UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

ON THE STABILITY OF 2-PERIODIC ORBITS

IN MINKOWSKI BILLIARD SYSTEMS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

Degree of

DOCTOR OF PHILOSOPHY

By

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Norman, Oklahoma

2024

ON THE STABILITY OF 2-PERIODIC ORBITS IN MINKOWSKI BILLIARD SYSTEMS

A DISSERTATION APPROVED FOR THE

DEPARTMENT OF MATHEMATICS

BY THE COMMITTEE CONSISTING OF

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A mis seres queridos.

Acknowledgments

I would like to begin by acknowledging my advisor, Dr. Pengfei Zhang for his guidance, wisdom, and patience during my graduate studies, and for introducing me to the playground that is mathematical billiards. Thanks for making time for me, even during your sabbatical, and for being a great advisor overall. Additionally, I want to thank the rest of my committee, Dr. Nikola Petrov, Dr. John Albert, and Dr. Robert Lewis-Swan for their support throughout this process. I had the pleasure of taking classes from most of you, and it was in these classes that my mathematical interests took shape. This is a testament not only to your talent as mathematicians, but your quality as individuals. I am also grateful to everyone in the Mathematics Department at the University of Oklahoma for creating a friendly learning environment for everyone.

I also want to acknowledge Dr. Susan Walden and the Bridge to Doctorate Fellowship, as well as TRAC-FLVN and the SMART scholarship for believing in me and providing me with amazing funding opportunities during my graduate studies. I never took these opportunities for granted, and it was often your belief in me that motivated me when I struggled to believe in myself.

During my time in Oklahoma, I was fortunate to cross paths with people whom I am privileged to call friends. I would like to take the time to mention Fahd, Celeste, Luke, Tyler, James, Amogh, Wes, Jessie, Skyler, Emily, Abood, Jaime, Myles, Kevin, and José. You all made the bad times bearable and the good times great. I also want to thank a few people from my time at Oklahoma City Community College. Namely Mr. Nunley for first inspiring me to study mathematics, and Christine Miller for being an amazing supervisor during my time at the Math Lab, and for always being available to write me a letter of recommendation. I don't know what lies you've told people, but it worked!

Por último, quiero agradecer a mi familia, que, aunque estén lejos, siempre me hacen sentir su amor y apoyo incondicional. Especialmente quiero agradecer a mis padres, Carlos y Patricia; mis hermanos, Alejandro y Héctor; mis abuelos, Carlos y María Enriqueta; mis tíos, Cuauhtémoc, Claudia, y Gerardo; mi primo Cuauhtémoc; y mi sobrino Víctor.

Abstract

We investigate the fundamental properties of Minkowski billiards and introduce a new coordinate system (s, u) on the phase space $\mathcal{M}$, under which the Minkowski billiard map $\mathcal{T}$ preserves the standard area form $\omega = ds \wedge du$. After, we derive an expression for the tangent map $D\mathcal{T}$ thus enabling the classification of periodic orbits as elliptic, parabolic, or hyperbolic. Then for a specific family of Minkowski billiards, we derive formulas for the first twist coefficient τ_1 for elliptic 2-periodic orbits in terms of the Minkowski norm and the geometric quantities of the billiard table. Additionally, we analyze the stability properties of these elliptic periodic orbits and provide some numerical simulations.

Contents

List of Figures

Chapter 1

Introduction

Dynamical billiards describe systems in which a particle moves within a container and reflects off its walls according to specific reflection rules. In [7, 8], Birkhoff studied dynamical billiards on convex domains with smooth boundaries and proved the existence of periodic orbits of period n for any $n \geq 2$. Later, Sinai pioneered the study of chaotic billiards where in his seminar paper [34], he investigated billiards with dispersing boundaries and proved the hyperbolicity and ergodicity of these systems. Afterward, Bunimovic observed in [10, 11, 12] that convex circular arcs could be used to construct chaotic billiards, leading to the introduction of the famous Bunimovic stadium. Subsequently, Wojtkowski significantly extended the construction of chaotic billiards in [39] by providing conditions under which a billiard table exhibits hyperbolic behavior. Since then, several classes of chaotic billiards have been discovered such as the elliptical stadium [27], track billiards [13], and asymmetric lemon billiards [14, 21]. An overview of chaotic billiards can be found in [17].

The study of stable billiard orbits is equally important. It is shown in [7, 33] that a dynamical system near an elliptic fixed point can be viewed as a perturbation of an integrable model i.e. the Birkhoff Normal Form. If the integrable model has nonzero twist coefficients, then the elliptic fixed point must be nonlinearly stable [29]. That is, on the phase space, the fixed point is contained within a nesting sequence of closed invariant disks with smooth boundaries. Generally, the twist coefficients can be expressed as the rational functions of the coefficients of the Taylor polynomials at the elliptic fixed point [30]. For billiards, Kamphorst and Pinto-de-Carvalho showed in [23] that the first twist coefficient may be written in terms of the geometric quantities of the billiard table such as the distances between the points of reflection and the curvature at these points. In [22], Jin and Zhang obtained an explicit formula of the second twist coefficient of periodic 2-orbits in terms of the same geometric quantities and provided applications of the formula to the study of stability properties of various billiard systems.

In [19], Gutkin and Tabachnikov expanded the theory of dynamical billiards by introducing billiards on Minkowski planes, and more generally, on Finsler manifolds. Unlike the Euclidean setting, these spaces are not equipped with an inner product, but rather a norm. Consequently, the concept of angles, which is essential in defining reflections in Euclidean billiards, no longer exists. Instead, Gutkin and Tabachnikov defined reflections through the fundamental concept of critical points of the trajectory length function. Since then, several notable results have emerged regarding Minkowski billiards (see [3, 4]). In this thesis, we investigate the properties of periodic orbits in dynamical billiards on Minkowski planes. That is, let $F : \mathbb{R}^2 \to \mathbb{R}$ be a Minkowski norm and $Q \subset \mathbb{R}^2$ be a connected, bounded domain with a piecewise smooth boundary. We consider a point mass moving freely inside Q, with reflections at the boundary determined by the Finsler reflection law (see chapter 3).

This thesis is structured as follows: In chapter 2 we define billiards systems on the Euclidean plane and present historical results on the stability of orbits in dynamical systems, emphasizing their applications to billiards. We discuss elliptic and hyperbolic fixed points, and for the former elaborate on how we can use the Birkhoff Normal Form to determine nonlinear stability. Additionally, we provide a brief introduction to Lyapunov exponents and show how they relate to chaotic billiards.

Chapter 3 introduces Finsler manifolds and Minkowski spaces, along with billiard systems on these spaces. Specifically, we state the Finsler reflection law as given in [19], and use it to reformulate the phase space $\mathcal{M}$, consisting of all unit vectors with foot at the boundary ∂Q that point inward toward Q. Although this definition aligns with that of Euclidean billiards, the conventional arc-length coordinate system (s, θ) is no longer applicable as angles are no longer defined. Therefore we introduce a new coordinate system (s, u) for the phase space $\mathcal{M}$ and demonstrate that the Minkowski billiard map $\mathcal{T}$ is symplectic using the generating function method. Subsequently, we derive an expression for the tangent map $D\mathcal{T}$ in terms of the Minkowski norm and the geometric quantities of the billiard table, thus enabling the classification of periodic orbits as elliptic, parabolic, or hyperbolic.

Finally, chapter 4 deals with the derivation of the first twist coefficient τ_1 for elliptic 2-periodic orbits. Together with Moser's twist mapping theorem, we use τ_1 to determine the nonlinear stability of periodic orbits for specific Minkowski billiard families. Additionally, we give a few numerical simulations for other periodic orbits whose stability properties are outside the scope of τ_1.

Chapter 2

Billiard Systems

2.1 Preliminaries

We begin with some standard terminology from dynamical systems and differential geometry. Great introductions to these subjects can be found in [9] and [26] respectively.

Given a non-empty set X and a function $f : X \to X$, a *discrete dynamical system* is a tuple (f, X) such that for $n \in \mathbb{N}$, the n-th iterate of f is given by n-fold composition $f^n = f \circ \cdots \circ f$ of f, and $f^0 = \mathrm{Id}$. If f is invertible, f^{-n} is the n-fold composition of f^{-1}. Given a dynamical system (f, X), we define for $x \in X$, the *forward orbit of* x as $\mathcal{O}_f^+(x) = \bigcup_{n\in\mathbb{N}} f^n(x)$, the *backward orbit of* x as $\mathcal{O}_f^-(x) = \bigcup_{n\in\mathbb{N}} f^{-n}(x)$ (if it exists), and the *orbit of* x as $\mathcal{O}(x) = \mathcal{O}_f^+(x) \cup \mathcal{O}_f^-(x)$. We say a point $x \in X$ is *periodic* if $f^N(x) = x$ for some $N > 0$, and the *period* is the smallest N satisfying this condition. If $f(x) = x$, then x is a *fixed point.* For a periodic point x, its orbit $\mathcal{O}(x)$ is a *periodic orbit.* Lastly, we say $A \subset X$ is *f-invariant* if $f(A) \subset A$, in which case we can consider the subsystem (A, f)

Let $(X, \mathcal{C}, \mu)$ be a probability space and $f : X \to X$ be a measurable map. We say is f is *μ-invariant* or *μ-preserving* if $\mu(T^{-1}(A)) = \mu(A)$. A measure preserving map f is said to be *ergodic* if for every $A \in \mathcal{C}$:

$$T^{-1}(A) = A \text{ if and only if } \mu(A) = 0 \text{ or } 1$$

Given a smooth manifold M, the *tangent space to M at x*, denoted T_xM, is the vector space composed of all tangent vectors to M at x. The disjoint union of all the tangent spaces of M is called the *tangent bundle of M*, namely $TM = \bigsqcup_{x \in M} T_xM$. For each $x \in M$ we can define the *cotangent space at x*, denoted T_x^*M, to be the dual space of T_xM, and their union $T^*M = \bigsqcup_{x \in M} T_x^*M$ to be the *cotangent bundle of M*. A *Riemannian metric g* on a smooth manifold M is a smooth variation of inner products $g_x : T_xM \times T_xM \to \mathbb{R}$ on each of the tangent spaces T_xM satisfying the following conditions:

1. $g_x(v, v) \geq 0$ for all $v \in T_xM$,

2. $g_x(u, v) = g_x(v, u)$ for all $u, v \in T_xM$,

3. $g_x(v, v) = 0$ if and only if $v = 0$.

Now let M be a differential manifold. A *differential k-form* is a continuous tensor field ω whose value at each point is an alternating k-tensor $\omega_p : T_pM \times \cdots T_pM \to \mathbb{R}$. That is,

$$\omega_p\left(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k\right) = -\omega_p\left(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k\right). \tag{2.1}$$

If $k = \dim M$, we say ω is a *volume form.*

Given a volume form ω on $\mathbb{R}^{2n}$, a linear map $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is *symplectic* if $\omega(Au, Av) = \omega(u, v)$ for all $u, v \in \mathbb{R}^{2n}$. Similarly, a differentiable map $f : U \to \mathbb{R}^{2n}$, with $U \subseteq \mathbb{R}^{2n}$, is symplectic if the tangent map Df is symplectic at every point. When $n = 1$, a symplectic map is equivalent to an area and orientation transformation, or one whose tangent map is an area and orientation preserving transformation.

2.2 Euclidean Billiard Systems

A *billiard system* consists of a billiard table, which is a domain $Q \subset \mathbb{R}^2$ bounded by a closed, piece-wise smooth curve γ such that $\gamma = \partial Q$, a point mass that moves within Q called the billiard ball, and a reflection law that governs how the billiard reflects off the boundary. It is also possible to consider higher dimensional billiards where $Q \subset \mathbb{R}^n$ is taken to be a piece-wise smooth boundary. However, we will not consider this case and throughout the paper we will consider all billiard systems to be planar.

Typically, the motion within Q is taken to be uniform so that the billiard ball travels with unit speed regardless of direction, and the reflection law is taken to be: the angle of incidence equals the angle of reflection. Regardless of the reflection law chosen, reflections off of cusps or corners (if any), are undefined. We will call a billiard system with the above properties a *Euclidean billiard system*, or just *Euclidean billiards* for short. The term Euclidean billiards is not standard. However we will use it repeatedly to distinguish it from a Finsler billiard system. To develop our understanding, we will assume that all billiard systems are Euclidean for the remainder of this chapter.

For a parameterized curve $\gamma = (x(t), y(t))$, we define (wherever possible) the *signed curvature* of γ at t as

$$\kappa(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}. \tag{2.2}$$

Further, the signed *radius of curvature* at t is given by

$$R(t) = \frac{1}{\kappa(t)}. \tag{2.3}$$

Geometrically, the radius of curvature is obtained by attaching to $\gamma(t)$ a circle of radius $R(t)$ such that their tangent lines agree. We say that γ is *convex* at t if $\kappa(t) > 0$, *concave* if $\kappa(t) < 0$, and *flat* if $\kappa(t) = 0$. By convention we will take the radius of curvature of a flat portion of γ to be ∞. If γ is a closed curve such as the boundary of a billiard table Q, and the center of the circle of radius $R(t)$ lies inside Q, then that portion of $\gamma(t)$ is convex, and concave if it lies outside of Q. If a billiard table is strictly convex we say the billiard system is *Birkhoff*. For more information about the curvature of planar curves see Chapter 3.

Consider the subset $\mathcal{M}$ of tangent bundle TQ given by

$$\mathcal{M} = \{(x, v) \in TQ : x \in \partial Q,\ ||v|| = 1,\ \text{and } v \text{ points inside } Q\}. \tag{2.4}$$

We call $\mathcal{M}$ the *phase space* of the billiard system, and physically, $\mathcal{M}$ represents all possible states of the billiard system. If we parameterize γ by arc-length s, and let $\theta \in [0, \pi]$ be the angle between v and $\gamma'(s)$, then each $(x, v) \in \mathcal{M}$ can be expressed in terms of new coordinates (s, θ), so that $\mathcal{M} = \{(s, \theta) : 0 \leq s \leq L,\ 0 \leq \theta \leq \pi\}$ where L is the length of γ. Since γ is a closed curve, we can identify $0 \sim L$, and view $\mathcal{M}$ as a cylinder.

