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AN ECONOMIC ANALYSIS OF A MARKOVIAN DEPLETION MODEL FOR PETROLEUM RESOURCE

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GRADUATE COLLEGE

AN ECONOMIC ANALYSIS OF A MARKOVIAN DEPLETION MODEL FOR PETROLEUM RESOURCE

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

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By

SOEMIRAT SLAMET

Norman, Oklahoma

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AN ECONOMIC ANALYSIS OF A MARKOVIAN

DEPLETION MODEL FOR PETROLEUM

RESOURCE

A DISSERTATION

APPROVED FOR THE DEPARTMENT OF ECONOMICS

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AN ECONOMIC ANALYSIS OF A MARKOVIAN DEPLETION MODEL
FOR PETROLEUM RESOURCE

By: Soemirat Slamet

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Uncertainty is one of the important topics in Natural Resource Economics that is under investigation. The purpose of this dissertation is to develop a Markovian depletion model for petroleum resource where the uncertainty is the geologic uncertainty. The analysis first derives the mathematical formulation for the expected net present value of a petroleum reserve expressed in monetary terms for an exploration in an unknown district. The computation uses a Markov process to quantify the uncertainty and the methodology is the spectral representation of a generator matrix. The price, wage and rental rates are assumed to be constant. It is also assumed that the firm that does the exploration is the firm that will produce the petroleum. The depletion model uses the Markovian exploration model as a constraint and computes the maximum discounted profit relative to the wage rate, using an optimum control method. The state variables are the net reserves and the total cumulative reserve additions per worker while the control variables are the capital labor ratios and the labor per capita ratio. A Cobb Douglas production function with constant returns to scale is assumed for the production of petroleum; the production and development cost is a linear function of capital and labor while the exploration cost is also dependent on the Markov process. The analysis indicates that under optimum production and when the stock of the resource is augmented by exploration, the change in the value of the deposit yields an interest rate on the scarcity rent equal to r, and the price of petro-

leum is equal to the marginal cost of production plus the scarcity rent. Also, the value of the capital used in exploration is proportional to the value of the petroleum produced. The analysis also shows that in general the Markovian model is a better representation for an exploration decision process than a decision tree process, and that the Markovian exploration model can be used in a three sector growth model. Finally, the Markovian exploration model may be used for any other mineral resource by modifying the exploration network.

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NOTATIONS

Due to the difficulties in writing subscripts for mathematical expressions in the computer print-out, notations for subscripts are written in the manner given in the following examples:

1. P_{ij} is written as P(ij); P_{5j} as P(5j); S_i as S(i).
2. $P_{5j}(t)$ is written as P(5j)(t); $B_k(ij)$ as B(k)(ij).

Hence, for example, the equation (3-8) is written as:

$$E(5)(X(t)) = \sum_j \sum_k e^{b(k)t} B(k)(5j)h(j)$$

must be read as:

$$E_5(X_t) = \sum_j \sum_k e^{b_k(t)} B_k(5j)h_j$$

In other words, one must read the letters or numbers within the first parentheses of a letter as a subscript of that letter, in particular in the discussion of the Markov chain and the Markov process. In some instances, however, letters within a parentheses indicate an argument of a function. For example, $x(g,t)$ indicates that x is a function of g and t; thse are found particularly in Chapter II.

LIST OF VARIABLES

The following list shows the important variables used in the analysis.

Chapter III:

P is the transition matrix of a Markov chain.

B is a sub-matrix of the transition matrix P in which the rows are the transient states and the coloumns are theabsorbing states.

Q is a sub-matrix of the transition matrix P which consists of all the transient states.

G is the matrix whose every element indicates the probability of the chain ever reaching an absorbing state from a transient state.

I is an identity matrix.

$S = (I-Q)^{-1}$, is a matrix whose element S(ij) also indicates the probability of the expected number of visits to a transient state j starting from a transient state i, before the chain eventually enters an absorbing state.

V(N(ij)) is the variance of S(ij).

W(t) is the sojourn time of a particular state.

S(i) is the parameter of the sojourn time distribution in state i.

P(t) is a matrix called the transition function of a Markov process.

A is the generator matrix of a Markov process.

b(j) is the j-th eigenvalue of the generator matrix A.

f(j) is the eigenvector associated with the j-th eigenvalue.

N is a matrix whose columns are the eigenvectors.

B(k) is the matrix formed by multiplying the column vector of the matrix N to the row vector of the inverse matrix N.

h(j) is a column vector whose j-th element shows a fixed rate of net reward of state j in exploration.

a is the discount rate.

T is the expected length of time for exploration.

g is the vector for the estimated amount of petroleum.

p is the expected price of petroleum.

p(j) is the imputed price of the information gained from state j.

c(j) is the amount of information gained from state i per unit of time.

L(j) is the amount of labor used in state j.

K(j) is the amount of capital used in state j.

w is the wage rate.

r is the rental rate.

d is a constant whose value is: $0 < d < 1$ and the value of which indicates the ease in which information can be obtained.

Chapter IV:

L is the total amount of labor used in a venture.

K is the total amount of capital used in a venture.

k(1) is the capital-labor ratio in the production of petroleum.

l(1) is the amount of labor used in production relative to L.

k(2) is the capital-labor ratio in development.

l(2) is the amount of labor used in development relative to L.

k(3) is the capital-labor ratio in exploration.

l(3) is the amount of labor used in exploration relative to L.

R is the net reserves per worker.

S is the cumulative reserve additions per worker.

s is the rate of reserve additions per worker.

q is the output of petroleum per worker per unit of time.

A is the technological constant of the Cobb Douglas production function in petroleum production.

a is the parameter of the Cobb Douglas production function.

h(1) is the scarcity rent.

h(2) is the marginal value of information.

h(3) is the shadow price of the overall capital.

V is the wage-rental ratio.

a(j) is the ratio of the amount of capital used in state j to the total amount of capital used in exploration.

c(j) is the ratio of the amount of labor used in state j to the total amount of labor used in exploration.

T is the expected length of time for exploration.

T(1) is the time of complete exhaustion of the deposit.

T(a) is the time at which the total amount of petroleum produced is equal to the total amount of discovery.

$$m = \sum_{j} \sum_{K} (B(k)(5j)a(j)/\delta(j))(e^{b(k)t} - 1)/b(k)$$

$$n = \sum_{j} \sum_{K} (B(k)(5j)c(j)/\delta(j))(e^{b(k)t} - 1)/b(k)$$

$$f = \sum_{j} \sum_{K} B(k)(5j)g(j)(e^{b(k)t} - 1)/b(k)$$

Chapter V:

$y(j)$ is the output of sector j per capita.

$A(j)$ is the technological constant of the Cobb Douglas production function of sector j.

$a(j)$ is the parameter of the Cobb Douglas production function of sector j.

u is the sum of the rate of labor increase and the rate of capital depreciation.

m is the rate of change in the labor force.

v is the wage-rental ratio.

LIST OF FIGURES

Chapter I

INTRODUCTORY SUMMARY

This dissertation attempts to analyse a Markovian exploration model and formulate a depletion model for a firm in which the exploration process is characterized by a Markovian model. Suggestions for the extension of the model into a growth model of several sectors are provided. This chapter contains a brief description of the background of the dissertation, the subject of the dissertation, the methodology of the analysis, and the organization of the report. The final section contains a summary of the conclusions.

1.1 INTRODUCTION

Optimization methods, in particular optimum control, have been used quite extensively in the analysis of problems in natural resource economics. They range from those analyzing problems at the firm level to those that analyze at the aggregate level; they include problems that are deterministic as well as those which are probabilistic. Among these are those analyzing models with exhaustible resources. One of the first analyses of exhaustible resource economics was made by Lewis Gray in 1914. However, the first analysis with optimum control of exhaustible resources was made by

Hotelling (1931). His model analyzed a firm that maximizes the present value of the utility obtained from consuming on the exhaustible resource, constrained by a finite, homogeneous, stock of the exhaustible resource in a single deposit.

After Hotelling's writing, there have been various modifications and extension on his model. Hartwick (1973) analyzed a firm with many deposits of homogeneous character; Schulze (1974) and Ulph (1978) analyzed a firm having deposits with multiple grades; Lewis (1977), Peterson (1978), and Pindyck (1978) provided models with stock augmentation of the exhaustible resource generated by exploration; Sweeny (1977), Lewis and Schmalensee (1980), Ulph and Folie (1980) analyzed the effects of market structures on optimum solutions. Other analyses on deterministic models include those made by Weinstein and Zeckhauzer (1978) and Mitra (1980). All the above analyses pertain to anlyses at the firm level.

Among the probabilistic models analyzing problems at the firm level, are those written by Lewis (1977), Hoel (1978), Loury (1978), Gilbert (1978), Kemp and Long (1978), Robson (1979), and Pindyck (1980). Lewis postulated a price uncertainty and analyzed the effects of a firm's behaviour toward risk; Hoel analyzed a problem when future costs were uncertain; Loury provided an analysis on optimal exploration on unknown reserves; Gilbert and Robson postulated that the stock of the exhaustibe resource was uncertain; Kemp and long assumed that the price and the stock of the exhaustible resource were uncertain; and Pindyck postulated that the demand and the stock of the exhaustible resource were uncertain and that the stock could be augmented by exploration. Most of these models used an expected utility function in their formulation and none of the above models used Markov process to control the uncertainty. Pindyck's model maximizes the discounted total profits of the firm, and hence does not use expected utility function in its approach. No explicit formulation is given for the cost of production, the cost of exploration and for the form of the discovery function. However, it provides conclusions on the effects of the uncertainty on price when the cost functions are linear or nonlinear. The uncertainty involved in Pindyck's model seems to suggest that it not only includes geologic uncertainty but also other uncertainties pertaining to future mineral discoveries. The uncertainties involved in Pindyck's model are controlled by a stochastic process with independent increment. As will be seen later, these characteristics of Pindyck's model differentiate it from the model that will be formulated and analyzed in this dissertation.

Most economic analyses on uncertainty in exploration use a decision tree process to solve the problem (Howe, 1979; pp.212-220; Raiffa, 1970; pp.1-50). This approach contains several disadvantages, among these are:

(1) The conditional probabilities are dependent upon the path of the branches to be passed;

(2) The path of the branches, at certain points, are controlled by deliberate decision actions, which are not incorporated into the computation. It is a given condition;

(3) There are no possibility for any reversibility of action.

On the other hand, a Markov chain and a Markov process, the stochastic processes that will be used in this analysis, have characteristics contrary to the above. Consequently, their characteristics are more suitable for an exploration decision process representation. The character and importance of these differences are fully explained in Chapter II and III.

1.2 THE OBJECTIVE AND SCOPE

The analysis is intended to formulize a depletion model at the firm level, where the uncertainty in the exhaustible resource discovery is quantified by a Markov process, and to suggest how a Markovian exploration model can be incorporated into a three sector growth model. First, the analysis tries to formulate a Markovian petroleum exploration model, where the uncertainty is a geologic uncertainty. The exploration activities involved are those activities generally performed in searching for petroleum resources in an unknown district. A Markov process then controls the transitions from one activity to another. The objective of the Markovian model is to find an expected value for the estimated petroleum resource in terms of monetary value. Second, the analysis uses the Markovian exploration model for formulating a depletion model. The depletion model itself is an optimum control problem, which maximizes the discounted profits per worker relative to the wage rate subject to the rate of change of net reserve and the expected cumulative reserve additions.

The model assumes that the price of petroleum, the wage rate and the rental rate are given. Consequently, it assumes that the firm is a price taker. The production function for petroleum exploitation is a Cobb Douglas production function subject to constant returns to scale.

Suggestions are made on extending the model to a three sector growth model where the first sector produces capital goods, the second sector produces consumption goods, and the third sector produces a resource commodity. A first suggested extension assumes that the price of the exhaustible resource, the wage and rental rates, are given. The model maximizes the production of the consumption goods. All production functions are of the Cobb Douglas type subject to constant returns to scale. A second suggested model assumes that the wage and rental rates are not fixed, but their changes are assumed not to affect the profitability of the exploration ventures. A brief discussion will be given for a suggested third model where the price of the exhaustible resource is not fixed, and where the change in the wage and rental rates affect the profitability of the exploration ventures.

1.3 THE METHODOLOGY

The methodology of the analysis consists of the followings:

(1) To use a Markov process to formulate a Markovian exploration model;

(2) To use optimum control to compute the maximum discounted total profit of the firm.

The step by step formulations are as follows:

(1) To define the geologic uncertainty by using the McKelvey diagram;

(2) To define a petroleum exploration network for an unknown district;

(3) To define a Markov chain for the transitions from one exploration activity to another in the exploration network, by which the probability of every exploration activity can be computed;

(4) To define a Markov process for the transitions from one exploration activity to another in the exploration network, and to compute the spectral representation of the generator matrix of the exploration activities;

(5) To define a net reward function for the exploration model. This net reward function is incorporated into the Markov process defined for the exploration activities

to establish a formula for computing the expected net present value of the exploration project;

(6) To define the functions of the production cost, the development cost and the expected exploration cost;

(7) To formulate the objective functional of the optimum control problem. This objective functional is the discounted total profit of the firm per worker relative to the wage rate;

(8) To formulate an average expected rate of discovery function and an expected cumulative reserve addition function;

(9) To formulate the optimization problem or the optimum control problem which consists of maximizing the objective functional above subject to the rate of change of the net reserve and the expected cumulative reserve addition;

(10) To evaluate the results of the optimum control problem and to compute the discounted total profit per worker relative to the wage rate;

(11) Finally, the analysis formulates a possible extension of the model into a three sector growth model, in which the Markovian exploration model is incorporated.

1.4 THE ORGANIZATION OF THE REPORT

The report of this analysis is organized in the following manner:

(1) Chapter 1 is an introductory summary;

(2) Chapter 2 contains a review of the literature on models in exhaustible resource economics, particularly about the optimum control models. It discusses the differences between these models and the model formulated in this analysis. It also discusses the differences between the decision tree process used in exploration and the Markov chain and the Markov process that are used in this analysis. Several suggestions are mentioned pertaining the growth model that incorporates the Markovian exploration model;

(3) Chapter 3 discusses several major topics. First, it discusses and describes the exploration activities utilized in a search for petroleum resource and defines a network for petroleum exploration in an unknown district. Second, it discusses a Markov chain which is used in the analysis of the exploration network and the limitation of the use of the Markov chain in exploration, and it also provides a computational procedure to compute the probability of every exploration activity. Third, it defines a Markov process that relaxes the limitation posted by the Markov chain. A computational procedure is provided for computing the net present va-

lue of the exploration project. This formula is basically the Markovian exploration model;

(4) Chapter 4 contains the analysis of the optimum control problem, which incorporates the Markovian exploration model developed in Chapter 3. It analyzes the optimum path for the state and control variables that maximize the discounted total profit per worker relative to the wage rate;

(5) Chapter 5 discusses a possible extension of the model into a three sector growth model, where an exhaustible resource is used as input in one of the sectors. The optimum control problem is to maximize the discounted production of the consumption goods per capita, subject to the rate of capital augmentation, the rate of change of the total net reserves, the expected total cumulative reserve additions in the economy, and the rate of change in the labor force. It also discusses briefly a third model, where the price of the exhaustible resource is not fixed and where change in the wage and rental rates affect the profitability of the exploration ventures;

(6) Finally, Chapter 6 provides the conclusions of the analysis.

1.5 A SUMMARY OF THE CONCLUSIONS

The analysis indicates that the use of a Markov chain and a Markov process is a more appropriate representation of the exploration decision process than the decision tree process. A Markov chain and a Markov process provide also a method for the modelling of exploration activities; by modifying the exploration network, such a model can be used for any mineral resource.

The Markovian exploration model that uses a Markov process, provides a model that computes the expected net present value of the estimated petroleum resource in monetary terms. It is based on a spectral representation computational procedure that provides a formula for computing the expected occurences of any exploration activity and a net reward function defined as the difference between the value of information gained from an exploration activity and the cost of performing that activity. It can also be based on an estimated amount of the petroleum resource present, instead of the information gained from the activities. In this case, a volumetric estimation method is required to compute the estimated amount of petroleum. The Markovian exploration model can practically be used for any type of mineral resource exploration.

The depletion model shows that the effect of a discovery is to extent the time of complete exhaustion. This seems to be obvious, but such a result may also be affected by the fact that the price of the exhaustible resource, the interest rate, the wage and rental rates are considered fixed. In other words, there is no variability of output that may be caused by a change in the price, the interest rate, or the wage and rental rates. The formulation of the depletion model is such that numerical data can be used to compute the maximum discounted total profits.

The analysis shows that an extension of the model into a three sector growth model is possible. Although the solutions to the problem are yet to be given, the Lagrange equation or the Hamiltonian for the problem is provided.

Chapter II

LITERATURE REVIEW

This chapter discusses the important models in mineral resource economics that have been previously developed, pertaining to the model that will be developed in this paper. It first reviews the previous models, then discusses the main differences between these models and the intended model. It also explains the nature of the uncertainty and elaborates the differences between the stochastic process used in the intended model and the tree decision process. Suggestions for the extensions of the growth model will be breifly mentioned.

Studies on natural resource economics of the Neoclassical models can basically be categorized into two major classes, namely:

(1) The first class are those describing the behavior of a firm engaged in extractive activities. They analyze the effects of external influences on the firm's behavior, such as market structure, taxes, and uncertainties;

(2) The second class are those describing a simple economy with natural resources treated as factor input. They have usually tried to analyze the implications of the exhaustibility of the non-renewable resources as input to

the process of economic growth and the evaluations of the welfare implications (Smith and Krutilla, 1978).

Both classes have generally accepted one of two ideas concerning the physical occurence of the natural resources. The first assumes a finite stock of resource with homogeneous quality and the second postulates an increasing supply of resources of progressively lower grade. However, Harris and Skinner (1979) do indicate that for some certain mineral resources, namely the less geochemically abundant minerals, once the richest grades are exploited, the quantity available decreases rather than increases at lower grades. So the hypothesis on the availability of natural resources may well include the opinion of Harris and Skinner. Also, uncertainty may be introduced in both classes; so models can further be classified into deterministic and probabilistic models.

2.1 THE MODELS OF THE FIRST CLASS CATEGORY

Within the first class deterministic models, analyses can be static or dynamic. Lewis Gray (1914) is the first writer to present a static analysis of a single firm exploiting a non-renewable resource. Basically, he is able to derive the properties of the time path of prices and the effects of taxes and the changes of price using static micro-economics theory (Gray, 1914). Other writings on static analysis include Frank Paish (1935); A.D Scott (1955); Anthony Fisher and John Krutilla (1975); and D.A. Brobst, W.P. Pratt, and

V.E. McKelvey (1973) write about the definitions of mineral reserves.

The analysis on the dynamic deterministic models using exhaustible resources is first done by Harold Hotelling (1931), followed by Richard Gordon (1967) and R.G. Cummings (1969). These analyses utilize cost functions and the calculus of variations to determine the optimal time path. Hotelling's model, which analyzes the behavior of the firm in competitive and monopoly market structures, is based on the maximization of the present value of profits. It can be written as:

$$\text{Maximize } J = \int_0^{\infty} qp(q)\, e^{-rt}\, dt$$

subject to $\int_0^T q\, dt = a$, and $q = f(pe^{rt}, t)$ where

q is quantity;

p is the difference in price and its unit cost;

r is the discount rate;

t is time;

T is the time of exhaustion;

a is a fixed value.

Hotelling evidently uses a fixed stock of a natural resource with homogeneous quality. The results of his analysis show that in a competitive industry, the exhaustibility of the mineral resource depends on the demand. If the demand is a negative exponential function of price, the exploitation will continue forever but at a diminishing rate. If the demand is linear, the resource will be depleted at a finite time. In a monopoly, the exploitation will be prolonged.

Modifications on Hotelling's model are quite extensive. Among the deterministic models, John M. Hartwick (1978) provides an analysis of a single firm having many deposits, while Alistair M. Ulph gives a resource depletion model with multiple grades analyzed for a competitive market structure (1978). Ulph's model maximizes the present value of profits and can be written as:

$$\text{Maximize } J = \int_0^\infty \left(\sum_{g=1}^{n} (p(t) - C(g)) X(g,t) \right) e^{-rt}\, dt$$

subject to:

$$dX(g,t)/dt = -\,x(g,t),\quad g = 1, 2, \ldots n$$

$$X(g,t),\ x(g,t) \geq 0 \text{ for all } g \text{ and } t,$$

where:

$X(g,t)$ is the stock of mineral resources of grade g, at time t;

$C(g)$ is the cost of producing one unit of the refined resource from resource of grade g.

His analysis concludes that exploitation must be sequential, beginning with the mineral resource of the highest grade.

Frederick M. Peterson (1978) provides a model of mining and exploration of an exhaustible resource. His model also maximizes the net present value of the profit streams and can be shown as follows:

$$\text{Maximize } J = \int_0^\infty (px(j) - M(j) - E(j))\, e^{-rt}\, dt$$

subject to:

$x(j)$, $D(j) \geq 0$; $X(j)(0) = 0$ and $D(j)(0) = 0$;

$D(j) - X(j) \geq 0$.

where:

$x(j) = dX(j)/dt$ is the extraction rate of firm j;

$M(j)$ is the extraction costs of firm j;

$X(j)$ is the cummulative extraction;

$D(j)$ is the cummulative discovery;

$E(j)$ is the exploration costs of firm j;

t is time.

His conclusions are that in a competitive industry, firms are prompted to explore rapidly; however, the existence of information spillover makes firms to delay exploration. In a monopoly, the monopolist tends to delay production and has excess reserves.

Other writings on deterministic models are those written by James L. Sweeney (1977); Robert S. Pindyck (1978); M. C. Weinstein and R. C. Zeckhauzer (1978); Tapan Mitra (1980).

Pindyck's analysis is concerned with the optimal exploration and production of an exhaustible resource. It treats the resource base as the basis for production and exploration activity as the means for increasing or maintaining the reserves. However, as depletion finally takes place, his analysis assumes that the exploration activities will result in smaller discoveries. These are the basic constraints imposed in the model. The model itself tries to determine the optimal rates for exploration and production in a competitive and monopolistic market structures. Pindyck's model can be written as follows:

$$\text{Maximize } W = \int_0^{\infty} (pq - C(1)(R)q - C(2)(w)) e^{-rt} \, dt$$

subject to:

$$dR/dt = dX/dt - q$$

$$dX/dt = f(w, X)$$

$$R,\ q,\ w,\ X \geq 0$$

where:

R is the proved reserves;

p is the price of the product;

q is the quantity produced;

X is the cummulative reserve addition;

C(1)(R) is the average production cost;

C(2)(w) is the cost of exploration efforts.

The Average production cost increases as the proved reserve is depleted; the cost of exploration efforts increases as the effort, w, increases. The results of his analysis indicate that if the initial reserve is small, the price path will have a U-shaped pattern, and not a steadily increasing path. This U-shaped path implies that price will be increasing over time in the later stage of resource exploitation. However, the exploration activities help to reduce such price increase. The pattern of optimal exploration activity depends on the initial reserve level and the rate of depletion; the determinant is the trade off between the cost savings in postponed exploration and the cost savings from lower extraction cost and the revenue gains from higher production when exploration results in greater amount of reserves.

M. C. Weinstein and R. J. Zeckhauzer have shown that the competitive rate of extraction is socially optimal while monopolistic rate is suboptimal; however, they indicate that if the net price, defined as the market price less the marginal cost of extraction, has constant elasticity then the competitive rate and the monopolistic rate are equal. While Sweeney's analysis concentrates on the possible bias from optimal competitive allocation; his analysis indicates that percentage depletion allowances, non-internalized externalities associated with extraction, and price controls, may lead to temporal biases. Percentage depletion allowances may lead to an over extraction of resources at the present time; and this current over extraction also occurs in the case of non- internalized externalities; on the other hand, price controls may lead to temporal biases, but their directions can not be predicted without additional information. The rate of change of the excess price seems to be a deterministic factor in predicting the direction of the bias due to price controls. In the case of monopoly, generally monopolies will lengthen the time of extraction until resources are ultimately depleted. However, the shape of the demand function influences the monopolist's behavior as whether to currently under-use or to currently over-use the resources.

It seems that most of the deterministic models of the first class use cost functions or combinations of price and cost functions to determine the optimum path for exploitation of the depletable resources.

Among the probabilistic models are those written by Tracy R. Lewis (1977); Micheal Hoel (1978); Glenn C. Loury (1978); M. C. Kemp and Ngo Van Long (1978); Richard Gilbert (1979); and Arthur J. Robson (1979); and Robert S. Pindyck (1980). Lewis analyzes the exploitation of an exhaustible resource when its price is uncertain. The problem is analyzed in a stochastic framework for monopolistic and competitive market structures. It shows that the rate of extraction varies directly with the willingness of the resource owner to accept risk. Risk-preferring owners will exploit the resource more rapidly than risk-neutral owners, and risk-averse owners will deplete the resource slower than the risk-neutral ones. Also, increasing the discount rate to account for risk will increase the rate of resource use, which according to Lewis' analysis is contrary to the desired policy, namely a slower rate of depletion. Hence, Lewis' analysis is directed towards the effect of attitudes toward risk in the exploitation of an exhaustible resource.

Most interesting is the analysis made by Pindyck; his model postulates that the market demand function and the reserve level are uncertain and that these uncertainties are controlled by stochastic processes with independent increment, called the Ito processes (Bath, 1972; pp.16, 183-184). The market demand function can be written as:

$$p = p(q, t) = y(t) f(q)$$

with $f'(q) < 0$, and $y(t)$ is a stochastic process of the form:

$$dy/y = a\ dt + s(1)\ dz(1)$$

$$dy/y = a\ dt + s(1)\ e(1)(t)\ \sqrt{dt} \quad \ldots(1)$$

and the reserves fluctuate randomly over time according to the stochastic process:

$$dR = -\ q dt + s(2)\ dz(2)$$

$$dR = -\ q dt + s(2)\ e(2)(t)\ \sqrt{dt} \quad \ldots(2)$$

where:

q is the rate of production;

s is the standard deviation;

e is a serially uncorrelated normal random variable with zero mean and unit variance.

The problem is then to maximize the expected value of the sum of the discounted profits as follows:

$$\text{Maximize } E(0) \int_0^T (y(t) f(q) - C(1)(R)) q e^{-rt} dt$$

subject to equations (1), (2), and $R \geq 0$.

Note that $C(1)(R)$ is the average production cost and $C'(1)(R) < 0$; $E(0)$ is the expected value function beginning from $t = 0$.

It is assumed that $E(e(1)(t) . e(2)(t)) = 0$, meaning that the stochastic process controlling the demand and reserves are independent for all t. The conclusions that Pindyck draws from the above model are the following:

(1) The expected rate of change of the competitive price differs from certainty case only if the production cost function C(1)(R) is nonlinear;

(2) In the case of a linear production cost function, the expected rate of change of price is the same as in the certainty case. However, the rate of change of production will be affected by the uncertainty.

Further, Pindyck has incorporated exploration efforts into the model and becomes:

$$\text{Maximize } E(0) \int_0^T (qp(q) - C(1)(R)q - C(2)(w)) e^{-rt} dt$$

subject to:

$$dR = -qdt + s(K)\ dz$$

$$dK = g(w)dt$$

where:

w is the level of exploration activity;

K is the stock of knowledge produced by w;

C(2)(w) is the average exploration cost;

dz is e(t)Vdt.

Further his analysis assumes that as exploration proceeds over time, it becomes more and more difficult to make new discoveries. The rate of discovery, dX/dt, depends upon the level of exploration effort, w, upon the cummulative discovery, X, and upon a parameter b which follows a stochastic process. Functionally, it can be written as:

$$x = dX/dt = f(w,X,b), \text{ with } f(w)>0,\ f(X)<0.$$

The dynamics of b is further specified as:

$$db = s(b)\ dz = s(b)\ e(t)\ V\ dt,$$

so that $E(db) = 0$. This means that the rate of discovery today, given w and X, is exactly known, but it is not known in the future. His model then becomes:

$$\text{Maximize } E(0) \int_0^T (qp-C(1)(R)-C(2)(w))e^{-rt}dt$$

subject to:

$$dR/dt = dX/dt - q$$

$$dX/dt = f(w,X,b)$$

$$db = s(b)e(t)Vdt$$

$$R,\ q,\ w,\ X \geq 0.$$

It is assumed that $C'(1)(R)<0$, $C'(2)(w)>0$, $C''(2)(w)>0$, and that the marginal discovery cost, $C'(2)(w)/f(w)$, increases with w. Note that in the above model, the demand uncertainty is assumed away; it is argued that the demand uncertainty has no effect on the dynamics of the price and the exploration as long as they are uncorrelated with the fluctuations in b (Pindyck, 1980). Several conclusions have been drawn from his analysis, among which are:

(1) When exploration is used to reduce reserve uncertainty:

(a) Extraction cost constant: reserve uncertainty has no effect on the expected rate of change in price;

(b) Extraction cost rises as R falls: (i) reserve uncertainty has no effect on the expected rate of change when extraction cost is linear; (ii) expected level of exploration effort rises overtime, falling abruptly when production stops;

(2) When exploration is used to accumulate reserves:

(a) Linear discovery function: uncertainty has no effect on either the price or exploration effort; (b) Nonlinear discovery function: the effect of uncertainty depends on the convexity of the marginal product of the exploration function.

Other stochastic models of resource exploration are those by Arrow and Chang (1978) and by Deshmukh and Pliska (1978) in which discrete increments of reserve discoveries occur stochastically via a Poisson process in proportion to the level of exploratory activity. Other probabilistic models that have been developed by authors mentioned previously, do not use Markov processes. Some other authors may use tree decision processes, which are based on certain probabilistic distribution functions or on subjective probabilistic values.

2.2 THE MAIN DIFFERENCES TO THE MODEL

The main differences between the model that will be developed in this paper, referred to as the model, and those using stochastic processes of the first class are:

(1) The uncertainty involved in the model is defined as the geologic uncertainty; one that will be encountered in any exploration activity of a mineral resource concerning the geologic information that is obtained from such activity. Pindyck's model stresses the future uncertainties involved in mineral discoveries, which may in-

clude not only geologic uncertainty but also other uncertainties due to the uncertain occurrence of future events;

(2) The sample space of the uncertainty of the model is the exploration activities of an exploration project, while others cover the whole resource base of the area. This means that in others, the stochastic processes are applied to the increment of mineral discovery of the whole resource base in that area; on the other hand, the stochastic process used in the model is intended to control the activities of an exploration project or a venture. Here, a venture is defined to be an exploration effort over a given limited area in which legal permission or rights have been granted for such effort to take place in that area.

(3) In particular to Pindyck's model, the model uses a Markov chain or a Markov process to be the stochastic process that quantifies the uncertainty. The Pindyck's model, as previously metioned, uses the Ito process.

As compared to other probabilistic models not using stochastic process, the difference is apparent, namely stochastic versus non-stochastic processes. Elaboration on the differences between the stochastic process used in the model and the tree decision process, which is usually used in solving the uncertainty in exploration, will be done in section 2.4 below.

2.3 THE UNCERTAINTY AND THE MCKELVEY DIAGRAM

In order to distinguish the nature and type of the uncertainty involved in the model, it is best to refer to a diagram, called the McKelvey diagram, described in Figure 2-1. This diagram basically classifies a mineral resource into several categories based on its economic feasibility and the degree of geologic certainty. The vertical axis of this diagram indicates the economic feasibility and the horizontal axis indicates the degree of geologic assurance. Based on the economic feasibility creterion, a mineral resource can be classified into (i) economic resource, and (ii) subeconomic resource. The latter can further be divided into paramarginal and submarginal. Based on the geologic assurance creterion, a mineral resource can be categorized into (i) identified resource, and (ii) undiscovered resource. The former can further be classified into measured, indicated, and inferred resources, while the latter into speculative, and hypothetical resources. The definitions for these terminologies are given in Appendix A. It is important to note that within the class of undiscovered resources, the geologic evaluations establish the degree of certainty of the existence of the resource, while within the class of identified resources, the geologic evaluations provide the degree of accuracy of estimates.

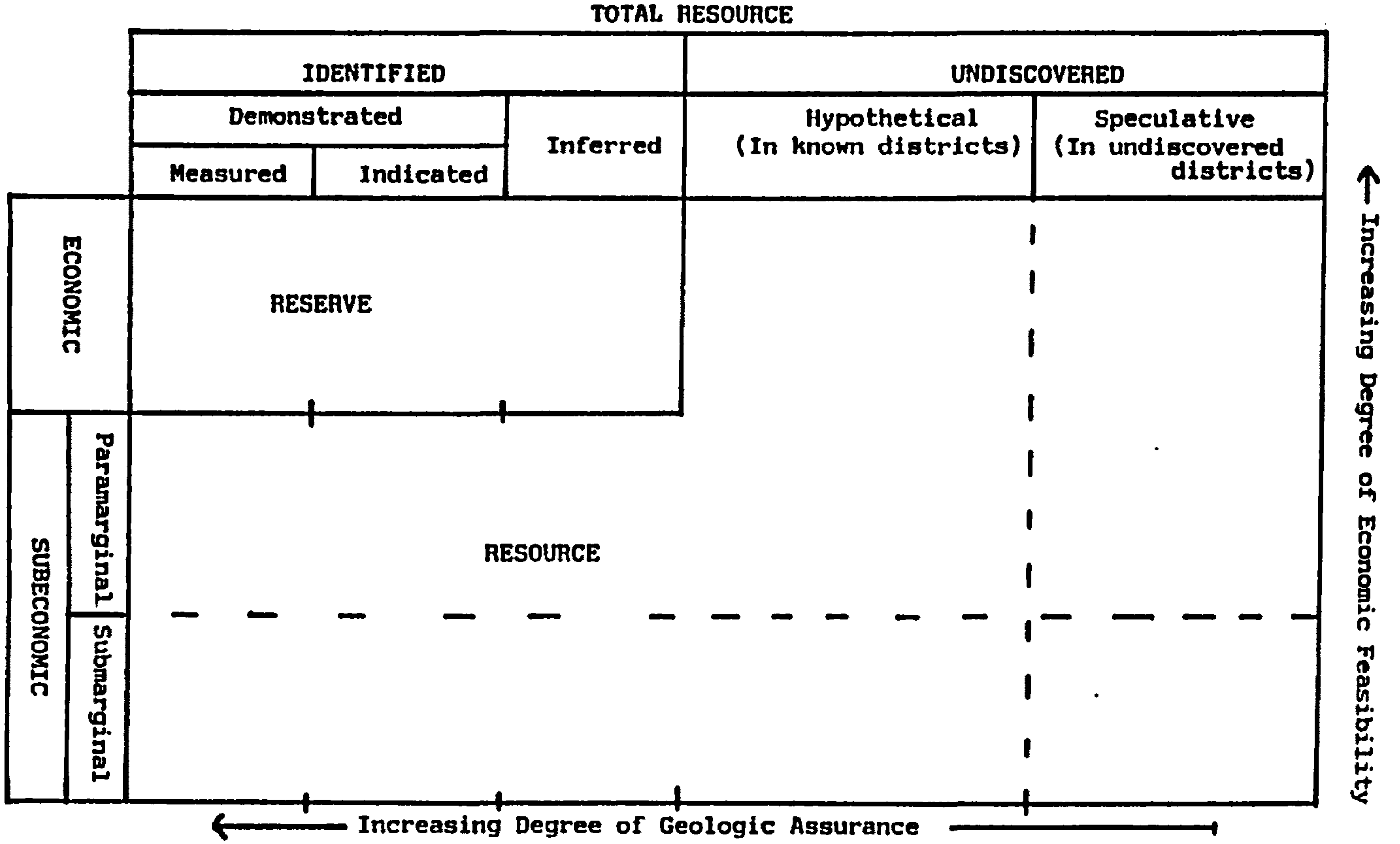


Figure 2-1: The McKelvey Diagram

In general, exploration uncertainties will mostly be encountered in the determination of undiscovered resource. However, geologic evaluations on identified resources are also subject to some degree of uncertainty due to the fact that no absolutely complete geologic information about the mineral resource is economically feasible. Such complete information will usually be obtained only after the mineral resource has been completely depleted.

