poorly understood (e.g. no explicit presentation is known), so a parametrization of their deep points eludes us. We are only able to circumvent this for the Markov cluster algebra via its upper cluster algebra, which is much better behaved (see next section).

The Markov upper cluster algebra

We now turn to the Markov upper cluster algebra. Letting x, y, z denote the initial cluster in the Markov cluster algebra $\mathcal{A}$, we refer to

$$M := \frac{x^2 + y^2 + z^2}{xyz}$$

as the **Markov element**. This element satisfies many remarkable properties. For example, it is independent on the choice of initial cluster; that is, for any other cluster $\{\overline{x}, \overline{y}, \overline{z}\}$,

$$M = \frac{\overline{x}^2 + \overline{y}^2 + \overline{z}^2}{\overline{xyz}}$$

As a consequence, M is Laurent in every cluster, and so it is in the upper cluster algebra $\mathcal{U}$.

Theorem 8.4.6 ([MM15, Theorem 6.2.2])**.** *Let $\mathcal{A}$ be the Markov cluster algebra, and let $\mathcal{U}$ be its upper cluster algebra. Then the initial cluster $\{x, y, z\}$ and the Markov element M generate $\mathcal{U}$, with presentation*

$$\mathcal{U} \simeq \mathbb{Z}[x, y, z, M]/\langle xyzM - x^2 - y^2 - z^2 \rangle$$

Thus, a homomorphism $\mathcal{U} \to \Bbbk$ is determined by its values on (x, y, z, M); conversely, any choice of values for (x, y, z, M) extends to a homomorphism $\mathcal{U} \to \Bbbk$ iff it kills $xyzM - x^2 - y^2 - z^2$. Therefore, $V(\mathcal{U}, \Bbbk)$ may be identified with the set of solutions to $xyzM = x^2 + y^2 + z^2$ inside $\Bbbk^4$.

The proof of Theorem 6.2.2 in [MM15] relied on a description of the deep locus of $\mathcal{U}$, albeit in a more algebraic language. We restate and reprove this description in the language of varieties.

Theorem 8.4.7. *Let $\mathcal{U}$ be the Markov upper cluster algebra.*

(A) For each $\alpha \in \mathbb{k}$, there is a deep point $p \in V(\mathcal{U}, \mathbb{k})$ which sends $(x, y, z, M) \mapsto (0, 0, 0, \alpha)$.

(X) For each $\beta, \gamma \in \mathbb{k}$ with $\beta^2 + \gamma^2 = 0$, there is a deep point $p \in V(\mathcal{U}, \mathbb{k})$ which sends

$$(x, y, z, M) \mapsto (0, \beta, \gamma, 0)$$

(Y) For each $\beta, \gamma \in \mathbb{k}$ with $\beta^2 + \gamma^2 = 0$, there is a deep point $p \in V(\mathcal{U}, \mathbb{k})$ which sends

$$(x, y, z, M) \mapsto (\beta, 0, \gamma, 0)$$

(Z) For each $\beta, \gamma \in \mathbb{k}$ with $\beta^2 + \gamma^2 = 0$, there is a deep point $p \in V(\mathcal{U}, \mathbb{k})$ which sends

$$(x, y, z, M) \mapsto (\beta, \gamma, 0, 0)$$

Every deep point in $V(\mathcal{U}, \mathbb{k})$ is one of these four types, which are pairwise disjoint except for the point which sends (x, y, z, M) to $(0, 0, 0, 0)$.

Proof. For any cluster $\{\overline{x}, \overline{y}, \overline{z}\}$, the mutation $\overline{x}'$ satisfies $\overline{x}' = \overline{yz}M - \overline{x}$. As a result, any point which kills $\overline{x}$ and either M or $\overline{yz}$ must also kill $\overline{x}'$. Iteratively applying this to the maps in the statement of the proposition shows that points of Type (A) kill every cluster variable, points of Type (X) kill every x-type variable, points of Type (Y) kill every y-type variable, and points of Type (Z) kill every z-type variable.

To show every deep point is of these types, consider a deep point $p \in V(\mathcal{U}, \mathbb{k})$ with $p(x) = 0$. The equation $xyzM = x^2 + y^2 + z^2$ implies that $p(y)^2 + p(z)^2 = 0$. If $p(y) = 0$, $p(z) = 0$, so p is of Type (A). If $p(y) \neq 0$, then $p(z) \neq 0$, and $p(x') = 0$. The equation $x' = yzM - x$ implies that $p(yzM) = 0$. Since $p(y)p(z) \neq 0$, $p(M) = 0$, and p is of Type (X). By a similar argument, deep points which kill y or z but not x must be of Types (Y) or (Z), respectively. $\square$

Deep points of the Markov cluster algebra

We are now in a position to classify deep points in the Markov cluster algebra itself, with Theorem 8.4.7 providing a source of deep points in $V(\mathcal{A}, \Bbbk)$ and and Lemma 8.4.3 providing a bound.