Suppose that $(s_0, v_0) \in \mathcal{M}$ is the initial position and direction of the billiard ball. Let $\gamma(s_1)$ be the first point of reflection, so that at $\gamma(s_1)$, the billiard has reflected direction θ_1. By repeating this process, we can define the *billiard map* $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ as the map such that $\mathcal{T}(s_j, \theta_j) = (s_{j+1}, \theta_{j+1})$ where $s_{j+1} = s_{j+1}(s_j, \theta_j)$ and $\theta_{j+1} = \theta_{j+1}(s_j, \theta_j)$. For strictly convex billiard tables whose boundary is C^k for $k \geq 2$, the billiard map $\mathcal{T}$ is of class C^{k-1} [18, 25]. In particular, if the boundary is smooth, then $\mathcal{T}$ is as well. By defining the billiard map, we can convert our billiard model to a discrete dynamical system whose orbits are given by $\mathcal{O}(s,\theta) = \{\mathcal{T}^j(s,\theta) : j \in \mathbb{Z}\}$. Moreover sometimes it will be more useful to use the coordinates (s,u) on $\mathcal{M}$ where $u = -\cos\theta$ so that $\mathcal{T}$ is symplectic.

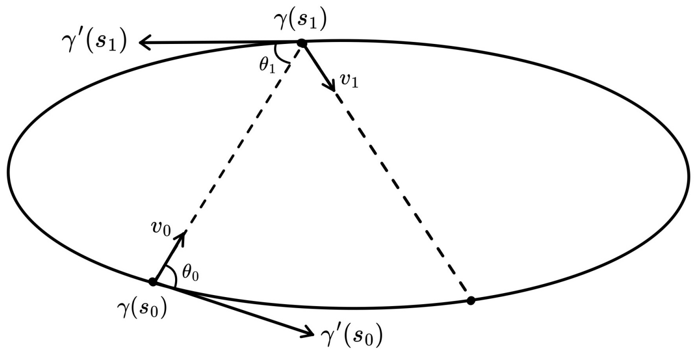


Figure 2.1: Euclidean Billiard Reflections

Theorem 2.1. *The area form given by* $\omega = \sin\theta ds \wedge d\theta$ *is invariant under the billiard map* $\mathcal{T}$ *i.e.* $\mathcal{T}^*\omega = \omega$. *Further, let* $L = L(s, s_1)$ *denote the distance between* $\gamma(s)$ *and* $\gamma(s_1)$, *and let* $\kappa(s)$ *and* $\kappa(s_1)$ *be the signed curvatures of* γ *at* s *and* s_1 *respectively. Then the differential of the billiard map* $D\mathcal{T} = \begin{bmatrix} \frac{\partial s_1}{\partial s} & \frac{\partial s_1}{\partial \theta} \\ \frac{\partial \theta_1}{\partial s} & \frac{\partial \theta_1}{\partial \theta} \end{bmatrix}$ *is given by*

$$D_{(s,\theta)}\mathcal{T} = \frac{1}{\sin\theta_1}\begin{bmatrix} L\kappa(s) - \sin\theta & L \\ L\kappa(s)\kappa(s_1) - \kappa(s_1)\sin\theta - \kappa(s)\sin\theta_1 & L\kappa(s_1) - \sin\theta_1 \end{bmatrix} \tag{2.5}$$

Proof. First note that $\sin\theta > 0$ so that ω is in fact an area form. Let $\gamma(s)$ and $\gamma(s_1)$ be distinct points on the boundary of the billiard table, and $L(s,s_1)$ be as above. Then $\frac{\partial L}{\partial s} = -\cos\theta$ and $\frac{\partial L}{\partial s_1} = \cos\theta_1$. It follows that

$$dL = \frac{\partial L}{\partial s}ds + \frac{\partial L}{\partial s_1}ds_1 = -\cos\theta ds + \cos\theta_1 ds_1 \tag{2.6}$$

$$d^2L = \sin\theta\ d\theta \wedge ds - sin\theta_1\ d\theta_1 \wedge ds_1 = 0 \tag{2.7}$$

thus proving the first part of the theorem. To compute the differential $D\mathcal{T}$, first note that since γ is parameterized by arc-length, $\gamma'(s)\cdot(\gamma(s_1)-\gamma(s)) = L(s,s_1)\cos\theta$ and $\gamma'(s_1)\cdot(\gamma(s_1)-\gamma(s)) = L(s,s_1)\cos\theta_1$. Hence

$$\cos\theta = \frac{1}{L(s,s_1)}\left(\gamma(s_1)-\gamma(s)\right)\cdot\dot{\gamma}(s) \tag{2.8}$$

$$\cos\theta_1 = \frac{1}{L(s,s_1)}\left(\gamma(s_1)-\gamma(s)\right)\cdot\dot{\gamma}(s_1). \tag{2.9}$$

Taking the differential of equation (2.8) yields

$$\begin{aligned}
-\sin\theta d\theta &= \left[-\frac{1}{L^2}\frac{\partial L}{\partial s}\left(\gamma(s_1)-\gamma(s)\right)\cdot\dot{\gamma}(s) + \frac{1}{L}\Bigg(\left(\gamma(s_1)-\gamma(s)\right)\cdot\gamma''(s) - \dot{\gamma}(s)\cdot\dot{\gamma}(s)\Bigg)\right]ds \\
&\quad + \left[-\frac{1}{L^2}\frac{\partial L}{\partial s_1}\left(\gamma(s_1)-\gamma(s)\right)\cdot\dot{\gamma}(s) + \right]ds_1 \\
&= \left[\frac{\cos^2\theta - 1}{L} + \kappa(s)\sin\theta\right]ds + \left[\frac{-\cos\theta\cos\theta_1}{L} + \frac{\cos(\theta+\theta_1)}{L}\right]ds_1 \\
&= \left[\frac{-\sin^2\theta}{L} + \kappa(s)\sin\theta\right]ds - \frac{\sin\theta\theta_1}{L}ds_1.
\end{aligned} \tag{2.10}$$

Solving for $\sin\theta_1 ds_1$ gives

$$\sin\theta_1 ds_1 = \left[L\kappa(s) - \sin\theta\right]ds + Ld\theta \tag{2.11}$$

from which we get

$$\frac{\partial s_1}{\partial s} = \frac{L(s,s_1)\kappa(s) - \sin\theta}{\sin\theta_1} \tag{2.12}$$

$$\frac{\partial s_1}{\partial \theta} = \frac{L(s,s_1)}{\sin\theta_1}. \tag{2.13}$$

Similarly, taking the differential of equation (2.9) yields

$$-\sin\theta_1 = \left[\frac{\cos\theta\cos\theta_1}{L} - \frac{\cos(\theta+\theta_1)}{L}\right] ds + \left[\frac{1-\cos^2\theta_1}{L} - \kappa(s_1)\sin\theta_1\right] ds_1. \quad (2.14)$$

Solving for $\sin\theta_1 d\theta_1$ gives

$$\sin\theta_1 d\theta_1 = -\frac{\sin\theta\sin\theta_1}{L} ds + \left[\kappa(s_1) - \frac{\sin\theta_1}{L}\right]\sin\theta_1 ds_1 \quad (2.15)$$

so that plugging in equation (2.11) into equation (2.15) results in

$$\sin\theta_1 d\theta_1 = \left[L\kappa(s)\kappa(s_1) - \kappa(s_1)\sin\theta - \sin\theta_1\kappa(s)\right] ds + \left[\kappa(s_1)L - \sin\theta_1\right] d\theta. \quad (2.16)$$

It follows that

$$\frac{\partial\theta_1}{\partial s} = \frac{1}{\sin\theta_1}\left[L(s,s_1)\kappa(s)\kappa(s_1) - \kappa(s_1)\sin\theta - \kappa(s)\sin\theta_1\right] \quad (2.17)$$

$$\frac{\partial\theta_1}{\partial\theta} = \frac{1}{\sin\theta_1}\left[L(s,s_1)\kappa(s_1) - \sin\theta_1\right] \quad (2.18)$$

□

The map $L(s, s_1)$ is called the generating function of the billiard in Q. In chapter 3 we will use this same approach to derive the differential of the billiard map in Finsler spaces. However, we will not have an inner product structure so the result will not be written in terms of angles.

2.3 Integrable Billiard Systems

A smooth function on the phase space of a billiard system is called an *integral* if it is invariant under the billiard map T. If such a function exists, we say that the billiard map is *integrable*. Loosely speaking, we will see that an integrable billiard system is one whose orbits are "predictable" even after a large number of reflections.

Recall the definitions of the three kinds of conic sections: Given two points F_1 and F_2 called the foci, an ellipse E is the set of all points $x \in \mathbb{R}^2$ such that the sum of the distances $|xF_1|$ and $|xF_2|$ is constant i.e. $E = \{x \in \mathbb{R}^2 : |xF_1| + |xF_2| = 2a\}$ for some $a > 0$. Similarly, a hyperbola is given by $H = \{x \in \mathbb{R}^2 : ||xF_1| - |xF_2|| = 2a\}$. A parabola is the set of all points that are equidistant from a fixed point F called the focus and a line d called the directrix i.e. $P = \{x \in \mathbb{R}^2 : |xF| = |xd|\}$. Further, we say that two conic sections are *confocal* if they have the same foci.

Some important facts about billiards on elliptical tables are as follows. The proof of these statements can be found in [37].

(E1) A billiard trajectory that passes through one focus reflects to a trajectory that passes through the other focus.

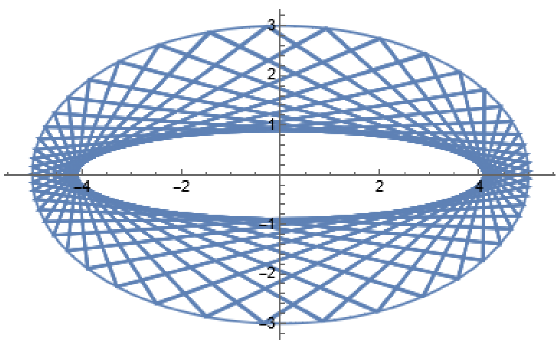
(E2) The billiard trajectory given in (E1) converges to the orbit that runs along the major axis.

(E3) Billiard trajectories are always tangent to a confocal conic.

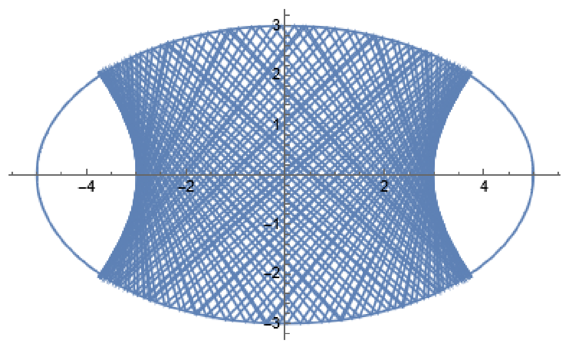
- (E3.1) If F_1 and F_2 are the foci of an ellipse, and any portion of a billiard trajectory does not intersect the segment F_1F_2, then the billiard trajectory will always be tangent to a confocal ellipse. In particular, the billiard trajectory will never intersect F_1F_2 (see Figure 2.2a).
- (E3.2) If a portion of a billiard trajectory intersects the segment F_1F_2, then the billiard trajectory will always be tangent to a confocal hyperbola. In particular, every reflection will intersect F_1F_2 (see Figure 2.2b).

(E4) The billiard map T is integrable.

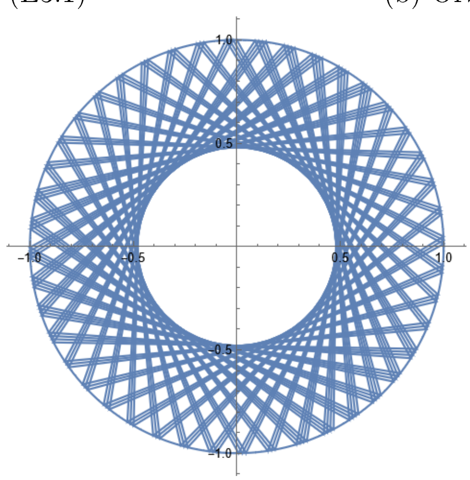
A **caustic** of a billiard system is a curve such that if a segment of a billiard trajectory is tangent to it, then every future reflection will also be tangent to said curve. From Figure 2.2 we can see how confocal ellipses and hyperbolas are caustics of elliptical billiard tables. Note that in the special case of periodic orbits traveling along the the major and minor axes, the segment F_1F_2 is the caustic, as it is the limit of the confocal ellipses. Further, if the billiard table is a circle, the caustics are either straight lines or concentric circles, including the degenerate case of the center as a circle with zero radius.



(a) Orbit of type (E3.1)



(b) Orbit of type (E3.2)



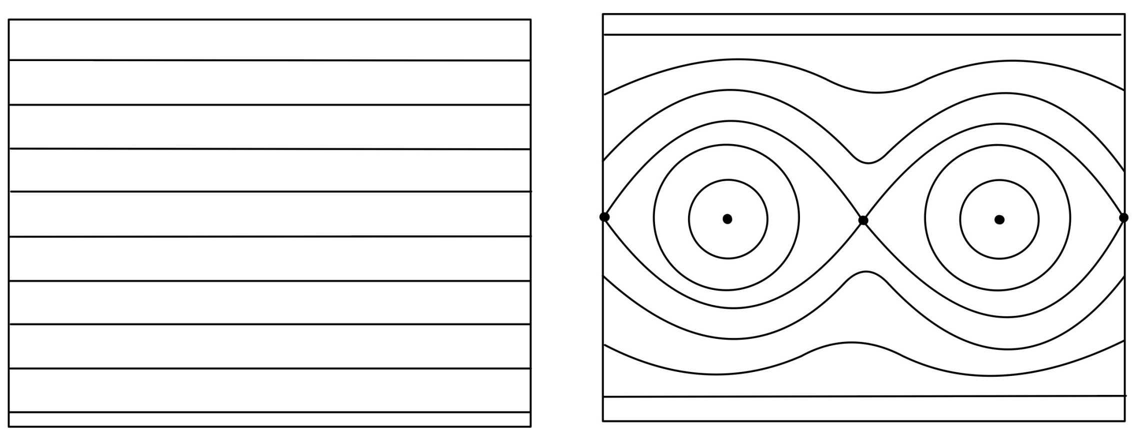
(c) Orbit inside a circle

Figure 2.2: Billiard orbits on elliptical tables.