In relation to the McKelvey diagram, the analysis will only be concerned with the portion of the resource that is economic and with the exploration activities to discover such resource. These activities include those that determine speculative resource and transfer it into a hypothetical one and further into an identified resource or a reserve. Within the identified resource, further exploration can transfer an inferred reserve into an indicated one, and further into a measured reserve. Since the analysis is only concerned with resources that are economically feasible at the time of determination, any other determinant, such as technological progress, increase in the price level, or change in the economic environment, that can change a sub-economic resource into an economic one, will not be incorporated into the model.

Hence, it is clear that the uncertainty involved in determining a mineral resource will be reflected on the degree of geologic assurance. In general, it can be said that the more intensive the exploration activities, the greater is the degree of geologic assurance. However, because the information gained from any exploration activity is subject to some degree of uncertainty, the decision to undertake an exploration activity which is generally based on the current geologic information available, will be subject to uncertainty too. It is this kind of uncertainty that will be discussed in the model and it will be argued that the Markovian approach is superior to other analytical techniques for modeling decision making involving this kind of uncertainty.

2.4 THE DIFFERENCES TO THE DECISION TREE PROCESS

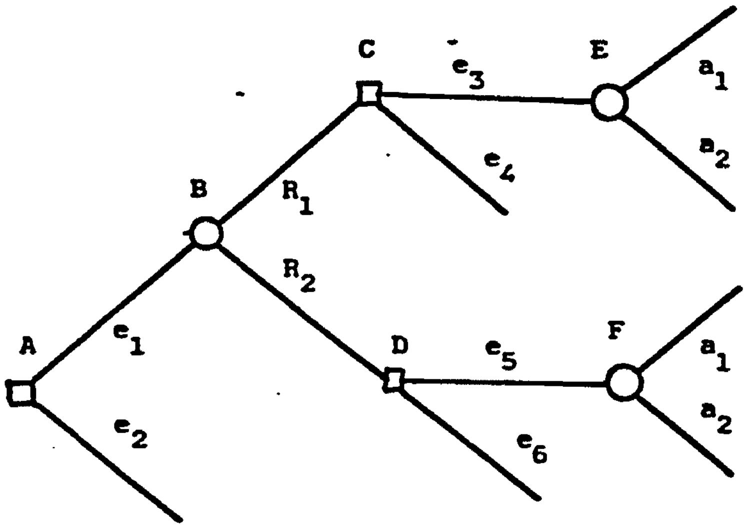
It is important to elaborate on the differences and the advantages between the stochastic process used in this model and the probabilistic approach to solve exploration uncertainties. The probabilistic approach which is usually used in exploration is the decision tree process (Raiffa, 1970; pp.1-50; Howe, 1979; pp.212-220). In the decision tree process, choices can either be controlled by the decision makers or by chance. It is usual to define a decision fork to be the point where choice is controlled by the decision makers and define a chance fork to be the point where choice is controlled by chance (Raiffa, 1970, pp.10-11). The example in Figure 2-2 shows a decision tree diagram; note that A, C, and D are decision forks, and B, E, and F are chance forks. Suppose the branches (e(1), R(1), e(3), a(1)) and (e(1), R(1), e(3), a(2)) are to be analyzed. Then the probability that a large productive well (a(1)) or a small productive well (a(2)) is found becomes a conditional probability in which the chance fork B has an outcome of R(1). Using Bayes' theorem the conditional probability can be expressed as:

$$P(a(1)|R(1)) = P(R(1) \text{ and } a(1))/P(a(1))$$

$$P(a(1)|R(1)) = P(R(1)|a(1))P(a(1))/(P(R(1)|a(1))P(a(1))$$
$$P(R(1)|a(2))P(a(2)))$$

and

$$P(a(2)|R(1)) = P(P(1)|a(2))P(a(2))/(P(R(1)|a(1))P(a(1))$$
$$P(R(1)|a(2))P(a(2)))$$



Legend: A, C, D = decision fork;
B, E, F = chance fork;

e_1: decision to drill;
e_2: decision not to drill;
R_1: the possibility of favorable outcome;
R_2: the possibility of unfavorable outcome;
e_3: decision to develop;
e_4: decision not to develop;
a_1: large productive well;
a_2: small productiverwell.

Figure 2-2: A Decision Tree Process

In general, when the branches of the chance fork E are a(1), a(2),...., a(n), then the conditional probability becomes:

$$P(a(j)|R(1)) = P(R(1)|a(j))P(a(j))/(\sum_{j=1}^{n} P(R(1)|a(j))P(a(j))$$

And it can be directly seen that $\sum_{j=1}^{n} P(a(j)|R(1)) = 1$.

Although the sum of the conditional probabilities is equal to one, a decision tree as described above can not be viewed as a Markov chain. In order to see this, one can analyze the previous example and let the tree diagram be drawn by omitting the decision fork as shown in Figure 2-3. Then the following conditional probability matrix can be formed:

$$P=\begin{bmatrix} P(R(1)|R(1)) & P(R(2)|R(1)) & P(a(1)|R(1)) & P(a(2)|R(1)) \\ P(R(1)|R(2)) & P(R(2)|R(2)) & P(a(1)|R(2)) & P(a(2)|R(2)) \\ P(R(1)|a(1)) & P(R(2)|a(1)) & P(a(1)|a(1)) & P(a(2)|a(1)) \\ P(R(1)|a(2)) & P(R(2)|a(2)) & P(a(1)|a(2)) & P(a(2)|a(2)) \end{bmatrix}$$

If R(1) is denoted as 1, R(2) as 2, a(1) as 3 and a(2) as 4, then the above matrix can be rewritten as:

$$P=\begin{bmatrix} P(11) & P(12) & P(13) & P(14) \\ P(21) & P(22) & P(23) & P(24) \\ P(31) & P(32) & P(33) & P(34) \\ P(41) & P(42) & P(43) & P(44) \end{bmatrix}$$

This matrix is similar to a transition matrix of a Markov chain. Note that a transition matrix of a Markov chain is defined as a square matrix whose elements represent transition probability values and that the sum of the elements of each row must be equal to one (Cinlar, 1975).

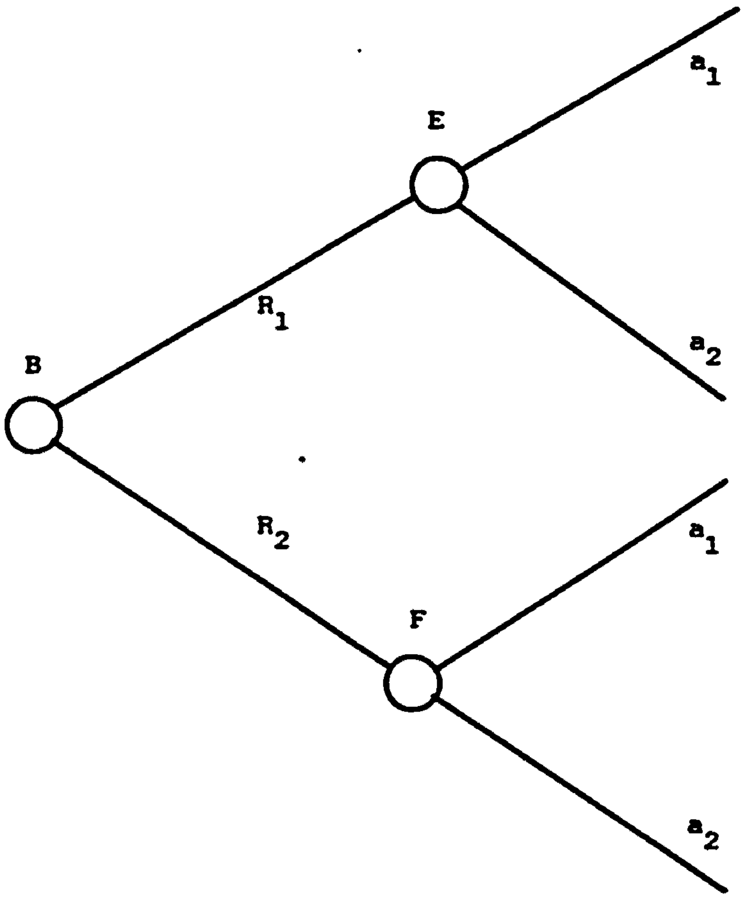


Figure 2-3: A Decision Tree

However, the sum of the conditional probabilities of every row in the above matrix is not equal to one. The value of P(R1|R1), interpreted as the probability that R(1) will occur if R(1) is true, is equal to one, and the value of P(R(2)|R(1)) is equal to zero because R(2) and R(1) are disjoint events. Since P(a(1)|R(1)) and P(a(2)|R(1)) are each not equal to zero, then the sum of the elements of the first row is not equal to one. By the same token, the values of P(R(2)|R(2)), P(a(1)|a(1)), and P(a(2)|a(2)) are each equal to one and the values of P(R(1)|R(2)), P(a(1)|a(2)), and P(a(2)|a(1)) are each equal to zero, because a(1) and a(2) are also disjoint events. However, the values of P(R(1)|a(1)), P(R(1)|a(2)), P(R(2)|a(1)), and P(R(2)|a(2)) are not equal to zero, hence, the sum of the conditional probabilities of the second row, the third row, and the fourth row, each can not be equal to one. Consequently, the above conditional matrix is not a Markov matrix. This is the first difference.

The second difference between the stochastic process, namely the Markov chain and the Markov process, and the decision tree process is that in the former processes the conditional probability is independent of the past history of the process. In a Markov chain, the chain can be mathematically expressed as:

$$P(X(n+1)=j|X(0)=a,\ldots\ldots X(n)=i) = P(X(n+1)|X(n)=i)$$

where X is the random variable, and n is the total steps accomplished by the chain and a,...i,j are the states. The above expression states that the probability that the chain will be in state j after (n+1) steps given that the chain was in state a at step 0, and in state b at step 1, and finally in state i at step n is equal to the probability that the chain will be in state j given that it is in state i at step n. The value of the conditional probability is not influenced by the fact that the chain was in different states before it reached state i. Also for a time homogeneous Markov chain, $P(X(n+1)=j|X(n)=i) = P(ij)$, which is independent of n (Cinlar, 1975). The term P(ij) is usually referred to as the transition probability; it is the probability that the chain will make a transition from state i to state j. For a Markov process, the transition probability can be written as $P(Y(t+s)=j|Y(t)=i) = P(ij)(s)$ and is also independent of the past history of the process. Unlike the Markov chain and the Markov process, the conditional probability of a decision tree does depend on the conditional probabilities of the previous branches. Using Bayes' theorem, the conditional probability of a certain branch can be related to the conditional probabilities of previous branches on its path. In order to show this, one can look at the following problem described by Raiffa (1970, pp.7-21; 36-38):

A collection of 1000 urns consists of 800 urns marked a(1) and 200 urns marked a(2). Each urn contains red balls, denoted as R, and black balls, denoted as B; a(1) urn contains 4 R and 6 B while a(2) urn contains 9 R and 1 B.

The following two questions are then asked:

(1) What is the probability of an R ball being taken from an urn a(1) ?

(2) Suppose the first draw is an R ball, denote this as R(1), and after a replacement, the second draw is also an R ball, denote this as R(2), then what is the probability that R(2) comes from an urn a(1) ?

For the first question, the probability tree diagram shown in Figure 2-4 can be formed. Note that A, B, and C are chance forks. It is then necessary to calculate the conditional probability P(a(1)|R). From the problem, the following conditional probabilities are known: P(a(1)) = 0.8; P(a(2)) = 0.2; P(R|a(1)) = 0.4; and P(R|a(2)) = 0.9. Since:

P(R|a(1)) = P(R and a(1))/P(a(1)) and

P(R|a(2)) = P(R and a(2))/P(a(2)), then:

P(R and a(1)) = P(R|a(1))P(a(1))

P(R and a(1)) = 0.4 x 0.8 = 0.32

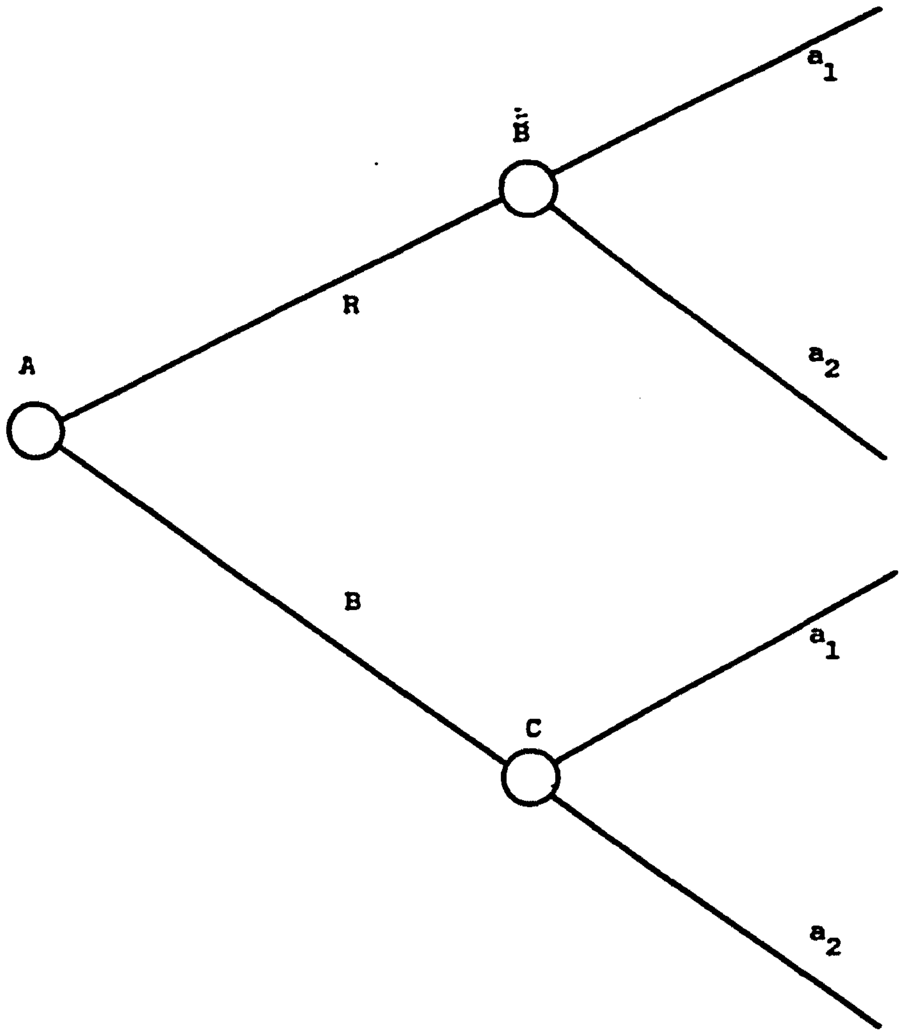
P(R and a(2)) = P(R|a(2))P(a(2))

P(R and a(2)) = 0.9 x 0.2 = 0.18

Since P(R) = P(R and a(1))+P(R and a(2)), then:

P(R) = 0.32 + 0.18 = 0.5

Hence, P(B) = 1-0.5 = 0.5



Legend: A, B, and C = chance fork.

Figure 2-4: The Probability Tree

By using Bayes' theorem, $P(a(1)|R)$ can be computed as follows:

$$P(a(1)|R) = P(R|a(1))P(a(1))/(P(R|a(1))P(a(1)) + P(R|a(2))P(a(2)))$$

$$P(a(1)|R) = 0.4 \times 0.8 /(0.4\times0.8+0.9\times0.2) = 0.64$$

So the probability that an R ball is taken from an urn a(1) is 0.64.

For the second question, the probability tree diagram shown in Figure 2-5 can be formed. Note that A, B, C, D, E, F, and G are chance forks. It is necessary to compute P(a(1) if R(1) and R(2)); this is the probability of a(1) if (R(1) and R(2)) are the branches. Note that the probability of the union (a(1) if R(1)) and R(2) is written as P(a(1) if R(1), and R(2)). Then:

$$P(a(1),R(2) \text{ and } R(1))=P(a(1))\times P(R(2) \text{ if } a(1))\times P(R(1) \text{ if } R(2) \text{ and } a(1))$$

$$P(a(1),R(2) \text{ and } R(1)) = 0.8\times0.4\times0.4 = 0.128$$

This is the path probability of (a(1),R(2),R(1)). Since there is a replacement, then:

$$P(R(1) \text{ if } R(2) \text{ and } a(1)) = P(R(2) \text{ if } a(1))$$

P(a(1),R(2) and R(1)) = P(R(1),R(2) and a(1)), namely the path probability of (a(1),R(2),R(1)) is equal to the path probability of (R(1),R(2),a(1)). From the first problem, P(R(1)) = 0.5 then:

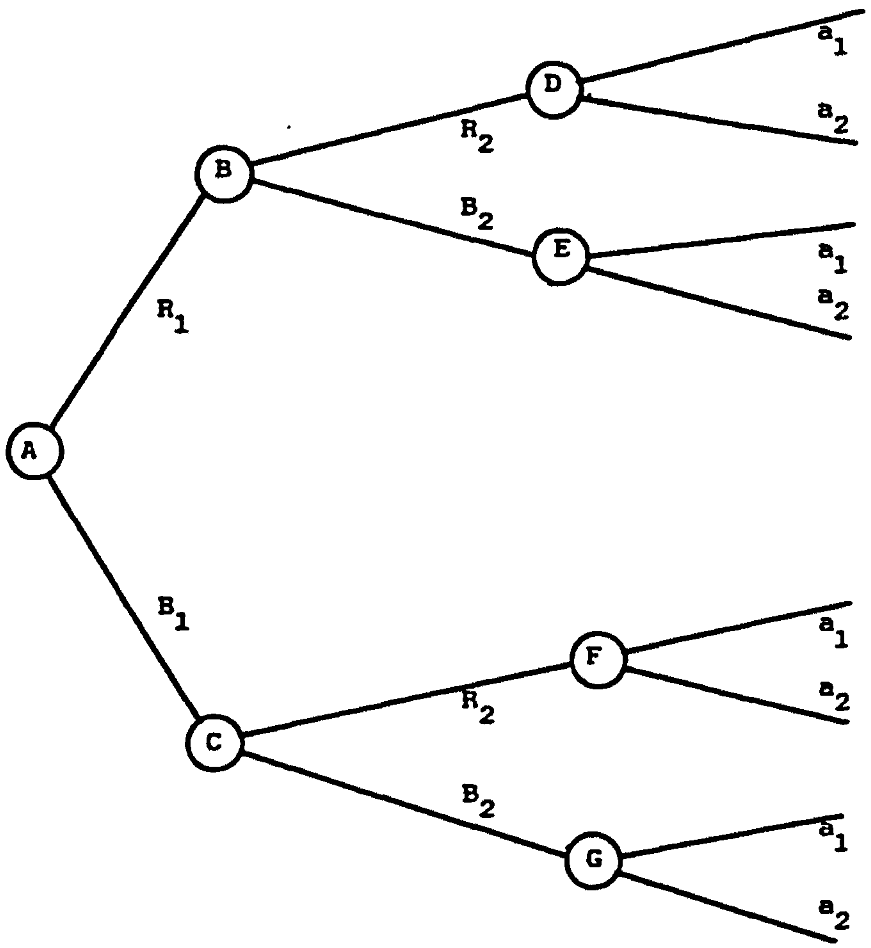
$$P(R(1),R(2) \text{ and } a(1)) = P(R(1))\times P(R(2) \text{ if } R(1))\times P(a(1) \text{ if } R(1) \text{ and } R(2))$$

$$P(R(2) \text{ if } R(1))\times P(a(1) \text{ if } R(1) \text{ and } R(2)) =$$

P(R(1),R(2) and a(1))/P(R(1)) = 0.128/0.5

= 0.256

One can look at the path (R(1),R(2),a(1)) of the tree diagram; the above value is the probability of the union of (R(2) if R(1)) and a(1) or P(R(2) if R(1), and a(1)) = 0.256.



Legend: A, B, C, D, F, G = chance fork.

Figure 2-5: The Probability Tree

From the path (a(2),R(2),R(1)) one obtains:

$$P(a(2),R(2) \text{ and } R(1)) = P(a(2)) \times P(R(2) \text{ if } R(1)) \times P(R(1) \text{ if } R(2) \text{ and } a(2))$$

$$P(a(2),R(2) \text{ and } R(1)) = 0.2 \times 0.9 \times 0.9 = 0.162$$

Since, $P(a(1),R(2) \text{ and } (R1)) = P(R(1),R(2) \text{ and } a(2))$ and $P(R(1)) = 0.5$, then:

$$P(R(1),R(2) \text{ and } a(2)) = P(R(1)) \times P(R(2) \text{ if } R(1)) \times P(a(2) \text{ if } R(1) \text{ and } R(2))$$

$$P(R(2) \text{ if } R(1)) \times P(a(2) \text{ if } R(1) \text{ and } R(2)) = P(R(1),R(2) \text{ and } a(2))/P(R(1)) = 0.162/0.5 = 0.324$$

As noted before, this value is equal to the probability of the union (R(2) if R(1)) and a(2), by looking at the path (R(1),R(2),a(2)). Hence:

$$P(R(2) \text{ if } R(1), \text{ and } a(2)) = 0.324$$

Since:

$$P(R(2) \text{ if } R(1)) = P(R(2) \text{ if } R(1), \text{ and } a(1)) \quad P(R(2) \text{ if } R(1), \text{ and } a(2))$$

Then : $P(R(2) \text{ if } R(1)) = 0.256 + 0.324 = 0.58$

Hence: $P(a(1) \text{ if } R(1) \text{ and } R(2)) = 0.256/0.58 = 0.44$

From the above computation, one can see that the first problem gives $P(a(1)|R) = 0.64$ and the second problem gives $P(a(1)|R(1) \text{ and } R(2)) = 0.44$. It is clear that the value of the conditional probability depends on the conditional probabilities at each chance fork along its path; in the second problem, it is along the parh (R(1),R(2),a(1)) and in the first problem, it is along the path (R,a(1)). Hence, stated generally, the conditional probability of a certain branch at a certain chance fork depends not only on the conditional probability of previous immediate branch but also on that of other previous branches along its path. This condition distinguishes the difference between the tree decision process and the Markov chain or the Markov process.

A last difference that is worth mentioning is that the tree decision process has at least one decision fork at which branches are divided by a deliberate decision action made by the decision makers. This means that once a certain branch has been chosen by the decision fork, other branches at that decision fork are omitted. In a Markov chain or in a Markov process, such omission is not necessary because all branhes are controlled by chance. There are transitions in a Markov chain or process that have similar behavior to decision forks in the tree decision process, for example the transition from a transient state to a recurrent state, but such transitions still have probability values. The decision fork, on the other hand, has no probability values, and in fact they are omitted from the computational procedure. This difference seems to have important bearings particularly on the evaluation of the whole system; but it becomes less important when a certain path of a particular decision branch is concerned. In such case, the decision fork may be omitted and all branches on this particular path are controlled by chance only. All these differences will become apparently important when discussing the Markovian wxploration model in Chapter III.

2.5 THE MODELS OF THE SECOND CLASS CATEGORY

With respect to the models of the second class category, namely those with growth aspects, numerous writings have been done in this field. Among the important writings are those done by Intriligator (1971), Solow (1974), Dasgupta and Heal (1974), Stiglitz (1974), Hu (1978), and Smith and Krutilla (1979). The model that will be looked at in this analysis is the model by Intriligator. His model is a two sector growth model, one sector produces capital goods and the other sector produces consumption goods. Expressed in terms of the capital-labor ratio and the wage-rental, his model maximizes the present value of the consumption per capita and can be written as:

$$\text{Maximize } J = \int_0^T e^{-rt}((k-k(1))/(k(2)-k(1)))f(2)(k(2))dt$$

subject to:

$$dk/dt = ((k(2)-k)/(k(2)-k(1)))f(1)(k(1))-ak$$

$$k(t(0)) = k(0)$$

$$k(j) = k(j)(v), \; j = 1, 2$$

$$v(2)(k) \leq v \leq v(1)(k)$$

$$k, k(1), k(2), v \geq 0$$

$v(t)$ is piecewise continuous.

where:

k is the overal capital-labor ratio;

$k(j)$ is the capital-labor ratio in sector j, $j = 1, 2$ and sector 1 produces capital goods and sector 2 produces consumption goods;

$v(t)$ is the wage-rental ratio; $v(j)$ is the wage-rental ratio in sector j, $j = 1, 2$;

a is a constant representing the sum of the capital good depreciation rate and the growth rate of the labor force.

$f(j)$ is the production function per capita of sector j, $j = 1, 2$.

The model has k as the state variable and v as the control variable.

The Hamiltonian for the above model is:

$$H = e^{-rt}(((k-k(1))/(k(2)-k(1)))f(2)(k(2))$$
$$q((k(2)-k)/(k(2)-k(1)))f(1)k(1)-ak))$$

where: $q(t)$ is the costate variable.

The canonical equations are:

$$dk/dt = ((k(2)-k)/(k(2)-k(1)))f(1)(k(1))-ak \text{ and}$$

$$dq/dt = (a+r)q-f(2)(k(2))/(k(2)-k(1))$$
$$qf(1)(k(1))/(k(2)-k(1))$$

The balance growth path of the above problem is given by the following solution:

$$f'(k(1)(v^*)) = a+r$$

$$k^* = k(2)(v^*))f(1)(k(1)(v^*))/(f(1)(k(1)(v^*))$$
$$a(k(2)(v^*))-k(1)(v^*)))$$

where k^* and v^* are the equilibrium values for the capital-labor ratio and for the wage-rental ratio. The equilibrium will be optimally maintained forever once it is obtained (Intriligator, 1971, pp.426-427). Figure 2-6 and 2-7 show the phase solution for the model. In Figure 2-6, where $k(2)>k(1)$, the balance growth path follows along the horizontal line $v = v^*$, where there is no change in v, and along the curve $k = k(v)$ where there is no change in k. This curve k is defined as:

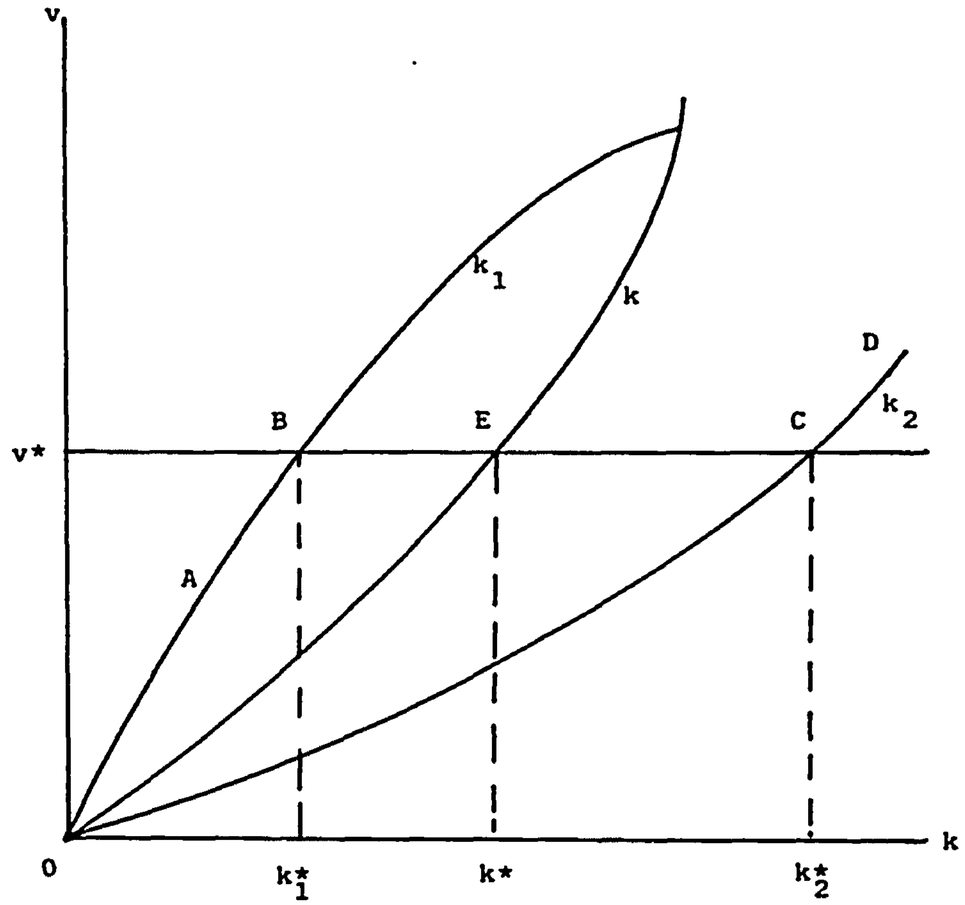
$$k = k(2)f(1)(k(1))/(f(1)(k)+a(k(2)-k(1)))$$

The horizontal line intersects with the curve k at the balance growth equilibrium, where $v = v^*$ and $k = k^*$. Intriligator gives the following interpretations for this balance growth path:

(1) If the initial stock of capital per capita is smaller than $k(1)^*$, then the economy specializes initially in the production of capital good, moving along the curve $k(1)$, increasing both k and v;

(2) When k reaches $k(1)^*$, then $v = v^*$; beyond this point, the economy will optimally produce capital goods as well as consumption goods, keeping the wage-rental ratio constant at v^*, and the economy approaches the balance growth equilibrium.

The reverse will happen if, initially the stock of capital per capita is greater than $k(2)^*$. It will initally produce consumer goods only, until it k reaches down at $k(2)^*$; afterward, both goods are produced at constant wage-rental ratio, at $v = v^*$. The economy then approaches the balance growth equilibrium at (k^*, v^*).



Legend: The Balanced growth path: ABE and DCE

Figure 2-6: The Phase Diagram, k'(2) > k'(1).

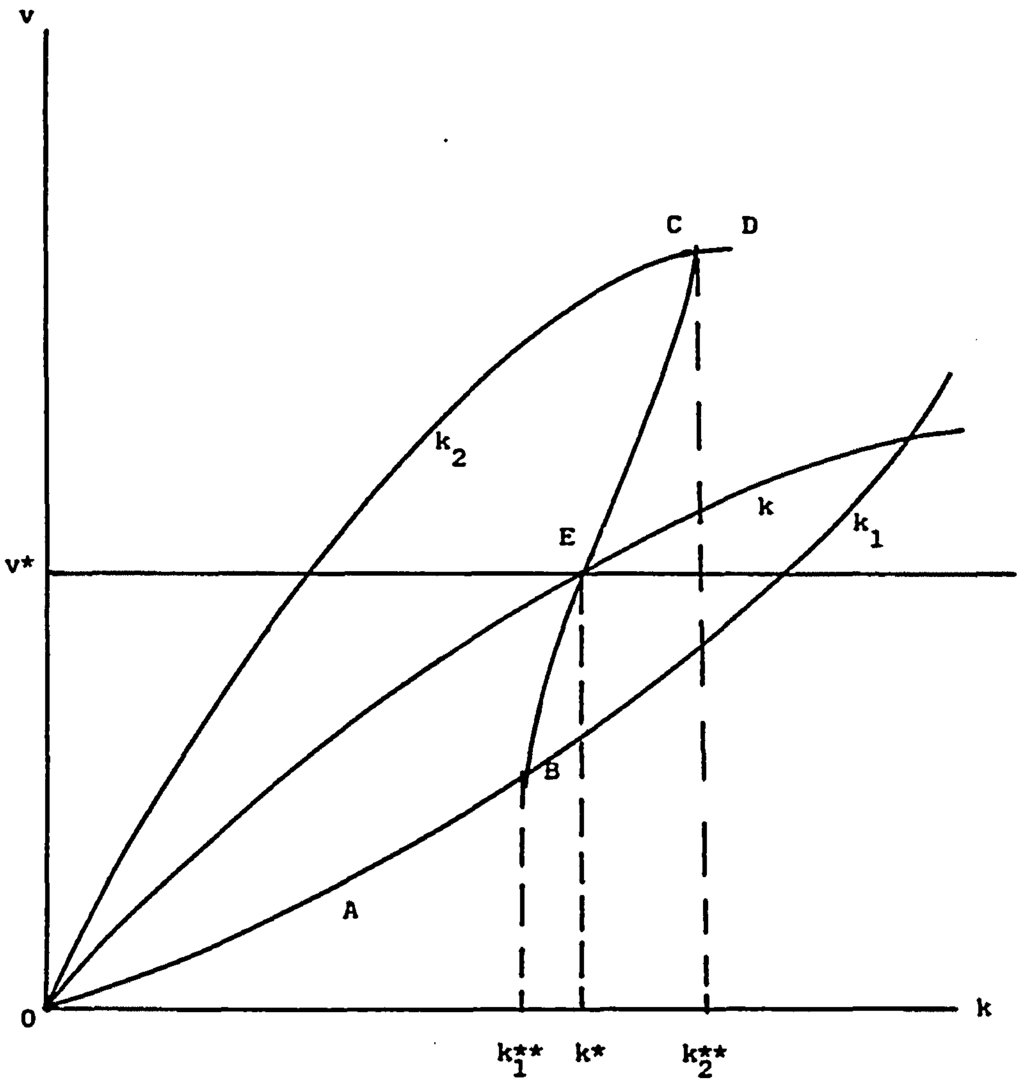
Source: Intriligator, 1971, p.427.

Figure 2-7 shows the phase plane solution when $k(1) > k(2)$. The conclusions that are drawn are:

(1) If initally the stock of capital per capita is smaller than $k(1)^{**}$, then the economy specializes initially in the production of capital goods, moving along the curve $k(1)$, increasing both k and v;

(2) Passing $k(1)^{**}$, the economy moves along a stable nonspecialized path, approaching the balance growth equilibrium at (k^*, v^*).

If initially the stock of capital is greater than $k(2)^{**}$, then the economy produces consumer goods, moving along the curve $k(2)$, reducing k and v. At points where $k < k(2)^{**}$, the economy moves along a stable nonspecialized path, reducing further k and v, and approaches the balance growth equilibrium at (k^*, v^*).



Legend: The balanced growth path: ABE and DCE

Figure 2-7: The Phase Diagram, k(1) > k(2).

Source: Intriligator, 1971, p.429.

An extension for the above model may include the following:

(1) The addition of another sector that produces a mineral resource commodity. This commodity is then used as input to produce capital goods;

(2) There will be uncertainty involved in the production of the mineral commodity. This uncertainty takes the form of an uncertainty in the mineral resource discovery;

(3) The uncertainty is controlled by a Markov process;

(4) A Cobb Douglas production function is assigned to each sector. These production functions are subject constant returns to scale.

A formulation for the above extension will be discussed in later chapter.