Proposition 8.4.8. *Let $\mathcal{A}$ be the Markov cluster algebra.*

- *(X) For each $\beta, \gamma \in \Bbbk$ with $\beta^2 + \gamma^2 = 0$, there is a unique deep point $p \in V(\mathcal{A}, \Bbbk)$ which kills every x-type cluster variable and sends (y, z) to (β, γ).*
- *(Y) For each $\beta, \gamma \in \Bbbk$ with $\beta^2 + \gamma^2 = 0$, there is a unique deep point $p \in V(\mathcal{A}, \Bbbk)$ which kills every y-type cluster variable and sends (x, z) to (β, γ).*
- *(Z) For each $\beta, \gamma \in \Bbbk$ with $\beta^2 + \gamma^2 = 0$, there is a unique deep point $p \in V(\mathcal{A}, \Bbbk)$ which kills every z-type cluster variable and sends (x, y) to (β, γ).*

Every deep point in $V(\mathcal{A}, \Bbbk)$ is one of these three types, which are pairwise disjoint except for the point which sends every cluster variable to 0.

Proof. A deep point in $V(\mathcal{A}, \Bbbk)$ which kills every x-type cluster variable must also kill $xx' = y^2 + z^2$, and so it must be of Type (X). To show such a deep point exists for all β, γ, observe that the deep point $p \in V(\mathcal{U}, \Bbbk)$ which sends (x, y, z, M) to $(0, \beta, \gamma, 0)$ (Type (X) in Theorem 8.4.7) restricts to a deep point in $V(\mathcal{A}, \Bbbk)$ which kills every x-type variable and sends (y, z) to (β, γ). Uniqueness follows from Lemma 8.4.3.2. Similar arguments hold for the other types.

By Lemma 8.4.3.1, every deep point of $\mathcal{A}$ must kill every cluster variable of one type, and so it must be one of the three types stated above. □

Remark 8.4.9. We now consider the geometric relation between $V(\mathcal{A}, \Bbbk)$ and $V(\mathcal{U}, \Bbbk)$. The map

$$V(\mathcal{U}, \Bbbk) \to V(\mathcal{A}, \Bbbk)$$

induces an isomorphism between deep points of Types (X), (Y), and (Z). However, since the deep points of Type (A) in $V(\mathcal{U}, \Bbbk)$ kill every cluster variable, they are all sent to the same point in $V(\mathcal{A}, \Bbbk)$: the point which sends every cluster variable to 0.

Example 8.4.10. Over the complex numbers (i.e. $\Bbbk = \mathbb{C}$), the solution set of $\beta^2+\gamma^2 = 0$ breaks apart into two lines $\gamma = \pm i\beta$. As a consequence, the deep locus in $V(\mathcal{U}, \mathbb{C})$ consists of seven lines, which are disjoint except for a common intersection point. Under the map

$$V(\mathcal{U}, \mathbb{C}) \to V(\mathcal{A}, \mathbb{C})$$

six of these deep lines are sent bijectively to the six deep lines in $V(\mathcal{A}, \mathbb{C})$, while the seventh is collapsed to the common intersection point.

Chapter 9

Folding

It is well-known that every multiply-laced or extended Dynkin diagram can be obtained by *folding* a simply-laced diagram [Ste08, Remark 5.4]; that is, by identifying the orbits of a particular automorphism of the diagram. In particular, the Lie algebras of type B_n, C_n, F_4, and G_2 can be constructed via folding of a simply-laced Coxeter-Dynkin diagram by a diagram automorphism [Kac90, Proposition 7.9].