A consequence of (E4) is that the phase portraits of elliptical and circular billiard systems are completely foliated by invariant curves of the billiard map $\mathcal{T}$. In the case of a circular table, the phase portrait consists of horizontal lines (see Figure 2.3b). Much more interesting however, is the phase portrait of an ellipse where we have three types of curves (see Figure 2.3b). The first type of curves consist of topological circles, also called *elliptic islands* that correspond to trajectories of type (E3.1). These elliptic islands surround two points corresponding to the orbit that travels along the minor axis. The second type of curves consist of the hourglass-shaped curves that surround the elliptic islands. These curves correspond to trajectories of type (E3.2). Lastly, the ∞-shaped curve corresponds to the billiard trajectory of type (E1). Recall that since the phase space is a cylinder, there are actually two singular points on the ∞-shaped curve corresponding to the orbit that travels along the major axis.

Shortly we will see how the shape of these invariant curves is of fundamental importance as it is related to the stability of billiard orbits. We finish this section by stating the Birkhoff conjecture for convex billiards:

> *The only integrable convex billiard systems are ellipses.*



(a) Circular billiard table (b) Elliptical billiard table

Figure 2.3: Phase space of elliptical billiard tables

2.4 Stability of Periodic Billiard Orbits

Now that we have introduced some basic properties of dynamical billiards, it is only natural to ask whether periodic orbits exist, and if so study their properties. Despite its simplicity, this is a surprisingly deep question. For example, in 1775 Fagnano proved that when the billiard table is an acute triangle, there always exists a periodic orbit - namely the *orthic triangle*. However, the case of an obtuse triangle has proven to be much more elusive. It wasn't until the last fifteen years that [32] proved the existence of periodic orbits when the billiard table is a triangle whose angles are all at most one hundred degrees.

In contrast, Birkhoff showed that every convex billiard table with a sufficiently smooth boundary has a periodic orbit. Therefore, we can shift our focus to studying the properties of these orbits instead. Given a periodic orbit a natural question to ask is: What if we varied the initial conditions, maybe ever so slightly? Do we obtain a trajectory that closely resembles the periodic orbit, or do we get something vastly different? And if the result is vastly different, is it at least in some sense 'predictable', or does it appear to be totally random?

A billiard orbit $\{(s_0, u_0), (s_1, u_1), ...\}$ is *periodic* if there exists a positive integer n such that for every $j \in \mathbb{Z}$, $s_j = s_{j+n}$. The minimal period is given by the smallest n satisfying this condition. If we take the arc length parameter s and mod it by the length of the boundary curve so that effectively, the length of ∂Q equals one, then we may take the billiard map $\mathcal{T} : \mathbb{R}/\mathbb{Z} \times (0, \pi) \to \mathbb{R}/\mathbb{Z} \times (0, \pi)$ and lift to it's universal cover $\tilde{\mathcal{T}} : \mathbb{R} \times (0, \pi) \to \mathbb{R} \times (0, \pi)$ where $\tilde{\mathcal{T}}$ satisfies $\tilde{\mathcal{T}}(s_j, \theta_j) = (s_{j+1}, u_{j+1})$ and $s_{j+m} = s_j + m$ for all integers m. The *rotation number* of an orbit of $\mathcal{T}$ is defined as

$$\lim_{j\to\infty} \frac{\tilde{\mathcal{T}}^j(s) - s}{j}, \tag{2.19}$$

and it is well known that for a periodic orbit, the rotation number is given by a rational number $\frac{m}{n}$ where n is the period of the orbit. Naturally, a billiard system doesn't contain any fixed points. However, a periodic orbit of period n will be a fixed point of $\mathcal{T}^n$. The following theorem whose proof is given in [7] shows that if we allow the boundary of a billiard table to be sufficiently smooth, then there exists periodic orbits of period n for all $n \geq 2$.

Theorem 2.2 (Birkhoff)**.** *Let Q be a Birkhoff billiard system such that the boundary of Q is at least C^2. Then for any $m/n \in (0,1) \cap \mathbb{Q}$ there exists at least two periodic orbits with rotation number m/n.*

Given an n-periodic orbit of a map $f : X \to X$, it represents a fixed point x^* of the iterated map f^n so that $D_{x^*}f^n$ fixes the origin. Depending on the eigenvalues λ_1 and λ_2 of $D_{x^*}f^n$, we can classify x^* as a *parabolic fixed point* if $\lambda_j \in \{0, 1\}$, or a *hyperbolic fixed point* if $\lambda_j \in \mathbb{R}$ are given by λ and $1/\lambda$ with $\lambda \neq \pm 1$, or an *elliptic fixed point* if λ_j are given by λ and $\overline{\lambda}$ with $\lambda = e^{i\phi}$ and $\lambda \neq \pm 1$. Shortly we will see how hyperbolic and elliptic fixed points are characterized by their stability - that is how the dynamical system behaves close to these fixed points. But first it will be useful to formally state what we mean by stability.

Let $f : X \to X$ be a dynamical system and x^* be a fixed point of f. We say that x^* is *Lyapunov stable* for $t > 0$ (future stability) if for every neighborhood U containing x^* there exists another neighborhood $V \subset U$ also containing x^* such that $f(x) \in V$ implies $f^t(x) \in U$ for all $t > 0$. We can similarly define Lyapunov stability for $t < 0$ (past stability). Intuitively, a fixed point being Lyapunov stable means that orbits which start close to the fixed point, remain forever close to it - assuming we're dealing with $t > 0$ (or $t < 0$ for past stability). There are different types of Lyapunov stable fixed points, such as *asymptotically stable* fixed points where orbits that start close not only remain close, but eventually converge to the fixed point, and *exponentially stable* fixed points if the orbits converge at an exponential rate.

2.4.1 Unstable Periodic Billiard Orbits

Having defined Lyapunov stability, we will begin with a short introduction to hyperbolic fixed points which, although are characterized by their (partial) instability, possess nice geometric properties that facilitate their study. For a more thorough introduction to hyperbolic fixed points, [40, 35, 24] are excellent places to start.

Let $A : V \to V$ be a linear map with eigenvalue λ. The λ-*generalized eigenspace* of A is given by $V_{[\lambda]} := \{v \in V : (A - \lambda I)^m v = \vec{0} \text{ for some } m > 0\}$.
The following are well known facts about generalized eigenspaces:

Proposition 2.3. *Suppose V is a finite dimensional complex vector space, $A : V \to V$ is a linear transformation, and $\lambda_1, ..., \lambda_k$ are distinct eigenvalues of A. Then*

1. *Each $V_{[\lambda]}$ is preserved by A*
2. $V = \bigoplus_{j=1}^{r} V_{[\lambda_j]}$
3. $\dim V = \sum_{j=1}^{r} M(\lambda_j)$ *where $M(\lambda_j)$ denotes the multiplicity of λ_j.*

Let $(0, 0)$ be a hyperbolic fixed point of the linear isomorphism $A : \mathbb{R}^n \to \mathbb{R}^n$, and E_+ and E_- be the generalized eigenspaces defined as follows:

$$E_+ = \bigoplus_{|\lambda_j|<0} \mathbb{R}^n_{[\lambda_j]}, \qquad E_- = \bigoplus_{|\lambda_j|>0} \mathbb{R}^n_{[\lambda_j]}. \tag{2.20}$$

For $\mathbb{R}^n = E_+ \oplus E_-$ we can decompose A as

$$A = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix} \tag{2.21}$$

where $A_+ = A|_{E_+} : E_+ \to E_+$ and $A_- = A|_{E_-} : E_- \to E_-$ are the restrictions of A onto E_+ and E_- respectively. Further, one can show that

$$|A_+^j x| \leq c r_+^j |x| \quad \forall j > 0,\ \forall x \in E_+ \tag{2.22}$$

$$|A_-^j x| \leq c r_-^j |x| \quad \forall j < 0,\ \forall x \in E_- \tag{2.23}$$

for some constant $c \geq 0$ and $0 < r_-, r_+ < 1$. Equations (2.22) and (2.23) show that for a hyperbolic fixed point, the linear system splits. In one direction the system experiences an exponential growth of vectors while in the other there is exponential decay. Hence the spaces E_+ and E_- may be characterized as

$$E_+ = \{x \in \mathbb{R}^n : A^j x \to \vec{0},\ j \to \infty\}, \tag{2.24}$$

$$E_- = \{x \in \mathbb{R}^n : A^j x \to \vec{0},\ j \to -\infty\}. \tag{2.25}$$

That is, E_+ and E_- are the subsets of $\mathbb{R}^n$ such that $x \in E_+$ converges to $\vec{0}$ under iterations of A, and $x \in E_-$ does the same but under iterations of A^{-1}. The behavior of the sets E_+ and E_- is not exclusive to linear maps, and can be expanded to diffeomorphisms as we shall now see.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism and x^* a hyperbolic fixed point of f. The *stable* and *unstable manifolds* of x^* are given by

$$W_+(f, x^*) = \{x \in \mathbb{R}^n : f^j(x) \to x^*, j \to \infty\} \tag{2.26}$$

$$W_-(f, x^*) = \{x \in \mathbb{R}^n : f^j(x) \to x^*, j \to -\infty\} \tag{2.27}$$

That is, the stable manifold is given by all points in $\mathbb{R}^n$ that converge to the fixed point x^* by iterating f, while the unstable manifold is given by those points which converge to x^* by iterating the inverse map f^{-1}. It is well know that W_+ and W_- are invariant under f. Further, by taking $f = A$ and $x^* = \vec{0}$, we can easily see that $W_+(A, \vec{0}) = E_+$ and $W_-(A, \vec{0}) = E_-$. The following theorem is taken from [24] and shows how the behavior of a dynamical system near a hyperbolic fixed point can be locally understood through it's linearization at that point.

Theorem 2.4 (Hartman-Grobman)**.** *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^n$ continuously differentiable, and $O = \vec{0} \in U$ a hyperbolic fixed point of f. Then there exists neighborhoods U_1, U_2, V_1, V_2 of O and a homeomorphism $h : U_1 \cup U_2 \to V_1 \cup V_2$ such that $f = h^{-1} \circ Df_0 \circ h$ on U_1, that is, the following diagram commutes:*

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "h"'] & U_2 \arrow[d, "h"] \\ V_1 \arrow[r, "Df_0"'] & V_2 \end{tikzcd}$$

Letting f be the billiard map $\mathcal{T}$ in the theorem above guarantees that the dynamics of $\mathcal{T}^n$ on a neighborhood of the fixed point P are the same as those of $D_P\mathcal{T}^n$ on a neighborhood of $(0, 0)$.

Perhaps the most important example of a hyperbolic billiard orbit is the 2-orbit that travels along the major axis of the ellipse, corresponding to a fixed point of the two-step billiard map $\mathcal{T}^2$. An important characteristic of this orbit is that it is both an attracting and repelling fixed point as billiard orbits that pass through a focus converge to the major axis, while trajectories that never intersect the segment F_1F_2 are always tangent to a confocal ellispe. Hence if we were to begin with the orbit along the major axis and slightly distort its direction so that it just misses the opposite vertex, the result would be a billiard orbit that is always tangent to a confocal ellipse

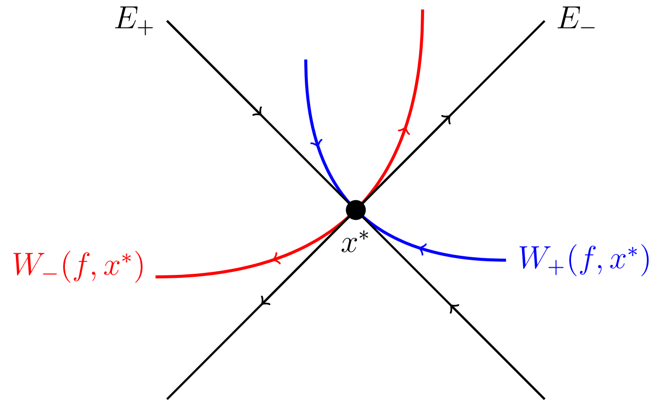


Figure 2.4: Hyperbolic fixed point

such as in Figure 2.2a. Lastly, recall that in the phase space of the elliptical billiard table, the ∞-shaped curve corresponds to billiard orbits that pass through one of the foci. This shows how the singular points corresponding to the orbit traveling along the major axis are fixed points of $\mathcal{T}^2$ and are precisely where the stable and unstable manifolds intersect.

2.4.2 Stable Periodic Billiard Orbits

Just as hyperbolic fixed points have the stable and unstable manifolds on the phase space as an important characteristic, we want to find a similar geometric characterization for elliptic fixed points. It turns out that given a diffeomorphism f with an elliptic fixed point x^*, then $D_{x^*}f$ acts as a rotation that fixes the origin so that $\vec{0}$ is surrounded by elliptic islands. However, these elliptic islands may not be present for the map f. This is in contrast to the hyperbolic case where f and D_xf always posses the same dynamic behavior. In the following, we give an example of an elliptic fixed point that is not stable, and a outline sufficient conditions for when the elliptic islands are inherited by f. For more details, see the classic text [29].

A fixed point x^* of $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is said to be *Moser stable* or *nonlinearly* *stable* if for $n \geq 1$ there exists a nested sequence of f-invariant neighborhoods U_n containing x^*, whose boundaries ∂U_n are invariant circles and $\bigcap_{n\geq 1} U_n = \{x^*\}$. We can see that nonlinear stability implies Lyapunov stability since the neighborhoods U_n are nested and invariant, hence we only need to choose neighborhoods $U_{n_1} \subset U_{n_0}$ of x^* and set $U = U_{n_0}$ and $V = U_{n_1}$ as in the definition of Lyapunov stability. However, ellipticity does not imply Lyapunov nor nonlinear stability as the following example demonstrates.