Chapter III

THE ANALYSIS OF THE MARKOVIAN EXPLORATION MODEL

This chapter attempts to analyze and formulate a Markovian Exploration model and in doing so shows why it is more appropriate than other uncertainty technique. It will analyze the following:

1. The Exploration Network;
2. The Markov Chain and the Markov Process as applied to Exploration;
3. The Net-reward Function and Its Use in the Markovian Exploration Model.

Each of the above categories is inter-related to each other and together they will provide the formulation of the Markovian exploration model. The discussion will proceed by analyzing (1) first, followed by (2) and (3). Much of the mathematical detail may be found in Appendix B.

3.1 <u>THE EXPLORATION NETWORK</u>

The discussion in this section proceeds by defining the uncer tainty, the resource classification, and the exploration net work. This explanation includes a brief elaboration upon the exploration activities which further provides a general picture of the exploration network.

The use of several geological terminologies in the explanation of the exploration activities is inevitable; however, several definitions for some of the most important terminologies are given. Finally, the analysis provides a diagrammatic representation of the exploration network.

3.1.1 <u>The Uncertainty and the Resource Classification</u>

In chapter 2 some explanation has already been given for the type and the nature of the uncertainty that is to be encoun terred in the analysis. A mineral resource classification diagram, called the Mc Kelvey diagram, is used to clarify the explanation.

The uncertainty involved is the geologic uncertainty; such uncertainty may be the result of the unknown occurence of mineral deposits. These deposits lie hidden beneath the earth's surface so that they are difficult to locate and to examine in a way which provides one with accurate knowledge of their extent and quality. An exploration, even with the use of adequate methods and equipment, may only yield results with a certain degree of certainty. The geologic information obtained from the exploration can be used to estimate the extent, the amount, and the quality of the deposit with some degree of uncertainty. It is this type of uncertainty that the analysis is concerned about.

Another source of uncertainty, but one with which the analysis does not deal, is the constant changing of the specifications of recoverable minerals. These changes are the result of technological advancement which changes the cost levels for exploitation and processing such that low grade mineral deposits, or deposits that are remotely located can be profitably exploited and processed. The knowledge about the occurence of such technological advancement in the future is usually incomplete and insufficient. The prediction of such occurence contains some degree of uncertainty.

The McKelvey diagram that is shown in Chapter 2 indicates a classification of a resource based on two parameters, namely (1) the degree of geologic certainty, and (2) the feasibility of economic recovery. The definitions of the terminologies used in that diagram are given in Appendix A. It should be noted that the word known in the definitions for reserve implies a certain degree of uncertainty. This word refers to the location, the size, and the quality of the deposit. The more intensive the exploration, the less is the degree of uncertainty. Within the reserve category the degree of certainty measures the accuracy of reserve determination. On the other hand, within the category of undiscovered resource, the degree of accuracy measures the accuracy of the occurence of a mineral deposit.

This analysis pertains only to the resource which is economically feasible at the time of determination. Such resource includes the identified as well as the undiscovered one; it does not include the paramarginal or the submarginal resource. Hence, the discovery of a mineral deposit that is too small or too remotely located to be economically feasible is not considered in the analysis. Referring to the Mc Kelvey diagram, the part that is to be analyzed is the following:

indentified resource	undiscovered resource
reserve	resource

In the case of petroleum resource, the determination of the quantity of the undiscovered one is done by the volumetric method. It is basically an extrapolation from a representative estimated mean concentration with unit volume of sediments to the volume of interest (McKelvey, 1973; pp.14-17). This volumetric method for quantifying undiscovered resource will be referred again in Chapter 4, in the discussion on the depletion model.

The exploration that is to be discussed in this chapter covers the exploration for the resource of undiscovered category. Hence, if such exploration confirms the existence of a petroleum resource, and if it further reveals the quantity and the quality, then the resource can be reclassified as reserve or identified resource.

3.1.2 The Explanation of the Exploration Network

The exploration network is defined as the organization and the inter-relationship of all exploration activities that form the exploration project needed to search for a particular mineral resource in commercial quantity. When the mineral resource is petroleum, then an exploration project to search for petroleum is an organized attempt intended to find petroleum in commercial quantities (Hamner and Loftis, 1961; pp.1-2). Hence, if such an attempt results in the discovery of petroleum in less than the commercial quantity, the exploration project is usually terminated. In this sense, the firm conducting the exploration is viewed as to have a profit maximizing behavior.

An exploration project usually consists of several or many exploration activities. For example, in the case of petroleum, the seismic reflection work is one of the exploration activities for finding petroleum. For further discussion, an exploration project is meant to be confined to a single venture. A venture is defined as the exploration activity efforts for finding a particular mineral resource over a given limited area or district in which legal permission or rights have been granted for such activities to take place in that area or district.

The types of the exploration activities that form the exploration project depend upon the kinds of minerals and deposits to be found. For example, the exploration activities for searching petroleum resource are different from those for searching uranium resource. Also a near surface deposit requires different exploration activities than a deep seated deposit. In fact these activities may change, not only form one type of mineral resource to another, but also from one district to another. This means, that even for a particular mineral resource, the exploration activities may differ from one deposit to another, if these deposits are not found in the same geological district; particularly for petroleum resource. The exploration activities for petroleum resource in known district, where established petroleum fields exist, are different than those in unknown district, where there are no known fields and where probably no wells have ever been drilled. Referring these explanations to the Mc Kelvey diagram which has been discussed earlier, the former is related to the hypothetical petroleum resource and the latter to the speculative one. In view of the above explanations, it is then impossible to give a general exploration network representation that will apply for every mineral resource. For this reason, this analysis is limited only to the discussion of exploration for petroleum resource. So, from this point on, the discussion is referred to this particular resource.

For the exploration in a known district, most of the petroleum companies have already the current subsurface geologic maps of the district. The project may then be restricted to geophysical exploration, such as reflection seismic activity to provide a more detailed subsurface map of the structure in greater scale, and possibly core drilling activity. The data obtained from these activities are combined with previous geologic interpretation of the district. However, in an unknown district, the exploration activities are conducted to obtain first the subsurface geologic map of the district. Depending on the geologic conditions of the district, such exploration activities vary in their scope. Usually, an interpretation of the subsurface geological conditions from geophysical data and studies of well logs and others, is determined by comparing them to the subsurface data with analogous geologic conditions that have been obtained from surface observations in the immediate area or in other district (Smith, and Wengerd, 1961; p.1). The discussion that follows concentrates on the analysis of the exploration network for the unknown district only. It first explains the exploration activities that form such exploration network, then followed by the diagrammatic representation of the network.

3.1.3 The Surface Geological Survey

The first phase of the exploration effort is in general the surface geological survey. It makes a comprehensive investigation of the geological phenomena exposed on the surface in the area under consideration. In particular, surface geology becomes of prime importance in new or unknown district, such as in large, undrilled structure. In such a district, the geologic conditions within the basin must be assumed, based on the exposed sections of the basin around the perimeter of the basin. Good surface structural data will establish trends and will serve as a guide for planning future seismograph work in that district without performing costly seismic reconnissance work (Smith, and Wengerd, 1951; p.1).

The stages of the field surface geology consist of:

(i) Reconnaissance geology;

(ii) Detailed reconnaissance geology;

(iii) Detailed geology. This includes laboratory analysis of rock samples.

The most important pre-field work that is usually done before actual field work is started may include:

(i) Examination of previous work done by the company;

(ii) Literature search. This provides information on the section to be surveyed;

(iii) Depending on the scope of work, air photographs must be prepared. Rapid reconnaissance of large area requires different types of air photographs than detailed geological surveys. Air photography is considered necessary for unknown districts where such activity has never been done before;

(iv) Sometimes reconnaissance flights to check the accessibility of the district are necessary.

Thus the entire surface geological survey will include the above pre-field work as well as the field surface geology itself. It should be noted that all data and samples taken from these surveys are analyzed in the laboratory.

3.1.4 The Geophysical Survey

The next exploration phase after the surface geology is the geophysical survey. The geophysical survey is used in petroleum exploration to obtain information about the possible location of subsurface petroleum accumulations and help determine the selection of drilling sites (Dyk, 1961; pp. 1-2). The geophysical exploration may include:

(i) Magnetic survey;

(ii) Gravity survey;

(iii) Seismic survey.

These methods are indirect, namely they do not determine the presence or absence of petroleum but rather measure parameters that may be related to favorable conditions for possible petroleum existence.

The magnetic survey is usually the first geophysical activity made in large areas where little data are available. It may reveal the approximate thickness of the sediments, localize an area for further exploration activity, and estimate scattered basement depths under less favorable conditions. The gravity survey, which is the second geophycal activity, may detect changes in rock density of the sedimentary basin; although its results may contain great ambiguity, it is often used, particularly in areas of easy accessibility, because of its relatively low cost of operation (Dyk, 1961; p.42). The third activity of this survey is the seismic one. Although this is the most widely used method of exploration, seismic surveys in unknown district still leave many facts unrevealed.

The possible results of the surface and geophysical surveys may include the following:

(1) Surface indication of petroleum:

(a) petroleum seepages;

(b) petroliferous rocks.

(2) The existence of favorable geological conditions for petroleum accumulations:

(a) potential source beds;

(b) potential reservoir beds.

(3) structural prospects:

(a) natural potential traps;

(b) potential drilling objectives.

(4) Potential areas for stratigraphic traps.

Whenever the results of the geophysical survey affirm the existence of structural prospects, then the next exploration phase is the exploration drilling.

3.1.5 The Exploration Drilling Phase

The wells that are drilled for petroleum can be divided into two categories, namely:

(1) The exploratory wells. These are wells that are drilled in searching for petroleum, either above or below known reservoirs in a known district, or at unknown districts;

(2) The development wells. These are wells that are drilled within the field boundaries of known reservoirs or near a producer well to extend the productive area of the known reservoir in a known district.

(Lahee, 1961; pp.1-3).

This discussion is only concerned with the first category. Further, the exploration wells can be subdivided into:

1-(a). Those which are drilled far from any petroleum district and which, if petroleum is found, will be the discovery well of the new field. Wells of this group are called the new field wildcats;

1-(b). Those which are drilled in relation to known producing areas of a known district. These can further be divided into:

(i) those located outside the known field boundaries of a partly developed reservoir of a known district; they may be drilled to extend the known reservoir, called the outpost wells, or to find a new reservoir, called the new reservoir wildcats;

(ii) those located within the known field boundaries; they may be drilled to search for a new reservoir, either below or above the known reservoir. Those drilled below are called deeper reservoir wells, and those above are called shallower reservoir wells.

A complete classification of exploratory wells is given in Figure 3-1.

Drilling objective	Classification when drilling is started	Classification after completion or abandonment	
		Successful	Unsuccessful
For long extension of partly developed reservoir	Outpost	Extension well	Dry outpost
For a new reservoir on a structure or in geological environment already productive			
(1) Within limits of proved reservoir for new reservoir			
(a) Above deepest productive reservoir	Shallower reservoir well	Shallower reservoir discovery well	Dry shallower reservoir well
(b) Below deepest productive reservoir	Deeper reservoir well	Deeper reservoir discovery well	Dry deeper reservoir well
(2) Outside limits of proved area of reservoir	New reservoir wildcat	New reservoir discovery wildcat	Dry new reservoir wildcat
For a new field	New field wildcat	New field discovery wildcat	Dry new field wildcat

Figure 3-1: Exploratory Wells Classification.
Source: Frederick H. Lahee, "Probability of Success in Exploration," in Graham B. Moody, Petroleum Exploration Handbook, McGraw Hill, New York, 1961. p.4-2.

3.1.6 The Diagrammatic Representation

The diagrammatic representation of an exploration project is formed by combining the exploration activities that are undertaken with the possible outcomes or results of these activities. Figure 3-2 shows a possible diagrammatic representation of a surface geology network of an exploration project for an unknown district. The related explanations of this diagram is given below:

A. Pre-field work and the reconnaissance surface geology are the first things to do. The results of the two activities may either affirm to do detailed reconnaissance surface geology directly or assert to do laboratory examinations of the field investigations before detailed reconnaissance is undertaken. Define then:

(i) state 1 to be the pre-field work and the reconnaissance survey;

(ii) state 2 to be the laboratory investigation;

(iii) state 3 to be the detailed reconnaissance surface geology;

The results of the detailed reconnaissance activity are usually examined through laboratory analysis and its outcome may either affirm further exploration, namely detailed geology, or continue detailed reconnaissaince work. Define then:

(iv) state 4 to be the laboratory activity for this stage;

(v) state 5 to be the detailed geology activity;

(vi) state 6 to be the laboratory activity at this stage.

Again, the results of the detailed geology activity must be checked and examined through laboratory analysis. The outcome of this analysis may either:

(a) affirm further exploration, namely geophysical survey, or

(b) continue detailed geology, or

(c) stop exploration work; this means that the project is abandoned.

At this stage, it is usually possible to interprete the general subsurface conditions of the district[1] however, if there are no indications whatsoever that a source rock or a reservoir rock is available, then it is usully suggested to stop the exploration work. Define then:

(vii) state 7 to be the stop activity.

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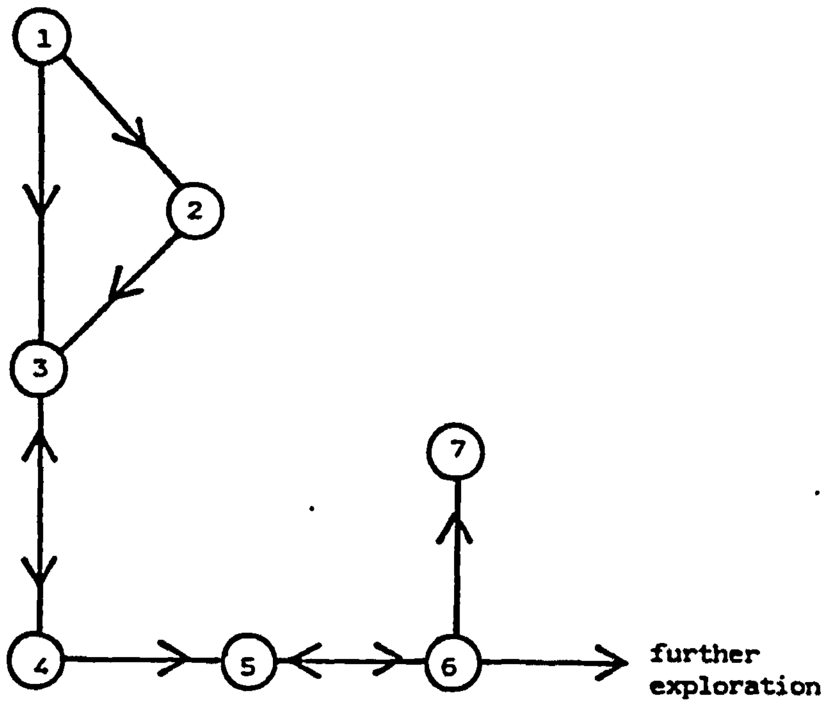
[1] Also at this stage, if required, an estimation of the petroleum resource by volumetric method can be conducted. The result, though, contains a large degree of uncertainty.



Legend: state 1: pre-field & reconnaissance geology activity;
state 2: laboratory activity;
state 3: detailed reconnaissance geology activity;
state 4: laboratory activity;
state 5: detailed geology activity;
state 6: laboratory activity;
state 7: termination.

Figure 3-2: The Surface Geology Network

B. For the diagrammatic representation of the geophysical survey refer to Figure 3-3. The magnetic survey is the first activity undertaken in the geophysical survey; the second is the gravity survey, and the third is the seismic survey. However, the laboratory interpretation of the magnetic survey may assert either to undertake the gravity survey in the next step, or to omit it and go directly to the seismic activity. The combined laboratory interpretation of the magnetic and seismic surveys, and possibly the gravity survey, may suggest in the following steps:

(a) to terminate the exploration work because no petroleum structural or stratigraphic traps are found, or

(b) to undertake more extensive magnetic survey, or

(c) to continue seismic survey, or

(d) to undertake further exploration, namely drilling activity.

The steps (b) or (c) are needed to further confirm the possible existence of a potential structural trap or a potential stratigraphic trap, while step (d) is undertaken if the laboratory interpretation indicates the possible existence of a petroleum trap. In this case, the geophysical survey also help to determine the location of the drilling sites. The results from the combined laboratory interpretation for the step on drilling can be classified into (i) less favorable, (ii) favorable, and (iii) very favorable. These are based on the information on potential traps as well as on the possible size of the traps. A less favorable outcome may indicate inconclusive information on the shape of the trap and the possiblity that it does not extend very far. On the other hand, a very favorable outcome may indicate a large extension of the trap with conclusive information on its size. The favorable outcome is considered to lie in between the two previous outcomes. Define then:

(i) state 8 to be the magnetic survey activity;

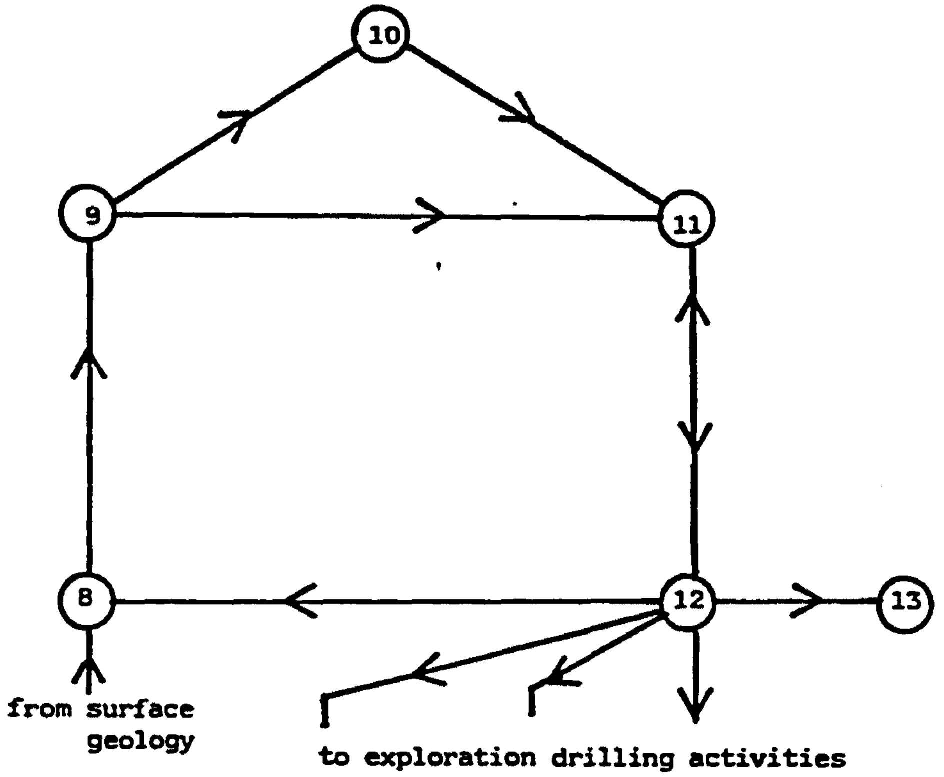
(ii) state 9 to be the laboratory interpretation activity at this stage;

(iii) state 10 to be the gravity activity;

(iv) state 11 to be the seismic survey activity;

(v) state 12 to be the combined laboratory interpretation activity;

(vi) state 13 to be the termination activity; should this be the step undertaken, then the project is usually abandoned.



Legend:

state 8 : magnetic survey activity;
state 9 : laboratory interpretation activity;
state 10: gravity survey activity;
state 11: seismic survey activity;
state 12: combined laboratory activity;
state 13: termination activity.

Figure 3-3: The Geophysical Network

C. As previously discussed, the drilling activity may be conducted under one of the three possible conditions set up by the geophysical survey, namely the less favorable, favorable, and the very favorable conditions. However, these conditions refer to structural conditions and do not refer to the presence of petroleum insitu.[2] So, even if very favorable conditions are to be expected, there is still a possibility that the wildcat drilling ends up with a dry well. On the other hand, less favorable conditions may result in a discovery well. It is of course rational to assume that the probability of striking petroleum when the structural conditions are very favorable is greater than when those conditions are less favorable or favorable. This assumption will be held in furtherdiscussion. The diagrammatic representation of the drilling activity network is provided in Figure 3-4.

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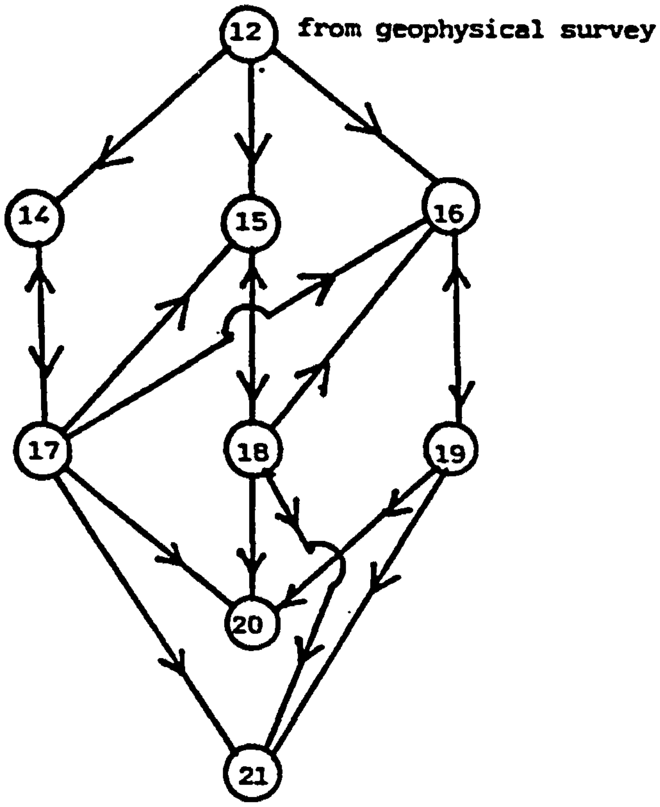
[2] Note that even if the presence of petroleum is not yet confirmed by drilling, one is not prevented to make an estimation of the amount of the possible petroleum resource by the volumetric method. In fact, if the project does not end up in state 13, a volumetric estimation at this stage will have less degree of uncertainty than the one done at the stage after the surface geological survey.



Legend:

state 14: wildcat drilling in less favorable condition;
state 15: wildcat drilling in favorable condition;
state 16: wildcat drilling in favorable condition;
state 17; 18, 19: laboratory activities;
state 20: discovery activity
state 21: termination activity (dry well)

Figure 3-4: The Drilling Activity Network

As the drilling proceeds and samples from the drillings are examined in the laboratory, more and more concrete information on the geological conditions is obtained. This may bring a possibility that the less favorable conditions as defined by the geophysical survey may become favorable or remain less favorable. Such possibility must also apply for the favorable conditions; they may become very favorable or remain favorable. However, the very favorable conditions as defined by the geophysical survey are usually not changed by the results of the drilling, as far as geological structures are concerned. This is because it is considered that the methods used in geophysical surveys are good enough to predict structural conditions, but they are not considered to be that good as to provide conclusive structural interpretation, especially in complex structures. The conditions for drilling may then be visualized as follows:

(i) wildcat drilling in a less favorable condition, after laboratory analysis, may result:

(a) drilling in the same geological condition, namely less favorable, or

(b) drilling in a favorable condition, or

(c) drilling in a very favorable condition.

All these three possibilities may end up with either a dry well or a discovery well;

(ii) wildcat drilling in a favorable condition, after laboratory analysis, may result:

(a) drilling in the same geological condition, namely favorable, or

(b) drilling in a very favorable condition.

Also the two possibilities may either end up with a dry well or a discovery well;

(iii) wildcat drilling in a very favorable condition, after laboratory analysis, usually the result confirms the geophysical interpretation.

The drilling in a very favorable condition may still end up with either a dry well or a discovery well. Define then:

(i) state 14 to be the wildcat drilling activity in less favorable conditions;

(ii) state 15 to be the wildcat drilling activity in favorable conditions;

(iii) state 16 to be the wildcat drilling activity in very favorable conditions;

(iv) state 17, 18, 19 to be laboratory activities;

(v) state 20 to be the discovery activity;

(vi) state 21 to be the dry well activity, which is also the termination activity.

Hence, the drilling activity network can be formed, and this is shown in Figure 3-4.

Combining the surface geology, geophysical, and drilling activity networks together, the exploration network of the project for a venture is obtained. This network is shown in Figure 3-5.

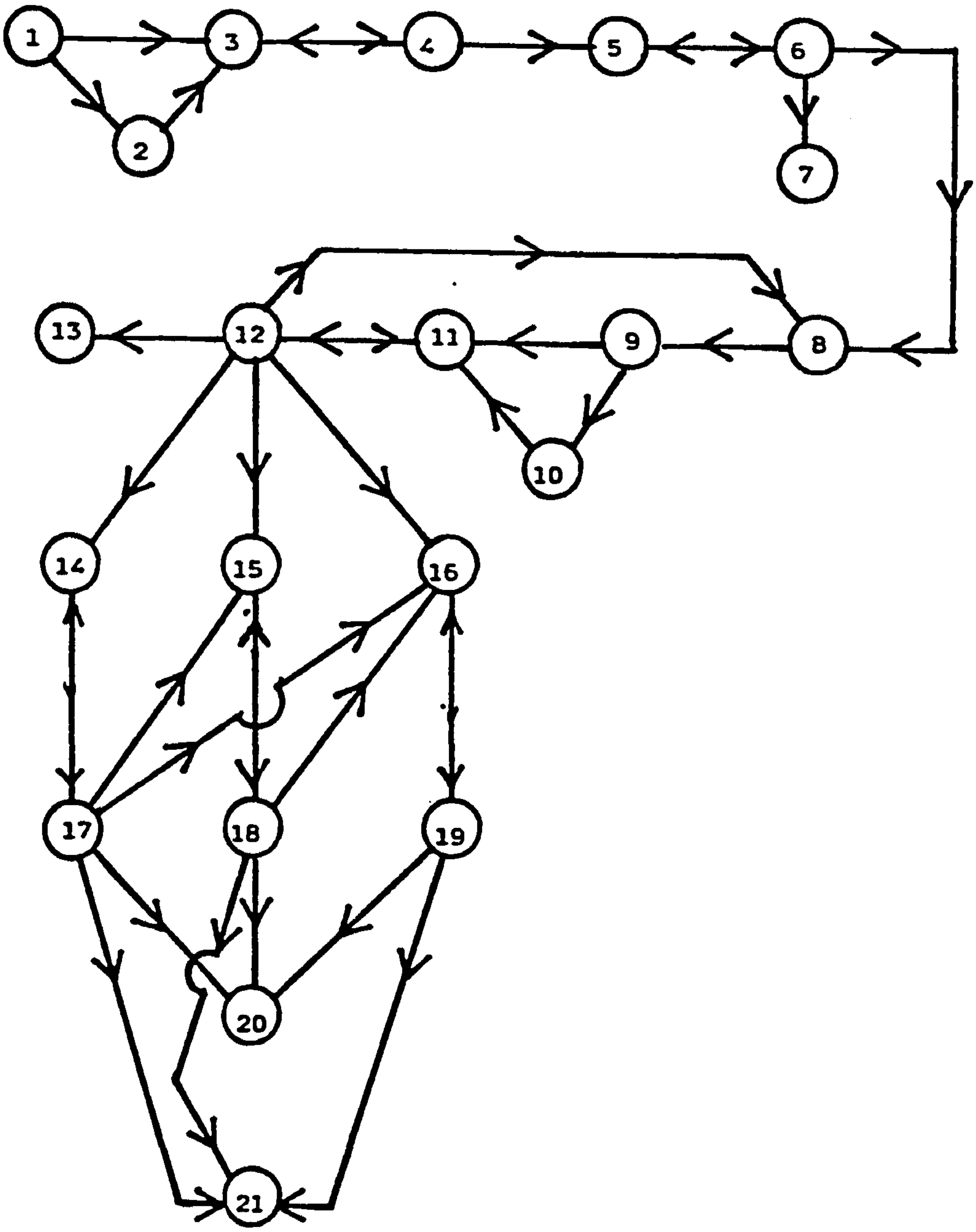


Figure 3-5: The Exploration Network

3.2 THE MARKOVIAN EXPLORATION MODEL

In this section the exploration network is modelled as a Markov chain and Markov process. The analysis will be related to the exploration project network discussed in the previous section. The analysis proceeds by discussing the Markov chain first, followed by the Markov process.

3.2.1 The Markov Chain

Definition:

A stochastic process with a state space E can be defined as a collection of random variables $\bar{X}(n)$ defined on the same probability space and having values in the state space E, where n is in T and T is a subset of $(-\infty,+\infty)$ (Cinlar, 1975; pp.6-9; Hoel et al, 1972; p.vii).

Heyman and Sobel (1982) define a Markov chain as follows:

> A Markov chain is a discrete-parameter stochastic process with a countably infinite state space and the property that the past history of the process is completely summarized by the present state.

With respect to this analysis, the state space is the set of all the exploration activities of an exploration project. Since the number of these activities in a particular exploration project is finite, the state space is finite. Suppose $(X(n), n = 1, 2, 3 \ldots)$ is a stochastic process that takes on a finite countable numbers of possible values $X(n)$, then $X(n)=i$ means that the process is in state i at time n. Whenever the chain is in state i at time n, then the probability that the chain will be in state j at time (n+1), is called a transition probability. This transition probability is independent of the time n, and is usually written as P(ij). The Markov chain can be mathematically defined as follows:

$$P\{X(n+1)=j|X(n)=i,\ X(n-1)=i(n-1),\ldots\ldots X(1)=i(1)\}=$$

$$P\{X(n+1)=j|X(n)=i\} = P\{X(2)=j|X(1)=i\} = P(ij) \quad \ldots\ldots(1)$$

(Ross, 1972; pp.84-85).

The above formulation states that the probability the chain will be in state j at time (n+1), given it was in state i, i(n-1)... i(1) at time n, (n-1),.....1 respectively, is equal to the probability that the chain will be in state j at the next step given that it is in state i now. Thus, the past history of the chain is completely summarized by the present state i. This property is very important in relation to its use in exploration. As is known, a decision to undertake an exploration activity in an exploration project depends only on the outcome of the last activity that has been undertaken. For example, the decision to undertake a wildcat drilling depends entirely on the outcome of the geophysical survey, and in turn, the decision to undertake a magnetic survey depends on the results of the surface geological survey.These decision steps are possible because all knowledge gathered from the past activities is being summarized in the activity that is currently undertaken. Hence, the use of the Markov chain to represent the process of exploration can be adequately justified. The probability of transferring from one exploration activity to the other becomes a transition probability of the Markov chain and the exploration network becomes a Markov chain system.

Referring to the exploration network of Figure 3-5, this network has twenty one states and the transition from one state to the other is a Markov chain transition. Each transition must be assigned a probability value, namely P(ij), and this probability is the transition probability which is independent of time n; so it is stationary. The Markov chain being considered is time homogeneous, namely it satisfies equation (1) (Cinlar, 1975; pp. 106-107). It is further required that:

$$P(ij) \geq 0, \ i, \ j > 0 \ \text{for} \ i=1,2,\ldots\ldots n. \quad \ldots\ldots\ldots\ldots (2)$$

And since the chain must make a transition into some state, it is true that:

$$\sum_{j}^{n} P(ij)=1 \quad \text{for} \ i=1,2,\ldots\ldots n. \quad \ldots\ldots\ldots\ldots (3)$$

It is customary to arrange the P(ij)'s into a square matrix, called the transition matrix, as follows:

$$P=\begin{bmatrix} P(11) & P(12) \ldots\ldots\ldots\ldots\ldots\ldots & P(121) \\ P(21) & P(22) \ldots\ldots\ldots\ldots\ldots\ldots & P(221) \\ P(31) & P(32) \ldots\ldots\ldots\ldots\ldots\ldots & P(321) \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots \\ & & \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots \\ P(211) & P(212) \ldots\ldots\ldots\ldots\ldots\ldots & P(2121) \end{bmatrix}$$

The sum of the probability values of each row must be equal to one. Define P(ij)=p(ij) for P(ij) not equal to zero or one, otherwise it is equal to zero or one. Then the transition matrix of the exploration network can be written as shown in Figure 3-6.

The probability P(ij) for i=j can be interpreted as the probability that the chain will continue to remain in state i after it has been in state i for one unit of time. For example, P(55) is the probability that the chain will still be in state 5 after the chain has been in state 5 for, say one month, if one month is defined to be a one unit of time. In other words, P(55) is the probability that it is still necessary to perform detailed surface geological survey after it has undertaken that activity for, say one month. However, if P(ij) for i=j is not equal to one, the chain has to leave state i eventually. For example, in the case of state 1, the chain must eventually make a transition, either to state 2 or state 3. In cases where P(ij)=1 for i=j, the chain will remain in state i forever once it reaches state i. This condition applies for states 7, 13, 20 and 21. Note that state 7, 13, and 21 are, infact, termination states, i.e. the exploration is unsuccessful; however these states are not classified in the same category because the knowledge accumulated at each of these states is different.

$$
P=\begin{array}{cc}
\left[\begin{array}{ccccccccccccccccccccc}
P_{11} & P_{12} & {}_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & P_{22} & P_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & P_{33} & P_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & P_{43} & P_{44} & P_{45} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_{55} & P_{56} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & P_{65} & P_{66} & P_{67} & P_{68} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{88} & P_{89} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{99} & P_{910} & P_{911} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{1010} & P_{1011} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{1111} & P_{1112} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{128} & 0 & 0 & P_{1211} & P_{1212} & P_{1213} & P_{1214} & P_{1215} & P_{1216} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{1414} & 0 & 0 & P_{1417} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{1515} & 0 & 0 & P_{1518} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{1616} & 0 & 0 & P_{1619} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{1714} & P_{1715} & P_{1716} & P_{1717} & 0 & 0 & P_{1720} & P_{1721} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P_{1815} & P_{1816} & 0 & P_{1818} & 0 & P_{1820} & P_{1821} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {}_{1916} & 0 & 0 & P_{1919} & P_{1920} & P_{1921} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
&
\begin{array}{c}
1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21
\end{array}
\\
\begin{array}{ccccccccccccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21
\end{array}
&
\end{array}
$$

Figure 3-6: The Transition Matrix

State 21 has more complete knowledge than that of state 13 or 17, while state 17 has more complete knowledge than state 13. State 20 is actually not a termination state but it is a state in which petroleum is discovered. The development stage will follow after the discovery; however, the development activities are considered not to be controlled by the Markov chain. Hence, as far as the Markov chain is concerned, the state 20 is a termination state, namely the exploration activity is terminated with a discovery.

Before further analyzing the exploration network problem, it seems appropriate to discuss the problems of assigning probability values to each transition in the section that follows.

3.2.1.1 Assigning Probability Value

There are basicallly two possible ways for assigning these values: (i) to derive a formulation which will show that these transitions have a certain probability distribution function, (ii) to provide subjective probability estimates. The first approach is usually mathematically more difficult than the second one. The difficulty in formulating such probability distribution functions is that the process that controlls the origin, accumulation, migration, and entrapment of petroleum reserve is not wholly known, so that the uncertainty in the decision for undertaking certain exploration activities may not obviously follow certain probability distribution functions. Unfortunately,there have been few studies in this area. Newendorp (1975; pp.324-327) shows that reserve distribution follow lognormal, binomial, multinomial, hypergeometric, or even exponential probability distribution functions.