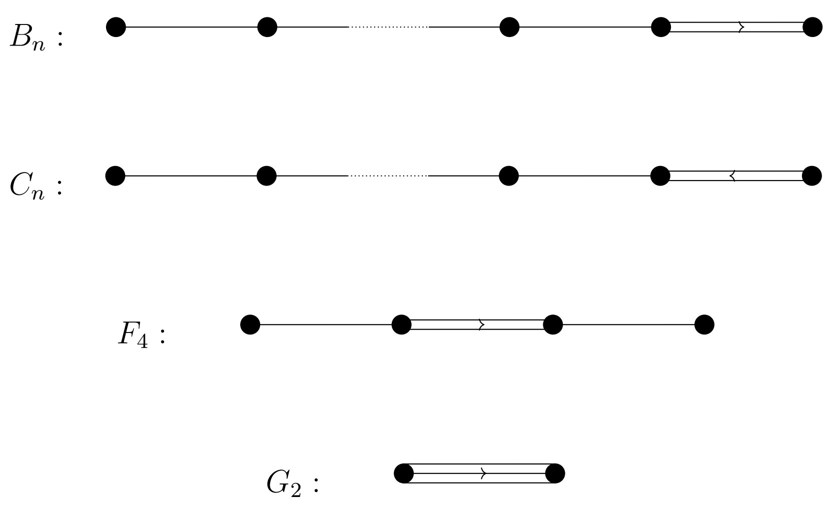


Figure 9.1: Dynkin diagrams of types B_n, C_n, F_4, and G_2

There have been several implementations of folding in the study of cluster algebras as a way to create skew-symmetrizable exchange matrices from skew-symmetric matrices [Dup08, FST11, FWZ21, Kau24]. We will follow the work of Dupont [Dup08], who first showed that the cluster algebras of type B_n, C_n, F_4, and G_2 can be constructed from foldings of orientations of simply-laced Coxeter-Dynkin diagrams by admissible automorphisms.

Given a particular folding of a quiver, it is natural to ask what relationship exists between the cluster algebras of the folded and unfolded quivers. In our current context, we would like to examine the relationship between the deep loci of the two cluster algebras.

9.1 Automorphisms of quivers

Let Q be a quiver with n vertices, all mutable, and let $B = (b_{i,j})$ be its associated skew-symmetric exchange matrix. The symmetric group $\mathfrak{S}_n$ acts on $n \times n$ matrices by setting

$$g.B = (b_{g^{-1}i,\, g^{-1}j}) \quad 1 \leq i, j \leq n$$

for any $g \in \mathfrak{S}_n$. Note that since B is skew-symmetric, $g.B$ will be skew-symmetric as well, so the action on B also induces an action $g.Q$ on the associated quiver. An element $g \in \mathfrak{S}_n$ is called an **automorphism** of B if $g.B = B$; equivalently, g is an automorphism if Q and $g.Q$ are isomorphic as quivers. A **group of automorphisms of** Q is a subgroup $G \subseteq \mathfrak{S}_n$ such that each $g \in G$ is an automorphism of Q.

An automorphism g of B is called **admissible** if $b_{i,j} = 0$ for every vertex j in the g-orbit of i; equivalently, an automorphism g of Q is admissible if there are no arrows between two vertices in the same g-orbit. A group of automorphisms G will be called admissible if every $g \in G$ is admissible; we then refer to the tuple

(Q, G) as an **admissible pair**. Our primary focus will be foldings of orientations of simply laced Coxeter-Dynkin diagrams, which are acyclic. By [Dup08, Lemma 3.3], if Q is acyclic and G is a group of automorphisms of Q, then G is admissible.

Let $\overline{Q}_0$ denote the set of G-orbits of the vertices of Q. The **quotient matrix** $\overline{B} = \left(\overline{b_{\bar{i},\bar{j}}}\right)_{\bar{i},\bar{j}\in\overline{Q}_0}$ is defined by

$$\overline{b_{\bar{i},\bar{j}}} = \sum_{k\in\bar{i}} b_{k,j}$$

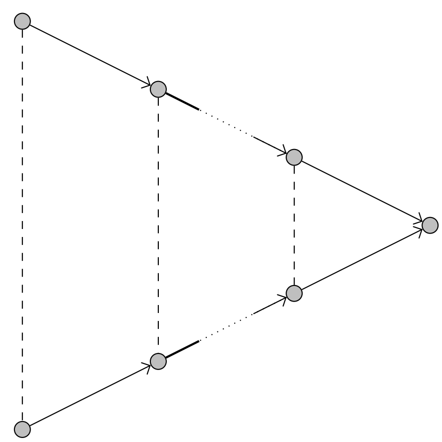
Equivalently, by [Dup08, Lemma 3.6], we have the following formulation:

$$\overline{b_{\bar{i},\bar{j}}} = \frac{1}{|\operatorname{stab}(i)|}\sum_{g\in G} b_{g.i,j}$$

In practice, one can think of folding a quiver as identifying the vertices in the same G-orbit to a single vertex. In Figures 9.2, 9.3, and 9.4, the vertices that are connected by dashed lines are identified by the folding action. If two or more vertices are identified, then all of the arrows incident to that vertex are identified in a neighborhood of the vertex, but the opposite end of the arrows are not necessarily identified. For example, in Figure 9.2, the right-most vertex of the A_{2n-1} quiver is invariant under the G-action and is not identified with any other vertices; as a result, there are two arrows arriving at vertex n from vertex $n-1$ in the B_n quiver, but only one arrow leaving vertex $n-1$ in the direction of vertex n.

Convention 9.1.1. If an arrow $i \to j$ is labeled with a tuple (d_i, d_j), this indicates that there are d_i arrows leaving vertex i and d_j arrows arriving at vertex j. If this label is omitted, the reader should assume that $(d_i, d_j) = (1, 1)$.