Consider lemon billiards first introduced in [20]. Let $Q(L)$ denote the class of lemon billiard tables formed by intersecting two unit circles and L represents the distance between the two centers. Denote by $\mathcal{O}_2 = \{P, \mathcal{T}(P)\}$ the periodic 2-orbit reflecting between $\gamma_0(0)$ and $\gamma_1(0)$ (See Figure 2.5). It is shown in [22] that $\mathcal{O}_2$ is elliptic for $L \in (0, 2)$, and nonlinearly stable for $L \in (0, 2)\backslash\{1\}$. However, numerical calculations in [15] demonstrate that the billiard map is ergodic on $Q(1)$.

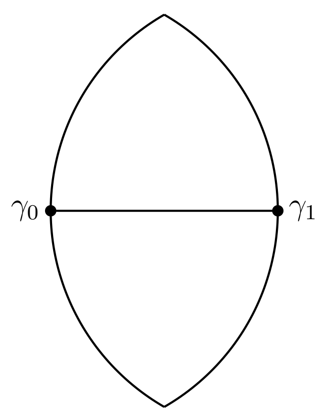


Figure 2.5: The lemon table $Q(1)$.

Together, the following proposition and theorem taken from [22] and [29] respectively, give sufficient conditions for when an elliptic fixed point of f is nonlinearly stable.

Proposition 2.5. *Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a symplectic (i.e. area preserving)* *map and $P=(0,0)\in U$ be an elliptic fixed point. Then the linearization $D_P f$ has eigenvalues of the form $\lambda=e^{i\theta}$ for $\theta\in(0,2\pi)/\{\pi\}$. If $\lambda^n\neq 1$ for all $n\geq 3$, then there exists an open neighborhood $U_N\subset U$ of P and an area preserving change of coordinates*

$$\begin{aligned} h_N : U_N &\to \mathbb{R}^2 \\ \begin{bmatrix} x \\ y \end{bmatrix} &\mapsto \begin{bmatrix} x+p_2(x,y)+\cdots+p_{2N+1}(x,y) \\ y+q_2(x,y)+\cdots+q_{2N+1}(x,y) \end{bmatrix} + h.o.t, \end{aligned} \tag{2.28}$$

where p_j and q_j are j^{th} degree polynomials for every $j=2,3,...,2N+1$, such that

$$h_N^{-1}\circ f\circ h_N\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\Theta(r^2) & -\sin\Theta(r^2) \\ \sin\Theta(r^2) & \cos\Theta(r^2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + h.o.t., \tag{2.29}$$

where $r^2=x^2+y^2$ and

$$\Theta(r^2)=\theta+\tau_1 r^2+\tau_2 r^4+\cdots+\tau_N r^{2N}. \tag{2.30}$$

Theorem 2.6 (Moser's twist mapping theorem)**.** *If for an area-preserving map of the form (2.29), the polynomial $\Theta(r^2)$ defined in (2.30) is not identically zero, then the origin is a nonlinearly stable fixed point.*

The symplectic transformation (2.28) is called the *Birkhoff transformation* and (2.29) is called the *Birkhoff normal form*. The cases such that $\lambda^n=1$ are called *resonances*, and τ_j is called the j^{th} *twist coefficient*. The name "twist coefficient" is given because geometrically, $\Theta(r^2)$ defined in (2.30) measures the amount of rotation about the fixed point.

Plenty has been written about the first twist coefficient. For example, [30] studied the behavior of the first twist coefficient near an elliptic fixed point for a one-parameter family of area-preserving diffeomorphisms and gave a method for calculating τ_1. Additionally, the application of twist coefficients to the case when f is taken to be the billiard map $\mathcal{T}$ was first given in [23], where the authors addressed the question of proving the stability of elliptic 2-periodic orbits for locally circular billiard tables. Further, [22] studied the Birkhoff normal form around elliptic periodic points for a large class of convex billiards and used it to give explicit formulas for the first two twist coefficients in terms of the geometric parameters of the billiard table. In both cases, the authors were able to apply the formulas of the twist coefficients to obtain characterizations of the nonlinear stability and local analytic integrability of billiards around periodic points.

As an example, consider the orbit on the elliptic billiard table given by $\frac{x^2}{\delta} + y^2 = 1$ for $\delta \in (0, 1)$ that travels along the minor axis. One can show (see chapter 4) that this orbit is an elliptic fixed point of $\mathcal{T}^2$ and it's respective first twist coefficient, first published in [22], is given by $\tau_1 = \frac{\delta}{2}$ so that $\Theta(r^2) \neq 0$. It follows from Moser's twist mapping theorem that this orbit is nonlinearly stable. Indeed, if we were to distort the orbit's initial direction so that the billiard ball misses the opposite vertex, the resulting orbit would be of type (E3.2) as long as the initial segment of the orbit intersects the segment F_1F_2 but doesn't go through any of the foci F_1 or F_2. Recall that in the phase space, these orbits are given by elliptic islands surrounding the elliptic fixed points as seen in Figure 2.3b.

2.5 Chaotic Billiards

By design, billiard systems are deterministic because a billiard's future state i.e. position and direction is completely determined by its initial conditions. However, despite their deterministic nature, billiards often exhibit chaotic behavior, meaning that an accurate description of its state in the near future is practically infeasible. For example, numerical descriptions of a systems future state may be inaccurate due to large round off errors, or a system's dynamics on its phase space may appear to be totally random. Although there is no formally agreed upon definition of chaos, an important characteristic of chaotic systems is their sensitivity to initial conditions, meaning that small changes in initial conditions produce large differences in future states. More precisely, we have that nearly orbits diverge at an exponential rate. For the rest of this chapter we outline the mechanisms needed for a system to exhibit sensitivity to initial conditions and apply them to the case of billiards. In fact, billiards were the testing ground in the development of many of the theorems presented below. For a much deeper understanding of chaotic billiards see [17].

Definition 2.7. *Let $f : X \to X$ be a measurable transformation that preserves a probability measure μ, and $A : X \to GL(n, \mathbb{R})$ a measurable function. For $m \in \mathbb{N}$, define $\mathcal{A}^m(x) : X \times \mathbb{Z} \to GL(n, \mathbb{R})$ as:*

$$\mathcal{A}^m(x) = A(f^{m-1}(x)) \cdots A(f(x))A(x) \tag{2.31}$$

$$\mathcal{A}^{-m}(x) = A(f^{-m}(x))^{-1} \cdots A(f^{-2}(x))^{-1}A(f^{-1}(x))^{-1} \tag{2.32}$$

$$\mathcal{A}^0(x) = Id \tag{2.33}$$

The pair $(f, \mathcal{A})$ is called a measurable cocycle and A the generator of $\mathcal{A}$.

A cocycle $\mathcal{A}$ induces a linear extension $\mathcal{F} : X \times \mathbb{R}^n \to X \times \mathbb{R}^n$ of f defined by $\mathcal{F}(x, v) = (f(x), A(x)v)$. This map defines a new dynamical system on the bundle $X \times \mathbb{R}^n$ whose iterates are given by $F^m(x, v) = (f^m(x), \mathcal{A}^m(x)v)$. The following theorem taken from [24] defines Lyapunov exponests, which we will see are quantities that measure the average exponential growth of the iterates of $\mathcal{F}$ along $\mathcal{F}$-invariant subspaces.

Theorem 2.8 (Osedelec Multiplicative Ergodic Theorem)**.** *Let $(f, \mathcal{A})$ be measurable cocycle on X with generator A. If*

$$\int_X \log^+ \|A(x)\| d\mu(x) < \infty \quad \textit{and} \quad \int_X \log^+ \|A^{-1}(x)\| d\mu(x) < \infty, \tag{2.34}$$

where $\log^+ s = \max\{\log s, 0\}$*, then there exists* $Y \subset X$ *with* $\mu(X/Y) = 0$ *such that for each* $x \in Y$ *the decomposition* $\mathbb{R}^n = \bigoplus_{j=1}^{k(x)} E_j(x)$ *is* $\mathcal{F}$*-invariant, and for each* $v \in E_j(x)$*, the limit*

$$\lim_{m \to \pm\infty} \frac{1}{m} \log \frac{\|\mathcal{A}^m(x)v\|}{\|v\|} = \pm\chi_j(x) \tag{2.35}$$

exists where $\chi_1(x) < \cdots < \chi_{k(x)}(x)$*.*

The values $\chi_j(x)$ from equation (2.35) are called *Lyapunov exponents* and their multiplicities are given by $\dim E_j(x)$. Given an n-dimensional Riemannian manifold M and a diffeomorphism $f : M \to M$, the pair $(f, D_x f)$ is an example of a measurable cocycle where $D_x f$ is the matrix representation of the derivative at x. Since $T_x M \cong \mathbb{R}^n$, the above decomposition applies to the tangent space. Furthermore, if we take f to be the billiard map $\mathcal{T}$, [17] shows how the conditions in equation (2.34) are satisfied, and thus Osedelec's Multiplicative Ergodic Theorem applies to billiards.

Physically, Lyapunov exponents measure the rate of separation between trajectories and thereby indicate stability or instability from small perturbations. To develop the intuition, consider the one-dimensional case where $f : \mathbb{R} \to \mathbb{R}$. Given initial points x_0 and $x_0 + \delta_0$, where δ_0 is *very* small, let δ_n be the separation between $f^n(x_0)$ and $f^n(x_0 + \delta_0)$. If $|\delta_n| \approx |\delta_0| e^{n\chi}$ and $\chi > 0$, then the trajectories separate at an exponential rate, indicating instability. Equation (2.35) can thus by obtained by noting that

$$\chi \approx \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \frac{1}{n} \ln \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right| = \frac{1}{n} \ln \left| \frac{d}{dx}\bigg|_{x_0} (f^n)(x) \right| \tag{2.36}$$

where in the last equality we have taken $\delta_0 \to 0$. If the limit of the last equality exists as $n \to \infty$, then χ is the Lyapunov exponent. Note that if $\chi = 0$, the distance between trajectories might still expand or contract but at a sub-exponential rate. For example, on a circular billiard table, Lyapunov exponents are identically zero at every point [17].

Let M be a Riemannian manifold. We say $x \in M$ is *hyperbolic* if Lyapunov exponents $\chi_1(x) > \cdots > \chi_{k(x)}(x)$ exists and $\chi_j(x) \neq 0$ for all $j = 1, ..., k(x)$. Similar to a hyperbolic fixed point, for a hyperbolic point $x \in M$ we can decompose its tangent space as $T_xM = E^u(x) \oplus E^s(x)$ where

$$E^s(x) = \bigoplus_{\chi_j(x)<0} E_j(x) \quad \text{and} \quad E^u(x) = \bigoplus_{\chi_j(x)>0} E_j(x). \tag{2.37}$$

The s and u in the superscripts denote *stable* and *unstable* because as before, if $\chi_j(x) > 0$ then $v_1 \in E_j(x)$ grows exponentially with rate $\chi_j(x)$. Similarly if $\chi_j(x) < 0$ then $v_2 \in E_j(x)$ contracts exponentially with rate $\chi_j(x)$. For the remainder of this section we focus on outlining two methods for proving hyperbolicity and apply them to the case of billiards. The first method, given in [38] is based off cocycles and it's statement is as follows:

Theorem 2.9 (Wojtkowski 1985). *Given a real matrix (a_{jk}), denote by $\mathscr{P} = \{(a_{jk}) \in GL(n, \mathbb{R})|\ a_{jk} \geq 0$ for all $j, k\}$ and $S\mathscr{P} = \{(a_{jk}) \in \mathscr{P}|\ \det(a_{jk}) = 1\}$. Further, Let $\mathscr{F} \subset \mathscr{P}$ be such that, for every $(a_{jk}) \in \mathscr{F}$ there exist no permutation matrices $(p_{jk}), (q_{jk})$ for which $(p_{jk})(a_{jk})(q_{jk})$ is upper triangular. Similarly, define $S\mathscr{F} = \{(a_{jk}) \in \mathscr{F}|\ \det(a_{jk}) = 1\}$. If $(f, \mathcal{A})$ be a measureable cocycle with generator $A : X \to S\mathscr{F}$, then $\max \chi_j(x) > 0$ almost everywhere.*

Note that the theorem above only shows that the maximal Lyapunov exponent is positive. But to show hyperbolicity, we need all Lyapunov exponents to be nonzero.

Fortunately this is the case for billiards, in fact for any symplectic map $f : M \to M$ where M is two-dimensional. That is for each $x \in M$ we have either two Lyapunov exponents $\chi_+(x)$ and $\chi_-(x)$ each with multiplicity one, or one Lyapunov exponent $\chi(x)$ with multiplicity two. In [17] the authors show that either $\chi_+(x) + \chi_-(x) = 0$ or $\chi(x) = 0$ up to a set of measure zero.

A class of billiard tables with hyperbolic behavior are the Sinai billiard tables which are given by piece-wise smooth curves whose non-flat components are concave. They were introduced along with their hyperbolic and ergodic behavior in [34]. One of the most known examples of these billiard tables consists of a square with a disk removed (see Figure 2.6). The reason for hyperbolicity is that nearby trajectories are scattered after being reflected off a concave portion of the boundary. For this reason, concave portions of a billiard table are commonly refereed to as *scattering*. Conversely, convex portions of a billiard table are called *focusing* given how nearby billiard trajectories converge at the center of the osculating circle that determines the radius of curvature. [1]

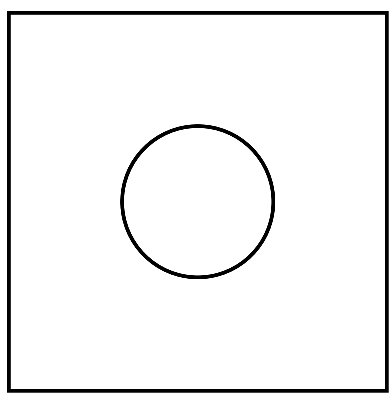


Figure 2.6: Sinai Billiard Table

Sinai billiards are not the only class of hyperbolic billiard tables. In [11] and [12]

[1]In fact, concave pieces of the boundary also have a focusing effect, but the point of focus occurs *outside* the billiard table thus producing the scattering effect.