Newendorp in his study assumes that decisions for drilling can be modelled using sampling without replacement or sampling with replacement (Newendorp, 1975; pp.227-349). In the first case, the probability of the outcome of a drilling venture is dependent upon the previous outcomes of the wells, while in the second case, the probability of a drilling venture is independent of previous outcomes. For the fisrt case, Newendorp suggests a hypergeometric distribution for the reserves distribution and for the second case, a binomial or multinomial distribution function with some geological restrictions. However, his results do not suggest what probability distribution function the transition activity will have. It is, of course, not obvious that the probability distribution function for the transition of an activity must be identical to the one for reserve distribution.

The subjective probability estimates are derived from subjective personal judgement. This subjective method is a personal opinion as to the likelihood that a given outcome will occur (Newendorp, 1975; pp.305-307). The opinion can be based on available statistical data, or it may be based on experience of the analysist. This method is usually less complicated than the first one and requires no complex mathematical formulation. However, it has certain limitations, namely (i) since it is a personal judgement, different analyses may give different probability estimates, even based on the same geological information, (ii) the judgement may be biased because of personal preferences toward risk; a risk averse person may give lower probability values than a risk neutral or a risk preferred person. In general, the analysis assumes that the person is a risk averse one. A person is said to be risk averse, if he will always prefer a sure consequence to any probabilistic mixture of consequences having the same mathematical expectation. On the other hand, a risk prefer one will prefer the probabilistic consequences to any sure consequence. A risk neutral person may be indifferent between the two kinds of consequences if their mathematical expectations are the same. In view of the existence of a large degree of uncertainty in quantifying an undiscovered resource, the subjective probability estimates may often give adequate results.

There are other methods for estimating probability values besides the above two methods, namely (i) the numerical grading techniques, and (ii) and the simulation techniques. These methods have been used to estimate the probability of reserve distributions (Newendorp, 1975; pp.310-311; 369-376). This analysis does not discuss as how these methods can be used to provide transition probability estimates, but it provides a brief description of these methods. The numerical grading techniques uses arbitrary grading scales for all factors which have a bearing on the possible presence of petroleum. The grading scale may range from one to four, where four is defined to be excellent and one to be poor. A weighting multiplier factor is assigned to those factors which are particularly important to petroleum accumulations. The weighting grades are then added and the result is compared to results of other alternative possible petroleum accumulations and the one that has the highest score is considered the most favorable. This method seems to have an advantage over previous methods because it includes most of the relevant factors that govern petroleum accumulation. However, since the numerical grading scale is quite arbitrarily chosen, results may differ from one analysis to the other. The second method, namely the simulation techniques, is basically a method that calculates the probability of petroleum accumulation by generating or simulating the distribution density functions of the random variables involved. To generate these functions, statistical data of the random variables are required. This method has certain advantages, namely (i) it allows professional judgement of one random variable at a time, (ii) it is conducive to sensitivity analysis, and (iii) the concept is so general that it often can be used for a variety of uncertainties.

However, since it needs statistical data, the method can not be used for unknown districts where such data are usually not present, unless an analogy can be made to a certain known district where the data are available. Also, the techniques are usually not mathematically simple.

Several remarks can be given when assigning probability values, namely:

(i) the probability values involved in this analysis are those assigned for the random variable(s) that govern(s) the transition of the chain from one state to another, rather than a single random variable determining the probability of reserve existence. All these values together, through the Markov chain, determine the probability of petroleum existence;

(ii) if a certain probability distribution function is to be assumed for the random variable that governs the transition of the chain, then such function must also be assumed for every transition in the network. This is basically a requirement of the Markov chain used in this analysis;

(iii) the network is designed for an exploration of an unknown district, where statistical data are absent. Methods using statistical data cannot be used, as previously explained.

In the discussion that follows, it is assumed that the probability values are determined already by one of the methods discussed above. The discussion on the classification of the states is then followed by the analysis on the computational procedure.

3.2.1.2 The Classification of the States

This section presents a classification of the states of the network into various types to simplify the computational procedure for finding the probability of being in any state. In order to do this, the states are renumberred and leave the position of the activities as they are. This means that the network remains the same, but the numbers that are assigned to each states are changed. The new diagram is shown in Figure 3-7 and this diagram indicates three important facts, namely:

(1) Every transition in the network is controlled by chance;

(2) The transition probability from one state to the next state does not depend on the transition probability of prior transitions;

(3) There is a possibility to repeat certain exploration activities. For example, the chain has the possibility to transfer from state 15 to state 11;

The above facts are usually not found in a decision tree process used in exploration.

The diagram in Figure 3-7 can also be simplified by grouping a number of states together as shown in Figure 3-8.

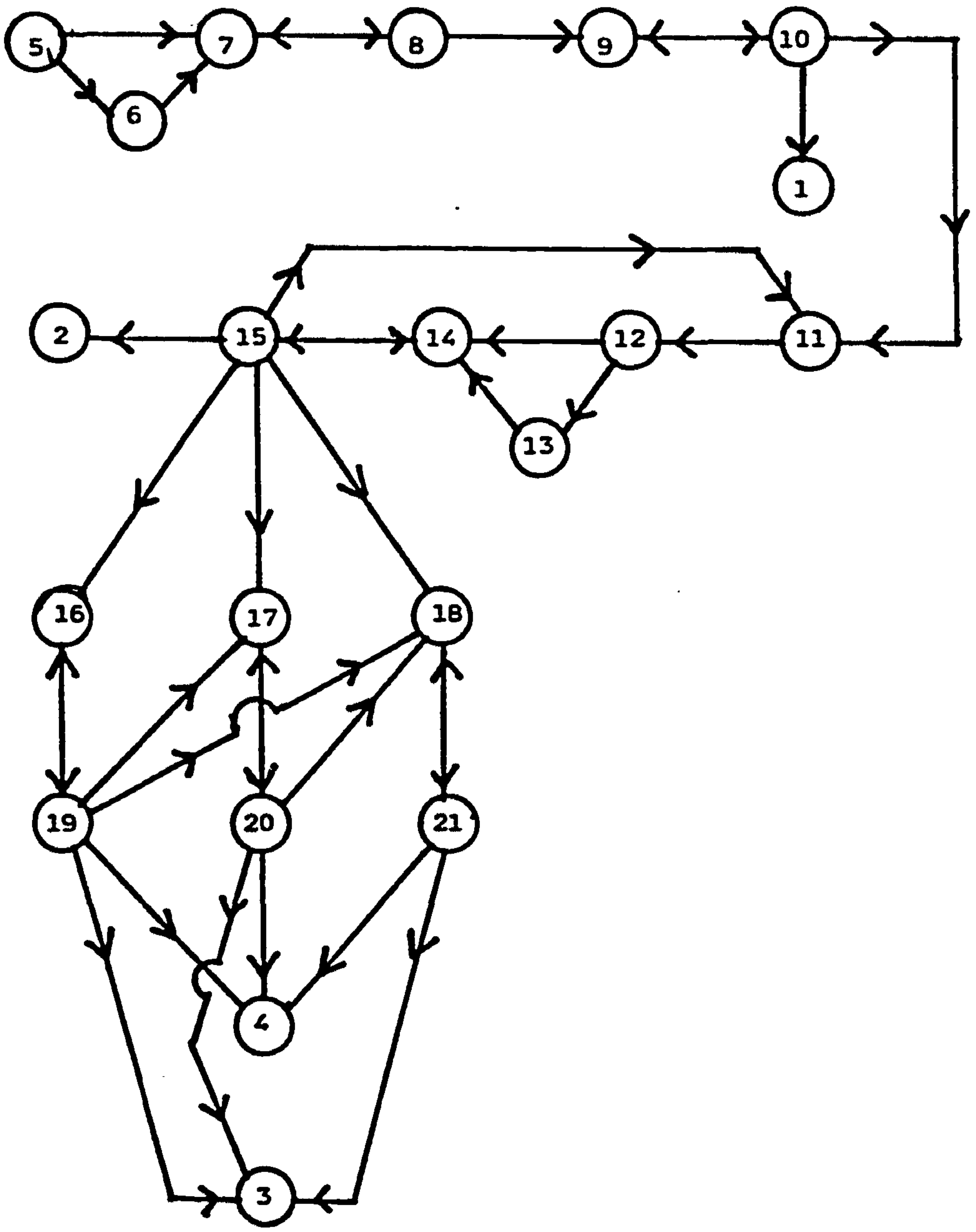


Figure 3-7: The New Exploration Network

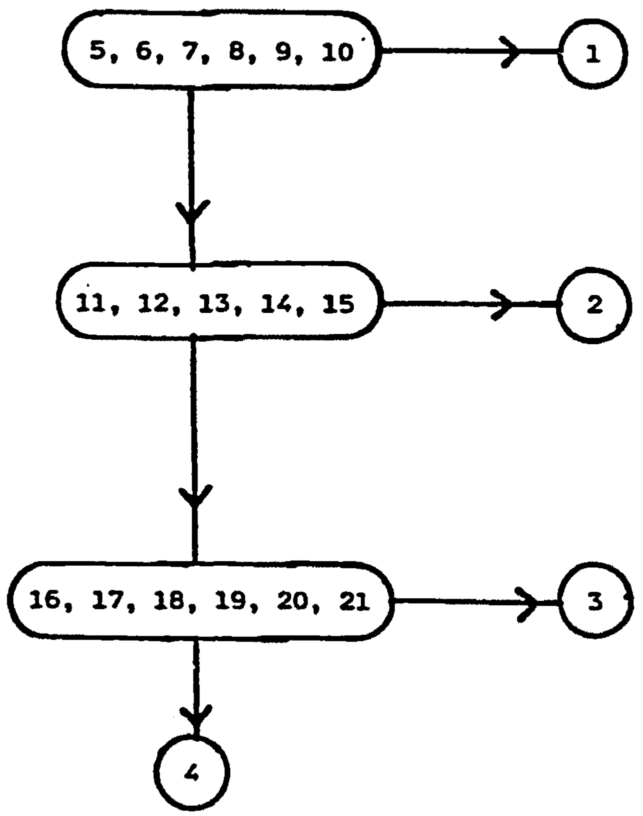


Figure 3-8: The Simplified Network

Several definitions are given below to enable one to classify the states.

Definitions:

(i) A state is called a recurrent state if the probability that the chain will revisit that state is equal to one;

(ii) A state is called a transient state if the probability that the chain will revisit that state is less than one;

(iii) A state is called an absorbing state if the probability that the chain will remain in that state forever, after reaching it, is equal to one.[3]

The mathematical definitions of the above are given in Appendix B.

The states can then be classified as follows:

(i) States 1, 2, 3, 4 are recurrent states, however they are also absorbing states; once the chain is in one of these states then it will remain there forever;

(ii) States 5, 7, 8, 9, 10 and 11, 12, 13, 14, 15 and 16, 17, 18, 19, 20, 21 are all transient states. The probability that the chain will revisit these states is less than one, which means that there is a possibility that the chain will never revisit these states again. Note that once the chain leaves the group

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[3] These definitions are given in a manner so that their meaning can be easily understood. A more precise definition, as defined in most stochastic processes theory, are given in Appendix B.

that contains state 5 the chain will never revisit the states in that group. This is also true for the two other groups that contain state 11 or 16.

The above classification implies that for the absorbing states, $P(jj)=1$ and for the transient states $P(jj)<1$. Let states 5, 6,7, 8, 10 together be called the first group;states 11, 12, 13, 14, 15 be the secondgroup; and states 17, 18, 19, 20, 21 be the third group. Then within the first group the chain may make a transition from state 7 to state 8 as well as from state 8 to state 7; likewise, from state 9 to state 10 and vice versa. However, eventually the chain must leave the first group and make a transition either to state 1 or state 11 in the second group. Within the second group the chain may make a transition from state 14 to state 15 as well as from state 15 to state 14, and also from state 15 back to state 11. When the chain does make a transition from state 15 to state 11, then the chain will have to undergo the transitions from state 11 to state 15 again. However, eventually the chain must leave the states of the second group and make a transition either to state 2 or to state 16, or 17, or 18 in the third group. Within the third group the chain may make a transition from state 16 to state 19 and vice versa, from state 17 to state 20 and vice versa, and from state 18 to state 21 and vice versa. Note that if the chain ever reaches state 19, then from there it can reach state 21 by two possible routes, namely: (i) via state 18 to state 21, or (ii) via state 17 to state 20, then to state 18, and finally to state 21. However, when the chain takes the second route, it has the possibility to make a transition from state 20 either to state 18 or to state 4, or back to state 17. On the other hand, when the chain takes the first route, is not possible to make other transitions except to go from state 18 to state 21. And finally, the chain must eventually leave the third group and make a transition either to state 3 or state 4. Note, whenever the chain reaches either state 1, 2, 3, or 4 the process is then terminated.

3.2.1.3 The Computational Procedure

It is now possible to define a computational procedure for calculating the probability of every exploration activity after classifying the states of the exploration network. The Markov matrix is then arranged according to the Markov diagram described in Figure 3-7, and it can be written as a partitioned matrix as follows:

$$P = \left[\begin{array}{c|c} I & 0 \\ \hline B & Q \end{array}\right]$$

where:

$$I = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

and

$$B=\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} P(51) & \cdot & \cdot & P(54) \\ P(61) & \cdot & \cdot & P(64) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ P(211) & \cdot & \cdot & P(214) \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 6 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 21 \end{array}$$

The entries of the sub-matrix B that have positive values but less than one are: P(101), P(152), P(193), P(194), P(203), P(204), P(213), P(214). Other entries are equal to zero.

$$Q=\begin{array}{c} \begin{array}{ccccc} 5 & 6 & 7 & & 21 \end{array} \\ \left[\begin{array}{ccccc} P(55) & P(56) & P(57) & \cdots\cdots & P(521) \\ P(65) & P(66) & P(67) & \cdots\cdots & P(621) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P(215) & P(216) & P(217) & \cdots\cdots & P(2121) \end{array}\right] \end{array} \begin{array}{c} \\ 5 \\ 6 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 21 \end{array}$$

The entries of the sub-matrix Q which are positive but less than one are:

(i) all diagonal entries, namely P(ii), i=5,....21;

(ii) among the first group: P(56), P(57), P(67), P(78) P(87), P(89), P(910), P(109), P(101), P(1011);

(iii) among the second group: P(1112), P(1213), P(1214) P(1314), P(1415), P(1514), P(1511), P(152), P(1516) P(1517), P(1518);

(iv) among the third group: P(1619), P(1720), P(1821), P(1916), P(1917), P(1918), P(193), P(194), P(2017), P(2018), P(203), P(204), P(2118), P(213), P(214).

Note that the diagonal entries above are interpreted as the probability that the chain, after being in state i for a certain time, will still be in state i after that time. Since all diagonal entries are considered to be positive, it is not possible for an exploration activity to be finished at the exact time stipulated by the decision makers. In view of the uncertainty involved, this seems to be a rational assumption. However, empirical analysis may assign relatively small values to some of these probabilities; this means that some of the activities so assigned may finish at a time near the time set by the decision makers. It should also be noted that the entries of the partitioned matrix P are no different than the original markov matrix P, except that the states in the partitioned matrix are arranged in different order.

The question is now, what is the probability that the chain, starting from state 5, will end up in state 1, or state 2, or state 3, or state 4 ? In other words, what is the probability that the exploration project will result in:

(i) termination of the exploration after the surface geology?

(ii) termination of the exploration after the geophysical survey ?

(iii) wildcat dry well ?

(iv) wildcat discovery well ?

First, in general the following can be written:

$$P^n = \left[\begin{array}{c|c} I & 0 \\ \hline B(n) & Q^n \end{array}\right] .$$

where: $B(n) = B + QB + Q^2B + Q^3B + \dots\dots\dots + Q^{n-1}B$,

$$B(n) = \sum_{k=0}^{n-1} Q^k B \qquad \dots\dots\dots\dots\dots\dots (3\text{-}1)$$

$B(n)(ij)$ is the probability that starting from state i, the chain enters the absorbing state j at or before the n-th step, where $i>4$ and $j=1,2,3,4$. For more detail discussion refer to Appendix B.

The following relationship can be written:

$$G = \lim_{n\to\infty} B(n) = SB \qquad \dots\dots\dots\dots\dots\dots (3\text{-}2)$$

(see Appendix B).

where $G(ij)$ is the probability that the chain ever reaches state j from state i;

$$S = (I-Q)^{-1} \qquad \dots\dots\dots\dots\dots\dots (3\text{-}3)$$

where I is an identity matrix.

Since the sub-matrices Q and B are known, the sub-matrices S and G can be computed. The above method of calculation is made possible using the limiting behavior of the Markov chain (see Appendix B).

By calculating the matrix G, the following entries of the matrix are known and these provide the solution of the problem:

(i) G(51), which is the probability that the project will result in the termination of the exploration after the surface geological survey;

(ii) G(52), which is the probability that the project will result in the termination of the exploration after the geophysical survey;

(iii) G(53), which is the probability that the project will result in a wildcat dry well;

(iv) G(54), which is the probability that the project will result in a wildcat discovery well.

Hence, the questions of the exploration problem have been answered by the computation of the matrix G.

Several other conclusions can be drawn from the use of the Markov chain computation procedure. Let N(ij) be a random variable denoting the number of times the chain visits state j starting in state i, before it eventually enters an absorbing state. So state i and j are transient states. Also let u(ij) be the expected number of visits to state j, E(N(ij)). Then the Theorem states:

> The entries of the matrix S, namely s(ij) are equal to u(ij).

(Bhat, 1972; pp.72-73).

Hence, the matrix S also indicates the possible values of the expected number of visits to state j, starting in state i, before the chain eventually enters an absorbing state. And from the formula for computing S above, the entries s(ij)'s are known. Let S(d) be a diagonal matrix whose entries are u(jj) or s(jj); let S(2) be a matrix whose entries are the square of u(ij) or s(ij). Then the Theorem states:

> The variance of the number of visits to state j from state i, denoted as V(N(ij)), is equal to (S(2S(d) - I) - S(2)).

(Bhat, 1972; p.74).

Hence, the formula for the variance can be written as:

$$V(N(ij)) = (S(2S(d) - I) - S(2)) \qquad \text{..........(3-4)}$$

Since the exploration work always begins in state 5, then i=5. Suppose that the unit of time spent for each activity is given in months. Then one might ask, what is the expected number of months that will be spent on detailed geological survey before a decision is reached either to stop or to continue further exploration ? And what is its variance ? The solution is given by computing:

(i) u(59) = s(59) which is also E(N(59)), the expected number of months spent in state 9 before entering either state 1 or state 11, via state 10;

(ii) V(N(59)), which is the variance of the expcted number computed in (i) above.

Note that the formula for computing the variance V(N(ij)) will result in matrix whose entries are V(N(ij)). So, V(N(59)) is the first row and the fifth column entry of that matrix; and state 5 becomes its first element.

Hence, one could also ask similar questions as above, for other states. The matrix S also indicates the expected number of visits to a state j, starting in state i, where i is not state 5. For instance, one might ask, what is the expected number of visits to state 15 when the chain starts in state 11, before it enters an absorbing state ? However, before providing an answer a careful evaluation must be done. If the exploration starts in state 11 instead of in state 5, it is assumed that all geological knowledge obtained from conducting exploration activities from state 5 to state 10 are already known. However, since the exploration starts in state 11, such activities are not actually done in that district. For an unknown district, such knowledge are usually obtained by the method of analogy to a similar known district; however, these are not usually better than those obtained by actual surveys. Hence, the risk involved in the exploration in unknown district will usually be greater when exploration starts in state 11 than when it starts in state 5. This implies that the transition probabilities given are no longer relevant in this case. So in order not to confuse the problem, this analysis assumes that for the particular exploration network given in Figure 3-7, the chain always starts in state 5.

3.2.2 The Markov Process

In the discussion of the Markov chains, it has been assumed that the time needed for each activity to take place is one unit time for every state. In most exploration activities, however, it is usually true that the length of time for an activity to take place is not completely known in advance. In order to incorporate the length of time of the activities as a random variable into the model, a Markov process must be defined.

In a Markov process, the time spent by the process in a certain state is a random variable. Mathematically, a Markov process can be defined as follows:

> The stochastic process $X = (X(t) ; t \text{ in } R(+))$ is said to be a Markov process with state space E, given that for any t, $s \geq 0$ and j in E, $P(X(t+s)=j|X(u) ; u \leq t) = P(X(t+s)=j|X(t))$.

(see Appendix B).

Note that R(+) denotes the set of all positive real numbers and t denotes the time.

When $P(X(t+s)=j|X(t)=i) = P(ij)(s)$ is independent of $t \geq 0$ for all i, j in E, and $s \geq 0$, the process X is said to be a time homogeneous Markov process. And $P(ij)(t)$ is called the transition function of the Markov process X and this function is dependent upon time t. In matrix notation, $P(ij)(t)$ is written as $P(t)$. For further discussion, the analysis limits itself to the time homogeneous Markov process.

To illustrate the Markov process that is used in this analysis, let the process be in state x(0) at time t=0. Suppose the process remains in state x(0) until some positive time t(1) at time which the process make a transition to another state x(1). If the process remains permanently in state x(0), then t(1) = infinity. However, if t(1) is finite, the process upon reaching state x(1) will remain there until some positive time t(2), where t(2)>t(1), when it makes a transition to another state x(2), or even back to state x(0). If the process never leaves state x(1) then t(2) = infinity. In general, the process can be illustrated as follows:

$$X(t) = \begin{cases} X(0), & 0<t<t(1) \\ x(1), & t(1)<t<t(2) \\ X(2), & t(2)<t<t(3) \\ \cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots & \end{cases}$$

An example of a Markov process realization is given in Figure 3-9.

Several theorems of the Markov process will be used and discussed in this section. The first theorem is the theorem about the so called sojourn time. Let W(t) be the length of time the process X remains in the state being occupied at the instant t, then W(t) is also called the sojourn time of that particular state. The Theorem states:

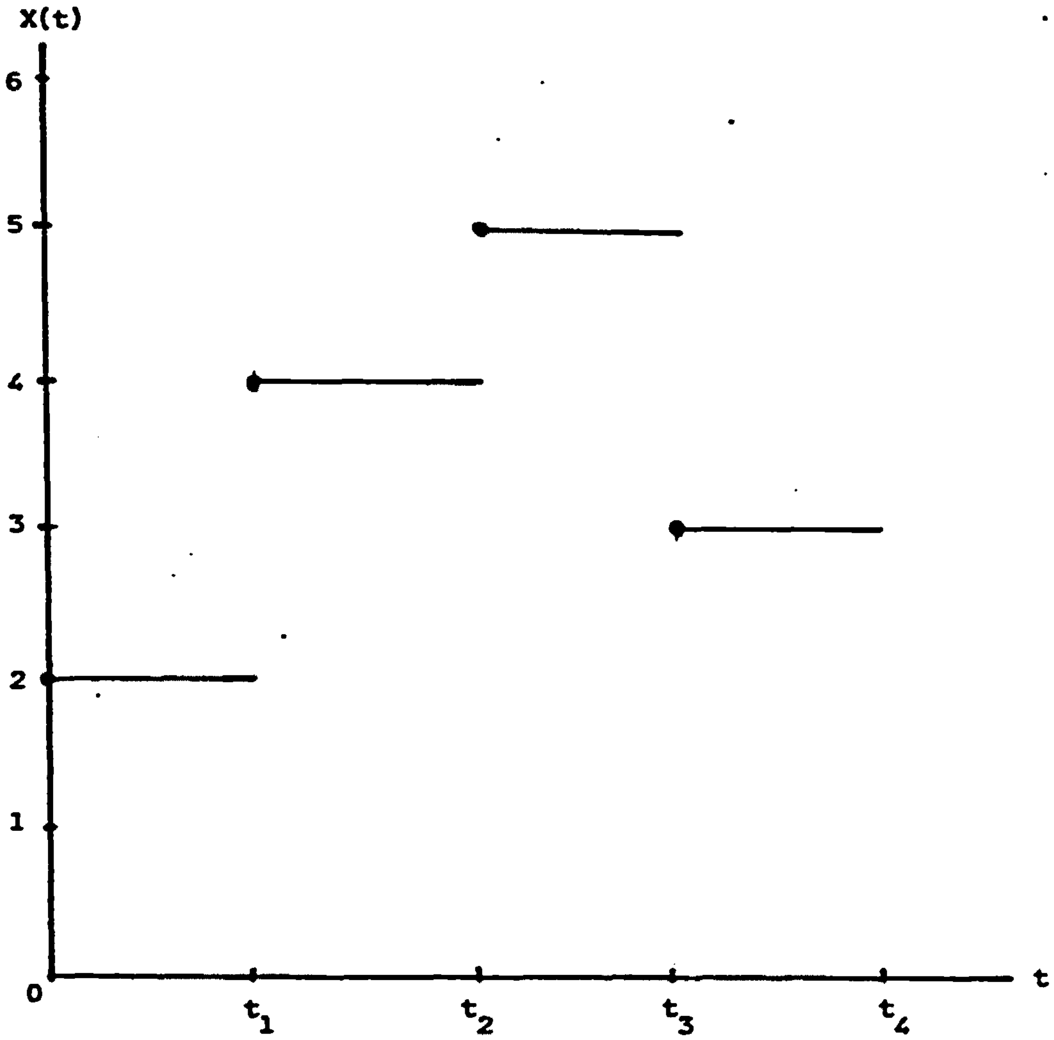
For any i in the state space E, and $t \geq 0$, then

$$P(W(t)>u \mid X(t)=i) = e^{-\mathcal{E}(i)u}, \quad u \geq 0 \text{ for some}$$

$$\mathcal{E}(i) \in (0, +\infty).$$

(Cinlar, 1975; p.242).

The sojourn time is then a random variable.



Legend: X(t) is the state.
t is the time.

Figure 3-9: A Markov Process Realization

The term $\varepsilon(i)$ is usually referred as the parameter of the sojourn time distribution in state i. Now define the times T(0), T(1), T(2) .... to be the instants of the transitions of the Markov process, and x(0), x(1), x(2), ... to be the successive states visited by the process, then:

$$P\{T(n+1)-T(n)\,|\,X(n)=i,\ X(n+1)=j\} = e^{-\varepsilon(i)u}$$

which is independent of state j. Hence, the probability distribution of the sojourn time depends only on the state in which the sojourn time is being evaluated, and not on the next state. The probability density function of the sojourn time is defined as:

$$f(i)(W(t)) = \begin{cases} \varepsilon(i)e^{-\varepsilon(i)u} & \text{, for } u\geq 0 \\ 0 & \text{, for } u<0. \end{cases}$$

where $\varepsilon(i)$ is the parameter of the density function.
Then the probability distribution function of the sojourn time is:

$$P\{W(t)>u\} = \int_0^{\infty} \varepsilon(j)e^{-\varepsilon(i)u}\,du = e^{-\varepsilon(i)u}\text{, for } u\geq 0.$$

This expression is identical to the one defined by the theorem.
From the above the mean of the exponential distribution $E(W(t))$ can be calculated. Let $t=W(t)$, then:

$$E(t) = \int_0^{\infty} t\ f(t)\ dt$$

$$E(t) = \int_0^{\infty}\varepsilon te^{-\varepsilon t}\,dt = \varepsilon\int_0^{\infty} te^{-\varepsilon t}\,dt$$

Integrating by parts yields:

$$E(t) = - te^{-\delta t} \Big|_0^\infty + \int_0^\infty e^{-\delta t}\, dt$$

$$E(t) = 1/\delta$$

For a particular state i, the mean sojourn time is

$$E(i)(t) = 1/\delta(i)$$

The Markov process also has the following properties:

(i) $P(ij)(t) \geq 0$;

(ii) $\sum_k P(ik)(t) = 1$;

(iii) $P(ij)(t+s) = \sum_{k \neq i} P(ik)(t)\ P(kj)(s)$, for s, $t \geq 0$.

It is assumed that the process will not stay in the transient states forever (Hoel et al, 1975; pp.84-85). This requires that the $\lim_{n \to \infty} t(n) = \infty$, where $t(n)$ is the time when the process make a transition to state n (see Appendix B). In fact, the process in the exploration network must eventually leave the transient states and enter one of the absorbing states. The equation (iii) above is usually called the Chapman-Kolomogorov equation. It states that, starting in state i, the process reaches state j at time (t+s), after it passes some intermediate state k at time t and from that state k it moves to state j at the remaining time s. This equation may be used to compute the probability values at a certain time. Note that the Markov process defined in this analysis satisfies the above mentioned properties.

3.2.2.1 Assigning Value to The Sojourn Time

The value of the mean sojourn time for a particular exploration activity must be estimated. This can be done by a statistical estimation; one computes the mean value of the length of time of a particular exploration activity from different exploration projects in unknown districts to obtain the mean sojourn time of that particular activity. The mean sojourn times for other exploration activities are calculated similarly. However, data for the above estimation are not always available or they are not compatible because of the unsimilarity of the districts. In such cases, subjective judgement is required to provide an estimated value of the mean sojourn time.

3.2.2.2 The Classification of States

The classification of states for the Markov process remains basicly the same as the one defined for the Markov chain, except that there are some other characteristics that must be added. Recall that:

$$P(W(t)>u|X(i)=i)=P(i)(W(t)>u)=e^{-\varepsilon(i)u}, \text{ for } u\geq 0$$

The value of $\varepsilon(i)$ ranges from 0 to infinity.

Then define the following:

(i) If $\varepsilon(i)=0$, $P(i)(W(t)>u) = 1$, then state i is an absorbing state. This also implies that the sojourn time of state i is infinitely large.

(ii) If $0<\varepsilon(i)<\infty$, $P(i)(W(t)>u) < 1$, then state i is called a stable state, which is also a transient state. This implies that the sojourn time of state i is positive but finite.

(iii) If $\delta(i)$ = infinity, $P(i)(W(t)>u) = 0$, then state i is called an instantaneous state. It implies that the sojourn time of state i is zero; in other words, the process makes a transition to another state as soon as it enters state i.

For the exploration network discussed in this analysis, the states are either absorbing or stable, no instantaneous states are present.

In the Markov chain the diagonal entries of the transition matrix Q of the transient states are set to be greater than zero. In the Markov process these diagonal entries must be equal to zero. The reason for this is that the actual time spent in a particular state is now incorporated into the process and is treated as a random variable. In other words, there is no predetermined sojourn time as is the case of the Markov chain. It is no longer necessary to state that the process will have a positive probability of staying in a particular state after the process has been in that state for a certain time.

Hence, referring to the Markov diagram of Figure3-5, the states 1, 2, 3, 4 are absorbing states, and the rest are all stable, or transient states.

3.2.2.3 The Computational Procedure

The computational procedure to derive the probability values of each of the exploration activities consists of (1) computing the generator matrix of the process, and (2) deriving the the spectral representation of the generator. The meaning of these terminologies will be explained later, as they become useful for the analysis.

Let Q be the transition matrix of the Markov process. This matrix is similar to the matrix P in the Markov chain. And let $\varepsilon(i)$ be the parameter of the exponential distribution function of the sojourn time in state i. The the Markov process satisfies the following integral equation:

$$P(ij)(t) = e^{-\varepsilon(i)t} I(ij) + \int_0^t \varepsilon(i) e^{-\varepsilon(i)s} \sum_{k \neq i} Q(ik) \cdot P(kj)(t-s)\,ds$$

(Cinlar, 1975; p.254; Hoel et al, 1972; p.87).

The importance of the above integral equation is that it expresses the transition function $P(ij)(t)$ in terms of the transition probability matrix Q and the parameter of the sojourn time distribution, ε. Since $P(ij)(t)$ is continuous, then it is differentiable and its derivative is also continuous (Cinlar, 1975; pp.255). Rewriting the integral equation as follows:

$$P(ij)(t) = e^{-\varepsilon(i)t} I(ij) + \varepsilon(i) e^{-\varepsilon(i)t} \int_0^t e^{-\varepsilon(i)s} \sum_{k \neq i} Q(ik) P(kj)(s)\,ds$$

and taking its derivative, one obtains:

$$P'(ij)(t) = -\varepsilon(i) P(ij)(t) + \varepsilon(i) \sum_{k \neq i} Q(ik) P(kj)(t)$$

For t=0, then:

$$P'(ij)(0) = -\varepsilon(i) I(ij) + \varepsilon(i) \sum_{k \neq i} Q(ik) P(kj)(0)$$

$$P'(ij)(0) = -\varepsilon(i) I(ij) + \varepsilon(i) \sum_{k \neq i} Q(ik) I(kj)$$

$P'(ij)(0) = -\delta(i)I(ij) + \delta(i)Q(ij)$

$P'(ij)(0)$ is called the generator of the process, $A(ij)$. The following can then be derived:

(a) For stable states: (i) $A(ij) = -\delta(i)$ for $i=j$, since $Q(ii)=0$, (ii) $A(ij) = \delta(i)Q(ij)$ for $i \neq j$;

(b) For absorbing states: (i) $A(ij)=0$ since $\delta(i)=0$, (ii) $Q(ij)=1$ for $i=j$ and $Q(ij)=0$ for $i \neq j$.

Hence, it is possible to derive the generator matrix A if the transition possibility matrix Q and the parameter δ are known. The transition matrix is usually known or assumed to be known, while the parameters of the distribution functions must be estimated. The mean values of the sojourn times for the activities can be determined by statistical estimation of different exploration projects in unknown districts or by subjective judgement. If for each exploration activity the estimated mean sojourn time is defined, then the parameter for the sojourn time of each activity can be computed. Hence, once Q and δ are known , the generator A can be computed.

The problem is now to compute the transitional function $P(ij)(t)$, or the matrix $P(t)$, from the generator matrix A. One method to compute this matrix $P(t)$ is to solve the following system of differential equations:

$P'(t) = A\,P(t)$ or

$P'(t) = P(t)\,A$ (Cinlar, 1975; p.255).

The first system is called the Kolomogorov backward equations and the second one is the Kolomogorov forward equations. Note that for both systems $P(0)=I$. This implies that:

$$P'(t)/P(t) = A$$

$$P'(t)/P(t) = \int A \, dt$$

$$P(t) = e^{At+C}$$

Since $P(0)=I$, then $C=0$. Hence:

$$P(t)=e^{At} \quad \text{and} \quad e^{At} = (t^{n}/n!)\, A^{n}.$$

Further it is necessary to use the concept of eigenvalues and eigenvectors to compute $P(t)$. The generator A for the exploration network is a 21 x 21 matrix. Let $b(i)$, $i=1, 2, \ldots, 21$ be the eigenvalues of A and $f(j)$, $j=1, 2, \ldots, 21$ be the eigenvectors associated to those eigenvalues. Thus, the $b(i)$'s are scalars and the $f(j)$'s are column vectors. Define the matrix $N = (f(1), f(2), \ldots\ldots, f(21))$ and the matrix $N^{-1}= C(j)$, $j=1, 2, \ldots, 21$ where $C(j)$'s are row vectors. Hence N and N^{-1} can be written as follows:

$$N=\begin{bmatrix} f(1)(1)\ldots\ldots f(21)(1) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ f(1)(21)\ldots f(21)(21) \end{bmatrix} \quad N^{-1}=\begin{bmatrix} C(1)(1)\ldots\ldots C(1)(21) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ C(21)(1)\ldots C(21)(21) \end{bmatrix}$$

If D is the diagonal matrix whose entries are the eigenvalues $b(i)$, $i=1, 2, \ldots, 21$, then $A = N\, D\, N^{-1}$. It follows that:

$A^k = N D^k N^{-1}$, for k=0, 1,...... Hence:

$P(t) = e^{At} = N e^{tD} N^{-1}$, where e^{tD} is the diagonal

matrix whose entries are $e^{b(j)t}$, j = 1, 2, ..., 21.