As implied by these figures, there may be multiple groups with admissible actions on a given quiver. Likewise, there may be distinct admissible pairs that produce isomorphic folded quivers.



descends to

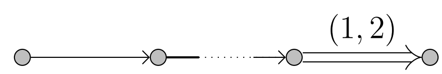
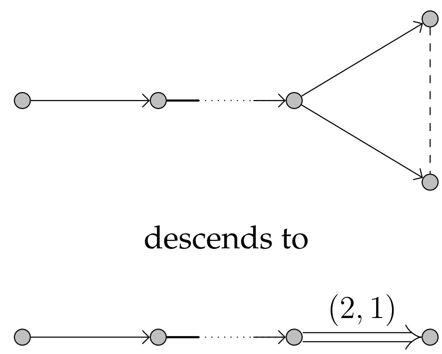
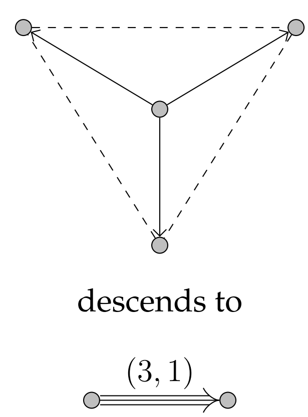


Figure 9.2: Folding of A_{2n-1} to B_n.



(a) Folding of D_{2n+1} to C_n.



(b) Folding of D_4 to G_2.

Figure 9.3: Foldings of D_n quivers.

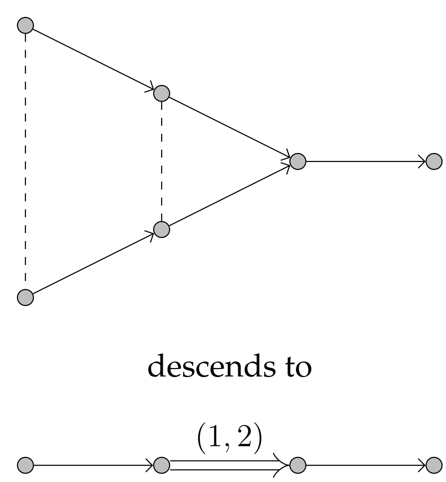


Figure 9.4: Folding of E_6 to F_4.

9.2 Cluster algebras and automorphisms of quivers

Given a seed $(Q, \mathbf{x})$ and an admissible group G, the action of G on vertices of Q induces an action on the cluster $\mathbf{x} = \{x_1, \dots, x_n\}$ by setting

$$g.x_i = x_{g.i} \quad \text{for any vertex } i \text{ and } g \in G$$

extended as a $\mathbb{Q}$-algebra homomorphism. This induces an action on the ring $\mathbb{Z}[\mathbf{x}^{\pm 1}] = \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of Laurent polynomials as well.

Lemma 9.2.1. *[Dup08, Lemma 5.2] Let Q be an acyclic quiver with an admissible automorphism group G. Then for any $g \in G$ and any cluster variable $x \in \mathcal{A}(Q)$, $g.x$ is a cluster variable in $\mathcal{A}(Q)$.*

Lemma 9.2.2. *[Dup08, Corollary 5.5] Fix an acyclic quiver Q endowed with a group of automorphisms G, and fix $\{i_1, \dots, i_k\}$ in a G-orbit of vertices of Q. Then for any $\sigma \in \mathfrak{S}_k$, we have*

$$\mu_{i_n} \circ \cdots \circ \mu_{i_1}(Q) = \mu_{i_{\sigma(n)}} \circ \cdots \circ \mu_{i_{\sigma(1)}}(Q).$$

That is, mutations at vertices in the same orbit can be performed in any order. Thus, for an admissible pair (Q, G) and Ω a G-orbit of vertices of Q, the product of mutations $\prod_{i \in \Omega} \mu_i(Q)$ is well defined and is called the **orbit mutation** $\mu_{\bar{i}}$ along Ω, or simply an *orbit mutation* $\mu_{\bar{i}}$.

Lemma 9.2.3. *[Dup08, Corollary 5.7] Let (Q, G) be an admissible pair. Fix a G-orbit $\{i_1, \dots, i_k\}$. Then $Q^{(k)} = \prod_{j=1}^{k} \mu_{i_j}(Q)$ is acyclic and G is an admissible group of automorphisms for $Q^{(k)}$.*

Remark 9.2.4. Note that in general, an automorphism g of Q is not necessarily an automorphism of every quiver mutation-equivalent to Q.

