71-12,560

CRAVER, Jr., William Lionel, 1939-
A METHOD FOR SELECTION OF SIGNIFICANT TERMS IN THE ASSUMED SOLUTION IN A RAYLEIGH-RITZ ANALYSIS.

The University of Oklahoma, Ph.D., 1970
Engineering Mechanics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

THIS DISSERTATION HAS BEEN MICROFILMED EXACTLY AS RECEIVED

Downloaded from SHAREOK. Persistent Handle URL: https://hdl.handle.net/11244/2913

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

A METHOD FOR SELECTION OF SIGNIFICANT TERMS IN THE ASSUMED SOLUTION IN A RAYLEIGH-RITZ ANALYSIS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

WILLIAM LIONEL CRAVER, JR.

Norman, Oklahoma

1970

A METHOD FOR SELECTION OF SIGNIFICANT TERMS IN THE ASSUMED SOLUTION IN A RAYLEIGH-RITZ ANALYSIS

APPROVED BY:

DISSERTATION COMMITTEE

ACKNOWLEDGEMENT

The author gratefully acknowledges the guidance of Dr. Davis M. Egle during this research. His rekindling of my enthusiasm was often needed and given.

The support and encouragement of Dr. Franklin J. Appl throughout the author's Ph.D. program is also gratefully acknowledged.

The National Defense Graduate Fellowship Program has provided financial support which made the author's attendance at the University possible.

My wife Becky's encouragement, understanding, and patience have been sustaining to me throughout this program.

TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENT	iii
LIST OF TABLES	v
LIST OF FIGURES	vii
LIST OF SYMBOLS	viii
Chapter	
I. INTRODUCTION	1
II. TAYLOR SERIES APPROXIMATION FOR EIGENVALUES AND EIGENVECTORS	4
Derivatives of Eigenvalues	12
Derivatives of Eigenvectors	16
III. FORMULATION OF METHOD TO CHOOSE TERMS FOR RAYLEIGH-RITZ ANALYSIS	19
Rayleigh-Ritz Analysis	19
New Method for Selecting Terms for Rayleigh-Ritz Analysis	22
Step by Step Procedure of New Method	29
IV. APPLICATIONS OF METHOD	32
Beam with Simple End Supports and Elastic Interior Supports	32
Discretely Stiffened Cylindrical Shell	55
V. CONCLUDING STATEMENTS	63
REFERENCES	65
APPENDIX	
COMPUTER PROGRAM DOCUMENTATION	67

LIST OF TABLES

Tables		Page
4-1.	Eigenvalues and Eigenvectors from Problem of 4 Terms - 1,2,3 and 6. Beam with Simple End Supports and 2 Elastic Interior Supports	36
4-2.	Calculated Second Order Changes in Lowest 4 Eigenvalues Due to Remaining 46 Terms. Beam with Simple End Supports and 2 Elastic Interior Supports	37
4-3.	Lowest 4 Eigenvalues with New Estimates for 4 by 4 Problem. Beam with Simple End Supports and 2 Elastic Interior Supports	40
4-4.	Lowest 4 Eigenvalues with New Estimates for 6 by 6 Problem. Beam with Simple End Supports and 2 Elastic Interior Supports	41
4-5.	Eigenvalues and Eigenvectors for Problem of First 5 Terms. Beam with Simple End Supports and 2 Elastic Interior Supports	43
4-6.	Calculated Second Order Changes in Lowest 3 Eigenvalues Due to Remaining 45 Terms Beam with Simple End Supports and 2 Elastic Interior Supports	44
4-7.	Lowest 3 Eigenvalues with New Estimates for 5 by 5 Problem. Beam with Simple End Supports and 2 Elastic Interior Supports	46
4-8.	Lowest 3 Eigenvalues with New Estimates for 10 by 10 Problem. Beam with Simple End Supports and 2 Elastic Interior Supports	47
4-9.	Lowest 3 Eigenvalues with New Estimates for 13 by 13 Problem. Beam with Simple End Supports and 2 Elastic Interior Supports	48

LIST OF TABLES (CONT'D.) Page

4-10. Calculation of Lowest Eigenvalue by Adding One Term at a Time to the Eigenvalue Problem Solved. Beam with Simple End Supports and 2 Elastic Interior Supports 50

4-11. Calculation of Second Lowest Eigenvalue by Adding One Term at a Time to the Eigenvalue Problem Solved. Beam with Simple End Supports and 2 Elastic Interior Supports 51

4-12. Eigenvalues and Eigenvectors from Problem of First 6 Terms. Beam with Simple End Supports and 5 Elastic Interior Supports 53

4-13. Lowest 6 Eigenvalues with New Estimates for 17 by 17 Problem. Beam with Simple End Supports and 5 Elastic Interior Supports 54

4-14. Eigenvalues and Eigenvectors from Problem of 3 Terms, 41, 42, and 43. Freely Supported Cylindrical Shell with 13 Equally Spaced Ring Stiffeners 57

4-15. Calculated Second Order Changes in Lowest 3 Eigenvalues Due to the 57 Remaining Terms. Freely Supported Cylindrical Shell with 13 Equally Spaced Ring Stiffeners 58

4-16. Lowest 3 Eigenvalues with New Estimates for 3 by 3 Problem. Freely Supported Cylindrical Shell with 13 Equally Spaced Ring Stiffeners 60

4-17. Lowest 3 Eigenvalues with New Estimates for 10 by 10 Problem. Freely Supported Cylindrical Shell with 13 Equally Spaced Ring Stiffeners 61

LIST OF FIGURES

Figures		Page
4-1.	Beam with Simple End Supports and Elastic Interior Supports	33
A-1.	Flow Diagram of AUTOEIG	70

LIST OF SYMBOLS

Symbol	Definition
a_{ij}	undetermined coefficient
$\underline{a}$	eigenvector of undetermined coefficients
A	cross-sectional area
A	matrix of undetermined coefficients in (3.10), (3.11)
b_{ij}	coefficient in series (2.36)
D	matrix in standard algebraic eigenvalue problem, (2.3); $[M]^{-1}[K]$
E	Young's modulus
I	area moment of inertia
I	identity matrix
k	stiffness
k_t	torsional stiffness
k_{ij}	element of stiffness matrix
K	stiffness matrix, Eq. (2.1), or matrix of coefficients for $\underline{a}$ in Eq. (3.12)
K'	matrix of coefficients for $\underline{a}'$ in Eq. (3.7)
ℓ	length of beam section, Figure 4-1
m	mass
m	in selection method, the integer number of terms in eigenvalue problem solved by Jacobi method
m_{ij}	element of mass matrix
M	mass matrix, Eq. (2.1), or matrix of coefficients for $\underline{a}$ in Eq. (3.12)
M'	matrix of coefficients for $\underline{a}'$ in Eq. (3.7)
n	number of term in series; maximum size of K and M matrices
p_i	design parameter
R	Rayleigh's quotient
T*	maximum kinetic energy/ω^2

V_{max}	maximum potential energy
w	deflection or displacement
$\underline{x}$	eigenvector
X	matrix of eigenvectors
δ_{ir}	Kronecker delta function
λ	eigenvalue
ρ	mass density
ϕ_j	assumed function
ω	natural frequency

CHAPTER I

INTRODUCTION

The analysis of structures for vibration and buckling often requires the solution of eigenvalue problems. Eigenvalue problems derived from continuous systems are often impossible to solve exactly, and all but the simplest problems require approximate methods to be employed. One of the most frequently used is the Rayleigh-Ritz method. In the Rayleigh-Ritz method the solution is represented by a linear combination of functions which satisfy the geometric boundary conditions of the system.

$$w_i = \sum_{j=1}^{n} a_{ij}\phi_j \quad . \tag{1.1}$$

In Eq. (1.1) w_i are the dependent variables, usually deflections, a_{ij} are undetermined coefficients, and ϕ_j are assumed functions, usually a complete set of linearly independent functions. The Rayleigh-Ritz analysis results in an eigenvalue problem of the form

$$[K]\underline{x} = \lambda[M]\underline{x} \quad . \tag{1.2}$$

The size of K and M is dependent on the number of terms used in the series (1.1). The more terms used in the series the more accurate will be the approximate solution to the eigenvalue problem.

The standard method of using the Rayleigh-Ritz analysis is to solve Eq. (1.2) several times adding terms to the series (1.1) each time until the addition of terms to the series (1.1) does not significantly affect the eigenvalues of interest from Eq. (1.2). This often involves solution of large eigenvalue problems which in turn requires large amounts of computer storage and time.

This research provides a method of selecting the terms in series (1.1) which significantly affect each eigenvalue of interest. Prior to applying the method, the general function, ϕ, must be chosen for Eq. (1.1), and at least one term, ϕ_j, that most closely approximates the mode of each eigenvalue of interest must be chosen. The method described here will then select the other terms in the series (1.1) which are significant to each eigenvalue of interest. This will necessarily reduce the size of the eigenvalue problem (1.2) required for obtaining accurate answers. The method will reduce the computer storage and computation time required. It should generate more confidence in the answers obtained because the effect of many of the terms omitted from the series (1.1) will be approximately known.

This research was originally directed toward the use of a Taylor series expansion of the eigenvalues as an aid to solving eigenvalue problems. The Taylor's series was an expansion of the eigenvalues as functions of the off-diagonal elements of the K and M matrices in Eq. (1.2). This investigation led to the discovery that the Taylor's series expansion of the eigenvalues could be used to select terms for the Rayleigh-Ritz analysis.

The method presented here uses explicit expressions for the derivatives of eigenvalues and eigenvectors with respect to elements of the K and M matrices in Eq. (1.2). The first derivative expressions, as will be discussed in Chapter II, have been derived and used elsewhere, but this is the first publication known to the author which derives and uses expressions for the second derivatives of the eigenvalues with respect to elements of the K and M matrices.

In Chapter II a discussion of the Taylor's series is presented and the equations for the derivatives are obtained. In Chapter III the Rayleigh-Ritz analysis is explained and the new approach for solution is formulated. The method is applied to various problems and the results are discussed in Chapter IV. Chapter V contains the concluding remarks about this research.

CHAPTER II

TAYLOR SERIES APPROXIMATION FOR EIGENVALUES AND EIGENVECTORS

In the design of structures, the dynamic analysis often involves solution of the algebraic eigenvalue problem:

$$[K]\underline{x} = \lambda[M]\underline{x} \quad . \tag{2.1}$$

The K and M matrices are n by n in size, often symmetric. The elements of the K and M matrices are functions of the design parameters of the structure.

Many times a structural system is analytically formulated, the eigenvalue problem, (2.1), is solved, then modifications are made to the system. Solution of (2.1) can be formidable, so it would be advantageous to be able to calculate some of the important natural frequencies of a modified structural system without completely solving Eq. (2.1) again. Perhaps many different changes in the structural system are possible and it is desirable to determine which change is optimum relative to the natural frequencies of the system. Solving Eq. (2.1) many times trying different combinations of the possible changes may be very time consuming and expensive.

Much work has been done over the past few years in determining the natural frequencies of modified structural systems from the eigenvalues of the original structural system without completely solving

the eigenvalue problem again. Local modifications of finite magnitude of a system such as the addition of a concentrated mass or linear spring have been solved successfully with the computation reduced compared to reworking the entire eigenvalue problem. For discussions and examples of this see Ref. [1,2,3].

If small changes are made in the parameters of the structure, an approximation to the i^{th} eigenvalue of the modified structure may be obtained by a Taylor expansion of the i^{th} eigenvalue

$$\lambda_i = (\lambda_i)_o + \frac{1}{1!}\left[\frac{\partial(\lambda_i)_o}{\partial p_1}(\Delta p_1) + \frac{\partial(\lambda_i)_o}{\partial p_2}(\Delta p_2) + \ldots\right]$$

$$+ \frac{1}{2!}\left[\frac{\partial^2(\lambda_i)_o}{\partial p_1^2}(\Delta p_1)^2 + \frac{\partial^2(\lambda_i)_o}{\partial p_2^2}(\Delta p_2)^2 + \ldots\right.$$

$$\left. + 2\frac{\partial^2(\lambda_i)_o}{\partial p_1 \partial p_2}(\Delta p_1)(\Delta p_2) + \ldots\right] + \ldots \qquad (2.2)$$

where p_1, p_2, ... are design parameters of the structure and $(\lambda_i)_o$ is the i^{th} eigenvalue of the unmodified system. The same type of expansion could be used to approximate the new eigenvectors of a modified structure.

The vital step in effecting an expansion of this type is the calculation of the derivatives of the eigenvalues and eigenvectors with respect to the design parameters of the structure.

Explicit expressions for the first derivatives of the eigenvalues and eigenvectors have been known for a number of years. In 1947 Coulson and Longuet-Higgins [4] used the derivatives in molecular

orbital theory, as did Brown and Bassett [5] in 1958. In 1952 Wittrick [6] presented the expression for the first derivative of an eigenvalue and demonstrated how it could be used in buckling and vibration problems. Recently several authors have used the first derivatives of eigenvalues and eigenvectors with respect to design parameters in "dynamic optimization" procedures. Zarghamee [7] used the expression for the first derivative of a frequency in a method to maximize the lowest frequency of a structural system of given weight varying only the stiffnesses. The derivative of frequency expression was used by Rubin [8,9] in an iteration type procedure for dynamic optimization of a complex structure. The optimization was for least weight while satisfying specific frequency requirements. Fox and Kapoor [10,11] have also used these expressions in a structural optimization procedure which designs for minimum weight while satisfying dynamic stress, displacement, and natural frequency restraints.

McCalley [12] used differential expressions for changes in eigenvalues and eigenvectors in an error analysis for eigenvalue problems. In a treatment of eigenvalue problems with statistical properties, Collins and Thomson [13] used the first partial derivative of eigenvalues and eigenvectors with respect to elements of the stiffness and mass matrices.

Two authors, Bellman [14] and Wilkinson [15] have discussed perturbations of eigenvalues and eigenvectors for the standard algebraic eigenvalue problem

$$[D]\underline{x} = \lambda[I]\underline{x} \quad . \tag{2.3}$$

The perturbations in the eigenvalues and eigenvectors were due to perturbations in the D matrix. The derivatives which we have discussed previously can be obtained from this perturbation theory.

Rogers [16] presented the derivative of an eigenvalue and an eigenvector for a very general system including matrices not symmetric, real, or hermitian.

In the publications previously mentioned, only the first derivatives of the eigenvalues and eigenvectors were used. In the procedures applying iteration techniques, the derivatives were used to predict the sign, positive or negative, and the relative magnitude of the changes in eigenvalues due to changes in various parameters of the system. This information gave an indication of what changes in the parameters would be most desirable. After making the changes to the parameters the new eigenvalues were calculated, not necessarily using the derivative expressions. However, in some of the procedures the first two terms of the Taylor series, Eq. (2.2), were used to calculate a final eigenvalue.

An infinite Taylor series is accurate over a certain interval of the independent variable, but this interval is not known in the expansion of eigenvalues discussed here. A truncated Taylor series is strictly accurate only for infinitesimal changes in the parameters, so if finite changes are considered in an expansion of eigenvalues, the final eigenvalue will be in error by some amount. As far as the author knows there is no simple method to determine the magnitude of the errors. Fox and Kapoor [10] do tabulate the errors for a special

case in which they calculated eigenvalues by using the derivatives, but this information cannot be used to determine the error for other problems.

In the original stages of this research it was hypothesized that if the derivatives of the eigenvalues with respect to design parameters could be calculated, a recursion relation might be found for the terms in the Taylor series (2.2). A general term might be found for the series and the values of Δp for which the series converged might be shown. Possibly a remainder could be calculated after the series was truncated. Substantiation of this hypothesis was not successful, but the work toward this substantiation led to the use of derivatives of eigenvalues and eigenvectors in chosing terms for a Rayleigh-Ritz analysis.