(Cinlar, 1975; pp.364-366).

However, the above formula is difficult to compute. To solve the problem, proceed as follows:

Since $N^{-1} N = I$, then: $C(j)f(k) = \begin{cases} 1, & \text{if } j=k \\ 0, & \text{if } j \neq k \end{cases}$

Next define B(k) to be the matrix obtained by multiplying the column vector f(k) to the row vector C(k), i.e. B(k)=f(k)C(k) and obtained the following matrix B(k):

$$B(k) = \begin{bmatrix} f(k)(1)C(k)(1) \ldots\ldots\ldots\ldots\ldots f(k)(1)C(k)(21) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f(k)(21)C(k)(1) \ldots\ldots\ldots\ldots\ldots f(k)(21)C(k)(21) \end{bmatrix}$$

where k=1, 2, ..., 21. In other words, there are twenty one B(k) matrices, each of which is formed in the above manner. It follows that:

$$B(j)B(k) = f(j)C(j)f(k)C(k)$$

$$B(j)B(k) = 0, \quad \text{if } j \neq k$$

$$B(j)B(k) = B(j), \text{ if } j=k$$

From $A = N D N^{-1}$, one obtains:

$$A(ij) = \sum_k N(ik)\ D(kk)\ N^{-1}(kj)$$

$$A(ij) = \sum_k f(k)(i)\ b(k)\ C(k)(j)$$

$$A(ij) = \sum_k b(k)\ B(k)$$

This result is: $A = b(1)B(1) + \dots\dots\dots + b(21)B(21)$ (Cinlar, 1975; p.368). This last representation of A in terms of its eigenvalues $b(j)$ and the matrices $B(j)$ is the spectral representation of A. It follows that:

$$A^k = b(1)^k B(1) + \dots\dots\dots\dots + b(21)^k B(21)$$

For k=0 : $I = B(1) + \dots\dots\dots\dots + B(21)$

Hence: $P(t) = e^{tA} = e^{tb(1)} B(1) + \dots\dots + e^{tb(21)} B(21)$

$$\dots\dots\dots\dots\dots\dots (3\text{-}5)$$

Hence, in this approach, the Kolomorgov backward equation is only used to derive the formula for P(t) and is not used to compute P(t) by solving the system of the differential equations. The only draw back to this procedure is that the computation of the eigenvalues and the eigenvectors is not easy. The EISPAC computer program which computes eigenvalues and eigenvectors of a matrix A can be used for this purpose (see Appendix C).

Having derived a computational form for P(t), it is possible to calculate the transition function for a particular t. The rule of thumb for computing t is as follows:

$$t = \sum_{i=1}^{n} 1/\delta(i) \qquad \dots\dots\dots\dots\dots\dots (3\text{-}6)$$

where n is the number of states that could possibly be visited by the process. For example, to compute P(54)(t), t can be approximated by:

$$t = 1/\delta(4) + 1/\delta(5) + \dots\dots\dots + 1/\delta(21)$$

Exclude state 1, 2, and 3 from the above computation. When making transitions from state 5 to state 4, the number of states that are possibly visited by the process, is eighteen. So the value of t for $P(54)(t)$ must be at least equal to the value of t computed as above. One may then ask the following questions:

(i) What is the probability that the exploration project will result in a discovery well after operating for t months ? Assume that the value of t is above the value set up by the above rule of thumb. This requires the computation of $P(54)(t)$. A similar question can be asked for a wildcat dry well; hence, compute $P(53)(t)$.

(ii) What is the probability that the exploration project will terminate after the geophysical survey, after t months of operation ? This requires the computation of $P(52)(t)$, where t is assumed to be equal to or greater than the following t:

$$t = 1/\delta(2) + 1/\delta(5) + 1/\delta(6) + \ldots\ldots + 1/\delta(15).$$

States 1, 3, 4, and 16-21 are not included in the above calculation, because these states will not be visited by the process, if it is required to stop in state 2. Refer to Figure 3-5 to visualize this situation.

(iii) What is the probability that the exploration project will terminate after the surface geological survey, after t months of operation ? This requires the calculation of $P(51)(t)$. Again t must be equal to or greater than the following t:

$$t = 1/\varepsilon(1)+1/\varepsilon(5)+\ldots\ldots+1/\varepsilon(10).$$

States 2, 3, 4, and 11-21 are not included in the computation of the approximated t above, since these states will not be visited by the process, if it is required to stop in state 1. A word of caution must be said here, namely if the given t to compute the probability is less than the value of t approximated by the rule of thumb, then the probability to be calculated should be considered zero; this is because the process will not be able to reach the designated state in the period provided by the given t.

Since the process always starts from state 5, then it is not necessary to compute the entire matrix $P(t)$ but only the entries of the fifth row of the transition function matrix $P(t)$. This greatly reduces the computational effort. This can be done as follows:

For k=1 : $B(1)(5j) = f(1)(5)C(1)(j)$, $j = 1, 2, \ldots, 21$.

For k=21: $B(21)(5j) = f(21)(5)C(21)(j)$, $j = 1, 2, \ldots, 21$.

For a given k, k = 1, 2, ..., 21, each $B(k)(5,j)$ is a row vector with twenty one elements or entries, each of which equals $f(k)(5)C(k)(j)$. Hence:

$$P(5j)(t) = \sum_k e^{tb(k)} B(k)(5j), \quad j = 1, 2, \ldots, 21. \qquad \ldots\ldots\ldots\ldots (3\text{-}7)$$

As t becomes larger and larger, and if state j is transient, then $P(5,j)(t)$ must become smaller and smaller, and in the limit, $\lim_{t\to\infty} P(5j)(t) = 0$. This is consistent with the requirement that the process can not remain in a transient state forever, and must eventually move to an absorbing state. On the other hand, if j is an absorbing state, and if $\lim_{t\to\infty} P(5j)(t)$ exists then $P(5,j)(t) = p(j)$, where $p(j)$ is a stationary probability value. In such case, the stationary probabilities are independent of the state in which the process starts. This condition can be interpreted that in the long run, the stationary probabilities reflect the most probable geological uncertainty. However, this interpretation must be carefully evaluated; computationally, since the stationary probability values are derived from the known transition probability values and the estimated parameters of the sojourn time distribution, they are dependent on these known and estimated values. So, it is more precise to say that the stationary probabilities reflect the most probable geological uncertainty under the given conditions.

3.3 THE NET REWARD FUNCTION

The net reward function can be defined as the difference between the value of the information obtained from a particular exploration activity and the cost of performing that activity. However, there are certain difficulties in evaluating the value of the information. First of all, some of the information gathered from the exploration may not have a market. For example, the results from a magnetic survey only, or from a reconnaissance surface geological survey only, may not have a market; in general, partial information seldom has any market. Secondly, most complete geological information has multiple usefulness. For example, a subsurface geological map resulted from a complete surface geological survey, or from a complete geophysical survey, and information on the rock formation obtained from drilling operation, may be used for other resource development, like water or other mineral resources. They may also be used for regional planning. Hence, the value of the information is not only tied to the expected price of petroleum but also to the prices of the information alternative uses. Thirdly, exploration information may have values to other firms, especially in the case where structural traps extend beyond the legal boundary of the firm doing the exploration. Fourthly, spill over of exploration information causes the information to becomes public property. For example, the discovery of petroleum can not be kept secret forever, and this may benefit other firms that want to buy mineral rights in areas near the area where such discovery is found. Hence, there is a public good aspect in exploration information. This means, that the firm producing the information can not prevent others from benefiting the information at zero cost. Hence, if the firm is willing to sell its exploration information, the value obtained is less than the benefits that are contained in producing that information.

In view of the above mentioned problems, it is assumed in this model that the firm doing the exploration is also the firm that is going to produce the petroleum whenever a discovery is found. So, the information gained is used by the firm to further explore the district, or to produce its own petroleum whenever a discovery is found. In this respect, the value of the information is just its value to the firm. Then the firm can do either of the following things: (i) To assign an imputed price for each piece of information which is related to the expected price of petroleum; the demand of information becomes a derived demand for the firm. This means, the higher the expected price of petroleum, the higher will be the demand for information, and vice versa. However, this analysis will not elaborate on how the imputed prices are derived. (ii) To use the information for estimating the amount of possible petroleum by the volumetric method.The value of the information is then the amount of possible petroleum times the expected price of petroleum. The expected price of petroleum becomes the shadow price for the information. The problem in this last approach is that volumteric estimation can usually be done only after a complete surface geological survey, or after a complete geophysical survey, or after a discovery. In other words, the results of an individual exploration activity can not be used for estimating the amount of petroleum by the volumetric method; it is the combined results of these activities that can be used by the volumetric method. Referring to the exploration network in Figure 3-5, this implies that once a volumetric estimate has been established, the value of the information obtained from the activities generating that estimate must banish. For example, if a volumetric estimate is done in state 10, namely after a complete surface geological survey, then the value of information in states 5 to 9 must be equal to zero. In other words, if a volumetric estimate is assigned to state 10, then no value of information must be assigned states 5 to 9. The reason for this is that the information obtained from states 5 to 9 has been used to generate the volumetric estimate that is assigned to state 10. A further discussion on this approach will be done in Chapter IV.

Whenever the firm uses the first approach then the information obtained from each activity has an imputed price. The value of the information obtained from an exploration activity is equal to the amount of information gathered from that activity times its imputed price. Algebraicly, then the values of the information from every state can be arranged as a column vector called the reward vector. The net reward vector is then a column vector whose entries are the difference between the value of information of an activity and the cost of performing that activity. In this respect, the analysis evaluates two kinds of cases, namely:

(a) The case where the net reward for every activity is fixed, but changes from one activity to another;

(b) The case where the net reward for every state has a cost function that depends on the sojourn time of the state.

The purpose of analyzing the net reward function is to compute the expected value of an exploration project. The analysis will first proceed by discussing the first case, and followed by the second one.

3.3.1 Case 1

In case 1, the net reward is fixed, and it can be expressed by a column vector h, whose entries h(j) are the rate of the net reward received when the process is in state j, where j = 1, 2, ..., 21 in the case of the exploration network. The value of h(j) can either be negative, positive, or zero.

The expected value of the exploration project is:

$$E(i)(h(X(t)) = \sum_j P(ij)(t)h(j)$$

Since the exploration starts from state 5, then the above can be written as:

$$E(5)(h(X(t)) = \sum_j P(5j)(t)h(j)$$

It is known from equation (3-5) that:

$$P(t) = P(ij)(t) = \sum_k e^{b(k)t} B(k)(ij)$$

$$P(5,j)(t) = \sum_k e^{b(k)t} B(k)(5,j), \quad j = 1, 2, ..., 21.$$

Hence:

$$\sum_j P(5,j)h(j) = \sum_j \sum_k e^{b(k)t} B(k)(5,j)h(j)$$

$$E(5)(X(t)) = \sum_j \sum_k e^{b(k)t} B(k)(5,j)h(j) \quad \ldots\ldots (3\text{-}8)$$

Note that B(k) and h(j) are independent of t.

The value of t is the total time required for the project to take place and this is not usually known. However, the value of t can be approximated by using the rule of thumb discussed in previous section. Let T be the total time required by the project as approximated by the rule of thumb and let a be the rate of discount, then the net present value of the project E(5)(NPV), can be formulized as follows:

$$E(5)(NPV) = \int_0^T e^{-at} \sum_j P(5,j)(t)h(j)\ dt,$$

$$E(5)(NPV) = \int_0^T e^{-at} \sum_j \sum_k e^{b(k)t} B(k)(5j)h(j)dt,$$

$$E(5)(NPV) = \sum_j \sum_k B(k)(5j)h(j) \int_0^T e^{(b(k)-a)t} dt.$$

Solving the integral term of the above expression gives:

$$\int_0^T e^{(b(k)-a)t} dt = (e^{(b(k)-a)t}/(b(k)-a)) \Big|_0^T$$

$$= (e^{(b(k)-a)T} - 1)/(b(k)-a)$$

Hence:

$$E(5)(NPV) = \sum_j \sum_k B(k)(5j)h(j)(e^{(b(k)-a)T} - 1)/(b(k)-a). \quad \ldots\ldots\ldots\ldots\ldots (3\text{-}9)$$

The expected net present value of the project can then be solved by computing the B(k)(5j)'s and the eigenvalues; the h(j)'s are given. Note that:

(a) if $e^{(b(k)-a)T}$ is less than one, then the associated term will become negative, if h(j) is positive;

(b) if $e^{(b(k)-a)T}$ is equal to one, then the associated term is equal to zero.

So the discounting of the expected value of the project may have the effect of reducing a term in E(5) (NPV) to zero or to a negative quantity. This is understandable, since the eigenvalues in the exponential function of E(5) (h(X(t)) express the rate of growth of the expected value of the project, while the discount rate is expressing the opposite. Hence, when the rate of growth is equal to the discount rate, the expected net present value of the term becomes equal to zero.

The ordinary rule for the net present value can be applied to select the most favorable project or to justify the economic feasibility of the project. Among the exploration projects of a venture, the project which gives the highest expected net present value is the most favorable one. For a single project, a positive expected net present value indicates that the project is economically justified to be undertaken. It should be noted that the evaluation of the project for different values of discount rates, provides different results concerning its economic feasibility.

3.3.2 Case 2

In the second case the net reward depends on the length of of the sojourn time; the longer the sojourn time the greater the total exploration cost for this particular activity. In general, it is also true that the longer the sojourn time the more complete the information gained. However, it is assumed that the value of the information gained from a certain activity is more affected by the type of the activity, than the length of time for such activity to take place. For example, the value of the information gained by the geophysical survey is greater than that obtained by the surface geological survey, and these values will be less affected by how long these activities have taken place. To formulate the net reward function, it is necessary to define the following:

Let:

L(i) be the amount of labor used in state i;

K(i) be the amount of capital used in state i;

r be the rental rate of capital used per unit of time;

w be the wage rate of labor used per unit of time;

W(i)(t) be the sojourn time in state i at time t;

c(i) be the amount of information gained from state i per unit of time;

p(i) be the imputed price of information gained from state i.

The imputed price is related to the expected price of petroleum and is considered to be given.

Suppose $C(i)$ is the total information gained from state i, then $C(i) = f(c(i), W(i)(t))$; it states, that the total information gained from state i is a function of the unit information gained and its sojourn time. Let the functional relationship be postulated as follows:

$$C(i) = c(i) . W(i)(t)^{d}$$

where d is a constant whose values is: $0 \leq d \leq 1$.

In case 1, the assumption that the net reward is fixed implies that the amount of information gained in each state must be independent of its sojourn time. This condition requires that the value of d must be equal to zero. In case 2, the value of d may range from zero to one. The value of d seems to indicate the ease in which information is gained in a certain district. For a given value of $c(i)$ and $W(i)(t)$, the largest value of $C(i)$ is obtained when d is equal to one. For this kind of a district, information gathering becomes relatively easy. When d is less than one, information gathering becomes relatively difficult, probably because of the remoteness of the district. However, when d is equal to zero, information gathering becomes independent of the sojourn time; this may indicate that given the available equipment and methods, the information has been exhausted.

Since $W(i)(t)$ is a random variable then $C(i)$ becomes randomly determined; so it is necessary to compute the expected value of $C(i)$. It is known that $W(i)(t)$ has an exponential probability density function, hence the expected value of $H(i)$ can be found by computing the the expected value of $W(i)(t)$.

$$E(W(i)(t)^{d}) = \int_0^{\infty} t^{d}\, \varepsilon(i) e^{-\varepsilon(i)t}\, dt.$$

The integral above may not converge absolutely. In such case using the expected mean value of the sojourn time may be more appropriate, namely:

$$E(C(i)) = c(i)\ (E(W(i)(t))^{d}$$

The expected net reward function, $E(NRF)$, can then be formulated follows:

$$E(NRF) = (p(i)c(i)(E(W(i)(t))^{d} - (K(i)r+L(i)w)E(W(i)(t))$$

$$E(NRF) = (p(i)c(i)/\varepsilon(i)^{d} - (K(i)r+L(i)w)/\varepsilon(i))$$

The expected net present value of the project is then:

$$E(5)(NPV) = \sum_j \sum_k B(k)(5j)(p(j)c(j)/\varepsilon(j)^{d} - (K(j)r+L(j)w)/\varepsilon(j)).\int_0^{T} e^{(b(k)-a)t}\, dt,$$

$$E(5)(NPV) = \sum_j \sum_k B(k)(5j)(p(j)c(j)/\varepsilon(j)^{d} - (K(j)r+L(j)w)/\varepsilon(j)).$$

$$(e^{(b(k)-a)T} - 1)/(b(k)-a). \qquad \text{.............} (3\text{-}10)$$

As in case 1, the value of the expected net present value may be positive or negative. Economic feasible ventures must have projects with positive expected net present values. For each venture, the firm can find the maximum expected net present value by optimizing over the choice of T, namely the length of time for the exploration project. Then it can rank order ventures by this maximum. No further elaboration of this problem will be done in this analysis.

Whenever the second approach is used, then the entries of the net reward vector must change. Entries h(10), h(15), and h(4) contain the difference of the value of the estimated amount of petroleum and the cost for performing the respective activity; the expected price of petroleum is used to give the value of the estimated amount of petroleum. All other entries of the net reward vector contain the cost of each respective activity, and they must have negative signs. Suppose g is the vector for the estimated petroleum, hence the positive elements are g(10), g(15), and g(4) while the others are zero. The formula for the second approach is:

$$E(5)(NPV) = \sum_j \sum_k B(k)(5,j)(pg(j)-(K(j)r+L(j)w)/\delta(j))$$
$$(e^{(b(k)-a)T}-1)/(b(k)-a) \qquad \ldots\ldots\ldots\ldots (3\text{-}11)$$

where p is the estimated price for petroleum.

In summary it can be concluded that the exploration network can be analyzed and fitted into a Markovian exploration model, either as a Markov chain or a Markov process. A modification of the exploration network will not change the nature and properties of the model; similar computational procedures can still be used. The net reward function is formulated by postulating an assumption that the firm generating the exploration is also the firm that will exploit the petroleum whenever a discovery is found. This simplifies the problem of exploration information. Evaluation on the values of exploration results can then be done either by using imputed prices or estimates on the amount of possible petroleum.In the following chapter, the Markovian exploration model will be incorporated into the optimization problem of the firm.

Chapter IV

DEPLETION MODEL

This chapter discusses and analyzes a depletion model for the firm incorporating the Markovian exploration model discussed in the previous chapter. To formulate a depletion model, it is necessary to define the optimization problem and solve it using the Hamiltonian approach. In other words, it is an optimum control problem. Embodied into this optimization problem are the following assumptions:

(1) The firm's behaviour is profit maximizing;

(2) The firm being analyzed is also the one that conduct the exploration discussed in the previous chapter;

(3) The decisions for investing on development and production are not subject to random variations; they are deterministic in nature;

(4) The evaluation on the exploration project is done by estimating the amount of possible petroleum by the volumetric method. The expected price of petroleum is used to derive the value of the estimation;

(5) The expected price of petroleum is given and constant overtime, and hence the firm is the price taker;

(6) The production function used in this model is a Cobb Douglas production function and it is subject to constant returns to scale;

(7) The wage rate and the rental rate are fixed; their respective values are determined by the labor market and the capital market.

The optimization problem consists of maximizing the net present value of the profit of the firm for each venture per worker relative to the wage rate, subject to several constraints. The function that is being maximized or minimized is usually referred in optimal control theory as the objective functional. To formulize the problem, a discussion on the items that form the objective functional will first be done, followed by the analysis on the optimization problem.

4.1 THE ELEMENTS OF THE OBJECTIVE FUNCTIONAL

Before discussing the elements of the objective functional it is necessary to define the following notations:

$L(t)$ is the total amount of labor used in one venture;

$K(t)$ is the total amount of labor used in one venture;

$L(1)(t)$ is the amount of labor used in production.

$L(2)(t)$ is the amount of labor used in development.

$L(3)(t)$ is the amount of labor used in exploration.

All the above are evaluated at time t.

$K(j)(t)$, $j = 1, 2, 3$ are defined similarly as above for capital.

Define: $l(j)(t) = L(j)(t)/L(t)$, for $j = 1, 2, 3$.

$k(j)(t) = K(j)(t)/L(j)(t)$, for $j = 1, 2, 3$.

Also:

$$L(1)(t)+L(2)(t)+L(3)(t) = L(t)$$

$$K(1)(t)+K(2)(t)+K(3)(t) = K(t)$$

Hence:

$$l(1)(t)+l(2)(t)+l(3)(t) = 1$$

$$l(1)k(1)+l(2)k(2)+l(3)k(3) = k$$

evaluated at t, and where $k=K(t)/L(t)$

4.1.1 The Total Revenue

The total revenue, TR, is usually defined as:

$$TR = PQ(t)$$

where: P is the price of petroleum;

Q(t) is the total output or production of petroleum per unit of time.

Using the Cobb Douglas production function, one can express the output as follows:

$$Q(t) = A\ K(1)(t)^{a}\ L(1)(t)^{b}\ R^{*}(t)^{1-a-b}$$

where: A is a technological coefficient and is constant; and

$R^{*}(1)(t)$ is the amount of reserves of the firm at time t.

Expressing the above production function per unit of worker, one obtains:

$$q(t) = Q(t)/L(t) = l(1)(t)\ Q(t)/L(1)(t)$$

$$q(t) = l(1)(t)\ A\ k(1)(t)^{a}\ R(t)^{1-a}$$

where: q(t) is the output per worker; and

R(t) is the amount of reserves per worker.

Also the price of petroleum, P, must be expressed in terms of the relative price with respect to the wage rate, namely p = P/w, where w is the wage rate. Hence, the total revenue per worker relative to the wage rate can be written as:

$$TRR(t) = p.q(t)$$

$$TRR(t) = p\ l(1)(t)Ak(1)(t)^{a}\ R(t)^{1-a} \qquad \text{............(4-1)}$$

4.1.2 The Total Costs

The total costs are devided into :

(1) The cost of production, TC(1) ;

(2) The cost of development, TC(2);

(3) The cost of exploration, TC(3);

All the above costs are expressed per unit of time.

4.1.2.1 The Production and Development Costs

One can formulize the production and development costs as the cost of using labor and capital in the respective activities. They can be written as:

$$TC(1)(t) = K(1)(t)r + L(1)(t)w$$

$$TC(2)(t) = K(2)(t)r + L(2)(t)w$$

where r is the rental rate and w is the wage rate.

The cost per worker relative to the wage rate can thus be computed as:

$$TCP(1)(t) = TC(1)(t)/L(1)(t)w$$

$$TCP(1)(t) = l(1)(t)(k(1)(t).V + 1)$$

$$TCP(2)(t) = TC(2)(t)/L(2)(t)w$$

$$TCP(2)(t) = l(2)(t)(k(2)(t).V + 1)$$

where the TCP's are the cost per worker relative to the wage rate;

V is the rental wage ratio, namely $V = r/w$.

For the purpose of this analysis, assume that $k(1) = k(2)$, for all t. Hence, the total costs of production and development per worker relative to the wage rate, TCR, is the following:

$$TCR(t) = l(1)(t)(k(1)(t)V+1) + l(2)(t)(k(2)(t)V+1)$$

$$TCR(t) = (k(1)(t)V+1)(l(1)(t)+l(2)(t))$$

$$TCR(t) = (k(1)(t)V+1)(1-l(3)(t)) \qquad \text{..............(4-2)}$$

4.1.2.2 The Exploration Cost

From previous discussion, it is known that the exploration activities are subject to random variations, controlled by either the Markov chain or the Markov process. Hence, the total cost of exploration must be an expected total cost. The expected total exploration cost of a project as defined in the previous chapter is:

$$TC(3)(t) = \sum_{j}\sum_{K} B(k)(5j)((K(j)(t)r+L(j)(t))/\delta(j)) \cdot \int_{0}^{T} e^{b(k)t}\,dt$$

where $K(j)(t)$ is the amount of capital used in state j at time t;

$L(j)(t)$ is the amount of labor used state j at time t.

The above cost function is derived from the formulation of the expected value of the project. Recall that the expected net present value of the project as given Chapter 3 is formulized as follows:

$$E(5)(NPV)=\sum_j \sum_K B(k)(5j)(pg(j)/\delta(j)^d-(K(j)(t)r+L(j)(t)/S(j))$$

$$\cdot \int_0^T e^{(b(k)-a)t}\,dt$$

Hence the expected value of the project is:

$$E(5)(NPV)= \sum_j \sum_K B(k)(5j)(pg(j)/\delta(j)^d-(K(j)(t)r+L(j)(t)/S(j))$$

$$\cdot \int_0^T e^{(b(k)t}\,dt$$

The negative term of the E(5)(P) is the expression for the expected exploration cost above.

To formulate the expected exploration cost for the optimization problem, it is assumed that there is a constant proportion of capital and labor used in each state to the total capital and labor utilized in the exploration. This proportionality can be written as follows:

$$a(j) = K(j)(t)/K(3)(t)$$

$$c(j) = L(j)(t)/L(3)(t)$$

The values of a(j) and c(j) are fixed and depend on the state j only. Note that: $0<a(j),c(j)<1$.

Hence:

$$K(j)(t) = a(j)K(3)(t)$$

$$L(j)(t) = c(j)L(3)(t)$$

The expected total exploration cost can then be expressed in terms of the total amount of capital and labor used in this section. The total expected exploration cost per worker relative to the wage rate can be computed as follows:

(1) Compute the term: $(K(j)(t)r+L(j)(t)w)/L(t).w$

This is equal to:

$= (a(j)K(3)(t)r+c(j)L(3)(t)w)/L(t).w$

$= (a(j)l(3)(t)k(3)(t)V+c(j)l(3)(t))$

$= l(3)(t)(a(j)k(3)(t)V+c(j))$

Recall that: $l(3)(t) = L(3)(t)/L(t)$

Hence:

$K(3)(t)/L(t) = K(3)(t)L(3)(t)/L(t)L(3)(t)$

$K(3)(t)/L(t) = l(3)(t)k(3)(t).$

(2) Substituting the above result in the total expected exploration cost, one obtains:

$TCR(3)(t) = TC(3)(t)/L(t).w$

$$TCR(3)(t) = \sum_j \sum_K B(k)(5j)l(3)(t)((a(j)k(3)(t)V +c(j))/\delta(j) \int_0^T e^{b(k)t} dt \quad \ldots\ldots(4\text{-}3)$$

where $TCR(3)(t)$ is the total expected exploration cost per worker relative to the wage rate.

4.1.3 The Total Profit

The total profit, TP, is usually formulized as the difference between the total revenue and the total costs; it can be written as:

$TP(t) = TR(t) - TC(t)$

The total profit per worker relative to the wage rate, TPR, is then:

$TPR(t) = (TR(t)-TC(t))/L(t).w$

The terms on the left hand side of the equation have already been computed, namely by equations (4-1), (4-2), (4-3). Combining these equations, one obtains:

$$TPR(t) = pl(1)(t)\,Ak(1)(t)^{a}R(t)^{1-a} - (k(1)(t)V+1)(1-l(3)(t))$$
$$- \sum_{j}\sum_{k} B(k)(5j)\,l(3)(t)\,(a(j)\,k(3)(t)V+c(j))/\delta(j)\,.$$
$$\cdot \int_{0}^{T} e^{b(k)t}\,dt$$

The TPR then forms the objective functional of the optimization problem.

4.2 A BRIEF THEORETICAL REVIEW

The maximization problem is usually formulated as maximizing a certain objective functional subject to certain constraints. The objective functional is usually characterized by two kinds of variables, namely the state variable and the control variable. The state variables represent the state or the condition of the objective functional and the control variables guide or control the state variable. The control variables are at one's disposal and can be controlled to maximized the objective functional.

In general the optimization problem can be written as follows:

$$\text{Maximize } J = \int_{0}^{T} e^{-rt}\, F(t, x, y)\, dt$$

subject to:

$$x(t(0)) = x(0),\quad x(T) = x(1),\quad x'(t) = y$$

where x is the state variable, and y is the control variable.

The Lagrange equation can be written as

$$L = (F(t,x,y) - h(x'(t) - y))\, e^{-rt}$$

where h is the Lagrange multiplier or the costate variable. The Hamiltonian can be formulated as follows:

$$H(d) = (F(t,x,y)+hy))\ e^{-rt}$$

$$H(d) = L+h\ x'(t)e^{-rt} \quad \text{and}\ H(d) = H\ e^{-rt}.$$

where H(d) is the discounted Hamiltonian and H is the current Hamiltonian.

The necessary conditions for maximization are defined by the Euler-Lagrange equations:

$$dpL/dpx - d(dpL/dpx')/dt = 0$$

$$dpL/dpy - d(dpL/dpy')/dt = 0$$

where $x' = dx/dt$, $y' = dy/dt$, and dp = partial derivative.

$$dpL/dpx = e^{-rt}\ (dpF/dpx);\ dpL/dpx' = -he^{-rt}$$

$$d(dpL/dpx')/dt= rhe^{-rt} -h'e^{-rt} =(rh-h')e^{-rt}.$$

Hence:

$$e^{-rt}\ (dpF/dpx)-(rh-h')e^{-rt} = 0$$

or $\quad dpF/dpx = rh-h'$

In terms of the Hamiltonian, one has:

$$dpH(d)/dpx=e^{-rt}\ (dpF/dpx)$$

Hence:

$$(dpH/dpx)e^{-rt} = e^{-rt}\ (dpF/dpx)$$

$$dpH/dpx = dpF/dpx$$

This implies that $dpH/dpx = dpL/dpx = rh-h'$. So the Hamiltonian, H, provides the same result as the Lagrange equation L.

$$dpL/dpy = yhe^{-rt} + (dpF/dpy)e^{-rt}$$

$$dpL/dpy' = 0, \text{ implying } d(dpL/dpy')/dt = 0$$

Hence:

$$dpL/dpy = e^{-rt}(hy + dpF/dpy) = 0$$

$$dpF/dpy = -hy$$

By the Hamiltonian one obtains:

$$dpH(d)/dpy = e^{-rt}(dpF/dpy + hy)$$

So $dpL/dpy = dpH(d)/dpy$ and

$$e^{-rt}(dpH/dpy) = e^{-rt}(dpF/dpy + hy)$$

$$dpH/dpy = dpF/dpy + hy$$

Since $dpH/dpy' = 0$, then $dpH/dpy = 0$, implying:

$$dpF/dpy + hy = 0 \text{ or}$$

$$dpF/dpy = -hy$$

So one obtains the same result by the Lagrange equation as by the Hamiltonian. One can then solve the problem by the equations obtained from the necessary conditions and the equation in the constraints.

Note, however, that equation (1) involves the solution of an ordinary differential equation; depending on the form of this equation, one can sometimes obtain a simple solution in one case but not in other cases. This holds true also for the differential equation in the constraint.

The condition indicated by derivative of the Hamiltonian with respect to the control variable characterizes a maximum for the Hamiltonian. In this respect, it can be said that if F is concave function of the control variables, then the Hamiltonian will also be concave function of the control variables. (Miller, 1978; pp.99-100). It is of course assumed that there is a maximum for J on the interval $[0, T]$. The above model can be extended into a model with several state and control variables and several constraints. This will be shown in the model discussed in the following section.

4.3 THE ANALYSIS OF THE MODEL

The optimization problem consists of maximizing the discounted profit per worker relative to the wage rate, subject to the rate of net reserve addition and the cumulative reserve addition generated by the exploration.

The objective functional to be maximized is:

$$J = \int_0^{T(1)} e^{-rt}\ TPB \qquad \text{or}$$

$$J = \int_0^{T(1)} e^{-rt}\ (pg-(k(1)V+1)(1-l(3))-\sum_j \sum_k B(k)(5,j)\,l(3)$$

$$(a(j)Vk(3)+c(j)/S(j)\ (e^{b(k)T}-1)/b(k))\,dt$$

(Note that the t is supressed in the above equation to simplify the notation).

The constraints are:

(1) The rate of net reserve addition per worker, R';

(2) The cumulative reserve additon per worker, S;

(3) The total capital used K.

The first constraint is: $R' = s - q$

where $s = dS/dt$, is the rate of reserve addition per worker generated by the exploration, and q is the output per worker, in which:

$$q = Al(1)k(1)^{a} R^{1-a}$$

Since the expected amount of petroleum from a discovery is:

$$\sum_j \sum_k B(k)(5,j)g(j) \int_0^T e^{b(k)t} dt,$$

where g is the vector of the estimated amount of petroleum, the rate of reserve addition per worker by exploration s, is:

$$s = \sum_j \sum_k B(k)(5,j)g(j)e^{b(k)t} /L \qquad \text{or}$$

$$s = l(3)/L(3) \sum_j \sum_k B(k)(5,j)g(j)e^{b(k)t}$$

Hence s is an exponential function depending upon the values of the eigenvalues b(k), and time t. One can then write the first constraint as:

$$R' = s - q$$

$$R' = l(3)/L(3) \sum_j \sum_k B(k)(5,j)g(j)e^{b(k)t} - q$$

The second constraint is:

$$S = \int_0^T s dt = l(3)/L(3) \sum_j \sum_k B(k)(5,j)g(j) \int_0^T e^{b(k)t} dt$$

However, the solution to the first constraint, the differential equation R', is difficult. A more feasible solution is to assume that the rate of reserve addition per worker is equal to an average value. This can be formulated as follows:

$$s=l(3)/L(3)\sum_j\sum_k B(k)(5,j)g(j)\int_0^T e^{b(k)t}\,dt/T$$

$$s=l(3)/L(3)\sum_j\sum_k B(k)(5,j)g(j)(e^{b(k)T}-1)/Tb(k)$$

At present, except for the ratio $l(3)/L(3)$, all other terms in s are known in advance. Consequently, the term:

$$\sum_j\sum_k B(k)(5,j)g(j)(e^{b(k)T}-1)/Tb(k)$$

is fixed. The second constraint becomes:

$$S=l(3)/L(3)\sum_j\sum_k B(k)(5,j)g(j)\int_0^T e^{b(k)t}\,dt$$

$$S=l(3)/L(3)\sum_j\sum_k B(k)(5,j)g(j)(e^{b(k)T}-1)/b(k)$$

Hence s is expressed in terms of an average value and S is the total value of this average value over the period (0, T). The differential equation can then be written as:

$$R' = l(3)C/L(3) - Al(1)k(1)^a R^{1-a}$$

$$R'+Al(1)k(1)^a R^{1-a} = l(3)C/L(3),$$

where $C = \sum_j\sum_k B(k)(5,j)g(j)(e^{b(k)T}-1)/Tb(k)$

Even this differential equation is still difficult to solve because of its non-linearity and non-homogeneity. However, some information can be obtained at the initial point and the terminal point. At these points s=0 and hence the differential equation at these points becomes:

$$R' + Al(1)k(1)^a R^{1-a} = 0$$

This equation may be solved if k(1) and l(1) are solved from other equations. This will be analyzed later.

The third constraint is:

$K(1)+K(2)+K(3) = K$, where K is constant overtime, or

$k(1)l(1)+k(2)l(2)+k(3)l(3) = k$

It is also assumed that L is constant overtime, hence $k = K/L$ is fixed. One must note that the model assumes that $k(1) = k(2)$.