Bunimovich constructed a class of hyperbolic billiard tables containing only convex or flat components. The most famous of these being the Bunimovich stadium which consists of a rectangle with two opposite sides replaced with half circles. In this case hyperbolicity occurs if enough time passes between reflections involving convex boundaries allowing for billiard trajectories to defocus (see Figure 2.7). Consider a segment of a billiard orbit on the Bunimovich stadium of length $L(s, s_1)$ that connects the two semicircles. We can use equation (2.5) along with the fact that $R = R_1 = 1$ to compute the measurable cocycle, which in this case is given by

$$D_{(s,\theta)}\mathcal{T} = \frac{1}{\sin\theta_1}\begin{bmatrix} L(s,s_1) - \sin\theta & L(s,s_1) \\ L(s,s_1) - \sin\theta - \sin\theta_1 & L(s,s_1) - \sin\theta_1 \end{bmatrix}. \tag{2.38}$$

Since the segment of the billiard orbit goes from one semicircle to the other, $L(s, s_1) - \sin\theta - \sin\theta_1 > 0$, so that all entries of $D_{(s,\theta)}\mathcal{T}$ are strictly positive. Hence we can apply Theorem 2.9 to show (s, θ) is a hyperbolic point.

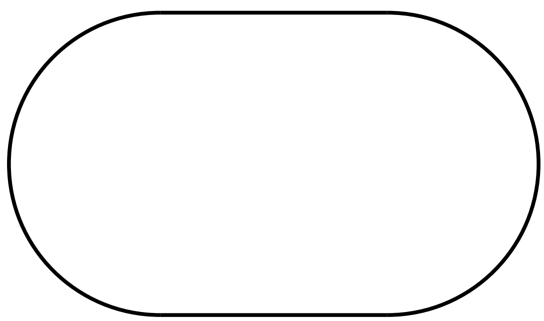


Figure 2.7: Bunimovich Stadium

The second, and more classical way of proving hyperboliciity is through what is known as the method of cones pioneered by V.M. Alekseev and improved upon by M. Wojtkowski in the case of symplectic maps. It turn out that the method of cones provides a superior framework for constructing billiard tables that exhibit hyperbolic behavior.

Let M be a two dimensional manifold and $x \in M$. We define a *cone* or $C(x) \subset T_xM$ as $C(x) = \{(\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2 : \tilde{x}_1\tilde{x}_2 \geq 0\}$ where $\tilde{x}_1$ and $\tilde{x}_2$ are an appropriate basis of T_xM. Given a probability measure μ, a cone $C(x)$ is said to be *preserved* by f if $D_xf(C(x)) \subseteq C(f(x))$ μ almost everywhere. The cone $C(x)$ is *strictly preserved* by f if it is preserved by f, and both boundary lines of $D_xf(C(x))$ are contained strictly within $C(f(x))$ almost everywhere. It is also possible for $C(x)$ to be *eventually strictly preserved* by f if it is preserved by f, and for almost every $x \in M$ there exists a natural number $n(x)$ such that $D_xf^{n(x)}(C(x)) \subset C(f^{n(x)}(x))$.

Geometrically, a cone is defined by boundary lines through the origin in T_xM with slopes $l(x) < r(x)$. It follows that the slopes of the lines contained within the cone form an interval $I(x) = \{t | l(x) \leq t \leq r(x), l(x) < r(x)\}$. Thus, the cone $C(x)$ being preserved by f is equivalent to the interval $I(f(x))$ determined by the cone $C(f(x))$ containing a subinterval $I_1(x)$ corresponding to $D_xf(C(x))$. That is, if $I_1(x) = \{t | l_1(x) \leq t \leq r_1(x),\ l_1(x) < r_1(x)\}$, then $l(f(x)) \leq l_1(x) < r_1(x) \leq r(f(x))$. The following theorem taken form [39] demonstrates how to use cones to show hyperbolicity:

Theorem 2.10 (Wojtkowski 1986)**.** *Let $(f, \mathcal{A})$ be a measurable cocycle as in Theorem 2.8. If there exists a cone that is eventually strictly preserved by f, then the Lyapunov exponent $\chi_+(x)$ of f is positive almost everywhere.*

From the perspective of billiards, we can define a cone $C(p) \subset T_p\mathcal{M}$ as follows: Let $p = (s, w)$ be a point on the phase space $\mathcal{M}$, and L be the line through $\gamma(s)$ that is parallel to w. An infinitesimal family of rays containing w with focus on a point $F \in L$ determines a vector $v \in T_p\mathcal{M}$. Thus, a collection of these focus points that form a segment on L correspond to a cone. Denote the signed distance from F to $\gamma(s)$ by L_F, where the sign is taken with respect to the family of rays. If $\gamma(s)$ lies

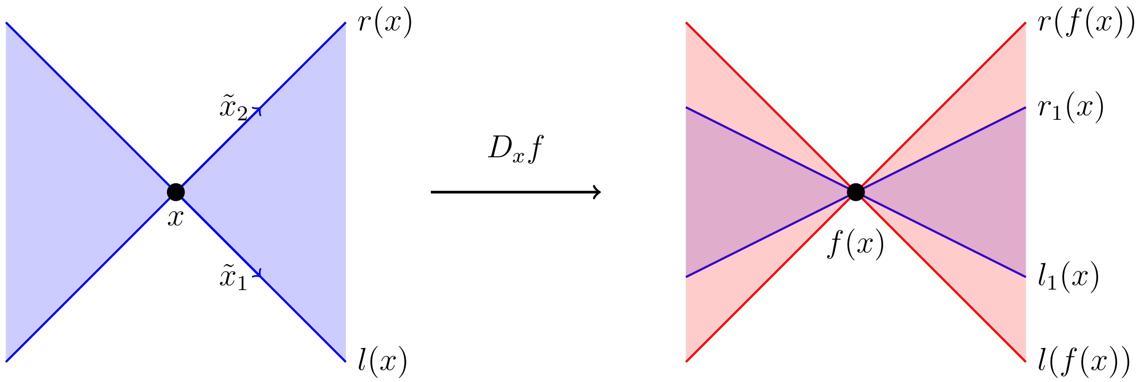


Figure 2.8: Preservation of Cones

on a convex portion of γ i.e. $\kappa(s) > 0$, $C(p)$ is given by $\{L_F | 0 \leq L_F \leq d\}$, where d is the length of the segment contained in a disk tangent to $\gamma(s)$ of radius $\frac{1}{2\kappa(s)}$. Alternatively, if $\gamma(s)$ belongs to a concave portion of γ so that $\kappa(s) < 0$, then we define $C(p)$ as $C(p) = \{L_F | -\infty \leq L_F \leq 0\}$.

In [39], Wojtkowski formalized conditions for which a large class of billiard tables exhibit hyperbolic behavior. The key insight being that the dynamics of a cone $C(p_0)$ can be obtained by the mirror equation

$$-\tilde{L}_{F_0}^{-1} + L_{F_1}^{-1} = \mathrm{sgn}\kappa_1 \frac{2}{d_1}, \tag{2.39}$$

where $\tilde{L}_{F_0}$ and L_{F_1} are the respective signed distances from $\gamma(s_1)$ to the focusing points F_0 and F_1 of the incident and reflected family of rays. Setting $\tilde{L}_{F_0} = L_{F_0} - L(s_0, s_1)$ in equation (2.39), yields [2]

$$L_{F_1} = \frac{d_1 L_{F_0} - d_1 L(s_0, s_1)}{2(\mathrm{sgn}\kappa_1) L_{F_0} - 2(\mathrm{sgn}\kappa_1) L(s_0, s_1) + d_1}. \tag{2.40}$$

which describes the action of $D\mathcal{T}$ on $C(p_0)$. Thus equation (2.40) can be used to formulate conditions on a billiard table for which $\mathcal{T}$ has nonzero Lyapunov exponents.

[2] For the sake of clarity, $L(s_0, s_1)$ denotes the length of a billiard segment so it is always positive. If there is a subscript, as in L_{F_0} then it represents a signed distance so the value may be negative.

That is, Assume that Q is such that $D\mathcal{T}$ preserves $C(p_0)$ so that Theorem 2.10 applies. We want to find the geometric conditions that are forced upon Q by this assumption.

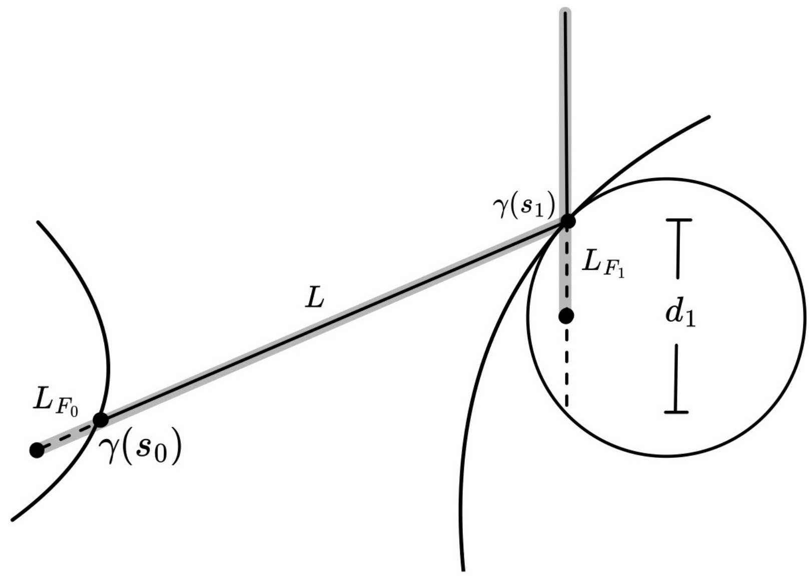


Figure 2.9: Mirror equation

For example, suppose we have two concave pieces γ_0 and γ_1 containing $\gamma(s_0)$ and $\gamma(s_1)$ respectively. From equation (2.40), $C(p_0)$ is preserved if $L_{F_0} \leq L(s_0, s_1)$. As γ_0 is concave, we already have that $L_{F_0} \leq 0$, so that no further conditions are required on the billiard table to apply Theorem 2.10 and hyperbolicity to occur. Next consider the case where γ_0 is convex and γ_1 is concave. It follows from the preservation of $C(p_0)$ that $L_{F_0} \leq L(s_0, s_1)$. However since now γ_0 is convex, we have $L_{F_0} \leq d_0$. Therefore the billiard table must satisfy the condition $d_0 \leq L(s_0, s_1)$ where the inequality is strict whenever $C(p_0)$ is strictly preserved. Lastly, consider the case where γ_0 and γ_1 are both convex. Let $D_2(s_1)$ be the disk tangent to $\gamma(s_1)$ of radius $\frac{1}{2\kappa_1}$. It follows from the mirror equation that Then $F_0 \notin D_2(s_1)$ if and only if $F_1 \in \mathring{D}_2(s_1)$. Hence the preservation of $C(p_0)$ implies that $d_0 + d_1 \leq L(s_0, s_1)$.

Geometrically, the condition $d_0 + d_1 < L(s_0, s_1)$ means that two convex pieces γ_0 and γ_1 of ∂Q that are far enough apart end up producing hyperbolic behavior.

For example, let Q be the Bunimovich stadium and consider a billiard segment on Q connecting the two semicircles. It is clear that $d_0 + d_1 < L(s_0, s_1)$ so that Theorem 2.10 applies. In fact, note that $d_j = \sin\theta_j$ so that this condition can be rewritten as $L(s_0, s_1) - \sin\theta_0 - \sin\theta_1 > 0$, which was the excat condition needed to show hyperbolicity using Theorem 2.9. However if γ_0 and γ_1 are part of the same smooth component of γ, it is also shown in [39] that $d_0 + d_1 < L(s_0, s_1)$ as long as $\frac{d^2R(s)}{ds^2} > 0$.

Chapter 3

Billiards in Finsler Manifolds and Minkowski Spaces

3.1 Finsler Geometry

Finsler geometry has its origin in the study of the calculus of variations. Consider the Lagrangian $ds = F(x(t), x'(t))$ where $x(t) = (x_1(t), \ldots, x_n(t))$ denotes position. Then the integral $\int_a^b F(x(t), x'(t))dt$ gives the distance traveled. Riemannian geometry considers the special case where F is independent of the direction vector $x'(t)$ and is given by

$$F^2(x) = g_{jk}(x)dx^j dx^j, \tag{3.1}$$

where as Finsler geometry deals with the general case $F(x, x'(t))$ without the quadratic restriction given in (3.1). As a result, Finsler geometry allows us to study more expansive cases such as objects traveling in anisotropic mediums where speed depends on the direction of travel. For a thorough introduction to Finsler Geometry see [6] and [16].

A *Finsler structure* on an n-dimensional C^∞ manifold M is a function $F : TM \to [0, \infty)$ such that it satisfies the following conditions:

(i) Regularity: F is C^∞ on $TM \backslash \{0\}$.

(ii) Positive homogeneity: $F(x, cv) = cF(x, v)$ for $c \in \mathbb{R}_+$ and $v \in T_xM$.

(iii) Strong convexity: The $n \times n$ Hessian matrix $(g_{jk}) := \left(\left[\frac{1}{2}F^2 \right]_{v^j v^k} \right)$ is positive definite at every point in $TM \backslash \{0\}$.

The pair (M, F) a *Finsler manifold*, and restricting a Finsler structure F to a tangent space T_xM gives a *Minkowski norm*. In other words, a Minkowski norm is a Finsler structure $F(v)$ that is independent of the point $x \in M$, and a Finsler manifold is a smooth variation of Minkowski norms which are not necessarily induced by an inner product. Therefore we can think of a Minkowski space as the Finsler counterpart to Euclidean space. On Finsler manifolds, geodesics are not necessarily straight lines. However, from [37], we have that in $\mathbb{R}^2$, Finsler structures $F(x_1, x_2, v_1, v_2)$, whose geodesics are given by straight lines satisfy

$$\frac{\partial^2 F}{\partial x_1 \partial v_2} = \frac{\partial^2 F}{\partial x_2 \partial v_1}. \tag{3.2}$$

Consequently geodesics on Minkowski planes are always straight lines.

Given a tangent space T_xM, the *indicatrix* $I_x \subseteq T_xM$ is the set of all Finsler unit vectors i.e. the unit sphere in the tangent space T_xM. Similarly, the *figuratrix* $J_x \subseteq T_x^*M$ is the unit sphere in the cotangent space. On Minkowski spaces, the indicatrices at every point are defined by parallel translations. For $x \in M$, define the *Lengendre transform* on the Minkowski space T_xM as the map $\mathcal{L}_x : I_x \to T_x^*M$ such that $\mathcal{L}_x(u)$ satisfies $\ker \mathcal{L}_x(u) = T_uI_x$ and $\mathcal{L}_x(u)(u) = 1$.