Calculation of the derivatives in the manner of previous derivations, for instance [10] becomes tremendously complex after the first derivative. The following method of obtaining the derivatives, which simplifies the algebra, was used. The design variables were simply to be the elements of the stiffness and mass matrices. Since the eigenvectors of an original problem are known, and the modal matrix may be used to diagonalize the original K and M matrices, the problem was formulated as follows. The original eigenvalue problem is:

$$\begin{bmatrix} k'_{11} & k'_{12} & \cdots & k'_{1n} \\ k'_{21} & k'_{22} & \cdots & - \\ - & & & \\ - & & & \\ - & & & \\ - & & & \\ k'_{n1} & & & k'_{nn} \end{bmatrix} \begin{Bmatrix} x'_1 \\ x'_2 \\ - \\ - \\ - \\ - \\ x'_n \end{Bmatrix} - \lambda \begin{bmatrix} m'_{11} & m'_{12} & \cdots & m'_{1n} \\ m'_{21} & m'_{22} & \cdots & - \\ - & - & & \\ - & - & & \\ - & - & & \\ - & - & & \\ m'_{n1} & - & & m'_{nn} \end{bmatrix} \begin{Bmatrix} x'_1 \\ x'_2 \\ - \\ - \\ - \\ - \\ x'_n \end{Bmatrix} = 0 \tag{2.4}$$

The proposed changes to be made to the K and M matrices are:

$$\Delta K' = \begin{bmatrix} \Delta k'_{11} & \Delta k'_{12} & \cdots & \Delta k'_{1n} \\ \Delta k'_{21} & \Delta k'_{22} & \cdots & - \\ - & - & & - \\ - & - & & - \\ - & - & & - \\ - & - & & - \\ \Delta k'_{n1} & - & \cdots & \Delta k'_{nn} \end{bmatrix}, \text{ and } \Delta M' = \begin{bmatrix} \Delta m'_{11} & \Delta m'_{12} & \cdots & \Delta m'_{1n} \\ \Delta m'_{21} & \Delta m'_{22} & \cdots & - \\ - & - & & - \\ - & - & & - \\ - & - & & - \\ - & - & & - \\ \Delta m'_{n1} & - & \cdots & \Delta m'_{nn} \end{bmatrix} \tag{2.5}$$

After the original eigenvalue problem (2.4) has been solved, the modal matrix may be used for a coordinate transformation. This diagonalizes the original K' and M' matrices and changes the values of the $\Delta K'$ and $\Delta M'$ matrices. The modal matrix is the matrix of eigenvectors,

$$[X] = [\underline{x}_1 \underline{x}_2 \underline{x}_3 \cdots x_n] \quad . \tag{2.6}$$

The coordinate transformation is made

$$
\begin{aligned}
[K] &= [X]^T[K'][X] \\
[M] &= [X]^T[M'][X] \\
[\Delta K] &= [X]^T[\Delta K'][X] \\
[\Delta M] &= [X]^T[\Delta M'][X] \quad .
\end{aligned}
\tag{2.7}
$$

We now have the original problem in a new form.

$$
\begin{bmatrix} k_{11} & 0 & 0 & \cdots & \cdots \\ 0 & k_{22} & 0 & \cdots & \\ 0 & & k_{33} & 0 & \\ - & & & - & \\ - & & & & - \\ - & & & & \\ 0 & \cdots & \cdots & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ - \\ - \\ - \\ - \\ x_n \end{Bmatrix} - \lambda \begin{bmatrix} m_{11} & 0 & 0 & & \\ 0 & m_{22} & 0 & \cdots & \\ 0 & 0 & m_{33} & 0 & \\ - & & & - & \\ - & & & & - \\ - & & & & \\ 0 & \cdots & \cdots & & m_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ - \\ - \\ - \\ - \\ x_n \end{Bmatrix} = 0
\tag{2.8}
$$

with changes

$$
\Delta K = \begin{bmatrix} \Delta k_{11} & \Delta k_{12} & \cdots & \Delta k_{1n} \\ \Delta k_{21} & \Delta k_{22} & & \\ - & & - & \\ - & & & - \\ - & & & \\ \Delta k_{n1} & & & \Delta k_{nn} \end{bmatrix}
\tag{2.9}
$$

$$
\Delta M = \begin{bmatrix} \Delta m_{11} & \Delta m_{12} & \cdots & \Delta m_{1n} \\ \Delta m_{21} & \Delta m_{22} & & \\ - & & - & \\ - & & & - \\ \Delta m_{n1} & & & \Delta m_{nn} \end{bmatrix}
$$

We seek the eigenvalues for the problem

$$[K+\Delta K]\underline{x} - \lambda[M+\Delta M]\underline{x} = 0 \ . \tag{2.10}$$

Equation (2.8) is an eigenvalue problem with n eigenvalues

$$(\lambda_1)_o = \frac{k_{11}}{m_{11}} \ , \ (\lambda_2)_o = \frac{k_{22}}{m_{22}}, \ \ldots \ (\lambda_n)_o = \frac{k_{nn}}{m_{nn}} \ . \tag{2.11}$$

The eigenvectors for Eq. (2.8) will be, if made M-orthonormal

$$\underline{X}_1 = \begin{Bmatrix} \frac{1}{\sqrt{m_{11}}} \\ 0 \\ \\ 0 \\ - \end{Bmatrix}, \underline{X}_2 = \begin{Bmatrix} 0 \\ \\ \frac{1}{\sqrt{m_{22}}} \\ 0 \\ - \end{Bmatrix} \quad \ldots \ \underline{X}_n = \begin{Bmatrix} 0 \\ 0 \\ - \\ - \\ - \\ - \\ \frac{1}{\sqrt{m_{nn}}} \end{Bmatrix} . \tag{2.12}$$

The approximation to the i^{th} eigenvalue of the modified problem, Eq. (2.10), will be

$$\lambda_i(k_{11},k_{12},\ldots k_{1n}\ldots,m_{11},m_{12},\ldots m_{1n},\ldots) =$$

$$(\lambda_i)_o + \frac{1}{1!}\left[\frac{\partial(\lambda_i)_o}{\partial k_{11}}(\Delta k_{11}) + \frac{\partial(\lambda_i)_o}{\partial k_{12}}(\Delta k_{12}) + \ldots \right.$$

$$\left. \frac{\partial(\lambda_i)_o}{\partial m_{11}}(\Delta m_{11}) + \frac{\partial(\lambda_i)_o}{\partial m_{12}}(\Delta m_{12}) + \ldots\right] + \frac{1}{2!}\left[\frac{\partial^2(\lambda_i)_o}{\partial k_{11}^2}(\Delta k_{11})^2 + \right.$$

$$\ldots 2\frac{\partial^2(\lambda_i)_o}{\partial k_{11}\partial k_{12}}(\Delta k_{11})(\Delta k_{12}) + \ldots + \frac{\partial^2(\lambda_i)_o}{\partial m_{11}^2}(\Delta m_{11})^2 + 2\frac{\partial^2(\lambda_i)_o}{\partial m_{11}\partial m_{12}}$$

$$\left.(\Delta m_{11})(\Delta m_{12}) + \ldots + 2\frac{\partial^2(\lambda_i)_o}{\partial k_{11}\partial m_{11}}(\Delta m_{11})(\Delta k_{11}) + \ldots\right] +$$

$$+ \frac{1}{3!}\left[\frac{\partial^3(\lambda_i)_o}{\partial k_{11}^3}(\Delta k_{11})^3 + \ldots\right] + \ldots \quad . \tag{2.13}$$

Now the derivatives of the eigenvalues with respect to the elements of the K and M matrices must be calculated.

Derivatives of Eigenvalues

The derivatives of the eigenvalues in which we are interested will be those from Eq. (2.8) in which the K and M matrices are diagonal. These derivatives are obtained as follows, see [10]. If λ_i and $\underline{x}_i$ are a corresponding eigenvalue and eigenvector

$$[K]\underline{x}_i - \lambda_i[M]\underline{x}_i = 0 \quad . \tag{2.14}$$

Premultiplication by $\underline{x}_i^T$ gives

$$\underline{x}_i^T[K]\underline{x}_i - \lambda_i \underline{x}_i^T[M]\underline{x}_i = 0 \quad . \tag{2.15}$$

Differentiation with respect to k_{rs} gives

$$\begin{aligned} &\frac{\partial \underline{x}_i^T}{\partial k_{rs}}[K]\underline{x}_i + \underline{x}_i^T\left[\frac{\partial K}{\partial k_{rs}}\right]\underline{x}_i + \underline{x}_i^T[K]\frac{\partial \underline{x}_i}{\partial k_{rs}} - \frac{\partial \lambda_i}{\partial k_{rs}}\underline{x}_i^T[M]\underline{x}_i \\ &- \lambda_i \frac{\partial \underline{x}_i^T}{\partial k_{rs}}[M]\underline{x}_i - \lambda_i \underline{x}_i^T\left[\frac{\partial M}{\partial k_{rs}}\right]\underline{x}_i - \lambda_i \underline{x}_i^T[M]\frac{\partial \underline{x}_i}{\partial k_{rs}} = 0 \quad . \end{aligned} \tag{2.16}$$

The first and third terms in Eq. (2.16) are equal as are the fifth and seventh terms because K and M are symmetric. These terms combine to give

$$2\frac{\partial \underline{x}_i^T}{\partial k_{rs}}[K-\lambda_i M]\underline{x}_i \quad ,$$

which is equal to zero by Eq. (2.14). The sixth term in Eq. (2.16) is zero. This leaves

$$\underline{x}_i^T[\frac{\partial K}{\partial k_{rs}}]\underline{x}_i - \frac{\partial \lambda_i}{\partial k_{rs}} \underline{x}_i^T[M]\underline{x}_i = 0 \quad . \tag{2.17}$$

Since the eigenvectors have been taken as M-orthonormal

$$\underline{x}_i^T[M]x_i = 1 \quad . \tag{2.18}$$

Therefore,

$$(\frac{\partial \lambda_i}{\partial k_{rs}})_o = (\underline{x}_i)_o^T[\frac{\partial K}{\partial k_{rs}}](\underline{x}_i)_o \tag{2.19}$$

Now to evaluate this further, $(x_i)_o$ is a vector with all terms zero except the i^{th} term, see Eq. (2.12). The term k_{rs} is any term in the K matrix, so $[\frac{\partial K}{\partial k_{rs}}]$ will have all terms zero except the terms (r,s) and (s,r), which will be 1.0. This leads to

$$\frac{\partial(\lambda_i)_o}{\partial k_{rs}} = (x_{si})_o(x_{ri})_o = \delta_{ir}\delta_{is}\frac{1}{m_{ii}} \quad . \tag{2.20}$$

$(x_{si})_o$ is the s^{th} term of the eigenvector $(\underline{x}_i)_o$. This means that

$$\frac{\partial(\lambda_i)_o}{\partial k_{rr}} = 0, \quad r \neq i \qquad \frac{\partial(\lambda_i)_o}{\partial k_{rs}} = 0, \quad r \neq s \tag{2.21}$$

and

$$\frac{\partial(\lambda_i)_o}{\partial k_{ii}} = \frac{1}{m_{ii}} \quad . \tag{2.22}$$

Equation (2.21) can be substantiated physically. In the series (2.13) any derivative containing an odd power of an off-diagonal

parameter must be zero. This is true because a change of coordinates will change the sign of several off diagonal elements Δk_{ij} and Δm_{ij}. If the derivatives containing odd powers of Δm_{ij} or Δk_{ij} were not zero, the terms containing these derivatives would change sign with a coordinate change. This means that the eigenvalues would change values with a change in coordinates which obviously cannot be correct.

In the same manner

$$\frac{\partial(\lambda_i)_o}{\partial m_{rr}} = 0, \quad r \neq i \qquad \frac{\partial(\lambda_i)_o}{\partial m_{rs}} = 0, \quad r \neq s \tag{2.23}$$

$$\frac{\partial(\lambda_i)_o}{\partial m_{ii}} = -\frac{\lambda_i}{m_{ii}} \quad . \tag{2.24}$$

To get the second derivative of the eigenvalue with respect to k_{rs}, differentiate Eq. (2.17) again.

$$2\frac{\partial \underline{x}_i^T}{\partial k_{rs}}\left[\frac{\partial K}{\partial k_{rs}}\right]\underline{x}_i + \underline{x}_i\left[\frac{\partial^2 K}{\partial k_{rs}^2}\right]\underline{x}_i - \frac{\partial^2 \lambda_i}{\partial k_{rs}^2}\underline{x}_i^T[M]\underline{x}_i -$$

$$- 2\frac{\partial \lambda_i}{\partial k_{rs}}\frac{\partial \underline{x}_i^T}{\partial k_{rs}}[M]\underline{x}_i - \frac{\partial \lambda_i}{\partial k_{rs}}\underline{x}_i^T\left[\frac{\partial M}{\partial k_{rs}}\right]\underline{x}_i = 0 \quad . \tag{2.25}$$

The second and fifth terms of Eq. (2.25) are zero.

$$2\frac{\partial \underline{x}_i^T}{\partial k_{rs}}\left[\frac{\partial K}{\partial k_{rs}}\right]\underline{x}_i - \frac{\partial^2 \lambda_i}{\partial k_{rs}^2}\underline{x}_i^T[M]\underline{x}_i - 2\frac{\partial \lambda_i}{\partial k_{rs}}\frac{\partial \underline{x}_i^T}{\partial k_{rs}}[M]\underline{x}_i = 0 \tag{2.26}$$

Applying Eq. (2.18),

$$\frac{\partial^2 \lambda_i}{\partial k_{rs}^2} = 2\frac{\partial \underline{x}_i^T}{\partial k_{rs}}\left[\frac{\partial K}{\partial k_{rs}}\right]\underline{x}_i - 2\frac{\partial \lambda_i}{\partial k_{rs}}\frac{\partial \underline{x}_i^T}{\partial k_{rs}}[M]\underline{x}_i = 0 \quad . \tag{2.27}$$

The derivatives of the eigenvectors with respect to k_{rs} and m_{rs} are shown in the next section. Substituting the derivative of the eigenvector with respect to k_{rs} into Eq. (2.27) and further evaluating as was done for Eq. (2.19) results in

$$\frac{\partial^2(\lambda_i)_o}{\partial k_{rs}^2} = \frac{2\,\delta_{si}}{m_{rr}m_{ss}[(\lambda_s)_o-(\lambda_r)_o]}, \quad s \neq r \quad . \tag{2.28}$$

$$\frac{\partial^2(\lambda_i)_o}{\partial k_{ii}^2} = 0 \quad . \tag{2.29}$$

In the same way

$$\frac{\partial^2(\lambda_i)_o}{\partial m_{rs}^2} = \frac{2\delta_{si}(\lambda_i)_o^2}{m_{rr}\,m_{ss}[(\lambda_s)_o-(\lambda_r)_o]}, \quad s \neq r \tag{2.30}$$

$$\frac{\partial^2(\lambda_i)_o}{\partial m_{ii}^2} = \frac{2(\lambda_i)_o}{(m_{ii})^2} \quad . \tag{2.31}$$

The mixed second partial derivatives are found to be zero

$$\frac{\partial^2(\lambda_i)_o}{\partial k_{rs}\partial k_{uv}} = 0 \tag{2.32}$$

$$\frac{\partial^2(\lambda_i)_o}{\partial m_{rs}\partial m_{uv}} = 0 \tag{2.33}$$

$$\frac{\partial^2(\lambda_i)_o}{\partial m_{rs}\partial k_{rs}} = 0 \qquad r \neq i \tag{2.34}$$

$$\frac{\partial^2(\lambda_i)_o}{\partial m_{ii}\partial k_{ii}} = -\frac{1}{(m_{ii})^2} \tag{2.35}$$

Following the same procedure higher derivatives can be evaluated but will not be needed here.