4.3.1 The Computation

The optimization problem can be formulated as follows:

$$\text{Maximize } J=\int_0^{T(1)} e^{-rt} \left(pAl(1)k(1)^a R^{1-a} - (k(1)V+1)(1-l(3)) - \sum_j \sum_k B(k)(5,j)(k(3)Va(j)+c(j))/S(j)\,(e^{b(k)T}-1)/b(k)\right)$$

Subject to:

$$R'=l(3)/L(3) \sum_j \sum_k B(k)(5,j)g(j)(e^{b(k)T}-1)/Tb(k) - Al(1)k(1)^a R^{1-a}$$

$$S=l(3)/L(3) \sum_j \sum_k B(k)(5,j)g(j)(e^{b(k)T}-1)/b(k)$$

$k(1)(l(1)+l(2))+k(3)l(3) = k$, or

$k(1)(1-l(3))+k(3)l(3) = k$

$S(0) = 0,\ R(0) = R(0),\ S(T(a))) = 0,\ R(T(1)) = 0$

$l(j),\ k(j) \geq 0,\ j=1, 2, 3$ and $S, R \geq 0$

The state variables are R and S and the control variables are l(1), l(3), k(1), k(3). Note that it is assumed that k(1)=k(2), as previously stated and T(a) is the time at which the total amount of petroleum produced is equal to the total amount of discovery and $T(1)$ is the time for complete exhaustion.

The Lagrange equation can be written as:

$$L=e^{-rt}((pAl(1)k(1)^{a}R^{1-a}-(k(1)V+1)(1-l(3)-l(3)$$

$$\sum_{j}\sum_{k}B(k)(5,j)(k(3)Va(j)+c(j))/\varepsilon(j)\ (e^{b(k)T}-1)/b(k))$$

$$-h(1)(R'-l(3)/L(3)\sum_{j}\sum_{k}B(k)(5,j)g(j)(e^{b(k)}-1)/Tb(k)$$

$$Al(1)k(1)^{a}R^{1-a})-h(2)(S-l(3)/L(3)\sum_{j}\sum_{k}B(k)(5,j)g(j)(e^{b(k)T}$$

$$-1)/b(k))+h(3)(k-k(1)(1-l(3))-k(3)l(3)))$$

The discounted Hamiltonian, $H(d)$ is:

$$H(d)=e^{-rt}((pAl(1)k(1)^{a}R^{1-a}-(k(1)V+1)(1-l(3))-l(3)$$

$$\sum_{j}\sum_{k}B(k)(5,j)(k(3)Va(j)+c(j))/\varepsilon(j)\ (e^{b(k)T}-1)/b(k)))$$

$$+h(1)(l(3)/L(3)\sum_{j}\sum_{k}B(k)(5,j)g(j)(e^{b(k)T}-1)/Tb(k))-$$

$$Al(1)k(1)^{a}R^{1-a})+h(2)(l(3)/L(3)\sum_{j}\sum_{k}B(k)(5,j)g(j)$$

$$(e^{b(k)T}-1)/b(k)))-h(3)(k(1)(1-l(3))+k(3)l(3)))$$

And $H(d)=e^{-rt}H$. The variables $h(1)$, $h(2)$ are the costate variables and $h(3)$ is Lagrange multiplier. Note that $h(3)$ is given a positive sign in the Lagrange equation in order to accomodate an economic interpretation for $h(3)$.

Define:

$$\sum_{j}\sum_{k}B(k)(5,j)a(j)/\varepsilon(j)\ (e^{b(k)T}-1)/b(k)=m$$

$$\sum_j \sum_k B(k)(5,j)c(j)/\delta(j) \quad (e^{b(k)T}-1)/b(k) = n$$

$$\sum_j \sum_k B(k)(5,j)g(j)(e^{b(k)T}-1)/b(k) = f$$

Then the Lagrange equation can be rewritten as:

$$L=e^{-rt}(pq-(Vk(1)+1)(1-l(3))-l(3)(mVk(3)+n)-h(1)(R'-$$

$$(l(3)f/L(3)T)+q)-h(2)(S-l(3)f/L(3))+h(3)(k-(1-l(3))k(1)$$

$$-l(3)k(3)))$$

The Hamiltonian is then:

$$H(d)=e^{-rt}(pq-(Vk(1)+1)(1-l(3))-l(3)(mVk(3)+n)+h(1)$$

$$((l(3)f/L(3)T)+q)+h(2)l(3)f/L(3)-h(3)((1-l(3))k(1)$$

$$k(3)l(3)))$$

The necessary conditions are:

(1) $dpL/dpR - d(dpL/dpR')/dt = 0$ or

$$dpL/dpR = d(dpL/dpR')/dt$$

$$dpL/dpR=(p(1-a)Al(1)k^{a}R^{-a}-h(1)(1-a)Al(1)k^{a}R^{-a})e^{-rt}$$

$$dpH(d)/dpR=(p(1-a)Al(1)k^{a}R^{-a}-h(1)(1-a)Al(1)k^{a}R^{-a})e^{-rt}$$

Hence,

$$dpL/dpR=dpH(d)/dpR=dpH/dpRe^{-rt}$$

$$dpL/dpR' = -h(1)e^{-rt}$$

$$d(dpL/dpR')/dt= rh(1)e^{-rt}-h'(1)e^{-rt}$$

$$=e^{-rt}(rh(1)-h'(1))$$

Hence,

$$dpHe^{-rt}/dpR = e^{-rt}(rh(1)-h'(1))$$

$dpH/dpR = (rh(1)-h'(1)$

So,

$$dpH/dpR=(p(1-a)Al(1)k^{a}(1)R^{-a} -h(1)(1-a)Al(1)k^{a}(1)R^{-a})$$

$$(p-h(1))(1-a)Al(1)k^{a}(1)R^{-a} =rh(1)-h'(1) \quad \ldots\ldots (4-4)$$

Note $h'(1) = dh/dt$

Since $dpq/dpR = (1-a)q/R$, then

$$(p-h(1))(1-a)q/R = rh(1)-h'(1) \quad \ldots\ldots\ldots\ldots (4-4a)$$

(2) $dpL/dpS - d(dpL/dps)/dt = 0$

$$dpL/dpS = -h(2)e^{-rt}$$

$$dpL/dps = h(1)e^{-rt} \quad (S' = s)$$

Note that $R' = s - q$ and that $d(\int sdt)/dt = 0$ because $\int sdt$ is considered to be only a function of $l(3)$, $L(3)$ once the integral has been evaluated at a fixed T. However, if T is not fixed, then $d(\int sdt)/ds) = s$. But this formulation will only complicate the problem without providing additional information.*

$$d(dpL/dps)/dt= h'(1)e^{-rt} -rh(1)e^{-rt}$$

$$d(dpL/dps)/dt=e^{-rt}(h'(1)-rh(1))$$

$$dpL/dpS = dpH(d)/dpS = -h(2)e^{-rt}$$

$$dpH/dpS\, e^{-rt} - e^{-rt}(h'(1)-rh(1)) = 0$$

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* If T is not fixed, then result of the necessary condition with respect to S is: $h'(1)-rh(1) = -sh'(2)+(rs-1-s')h(2)$. This differential equation is unsolveable without further information on $h(1)$ and $h(2)$.

Hence, $-h(2) = (h'(1)-rh(1))$

$$h(2) = rh(1)-h'(1) \qquad \text{................(4-5)}$$

(3) $dpL/dpk(1) - d(dpL/dpk'(1))/dt = 0$

$$dpL/dpk(1)=e^{-rt}(pdpq/dpk(1)-V(1-l(3))-h(1)dpq/dpk(1)-h(3)(1-l(3)))$$

$dpL/dpk'(1) = 0$, since L is not a function of $k'(1)$.

$d(dpL/dpk'(1))/dt = 0$

$H(d)/dpk(1) = dpL/dpk(1)$

Hence,

$$e^{-rt}(pdpq/dpk(1)-V(1-l(3))-h(1)dpq/dpk(1)-h(3)(1-l(3))) = 0$$

Since $dpq/dpk(1) = aq/k(1)$, then:

$$aqh(1)/k(1) +(1-l(3))h(3)=apq/k(1) -V(1-l(3)) \text{..(4-6)}$$

(4) $dpL/dpl(1) - d(dpL/dpl'(1))/dt = 0$

$$dpL/dpl(1)=e^{-rt}(pdpq/dpl(1)-(Vk(1)+1)-(fh(1)/L(3)T)-(fh(2)/L(3))+(k(3)-k(1))h(3))$$

Note that $(1-l(3)) = (l(1)+l(2))$

$dpL/dpl(1) = 0$, since L is not a function of $l(1)$.

$$dpH(d)/dpl(1)=dpL/dpl(1)=dpH/dpl(1)\ e^{-rt}$$

Since $dpq/dpl(1) = q/l(1)$, then

$$(fh(1)/l(3)T)+(fh(2)/L(3))+(k(1)-k(3))h(3) = (pq/l(1))-(Vk(1)+1) \qquad \text{................(4-7)}$$

(5) $dpL/dpk(3) - d(dpL/dpk'(3))/dt = 0$

$dpL/dpk(3) = aVl(3) - l(3)h(3)$

$dpL/dpk'(3)=0$, since L is not a function of $k'(3)$.

$d(dpL/dpk'(3))/dt = 0$

$dpH(d)/dpk(3) = dpL/dpk(3) = (dpH/dpk(3))e^{-rt}$

Hence:

$mVl(3)-l(3)h(3) = 0$ ....(4-8)

(6) $dpL/dpl(3) - d(dpL/dpl'(3))/dt = 0$

$dpL/dpl(3) = (Vk(1)+1)-(mVk(3)+n)+(fh(1)/L(3)T)+(fh(2)/L(3)) + (k(1)-k(3))h(3)$

$dpL/dpl'(3) = 0$, since L is not a function of $l'(3)$

$d(dpL/dpl'(3))/dt = 0$

Hence:

$(fh(1)/L(3)T)+(fh(2)/L(3))+(k(1)-k(3))h(3) =$

$(mVk(3)+n)-(Vk(1)+1)$ ....(4-9)

Rewriting the results of the Euler-Lagrange equations together, one has the following:

$(p-h(1))(1-a)q/R = rh(1)-h'(1)$ ....(4-4a)

$h(2) = rh(1)-h'(1)$ ....(4-5)

$(aqh(1)/k(1))+(1-l(3))h(3) = (apq/k(1))-V(1-l(3))$

...............(4-6)

$(fh(1)/L(3)T)+(fh(2)/L(3))+(k(1)-k(3))h(3) =$

$(pq/l(1))-(Vk(1)+1)$ ....(4-7)

$mVl(3)-l(3)h(3) = 0$ ....(4-8)

$(fh(1)/L(3)T)+(fh(2)/L(3))+(k(1)-k(3))h(3) =$

$(mVk(3)+n)-(Vk(1)+1)$ ....(4-9)

From equation (4-8) one obtains:

$h(3) = +mV$ ....(4-10)

From equations (4-7) and (4-9) one obtains:

$(mVk(3)+n) = pq/l(1)$ ....(4-11)

Equation (4-4a) shows that the change in the production cost due to the change in the net reserve, and the change in the value of the deposit in the future, yield an interest rate on the scarcity rent equal to r. This condition characterizes the optimal net reserve in situ resource. Equation (4-5) indicates that when the stock of the resource can be supplemented by exploration, the change in the value of the deposit yield an interest rate on the scarcity rent equal to r. In this respect, h(1) is interpreted to be the scarcity rent and h(2) to be the marginal value of information. Equation (4-6) shows that under the optimum production and exploration, the price of petroleum will be equal to the marginal cost of production plus the scarcity rent. Equation (4-11) shows that the value of the capital used in exploration is proportional to the value of petroleum produced.

To find the relationship between h(1) and h(2), substitute h(3) from equation (4-10) into equation (4-7):

$$(fh(1)/L(3)T)+(fh(2)/L(3))+mV(k(1)-k(3))=(pq/l(1))-(Vk(1)+1)$$

$$(f/L(3))((h(1)/T)+h(2))=(pq/l(1))+mV(k(3)-k(1))-(Vk(1)+1)$$

$$((h(1)/T)+h(2)) = (L(3)/f)((pq/l(1))+mV(k(3)-k(1))-(Vk(1)+1))$$

$$((h(1)/T)+h(2)) = (L(3)/f)((mVk(3)+n)+mVk(3)-Vk(1)(m+1)-1)$$

$$((h(1)/T)+h(2)) = (L(3)/f)((n-1)+2mVk(3)-Vk(1)(m+1) \qquad \text{..........}(4-12)$$

To find the expression for h(1), solve the differential equation (4-4a) as follows:

$$p(1-a)q/R = -h'(1)+(r+(1-a)q/R)h(1), \text{ or}$$

$$-p(1-a)q/R = h'(1)-(r+(1-a)q/R)h(1)$$

The solution for the above differential equation is:

$$h(1)e^{-(rt+(1-a)\int q/R\,dt)} = -p(1-a)\int q/R\,dt \quad \ldots\ldots (4-13)$$

Further substitute h(2) from equation (4-5) into equation (4-12):

$$(h(1)/T)+rh(1)-h'(1)=(L(3)/f)((n-1)+2mVk(3)-Vk(1)(m+1))$$

$$h'(1)-((1/T)+r)h(1)=(L(3)/f)(Vk(1)(m+1)-(n-1)-2mVk(3))$$

The solution for the above differential equation is:

$$h(1)e^{-((1/T)+r)t} = (L(3)/f)\int(Vk(1)(m+1)-(n-1)-2mVk(3))\,dt \quad \ldots\ldots\ldots\ldots\ldots\ldots (4-14)$$

Equations (4-13) and (4-14) are identical if and only if the RHS and the LHS terms of both equations are equal, namely:

$$e^{-(rt+(1-a)\int q/R\,dt)} = e^{-((1/T)+r)t}, \text{ and}$$

$$-p(1-a)\int q/R\,dt = (L(3)/f)\int(Vk(1)(m+1)-(n-1)-2mVk(3))\,dt \quad \ldots\ldots\ldots\ldots\ldots (4-14a)$$

To solve equation (4-13), it is necessary that the above conditions hold. Hence:

$$(rt+(1-a)\int q/R\,dt) = ((1/T)+r)t$$

$$\int q/R\,dt = t/(T(1-a)) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4-15)$$

Subtituting the above result into equation (4-13) one obtains:

$$h(1)e^{-(r+(1/T))t} = -pt/T$$

$$h(1) = -(pt/T)e^{(r+(1/T))t} \quad \text{................} (4\text{-}16)$$

This is the optimum path of h(1) when equation (4-14a) is satisfied. To solve for h(2), first take the derivative of h(1) with respect to t:

$$h'(1) = -(r+(1/T))(pt/T)e^{(r+(1/T))t} - (p/T)e^{(r+(1/T))t}$$

$$h'(1) =-((r+(1/T))t+1)(p/T)e^{(r+(1/T))t} \quad \text{.........} (4\text{-}16a)$$

Then solve h(2) by substituting h'(1) and h(1) above into equation (4-5):

$$h(2) = rh(1)-h'(1)$$

$$h(2)=((r+(1/T))t+1)(p/T)e^{(r+(1/T))t}-(rpt/T)e^{(r+(1/T))t}$$

$$h(2) = (1+(t/T))(p/T)e^{(r+(1/T))t} \quad \text{............} (4\text{-}17)$$

To solve the differential equation of the constraint, differentiate equation (4-14a) with respect to t:

$$d(\int q/R\ dt)/dt = d(t/(1-a)T)/dt$$

$$q/R = 1/(1-a)T \text{ or}$$

$$Al(1)k(1)^{a}R^{-a} = 1/(1-a)T \quad \text{...............} (4\text{-}18)$$

Recall that the constraint differential equation is:

$$R' + Al(1)k(1)^{a}R^{1-a} = l(3)C/L(3)$$

where $C = \sum_j \sum_k B(k)(5,j)g(j)e^{b(k)t}$

Substitute equation (4-18) into the above differential equation:

$$R' + R/(1-a)T = l(3)C/L(3)$$

The above equation is a non-homogeneous first order linear differential equation. Since it is assumed that the total amount of workers, L, is fixed, and that only reallocation of workers is possible, then $l(3)/L(3) = 1/L$ is also fixed, the above differential equation has the following solution:

$$R = e^{-t/(1-a)T} \; 1/L \int C\, dt$$

$$R=e^{-t/(1-a)T} \sum_{j} \sum_{k} B(k)(5,j)g(j)(e^{b(k)T}-1)/b(k)L+R(0)$$

.... (4-19)

The above equation can also be written as:

$$R = (f/L)e^{-t/((1-a)/T)} + R(0) \qquad \text{..............} (4\text{-}19a)$$

Note that the function R is not continuous at t = 0, since the first term on the RHS of the above equation is a decreasing function for every value of t. In other words, when t = 0, R = R(0), but whenever t > 0, the value of R suddenly jumps into a value expressed in equation (4-19) or (4-19a). The value of f must be positive and results in a positive present value of the net reward function; otherwise the exploration venture is not economically feasible (see Chapter 3 on Net reward function).

Since the first term on the RHS of equation (4-19) is a decreasing function, it eventually becomes very small; it becomes zero only when t is equal to infinity. However, for practical purposes, if this term reaches a certain value Z, where Z can be defined as the amount of reserve where the explotation of which can no longer be profitable, the term can be assumed to be zero. The value of Z is determined by the recovery ratio. Suppose that the time at which the first term of the RHS of equation (4-19) becomes zero is at $t = T(a)$, then at this point, the net reserve R is equal to $R(0)$ and $C = 0$. Then the differential equation for R, becomes a homogeneous first order linear differential equation. Its solution is:

$$R = R(0)e^{-t/(1-a)T}, \text{ for } t > T(a) \qquad \text{..........(4-20)}$$

Hence, for $t > T(a)$ the depletion proceeds according to equation (4-20). Note that there is no relationship between T and T(a). T(a) can be defined as the time at which the total amount of petroleum produced is equal to the total amount of discovery. And T is the estimated length of time for exploration.

One can then conclude that for $0 < t \leq T(a)$, the optimum path for R follows equation (4-19), and for $t > T(a)$, it follows equation (4-20). In general there is a kinked point on the path for R at $t = T(a)$.

There are four possible depletion cases, namely:

(1) When $q < s$;

(2) When $q = s$;

(3) When $q > s$;

(4) when $s = 0$ (no discovery)

For the same amount of total discovery, the time for complete exhaustion, T(1), is the greatest in case (1). Also the time T(a) is the longest in case (1). Note that T(a) = 0 in case (4), since there is no discovery.

To solve for k(1) substitute h(3) from equation (4-10) and h(1) from equation (4-16) into equation (4-6):

$$(aq/k(1))(p-h(1)) = V(1-l(3))+mV(1-l(3))$$

$$(aq/k(1))(p-h(1)) = V(1-l(3))(1+m)$$

$$k(1) = (aq(p-h(1))/(V(1-l(3))(1+m))$$

$$k(1) = (apq)(1+(t/T))e^{(r+(1/T))t}/(V(1-l(3))(1+m)) \qquad \text{.............. (4-21)}$$

To solve for k(3) substitute h(1) from equation (4-16) into equation (4-14), one obtains:

$$-pt/T = (L(3)/f) \int (Vk(1)(m+1)-(n-1)-2mVk(3))\,dt$$

Differentiate the above equation with respect to t:

$$-p/T = (L(3)/f)(Vk(1)(m+1)-(n-1)-2mVk(3))$$

$$-(fp/L(3)T) = Vk(1)(m+1)-(n-1)-2mVk(3)$$

Substitute the value of k(1) from equation (4-21) into the above equation:

$$-(fp/L(3)T) = (apq(1+(t/T))e^{(r+(1/T))t}/(1-l(3)))-(n-1)$$
$$-2mVk(3)$$

$$k(3) = (apq(1+(t/T))e^{(r+(1/T))t}/2mV(1-l(3)))+$$
$$(fp/2L(3)T)-(n-1)/2mV \qquad \text{................ (4-22)}$$

To solve for l(1), use equation (4-11):

$$(mVk(3)+n) = pq/l(1)$$

$$l(1) = pq/(mVk(3)+n) \qquad \text{................ (4-23)}$$

Substituting the value of k(3) from equation (4-22) into equation (4-23), one obtains:

$$l(1) = 2pTL(3)q(1-l(3))/(aLTL(3)q(1+(t/T))e^{(r+(1/T))t}$$
$$+fp(1-l(3))+(n-1)(1-l(3))TL(3)) \quad \text{........} (4\text{-}24)$$

Equation (4-21), (4-22), and (4-24) provide the optimum solutions for k(1), k(3), and l(1), in terms of q, t, and l(3). To solve for q, take the derivative of R with respect to t from equation (4-19a) and (4-20):

$$R' = -fe^{-t/(T(1-a))}/(TL(1-a)), \quad \text{(from (4-19a))}$$

$$R' = -R(0)e^{-t/(T(1-a))}/(T(1-a)), \quad \text{(form (4-20))}$$

Substitute the above results, each into the contraint equation: R' = s - q, or R' = (f/LT)-q, one obtains:

$$q = (f/LT)(1+e^{-t/(T(1-a))}/(1-a)), \quad 0 < t \leq T(a)$$
$$\text{.............} (4\text{-}25)$$

$$q = R(0)e^{-t/(T(1-a))}/(T(1-a)), \quad t > T(a) \quad \text{.....} (4\text{-}25a)$$

Recall that the equations for k(1), k(3), and l(1) still contain l(3) as its variable. Since there are only three equations, namely equations (4-6), (4-7), and (4-9), to solve for k(1), k(3), l(1), and l(3), then one is forced to assume a condition for the variable l(3). If l(3) is assumed to be an exogeneous variable, then the system can be solved. The control variables k(1), k(3), and l(1) can be solved by equations (4-21), (4-22), and (4-24), respectively. Note that one needs to substitute the values of q from equations (4-25) and (4-25a) into equations (4-21), (4-22), and (4-24) for the appropriate time periods, in order to solve for the optimum paths of the control variables. Hence, it is necessary to assume l(3) as an exogeneous variable in order to provide solutions to the problem. One must then interprete the problem, that for each value of l(3), there is one optimization problem to solve. The solutions for the state and control variables obtained, maximize the objective functional of that particular problem.

4.3.2 The Evaluation of The Results

The results can be summarized as follows:

(1) The state variables:

$$R = (f/L)e^{-t/((1-a)/T)} + R(0), \quad \text{for } 0 < t \leq T(a)$$

$$R = R(0)e^{-t/(1-a)T}, \text{ for } t > T(a)$$

$$S = (f/L)$$

(2) The control variables:

$$k(1) = (apq)(1+(t/T))e^{(r+(1/T))t}/(V(1-l(3))(1+m))$$

$$k(3) = (apq(1+(t/T))e^{(r+(1/T))t}/2mV(1-l(3))) + (fp/2L(3)T) - (n-1)/2mV$$

$$l(1) = 2pTL(3)q(1-l(3))/(aLTL(3)q(1+(t/T))e^{(r+(1/T))t} + fp(1-l(3)) + (n-1)(1-l(3))TL(3))$$

(3) The costate variables:

$$h(1) = -(pt/T)e^{(r+(1/T))t}$$

$$h(2) = (1+(t/T))(p/T)e^{(r+(1/T))t}$$

$$h(3) = mV$$

The model requires that K and L are constant overtime, and that l(3) be an exogeneous variable. Each value of l(3) provides a separate optimization problem. Also it is assumed that k(1) = k(2). These restrictions are necessary in order to solve the optimization problem and provide feasible solutions.

The net reserve R, changes negative exponentially overtime. However, a discovery delays the time of complete exhaustion T(1). Due to a discovery, the net reserve changes between t = 0 and t = T(a) according to equation (4-19). At t = T(a), when the total amount of the discovery is equal to the total amount of the resource used, the net reserve begins to change according to equation (4-20). At this stage, the effect of discovery on depletion no longer exist and the resource begins to deplete. In general, the optimum path of the net reserve makes a kinked point at t = T(a). Figure 4-1 shows the optimum paths for R in the four possible cases, namely when q < s, q = s, q > s, and s = 0.

Figure 4-2 indicates the optimum path for q, at $0 < t \leq T(a)$ and at $t > T(a)$.

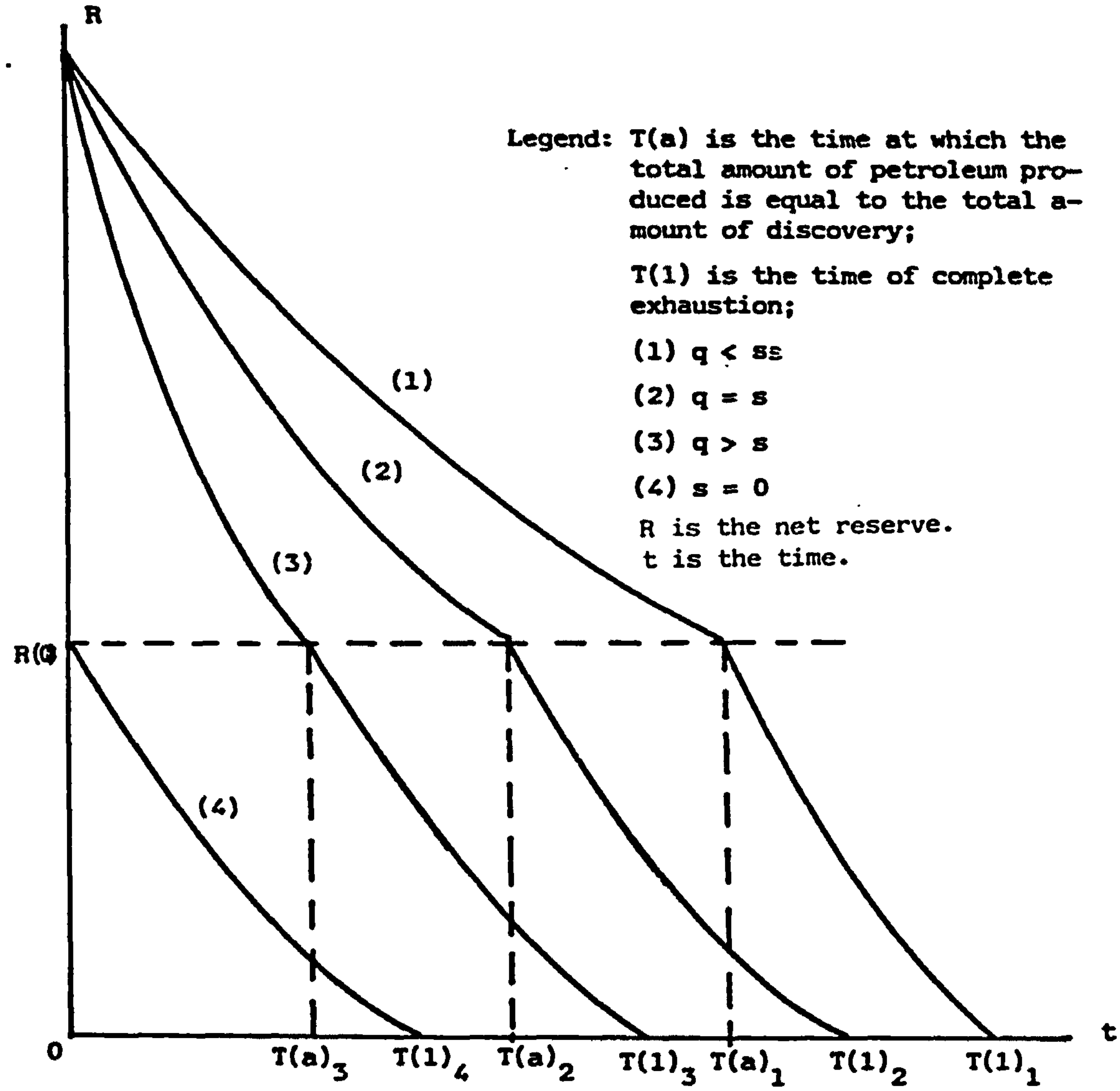

Figure 4-1: The Optimum Path for R

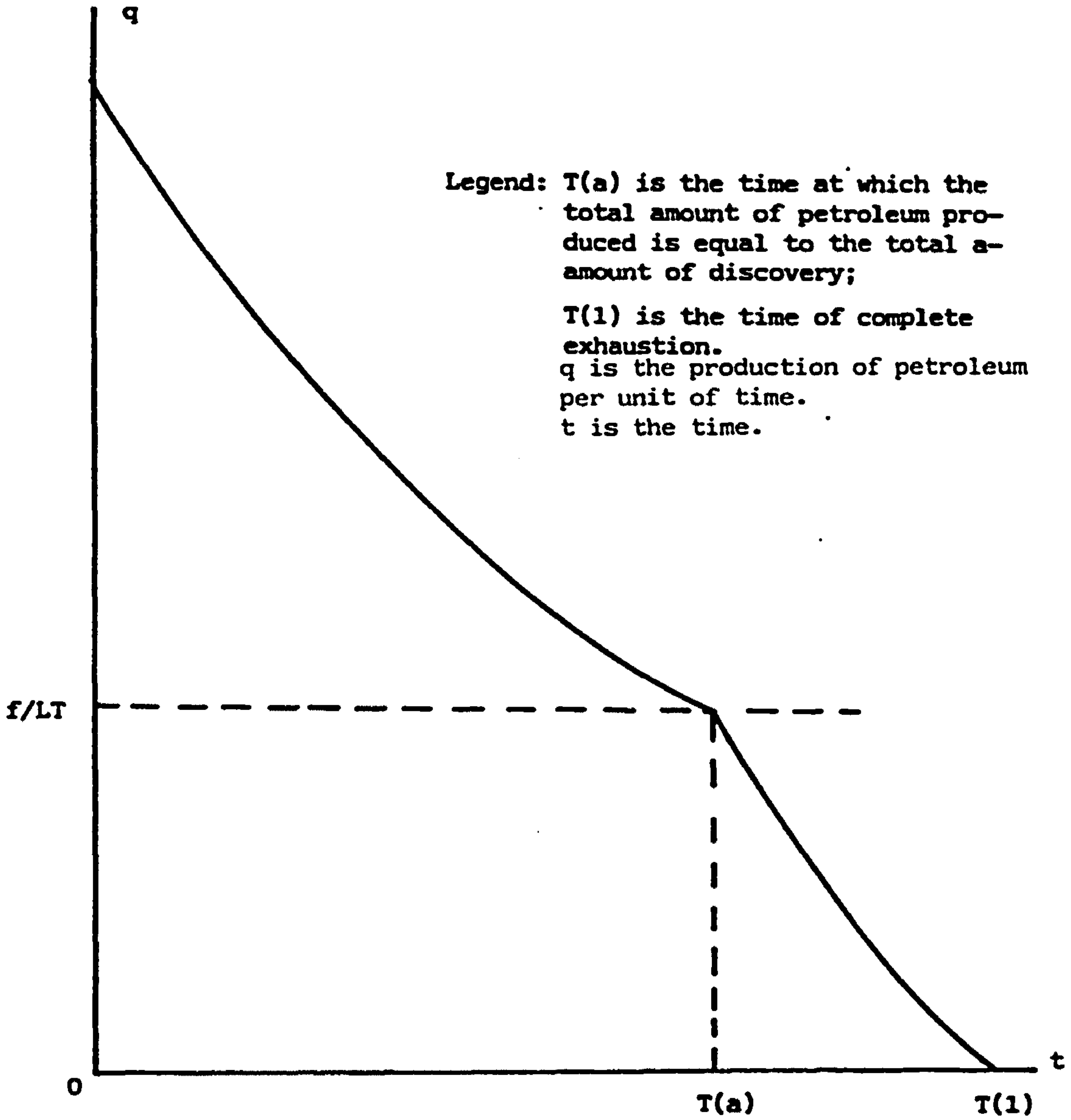

Figure 4-2: The Optimum Path for q

To derive the graph for the optimum path for $k(1)$, one has to evaluate the value:

$$q(1+(t/T))e^{(r+(1/T))t}$$

For $0 < t \leq T(a)$, the value of q contains a constant (see equation (4-25)); although q is actually decreasing, the above multiplication may be increasing. If this is the case, then $k(1)$ is increasing at $0 < t \leq T(a)$. For $t > T(a)$, the resulting exponential multiplication is:

$$e^{(r+(1/T)-1/(T(1-a)))t}(1+(t/T))$$

If $rT+1 < 1/(1-a)$, then the above multiplication is decreasing. The value of t/T increases less than the exponential decrease at larger t. If this is the case, then $k(1)$ decreases at $t > T(a)$. Figure 4-3 shows a possible optimum path for $k(1)$. However, if $rT+1 > 1/(1-a)$, the $k(1)$ will be increasing at $t > T(a)$.

The trend of the optimum path for k(3) is similar to the one for k(1); Figure 4-4 shows the optimum path for k(3).