Some examples of Finsler manifolds are given below, with the last example being the main focus of chapter 4.

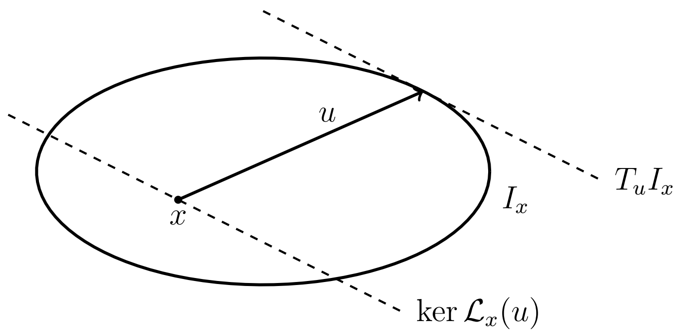


Figure 3.1: Legendre Transform

Riemannian manifolds

Recall that given a smooth manifold M, a Riemannian metric on M is a smooth-covariant 2-tensor field g whose value g_x at $x \in M$ is an inner product on T_xM. If $(x^1, \dots x^n)$ are local coordinates on an open subset $U \subseteq M$ then we may write $g = g_{jk} dx^j dx^k$ where $g_{jk}(x) = g_x \left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right\rangle$. Thus g gives a Finsler structure on M by setting $F(x, v) := \sqrt{g_x \langle v, v \rangle}$.

Randers Spaces

Minkowski norm F is called a *Randers norm* if it is of the form $F(x, v) = \tilde{g}(x, v) + h(x, v)$ where $\tilde{g}(x, v) = \sqrt{\tilde{g}_{jk}(x) v^j v^k}$ is a Riemannian metric and $h(x, v) = h_j(x) v^j$ is a 1-form on M with $\|h_x\|_{\tilde{g}} = \tilde{g}^{jk}(x) h_j(x) h_k(x) < 1$ for any $x \in M$.

Let (M, g) be a Riemannian manifold. Supposed that an object on M is propelled by an internal force with velocity u_x such that $g(x, u_x) = 1$. Without an external force acting on an object, a path of shortest time is the one of shortest length. However if we have an external force w_x (such as wind) acting on the object, we now want to determine the path of shortest time between two points. This problem is called Zermelo's navigation problem, and it is shown in [16], that a Minkowski norm is a Randers norm if and only if it is a solution to Zermelo's navigation problem on a

Riemannian manifold. Thus for any smooth curve γ in M, the Randers length of γ is the time it takes an object to travel along it. To obtain an explicit expression for F in terms of g and w, note that for any $v \in T_xM/\{0\}$, there is a unique solution F to the equation

$$\left\| \frac{v}{F(x,v)} - w_x \right\|_g = \sqrt{g_{jk}\left(\frac{v^j}{F} - w_x^j\right)\left(\frac{v^k}{F} - w_x^k\right)} = 1, \tag{3.3}$$

which we can solve for F to get

$$F(x,v) = \frac{\sqrt{\langle v, w_x\rangle_g^2 + \|v\|_g^2(1-\|w_x\|_g^2)}}{1-\|w_x\|_g^2} - \frac{\langle v, w_x\rangle_g}{1-\|w_x\|_g^2} \quad \text{for } v \in T_xM. \tag{3.4}$$

From equation (3.3) we can see that the indicatrix of of a Randers space is given by a shifted ellipsoid. Further, we can use the first and second terms in equation (3.4) obtain expressions of $\tilde{g}$ and h respectively:

$$\tilde{g}(x) = \frac{g_{jk}(x)w_x^j}{1-\|w_x\|_g^2} \cdot \frac{g_{jk}(x)w_x^k}{1-\|w_x\|_g^2} + \frac{g_{jk}(x)}{1-\|w_x\|_g^2}, \tag{3.5}$$

$$h_j(x) = -\frac{g_{jk}(x)w_x^k}{1-\|w_x\|_g^2}. \tag{3.6}$$

Mixed norm spaces

Next, we consider the Minkowski space where the norm is given by

$$F_{a,b,2k}(v) = a\|v\|_2 + b\|v\|_{2k} \qquad \text{for } a > 0,\ b \geq 0 \tag{3.7}$$

where $\|v\|_2 = (v_1^2 + v_2^2)^{\frac{1}{2}}$ and $\|v\|_{2k} = \left(v_1^{2k} + v_2^{2k}\right)^{\frac{1}{2k}}$. Clearly, $F_{1,0,k}(v) = \|v\|_2$ is just the usual euclidean norm on $\mathbb{R}^2$.

Proposition 3.1. *$(F_{a,b,2k}, \mathbb{R}^2)$ is a Minkowski space.*

Proof. We only need to check that condition (iii) in Definition **??** is satisfied as the first two are easy to see. Let $F = F_{a,b,2k}$ and $(g_{jk}) = \left(\left[\frac{1}{2}F^2\right]_{v_j v_k}\right)$ be the Hessian matrix of $\frac{1}{2}F^2$ whose entries are given by

$$g_{11} = (F_{v_1})^2 + FF_{v_1v_1}, \tag{3.8}$$

$$g_{12} = g_{21} = F_{v_1}F_{v_2} + FF_{v_1v_2}, \tag{3.9}$$

$$g_{22} = (F_{v_2})^2 + FF_{v_2v_2}. \tag{3.10}$$

It suffices to show that $g_{11} > 0$ and $\det(g_{jk}) > 0$. The first and second order partials of F are given by

$$F_{v_1} = av_1(v_1^2 + v_2^2)^{-\frac{1}{2}} + bv_1^{2k-1}(v_1^{2k} + v_2^{2k})^{\frac{1}{2k}-1}, \tag{3.11}$$

$$F_{v_2} = av_2(v_1^2 + v_2^2)^{-\frac{1}{2}} + bv_2^{2k-1}(v_1^{2k} + v_2^{2k})^{\frac{1}{2k}-1}, \tag{3.12}$$

$$F_{v_1v_1} = \alpha v_2^2 + (2k-1)\beta v_1^{2k-2}v_2^{2k}, \tag{3.13}$$

$$F_{v_2v_2} = \alpha v_1^2 + (2k-1)\beta v_1^{2k}v_2^{2k-2}, \tag{3.14}$$

$$F_{v_1v_2} = -v_1v_2\left[\alpha + \beta(2k-1)(v_1v_2)^{2k-2}\right], \tag{3.15}$$

where $\alpha = a(v_1^2 + v_2^2)^{-\frac{3}{2}}$ and $\beta = b(v_1^{2k} + v_2^{2k})^{\frac{1}{2k}-2}$. Note $F_{v_1v_1}, F_{v_2v_2} > 0$ implies $g_{11} > 0$. Further, a small calculation gives $(F_{v_1v_2})^2 = F_{v_1v_1}F_{v_2v_2}$, which we can use to show

$$\det(g_{jk}) = F\left[(F_{v_1})^2F_{v_2v_2} - 2F_{v_1}F_{v_2}F_{v_1v_2} + F_{v_1v_1}(F_{v_2})^2\right] \tag{3.16}$$

$$= F\left(F_{v_1}\sqrt{F_{v_2v_2}} - F_{v_2}\sqrt{F_{v_1v_1}}\right)^2 > 0. \tag{3.17}$$

□

3.2 Finsler Billiards

Finsler billiard systems were introduced in [19] as a natural generalization of Euclidean billiards that use a Finsler structure to represent a billiard traveling on a anisotropic and inhomogeneous medium. On Euclidean billiard systems, a billiard travels with uniform motion at every point on the interior of the billiard table; a consequence of the indicatrix of the Euclidean metric being a unit circle at each point. Although the indicatrix of a Finsler structure is strictly convex and smooth, it is not necessarily symmetric and may vary from point to point. Hence the motion of the billiard depends on its location and direction. We will restate the reflection law for Finsler billiard systems given the lack of angles in Finsler geometry, and prove how under an appropriate set of coordinates, the billiard map is symplectic in the Minkowski setting.

3.2.1 The Reflection Law on Finsler Manifolds

Let Q be a 2-dimensional Finsler manifold with boundary ∂Q. As usual, Q is the billiard table and ∂Q is parameterized by a piece-wise smooth curve γ. Further, let y_0 and y_1 be points on the interior of Q and x be a point on the boundary. Denote the Finsler distances from y_0 to x and from x to y_1 as $F(x - y_0)$ and $F(y_1 - x)$ respectively. If y_0x is a geodesic ray with reflection at x, then xy_1 is the reflected ray if x is a critical point of the distance function $F(x - y_0) + F(y_1 - x)$. In [19] Gutkin and Tabachnikov established the reflection law for Finsler billiard systems in terms of the indicatrix as follows:

Theorem 3.2 (Finsler billiard reflection law)**.** *Let y_0, y_1, and x be as above so that xy_1 is the billiard reflection of y_0x, and let $u, v \in I_x$ travel along y_0x and xy_1 respectively.*

Then $\mathcal{L}(v) - \mathcal{L}(u)$ is conormal to $T_x\partial Q$ i.e. $(\mathcal{L}(v) - \mathcal{L}(u))(w) = 0$ for $w \in T_x\partial Q$.

Given an incident billiard trajectory, we can use Theorem 3.2 to find the reflected trajectory in the following manner:

(1) If T_uI_x is not parallel to $T_x\partial Q$, the line T_uI_x intersects $T_x\partial Q$ at a unique point, say $w \in T_x\partial Q \cap T_uI_x$. Note that w spans $T_x\partial Q$ and $\mathcal{L}(u)(w) = 1$. The Finsler reflection law states that the reflected vector $v \in T_xI$ must satisfy $\mathcal{L}(v)(w) = 1$. Therefore $w \in T_vI_x$. This determines v (see Figure 3.2).

(2) If T_uI_x is parallel to $T_x\partial Q$, then $\mathcal{L}(u)$ is proportional to a covector that is conormal to $T_x\partial Q$. The Finsler reflection law states that $\mathcal{L}(v)$, where v is the reflected ray, is also be proportional to $\mathcal{L}(u)$. Hence T_vI_x is parallel to T_uI_x. This determines the vector v.

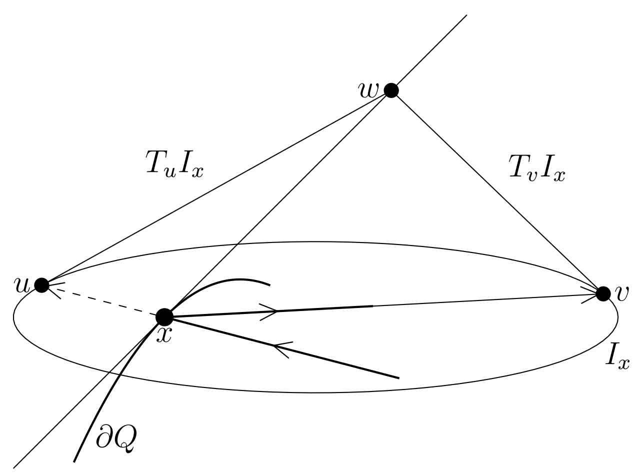


Figure 3.2: An illustration of the Finsler reflection law.

It is important to note that Finsler billiard systems are not generally reversible i.e. given an incident and reflected billiard trajectory, we cannot reverse their roles unless F is *reversible*. That is, F is radially symmetric about the origin so that it satisfies $F(x, -\tilde{v}) = F(x, \tilde{v})$ for $\tilde{v} \in T_xM$. Lastly, it can be shown that when the indicatrix is a circle, the reflection is that of equal angles as expected.

In [36] and [31] Randnović and Tabachnikov independently proved that the Finsler billiad reflection law is the same as the Euclidean reflection law for Randers spaces where the indicatrix is given by an ellipse with focus at the origin. Moreover, Radnović showed how when the indicatrix is an ellipse, with focus not necessarily at the origin, there is a linear mapping of the Randers space which transforms the Finsler billiard system to a trajectorially equivalent Euclidean billiard system. Recently, Akopyan and Karasev gave conditions in [1] for which two different Minkowski norms define the same billiard reflections:

Theorem 3.3 (Akopyan, Karasev)**.** *Let I be the indicatrix of a Minkowski norm, and J be its convex image under a projective transformation that maps each line passing through the origin to itself, preserving the orientation at the origin. Then I and J determine the same billiard reflections.*

It [2] it is shown that these transformations are precisely those who map a circle to an ellipse with one foci at the origin.

3.2.2 The Billiard Map in Minkowski Planes

Let γ with parameter s define the boundary of a billiard table Q. The phase space $\mathcal{M}$ is given by all points (x, v) s.t. $x \in \partial Q$, and $v \in I_x$ points to the interior of Q, as we had in Euclidean billiards. The Finsler billiard map $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is defined using the new reflection law in Theorem 3.2. That is, let $x_0 = \gamma(s_0)$ and $x_1 = \gamma(s_1)$ be points on ∂Q, $v_0 \in I_{x_0}$ be tangent to the billiard flow from x_0 to x_1, and $v_1 \in I_{x_1}$ point in the direction of the reflected billiard trajectory. Then the Minkowski billiard map $\mathcal{T}$ is given by $\mathcal{T} : (x_0, v_0) \mapsto (x_1, v_1)$.

As in the Euclidean case, we can express the Minkowski billiard map with new coordinates using the generating function method. Let $Q \subset (\mathbb{R}^2, F)$ be a compact and convex domain with piece-wise smooth boundary ∂Q, and $\gamma(s)$ be a parameterization of ∂Q with F-arc-length parameter s in the sense that $F(\gamma'(s)) = 1$. Further, let $\ell(s, s_1) = F(\gamma(s_1) - \gamma(s))$ denote the Minkowski length from $\gamma(s)$ to $\gamma(s_1)$. Define a new variable as

$$u = u(s, s_1) := \frac{\partial \ell}{\partial s}(s, s_1). \tag{3.18}$$

It is easy to see that this introduces a new coordinate system (s, u) on $\mathcal{M}$. Applying the above definition to the next iterate of the billiard orbit gives $u_1 = u_1(s_1, s_2) = \frac{\partial \ell}{\partial s_1}(s_1, s_2)$. Then we can express the Minkowski billiard map as $\mathcal{T} : (s, u) \mapsto (s_1, u_1)$. With these new coordinates we can restate Theorem 2.1 but in the Minkowski setting.