Derivatives of Eigenvectors

Since the eigenvectors form a complete set of vectors any derivatives of the eigenvector can be represented by a linear combination of the eigenvectors, as in [7,10]

$$\frac{\partial \underline{x}_i}{\partial k_{rs}} = \sum_{j=1}^{n} b_{ij}\underline{x}_j \quad . \tag{2.36}$$

Differentiating Eq. (2.14) with respect to k_{rs}.

$$[\frac{\partial K}{\partial k_{rs}}]\underline{x}_i + [K]\frac{\partial \underline{x}_i}{\partial k_{rs}} - \frac{\partial \lambda_i}{\partial k_{rs}}[M]\underline{x}_i - \lambda_i[\frac{\partial M}{\partial k_{rs}}]\underline{x}_i - \lambda_i[M]\frac{\partial \underline{x}_i}{\partial k_{rs}} = 0 \; . \tag{2.37}$$

The fourth term in Eq. (2.37) is zero.

$$[K-\lambda_i M]\frac{\partial \underline{x}_i}{\partial k_{rs}} = -[\frac{\partial K}{\partial k_{rs}} - \frac{\partial \lambda_i}{\partial k_{rs}} M]\underline{x}_i \quad . \tag{2.38}$$

Substituting Eq. (2.36) into Eq. (2.38)

$$[K-\lambda_i M]\sum_{j=1}^{n} b_{ij}\underline{x}_j = -[\frac{\partial K}{\partial k_{rs}} - \frac{\partial \lambda_i}{\partial k_{rs}} M]\underline{x}_i \; . \tag{2.39}$$

Premultiplying both sides by $\underline{x}_k^T$

$$\underline{x}_k^T[K]\sum_{j=1}^{n} b_{ij}\underline{x}_j - \lambda_i \underline{x}_k^T[M]\sum_{j=1}^{n} b_{ij}\underline{x}_j = -\underline{x}_k^T[\frac{\partial K}{k_{rs}}]x_i + \underline{x}_k^T[\frac{\partial \lambda_i}{k_{rs}} M]\underline{x}_i \tag{2.40}$$

This simplifies to

$$b_{ik}\lambda_k - b_{ik}\lambda_i = -\underline{x}_k^T[\frac{\partial K}{\partial k_{rs}} - \frac{\partial \lambda_i}{\partial k_{rs}} M]\underline{x}_i \tag{2.41}$$

$$b_{ik} = \frac{\underline{x}_K^T[\frac{\partial K}{\partial k_{rs}} - \frac{\partial \lambda_i}{\partial k_{rs}} M]\underline{x}_i}{\lambda_i - \lambda_k}, \quad k \neq i \quad . \tag{2.42}$$

Now differentiating Eq. (2.18) with respect to k_{rs} gives

$$2\underline{x}_i^T[M] \frac{\partial \underline{x}_i}{\partial k_{rs}} + \underline{x}_i^T[\frac{\partial M}{\partial k_{rs}}]\underline{x}_i = 0 \quad . \tag{2.43}$$

Substituting Eq. (2.36) into (2.43)

$$\underline{x}_i^T[M] \sum_{j=1}^{n} b_{ij}\underline{x}_j = 0 \tag{2.44}$$

$$b_{ii}\, \underline{x}_i^T[M]\underline{x}_i = 0 \tag{2.45}$$

Therefore $$b_{ii} = 0 \quad . \tag{2.46}$$

Equation (2.36) can now be written as

$$\frac{\partial \underline{x}_i}{\partial k_{rs}} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\underline{x}_j^T[\frac{\partial K}{\partial k_{rs}} - \frac{\partial \lambda_i}{\partial k_{rs}} M]\underline{x}_i}{(\lambda_i - \lambda_j)} \underline{x}_j \tag{2.47}$$

Further evaluation gives the result

$$\frac{\partial (x_{ii})_o}{\partial k_{rs}} = 0 \tag{2.48}$$

$$\frac{\partial (x_{mi})_o}{\partial k_{rs}} = \frac{\delta_{rm}\,\delta_{si}}{[(\lambda_s)_o - (\lambda_r)_o]\, m_{rr}\sqrt{m_{ss}}}, \quad m \neq i \quad . \tag{2.49}$$

In the same manner the derivative of the eigenvectors with respect to m_{rs} can be obtained.

$$\frac{\partial (x_{ii})_o}{\partial m_{rs}} = - \frac{\delta_{ir}\, \delta_{is}}{2(m_{ii})^{3/2}} \tag{2.50}$$

$$\frac{\partial (x_{mi})_o}{\partial m_{rs}} = - \frac{\lambda_i \;\; \delta_{mr} \;\; \delta_{is}}{[(\lambda_s)_o - (\lambda_r)_o] m_{rr} \sqrt{m_{ss}}} \quad , \quad m \neq i \; . \tag{2.51}$$

Higher derivatives of the eigenvectors can also be obtained, but are not needed here.

The explicit expressions for the second derivatives of the eigenvalues with respect to elements of the K and M matrix have been derived here for the special case of a diagonal K and M matrix. In this special case the second derivatives are needed because the first derivatives are zero. In the next chapter the derivatives, Equations (2.28), (2.30), and (2.51) are used in the method of selecting terms for a Rayleigh-Ritz analysis.

CHAPTER III

FORMULATION OF METHOD TO CHOOSE TERMS FOR RAYLEIGH-RITZ ANALYSIS

As noted in the introduction, the Rayleigh-Ritz method is an approximate method often used in solving eigenvalue problems for structural systems. Strictly, it is a general procedure used for solving various types of variational problems, see [17,18], but here the interest is in eigenvalue problems for structural systems.

Rayleigh-Ritz Analysis

For the Rayleigh-Ritz method the assumed deflection solution is represented by a linear combination of functions which satisfy the geometric boundary conditions of the system.

$$w_i = \sum_{j=1}^{n} a_{ij}\phi_j \quad , \tag{3.1}$$

where w_i is the deflection, a_{ij} are undetermined coefficients, and ϕ_j are the assumed functions.

Rayleigh's Principle states that a conservative system vibrating about an equilibrium position has a stationary value of frequency in the neighborhood of a natural mode. In the neighborhood

of the fundamental mode, the stationary value is a minimum. Rayleigh's quotient may be expressed in the form, see [19],

$$\lambda = R(w) = \frac{V_{max}}{T^*} , \tag{3.2}$$

where $\lambda = \omega^2$, V_{max} is the maximum potential energy of the system in free vibration, and T^* is the maximum kinetic energy of the system in free vibration divided by ω^2. V_{max} and T^* are functions of the spatial coordinates of the system. In the Rayleigh-Ritz method, the series Eq. (3.1) is substituted into Eq. (3.2) making Eq. (3.2) a function of the unknown coefficients, a_{ij}. Since Rayleigh's quotient always yields an upper bound on the fundamental frequency of the system, it is desirable to have a minimum of Rayleigh's quotient. The essence of the Rayleigh-Ritz method is to secure a minimum of the Rayleigh quotient with respect to the undetermined coefficients. Therefore to secure this minimum the partial derivatives of Eq. (3.2) are taken with respect to the undetermined coefficients. These partial derivatives are set equal to zero.

$$\frac{\partial \lambda}{\partial a_n} = \frac{\partial \left(\frac{V_{max}}{T^*}\right)}{\partial a_n} = 0 \quad , \qquad n = 1,2,3,\ldots . \tag{3.3}$$

$$\frac{\partial \left(\frac{V_{max}}{T^*}\right)}{\partial a_n} = T^* \frac{\partial V_{max}}{\partial a_n} - V_{max} \frac{\partial T^*}{\partial a_n} = 0 \tag{3.4}$$

$$\frac{\partial V_{max}}{\partial a_n} - \frac{V_{max}}{T^*} \frac{\partial T^*}{\partial a_n} = 0 \tag{3.5}$$

$$\frac{\partial V_{max}}{\partial a_n} - \lambda \frac{\partial T^*}{\partial a_n} = 0 , \qquad n = 1,2,3,\ldots . \tag{3.6}$$

Equation (3.6) is a set of n equations which can be represented in matrix form as

$$[K']\underline{a}' = \lambda[M']\underline{a}' \quad . \tag{3.7}$$

This is a standard eigenvalue problem. The solution of this problem will yield approximations to the natural frequencies of the system and the modes of the system in terms of the assumed functions ϕ_j. If ϕ_j is a complete set of functions, the solution of Eq. (3.7) will be an exact solution of the eigenvectors and eigenvalues of the system, provided an infinite number of terms could be taken in the series Eq. (3.1). However, Eq. (3.1) must be truncated. The more accurately the functions, ϕ_j, define the natural modes of the system, the fewer the number of terms needed in (3.1) to get accurate values for the eigenvalues. In fact, if an exact mode is chosen for ϕ_j, only one term will be needed to yield the eigenvalue for that mode. Since the natural modes are normally not known, the more terms included in (3.1) the more accurate will be the natural frequencies obtained, but the eigenvalue problem (3.7) will become larger.

The usual way to apply the Rayleigh-Ritz method is to solve the problem using a certain number of terms in the series, Eq. (3.1), then repeat the solution, adding another term to the series until the eigenvalues of interest show a relatively small change when another term is added to the series. The number of terms which can be included in the series is usually limited by either computer storage or computer time required to solve the eigenvalue problem. When the procedure of adding terms to the series is stopped, in effect the eigenvalue problem, Eq. (3.7), is truncated. There is no information about the effect of the terms omitted from the series. There has been no judicious selection of the terms in the series.

Many terms included may have little effect on the eigenvalues in question, and important terms may have been omitted.

A method which does employ some selection of terms has been presented by Kukudzhanov [20]. This method has the following steps: (1) solution of all possible 2 by 2 eigenvalue problems, (2) selection of the 2 by 2 problem which gives the lowest frequency, (3) then solution of all possible 3 by 3 problems which result from adding a term to the chosen 2 by 2 problem, (4) a repetition of this procedure of building the size of the eigenvalue problem one term at a time until a satisfactory solution is obtained. This method selects terms, but many eigenvalue problems must be solved, although the size of these eigenvalue problems will probably be smaller than with the standard approach.

New Method for Selecting Terms for Rayleigh-Ritz Analysis

A method is presented here which selects significant terms in series (3.1) to be used in solving the Rayleigh-Ritz problem. Before using this method, the general function, ϕ, must be chosen for Eq. (3.1). The K and M matrices are then calculated for n terms in the assumed series. For each eigenvalue of interest, one term which most closely approximates the mode of interest must be selected, for a total of m terms. This method then selects the terms which are significant to the eigenvalues of interest. The terms are selected by using the partial derivatives of the eigenvalues with respect to the elements of the K and M matrices of Eq. (3.7). An eigenvalue problem of size

m is then solved for each eigenvalue of interest. m is an arbitrary number depending on the problem, but m will likely be much smaller than n. Information is calculated about the approximate effect of the n-m terms omitted from the eigenvalue problem. A detailed explanation of the method now follows.

A problem, (3.7), is formulated with n terms. From these n terms, m terms are selected as those terms representing the best approximation to the eigenvalues to be calculated. The eigenvalue problem for them is then partitioned as shown. The m by m submatrices in the upper left hand corners of K and M contain the m best approximation terms.

$$\left[\begin{array}{ccc|c} k'_{11}k'_{12} & \cdots & k'_{1m} & \cdots\ k'_{1n} \\ k'_{21}k'_{22} & \cdots & & \\ \vdots & & & \\ k'_{m1}\cdots & k'_{mm} & \cdots & \\ \hline \vdots & & & \\ k'_{n1} & \cdots & \cdots & \cdots\ k'_{nn} \end{array}\right] \left\{\begin{array}{c} a_1 \\ - \\ \vdots \\ a_m \\ -- \\ \vdots \\ a_n \end{array}\right\} = \lambda \left[\begin{array}{ccc|c} m'_{11}m'_{12} & \cdots & m'_{1m} & \cdots m'_{1n} \\ m'_{21}m'_{22} & & & \\ \vdots & & & \\ m'_{m1} & m'_{mm} & \cdots & \\ \hline & & & \\ & & & \\ m'_{n1} & \cdots & \cdots & \cdots m'_{nn} \end{array}\right] \left\{\begin{array}{c} a_1 \\ - \\ \vdots \\ a_m \\ -- \\ \vdots \\ a_n \end{array}\right\} \qquad (3.8)$$

If the eigenvalue problem is solved for m terms, eigenvalues and eigenvectors are obtained:

$$(\lambda_1)_m,\ (\lambda_2)_m,\ \ldots,\ (\lambda_m)_m,\ (\underline{a}_1)_m,\ (\underline{a}_2)_m,\ \ldots,\ (\underline{a}_m)_m\ .$$

Now it is desirable to know the effect of adding terms m+1 through n to the eigenvalues in question. As noted in the last chapter, an approximation to this may be obtained by a Taylor expansion about the

eigenvalue in the m by m problem.

$$\lambda_i = (\lambda_i)_m + \frac{1}{1!}\left[\frac{\partial(\lambda_i)_m}{\partial k_{i(m+1)}} k_{i(m+1)} + \frac{\partial(\lambda_i)_m}{\partial m_{i(m+1)}} m_{i(m+1)} + \right.$$

$$\left. + \ldots \frac{\partial(\lambda_i)_m}{\partial k_{i(m+2)}} k_{i(m+2)} + \frac{\partial(\lambda_i)_m}{\partial m_{i(m+2)}} m_{i(m+2)} \ldots\right] + \ldots \qquad (3.9)$$

$$+ \frac{1}{2!}\left[\frac{\partial^2(\lambda_i)_m}{\partial k^2_{i(m+1)}} (k_{i(m+1)})^2 + \frac{\partial^2(\lambda_i)_m}{\partial m^2_{i(m+1)}} (m_{i(m+1)})^2 + \ldots\right] + \ldots$$

The coordinates in Eq. (3.8) can be changed using the modal matrix for the m terms for which the eigenvalue problem was solved. The modal matrix will be

$$A = \left[\begin{array}{cccc:cc} a_{11} & a_{12} & \cdots & a_{1m} & 0 & \cdots\ 0 \\ a_{21} & a_{22} & & - & & \\ a_{31} & a_{32} & & - & & \\ a_{m1} & a_{m2} & & a_{mm} & 0 & 0 \\ \hdashline 0 & 0 & & 0 & 1 & 0 \\ - & - & & - & 0 & 1 \\ - & - & & - & - & \\ 0 & 0 & & 0 & & \cdots\ 1 \end{array}\right] \qquad (3.10)$$

The transformation of coordinates is

$$[K] = [A]^T[K'][A]$$
$$[M] = [A]^T[M'][A] \quad . \qquad (3.11)$$

The eigenvalue problem is now the same as Eq. (3.8) except for a transformation of coordinates.

$$[K]\underline{a} = \lambda[M]\underline{a} \qquad (3.12)$$

This can be put into the form

$$\left[[K] + [\Delta K]\right] \underline{a} = \lambda \left[[M] + [\Delta M]\right] \underline{a} \tag{3.13}$$

or

$$\begin{bmatrix} k_{11} & 0 & 0 & \cdots & 0 \\ 0 & k_{22} & & & \\ 0 & & k_{mm} & & \\ \vdots & & & & \\ 0 & & & & k_{nn} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_m \\ \vdots \\ a_n \end{Bmatrix} = \lambda \begin{bmatrix} m_{11} & 0 & \cdots & \cdots & 0 \\ 0 & m_{22} & \cdots & \cdots & \\ & & m_{mm} & & \\ \vdots & & & & \\ 0 & & & & m_{nn} \end{bmatrix} \begin{Bmatrix} a_2 \\ a_2 \\ a_m \\ \vdots \\ a_n \end{Bmatrix}$$

with changes

$$\Delta K = \begin{bmatrix} 0 & \cdots & 0 & k_{1(m+1)} & k_{1(m+2)} & \cdots & k_{1n} \\ \vdots & & \vdots & \vdots & & & \\ 0 & \cdots & 0 & k_{m(m+1)} & \cdots & \cdots & k_{mn} \\ k_{(m+1)1} & \cdots & k_{(m+1)m} & 0 & k_{(m+1)(m+2)} & \cdots & k_{(m+1)n} \\ \vdots & & \vdots & & 0 & & \\ k_{n1} & \cdots & & & & & 0 \end{bmatrix} \tag{3.14}$$

$$\Delta M = \begin{bmatrix} 0 & \cdots & 0 & m_{1(m+1)} & m_{1(m+2)} & \cdots & m_{1n} \\ \vdots & & \vdots & \vdots & & & \\ 0 & \cdots & 0 & m_{m(m+1)} & & & m_{mn} \\ m_{(m+1)1} & & m_{(m+1)m} & 0 & m_{(m+1)(m+2)} & \cdots & m_{(m+1)n} \\ \vdots & & \vdots & & & & \\ m_{n1} & \cdots & & & & & 0 \end{bmatrix} \tag{3.15}$$

Equations (3.14) and (3.15) are in exactly the same form as Eqs. (2.8) and (2.9). The derivatives in Chapter II have been obtained for this type of matrix, Eq. (3.14), with changes (3.15). Using the fact that the first partial derivatives and mixed second partial derivatives are zero, Eqs. (2.23) and (2.32) to (2.34), the new eigenvalues can be approximated by the Taylor series.

$$\lambda_i = (\lambda_i)_m + \frac{1}{2!}\left[\frac{\partial^2(\lambda_i)_m}{\partial k_{i(m+1)}^2}(k_{i(m+1)})^2 + \frac{\partial^2(\lambda_i)_m}{\partial k_{i(m+2)}^2}(k_{i(m+2)})^2\right.$$

$$+ \ldots \frac{\partial^2(\lambda_i)_m}{\partial k_{in}^2}(k_{in})^2 + \frac{\partial^2(\lambda_i)_m}{\partial m_{i(m+1)}^2}(m_{i(n+1)})^2 + \ldots \qquad (3.16)$$

$$\left. + \frac{\partial^2(\lambda_i)_m}{\partial m_{i(m+2)}^2}(m_{i(n+2)})^2 + \ldots \frac{\partial^2(\lambda_i)_m}{\partial m_{in}^2}(m_{in})^2 + \ldots\right] .$$

As pointed out in Chapter II, this truncated equation is accurate only for very small changes in the design parameters. If the significant terms are included in the original m by m problem the terms left are insignificant and Eq. (3.16) should give a good estimate of the i^{th} eigenvalue for the n by n problem. When all significant terms have not been included in the m by m problem, an excellent indication of the terms which are significant to the i^{th} eigenvalue are given by the individual terms from Eq. (3.16)

$$\frac{1}{2!}\left[\frac{\partial^2(\lambda_i)_m}{\partial^2 k_{is}}(k_{is})^2 + \frac{\partial^2(\lambda_i)_m}{\partial m_{is}^2}(m_{is})^2\right], \qquad (3.17)$$

where s is any of the terms m+1 through n. The individual terms Eq. (3.17) do not necessarily give a good estimate of the actual change caused to the i^{th} eigenvalue when the s^{th} term is added to the eigenvalue problem, it only gives an excellent indication of the relative significance of the terms m+1 through n on the i^{th} eigenvalue. When all significant terms have been included in the m by m problem, the terms (3.17) do give a good estimate of the actual change.