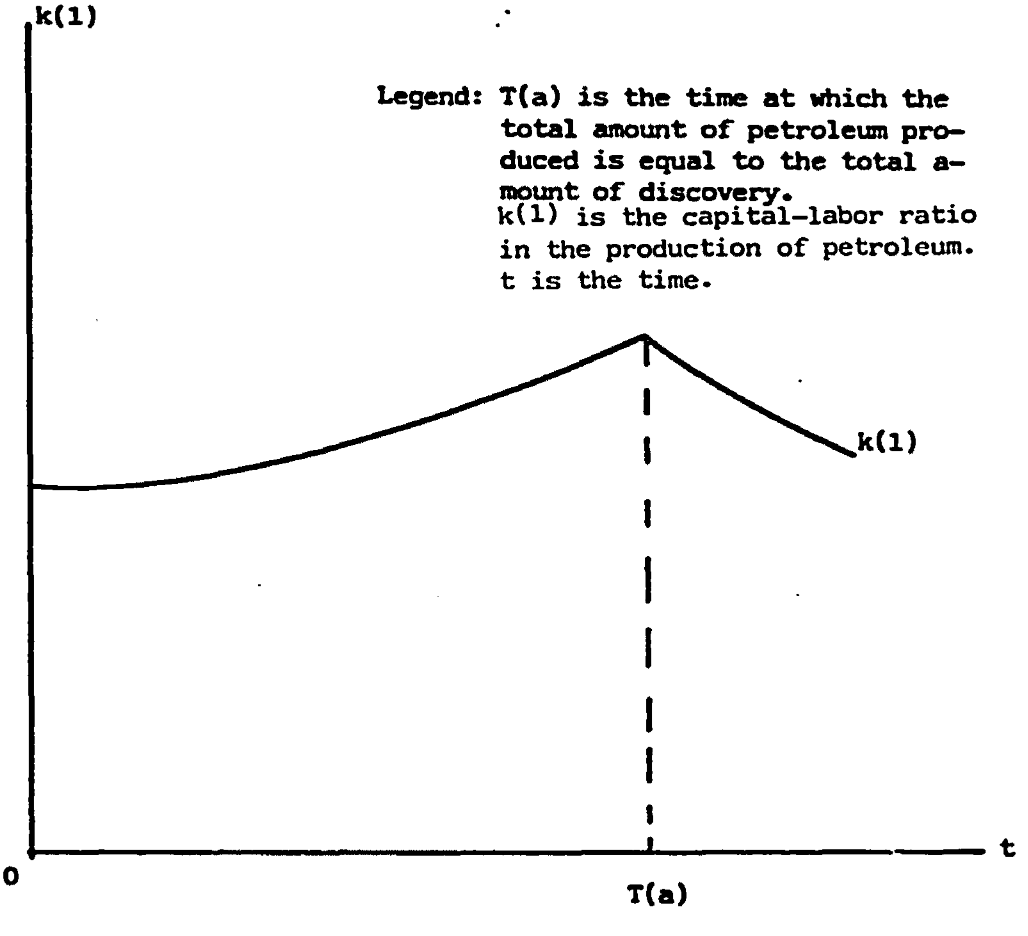

Figure 4-3: The Optimum Path for k(1)

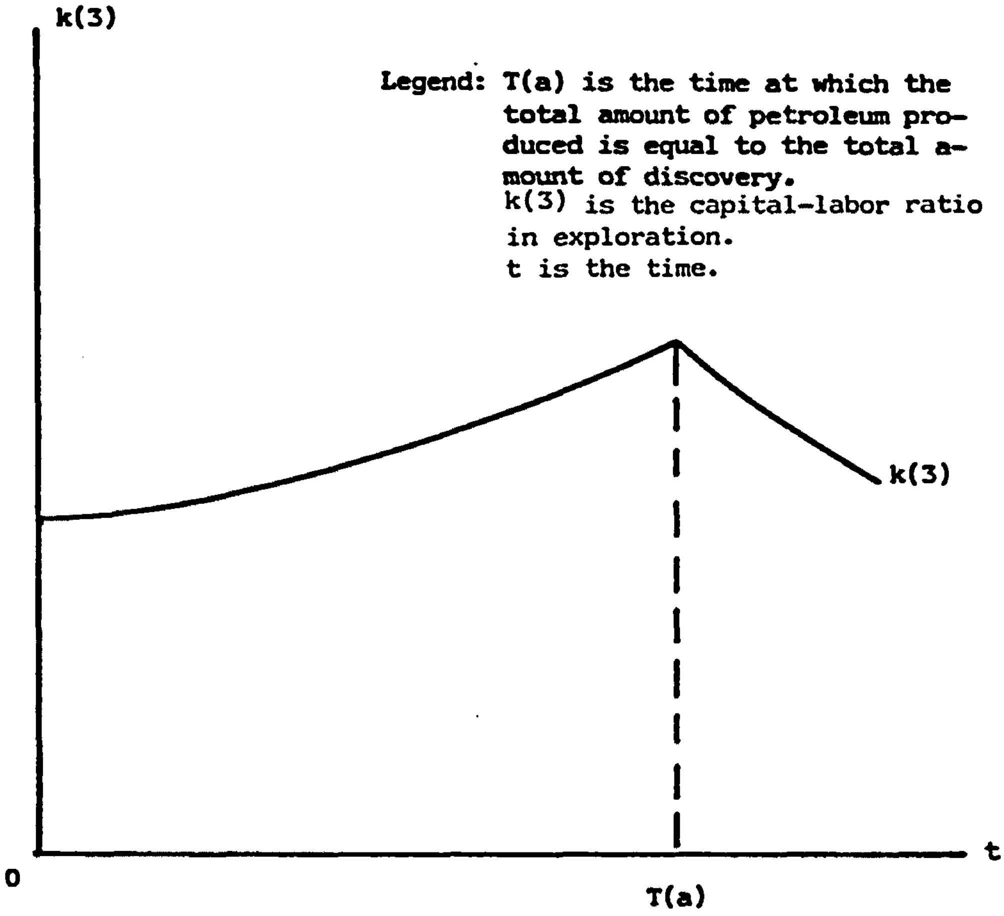

Figure 4-4: The Optimum Path for k(3)

Since k(3) must be positive, then it is required that:

$$(apq(1+(t/T))e^{(r+(1/T))t}/(1-l(3)))+(fp/L(3)T) > (n-1)$$

However, since f contains the reward vector and n is only a function of the Markov process, then it is likely that (fp/L(3)T) > (n-1), and so k(3) is definitely positive.

The equation for l(1) shows, that since q is decreasing, the nominator of the equation is decreasing for all t. For $0 < t \leq T(a)$, the first term of the denominator of the equation for l(1) increases (see evaluation for k(1)); consequently, since the nominator decreases and the denominator increases, l(1) must be decreasing at $0 < t \leq T(a)$. For t > T(a), the denominator of the equation for l(1) may decrease or increase. However, if it decreases, then the rate of decrease of the nominator is greater than the one of the denominator. This can be shown as follows:

The exponential power of the nominator is -t/(T(1-a)), while of the denominator is -t/(T(1-a))+(r+(1/T))t. Since r and T are positive, then the exponential power of the nominator is negatively greater than that of the denominator. This implies that the rate of decrease of the nomininator is greater than that of the denominator. Consequently, l(1) decreases at t > T(a). Figure 4-5 shows an optimum path for l(1).

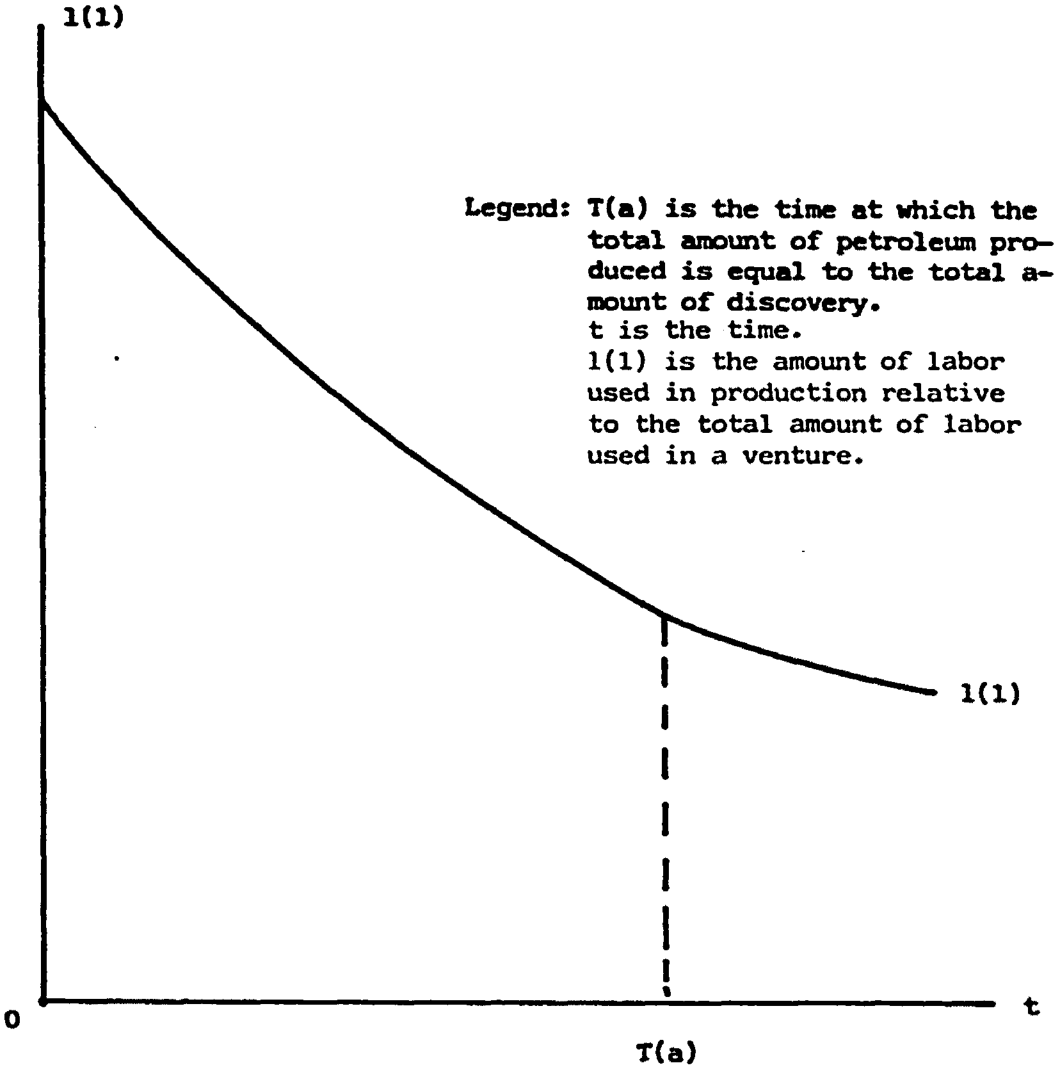

Figure 4-5: The Optimum Path for l(1)

The above evaluations for k(1), k(3), and l(1) indicate that the use of capital for production is increasing until the net reserve left is equal to the initial reserve. After this point, the capital used for production may increase or decrease. If at $t > T(a)$, k(1) actually decrease, and its rate of decrease is greater than that of l(1), the use of capital over this period decreases. Otherwise, the use of capital must increase.

The use of capital for exploration increases at $0 < t \leq T(a)$, because more intensive exploration methods have to be used as it approaches a discovery. In general, more intensive exploration methods are more capital intensive, hence they require more capital. At $t > T(a)$, the use of capital may increase or decrease. If k(3) actually decreases, and since l(3) is given, then the use of capital in exploration at $t > T(a)$ must decrease. However, if one implies that no exploration activities exist at $t > T(a)$, then the use of capital stops at $t = T(a)$. Note that in the computation for the discounted profit that will be presented in the next section, one assumes that exploration still proceeds at $t > T(a)$ but with no possible discovery. In such case one must extend the use of capital at $t > T(a)$.

The costate variable h(1) is negative and it can be interpreted as the scarcity rent, namely the user cost of the marginal unit being extracted at any point of time. The user cost is defined as the present value of all future sacrifices associated with the use of a particular unit of in-situ petroleum resource (Howe, 1979; p.78). Hence h(1) is basically a cost and so it has a negative sign. The value of h(1) increases negatively overtime. The multiplier h(3) is positive and constant; it can be interpreted as the shadow price for the overal capital. The costate variable h(2) is positive and is increasing exponentially. This costate variable can be interpreted as the marginal value of the information resulted from exploration (Howe, 1979, pp.218-219). Its value increases overtime as exploration becomes more difficult as it approaches a discovery or after a discovery has been made.

Figure 4-6 shows an optimum path for h(1), h(2), .

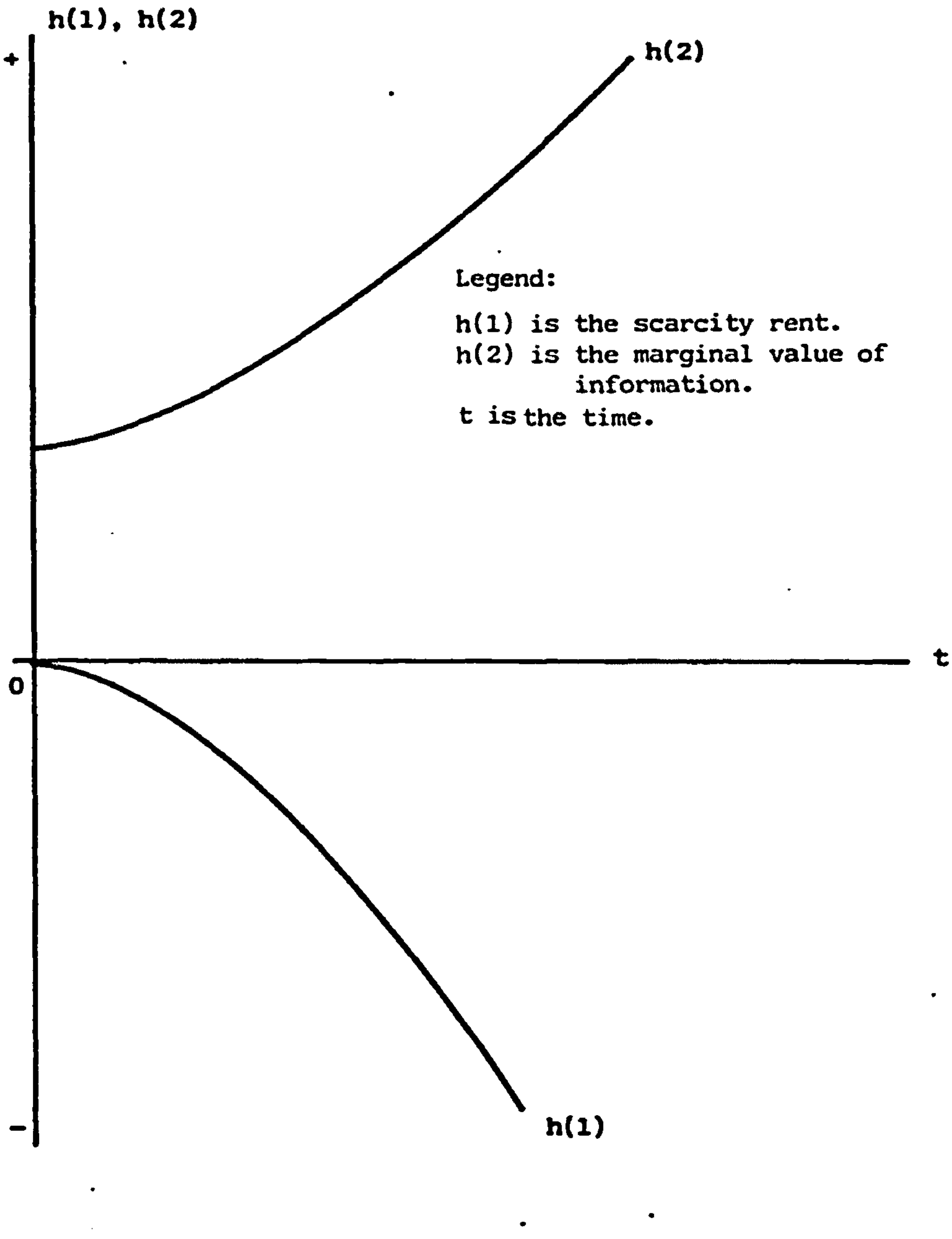

Figure 4-6: The Optimum Path for the Costate Variables

4.3.3 The Discounted Total Profit

Finally one can compute the discounted total profit by substituting the values for q, k(1), and k(3) into the objective functional. The objective function becomes a fuction of t and the integration can then be performed. The computation for the discounted total profit is as follows:

Recall that the objective function is:

$$J = \int_0^{T(1)} e^{-rt} \Big(pq - (Vk(1)+1)(1-l(3)) - \sum_j \sum_k B(k)(5,j)\,l(3)\,(a(j)Vk(3)+c(j))/\delta(j)\,(e^{b(k)T}-1)\Big)\,dt$$

Compute the above integral term by term. It is assumed that the exploration still continue at t > T(a), even without any possible discovery. One must note that for the following computation, any term containing q must be evaluated for the following values of q:

$$q = (f/LT)(1+e^{-t/(T(1-a))}/(1-a)), \quad 0 < t \leq T(a)$$

(see equation (4-22)), and :

$$q = (R(0)/(T(1-a)))e^{-t/(T(1-a))}, \quad t > T(a)$$

(see equation (4-25a)).

The computations are as follows:

$$(1)\ \int_0^{T(1)} e^{-rt}\,pq\,dt = \int_0^{T(a)} e^{-rt}\,pq\,dt + \int_{T(a)}^{T(1)} e^{-rt}\,pq\,dt$$

$$A = \int_0^{T(a)} pqe^{-rt}dt; \quad B = \int_{T(a)}^{T(1)} pqe^{-rt}$$

$$A = (fp/LT)\int_0^{T(a)} (e^{-rt} + e^{-(1+(1-a)rT)t/(T(1-a))})\,dt$$

$$A = (fp/LT)(((1-e^{-rT(a)})/r) + ((1-e^{-cT(a)})/c))$$

$$\ldots\ldots\ldots\ldots (4\text{-}26)$$

where $c = (1+(1-a)rT)/(T(1-a))$

$$B = \int_{T(a)}^{T(1)} pqe^{-rt}dt = (pR(0)/(T(1-a)))\int_{T(a)}^{T(1)} e^{-ct}dt$$

$$B = (pR(0)/(cT(1-a)))(e^{-cT(a)} - e^{-cT(1)}) \;\ldots\ldots (4\text{-}27)$$

Hence:

$$\int_0^{T(1)} e^{-rt} pq\, dt = (A + B)$$

$$(2) \int_0^{T(1)} e^{-rt} (Vk(1)+1)(1-l(3))dt =$$

$$(1-l(3))V\int_0^{T(1)} e^{-rt} k(1)\, dt + (1-l(3))\int_0^{T(1)} e^{-rt}\, dt$$

$$D = (1-l(3))\int_0^{T(1)} e^{-rt} dt = (1-l(3))(1-e^{-rT(1)})/r \;\ldots (4\text{-}28)$$

$$E = V(1-l(3))\int_0^{T(1)} e^{-rt} k(1)\, dt$$

$$E = V(1-l(3))\left(\int_0^{T(a)} e^{-rt} k(1)\, dt + \int_{T(a)}^{T(1)} e^{-rt} k(1)\, dt\right)$$

$$E(1) = V(1-l(3))\int_0^{T(a)} e^{-rt} dt$$

$$E(1) = (af/(LT(1+m)))\int_0^{T(a)} (e^{t/T} + e^{-at/(T(1-a))})$$
$$(1+(t/T))\, dt$$

$$E(1) = (af/(LT(1+m)))(T(e^{T(a)/T}-1) + (1-a)T(1-$$
$$e^{-aT(a)/(T(1-a))}) + (T(a)/T)e^{T(a)/T} - (e^{T(a)/T}-1))$$

$$\ldots\ldots\ldots\ldots (4\text{-}29)$$

$$E(2) = V(1-l(3))\int_{T(a)}^{T(1)} e^{-rt} k(1)\, dt$$

$$E(2) = (aR(0)/(T(m+1)(1-a)))\int_{T(a)}^{T(1)} e^{-at/(T(1-a))} (1$$
$$(t/T))\, dt$$

$$E(2) = (aR(0)/(T(m+1)(1-a)))((e^{-aT(a)/(T(1-a))} - e^{-aT(1)/(T(1-a))})(1-a)((T+1)/a) + (T(1)e^{-aT(1)/(T(1-a))} - T(a)e^{-aT(a)/(T(1-a))})/T) \quad \ldots\ldots\ldots\ldots\ldots (4\text{-}30)$$

$$E = E(1)+E(2)$$

$$\int_0^{T(1)} e^{-rt}(VK(1)+1)(1-l(3))\,dt = D+E = D+E(1)+E(2)$$

$$(3)\int_0^{T(1)} e^{-rt} l(3)(mVk(3)+n)\,dt = mVl(3)\int_0^{T(1)} e^{-rt}k(3)\,dt + nl(3)\int_0^{T(1)} e^{-rt}\,dt$$

$$F = nl(3)\int_0^{T(1)} e^{-rt}dt = (nl(3)/r)(1-e^{-rT(1)}) \quad \ldots\ldots (4\text{-}31)$$

$$G = mVl(3)\int_0^{T(1)} e^{-rt}k(3)\,dt$$

$$G = (apl(3)/2(1-l(3)))(\int_0^{T(1)} e^{t/T}q(1+(t/T))\,dt$$

$$(al(3)/2)((fp/L(3)T)-n+1)\int_0^{T(1)} e^{-rt}\,dt)$$

$$G(1) = (al(3)/2)((fp/L(3)T)-n+1)(\int_0^{T(1)} e^{-rt}\,dt)$$

$$G(1) = (al(3)/2)((fp/L(3)T)-n+1)(1-e^{-rT(1)}) \quad \ldots (4\text{-}32)$$

$$G(2) = (apl(3)/(2(1-l(3))))(\int_0^{T(1)} e^{t/T}q(1+(t/T))\,dt)$$

$$G(2) = (apl(3)/(2(1-l(3))))(\int_0^{T(a)} e^{t/T}q(1+(t/T))dt + \int_{T(a)}^{T(1)} e^{t/T}q(1+(t/T))dt)$$

$$G(3) = (apl(3)/(2(1-l(3))))(\int_0^{T(a)} e^{t/T}q(1+(t/T))\,dt)$$

$$G(3) = (apl(3)f/(2LT(1-l(3))))(T(e^{T(a)/T}-1)+(1-a)T$$

$$(1-e^{-aT(a)/(T(1-a))})/a + (T(a)/T)e^{T(a)/T} -$$

$$(e^{T(a)/T}-1)) \qquad \ldots\ldots\ldots\ldots\ldots\ (4\text{-}33)$$

$$G(4) = (apl(3)/(2(1-l(3)))(\int_{T(a)}^{T(1)} e^{t/T} q(1+(t/T))\ dt)$$

$$G(4) = (apl(3)R(0)/(2T(1-l(3))(1-a)(e^{-aT(a)/(T(1-a))} -$$

$$e^{-aT(1)/(T(1-a))})(T+1)((1-a)/a)+(1/T)(T(1)$$

$$e^{-aT(1)/(T(1-a))} - T(a)e^{-aT(a)/(T(1-a))}))$$

$$\ldots\ldots\ldots\ldots\ldots\ (4\text{-}34)$$

$$G(2) = G(3)+G(4)$$

$$G = G(1)+G(2) = G(1)+G(3)+G(4)$$

Hence:

$$\int_0^{T(1)} e^{-rt} l(3)(mVk(3)+n)\ dt = F+G = F+G(1)+G(3)+G(4)$$

The maximum discounted total profit is:

$$J = (A+B) - (D+E) - (F+G)$$

$$J = (A+B) - (D+E(1)+E(2)) - (F+G(1)+G(3)+G(4))$$

4.3.4 The Model Revisited

The analysis has assumed that the expected price of petroleum and the state of technology are considered constant. Under such assumptions several conclusions have been drawn from the model. Suppose one were to ask the question: what inferences can be drawn from the model in the case of an exogeneously determined price change or technological change ? Several inferences on the short run effects of such changes can be drawn from the model.

First, suppose there is an increase in the price of petroleum; then the following inferences about the state, control, and costate variables can be drawn as follows:

(1) Equation (4-22) for the exploration capital labor ratio k(3) indicates that k(3) is directly related to the price of petroleum p. Hence, as p increases, other things remain the same, then k(3) increases. Since l(3) is exogeneously determined and is considered fixed, the amount of capital being used in exploration increases as k(3) increases due the increase in p. This implies that exploration efforts increase.

(2) As exploration efforts increase, the possibility of finding more petroleum resources increases. This will be reflected by the increase in the estimated amount of petroleum calculated by the volumetric method and this is shown by the increase of the value of the vector g(j). From the equation for f on page 138, one may notice that f is directly related to g(j); hence, as g(j) increases, f increases too. Equation (4-19a) for the net reserve R shows that R is directly related to f; this means that as f increases, the net reserve R increases. Graphically, this implies that the curve for the net reserve R, as shown in Figure 4-1, will have to shift to the right as the price increases. Consequently, the time for complete exhaustion, T(1), will be longer.

(3) Equation (4-21) for the capital labor ratio in production, k(1), shows that k(1) is directly related to the price p. So an increase in p will have to increase k(1). The curve for k(1), as shown in Figure 4-3, will shift upward. Equation (4-23) and (4-24) for the relative amount of labor used in production, l(1), shows that an increase in the price p will have an ambiguous effect on the value of l(1). The value of l(1) may increase or decrease or remain constant depending on the actual numerical values of the variables in the equation for l(1).

(4) Equation (4-21) for the amount of production q, shows that q is directly related to f; so as p increases, f increases, and in turn q increases. This means that the curve for q, as shown in Figure 4-2, will shift to the right to the right whenever the price p increases. In other words, the production of petroleum will increase if the price of petroleum increases. This situation is consistent to the behavior of firm in a competitive market structure and one may recall that the model is designed for such market structure. It is particularly in monopoly that one finds a retardation of production as the price of the resource commodity increases.

(5) Equation (4-16) for the costate variable h(1) indicates that an increase in the price p will cause the slope of the curve for h(1) to increase. This means that the rate of change in the scarcity rent increases negatively greater

over time. Also equation (4-17) for the costate variable h(2) shows that the slope of the curve for h(2) increases as the price p increases. Hence, the rate of change in the marginal value of information is greater over time. While the shadow price for the overal capital h(3) remains unchanged as the price p changes. This is shown by equation (4-10).

When there is a technological change in the extraction industry and if such change does not affect the reclassification of the submarginal resources into economic ones, then several inferences about such change can be drawn from the model. The change in the state of technology in petroleum production can be reflected by the change in the parameter A of the Cobb Douglas production function and technological progress will increase the value of A. An increase in the value A means that more petroleum can be produced per unit of time from the same amount of input resources used. Hence, as A increases, then q increases. The inferences that can be drawn from a change in technology are similar to that from the price increase. As production becomes relatively cheaper due to a technological progress in production, firms are stimulated to increase their exploration efforts because of the relatively higher profit that can be obtained from producing petroleum. This implies also that the net reserve R will have to increase. Equation (4-11) confirms the above conclusion. This equation indicates that for any given price p, the value of the petroleum produced increases as the production per unit of time increases and this value is proportional to the value of the capital used in exploration. Hence, as q increases, the capital used in exploration increases. The inferences about k(1) and l(1) are similar to that from the price increase. The equations for the costate variables indicate that these variables are not affected by the change in technology.

If the technological change affects the reclassification of the submarginal resources into reserves, then it is practically difficult to draw any inference from the model. The reason for this is that the model does not include any computational procedure for selecting and for quantifying submarginal resources. In other words, the analysis for such technological change is beyond the scope of this dissertation. In summary, one can conclude that the short run effects of price increase on the state and control variables are similar to that of a technological change.

Chapter V

AN EXTENSION OF THE MODEL

In Chapter IV, the Markovian exploration model has been used to analyze the behavior of the profit maximizing firm constrained by the exhaustibility of the resource. In this chapter, the analysis tries to suggest that the Markovian model can also be used in a simple three sector growth model, where an exhaustible resource is used as an input in the production function of one of the sectors. Like the depletion model discussed in Chapter IV, the uncertainty aspect is quantified by a Markov process and the entire model can be treated as a deterministic model. The model that this analysis suggests, is a three sector growth model that is based on the Intriligator's two sector growth model. Several suggestions on the possible extension of the Intriligator's model have already been mentioned in Chapter II. The analysis will discuss and present the Hamiltonian equations of the suggested models, but it will not provide solutions to the problems.

The first model is based on the assumptions that the price of the exhaustible resource, the wage and the rental rates are fixed. Other assumptions follow the suggestions already mentioned in Chapter 2. The second model, will relax the assumption on the wage and rental rates. Finally, a brief discussion will be presented for the model when the price of the exhaustible resource is not fixed and when the change in the wage rental ratio effects the profitability of of the exploration ventures.

5.1 THE FIRST MODEL

All the suggested models consist of an economy of three sectors, namely:

(1) Sector 1: produces capital goods

(2) Sector 2: produces consumption goods

(3) Sector 3: produces resource commodity

Their production functions are as follows:

(1) Sector 1: $y(1) = A(1)\,l(1)\,k(1)^{a(1)}\ y(4)^{1-a(1)}$

(2) Sector 2: $y(2) = A(2)\,l(2)\,k(2)^{a(2)}\ y(5)^{1-a(2)}$

(3) Sector 3: $y(3) = A(3)\,l(3)\,k(3)^{a(3)}\ R^{1-a(3)}$

(Note that the t is supressed from the above equations to simplify the notations).

where

$l(j)$ is the ratio of the workers used in sector j over the total workers available in the economy, defined as $l(j) = L(j)/L$;

$k(j)$ is the capital labor ratio of sector j, defined as $k(j) = K(j)/L(j)$;

R is the net reserve of the exhaustible resource

$y(j)$ is the output of sector j per capita, j =1, 2, 3.

$y(4) = b\ y(3)$; $y(5) = c\ y(3)$; and $b+c = 1$

From the definitions of $k(j)$ and $l(j)$ one can write that:

$$k(1)l(1) + k(2)l(2) + k(3)l(3) = k$$

$$l(1) + l(2) + l(3) = 1$$

where

k is the overal capital labor ratio of the economy, defined as $k = K/L$

$K(t)$ is the total aggregate stock of capital at time t;

$L(t)$ is the total labor force available at time t;

$K(j(t))$ is the amount of capital used in sector j at time t;

$L(j(t))$ is the amount of labor used in sector j at time t;

It is also known that:

$$k' = y(1) - uk$$

$$R' = s - y(3)$$

$$R(0) = R(0);\ S(0) = 0$$

$$L = mL'$$

where

k' is the rate of capital goods augmentation per capita;

u is the summation of the rate of the labor increase and the rate of the capital depreciation;

s is the rate of total reserves addition per capita;

R' is the rate of total net reserves additon;

R is the total net reserves;

S is total cumulative reserve additon per capita for the economy;

m is the rate of change in the labor force.

Assume that in the first model there are n mineral ventures, each has a rate of total reserve additon per capita equal to the following:

$$s(i)=1/L \sum_j \sum_k B(k)(5,j)g(j)(e^{b(k)T(i)}-1)/b(k)T(i), \quad i=1, \ldots n$$

and a total cumulative reserve additon equal to:

$$S(i)=1/L \sum_j \sum_k B(k)(5,j)g(j)(e^{b(k)T(i)}-1)/b(k), \quad i=1, \ldots\ldots n$$

Note that each venture may have different exploration network.

The rate of total reserve addition per capita for the economy is the average value of the rates of all ventures, namely:

$$s = 1/n \sum_{i=1}^{n} s(i).$$

And the total cumulative reserve addition for the economy is:

$$S = \sum_{i=1}^{n} S(i).$$

The optimization problem of the first model can be formulated as follows:

$$\text{Maximize } J = \int_0^{T(1)} e^{-rt}\, y(2)\; dt$$

Subject to:

$$k' = y(1) - uk$$

$$R' = 1/n \sum_{i=1}^{n} (1/L \sum_j \sum_k B(k)(5,j)g(j)(e^{b(k)T(i)}-1)$$
$$/T(i)b(k) - y(3)$$

$$S = \sum_{i=1}^{n} (1/L \sum_j \sum_K B(k)(5,j)g(j)(e^{b(k)T(i)} - 1)/b(k)$$

$$L' = mL$$

$$y(1) = A(1)l(1)k(1)^{a(1)} y(4)^{1-a(1)}$$

$$y(2) = A(2)l(2)k(2)^{a(2)} y(5)^{1-a(2)}$$

$$y(3) = A(3)l(3)k(3)^{a(3)} R^{1-a(3)}$$

$$l(1) + l(2) + l(3) = 1$$

$$k(1)l(1) + k(2)l(2) + k(3)l(3) = k$$

$$R(0) = R(0);\ S(0) = 0$$

$$k(j),\ l(j) \geq 0,\ j = 1, 2, 3.$$

The control variables are: $k(1)$, $K(3)$, $l(1)$, $l(3)$.

The state variables are: k, R, S.

One can further assume that $k(2) = C(1)k(1)$, $0 < C(1) < 1$ and $l(2) = C(2)l(1)$, $0 < C(2) < 1$.

The Lagrange equation can then be formulated as follows:

$$L = e^{-rt} (y(2) - h(1)(k' - y(1) + uk) - h(2)(R' - 1/n \sum_{i=1}^{n} l(3)/L(3)$$

$$\sum_j \sum_K B(k)(5,j)g(j)(e^{b(k)T(i)} - 1)/T(i)b(k) + y(3)) - h(3)$$

$$(S - \sum_{i=1}^{n} l(3)/L(3) \sum_j \sum_K B(k)(5,j)g(j)(e^{b(k)T(i)} - 1)/b(k))) -$$

$$h(4)(L' - mL) - h(5)(y(1) - A(1)l(1)k(1)^{a(1)} y(4)^{1-a(1)}) -$$

$$h(6)(y(2) - A(2)l(2)k(2)^{a(2)} y(5)^{1-a(2)}) - h(7)(y(3)A(3)$$

$$l(3)k(3)^{a(3)} R^{1-a(3)}) - h(8)(1 - l(1) - C(2)l(1) - l(3)) - h(9)$$

$$(k - k(1)l(1) + C(1)C(2)k(1)l(1) + k(3)l(3)))$$

The Hamiltonian can be formed in a similar way discussed in Chapter IV. The Euler-Lagrange necessary conditions will consist of sixteen equations, namely seven equations from the derivatives with respect to the state and control variables and nine equations from the derivatives with respect to the costate variable. The solution to the above problem may require the solution to non-linear and non-homogeneous equations. If this is the case, then a numerical method may be required to solve such equation (Ames, 1968; pp.209- 228).

5.2 THE SECOND MODEL

When the wage and the rental rates are not fixed, one obtains the advantage that there is a one to one relationship between the wage rental ratio and the capital labor ratio. This approach is similar to the one done by Intriligator on his two sector growth model mentioned in Chapter II. The change in the wage rental ratio may have some effects on the profitability of the exploration ventures, depending on the nature and the magnitude of the change. However, in the second model the analysis assumes that the change in the wage rental ratio does not effect the profitability of the ventures. This is possible, if one can define a value for the wage rental ratio, above which at least one venture is non-profitable. The wage rental ratio is then allowed to change only up to the above defined value. The disadvantage of this assumption is that it disregards the possibility of investment on ventures that are still profitable above the defined wage rental ratio. The problem of incorporating the effect of the change in wage rental ratio on the profitability will be discussed in the third model.

A possible formulation of the second model can be described as follows:

Let r be the competitive rental rate and w be the competitive wage rate. In a competitive economy, all factors of production earn a return equal to their marginal product.

Define : $f(1) = A(1)k(1)^{a(1)}y(4)^{1-a(1)}$;

$$f(2) = A(2)k(2)^{a(2)}y(5)^{1-a(2)}$$

$$f(3) = A(3)k(3)^{a(3)}R^{1-a(3)}$$

$$y(4) = b\ y(3);\ y(5) = c\ y(3);\ \text{and } b+c = 1$$

Then:

$$r = dpf(1)/dpk(1) = a(1)A(1)k(1)^{a(1)-1}y(4)^{1-a(1)}$$

$$r=p(1)dpf(2)/dpk(2)=p(1)a(2)A(2)k(2)^{a(2)-1}y(5)^{1-a(2)}$$

$$r=p(2)dpf(3)/dpk(3)=p(2)a(3)A(3)k(3)^{a(3)-1}R^{1-a(3)}$$

where p(1) is the relative price of the consumption good with respect to the price of capital; and p(2) is the relative price of the resource commodity with respect to the price of capital.

Also one can have the following relations:

$$w=f(1)-k(1)f'(1)=(1-a(1))A(1)k(1)^{a(1)}y(4)^{1-a(1)}$$

$$w=p(1)(f(2)-k(2)f'(2))=p(1)(1-a(2))A(2)k(2)^{a(2)}y(5)^{1-a(2)}$$

$$\bar{w}=p(2)(f(3)-k(3)f'(3))=p(2)(1-a(3))A(3)k(3)^{a(3)}R^{1-a(3)}$$

(Intriligator, 1971, pp.419-420).