Proposition 9.2.5. *[Dup08, Proposition 5.9] Let Q be an acyclic quiver with associated matrix B endowed with an admissible group of automorphisms G. For any vertex k in Q and its G-orbit $\bar{k}$, we have*

$$\overline{\prod_{l \in \bar{k}} \mu_l(B)} = \mu_{\bar{k}}\left(\overline{B}\right)$$

Further, this mutation extends to cluster variables. Define a projection map π on upper cluster algebras as

$$\pi : \begin{cases} \mathbb{Z}\left[u_i^{\pm 1},\, i \in Q_0\right] & \to \ \mathbb{Z}\left[u_{\bar{i}}^{\pm 1},\, \bar{i} \in \overline{Q}_0\right] \\ \qquad u_i & \mapsto \qquad u_{\bar{i}} \end{cases}$$

and a corresponding map on G-invariant seeds $\left(\mathbf{x}, \overline{B}\right)$ as

$$\widetilde{\pi}\left(\left(x_i,\, i \in Q_0\right), B\right) = \left(\left(\pi(x_i),\, \bar{i} \in \overline{Q}_0\right), \overline{B}\right).$$

Using these maps, we are able to relate the cluster algebra of a quiver with the cluster algebra of its folding by the action of G.

Theorem 9.2.6. *[Dup08, Theorem 5.16] Let Q be an acyclic quiver with admissible group of automorphisms G. Then every seed in $\mathcal{A}\left(\overline{Q}\right)$ is the image under $\widetilde{\pi}$ of a seed in $\mathcal{A}(Q)$. In particular, every cluster in $\mathcal{A}\left(\overline{Q}\right)$ is the image of a cluster in $\mathcal{A}(Q)$ and every cluster variable in $\mathcal{A}\left(\overline{Q}\right)$ is the projection of a cluster variable in $\mathcal{A}(Q)$. Thus, $\mathcal{A}\left(\overline{Q}\right)$ is a $\mathbb{Z}$-subalgebra of the algebra $\pi\left(\mathcal{A}(Q)\right)$.*

While this does not always give equality of cluster algebras, it does when Q is an orientation of a simply-laced Coxeter-Dynkin diagram.

Theorem 9.2.7. *[Dup08, Theorem 7.3] Let Q be an orientation of a simply laced Coxeter-Dynkin diagram with an admissible group of automorphisms G, and $\overline{Q}$ the corresponding non-simply laced quiver. Then*

$$\pi\left(\mathcal{A}\left(Q\right)\right) = \mathcal{A}\left(\overline{Q}\right).$$

In particular, Dupont proves in this limited case that the image of every cluster variable of $\mathcal{A}(Q)$ is a cluster variable of $\mathcal{A}\left(\overline{Q}\right)$. This is not true for more general classes of quivers, however, as is demonstrated in [Dup08, Section 8].

Despite this limitation, Theorem 9.2.7 does allow us to investigate the action of folding on deep points of A_n, D_n, and E_6 quivers.

9.3 Type B_n

First, we examine the type B_n cluster algebras for $n \geq 2$, which arise as foldings of cluster algebras of type A_{2n-1}, as was previously demonstrated in Figure 9.2.

Proposition 9.3.1. *Let Q be the quiver*

$$\begin{tikzcd} x_1 \arrow[r] & x_2 \arrow[r, dotted] & x_{n-1} \arrow[r] & x_n & x_{n+1} \arrow[l] & x_{2n-2} \arrow[l, dotted] & x_{2n-1} \arrow[l] \end{tikzcd}$$

of type A_{2n-1} with admissible group of automorphisms

$$G = \left\langle \prod_{i=1}^{n-1} (i \quad 2n-i) \right\rangle \cong \mathbb{Z}_2$$

identifying x_i with x_{2n-i}, and $\overline{Q}$ the corresponding non-simply laced quiver

$$\begin{tikzcd} z_1 \arrow[r] & z_2 \arrow[r, dotted] & z_{n-1} \arrow[r, Rightarrow, "{(1,2)}"] & z_n \end{tikzcd}$$

of type B_n with $z_j \coloneqq x_{\bar{j}}$. Let $p \in V(\mathcal{A}(Q), \Bbbk)$ be a deep point that is constant on G-orbits; that is, if i and j are in the same G-orbit, then $p(x_i) = p(x_j)$. Define $\bar{p}\colon \mathcal{A}\left(\overline{Q}\right) \to \Bbbk$ as

$$\bar{p}(z_i) = \bar{p}(\pi(x_i)) = p(x_i).$$

Then $\bar{p}$ is a deep point of $V\left(\mathcal{A}\left(\overline{Q}\right), \Bbbk\right)$.