The prime objective of this method is to determine from the K and M matrices of an n by n problem the terms which are significant in calculating the i^{th} eigenvalue. For any Rayleigh-Ritz problem of size n by n, the i^{th} eigenvalue, λ_i being one of the lowest eigenvalues, can be calculated accurately by solving only an m by m eigenvalue problem. As was noted before one of the most significant terms for each of the low frequencies of interest must be included in the original m by m problem. This is not an unreasonable limitation since the geometry of the structure will almost always provide insight for this choice.

Rayleigh's quotient can also be used to give a good approximation of the i^{th} eigenvalue of the problem defined by Eqs. (3.14) and (3.15). For Eq. (3.14) the i^{th} eigenvalue has the value

$$(\underline{a}_i)_m = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ \frac{1}{\sqrt{m_{ii}}} \\ 0 \\ 0 \\ \vdots \end{Bmatrix} \tag{3.18}$$

Now using the derivatives of the eigenvectors with respect to elements of the K and M matrices, Eqs. (2.50) and (2.51), the eigenvector $\underline{a}_i$ can be approximated including all the terms from m+1 to n.

$$\underline{a}_i = (\underline{a}_i)_m + \frac{1}{1!}\left[\frac{\partial(\underline{a}_i)_m}{\partial k_{i(m+1)}} k_{i(m+1)} + \frac{\partial(\underline{a}_i)_m}{\partial m_{i(m+1)}} m_{i(m+1)} + \right.$$

$$\left. \frac{\partial(a_i)_m}{\partial k_{i(m+2)}} k_{i(m+2)} + \frac{\partial(\underline{a}_i)_m}{\partial m_{i(m+2)}} m_{i(m+2)} + \ldots\right] + \ldots \tag{3.19}$$

If the significant terms are included in the m by m problem, Eq. (3.10) should give a better approximation of the eigenvector, $\underline{a}_i$. Rayleigh's quotients using the new eigenvector should give a better approximation to the i^{th} eigenvalue.

$$\lambda_i = \frac{\underline{a}_i^T [K] \underline{a}_i}{\underline{a}_i^T [M] \underline{a}_i} \quad . \tag{3.20}$$

When all the significant terms have been included in the m by m problem, λ_i and $(\lambda_i)_m$ from Eq. (3.16) and λ_i from Eq. (3.20) should converge to the same value. Let us now restate the method step by step.

Step by Step Procedure of New Method

An eigenvalue problem is to be solved by the Rayleigh-Ritz method. Taking n terms in the series (3.1), elements have been calculated for the K and M matrix in Eq. (3.7).

$$[K']\underline{a}' = \lambda[M']\underline{a}' \quad . \tag{3.7}$$

Step 1. For each of the eigenvalues of interest, choose at least one term in the series (3.1) which most closely represents the mode of interest. The geometry of the structure will be important in determining which terms to include. Experience has shown that the terms which have the lowest values of k_{ii}/m_{ii} from Eq. (3.7) are often the most significant terms in the lower frequencies. If there is any doubt, the terms with the lowest values of k_{ii}/m_{ii} should be included.

Step 2. The terms which have been selected in Step 1 make up an eigenvalue problem of size m by m. This problem should be solved, providing m eigenvalues and eigenvectors.

Step 3. A coordinate transformation is now effected using the m eigenvectors from Step 2, see Eqs. (3.10) and (3.11). The problem can now be visualized as

$$\Big[[K]+[\Delta K]\Big]\ \underline{a} = \lambda\Big[[M]+[\Delta M]\Big]\ \underline{a} \quad . \tag{3.13}$$

Step 4. A new approximation for the i^{th} eigenvalue can be calculated using the Taylor series (3.16) rearranged here

$$\lambda_i = (\lambda_i)_m + \frac{1}{2!}\left[\frac{\partial^2(\lambda_i)_m}{\partial k^2_{i(m+1)}}\left(k_{i(m+1)}\right)^2 + \frac{\partial^2(\lambda_i)_m}{\partial m^2_{i(m+1)}}\left(m_{i(m+1)}\right)^2\right]$$

$$+ \frac{1}{2!}\left[\frac{\partial^2(\lambda_i)_m}{\partial k^2_{i(m+2)}}\left(k_{i(m+2)}\right)^2 + \frac{\partial^2(\lambda_i)_m}{\partial m^2_{i(m+2)}}\left(m_{i(m+2)}\right)^2\right] + \cdots \tag{3.21}$$

A new approximation to the i^{th} eigenvalue can also be calculated using expressions (3.19) and (3.20). The i^{th} eigenvalue in these equations refers to the eigenvalue associated with the i^{th} term in the problem (3.13). Most likely the lowest eigenvalue is of interest. Let us assume that the lowest eigenvalue is associated with the i^{th} term, i.e., after the coordinate transformation k_{ii}/m_{ii} is the smallest of these possible terms from (3.13). If λ_i from Eq. (3.21), λ_i from Eq. (3.20), and $(\lambda_i)_m$ from Eq. (3.21) are nearly equal, then λ_i will be nearly as accurate as if the n by n eigenvalue problem had been solved. An evaluation of "nearly equal" and "nearly as accurate as" from the previous sentence will be made in the next chapter.

If the λ_i's from Eqs. (3.20) and (3.21) are not reasonably equal, then there are terms which are significant to the i^{th} eigenvalue which have not been included in the m by m problem.

Step 5. The significant terms are indicated by the individual terms in Eq. (3.21). The r most significant of these terms should be chosen and added to the m by m eigenvalue problem. The procedure

outlined here should be repeated starting with Step 2. The new problem in Step 2 will be of size (m+r) by (m+r). r is, of course, of arbitrary size. The procedure should be repeated from Step 2 until an accurate value of λ_i is obtained.

The procedure should be carried out from Step 4 for each eigenvalue of interest.

Completion of this method will yield the eigenvalues of interest calculated to nearly the same accuracy as the n by n problem by solving only a few (m+r)by(m+r) eigenvalue problems, where (m+r)<n. The relative magnitudes of m+r and n depend upon the problem and the accuracy desired.

The step by step procedure has been computer programmed. An explanation and listing of the program is in the Appendix. The step by step procedure listed here is not necessarily rigid. The basic idea is to use the individual terms in Eq. (3.21) to indicate which significant terms have been left of the problem for the eigenvalues of interest. Many different procedures could be used and may have been, but the procedure listed here and programmed seemed to be a good general procedure. The next chapter will present some applications of this method.

CHAPTER IV

APPLICATIONS OF THE METHOD

An evaluation and investigation of this technique has been conducted and some of the results are presented here.

Beam with Simple End Supports and Elastic Interior Supports

The first problem is as shown in Fig. 4-1, a beam on simple end supports with elastic interior supports. This problem is typical of problems which might be solved by a Rayleigh-Ritz analysis. Some problems of this type are solved in Ref. [21]. If w is the deflection of the beam in the y direction, the expression for potential energy of the beam is

$$V_{max} = \frac{EI}{2}\int_0^{3\ell}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \tfrac{1}{2}k[w(\ell)]^2 + \tfrac{1}{2}k[w(2\ell)]^2$$

$$+ \tfrac{1}{2}k_t\left[\frac{\partial w(\ell)}{\partial x}\right]^2 + \tfrac{1}{2}k_t\left[\frac{\partial w(2\ell)}{\partial x}\right]^2 \quad . \tag{4.1}$$

For the kinetic energy of the beam

$$T^* = \tfrac{1}{2}\rho A\int_0^{3\ell} w^2 dx \quad . \tag{4.2}$$

For the modal function a sine series was assumed

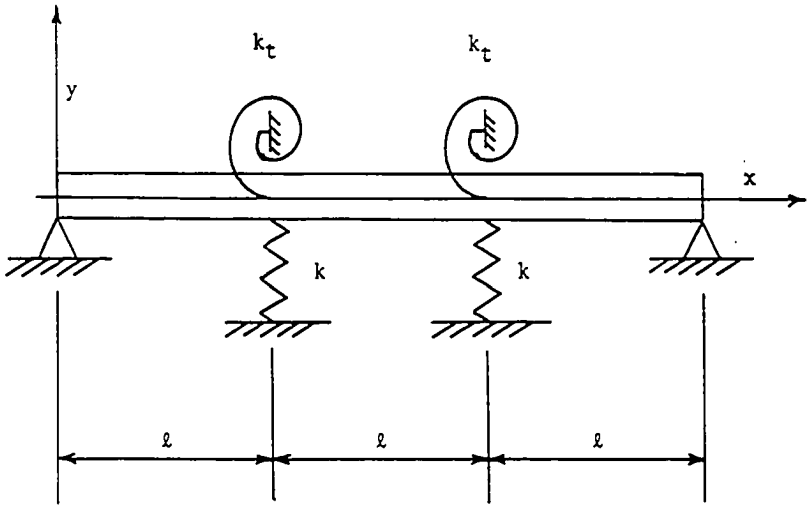


Figure 4-1- Beam with Simple End Supports and 2 Elastic Interior Supports.

$$w = \sum_{j=1}^{n} a_j \sin \frac{j\pi x}{3\ell} \quad . \tag{4.3}$$

Non-dimensionalizing the equations of motion for this beam gives the frequency parameter or eigenvalue as

$$\lambda = \frac{\omega^2 \rho A \ell^4}{EI} \quad . \tag{4.4}$$

The spring stiffness parameters to be varied are

$$\frac{\ell^3 k}{EI} \qquad \text{and} \qquad \frac{\ell k_t}{EI} \quad .$$

The Rayleigh-Ritz analysis presents the eigenvalue problem

$$[K]\underline{a} - \lambda[M]\underline{a} = 0 \tag{4.5}$$

with

$$k_{nm} = \frac{k\ell^3}{EI}\left[\sin\frac{n\pi}{3}\sin\frac{m\pi}{3} + \sin\frac{2n\pi}{3}\sin\frac{2m\pi}{3}\right]$$

$$+ \frac{k_t\ell}{EI}\left(\frac{\pi}{3}\right)^3 nm\left[\cos\frac{n\pi}{3}\cos\frac{m\pi}{3} + \cos\frac{2n\pi}{3}\cos\frac{2m\pi}{3}\right] \tag{4.6}$$

$$n \neq m$$

$$k_{nn} = \frac{3}{2}\left(\frac{\pi n}{3}\right)^4 + \frac{k\ell^3}{EI}\left[\sin^2\frac{n\pi}{3} + \sin^2\frac{2n\pi}{3}\right]$$

$$+ \frac{k_t\ell}{EI}\left(\frac{\pi n}{6}\right)^2\left[\cos^2\frac{n\pi}{3} + \cos^2\frac{2n\pi}{3}\right] \quad . \tag{4.7}$$

For this problem the mass matrix is diagonal

$$M_{nm} = 0, \qquad n \neq m$$
$$M_{nn} = 3/2 \quad . \tag{4.8}$$

The first case run for this problem was relatively simple with no torsional spring and very stiff displacement springs.

$$\frac{\ell^3 k}{EI} = 10000$$
$$\frac{\ell k_t}{EI} = 0 \quad . \tag{4.9}$$

The first 50 terms were included in the series (4.3) and the lowest 3 eigenvalues were of interest. To initially include the most significant term for each of the lowest 3 eigenvalues, terms 1, 2, 3 and 6 were chosen. The results of this 4 by 4 problem are shown in Table 4-1. This completes steps 1 and 2 in the procedure.

A coordinate change was made using the modal matrix, step 3. The second order changes as shown in Eq. (3.21) were then calculated to determine the relative effects of the remaining 46 terms on the lowest 4 eigenvalues. Since 4 terms were chosen in the first step, the eigenvalues must be determined for all 4 of these because it was not known which of these 4 terms were most significant for the lowest 3 eigenvalues. These effects are shown in Table 4-2. The addition of all these effects for each eigenvalue to the value from Table 4-1 gives a new approximation to

Table 4-1. Eigenvalues and Eigenvectors from Problem of 4 Terms - 1, 2, 3 and 6

Beam with Simple End Supports and 2 Elastic Interior Supports.

$$\frac{\ell^3 k}{EI} = 10000 \ , \qquad \frac{\ell k_t}{EI} = 0$$

Number	Eigenvalue
1	97.41
2	1558.54
3	10001.12
4	10019.23

	Eigenvector Components			
Term / Number	1	2	3	4
1	0.15×10^{-5}	-0.22×10^{-5}	1.00	0.63×10^{-12}
2	0.58×10^{-5}	-0.15×10^{-5}	-0.12×10^{-10}	1.00
3	1.00	0.59×10^{-8}	-0.15×10^{-5}	-0.58×10^{-5}
4	-0.59×10^{-8}	1.00	0.22×10^{-5}	0.15×10^{-5}

Table 4-2. Calculated Second Order Changes in Lowest 4 Eigenvalues Due to Remaining 46 Terms.

Beam with Simple End Supports and 2 Elastic Interior Supports.

$$\frac{\ell^3 k}{EI} = 10000. \quad , \qquad \frac{\ell k_t}{EI} = 0.0$$

Term Added	Predicted Change in Eigenvalue			
	Lowest Eigenvalue	2nd Lowest Eigenvalue	3rd Lowest Eigenvalue	4th Lowest Eigenvalue
4	All less than 1.0×10^{-10}	All less than 1.0×10^{-7}	*	-346500.
5			-133300.	-
7			-34650.	-
8			-	-20380.
9			-	-
10			-	-8329.
11			-5680.	-
12			-	-
13			-2912.	-
14			-	-2165.
15			-	-
16			-	-1269.
17			-996.	-
18			-	-
19			-638.	-
20			-	-520.
21			-	-
22			-	-355.
23			-297.	-
24			-	-
25			-213.	-

Table 4-2 (continued)

Table 4-2 (continued)

Term Added	Predicted Change in Eigenvalue			
	Lowest Eigenvalue	2nd Lowest Eigenvalue	3rd Lowest Eigenvalue	4th Lowest Eigenvalue
26	All less than 1.0×10^{-10}	All less than 1.0×10^{-7}	-	-182.
27			-	-
28			-	-135.
29			-118.	-
30			-	-
31			-90.	-
32			-	-79.
33			-	-
34			-	-62.
35			-55.	-
36			-	-
37			-44.	-
38			-	-40.
39			-	-
40			-	-32.
41			-29.	-
42			-	-
43			-24.	-
44			-	-22.
45			-	-
46			-	-18.
47			-17.	-
48			-	-
49			-14.	-
50			-	-13.