Proposition 3.4. *The Minkowski billiard map $\mathcal{T}$ preserves the area form $\omega = du \wedge ds$. Moreover, $D_{(s,u)}\mathcal{T} = \begin{bmatrix} \frac{\partial s_1}{\partial s} & \frac{\partial s_1}{\partial u} \\ \frac{\partial u_1}{\partial s} & \frac{\partial u_1}{\partial u} \end{bmatrix}$ with entires given by*

$$\frac{\partial s_1}{\partial s} = \frac{\sum_{i,j=1}^{2} F_{v_j v_k}(v)\gamma_j'(s)\gamma_k'(s) - \sum_{i=1}^{2} F_{v_j}(v)\gamma_j''(s)}{\sum_{i,j=1}^{2} F_{v_j v_k}(v)\gamma_j'(s)\gamma_k'(s_1)}, \tag{3.19}$$

$$\frac{\partial s_1}{\partial u} = \frac{-1}{\sum_{i,j=1}^{2} F_{v_j v_k}(v)\gamma_j'(s)\gamma_k'(s_1)}, \tag{3.20}$$

$$\frac{\partial u_1}{\partial s} = \sum_{i,j}^{2} F_{v_j v_k}(v)\gamma_j'(s_1)\gamma_k'(s) - \Bigg(\sum_{i,j=1}^{2} F_{v_j v_k}(v)\gamma_j'(s_1)\gamma_k'(s_1)$$

$$+\sum_{i=1}^{2} F_{v_j}(v)\gamma_j''(s_1)\Bigg)\left(\frac{\sum\limits_{i,j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s)\gamma_k'(s) - \sum\limits_{i=1}^{2} F_{v_j}(v)\gamma_j''(s)}{\sum\limits_{i,j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s)\gamma_k'(s_1)}\right), \quad (3.21)$$

$$\frac{\partial u_1}{\partial u} = \frac{\sum\limits_{i,j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s_1)\gamma_k'(s_1) + \sum\limits_{i=1}^{2} F_{v_j}(v)\gamma_j''(s_1)}{\sum\limits_{i,j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s)\gamma_k'(s_1)}. \quad (3.22)$$

Proof. We proceed in much the same was as in Theorem 2.1. Let $\gamma(s), \gamma(s_1)$, and $\gamma(s_2)$ be points on the boundary of the billiard table such the the segment $\gamma(s_1)\gamma(s_2)$ is the reflected billiard trajectory of $\gamma(s)\gamma(s_1)$, and set $u_1(s_1, s_2) = \frac{\partial \ell}{\partial s_1}(s_1, s_2)$ and $\tilde{u}_1 = \frac{\partial \ell}{\partial s_1}(s, s_1)$. By the billiard reflection law, s_1 is a critical point of the distance function $\ell(s, s_1) + \ell(s_1, s_2)$ so that $\frac{\partial}{\partial s_1}\left[\ell(s, s_1) + \ell(s_1, s_2)\right] = \tilde{u}_1(s, s_1) + u_1(s_1, s_2) = 0$ implies $u_1(s_1, s_2) = -\tilde{u}_1(s, s_1)$. Using this relationship we can write

$$d\ell(s, s_1) = u(s, s_1)ds + \tilde{u}(s, s_1)ds_1 = u(s, s_1)ds - u_1(s_1, s_2)ds_1. \quad (3.23)$$

Note that $dd\ell = du \wedge ds - du_1 \wedge ds_1 = 0$, thus proving the first claim. To derive the differential of the billiard map, let $v = \gamma(s_1) - \gamma(s)$ so that we may write

$$u = \frac{\partial \ell}{\partial s}(s, s_1) = -\sum_{j=1}^{2} F_{v_j}(v)\gamma_j'(s) \quad (3.24)$$

$$u_1 = \frac{\partial \ell}{\partial s_1}(s_1, s_2) = -\frac{\partial \ell}{\partial s_1}(s, s_1) = -\sum_{j=1}^{2} F_{v_j}(v)\gamma_j'(s_1) \quad (3.25)$$

Taking the differentials of u and u_1 yield

$$du = \left[\sum_{j,k=1}^{2} F_{v_jv_k}(v)\gamma_j'(s)\gamma_k'(s) - \sum_{j=1}^{2} F_{v_j}(v)\gamma_j'(s)\right] ds$$

$$- \sum_{j,k=1}^{2} F_{v_jv_k}(v)\gamma_j'(s)\gamma_k'(s_1)\ ds_1 \quad (3.26)$$

$$du_1 = \sum_{j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s_1)\gamma_k'(s)\ ds - \Bigg[\sum_{j,k=1}^{2} F_{v_jv_k}(v)\gamma_j'(s_1)\gamma_k'(s_1)$$

$$+ \sum_{j=1}^{2} F_{v_j}(v)\gamma_j''(s_1)\Bigg] ds_1. \tag{3.27}$$

Solving for ds_1 in equation (3.26) gives

$$ds_1 = \frac{\sum\limits_{i,j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s)\gamma_k'(s) - \sum\limits_{i=1}^{2} F_{v_j}(v)\gamma_j''(s)}{\sum\limits_{i,j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s)\gamma_k'(s_1)}\ ds - \frac{du}{\sum\limits_{i,j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s)\gamma_k'(s_1)}. \tag{3.28}$$

Substituting equation (3.28) into equation (3.27) and collecting ds and du terms yields

$$du_1 = \Bigg[\sum_{i,j}^{2} F_{v_jv_k}(v)\gamma_j'(s_1)\gamma_k'(s) - \bigg(\sum_{i,j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s_1)\gamma_k'(s_1)$$

$$+ \sum_{i=1}^{2} F_{v_j}(v)\gamma_j''(s_1)\bigg)\left(\frac{\sum\limits_{i,j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s)\gamma_k'(s) - \sum\limits_{i=1}^{2} F_{v_j}(v)\gamma_j''(s)}{\sum\limits_{i,j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s)\gamma_k'(s_1)}\right)\Bigg]\ ds$$

$$+ \frac{\sum\limits_{i,j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s_1)\gamma_k'(s_1) + \sum\limits_{i=1}^{2} F_{v_j}(v)\gamma_j''(s_1)}{\sum\limits_{i,j=1}^{2} F_{v_jv_k}(v)\gamma_j'(s)\gamma_k'(s_1)} du. \tag{3.29}$$

The entries of the tangent map $D\mathcal{T}$ follow from equations (3.28) and (3.29). $\square$

3.3 Curvature in Normed Spaces

As summarized in Chapter 2, the stability of a billiard orbit depends on the geometry of the billiard table, namely the curvature at the points of reflection and the distance between them. To analyze the stability of billiard orbits on Minkowski planes, it is necessary to extend the concept of curvature from Euclidean space to the Minkowski setting. In this section we state how different curvature concepts extend to normed planes i.e. Minkowski planes where the indicatrix is symmetric about the origin.

On the Euclidean plane, given a curve γ with Euclidean length L, we can intuitively think of its curvature as how fast the tangent field varies with respect to the distance traveled on γ. Formally we define curvature as follows: Let $\gamma : [0, L] \to \mathbb{R}^2$ be a smooth curve with arc-length parameter s and $\nu : [0, L] \to \mathbb{R}$ be the function that maps s to $2A_s$ where A_s is the area of the sector in the unit circle between $\gamma'(0)$ and $\gamma'(s)$. We define the *signed curvature* of γ at s as $\kappa(s) = \nu'(s)$. Equivalently, since γ'' is normal to γ', we can define $\kappa(s)$ in terms of the positively oriented vector field $n(s)$ of unit length:

$$\gamma''(s) = \kappa(s)n(s). \tag{3.30}$$

Geometrically, this means that the area of the unit circle determined by the normal field $n(s)$ is the same as the area determined by the tangent field γ'.

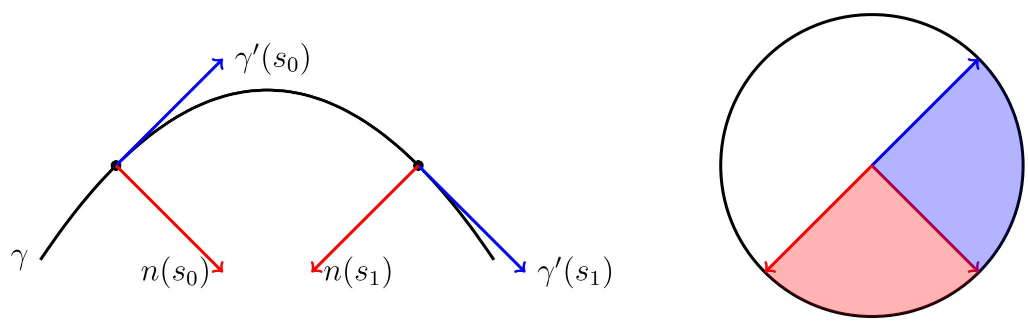


Figure 3.3: Euclidean Curvature

Lastly, as mentioned in chapter 2, one can show that $\kappa(s) = \frac{1}{R(s)}$, where $R(s)$ is the radius of the circle attached to $\gamma(s)$ such that their tangent lines agree. However, When generalizing these equivalent definitions of curvature to a normed plane, [5] shows that they no longer yield the same curvature. The difference arises from using a new definition of orthogonality that is not necessarily symmetric due to the lack of an inner product structure. For the rest of this section, let $(V, \|\cdot\|)$ be a normed space with $\|\cdot\| = F(\cdot)$ and I denote the indicatrix at an arbitrary point.

Given $v, w \in V$, we say that v is *Birkhoff orthogonal* to w, denoted $v \dashv_B w$ if $\|v\| \leq \|v+tw\|$ for all $t \in \mathbb{R}$. Geometrically, $v \dashv_B w$ indicates that $T_{v/\|v\|}I$ is parallel to w (See Figure 3.4). Birkhoff orthogonality is not symmetric, and reversing it involves the use of a different norm. However, closed curves that do reverse Birkhoff orthogonality are called *Radon Curves*. The norm needed to reverse Birkhoff orthogonality is defined as follows: Fix a determinant form $[\cdot,\cdot]$ on V i.e. a non-degenerate bi-linear form such that $[v, v] = 0$ for all v. We define the *anti-norm* on V as

$$\|v\|_a := \sup\{|[w, v]| : \|w\| = 1\}. \tag{3.31}$$

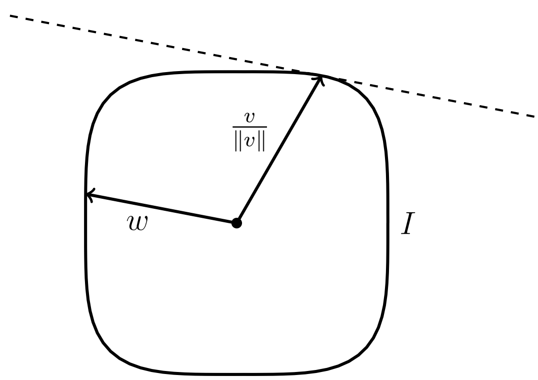


Figure 3.4: Birkhoff orthogonality

In $\mathbb{R}^2$, a determinant form is unique up to a constant multiple since it amounts to fixing an orientation with unit area. Further, although $\dim V = \dim V^*$, there is no canonical isomorphism between V and V^*. However, any isomorphism is given by an identification $v \mapsto [\cdot, v]$ for any $v \in V$, where $[\cdot,\cdot]$ is a non-degenerate bilinear form. It follows from equation (3.31) that $|[w, v]| \leq \|w\|\|v\|_a$, and it is shown in [28] that $w \dashv_B v$ if and only if $v \dashv_B^a w$, where $\dashv_B^a$ denotes Birkhoff orthogonality in the antinorm. Note that whenever $w \dashv_B v$ and $\|w\| = 1$, equation (3.31) is equal to the dual norm of the functional $[\cdot, v] \in V^*$. So for example, whenever $\frac{1}{p} + \frac{1}{q} = 1$, the norms on ℓ_p and ℓ_q reverse Birkhoff orthogonality.

Having redefined orthogonality we can now generalize the normal field and the first curvature concept, where curvature was given in terms of the area swept within the unit circle by the tangent field γ'. Let $\gamma : [0, \ell] \to \mathbb{R}^2$ be a smooth curve of normed length ℓ, and s be an arc length parameter of γ also in the norm. The *right normal field to* γ is the vector field $n_\gamma : [0, \ell] \to V$ such that each $s \in [0, \ell]$ gets mapped to the unique vector such that $\gamma'(s) \dashv_B n_\gamma(s)$ and $[\gamma'(s), n(s)] = 1$. It is shown in [5] how $n_\gamma(s)$ is unit in the anti-norm for all s.

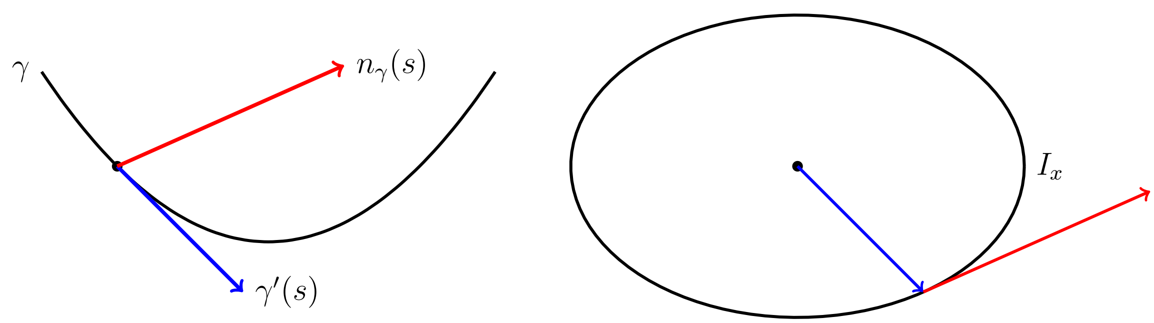


Figure 3.5: The right normal field n_γ

Let $\varphi : [0, 2A(I)] \to \mathbb{R}^2$ be a parameterization of the indicatrix I by twice the area of its sectors. Identify the tangent field γ' within the indicatrix so that

$$\gamma'(s) = \varphi(\nu(s)), \tag{3.32}$$

where $\nu : [0, \ell] \to \mathbb{R}$ maps the arc-length of γ to twice the area of the sectors spanned by $\gamma'(0)$ and $\gamma'(s)$. We define the *Minkowski curvature* of γ at $\gamma(s)$ as

$$\kappa_m := \nu'(s). \tag{3.33}$$

The analogous definition to equation (3.33), but where the area is spanned by n_γ

within the indicatrix of the anti-norm I_a is called the *normal curvature* and is denoted κ_n. Differentiating equation (3.32) yields

$$\gamma''(s) = \nu'(s)\frac{d\varphi}{d\nu}(\nu(s)) = \kappa_m(s)n_\gamma(s). \tag{3.34}$$

Geometrically, this means that as γ' rotates about the indicatrix I, the right normal field n_γ rotates about the indicatrix of the anti-norm I_a. However, since the indicatrix in the norm and anti-norm are generally not the same, γ' and n_γ generally sweep different areas and thus yield different values for $\kappa_m(s)$ and $\kappa_n(s)$.