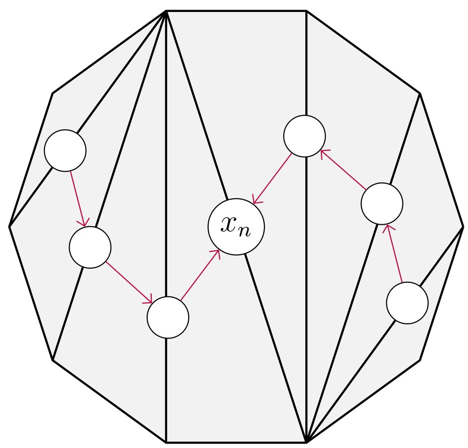
Proof. First, we need to check that $\bar{p}$ is a well-defined ring homomorphism. Since the underlying diagram is acyclic, we have the presentation

$$\mathcal{A}\left(\overline{Q}\right) \cong \langle z_1, z_2, \ldots, z_{n-1}, z_n, z_1', z_2', \ldots, z_{n-1}', z_n' \rangle / R$$

where

$$R = \langle z_1 z_1' - z_2 - 1,\ z_2 z_2' - z_1 - z_3,\ \ldots,\ z_{n-1} z_{n-1}' - z_{n-2} - z_n,\ z_n z_n' - z_{n-1}^2 - 1 \rangle.$$

Observe that the quiver Q corresponds to a triangulation of a $(2n+2)$-gon such as the one pictured here:



Since $\mathcal{A}(Q) \cong \mathcal{A}(A_{2n-1})$, by Theorem 4.3.5, a deep point $p \in V(\mathcal{A}(Q), \Bbbk)$ is necessarily of the form

$$p(x_i) = p(x_{2n-i}) = \begin{cases} 0 & i \text{ odd} \\ (-1)^{i/2} & i \text{ even} \end{cases}$$

for $1 \leq i \leq n$, and

$$p(x_i') = \begin{cases} (-1)^{n/2} & i = n \\ 0 & \text{otherwise.} \end{cases}$$

First, suppose that n is even. Note that p is constant on G-orbits, and satisfies the relations

$$x_1 x_1' = x_2 + 1,\ x_2 x_2' = x_1 + x_3,\ \ldots,\ x_{n-1} x_{n-1}' = x_{n-2} + x_n,\ x_n x_n' = x_{n-1}^2 + 1.$$

Hence, $\bar{p}$ satisfies the relations of R, so it is a well-defined ring homomorphism. Next, suppose that $\mathrm{char}(\Bbbk) = 2$ and n is odd. Again, note that p is constant on G-orbits and satisfies the same relations, so $\bar{p}$ is well-defined. Therefore, if $p \in V(\mathcal{A}(A_{2n-1}), \Bbbk)$ is deep, $\bar{p}$ is a well-defined ring homomorphism $\mathcal{A}(B_n) \to \Bbbk$.

Since p kills at least one cluster variable in each cluster of $\mathcal{A}(Q)$, and every cluster of $\mathcal{A}(\overline{Q})$ is the image of a cluster in $\mathcal{A}(Q)$, $\bar{p}$ kills at least one cluster variable in every cluster of $\mathcal{A}(\overline{Q})$. Thus, $\bar{p}$ is deep. □

Corollary 9.3.2. *The deep locus of* $V(\mathcal{A}(A_{2n-1}), \Bbbk)$ *embeds into the deep locus of* $V(\mathcal{A}(B_n), \Bbbk)$.

Proof. The deep locus of $V(\mathcal{A}(A_{2n-1}), \Bbbk)$ is either empty or a single point. If it is a single point p, it is constant on G-orbits, so $\bar{p}$ is deep by Proposition 9.3.1. □

Remark 9.3.3. The map on deep points $p \mapsto \bar{p}$ defined above is not necessarily an isomorphism. For example, if $\mathrm{char}(\Bbbk) = 2$, then $V(\mathcal{A}(B_2), \Bbbk)$ has three deep points [BM25, Corollary 4.6]. Only one of those deep points is inherited from $V(\mathcal{A}(A_3), \Bbbk)$, however.

9.4 Type G_2

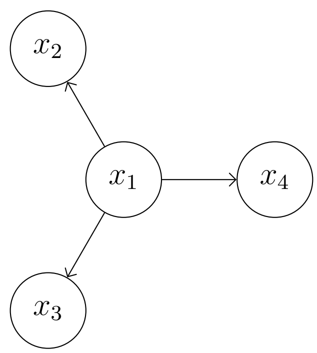
Next, we examine the type G_2 cluster algebra, which arises as a folding of the type D_4 cluster algebra, previously depicted in Figure 9.3b. Before considering the folding, we immediately recognize that it is impossible for the deep locus of the D_4 cluster variety to embed into the deep locus of the G_2 cluster algebra.

Proposition 9.4.1. *The deep locus of* $V(\mathcal{A}(D_4), \Bbbk)$ *does not embed into the deep locus of* $V(\mathcal{A}(G_2), \Bbbk)$.

Proof. From [BM25, Corollary 4.6], the deep locus of the G_2 cluster variety has a single point if $\mathrm{char}(\Bbbk) \neq 3$, and four points if $\mathrm{char}(\Bbbk) = 3$. From Remark 7.3.2, the deep locus of the D_4 cluster variety contains six lines. By cardinality, there is no injective map sending the deep locus of $V(\mathcal{A}(D_4), \Bbbk)$ to the deep locus of $V(\mathcal{A}(G_2), \Bbbk)$. $\square$

As in the B_n case above, the unique deep point that is constant on G-orbits does embed into the deep locus, however.