*Blank spaces have values less than 1.0×10^{-6}.

each of the 4 lowest eigenvalues, see Eq. (3.21). These new estimates are shown in Table 4-3.

The first order changes, due to the remaining 46 terms, are calculated for the eigenvectors of the 4 lowest eigenvalues, see Eq. (3.19). The modified eigenvectors are used in Rayleigh's quotient, Eq. (3.20), to calculate a new approximation to the lowest 4 eigenvalues. These estimates are also in Table 4-3. This completes step 4 in the procedure.

The results in Table 4-2 and 4-3 need to be discussed now. For the lowest 2 eigenvalues the 3 estimates from Table 4-3 are exactly the same, and the changes from Table 4-2 are very small. This was to be expected, at least for the lowest eigenvalue because of our knowledge about the physical aspects of the problem. The eigenvalues designated 3^{rd} and 4^{th} lowest have certainly not been approximated very closely as can be seen in Table 4-3. Step 5 in the procedure is the selection of the terms which are significant to each eigenvalue of interest. The terms are selected as the largest terms from the second order changes in Table 4-2.

Including the initial 4 terms, a total of 6 terms were chosen to determine a better value for the lowest 4 eigenvalues. Terms in the original 4 terms which were insignificant were omitted from the 6 by 6 problem. Six terms were chosen for each of the 4 eigenvalues. Solutions to these 6 by 6 problems with the new approximations from the Taylor series, and from Rayleigh's quotient are shown in Table 4-4.

Table 4-3. Lowest 4 Eigenvalues with New Estimates for 4 by 4 Problem.

Beam with Simple End Supports and 2 Elastic Interior Supports.

$$\frac{\ell^3 k}{EI} = 10000. \quad , \qquad \frac{\ell k_t}{EI} = 0.0$$

	Eigenvalue from 4 by 4 Problem	Eigenvalue with Second Order Approximations for Remaining 46 Terms	Rayleigh's Quotient for New Eigenvector Approximation
Lowest Eigenvalue	97.41	97.41	97.41
2nd Lowest Eigenvalue	1559.	1559.	1559.
3rd Lowest Eigenvalue	10001.	-169000.	15900.
4th Lowest Eigenvalue	10019.	-370100.	11690.

Table 4-4. Lowest 4 Eigenvalues with New Estimates for 6 by 6 Problems.

Beam with Simple End Supports and 2 Elastic Interior Supports.

$$\frac{\ell^3 k}{EI} = 10000. \quad , \qquad \frac{\ell k_t}{EI} = 0.0$$

	Eigenvalue from 6 by 6 Problem	Eigenvalue with Second Order Approximations for Remaining Terms	Rayleigh's Quotient for New Eigenvector Approximations	Maximum* Difference of Value in Columns 1,2, or 3 in % of Column 1	Correct Answer From 50 by 50 Problem
Lowest Eigenvalue	97.41	97.41	97.41	0	97.41
2nd Lowest Eigenvalue	166.8	158.6	165.8	4.9	158.9
3rd Lowest Eigenvalue	338.1	334.9	336.0	1.0	334.4
4th Lowest Eigenvalue	1559.	1559.	1559.	0	1559.

*From the values in Columns 1, 2, and 3, $\frac{\text{(maximum value - minimum value)}}{\text{(value in Column 1)}} \times 100.$

The approximations for the lowest 3 eigenvalues have converged, which means that the significant terms of the 50 have been included in the eigenvalue problem solved for each eigenvalue of interest. The eigenvalue which was originally thought to be the second lowest eigenvalue, Table 4-1, is found to be the fourth largest or even higher, Table 4-4. This was precisely the reason it was noted earlier that all four terms originally chosen must be made to converge.

This was a fairly simple problem in that it did not take many terms to calculate accurate eigenvalues, and most of these were in the first few terms. A slightly more difficult problem is now solved for the same beam with different spring constants.

$$\frac{\ell^3 k}{EI} = 2000. \,, \qquad \frac{\ell k_t}{EI} = 200.$$

Fifty terms were included in the series, and initially the first five terms in the series were chosen. The results of this problem are shown in Tables 4-5 through 4-9. For each eigenvalue of interest a 10 by 10 problem, Table 4-8, and finally a 13 by 13 problem, Table 4-9, was solved. The terms for these 10 by 10 and 13 by 13 problems were selected as the largest values for the second order changes in Table 4-6.

Table 4-5. Eigenvalues and Eigenvectors for Problem of First 5 Terms.

Beam with Simple End Supports and 2 Elastic Interior Supports.

$$\frac{\ell^3 k}{EI} = 2000. \quad , \qquad \frac{\ell k_t}{EI} = 200.$$

Number	Eigenvalue
1	263.3
2	1364.
3	2768.
4	4425.
5	6351.

	Eigenvector Components				
Term Number	1	2	3	4	5
1	-0.271	-0.146×10^{-5}	-0.474	-0.320×10^{-5}	0.8375
2	0.181×10^{-5}	-0.556	0.158×10^{-5}	0.831	-0.266×10^{-5}
3	-0.738	0.547×10^{-6}	0.662	0.115×10^{-5}	0.136
4	0.353×10^{-5}	0.831	0.196×10^{-5}	0.556	0.158×10^{-5}
5	0.618	-0.310×10^{-5}	0.581	-0.284×10^{-5}	0.529

Table 4-6. Calculated Second Order Changes in Lowest 3 Eigenvalues Due to Remaining 45 Terms

Beam with Simple End Supports and 2 Elastic Interior Supports.

$$\frac{\ell^3 k}{EI} = 2000. \quad , \quad \frac{\ell k_t}{EI} = 200.$$

Term Added	Predicted Change in Eigenvalue		
	Lowest Eigenvalue	2nd Lowest Eigenvalue	3rd Lowest Eigenvalue
6	-*	-1085.	-
7	-0.527	-	-2544.
8	-	-778.	-
9	-2.67	-	-976.
10	-	-263.	-
11	-4.47	-	-86.9
12	-	-708.	-
13	-0.019	-	-677.
14	-	-335.	-
15	-1.84	-	-630.
16	-	-163.	-
17	-1.65	-	-88.0
18	-	-477.	-
19	-0.087	-	-296.
20	-	-175.	-
21	-1.26	-	-425.
22	-	-102.	-
23	-0.795	-	-67.5
24	-	-329.	-
25	-0.096	-	-161.

Table 4-6 Continued

Table 4-6 (continued)

Term Added	Predicted Change in Eigenvalue		
	Lowest Eigenvalue	2nd Lowest Eigenvalue	3rd Lowest Eigenvalue
26	-	-105.	-
27	-0.883	-	-298.
28	-	-68.8	-
29	-0.454	-	-50.7
30	-	-235.	-
31	-0.086	-	-100.
32	-	-69.6	-
33	-0.644	-	-217.
34	-	-49.1	-
35	-0.289	-	-38.7
36	-	-175.	-
37	-0.074	-	-67.9
38	-	-49.3	-
39	-0.486	-	-164.
40	-	-36.6	-
41	-0.199	-	-30.3
42	-	-134.	-
43	-0.062	-	-48.9
44	-	-36.6	-
45	-0.378	-	-127.
46	-	-28.3	-
47	-0.144	-	-24.3
48	-	-106.	-
49	-0.052	-	-36.8
50	-	-28.3	-

*Blank spaces have values less than 1.0×10^{-6}

Table 4-7. Lowest 3 Eigenvalues with New Estimates for 5 by 5 Problem.

Beam with Simple End Supports and 2 Elastic Interior Supports.

$$\frac{\ell^3 k}{EI} = 2000. \quad , \qquad \frac{\ell k_t}{EI} = 200.$$

	Eigenvalue from 5 by 5 Problem	Eigenvalue with Second Order Approximations for Terms 6 through 50	Rayleigh's Quotient for New Eigenvector Approximation
Lowest Eigenvalue	263.3	246.2	286.9
2nd Lowest Eigenvalue	1364.1	-4172.	16530.
3rd Lowest Eigenvalue	2768.5	-4385.	15000.

Table 4-8. Lowest 3 Eigenvalue with New Estimates for 10 by 10 Problem.

Beam with Simple End Supports and 2 Elastic Interior Supports.

$$\frac{\ell^3 k}{EI} = 2000. \quad , \qquad \frac{\ell k_t}{EI} = 200.$$

	Eigenvalue from 10 by 10 Problem	Eigenvalue with Second Order Approximations from Remaining 40 Terms	Rayleigh's Quotient for New Eigenvector Approximation	Maximum* Difference of Values in Columns 1, 2, or 3 in % of Column 1
Lowest Eigenvalue	234.7	231.2	234.0	1.5
2nd Lowest Eigenvalue	239.5	231.5	241.5	4.2
3rd Lowest Eigenvalue	605.5	539.5	529.2	12.6

*From the values in column 1,2,and 3, $\frac{\text{(maximum value-minimum value)}}{\text{(value in column 1)}} \times 100.$

Table 4-9. Lowest 3 Eigenvalues with New Estimates for 13 by 13 Problem.

Beam with Simple End Supports and 2 Elastic Interior Supports.

$$\frac{\ell^3 k}{EI} = 2000. \quad , \qquad \frac{\ell k_t}{EI} = 200.$$

	Eigenvalue from 13 by 13 Problem	Eigenvalue with Second Order Approximations for Remaining 37 Terms	Rayleigh's Quotient for New Eigenvector Approximations	Eigenvalue from Solution of 50 by 50 Problem	Maximum Difference* of values in Columns 1, 2, or 3 in % of Column 1
Lowest Eigenvalue	233.1	230.9	232.4	230.6	1.0
2nd Lowest Eigenvalue	235.5	231.3	234.7	231.2	1.8
3rd Lowest Eigenvalue	480.5	467.3	469.7	464.4	2.7

*From the values in Columns 1, 2, and 3, $\frac{\text{(maximum value-minimum value)}}{\text{(value in Column 1)}} \times 100.$

To give an indication of how the approximations to the eigenvalues converge, the lowest and 2nd lowest eigenvalues were calculated adding one term at a time to the eigenvalue problem solved. After each solution the estimated changes for each of the terms remaining in the 50 term series was calculated. Also after each solution a new estimate of the eigenvector was calculated and the eigenvalue was again approximated using Rayleigh's quotient. This information is shown in Tables 4-10 and 4-11. The initial eigenvalues were obtained by using the first 5 terms in the series. From this information and other problems worked, the maximum difference of the 3 approximations calculated gives a good indication of how close the approximations are to the actual solutions from the n by n problem.

A problem is now shown which requires more initial terms to be chosen and requires more terms in the eigenvalues to be solved.

The problem is a beam with simple end supports and 5 equally-spaced elastic interior supports, the same as the supports in Fig. 4-1. For the modal function a sine series was used

$$w = \sum_{j=1}^{n} a_j \sin \frac{j\pi x}{6\ell} \quad . \tag{4.10}$$

The spring stiffness parameters used were

$$\frac{\ell^3 k}{EI} = 2000. \quad , \qquad \frac{\ell k_t}{EI} = 200.$$

Table 4-10. Calculation of Lowest Eigenvalue by Adding One Term at a Time to the Eigenvalue Problem Solved.

Beam with Simple End Supports and 2 Elastic Interior Supports.

$$\frac{\ell^3 k}{EI} = 2000., \qquad \frac{\ell k_t}{EI} = 200.$$

Number of Terms in Solved Eigenvalue Problem	Term added to Eigenvalue Problem	Eigenvalue from Solved Problem	Eigenvalue Estimated from Effect of Remaining Terms	Eigenvalue from Rayleigh's Quotient	Estimated Change in Eigenvalue by Adding Last Term Included	Actual Change in Eigenvalue by Adding Last Term Included	Maximum Difference of alues in Column 1, 2, and 3 in % of Column 1	Col.3-Correct Value / Correct Value x 100
5	1,2,3 4,5	263.33	246.2	286.9			15.5	14.2
6	11	256.21	243.9	270.5	-4.47	-7.12	10.4	11.1
7	9	244.94	234.8	253.5	-2.61	-11.27	7.7	6.2
8	15	240.80	232.6	244.8	-1.97	-4.14	5.1	4.0
9	17	238.94	232.3	241.2	-1.52	-1.86	3.7	3.6
10	21	236.83	231.6	237.2	-1.36	-2.11	2.4	2.7
11	27	235.54	231.2	235.0	-0.96	-1.29	1.8	2.2
12	23	234.72	231.2	233.9	-0.73	-0.82	1.5	1.8
13	33	233.87	231.0	232.8	-0.70	-0.85	1.2	1.4
14	39	233.25	230.9	232.1	-0.53	-0.62	1.1	1.2
15	29	232.80	230.9	231.8	-0.42	-0.45	0.8	0.9
16	45	232.34	230.9	231.4	-0.41	-0.46	0.6	0.8
17	35	232.05	230.8	231.2	-0.27	-0.29	0.6	0.6
18	41	231.85	230.8	231.1	-0.19	-0.20	0.5	0.5
19	19	231.64	230.8	230.9	-0.18	-0.19	0.4	0.4

Correct value for 50 terms = 230.62.

Table 4-11. Calculation of Second Lowest Eigenvalue by Adding One Term at a Time to the Eigenvalue Problem Solved.

Beam with Simple End Supports and 2 Elastic Interior Supports.

$$\frac{\ell^3 k}{EI} = 2000. \quad , \qquad \frac{\ell k_t}{EI} = 200.$$

Number of Terms in Solved Eigenvalue Problems	Term Added to Eigenvalue Problem	Eigenvalue from Solved Problem	Eigenvalue Estimated from Effect of Remaining Terms	Eigenvalue from Rayleigh's Quotient	Estimated Change in Eigenvalue by Adding Last Term Included	Actual Change in Eigenvalue by Adding Last Term Included	Maximum Difference of Values in Column 1,2, and 3 in % of Column 1	Col.3-Correct Value / Correct Value x 100
5	1,2,3 4,5	1364.12	-4172.	16530.				
8	6,8,10	265.11	235.5	314.2			29.7	14.7
9	12	251.11	232.6	271.4	-5.88	-14.00	15.5	8.7
10	14	246.83	232.5	259.7	-3.24	-4.28	11.1	6.8
11	18	242.01	231.7	247.3	-2.90	-4.82	6.5	4.7
12	24	239.51	231.5	241.5	-1.80	-2.50	4.2	3.6
13	20	238.03	231.5	238.9	-1.28	-1.48	3.1	3.0
14	30	236.54	231.4	236.2	-1.19	-1.49	2.2	2.3
15	36	235.54	231.3	234.7	-0.85	-1.00	1.8	1.9
16	26	234.81	231.3	233.9	-0.67	-0.73	1.5	1.6
17	42	234.10	231.3	233.0	-0.63	-0.71	1.2	1.3
18	48	233.56	231.3	232.5	-0.49	-0.54	1.0	1.0
19	32	233.13	231.3	232.2	-0.41	-0.43	0.8	0.9

Correct answer for 50 terms = 231.17.

The first 50 terms were used in the series and the lowest 3 eigenvalues were of interest. The first 6 terms were initially chosen to include the most significant term in the first 3 eigenvalues. The solution for the original 6 by 6 eigenvalue problem is shown in Table 4-12. These original 6 eigenvalues converged after 17 terms were selected for each. These answers are shown in Table 4-13. The answers are a result of the solution of six 17 by 17 eigenvalues problems. Although this is quite a few terms and requires the solution of 6 eigenvalue, the storage required is much less than for the 50 by 50 problem and the computation time is also less.

Table 4-12. Eigenvalues and Eigenvectors from Problem of First 6 Terms.

Beam with Simple End Supports and 5 Elastic Interior Supports.

$$\frac{\ell^3 k}{EI} = 2000. \quad , \qquad \frac{\ell k_t}{EI} = 200.$$

Number	Eigenvalue
1	2002.
2	2010.
3	2262.
4	2531.
5	3068.
6	3598.