Let v be the wage rental ratio defined as $v = w/r$. Hence:

$$v = ((1-a(1))/a(1))k(1) = p(1)((1-a(2))/a(2))k(2)$$
$$= p(2)((1-a(3))/a(3))k(3).$$

Since a(1), a(2), and a(3) are the parameters of the production functions, which are considered given, then the above equations for v indicates that there is a linear relationship between the wage rental ratio and the capital labor ratio. Also note that:

$$dpv/dpk(1) = (1-a(1))/a(1) > 0$$

$$dpv/dpk(2) = (1-a(2))/a(2) > 0$$

$$dpv(dpk(2) = (1-a(3))/a(3) > 0$$

The above results imply that once one assumes a condition for the relationship between k(1), k(2), and k(3), there will be no possibility for any factor intensity reversal to occur. If one assumes that $k(1) > k(2) > k(3)$, then this condition will hold for any value of v, except for $v = 0$. Since $l(1)+l(2)+l(3) = 1$ and $l(1)k(1)+l(2)k(2)+l(3)k(3)=k$ and one assumes that $k(2) = c(1)k(1)$ and $l(2) = c(2)l(1)$ and if one defines that:

$$1+c(1)c(2) = c(3) \text{ and } 1+c(2) = c(4)$$

then one can derive that:

$$l(1) = (k-k(3))/(c(3)k(1)-c(4)k(3))$$

$$l(2) = (c(2)(k-k(3)))/(c(3)k(1)-c(4)k(3))$$

$$l(3) = (c(3)k(1)-c(4)k)/(c(3)k(1)-c(4)k(3))$$

The production functions can then be expressed in terms of the capital intensities as follows:

$$y(1) = ((k-k(3))/(c(3)k(1)-c(4)k(3)))A(1)k(1)^{a(1)}y(4)^{1-a(1)}$$

$$y(2) = (c(2)(k-k(3))/(c(3)k(1)-c(4)k(3)))A(2)$$
$$k(2)^{a(2)}y(5)^{1-a(2)}$$

$$y(3) = ((c(3)k(1)-c(4)k)/(c(3)k(1)-c(4)k(3)))A(3)$$
$$k(3)^{a(3)}R^{1-a(3)}$$

The optimization problem can then be formulated as follows:

$$\text{Maximize } J = \int_0^{T(1)} e^{-rt}(c(2)(k-k(3))/(c(3)k(1)-c(4)k(4)))$$
$$A(2)k(2)^{a(2)}y(4)^{1-a(2)}\,dt$$

subject to:

$$k' = ((k-k(3))/(c(3)k(1)-c(4)k(4)))A(1)k(1)^{a(1)}y(4)^{1-a(1)}-uk$$

$$R' = (1/n)\sum_{i=1}^{n}G(i)-((c(3)k(1)-c(4)k)/(c(3)k(1)-c(4)k(3)))$$
$$A(3)k(3)^{a(3)}R^{1-a(3)}$$

$$L' = mL$$

$$R = R(0),\ \text{at } t = 0,\ \text{and } k(0) = 0$$

$$R(T) = 0,\ k(T) = 0,\ \text{and } R,\ k \geq 0,$$

where:

$$G(i) = (1/L)\sum_j\sum_k B(k)(5,j)g(j)(e^{b(k)T(i)}-1)/T(i)b(k)$$

which is the rate of reserve addition of venture i per capita;

m is the rate of change of the labor force.

The state variable is k and L while the control variable is v.

The Lagrange equation is:

$$L = e^{-rt}(y(2)-h(1)(k'-y(1)+uk)-h(2)(R'-(1/n)\sum_{i=1}^{n}G(i)+y(3))$$
$$-h(3)(L'-mL))$$

where y(1), y(2), and y(3) must be expressed in terms of the capital intensities and y(4) and y(5) be expressed in terms of y(3).

The discounted Hamiltonian can be formulated in a similar way explained in Chapter IV. Note that the result for the optimum path for the wage rental ratio v, must be upper bounded by a predetermined v value, above which at least one venture becomes non-profitable.

The above model seems to be simplier than the first model, since it contains only two state variables and one control variable. But this does not suggest that the solution to the problem is consequently simplier.

5.3 THE THIRD MODEL

Another possible extension to the first and second model is to relax the assumption that the price of the resource commodity is fixed. When the price of the resource commodity is not fixed then its change effects the profitability of the exploration ventures. The optimization problem then becomes a feed back control problem or a closed loop control problem (Intriligator, 1971, pp. 299-302). The difficulty arise not only from formulating the optimization problem, but also from formulating the problem of profitability of the ventures in the aggregate form. Note that it is necessary to incorporate the profitability expression into the model. Also the extension of the second model into a model that incorporates the profitability of the exploration ventures, will face similar difficulties. Such an extension is also a feed back control problem. Further extension of the model, is to assume that the wage rates in sector 2 is not equal to the wage rate in sector 1 and 3. This will lead to the problem of immiserizing growth in the dynamic sense.

Chapter VI

THE CONLUSIONS

This chapter contains the conclusions that can be drawn from this analysis; its contetnts are divided into two parts: the first part contains the conclusions that can be deduced from the literature study, and the second part contains the conclusions that can be drawn from the analysis of the model.

6.1 THE CONCLUSIONS OF THE LITERATURE STUDY

The first analysis on depletion models using optimum control theory concerning an exhaustible resource is done by Hotelling (1931). He analyzes the behavior of a firm maximizing its present value of profits subject to a fixed amount of stock of an exhaustible resource with homogeneous quality in a competitive and a monopoly market structures. Modifications on Hotelling's model that use optimum control formulation, include the analyses on a firm having many deposits with homogeneous quality (Hartwick, 1978), or on a firm having deposits with multiple grades (Sulph, 1978), or analyses that incorporate exploration activities into the models (Peterson, 1978; Pindyck, 1978). It seems that all the deterministic models that have been reviewed analyze a firm behavior when the present value of the profits is maximized. However, it is unfortunate that these analyses do not specify explicitly the cost functions.

Most of the probabilistic models that have been reviewed that analyze the behavior of a firm, use an expceted utility function to define the uncertainty; indeed, such an approach has become a general method in solving the problem of uncertainty in economics. While using expected utility functions, these models do not specify explicitly the forms of the cost functions being used. In so doing, the conclusions that may be drawn from these models are general in nature; also that the problems of computation do not arise as it will if the cost functions are explicitly defined. A probabilistic model that does not use an expected utility is the model developed by Pindyck (1980); it maximizes the expected present value of profit subject to a random fluctuation in demand, reserve. The uncertainty is controlled by a stochastic process with independent increment or the Ito process. Although Pindyck does explain the character of the cost functions, he does not specify these functions explicitly. Again, the conclusions are general in nature and no computational problems are involved with respect to these functions. Hence, while these models are useful to describe the general trend of a firm behavior, they lack the capaibilty for numerical evaluations.

The main differences between the Markovian model and the probabilitic models decribed in this analysis are:

(1) The uncertainty of the Markovian model is a geologic uncertainty; this is explained by the McKelvey diagram. Other models may have uncertainty in price, in cost, or in the reserves. In the case of Pindyck's model, the uncertainty in reserve or discovery contains not only geologic uncertainty but also the uncertainty of the occurence of future events;

(2) The Markovian model uses a Markov process to control the uncertainty; it defines the probability of the transition from one exploration activity to another. Other probabilistic models do not use a Markov process and also they do not define such transition probability. In the case of Pindyck's model, the stochastic process controls the random fluctuations of reserves and discoveries and uses the whole resource base of the area as the sample space. In the Markovian model, the sample space is the total exploration activities involved in the search for a petroleum resource.

(3) The Markovian model particularly analyzes the exploration for petroleum resource; this analysis is more specific than other models that have been reviewed.

As compared to the decision tree process, the Markovian model is a better representation for describing the decision process in the exploration of a mineral resource. First, it is because the transition probability in a Markov chain or a Markov process depends only on the state where the chain or process is currently. This situation is similar to a decision process in exploration where the undertaking of a certain exploration activity depends only on the outcome of the last previous activity. On the other hand, the conditional probability in a decision tree process depends on the conditional probabilities of the branches on the path being passed by the decision process. Second, in a Markov chain or a Markov process, the transfer from one activity to another is controlled by chance; while in a decision tree process, there is at least one point at which the transfers are not controlled by chance but by a deliberate action made by the decision makers. This decision process is not incorporated into the computation but rather is taken as a given condition. This also implies, that once a decision is made to take a particular branch at a decision fork, all other branches at this point are disregarded. Third, a decision tree process is irreversible; it means that there is no possibility for a particular action to be repeated. In fact, the assumption that is postulated for a decision tree process in exploration is that the exploration proceeds always from a less intensive exploration activity to a more intensive one. Such a condition does not need to be imposed for a Markov chain and a Markov process. Hence, the second and the third differences above limit the usefulness of the decision tree process as a model in exploration. Four, a decision tree process is not able to incoporate the length of time for an exploration activity to take place. But a Markov process can incorporate these times in the computation of the present value of a petroleum resource. Hence, in general one can conclude that the Markovian model is a better representation for an exploration decision process than a decision tree process.

6.2 THE CONCLUSIONS OF THE ANALYSIS

The Markovian exploration model is designed to compute the expected present value of a petroleum deposit. It is based on:

(1) An exploration network for a search of petroleum resource in an unknown district;

(2) A Markov process.

Exploration activities are in general very diverse in nature; to generalize an exploration network that will apply for every mineral resource is impossible. Exploration for petroleum resource is in general more established and less diverse than that for other mineral resources. The exploration network that has been designed may represent most of the exploration activities in seraching for petroleum. The selection of an unknown district is not only consistent to the nature of the uncertainty defined in the model but also it allows one to design a network in a more general manner.

Basically the exploration activities can be divided into three major activities, namely:

(1) The surface geological survey;

(2) The geophysical survey;

(3) The exploration drilling.

Each of the above major activities is subdivided into several exploration activities. The exploration network is designed to accomodate for possible repeated work for several exploration activities; it also accomodates the possible failure and success for each of the above major activities. In this way, the network represent a complete possible exploration work that covers all possible outcomes.

Since the exploration network is designed for an unknown district, no statistical data concerning geological or exploration activities are available for the concerned area. Hence, when using a Markov chain to control the uncertainty, methods for assigning transition probability values that use statistical data can not be used. The subjective method for assigning transition probability values may be used, and in view of the large degree of uncertainty involved in an unknown district, this method may provide adequate results. The disadvantages of the subjective method are: (i) subjective method is a personal judgement, so that different analysis may give different estimates, even based on the same geological information; (ii) judgement may be biased because of personal preference towards risk; a risk averse person will usually give lower probability estimates than a risk neutral or a risk preferred person. In the attempt of assigning transition probability values, if a certain probability distribution function is to be assumed for the random variable that governs the transitions, then a Markov chain requires that such function is also assumed for every transition in the network.

The analysis on the use of a Markov chain for the model indicates that certain important questions pertaining to exploration can be answered adequately. One can compute the probability of the outcomes of the major activities mentioned previously, or the probability of failure or success of the exploration project (refer to equation (3-2)). In fact, one can compute the probability of the occurence of every activity in the network. Besides, one can also compute the probability of the accurrence of an activity when time is involved; however, one must assume that the length of time for every activity to take place is one unit time.

The computation procedure requires that the activities be classified; the classification indicates that the activities are either an absorbing state or a transient state. The transition matrix of the Markov chain is then written as a partitioned matrix consisting of sub-matrices of the absorbing states and the transient states. The computation is then performed based on this transition matrix; it requires the invertibility of the (I-Q) matrix, where Q is the sub- matrix of the transient states.

In most exploration, the length of time for any exploration activity to take place is not the same for every activity and they are usually not one unit time. So, the use of a Markov chain in the model has certain limitation. In a Markov process, however, the length of time of an activity to take place is a random variable with a negative exponential distribution. Hence, by using a Markov process one can extend the model into a more realistic one. The classification of the activities in a Markov process is based on the value of the parameter of the exponential distribution function of the sojourn time. The classification indicates that the activities in the network are either an absorbing state or a transient state, and none is an instantaneous state. The parameter of the exponential distribution function of the sojourn time must be estimated from the expected mean value of the sojourn time. The expected mean value of the sojourn time for a particular activity can be either inferred from statistical data of sojourn times of different exploration projects or be subjectively determined.

The computational procedure consists of first calculating the spectral representation of the generator matrix of the Markov process and second, estimating the net reward vector. The former requires the computation of the eigenvalues and the eigenvectors of a certain matrix; to do this, the EISPAC subroutine can be used for such computation. The net reward vector or function is the difference between the value of the information gained from an activity and the cost of performing that activity. It is assumed that the firm performing the exploration is also the firm that will produce the petroleum if a discovery is found. This assumption will eliminate the complexity of information valuation. The net reward vector can either be assumed fixed or be influenced by the sojourn time. In the former, the value of the information gained is not influenced by the length of time for the activities to take place. This situation may be interpreted that the information has been exhausted. The model that computes the expected present value of the petroleum resource is relatively simple and it is defined by equation (3-9). In the latter case, one has to assume a value for d, which defines the ease of the information gathering. The value of d ranges from zero to one. When d is zero, then one has the model with the fixed net reward vector. The value of d must be interpreted that the greater the value of d, the easier the information gathering. When the values of information are used as the outcomes of the exploration activities, then imputed prices of the information gained, which are related to the expected price of petroleum must be used. The expected present value of the petroleum resource is then defined by equation (3-10). On the other hand, when estimated values of the amount of petroleum present are used as the outcomes of the major exploration activities, the expected price of petroleum must be used. This expected price of petroleum becomes the shadow price for the information. The expected present value of the petroleum resource is then defined by equation (3-11). One may recall that the elements h(10), h(15), and h(4) of the net reward vector h, contain the difference of the value of the estimated amount of petroleum and the cost of performing the respective activity; all other entries of the net reward vector contain only the cost of performing each respective activity and these must have negative signs. The method for estimating the amount of petroleum is by the volumetric method. In all cases, the expected price for petroleum is fixed and given.

The analysis of the depletion model gives the following results:

(1) Under optimum production, the change in the value of the deposit and the change in the future production cost yield an interest rate on the scarcity rent equal to r. This is indicated by equation (4-4a). However, when the stock of the resource is supplemented by exploration, the change in the value of the deposit yields an interest rate on the scarcity rent equal to r. This shown by equation (4-5);

(2) Under the optimum production and exploration, the price of the petroleum resource will be equal to the marginal cost of production and the scarcity rent. This is indicated by equation (4-6);

(3) The value of the capital used in exploration is proportional to the value of the petroleum produced. This is shown by equation (4-11).

The scarcity rent being a cost for the firm, will have a negative sign and is increasing negatively overtime. The costate variable h(1) is interpreted to be the scarcity rent. This means that the present value of all future sacrifices associated with the use of a particular unit of insitu resources increases as more petroleum is produced. The marginal value of information, namely the costate variable h(2), increases overtime as the resource is being extracted. The Lagrange multiplier h(3), which denotes a shadow price for capital, is constant overtime.

As production proceeds, the use of capital for production increases, but after the amount of petroleum produced is equal to the amount of the discovery, the use of capital for production depends on the rate of decrease of the labor used in this sector. The use of labor for production decreases overtime and if its rate of decrease is greater than that of the capital labor ratio, then the use of capital increases. Otherwise, it decreases. The increase of the use of capital in production indicates that as more petroleum is being produced, production becomes more energy intensive, and more capital is required to produce the same amount of petroleum per unit of time. The use of capital in exploration also increases overtime, as more intensive methods are being used. The decrease in the use of capital in exploration occurs when the effect of discovery on production has ceased to exist and no discovery is then possible.

As compared to the results obtained by the models described in Chapter II, there is one consistent result that this analysis provides, namely that the mineral resource will be depleted at a finite time. Hotelling has argued that such finding is the result of a linear demand. In this analysis, the reasons for having a finite time for complete exhaustion are several: (1) the constant price for the resource commodity, (2) the exponential functions of the costate variables (h(1) and h(2)), and (3) the assumption that exploitation must stop after the reserve becomes relatively very small, because production is no longer economically and technologically feasible at this stage. With respect to Pindyck's probabilistic model, no comparable result can be drawn because different assumptions are being used for the cost functions. Pindyck uses either a linear or a non-linear cost function while this model uses both linear and and non-linear cost functions. Moreover, Pindyck's model does not impose a fixed resource commodity price.

The analysis on the growth models, shows that the Markovian model can be used in such models. It indicates that the Markovian model can be used in a three sector growth model; the growth model can be a simple one, where price of petroleum and the wage and rental rates are fixed, or it can be a more complex growth model, where the price of petroleum or the wage and rental rates are not constant and they affect the profitability of the exploration ventures.

Finally, one can conlude that although the Markovian exploration model is designed for petroleum resource, this model may be used for any other mineral resource by modifying the exploration network.

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Appendix A

RESOURCE DEFINITIONS

The following definitions are provided in conjunction to the McKelvey Diagram discussed in Chapter 2 and 3.

Resource is a concentration of naturally occurring solid, liquid, or gaseous materials in or on the earth's crust in such form that economic extraction of a commodity is currently or potentially feasible.

Identified resources are specific bodies of mineral-bearingmaterial whose location, quality, and quantity are known from geologic evidence supported by engineering measurements.

Undiscovered resources are unspecified bodies of mineral-bearing material surmised to exist on the basis of broad geologic knowledge and theory.

Hypothetical resources are undiscovered resources that may be reasonably be expected to exist in a known mining district under known geologic conditions. Exploration that confirms their existence and reveals the quantityand the quality will permit their reclassification as reserve or identified subeconomic resources.

Speculative resources are undiscovered resources that may occur in known types of deposits in a favorable geologic setting where no discoveries have been made, or in as yet unknown types of deposits that remain to be recognized. Exploration that confirms their existence and reveals their quantity and quality will permit their reclassification as reserve or identified subeconomic resources.

Reserves or economic resources are that portion of the identified resources from which usable minerals or energy commodities can be economically and legally extracted at the time of determination.

Paramarginal resources are the portion of the subeconomic resources that border on being economically producible or are not commercially available solely because of legal or political circumstances. Submarginal resources are the portion of the subeconomic resources which would require a substantial higher price (more than 1.5 times the price at time of determination) or a major cost-reducing technology to be economically feasible.

(McKelvey, 1973; McKelvey et al, 1973).

Appendix B

MARKOV CHAINS AND MARKOV PROCESSES

This appendix contains definitions and theorems of Markov chains and Markov processes that pertain to the discussion in the text.

B.1 MARKOV CHAINS

Definition: A stochastic process $X=(X(n); n \text{ in } N)$ is called a Markov chain provided that:

$$P(X(n+1)=j \mid X(0), X(1), \ldots, X(n)) = P(X(n+1) \mid X(n))$$

for all j in E and n in N, where E is a countable state space and N is the set of all positive integers.

Hence, a Markov chain is a sequence of random variables such that for any n, $X(n+1)$ is conditionally independent of $X(0), \ldots X(n-1)$ (Cinlar, 1975; p.107). This means, that the state $X(n+1)$ visited by the chain at time $(n+1)$ is independent of the states visited by chain in the past.

A time homogeneous Markov chain satisfies the following conditional probability statement:

$$P(X(n+1)=j \mid X(n)=i) = P(ij), \text{ for all } i,j \text{ in } E,$$

independent of n (Cinlar, 1975; p.107).

Definition: A square matrix P with entries $P(ij)$ for all j, i in E, is called a Markov matrix provided that:

(a) for any i,j in E, $P(ij) \geq 0$;

(b) for each i in E, $\sum_j P(ij) = 1$

(Cinlar, 1975; pp.108).

The probability $P(ij)$ is called a transition probability, and the Markov matrix is also called a transition matrix.

Proposition: For any m in N,

$P(X(n+m)=j|X(n)=i) = P(ij)^m$, for all i, j in E and n in N.

In particular for m=0, $P^0 = I$. In other words, the probability that the chain moves from state i to state j in m steps is the (ij)th entry of the mth power of the transition matrix P. Consequently:

$P^{m+n} = P^m P^n$, for any n, m in N.

The Chapman-Kolomogorov equations can be written as follows:

$P^{m+n}(ij) = \sum_{k \neq i} P^m(ik) P^n(kj)$, for i, j in E.

Theorem: For fixed n in N, Y is a bounded function of the random variables X(n), X(n+1),........, then:

$P(Y|X(0),\ldots.X(n)) = E(Y|X(n))$.

This means that the past and the future of a Markov chain are independent, provided that the present state is known; the present state is meant to be a fixed time n (Cinlar, 1975; p.116).

Proposition: For any i in E and finite integer k, m $\geq$ 1, and if T(m) is the stopping time for state j, then:

$$P(i)(T(m+1)-T(m)=k \mid T(1),\ldots T(m)) = \begin{cases} P(j)(T(1)=k), & T(m)<\infty \\ 0, & T(m)=\infty \end{cases}$$

Let $P(i)(T(1)=k) = F(ij)(k)$, then:

$$F(ij)(k) = \begin{cases} P(ij), & \text{for } k=1 \\ \sum_{b \in E-j} P(ib)F(bj)(k-1), & \text{for } k \geq 2 \end{cases}$$

where $F(ij)(k)$ is the probability of the first visit of the chain to state j occurring at time k, starting in state i. When $F(ij)$ is defined to be the probability that, starting in state i, the chain ever visits state j, then:

$$F(ij) = P(ij) + \sum_{b \in E-j} P(ib)F(bj).$$

__Proposition__: If $N(j)$ is the total number of visits to j, then:

$$P(j)(N(j)=m) = F^{m-1}(jj).(1-F(jj)), \quad m = 1, 2, \ldots.$$

and for $i \neq j$:

$$P(i)(N(j)=m) = \begin{cases} 1-F(ij), & m=0 \\ F(ij)F^{m-1}(ij).(1-F(jj), & m=1,2\ldots \end{cases}$$

__Corollary:__

$$P(j)(N(j)<\infty) = \begin{cases} 1, & \text{for } F(jj)<1 \\ 0, & \text{for } F(jj)=1 \end{cases}$$

If $F(jj) = 1$, then the probability that $N(j) = +\infty$ is equal to one; therefore $E(j)(N(j)) = +\infty$. If $F(jj)<1$, $N(j)$ has the geometric distribution with success probability $p=1-F(jj)$, starting in state j; $E(j)(N(j))=1/p = 1/(1-F(jj))$.

Let $R(ij)$ be the expected value of the total number of visits by the chain to state j, starting at state i, i.e. $R(ij)=E(i)(N(j))$, then:

$R(jj) = 1/(1-F(jj))$

$R(ij) = F(ij)R(jj)$, for $i\neq j$.

The matrix R whose (ij) entry is $R(ij)$ is called the potential matrix of the Markov chain.

<u>Definitions</u>: If T is the time of first visit to state state j by the Markov chain, then:

(a) State j is called recurrent, if $P(j)(T<\infty) = 1$.

(b) State j is called transient, if $P(T=+\infty)>0$.

(c) A recurrent state j is called null, if $E(j)(T)=$; otherwise,it is called non-null.

(d) A recurrent state j is said to be periodic with period d, if $d\geq 2$ is the largest integer for which $P(j)(T=nd, n\geq 1)=1$; otherwise, if there is no $d\geq 2$, the state j is called aperiodic.

Note that if j is recurrent, then $F(jj)=1$, $R(jj)=\infty$; if j is transient $F(jj)<1$, $R(jj)=1/(1-F(jj))$, and $R(ij)=F((ij)R(jj)$.

Also:

$\sum_{n}^{\infty} P^n(ij) = \infty$, j=recurrent and $\sum_{n}^{\infty} P^n(ij) < \infty$, $\lim_{n\to\infty} P(ij)=0$, j=transient. While for a Markov chain having finite states, no state is null recurrent (Cinlar, 1975; pp.125-127).

Definitions:

(a) A set of states is said to be closed if no state outside that set can be reached from any state in that set.

(b) A state is forming a closed set by itself is called an absorbing state. Note: this requires P(jj)=1.

(c) A closed set is irreducible if no proper subset of it is closed.

(d) A Markov chain is called irreducible if its only closed set is the set of all sets.

(Cinlar, 1975; p.125; Ross, 1972; pp.89-80).

Note that all states of an irreducible Markov chain can be reached from each other; such a Markov chain represents a Markov matrix.

Proposition: If state j can be reached from i, and state k from state j, then state k can be reached from state i.

Lemma: If j is a recurrent state and k can be reached from state j, then state j can be reached from state k.

Theorem: From a recurrent state, only recurrent states can be reached.

__Theorem:__ If X is an irreducible Markov chain, then either all states are transient, or all states are recurrent, or all are recurrent non-null. Also, either all states are aperiodic, or else if one is periodic with period d, then all states are periodic with period d.

__Corollary:__ If C is a set of an irreducible closed set of finitely many states, then no state in C is recurrent null.

Any Markov matrix can be partitioned as follows:

$$P = \begin{bmatrix} K & | & 0 \\ --- & + & --- \\ L & | & Q \end{bmatrix} \begin{array}{l} \text{recurrent states} \\ \\ \text{transient states} \end{array}$$

where the sub-matrix K represents all of the recurrent states, and the sub-matrix Q represents all transient states. It follows that:

$$P^n = \begin{bmatrix} K^n & | & 0 \\ ---- & + & --- \\ L(n) & | & Q^n \end{bmatrix} \qquad R = \begin{bmatrix} \sum_n^\infty K^n & | & 0 \\ ------- & + & --------- \\ \sum_n^\infty L(n) & | & \sum_n^\infty Q^n \end{bmatrix}$$

where K^n and Q^n are the nth power of K and Q respectively, but not for L .

If $S = \sum_n^\infty Q^n$, then $S = I + Q + Q^2 + \ldots\ldots$

Multiplying both sides by Q : $QS = Q + Q^2 + \ldots. = S-I$. If the number of transient states is finite, then the matrix (I-Q) will have a full rank, therefore:

$$S = (I-Q)^{-1}$$

A Markov matrix can also be partitioned as follows:

$$P = \left[\begin{array}{c|c} I & 0 \\ \hline B & Q \end{array}\right]$$

where Q is the sub-matrix of all transient states, while B is defined to be the sub-matrix whose columns are b(1), b(2),....... , namely:

$$B(ij) = \sum_{k} P(ik), \quad i \text{ in } D, \; j = 1, 2, ...$$

where D is the set of all transient states.

Then for n>0, the nth power of P is:

$$P^{n} = \left[\begin{array}{c|c} I & 0 \\ \hline B(n) & Q^{n} \end{array}\right]$$

where $B(n) = (I + Q + Q^{2} + ... + Q^{n-1})B$. In fact B(n)(ij) is the probability that, starting in state i, the chain enters a set of recurrent states C(j) at or before the nth step. Hence:

$$G = \lim_{n \to \infty} B(n) = \left(\sum_{k}^{\infty} Q^{k} \right) B = S.B,$$

where G(ij) is the probability of the chain ever reaching the set C(j) from a transient state i. Note that the set C(j) in the exploration model is every absorbing state.

The matrix S can be used to analyze several exploration problems; for example, one can compute the expected number of visits to a state j, starting in state i, before the chain enters a recurrent state or an absorbing state. Or, one can compute the variance of such visits. Here, state i and j are transient. If the (ij) entry of matrix S is s(ij), then s(ij)=u(ij)=E(N(ij)), where u(ij) is the expected number of visits to state j, starting in state i, before the chain enters a recurrent state or an absorbing state. N(ij) is the random variable denoting the above number of visits. The variance of the above expected value can be calculated by using the following formula:

$$V(N(ij)) = (S(2S(d) - I) - S(2)),$$

where V(N(ij)) is the variance N(ij); S(d) is the diagonal matrix whose (jj) entry is s(jj); and S(2) is a matrix whose (ij) entry is the square of s(ij) or u(ij) (Bhat, 1972; pp.72-73). The derivation of the above variance formula is not given here, but it can be found in Bhat (1972; pp.72-78).

B.2 <u>MARKOV PROCESSES</u>

In a Markov chain, no consideration is given to the length of time spent by the chain in each state. In fact such length of time is considered to be one unit for every state. However, in a Markov process, the length of time spent by the process in any state is a random variable with a negative exponential distribution. The Markov process to be used in this analysis is a Markov pure jump process (Hoel et al, 1972; pp.84-89). Mathematically, the Markov process can be defined as follows:

The stochastic process $X=(X(t)\,;\ t \text{ in } R(+))$ is said to be a Markov process with state space E, if for any t, $s\geq 0$ and j in E,

$$P(X(t+s)=j|X(u)\,;\ u<t) = P(X(t+s)=j|X(t))$$

(Cinlar, 1975; p.233).

Note that R(+) is the set of all positive real numbers.

The process X is said to be time homogeneous, when:

$$P(X(t+s)=j|X(t)=i) = P(ij)(s)$$

is independent of t, for s, $t\geq 0$ and for all i, j in E. The analysis is concerned only with the time homogeneous Markov process.

The Markov process can be illustrated as follows:

$$X(t) = \begin{cases} x(0), & 0<t<t(1) \\ x(1), & t(1)<t<t(2) \\ x(2), & t(2)<t<t(2) \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \end{cases}$$

In order that the process will not remain in a transient state forever, then $\lim_{n\to\infty} t(n) = \infty$ (Hoel et al, 1972; pp.84-85). For example, if:

$$t(n) = 2 - 1/(2^{n-1}),$$

then $\lim_{n\to\infty} t(n) = 2$. This means, if t eventually equals to two, the process will remain in X(n) forever. Hence, it is assumed that the process in exploration will eventually move to an absorbing state.

In a Markov process, the transition function, written as $P(ij)(t)$ satisfies the following properties:

(i) $P(ij)(t) \geq 0$;

(ii) $\sum_{k} P(ik)(t)=1$; $i=1, 2, \ldots$

(iii) $\sum_{k \neq i} P(ik)(t)P(kj)(s)=P(ij)(t+s)$

where (iii) is known as the Chapman-Kolomogorov equation.

<u>Theorem</u>: For any i in E and $t \geq 0$,

$$P\{W(t)>u \mid X(t)=i\} = e^{-\delta(i)u}, \quad u \geq 0$$

for some number $\delta(i)$ in $(0, +\infty)$.

where $W(t)$ is the length of time the process remains in the state being occupied at the instant t; $W(t)$ is also called the sojourn time. For a particular state i, the sojourn time is written as $W(i)(t)$. $\delta(i)$ is a parameter of the density function of the sojourn time. For a particular state i this function, $f(i)(W(t))$, has the following form:

$$f(i)(W(t)) = \begin{cases} \delta(i)e^{-\delta(i)u} & \text{, for } u \geq 0 \\ 0 & \text{, for } u<0 \end{cases}$$

Hence:

$$P(i)\{W(t)>u\} = \int_{0}^{\infty} \delta(i)e^{-\delta(i)u}\,du = e^{-\delta(i)u}.$$

If the times $T(0)$, $T(1)$, $T(2)$, ...are the instants of transition for a Markov process, and $X(0)$, $X(1)$, $X(2)$, the successive states visited by the process, then:

$P(T(n+1)-T(n)>u \mid X(n)=i,\ X(n+1)=j)=e^{-\varepsilon(i)u}$, $u \geq 0$. is independent of state j. It means, that the distribution of the sojourn time depends only on the state in which the sojourn time is being evaluated, and it does not depend on the next state.

A Markov process also satisfies the following integral equation:

$$P(ij)(t)=e^{-\varepsilon(i)t}I(ij) + \int_0^t \varepsilon(i)e^{-\varepsilon(i)s} \sum_{k\neq i} Q(ik)\cdot P(kj)(t-s)\,ds$$

(Cinlar, 1975; p.254).

If state i is an absorbing state, i.e. $\varepsilon(i)=0$, the above integral equation reduces to: $P(ij)(t) = I(ij)$. If state i is a transient or stable state, and $T(1)$ is the time when the process is in an intermediate state k, then the event $(T(1)\leq t,\ X(T(1))=k$ and $X(t)=j)$ occurs if and only if the first move occurs at some time $s\leq t$ and takes the process to state k, and goes to state j from state k in the remaining $(t-s)$ units of time. Hence:

$$P(i)(T(1)\leq t,\ X(T(1))=k,\ X(t)=j) = \int_0^t \varepsilon(i)e^{-\varepsilon(i)s} Q(ik)P(kj)(t-s)ds$$

$$P(i)(T(1)\leq t,\ X(t)=j)= \sum P(i)(T(1)<t,\ X(T(1))=k,\ X(t)=j)= \int_0^t \varepsilon(i)e^{-\varepsilon(i)s} \sum_{k\neq i} Q(ik)P(kj)(t-s)ds$$

Also:

$$P(i)(T(1)>t,\ X(t)=j)=I(ij)P(i)(T(1)>t)=e^{-\varepsilon(i)t}I(ij)$$

Since $P(ij)(t)=P(i)(T(1)>t, \quad X(t)=j)+P(i)(T(1)\leq t, \quad X(t)=j)$ then:

$$P(ij)(t)=e^{-\delta(i)t}I(ij)+\int_0^t \delta(i)e^{-\delta(i)s}\sum_{k\neq i}Q(ik).$$
$$.P(kj)(t-s)ds.$$

(Cinlar, 1975; p.254).

In the Markov process used in the analysis, the sojourn times are conditionally independent of each other, given successive states being visited; and as shown, each sojourn time has an exponetial distribution function with a parameter dependent on the state being visited (Cinlar, 1975; p.248). Hence:

$$P(T(1)-T(0)>u(1),\ldots\ldots T(n)-T(n-1)>u(n)\,|\,X(0)=j(0),\ldots.$$
$$X(n)=j(n)) = e^{-\delta(0)u(1)}\ldots.e^{-\delta(n-1)u(n)}.$$

Appendix C

THE EISPAC SUBROUTINE

```
C       THIS PROGRAM COMPUTES THE EIGENVALUES AND
C       EIGENVECTORS OF AN N BY N MATRIX USES EISPAC
C       FIRST CARD N PROB
C       NEXT CARD N
C       NEXT CARD A BY ROWS
 1000 FORMAT('0')
  105 FORMAT('1     PROBLEM     ',I1,2X,'OF  ',I1)
  120 FORMAT('0     MATRIX SIZE  'I2,2X,'BY ',I2)
  130 FORMAT('0     EIGENVALUE NUMBER ',I2)
  140 FORMAT('0     EIGENVECTOR NUMBER ',I2)
      REAL*8 A(25,25),WR(25),WI(25)
      REAL*8 ZP(25,25),ZI(25,25)
      DATA LDA /25/
      READ(5,*) NPROB
      DO 1 IPROB=1,NPROB
      WRITE(6,105) IPROB, NPROB
      READ(5,*) N
      WRITE(6,120) N,N
      WRITE(6,1000)
      DO 10 I=1,N
      READ(5,*) (A(I,J),J=1,N)
      WRITE(6,*) (A(I,J),J=1,N)
   10 CONTINUE
      CALL EISPAC(LDA,N,MATRIX('REAL',A),
      VALUES(WR,WI),VECTOR(ZP))
      DO 150 K=1,N
      IF (WI(K).NE.0.0) GO TO 110
         DO 100 J=1,N
         ZI(J,K)=0.0
  100    CONTINUE
         GO TO 150
  110 IF (WI(K).LT.0.0) GO TO 1300
         DO 1200 J=1,N
         ZI(J,K)=ZP(J,K+1)
 1200    CONTINUE
      GO TO 150
 1300 DO 1400 J=1,N
      ZP(J,K)=ZP(j,K-1)
      ZI(J,K)=-ZI(J,K-1)
 1400 CONTINUE
  150 CONTINUE
      DO 20 I=1,N
      WRITE(6,130) I
      WRITE(6,*) WR(I),WI(I)
```

```
      WRITE(6,140) I
         DO 30 J=1,N
         WRITE(6,*) ZP(J,I),ZI(I,J)
   30    CONTINUE
   20 CONTINUE
    1 CONTINUE
      STOP
      END
```