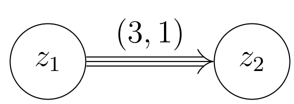
Proposition 9.4.2. *Let Q be the quiver*



of type D_4 with admissible group of automorphisms

$$G = \langle\, (2 \quad 3 \quad 4)\,\rangle \cong \mathbb{Z}_3$$

identifying x_2, x_3, and x_4, and $\overline{Q}$ the corresponding non-simply laced quiver



of type G_2 with $z_j \coloneqq x_{\bar{j}}$. Let $p \in V(\mathcal{A}(Q), \Bbbk)$ be a deep point that is constant on G-orbits; i.e. if i and j are in the same G-orbit, then $p(x_i) = p(x_j)$. Define $\bar{p} \colon \mathcal{A}\left(\overline{Q}\right) \to \Bbbk$ as

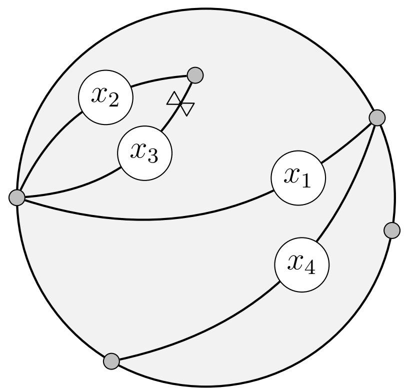
$$\bar{p}(z_i) = \bar{p}(\pi(x_i)) = p(x_i).$$

Then $\bar{p}$ is a deep point of $V\left(\mathcal{A}\left(\overline{Q}\right), \Bbbk\right)$.

Proof. First, we need to check that $\bar{p}$ is a well-defined ring homomorphism. Since the underlying diagram is acyclic, we have the presentation

$$\mathcal{A}\left(\overline{Q}\right) \cong \langle z_1, z_2, z_1', z_2' \mid z_1 z_1' - z_2^3 - 1, z_2 z_2' - z_1 - 1 \rangle.$$

Observe that the quiver Q corresponds to the tagged triangulation of the once-punctured disk pictured here:



From Remark 7.3.2, we know that $\mathcal{A}(Q) \cong \mathcal{A}(D_4)$ has six deep lines which intersect at a single point. However, the only deep point that is constant on G-orbits is the point of intersection. In terms of this initial cluster, the point of intersection sends x_1 and x_1' to -1 and kills all other generators. Sending z_1 and z_1' to -1 and killing all other generators of $\mathcal{A}\left(\overline{Q}\right)$, we see that $\bar{p}$ does satisfy the relations in our presentation of $\mathcal{A}\left(\overline{Q}\right)$, so $\bar{p}$ is a well-defined ring homomorphism.

Since p kills at least one cluster variable in each cluster of $\mathcal{A}(Q)$, and every cluster of $\mathcal{A}\left(\overline{Q}\right)$ is the image of a cluster in $\mathcal{A}(Q)$, $\bar{p}$ kills at least one cluster variable in every cluster of $\mathcal{A}\left(\overline{Q}\right)$. Thus, $\bar{p}$ is deep. $\square$

9.5 Type C_n

Now we will examine the type C_n cluster algebras for $n \geq 3$, which arise as foldings of cluster algebras of type D_{n+1}, previously depicted in Figure 9.3a.

Proposition 9.5.1. *Let Q be the quiver*

$$\begin{tikzcd}
& & & & x_n \\
x_1 \arrow[r] & x_2 \arrow[r, dotted] & x_{n-2} \arrow[r] & x_{n-1} \arrow[ur] \arrow[dr] & \\
& & & & x_{n+1}
\end{tikzcd}$$

of type D_{n+1} with admissible group of automorphisms

$$G = \langle\, (n \quad n+1)\, \rangle \cong \mathbb{Z}_2$$

identifying x_n with x_{n+1}, and $\overline{Q}$ the corresponding non-simply laced quiver

$$\begin{tikzcd}
z_1 \arrow[r] & z_2 \arrow[r, dotted] & z_{n-2} \arrow[r] & z_{n-1} \arrow[r, Rightarrow, "{(2,1)}"] & z_n
\end{tikzcd}$$

of type C_n with $z_j \coloneqq x_{\bar{j}}$. Let $p \in V(\mathcal{A}(Q), \Bbbk)$ be a deep point that is constant on G-orbits; i.e. if i and j are in the same G-orbit, then $p(x_i) = p(x_j)$. Define $\bar{p} \colon \mathcal{A}\left(\overline{Q}\right) \to \Bbbk$ as

$$\bar{p}(z_i) = \bar{p}(\pi(x_i)) = p(x_i).$$

Then $\bar{p}$ is a deep point of $V\left(\mathcal{A}\left(\overline{Q}\right), \Bbbk\right)$.