Eigenvector Components

Term / Number	1	2	3	4	5	6
1	0.938	-0.925×10^{-4}	0.298	-0.440×10^{-4}	0.175	-0.297×10^{-4}
2	0.102×10^{-3}	0.870	0.284×10^{-4}	0.413	0.168×10^{-4}	0.270
3	-0.340	0.499×10^{-5}	0.890	-0.668×10^{-6}	0.305	0.227×10^{-6}
4	-0.455×10^{-6}	-0.484	-0.405×10^{-5}	0.823	-0.283×10^{-5}	0.298
5	-0.0646	-0.138×10^{-5}	-0.345	-0.352×10^{-5}	0.936	0.154×10^{-5}
6	0.827×10^{-6}	-0.0992	0.346×10^{-6}	-0.390	-0.943×10^{-9}	0.916

Table 4-13. Lowest 6 Eigenvalues with New Estimates for 17 by 17 Problem.

Beam with Simple End Supports and 5 Elastic interior Supports

$$\frac{\ell^3 k}{EI} = 2000., \qquad \frac{\ell k_t}{EI} = 200.$$

	Eigenvalue from 17 by 17 Problem	Eigenvalue with Second Order Approximations for Remaining Terms	Rayleigh's Quotient for New Eigenvector Approximation	Eigenvalue from Solution of 50 by 50 Problem	Maximum* Difference of Values in Columns 1, 2, or 3 in % of Column 1
Lowest Eigenvalue	237.4	236.6	236.5	236.1	0.4
2nd Lowest Eigenvalue	238.0	236.6	236.7	236.2	0.6
3rd Lowest Eigenvalue	435.2	434.5	434.6	434.1	0.2
4th Lowest Eigenvalue	473.6	468.4	478.8	465.1	1.2
5th Lowest Eigenvalue	542.5	516.5	528.3	513.9	4.8
6th Lowest Eigenvalue	573.5	560.6	559.8	554.7	2.4

Discretely Stiffened Cylindrical Shell

The problem that was the motivation for seeking a method to choose terms for a Rayleigh-Ritz analysis was that of free vibration of discretely stiffened cylindrical shells with arbitrary end conditions, see [24]. This reference develops an analysis to determine the free vibrational characteristics of a thin uniform cylindrical shell with arbitrary end conditions with an arbitrary number of ring and stringer stiffeners. A Rayleigh-Ritz analysis is used to obtain approximate solutions. Kinetic energy and potential energy expressions were derived for the cylinder, stringers, and rings in terms of the displacement of the middle surface of the cylinder. A finite series was assumed for the deflection shapes and the eigenvalue problem was formulated similarly to those explained previously in this dissertation.

In this problem there were 3 displacement variables. Longitudinal and circumferential mode shapes had to be assumed. Unsymmetrical stringer combinations made antisymmetric mode shapes necessary. The resulting eigenvalue problem became large with only a few assumed terms. Because of the large eigenvalue problems needed to obtain good approximations to the eigenvalues of interest, much time and effort was expended in trying to solve the eigenvalue problems. Accurate answers were obtained in many cases only after 60 by 60 eigenvalue problems were solved. The difficulties in obtaining accurate answers are discussed in Ref. [25], page 61.

To solve the 60 by 60 eigenvalue problems, three 60 by 60 double precision matrices were necessary. Using the method of selecting terms, the 60 by 60 K and M matrices were needed in single precision only, and three double precision matrices of only 15 by 15 or less were needed. Therefore the core storage required using the method of selecting terms was less than half of the standard method. The sizes of the eigenvalue problems using the double precision routine were much smaller, which means that computation time was reduced considerably.

An example is now shown of a typical solution using the new method. The case is for a freely supported cylindrical shell with 13 equally spaced rings. The natural frequencies of interest are for a circumferential wave number of 10. The lowest radial frequency in this mode is desired. The 60 by 60 K and M matrices were calculated and the new method was applied. The most significant term in the lowest frequency was known to be the 41, 42, or 43 term. The solution to this 3 by 3 problem is in Table 4-14. The estimated effect of the remaining 57 terms on the 3 lowest eigenvalues is shown in Table 4-15. In Table 4-16 the new approximations for the eigenvalues are given. The approximations are not adequate. Significant terms were selected from Table 4-15, and a 10 by 10 problem was worked for each of the 3 lowest eigenvalues. The lowest eigenvalues have converged at this point. It is seen from Table 4-17 that using the new method a 10 by 10 eigenvalue problem gave an approximate answer to the lowest eigenvalue which

Table 4-14. Eigenvalues and Eigenvectors from Problem of 3 Terms, 41, 42, and 43.

Freely Supported Cylindrical Shell with 13 Equally Spaced Ring Stiffeners.

Circumferential Wave Number = 10

Number	Eigenvalue
1	2.882
2	2.891
3	2.910

	Eigenvector Components		
Term / Number	41	42	43
1	0.9987	-0.0455	-0.0250
2	0.0419	0.9907	-0.1292
3	0.0306	0.1280	0.9913

Table 4-15. Calculated Second Order Changes in Lowest 3 Eigenvalues Due to the 57 Remaining Terms.

Freely Supported Cylindrical Shell with 13 Equally Spaced Ring Stiffeners.

Circumferential Wave Number = 10

Term Added	Predicted Change in Eigenvalue		
	Lowest	2nd Lowest	3rd Lowest
1	-0.782×10^{-3}	-0.138×10^{-5}	-0.735×10^{-6}
2	-0.124×10^{-4}	-0.587×10^{-2}	-0.978×10^{-4}
3	-0.798×10^{-5}	-0.213×10^{-3}	-0.0125
4 ↓ 20	0 ↓ 0	0 ↓ 0	0 ↓ 0
21	-0.951	-0.167×10^{-2}	-0.892×10^{-3}
22	-0.196×10^{-2}	-0.928	-0.0154
23	-0.583×10^{-3}	-0.0155	-0.913
24 ↓ 29	negligible	negligible	negligible
30	-0.149×10^{-3}	-0.397×10^{-2}	-0.233
31	-0.467×10^{-3}	-0.221	-0.367×10^{-2}
32	-0.212	-0.374×10^{-3}	-0.199×10^{-3}
33	-0.200	-0.352×10^{-3}	-0.188×10^{-3}
34	-0.392×10^{-3}	-0.185	-0.308×10^{-2}
35	-0.111×10^{-3}	-0.297×10^{-2}	-0.174
36 ↓ 40	negligible	negligible	negligible

Table 4-15 Continued

Table 4-15 (Cont'd.)

Term Added	Predicted Change in Eigenvalue		
	Lowest	2nd Lowest	3rd Lowest
44	-0.267×10^{-4}	-0.341×10^{-3}	-0.298×10^{-2}
45	-0.274×10^{-4}	-0.313×10^{-3}	-0.209×10^{-2}
46	-0.272×10^{-4}	-0.300×10^{-3}	-0.181×10^{-2}
47	-0.271×10^{-4}	-0.293×10^{-3}	-0.168×10^{-2}
48	-0.269×10^{-4}	-0.288×10^{-3}	-0.160×10^{-2}
49	-0.268×10^{-4}	-0.284×10^{-3}	-0.155×10^{-2}
50	-0.140	-0.370	-0.214
51	-0.0375	-17.09	-0.261
52	-14.71	-0.0216	-0.730×10^{-2}
53	-12.97	-0.0282	-0.0223
54	-0.0225	-0.116×10^{-2}	-0.231
55	582×10^{-2}	-0.165	-10.94
56	56×10^{-4}	-0.266×10^{-3}	-0.139×10^{-2}
57	53×10^{-4}	-0.264×10^{-3}	-0.137×10^{-2}
58	-0.251×10^{-4}	-0.261×10^{-3}	-0.136×10^{-2}
59	-0.249×10^{-4}	-0.258×10^{-3}	-0.134×10^{-2}
60	-0.246×10^{-4}	-0.256×10^{-3}	-0.133×10^{-2}

Table 4-16. Lowest 3 Eigenvalues with New Estimates for 3 by 3 Problem.

Freely Supported Cylindrical Shell with 13 Equally Spaced Ring Stiffeners.

Circumferential Wave Number = 10

	Eigenvalue from 3 by 3 Problem	Eigenvalue with Second Order Approximations from Remaining 57 Terms	Rayleigh's Quotient for New Eigenvectors Approximation
Lowest Eigenvalue	2.88	-26.2	4.30
2nd Lowest Eigenvalue	2.89	-27.8	4.24
3rd Lowest Eigenvalue	2.91	-31.3	4.11

Table 4-17. Lowest 3 Eigenvalues with New Estimates for 10 by 10 Problem.

Freely Supported Cylindrical Shell with 13 Equally Spaced Ring Stiffeners.

Circumferential Wave Number = 10

	Eigenvalue from 10 by 10 Problem	Eigenvalue with 2nd Order Approximations from Remaining 40 Terms	Rayleigh's Quotient for New Eigenvector Approximations	Maximum* Difference of values in Columns 1,2, and 3 in % of Column 1	Correct answer from 60 by 60 problem
Lowest Eigenvalue	0.2888	0.2846	0.2846	1.5	0.2840
2nd Lowest Eigenvalue	0.3044	0.2887	0.2885	5.2	0.2854
3rd Lowest Eigenvalue	0.3230	0.2956	0.2960	8.4	0.2911

*Of the values in Columns 1, 2, and 3, $\frac{\text{(maximum value-minimum value)}}{\text{(value in Column 1)}} \times 100$.

is nearly as accurate as that from the 60 by 60 eigenvalue problem solved previously. In an analysis like this where many eigenvalue problems need to be solved, this new method should help reduce computer storage needed and computation costs.

CHAPTER V

CONCLUDING STATEMENTS

A method has been described here which can select terms to be used in a Rayleigh-Ritz analysis of an eigenvalue problem. The method uses explicit expressions for derivatives of the eigenvalues with respect to elements of the K and M matrices. Derivations of these expressions have been presented, the method has been described, and some applications have been shown.

The method has been programmed for a computer, and the documentation for this computer program is in the Appendix. The documentation is written so that the program can be used, directly as documented, in solving Rayleigh-Ritz eigenvalue problems. In the applications only the eigenvalues have been dicusssed, but the eigenvectors of these Rayleigh-Ritz problem can also be calculated as explained in the Appendix.

From the applications presented in Chapter IV it can be seen that use of this method can significantly reduce the computer storage and computation time needed for some Rayleigh-Ritz eigenvalue problems. The method will be most useful in helping solve problems in which the lower eigenvalues can be approximated by

only a few terms in the series, but it is not known which terms are significant. This type of problem cannot necessarily be identified before solution, and sometimes the method might not provide a savings.

This method can possibly be improved, and some of these possibilities are worth mentioning. A prerequisite for using the method is some knowledge about the most significant term for each eigenvalue of interest. This is a limitation, which if removed, would make the method useful to more problems.

The approximations made by the Taylor series and Rayleigh's quotient may be either greater or less than the actual eigenvalue. Only the Rayleigh quotient approximation to the lowest eigenvalue is known to be an upper bound. It would be advantageous to be able to establish whether all the approximations are upper or lower bounds.

REFERENCES

1. Weissenburger, J. T., "Effect of Local Modifications on the Vibration Characteristics of Linear Systems", Journal of Applied Mechanics, 35, Series E, No. 2, 327-332 (June 1968).

2. Lax, Melvin, "Localized Perturbations", The Physical Review, 94, 139-1392 (1954).

3. Thomson, William T., Vibration Theory and Applications, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1965).

4. Coulson, C. A. and Longuet-Higgins, H. C., "The Electronic Structure of Conjugated Systems", Proceeding of the Royal Society of London, Series A, 191, 39-60 (December 1947).

5. Brown, R. D. and Bassett, J. M., "A Method for Calculating the First Order Perturbation of an Eigenvalue of a Finite Matrix", Proceedings of the Physical Society of London, 71, 724-732 (1958).

6. Wittrick, W. H., "Rates of Change of Eigenvalues, with Reference to Buckling and Vibration Problems", Journal of the Royal Aeronautical Society, 66, 590-591 (September 1962).

7. Zarghamee, Mehdi S., "Optimum Frequency of Structures", AIAA Journal, 6, No. 4, 749-750 (April 1968).

8. Rubin, Charles P., "Dynamic Optimization of Complex Structures", AIAA Structural Dynamics and Aeroelasticity Specialist Conference, 9-14 (April 1969).

9. Rubin, Charles P., "Minimum Weight Design of Complex Structures Subject to a Frequency Constraint", AIAA Journal, 8, No. 5, 923-927 (May 1970).

10. Fox, R. L. and Kapoor, M. P., "Rates of Change of Eigenvalues and Eigenvectors", AIAA Journal, 6, No. 12, 2426-2429 (December 1968).

11. Fox, R. L. and Kapoor, M. P., "Structural Optimization in the Dynamics Response Regime: A Computational Approach", AIAA Structural Dynamics and Aeroelasticity Specialist Conference, 15-20 (April 1969).

12. McCalley, Robert B. Jr., "Error Analysis for Eigenvalue Problems", 2nd Conference on Electronic Computation (ASCE), 523-550.

13. Collins, Jon D. and Thomson, William T., "The Eigenvalue Problem for Structural Systems with Statistical Properties", AIAA Journal, 7, No. 4, 642-648 (April 1969).

14. Bellman, Richard, Introduction to Matrix Analysis, McGraw-Hill Book Co., Inc., New York (1960).

15. Wilkinson, J. H., The Algebraic Eigenvalue Problem, Clarendon Press, Oxford (1965).

16. Rogers, Lynn C., "Derivatives of Eigenvalues and Eigenvectors", AIAA Journal, 8, No. 5, 943-944 (May 1970).

17. Hildebrand, Francis B., Methods of Applied Mathematics, Second edition, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1965).

18. Schechter, R. S., The Variational Method in Engineering, McGraw-Hill Book Co., New York (1967).

19. Mierovitch, Leonard, Analytical Methods in Vibrations, The Macmillan Co., New York (1967).

20. Kukudzhanov, S. N., "On the Best Initial Approximations in the Eigenvalue Problem of the Ritz and Bubnov-Galerkin Methods", Referativnyi Zhurnal Mekhanika, USSR, 122, No. 12 (1964), English translation - NASA TT F-341, 568-572 (1966).

21. Lin, Y. K., "Free Vibrations of a Continuous Beam on Elastic Supports", International Journal of Mechanical Sciences, Pergamon Press Ltd., 4, 409-423 (1962).

22. System/360 Scientific Subroutine Package Version III Programmer's Manual, Form H20-0205, International Business Machines Corporation (1968).

23. Bishop, R.E.D., Gladwell, G.M.L. and Michaelson, S., The Matrix Analysis of Vibration, Cambridge University Press (1965).

24. Egle, D. M. and Soder, K. E. Jr., "A Theoretical Analysis of the Free Vibration of Discretely Stiffened Cylindrical Shells with Arbitrary End Conditions", NASA CR-1316 (June 1969).

25. Soder, K. E. Jr., "An Analysis of Free Vibration of Thin Cylindrical Shells with Rings and Stringers Treated As Discrete Elements which May Be Nonsymmetric, Eccentric, and Arbitrarily Spaced", Ph.D. Dissertation, School of Aerospace and Mechanical Engineering, University of Oklahoma, Norman, Oklahoma (1968).

APPENDIX A

COMPUTER PROGRAM DOCUMENTATION

In this appendix the computer programs used for the applications in Chapter IV are detailed. Using this documentation, one should be able to use the programs as they are listed here in solving a Rayleigh-Ritz problem.

The computer programs are written in G-level Fortran IV, and were run on a IBM System 360, Model 50 computer. The main program and 2 subroutines are

Program AUTOEIG
Subroutine EIGN 2
Subroutine DEIGEN

A flow chart for the main program is shown in Fig. A-1, and a brief explanation of the main program and 2 subroutines follows.

Program AUTOEIG

The main program reads the input data, follows the routine shown in the flow diagram, Fig. A-1, and prints out the necessary information. Each time an eigenvalue problem is to be solved this program calls EIGN2.

The applications in Chapter IV discussed only the eigenvalues, but this computer program will also calculate the eigenvectors if necessary. Another single precision array of size n by n is needed in the program. The eigenvectors are determined just as in a Jacobi rotation method. Each time a transformation of coordinates is made, the eigenvector matrix is post-multiplied by the transformation matrix. To use the eigenvector calculation part of program AUTOEIG remove the C in the first column of all statements with VEC in columns 73-75.

Subroutine EIGN2

This subroutine computes the matrix

$$[D] = [L^{-1}][K][L^{-1}]^T$$

where

$$[M] = [L][L]^T \quad ,$$

and transforms [D] into compressed storage mode, see [22]. The method used to compute [D] is shown in Ref. [23], page 295 or in Ref. [15], page 229. This subroutine calls DEIGEN to solve for the eigenvalues and eigenvectors of [D].

Subroutine DEIGEN

This subroutine is from the IBM Scientific Subroutine Package [22]. It solves eigenvalue problems of the type

$$[D]\underline{x} - \lambda[I]\underline{x} = 0$$

by the Jacobi method.

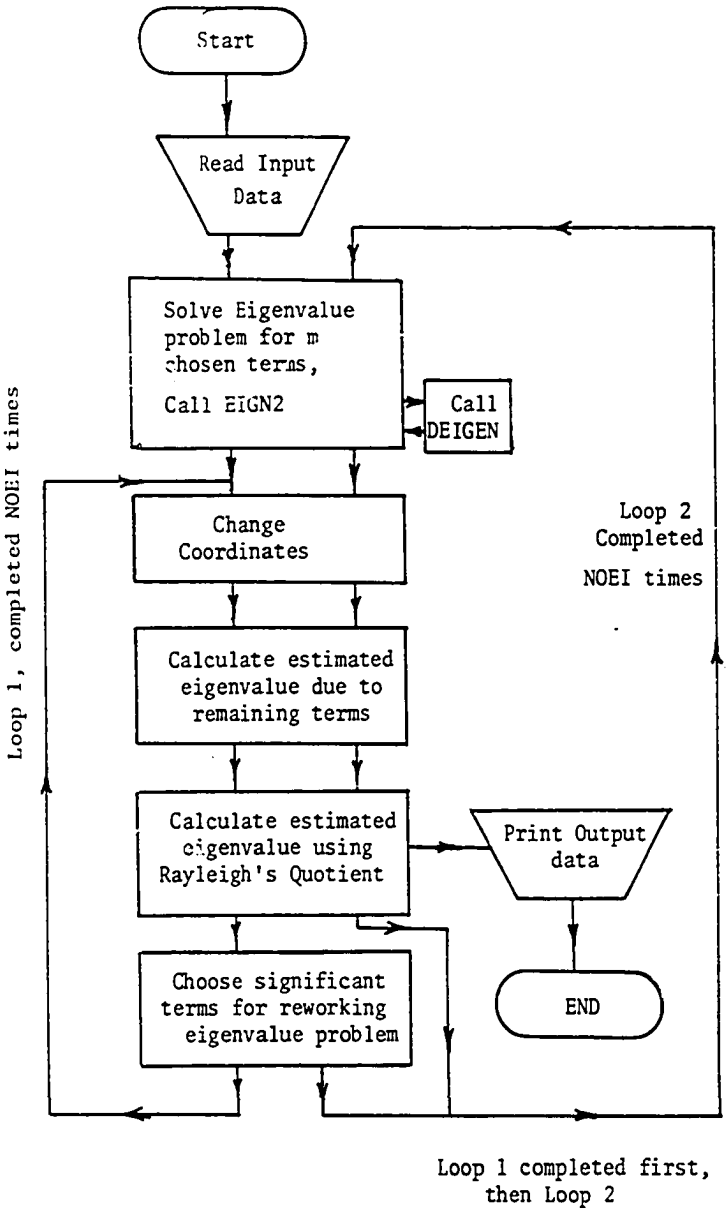


Figure A-1. Flow Diagram of AUTOEIG.

Input for Program AUTOEIG

1. Format (16I5) N,M,NLOE,NOEI,IFP

   N = Total number of terms in assumed series; size of input K and M matrices.

   M = Number of terms in initial problem worked. At least one of most significant terms for each eigenvalue of interest.

NLOE = Number of terms in each eigenvalue problem solved after significant terms are selected.

NOEI = Number of eigenvaiue problems of size NLOE solved, see Fig. A-1. Most likely same as M.

 IFP = Indicator; if IFP = 0 normal printout occurs; if IFP>0 extra information is printed, see program listing.

2. FORMAT(16I5) L01(M)

   L01(M) = Number of each M term chosen for initial problem.

3. FORMAT(5E16.8) S(N,N), EM(N,N).

   S(N,N) = Input K matrix.