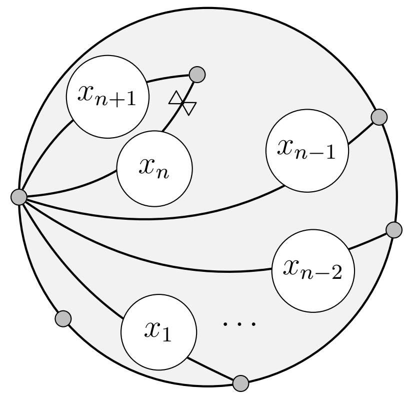
Proof. First, we need to check that $\bar{p}$ is a well-defined ring homomorphism. Since the underlying diagram is acyclic, we have the presentation

$$\mathcal{A}\left(\overline{Q}\right) \cong \langle z_1, z_2, \ldots, z_{n-1}, z_n, z_1', z_2', \ldots, z_{n-1}', z_n' \rangle / R$$

where

$$R = \langle z_1 z_1' - z_2 - 1,\ z_2 z_2' - z_1 - z_3,\ \ldots,\ z_{n-1} z_{n-1}' - z_{n-2} - z_n^2,\ z_n z_n' - z_{n-1} - 1 \rangle.$$

Observe that the quiver Q corresponds to the tagged triangulation of the once-punctured disk pictured here:



Before proceeding, we need to prove the following claim.

- **Claim:** A deep point $p \in V(\mathcal{A}(Q), \Bbbk)$ is constant on G-orbits if and only if p kills all radii.

  Proof of Claim: From Remarks 7.3.2 and 7.4.2, if p spares any radius, then it spares alternating radii of one tagging and kills all radii of the other tagging. Expressing x'_n and x'_{n+1} as Laurent polynomials in the initial cluster, we have

  $$x'_n = \frac{x_{n-1}+1}{x_n} \quad \text{and} \quad x'_{n+1} = \frac{x_{n-1}+1}{x_{n+1}},$$

  so x'_n and x'_{n+1} are in the same G-orbit. Since p must kill all radii of one of the taggings, if p is constant on G-orbits, then it must kill all radii of both taggings.

  Further, $\mathcal{A}(D_{n+1})$ has $(n+1)^2$ cluster variables and $\mathcal{A}(C_n)$ has $n(n+1)$ cluster variables [FWZ21, Figure 5.17], so there are precisely $n+1$ orbits of size two. As noted in the proof of [FWZ21, Theorem 5.5.1], these orbits of size two correspond to the two taggings of each radius in the once-punctured $(n+1)$-gon. Consequently, totally radial points are constant on G-orbits. $\square$

Now, we can show the totally-radial deep points give rise to well-defined ring homomorphisms $\mathcal{A}\left(\overline{Q}\right) \to \Bbbk$.

- Suppose that $n \geq 4$. From Remark 7.4.2, we know that the deep locus of $V(\mathcal{A}(D_{n+1}), \Bbbk)$ contains a component of dimension $n-2$ corresponding to the totally radial points, which are constant on G-orbits.

  In $V(\mathcal{A}(Q), \Bbbk)$ with the initial cluster as pictured, a totally radial deep point p must kill x_n, x_{n+1}, x'_n, and x'_{n+1}, and send x_{n-1} to -1. Note that these values satisfy relations of R, so the totally radial deep points give rise to well-defined homomorphisms $\mathcal{A}\left(\overline{Q}\right) \to \Bbbk$.
- Suppose that $n = 3$. From Remark 7.3.2, we know that the deep locus of $V(\mathcal{A}(D_4), \Bbbk)$ is comprised of 6 lines which intersect at a single point. Two of these lines, along with the point of intersection, correspond to totally radial points, which are constant on G-orbits. As above, these satisfy the relations of R and give rise to well-defined ring homomorphisms $\mathcal{A}\left(\overline{Q}\right) \to \Bbbk$.

Since p kills at least one cluster variable in each cluster of $\mathcal{A}(Q)$, and every cluster of $\mathcal{A}(\overline{Q})$ is the image of a cluster in $\mathcal{A}(Q)$, $\bar{p}$ kills at least one cluster variable in every cluster of $\mathcal{A}(\overline{Q})$. Thus, $\bar{p}$ is deep. $\square$

In particular, we have the following result:

Porism 9.5.2. *The totally radial points of $V(\mathcal{A}(D_{n+1}), \Bbbk)$ embed into the deep locus of $V(\mathcal{A}(C_n), \Bbbk)$.*

9.6 Type F_4

Finally, we mention the folding depicted in Figure 9.4. By [CGSS24, Theorem 1.5], the deep locus of $V(\mathcal{A}(E_6), \Bbbk)$ is empty. As such, if $V(\mathcal{A}(F_4), \Bbbk)$ has any deep points, they do not arise from a folding of the E_6 quiver.

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