  EM(N,N) = Input M matrix.

```
C       PROGRAM AUTOEIG
        DOUBLE PRECISION SP(20,20),EMP(20,20),EVEC(20,20),EVAL(20)
        DIMENSION S(50,50),EM(50,50),DK2(50),DK3(50),LO(5,20)
        DIMENSION CO(20,2)),CM(20,20),VEC(50),B(50)
        DIMENSION LO1(20)
        DIMENSION ERR(50)
C       DIMENSION X(50,50),CVEC(8)                                          VEC
    8 FORMAT(1H0,I5,'EIGENVALUE =',E16.8)
    9 FORMAT(1H0,'SECOND ORDER CHANGES ARE')
   10 FORMAT(10E12.4)
   11 FORMAT(16I5)
   12 FORMAT(10E12.5)
   13 FORMAT(1H0,'NEW APPROXIMATION =', E16.8)
   14 FORMAT(1H0,'RAYLEIGH QUOTIENT APPROXIMATION =',E16.8)
   15 FORMAT(1H0,'SIGNIFICANT TERMS ARE')
   16 FORMAT(1H0,'TERMS FOR EIGENVALUE PROBLEM ARE')
   17 FORMAT(1H0)
   18 FORMAT(1H1)
   19 FORMAT(5I5,6E16.5)
   20 FORMAT(1H0,5X,I5,E16.8)
   21 FORMAT(1H0,'NEW EIGENVECTOR APPROXIMATION')
  102 FORMAT(10X,I3,3X,E26.16,E20.3)
  103 FORMAT(1H0,11X,1HN,10X,11HEIGENVALUES,'       RELATIVE ERROR'/)
  140 FORMAT(1H0,'SP MATRIX')
  141 FORMAT(1H0,'EMP MATRIX')
  142 FORMAT(1H0,'EIGENVECTOR',I5)
  143 FORMAT(1H0,'CORRECT EIGENVECTORS')
  801 FORMAT(/  3X11HEIGENVECTOR,I3)
        READ(1,11)N,M,NLOE,NOEI,IEP
        READ(1,11)(LO1(I),I=1,M)
        MI = M
        KKK = 1
        NOT = 1
        NO2 = 0
        NLO = NLOE
        DO 60 K=1,M
```

```
   60 LO(1,K) = LOI(K)
C     DO 311 I=1,N                                                    VEC
C     DO 310 J=1,N                                                    VEC
C 310 X(I,J) = 0.0                                                    VEC
C 311 X(I,I) = 1.0                                                    VEC
      WRITE(3,18)
      IF(IFP)52,100,52
   52 WRITE(3,12)S
      WRITE(3,17)
      WRITE(3,12)EM
C
C     SET UP  SP  AND  EMP  MATRICES
C
  100 CONTINUE
      WRITE(3,18)
      WRITE(3,16)
      WRITE(3,11)(LO(KKK,K),K=1,MI)
      WRITE(3,17)
      DO 40 K=1,MI
      L = LO(KKK,K)
      DO 40 I=1,MI
      J = LO(KKK,I)
      SP(I,K) = S(J,L)
   40 EMP(I,K) = EM(J,L)
      IF(IFP) 38,39,38
   38 WRITE(3,140)
      DO 37 I=1,MI
   37 WRITE(3,10)(SP(I,J),J=1,MI)
      WRITE(3,17)
      WRITE(3,141)
      DO 33 I=1,MI
   33 WRITE(3,10) (EMP(I,J),J=1,MI)
      WRITE(3,17)
   39 CONTINUE
      CALL EIGN2(SP,EMP,EVEC,EVAL,MI,NERR,ERR)
      WRITE(3,103)
```

```
      WRITE(3,102)(I,EVAL(I),ERR(I),I=1,MI)
      DO 4 I=1,MI
      WRITE(3,901)I
    4 WRITE(3, 10)(EVEC(J,I),J=1,MI)
      WRITE(3,17)
C
C     CHANGE COORDINATES
C
C
C     RESET SP AND EMP MATRICES
C
      DO 44 K=1,MI
      L = LO(KKK,K)
      DO 44 I=1,MI
      J = LO(KKK,I)
      SP(I,K) = S(J,L)
   44 EMP(I,K) = EM(J,L)
      P = 0.0
      PM = 0.0
      DO 42 I=1,N
      DO 48 NK=1,MI
      NKO = LO(KKK,NK)
      IF(NKO-I)48,50,48
   48 CONTINUE
      DO 41 JO =1,MI
      J = LO(KKK,JO)
      DO 49 KN = 1,MI
      KA = LO(KKK,KN)
      P = P + EVEC(KN,JO)*S(I,KA)
      PM = PM + EVEC(KN,JO)*EM(I,KA)
   49 CONTINUE
      DK2(JO) = P
      DK3(JO) = PM
      PM = 0.0
      P = 0.0
   41 CONTINUE
```

```
      DO 45 JO=1,MI
      J = LO(KKK,JO)
      S(J,I) = DK2(JO)
      EM(J,I) = DK3(JO)
      S(I,J) = S(J,I)
   45 EM(I,J) = EM(J,I)
   50 CONTINUE
   42 CONTINUE
C
C     EIGENVECTOR CALCULATIONS
C
C     DO 315 JA =1,N                                                    VEC
C     DO 313 L=1,MI                                                     VEC
C     LA = LO(KKK,L)                                                    VEC
C     DK2(LA) = 0.0                                                     VEC
C     DO 312 MK =1,MI                                                   VEC
C     LOD = LO(KKK,MK)                                                  VEC
C 312 DK2(LA) = X(JA,LOD)*EVEC(MK,L)+DK2(LA)                            VEC
C 313 CONTINUE                                                          VEC
C     DO 314 LK =1,MI                                                   VEC
C     LA = LO(KKK,LK)                                                   VEC
C 314 X(JA,LA) = DK2(LA)                                                VEC
C 315 CONTINUE                                                          VEC
C     DO 370 I=1,NOEI                                                   VEC
C 370 EVEC(I) = LO(KKK,I)                                               VEC

C     DIAGONALIZE  SP AND EMP  MATRICES
C
      DO 23 I=1,MI
      DO 23 J=1,MI
      CO(I,J) = 0.0
      CM(I,J) = 0.0
      DO 23 K=1,MI
      CM(I,J) = CM(I,J) + EVEC(K,I)*EMP(K,J)
   23 CO(I,J) = CO(I,J) + EVEC(K,I)*SP(K,J)
      DO 24 I=1,MI
      DO 24 J=1,MI
```

```
      EMP(I,J) = 0.0
      SP(I,J) = 0.0
      DO 24 K=1,MI
      EMP(I,J) = EMP(I,J) + CM(I,K)*EVEC(K,J)
   24 SP(I,J) = SP(I,J) + CO(I,K)*EVEC(K,J)
      DO 25 K=1,MI
      L = LO(KKK,K)
      DO 25 I=1,MI
      J = LO(KKK,I)
      S(J,L) = SP(I,K)
   25 EM(J,L) = EMP(I,K)
      WRITE(3,17)
      KOK = KKK
C
C     CALCULATE CHANGES IN EIGENVALUES
C
      IF(NOP) 201,200,201
  201 KKK = 0
  202 KKK = KKK+1
  200 CONTINUE
      MU = KKK
      KU = LO1(MU)
      ELAM = EVAL(MU)
      DO 30 I=1,N
      DO 36 J=1,MI
      JO = LO1(J)
      IF(I-JO) 36,35,36
   36 CONTINUE
      DK2(I) = (S(KU,I)*S(KU,I)+EVAL(MU)*EVAL(MU)*EM(KU,I)*EM(KU,I))/
     1(EM(KU,KU)*EM(I,I)*(EVAL(MU)-(S(I,I)/EM(I,I))))
      GO TO 30
   35 DK2(I) = 0.0
   30 ELAM = ELAM + DK2(I)
      WRITE(3,17)
      WRITE(3,8) KKK,EVAL(MU)
      WRITE(3,9)
```

```
      WRITE(3,10)(DK2(J),J=1,N)
      WRITE(3,13) ELAM
C
C     USING RAYLEIGH'S QUOTIENT
C
      KA = KKK
      KU = LOI(KA)
      DO 97 J=1,N
      DO 93 JO=1,MI
      KO = LOI(JO)
      IF(KO-J)93,94,93
   93 CONTINUE
      GO TO 96
   94 VEC(J) = 0.0
      GO TO 97
   95 VEC(J) = (S(KJ,J)-EVAL(KA)*EM(KU,J))/((EVAL(KA)-(S(J,J)/EM(J,J)))
     1*EM(J,J)*(EM(KU,KJ)**0.5))
   97 CONTINUE
      VEC(KU) = 1.0/(EM(KU,KU)**0.5)
      IF(IFP) 83,82,83
   83 WRITE(3,21)
      WRITE(3,10)(VEC(I),I=1,N)
   82 CONTINUE
      DO 91 I=1,N
      B(I) = 0.0
      DO 91 J=1,N
   91 B(I) = B(I) + VEC(J)*S(J,I)
      ENUM = 0.0
      DO 98 I=1,N
   98 ENUM = ENUM + B(I)*VEC(I)
      DO 95 I=1,N
      B(I) = 0.0
      DO 95 J=1,N
   95 B(I) = B(I) + VEC(J)*EM(J,I)
      DNOM = 0.0
      DO 99 I=1,N
```

```
   99 DNOM = DNOM + B(I)*VEC(I)
      ELAM = ENUM/DNOM
      WRITE(3,17)
      WRITE(3,14) ELAM
      IF(NOP) 203,85,203
  203 IF(NOEI-KKK) 202,204,202
  204 KKK = KOK
      GO TO 80
C
C     CHOOSE TERMS FOR REWORKING EIGENVALUE PROBLEM
C
   85 CONTINUE
      WRITE(3,15)
      DO 87 J=1,KKK
      LO(KKK,J) = LO1(J)
   87 EVEC(J,KKK) = 0.000
      NU = KKK
      DO 86 I=1,MI
      VAL = DABS(EVEC(I,KA))
      IF(VAL-0.01)86,86,88
   88 NU = NU + 1
      LO(KKK,NU) = LO1(I)
   86 CONTINUE
      PER = NPER
      MOP = NU+1
      DO 32 K=MOP,NLO
      GUES = ABS(DK2(1))
      NOGU = 1
      DO 31 I=2,N
      IF(GUES-ABS(DK2(I)))34,31,31
   34 GUES = ABS(DK2(I))
      NOGU = I
   31 CONTINUE
      LO(KKK,K) = NOGU
      DK2(NOGU) = 0.0
      WRITE(3,20)LO(KKK,K),GUES
```

```
   32 CONTINUE
      IF(NOP) 75,76,75
   76 IF(NOEI-NOT) 77,78,77
   77 KKK = KKK+1
      NOT = NOT +1
      GO TO 200
   78 KKK = 0
      NOP = 1
      MI = NLO
   75 CONTINUE
   80 CONTINUE
      KKK = KKK+1
      DO 84 K=1,NLO
   84 LOI(K) = LO(KKK,K)
      IF(NOEI-KKK)999,100,100
  999 CONTINUE
C     WRITE(3,18)                                                    VEC
C     WRITE(3,143)                                                   VEC
C     DO 950 I=1,NOEI                                                VEC
C     WRITE(3,142)I                                                  VEC
C     LAM = CVEC(I)                                                  VEC
C     WRITE(3,10)(X(J,LAM),J=1,N)                                    VEC
C 950 CONTINUE                                                       VEC
      END
```

```
      SUBROUTINE EIGN2(A,B,C,EVAL,N,NERR,ERR)
      DOUBLE PRECISION A(20,20),B(20,20),C(20,20),EVAL(20),SUM
      DIMENSION ERR(1)
C
C     TRIANGULARIZE B
C
      NERR=0
      IF(B(1,1).LE.0.D0)  GO TO 990
      C(1,1)=DSQRT(B(1,1))
      DO 1 I=2,N
    1 C(I,1)=B(I,1)/C(1,1)
      DO 2 J=2,N
      DO 2 I=1,N
      IF(I-J)3,4,5
    3 C(I,J)=0.D0
      GO TO 2
    4 KK=J-1
      SUM =0.D0
      DO 6 K=1,KK
    6 SUM=SUM+C(J,K)**2
      IF(B(J,J).LT.SUM) GO TO 991
      C(I,J)=DSQRT(B(J,J)-SUM)
      GO TO 2
    5 KK=J-1
      SUM=0.D0
      DO 7 K=1,KK
    7 SUM=SUM+C(I,K)*C(J,K)
      IF(C(J,J).EQ.0.D0) GO TO 992
      C(I,J)=(B(I,J)-SUM)/C(J,J)
    2 CONTINUE
C
C     INVERT THE TRIANGJLAR MATRIX
C
      DO 11 I=1,N
      DO 11 J=1,N
      IF(I-J)12,13,14
```

```
   12 B(I,J)=0.D0
      GO TO 11
   13 B(J,J)=1.D0/C(J,J)
      GO TO 11
   14 KL=J+1
      KU=I-1
      IF(KU-KL)15,16,16
   15 B(I,J)=-C(I,J)*B(J,J)/    C(I,I)
      GO TO 11
   16 SUM=0.D0
      DO 17 K=KL,KU
   17 SUM=SUM+C(I,K)*B(K,J)
      B(I,J)=-(C(I,J)*B(J,J)+SUM)/C(I,I)
   11 CONTINUE
C
C     COMPUTE THE NEW DYNAMIC MATRIX
C
      DO 21 I=1,N
      DO 21 J=1,N
      C(I,J)=0.D0
      DO 21 K=1,N
   21 C(I,J)=C(I,J)+B (I,K)*A(K,J)
      DO 22 I=1,N
      DO 22 J=1,N
      A(I,J)=0.D0
      DO 22 K=1,N
   22 A(I,J)=A(I,J)+C(I,K)*B(J,K)
C
C     CONVERT TO VECTOR STORAGE MODE 1
C
      K=0
      L=1
      DO 41  J=1,N
      DO 41  I=1,J
      K=K+1
   39 IF(K-20)41,41,40
```

```
   40 K=1
      L=L+1
   41 A(K,L)=A(I,J)
C
C     SOLVE FOR EIGENVALUES AND EIGENVECTORS OF A
C
      CALL DEIGEN(A,C,N,0,NR,ERR)
      DO 44 I=1,N
      II=I*(I-1)/2 +I
      IC=1+(II-1)/20
      IR=II-20*((II-1)/20)
      NA = N-I+1
   44 EVAL(NA) = A(IR,IC)
      IJ=0
      DO 47 J=1,N
      DO 47 I=1,N
      IJ=IJ+1
      IC=1+(IJ-1)/20
      IR=IJ-20*((IJ-1)/20)
   47 A(I,J)=C(IR,IC)
      DO 48 I=1,N
      DO 48 J=1,N
   48 C(I,J)=A(I,J)
C
C     COMPUTE EIGENVECTORS OF ORIGINAL PROBLEM
C
   23 DO 24 I=1,N
      DO 24 J=1,N
      A(I,J)=0.D0
      DO 24 K=1,N
   24 A(I,J)=A(I,J)+B(K,I)*C(K,J)
C
C     NORMALIZE EIGENVECTORS
C
      DO 26 J=1,N
      SUM=0.D0
```

```
      DO 27 I=1,N
   27 SUM=SUM+A(I,J)*A(I,J)
      SUM=DSQRT(SUM)
      IF(SUM)29,26,29
   29 DO 28 I=1,N
      NA = N-J+1
   28 C(I,NA) = A(I,J)/SUM
   26 CONTINUE
      GO TO 25
  990 NERR=1
      GO TO 993
  991 NERR=2
      GO TO 993
  992 NERR=3
  993 WRITE(3,1000) NERR
 1000 FORMAT(//' MATRIX B CANNOT BE TRIANGULARIZED.  NERR =', I2//)
   25 RETURN
      END
```

```
C                                                                       EIGEN001
C     ...................................................................EIGEN002
C                                                                       EIGEN003
      SUBROUTINE DEIGEN                                                 EIGEN004
C                                                                       EIGEN005
C     PURPOSE                                                           EIGEN006
C        COMPUTE EIGENVALUES AND EIGENVECTORS OF A REAL SYMMETRIC       EIGEN007
C        MATRIX                                                         EIGEN008
C                                                                       EIGEN009
C     USAGE                                                             EIGEN010
C        CALL DEIGEN(A,R,N,MV)                                          EIGEN011
C                                                                       EIGEN012
C     DESCRIPTION OF PARAMETERS                                         EIGEN013
C        A - ORIGINAL MATRIX (SYMMETRIC), DESTROYED IN COMPUTATION.     EIGEN014
C            RESULTANT EIGENVALUES ARE DEVELOPED IN DIAGONAL OF         EIGEN015
C            MATRIX A IN DESCENDING ORDER.                              EIGEN016
C        R - RESULTANT MATRIX OF EIGENVECTORS (STORED COLUMNWISE,       EIGEN017
C            IN SAME SEQUENCE AS EIGENVALUES)                           EIGEN018
C        N - ORDER OF MATRICES A AND R                                  EIGEN019
C        MV- INPUT CODE                                                 EIGEN020
C                  0   COMPUTE EIGENVALUES AND EIGENVECTORS            EIGEN021
C                  1   COMPUTE EIGENVALUES ONLY (R NEED NOT BE         EIGEN022
C                      DIMENSIONED BUT MUST STILL APPEAR IN CALLING    EIGEN023
C                      SEQUENCE)                                       EIGEN024
C                                                                       EIGEN025
C     REMARKS                                                           EIGEN026
C        ORIGINAL MATRIX A MUST BE REAL SYMMETRIC (STORAGE MODE=1)      EIGEN027
C        MATRIX A CANNOT BE IN THE SAME LOCATION AS MATRIX R            EIGEN028
C                                                                       EIGEN029
C     SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED                     EIGEN030
C        NONE                                                           EIGEN031
C                                                                       EIGEN032
C     METHOD                                                            EIGEN033
C        DIAGONALIZATION METHOD ORIGINATED BY JACOBI AND ADAPTED        EIGEN034
C        BY VON NEUMANN FOR LARGE COMPUTERS AS FOUND IN 'MATHEMATICAL   EIGEN035
C        METHODS FOR DIGITAL COMPUTERS', EDITED BY A. RALSTON AND       EIGEN036
```

```
C           H.S. WILF, JOHN WILEY AND SONS, NEW YORK, 1962, CHAPTER 7   EIGEN037
C                                                                       EIGEN038
C     ..................................................................EIGEN039
C                                                                       EIGEN040
      SUBROUTINE DEIGEN(A,R,N,MV,NR,ERR)
      DIMENSIONA(1),R(1),ERR(1)
C                                                                       EIGEN043
C        ...............................................................EIGEN044
C
C        IF A DOUBLE PRECISION VERSION OF THIS ROUTINE IS DESIRED, THE  EIGEN046
C        C IN COLUMN 1 SHOULD BE REMOVED FROM THE DOUBLE PRECISION      EIGEN047
C        STATEMENT WHICH FOLLOWS.                                       EIGEN048
C                                                                       EIGEN049
      DOUBLE PRECISION A,R,ANORM,ANRMX,THR,X,Y,SINX,SINX2,COSX,         EIGEN050
     1                 COSX2,SINCS                                    EIGEN051
C                                                                       EIGEN052
C        THE C MUST ALSO BE REMOVED FROM DOUBLE PRECISION STATEMENTS    EIGEN053
C        APPEARING IN OTHER ROUTINES USED IN CONJUNCTION WITH THIS      EIGEN054
C        ROUTINE.                                                       EIGEN055
C                                                                       EIGEN056
C        THE DOUBLE PRECISION VERSION OF THIS SUBROUTINE MUST ALSO      EIGEN057
C        CONTAIN DOUBLE PRECISION FORTRAN FUNCTIONS.  SQRT IN STATEMENTSEIGEN058
C        40, 68, 75, AND 78 MUST BE CHANGED TO DSQRT.  ABS IN STATEMENT EIGEN059
C        62 MUST BE CHANGED TO DABS.                                    EIGEN060
C                                                                       EIGEN061
C        ...............................................................EIGEN062
C                                                                       EIGEN063
C        GENERATE IDENTITY MATRIX                                       EIGEN064
C                                                                       EIGEN065
      IF(MV-1) 10,25,10                                                 EIGEN066
   10 IQ=-N                                                             EIGEN067
      DO 20 J=1,N                                                       EIGEN068
      IQ=IQ+N                                                           EIGEN069
      DO 20 I=1,N                                                       EIGEN070
      IJ=IQ+I                                                           EIGEN071
      R(IJ)=0.D+00                                                      EIGEN072
```

```
      IF(I-J) 20,15,20                                                  EIGEN073
   15 R(IJ)=1.D+00                                                      EIGEN074
   20 CONTINUE                                                          EIGEN075
C                                                                       EIGEN076
C        COMPUTE INITIAL AND FINAL NORMS (ANORM AND ANORMX)             EIGEN077
C                                                                       EIGEN078
   25 ANORM=0.D+00                                                      EIGEN079
      DO 35 I=1,N                                                       EIGEN080
      DO 35 J=I,N                                                       EIGEN081
      IF(I-J) 30,35,30                                                  EIGEN082
   30 IA=I+(J*J-J)/2                                                    EIGEN083
      ANORM=ANORM+A(IA)*A(IA)                                           EIGEN084
   35 CONTINUE                                                          EIGEN085
      IF(ANORM) 165,165,40                                              EIGEN086
   40 ANORM=1.414D+00*DSQRT(ANORM)                                      EIGEN087
      ANRMX=ANORM*1.0D-12                                               EIGEN088
C                                                                       EIGEN089
C        INITIALIZE INDICATORS AND COMPUTE THRESHOLD, THR               EIGEN090
C                                                                       EIGEN091
      NR=0
      IND=0                                                             EIGEN092
      THR=ANORM                                                         EIGEN093
   45 THR=THR/FLOAT(N)                                                  EIGEN094
   50 L=1                                                               EIGEN095
   55 M=L+1                                                             EIGEN096
C                                                                       EIGEN097
C        COMPUTE SIN AND COS                                            EIGEN098
C                                                                       EIGEN099
   60 MQ=(M*M-M)/2                                                      EIGEN100
      LQ=(L*L-L)/2                                                      EIGEN101
      LM=L+MQ                                                           EIGEN102
   62 IF(DABS(A(LM))-THR) 130,65,65                                     EIGEN103
   65 IND=1                                                             EIGEN104
      NR=NR+1
      LL=L+LQ                                                           EIGEN105
      MM=M+MQ                                                           EIGEN106
```

```
      X=0.5D+00*(A(LL)-A(MM))                                          EIGEN107
   68 Y=-A(LM)/DSQRT(A(LM)*A(LM)+X*X)                                  EIGEN108
      IF(X) 70,75,75                                                   EIGEN109
   70 Y=-Y                                                             EIGEN110
   75 SINX=Y/DSQRT(2.D+00*(1.D+00+(DSQRT(1.D+00-Y*Y))))                EIGEN111
      SINX2=SINX*SINX                                                  EIGEN112
   78 COSX=DSQRT(1.0D+00-SINX2)                                        EIGEN113
      COSX2=COSX*COSX                                                  EIGEN114
      SINCS =SINX*COSX                                                 EIGEN115
C                                                                      EIGEN116
C         ROTATE L AND M COLUMNS                                       EIGEN117
C                                                                      EIGEN118
      ILQ=N*(L-1)                                                      EIGEN119
      IMQ=N*(M-1)                                                      EIGEN120
      DO 125 I=1,N                                                     EIGEN121
      IQ=(I*I-I)/2                                                     EIGEN122
      IF(I-L) 80,115,80                                                EIGEN123
   80 IF(I-M) 85,115,90                                                EIGEN124
   85 IM=I+MQ                                                          EIGEN125
      GO TO 95                                                         EIGEN126
   90 IM=M+IQ                                                          EIGEN127
   95 IF(I-L) 100,105,105                                              EIGEN128
  100 IL=I+LQ                                                          EIGEN129
      GO TO 110                                                        EIGEN130
  105 IL=L+IQ                                                          EIGEN131
  110 X=A(IL)*COSX-A(IM)*SINX                                          EIGEN132
      A(IM)=A(IL)*SINX+A(IM)*COSX                                      EIGEN133
      A(IL)=X                                                          EIGEN134
  115 IF(MV-1) 120,125,120                                             EIGEN135
  120 ILR=ILQ+I                                                        EIGEN136
      IMR=IMQ+I                                                        EIGEN137
      X=R(ILR)*COSX-R(IMR)*SINX                                        EIGEN138
      R(IMR)=R(ILR)*SINX+R(IMR)*COSX                                   EIGEN139
      R(ILR)=X                                                         EIGEN140
  125 CONTINUE                                                         EIGEN141
      X=2.0+00*A(LM)*SINCS                                             EIGEN142
```

```
      Y=A(LL)*COSX2+A(MM)*SINX2-X                                       EIGEN143
      X=A(LL)*SINX2+A(MM)*COSX2+X                                       EIGEN144
      A(LM)=(A(LL)-A(MM))*SINCS+A(LM)*(COSX2-SINX2)                     EIGEN145
      A(LL)=Y                                                           EIGEN146
      A(MM)=X                                                           EIGEN147
C                                                                       EIGEN148
C        TESTS FOR COMPLETION                                           EIGEN149
C                                                                       EIGEN150
C        TEST FOR M = LAST COLUMN                                       EIGEN151
C                                                                       EIGEN152
  130 IF(M-N) 135,140,135                                               EIGEN153
  135 M=M+1                                                             EIGEN154
      GO TO 60                                                          EIGEN155
C                                                                       EIGEN156
C        TEST FOR L = SECOND FROM LAST COLUMN                           EIGEN157
C                                                                       EIGEN158
  140 IF(L-(N-1)) 145,150,145                                           EIGEN159
  145 L=L+1                                                             EIGEN160
      GO TO 55                                                          EIGEN161
  150 IF(IND-1) 160,155,160                                             EIGEN162
  155 IND=0                                                             EIGEN163
      GO TO 50                                                          EIGEN164
C                                                                       EIGEN165
C        COMPARE THRESHOLD WITH FINAL NORM                              EIGEN166
C                                                                       EIGEN167
  160 IF(THR-ANRMX) 165,165,45                                          EIGEN168
C                                                                       EIGEN169
C        SORT EIGENVALUES AND EIGENVECTORS                              EIGEN170
C                                                                       EIGEN171
  165 IQ=-N                                                             EIGEN172
      DO 185 I=1,N                                                      EIGEN173
      IQ=IQ+N                                                           EIGEN174
      LL=I+(I*I-I)/2                                                    EIGEN175
      JQ=N*(I-2)                                                        EIGEN176
      DO 185 J=I,N                                                      EIGEN177
      JQ=JQ+N                                                           EIGEN178
```

```
      MM=J+(J*J-J)/2                                                    EIGEN179
      IF(A(LL)-A(MM)) 170,185,185                                       EIGEN180
  170 X=A(LL)                                                           EIGEN181
      A(LL)=A(MM)                                                       EIGEN182
      A(MM)=X                                                           EIGEN183
      IF(MV-1) 175,185,175                                              EIGEN184
  175 DO 180 K=1,N                                                      EIGEN185
      ILR=IQ+K                                                          EIGEN186
      IMR=JQ+K                                                          EIGEN187
      X=R(ILR)                                                          EIGEN188
      R(ILR)=R(IMR)                                                     EIGEN189
  180 R(IMR)=X                                                          EIGEN190
  185 CONTINUE                                                          EIGEN191
C     COMPUTE ERROR IN EIGENVALUES
C
      DO 54 I=1,N
      ERR(I)=0.0
      DO 54 J=1,N
      IF(I-J) 51,54,51
   51 CALL LOC(I,I,II,N,N,1)
      CALL LOC(J,J,JJ,N,N,1)
      CALL LOC(I,J,IJ,N,N,1)
      DEM=DABS((A(II)-A(JJ))*A(II))
      IF(DEM) 52,53,52
   53 DEM=2.0*A(IJ)
   52 ERR(I)=ERR(I)+2.0*(A(IJ  ) ) **2/DEM
   54 CONTINUE
      RETURN                                                            EIGEN192
      END
```