Using equation (3.34) we can rewrite $D\mathcal{T}$ given in equations (3.19)-(3.22) in therms of the Minkowski curvature and right normal field of the billiard table. However, this is far from practical. Fortunately it is shown in [5] how κ_m and n_γ may be expressed in terms of a Euclidean structure. Given a parameterization $r(\theta(s))$ of the indicatrix I such that $\gamma'(s) = r(\theta(s)) \cdot (\cos\theta(s), \sin\theta(s))$ for $\theta(s) \in [0, 2\pi]$, we can write

$$\kappa_m(s) = \kappa_e(s)r(\theta(s))^3, \tag{3.35}$$

and

$$\begin{aligned} n_\gamma(s) = &\frac{1}{r(\theta(s))^2}\frac{dr}{d\theta}(\theta(s)) \cdot (\cos\theta(s), \sin\theta(s)) \\ &+ \frac{1}{r(\theta(s))} \cdot (-\sin\theta(s), \cos\theta(s)). \end{aligned} \tag{3.36}$$

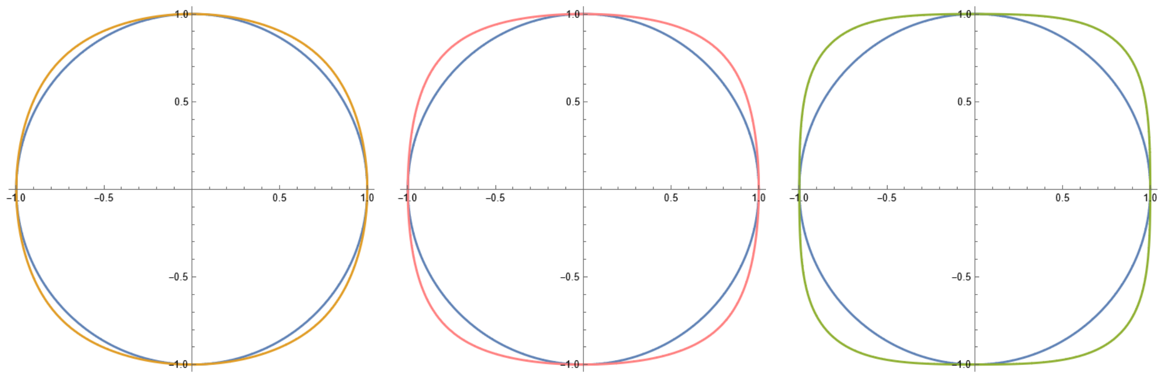
Chapter 4

Billiards in Mixed Norm Spaces

4.1 Class of Billiard Systems

In this chapter, we study the stability properties of certain periodic 2-orbits of (locally defined) billiard tables in the Minkowski plane $(\mathbb{R}^2, F)$ where $F(v) = a\|v\|_2 + b\|v\|_4$ and $a > 0$, $b \geq 0$, and $a + b = 1$. We showed that $(\mathbb{R}^2, F)$ is a Minkowski space in Proposition 3.1, and in fact is a normed space so that the indicatrix given by the level curve $F(v) = 1$ is symmetric about the origin. The condition $a + b = 1$ not only makes it so that the calculations for the twist coefficient τ_1 are easier to manage, but make it so that the billiard has identical motion to a Euclidean billiard system along the horizontal and vertical axes.

Let $\alpha(t) = \sum\limits_{n\geq 1} \alpha_{2n} t^{2n}$ and $\beta(t) = \sum\limits_{n\geq 1} \beta_{2n} t^{2n}$ with $t \in (-\epsilon, \epsilon)$, be two even functions for some $\epsilon > 0$, and $\gamma_0 = (\alpha(t), t)$ and $\gamma_1 = (L - \beta(t), -t)$ be two curves on $\mathbb{R}^2$. The class of billiard tables $Q(L, \alpha(t), \beta(t))$ we will consider are those whose boundary ∂Q is given by a piece-wise smooth curve γ consisting of two horizontal line segments connecting the curves γ_0 and γ_1 at their endpoints (see Figure 4.2). Rather than parameterizing γ with a global Minkowski arc-length parameter, we will use a local



(a) $a = 0.7$ and $b = 0.3$ (b) $a = 0.4$ and $b = 0.6$ (c) $a = 0.1$ and $b = 0.9$

Figure 4.1: Indicatricies such that $a + b = 1$

arc-length parameter s on each of the curves γ_j such that $F(\gamma'(s))$ for all s and $s(\gamma_j(0)) = 0$ for $j = 0, 1$. It is clear from the Finsler reflection law that there exists a periodic 2-orbit $\mathcal{O}_2$, bouncing back and forth between $(0, 0)$ and $(L, 0)$ where L is the Euclidean length between the two reflection points.

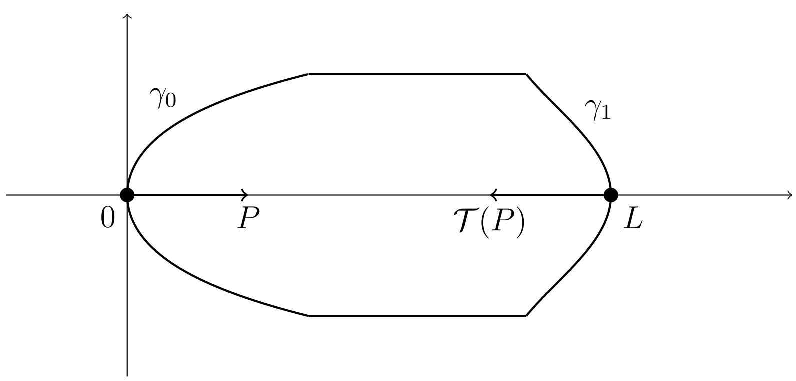


Figure 4.2: An illustration of the billiard table.

The phase space $\mathcal{M}$ is the set of unit vectors with foot on ∂Q that point inside Q. However, in chapter 3 we introduced a new set of coordinates (s, u) on $\mathcal{M}$, where $u = \frac{\partial F}{\partial s}(s, s_1)$ and $F(s, s_1) = F(\gamma(s_1) - \gamma(s))$ is the Finsler length from $\gamma(s)$ to $\gamma(s_1)$. Therefore the billiard map $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is given by $(s_1, u_1) = \mathcal{T}(s, u)$ with $u_1 = \frac{\partial F}{\partial s_1}(s_1, s_2)$. For example, the periodic 2-orbit in Figure 4.2 is given by $\mathcal{O}_2 = \{P, \mathcal{T}(P)\}$ where $P = (0, 0)$ and $\mathcal{T}(P) = (0, 0)$.

4.2 The First Twist Coefficient for Symmetric Billiard Tables

In chapter 2 we stated how Moser's twist mapping theorem (Theorem 2.6) utilizes the twist coefficients τ_j in the Birkhoff normal form of a symplectic function given in equation (2.29), to prove the nonlinear stability of elliptic periodic billiard orbits. Having reformulated the billiard map $\mathcal{T}$ on Minkowski planes in chapter 3, we now construct its Birkhoff transformation and give an explicit formula for the first twist coefficient in terms of the geometric properties of the billiard table. We will then use the first twist coefficient to establish descriptions of the nonlinear stability of certain Minkowski billiard systems.

We will first calculate the first twist coefficient $\tau_1(\mathcal{T}^2, P)$ of the two-step billiard map $\mathcal{T}^2$ about the fixed point P in the case where $\alpha(t) = \beta(t)$, so that the billiard table $Q(L, \alpha(t), \beta(t))$ is symmetric about the line $x = L/2$. From the symmetry, we get that the radius of curvature functions $R_j(s)$ on γ_j are identical, so we set $R(s) = R_0(s) = R_1(s)$. Moreover, let Rot_π be the rotation of $Q(L, \alpha(t), \beta(t))$ about the point $(\frac{L}{2}, 0)$ by π. Since $F(v) = F(-v)$ then Rot_π identifies the phase space $\mathcal{M}_1$ of γ_1 with the phase space $\mathcal{M}_0$ of γ_0.[1]

Pick a small neighborhood $U_0 \subset \mathcal{M}_0$ of $P \in \mathcal{M}_0$. Then $U_1 = \mathrm{Rot}_\pi(U_0) \subset \mathcal{M}_1$ is a small neighborhood of $\mathcal{T}(P) \in \mathcal{M}_1$. Let $\mathcal{T}_j = \mathcal{T}|_{U_j} : U_j \to \mathcal{M}_{1-j}$ be the restriction of the billiard map $\mathcal{T}$ to U_j, $j = 0, 1$. Observe that $\mathrm{Rot}_\pi \circ \mathcal{T}_0 = \mathcal{T}_1 \circ \mathrm{Rot}_\pi$. It follows that restricting on U_0,

$$\mathcal{T}^2 = \mathcal{T}_1 \circ \mathcal{T}_0 = \mathrm{Rot}_\pi \circ \mathcal{T}_0 \circ \mathrm{Rot}_\pi \circ \mathcal{T}_0 = (\mathrm{Rot}_\pi \circ \mathcal{T}_0)^2 . \tag{4.1}$$

[1]This is false for general Minkowski norms that don't satisfy $F(-v) = F(v)$.

It follows from equation (2.29) that $\tau_1(\mathcal{T}^2, P) = 2\tau_1(\mathrm{Rot}_\pi \circ \mathcal{T}_0, P)$. Identifying $\mathcal{M}_1$ with $\mathcal{M}_0$ using Rot_π, we can identify the one-step billiard map $\mathcal{T}_0$ with the abstract self-map $\mathrm{Rot}_\pi \circ \mathcal{T}_0 : U_0 \to \mathcal{M}_0$. In the following, we will abuse our notation and use $\mathcal{T}$ for the abstract map $\mathrm{Rot} \circ \mathcal{T}_0$. In particular $\tau_1(\mathcal{T}^2, P) = 2\tau_1(\mathcal{T}, P)$.

To find the elliptic conditions of our periodic orbit, we begin with the calculation of the tangent matrix of the one-step billiard map $\mathcal{T}$. To do so, we make use of equations (3.19) - (3.22) along with the first and second order partial derivatives of $F(v) = F(v_1, v_2)$ given by

$$F_{v^1} = \frac{av_1}{\sqrt{v_1^2 + v_2^2}} + \frac{bv_1}{(v_1^4 + v_2^4)^{\frac{3}{4}}},$$

$$F_{v^2} = \frac{av_2}{\sqrt{v_1^2 + v_2^2}} + \frac{bv_2}{(v_1^4 + v_2^4)^{\frac{3}{4}}},$$

$$F_{v^1v^1} = \frac{av_2^2}{(v_1^2 + v_2^2)^{\frac{3}{2}}} + \frac{3bv_1^2v_2^4}{(v_1^4 + v_2^4)^{\frac{7}{4}}},$$

$$F_{v^1v^2} = -\frac{av_1v_2}{(v_1^2 + v_2^2)^{\frac{3}{2}}} - \frac{3bv_1^3v_2^3}{(v_1^4 + v_2^4)},$$

$$F_{v^2v^2} = \frac{av_1^2}{(v_1^2 + v_2^2)^{\frac{3}{2}}} + \frac{3bv_1^4v_2^2}{(v_1^4 + v_2^4)^{\frac{7}{4}}}.$$

Note that the tangent vector at $(0,0)$ and $(L,0)$ are given by $\gamma_0'(0) = \langle 0, -1\rangle$ and $\gamma_1'(0) = \langle 0, 1\rangle$ respectively. Recall $\gamma''(s) = \kappa_m(s)n_\gamma(s)$ where κ_m and n_γ can be written in terms of a parameterization $r(\theta(s))$ of the indicatrix as shown in equations (3.35) and (3.36). Given the norm $F(v) = a\|v\|_2 + b\|v\|_4$ for $a > 0$ and $b \geq 0$, the parameterization is given by

$$r(\theta(s)) = \frac{1}{a + b\left(\cos^4\theta(s) + \sin^4\theta(s)\right)^{\frac{1}{4}}}. \tag{4.2}$$

Moreover, from equation (3.36) we see that κ_m may be written in terms of the Euclidean radius of curvature since $\kappa_e(s) = \frac{1}{R(s)}$. Letting $R(0) = R$ and $a + b = 1$, equations (3.19)-(3.22) at $P = (0,0)$ simplify to:

$$\frac{\partial s_1}{\partial s}(0,0) = \frac{L}{aR} - 1, \tag{4.3}$$

$$\frac{\partial s_1}{\partial u}(0,0) = \frac{L}{a}, \tag{4.4}$$

$$\frac{\partial u_1}{\partial s}(0,0) = \frac{L-2aR}{aR^2}, \tag{4.5}$$

$$\frac{\partial u_1}{\partial u}(0,0) = \frac{L}{aR} - 1. \tag{4.6}$$

The tangent matrix of the one-step billiard map $\mathcal{T}$ at P is thus given by

$$D_p\mathcal{T} = \begin{bmatrix} \tilde{a}_{10} & \tilde{a}_{01} \\ \tilde{b}_{10} & \tilde{a}_{01} \end{bmatrix} := \begin{bmatrix} \frac{L}{aR} - 1 & \frac{L}{a} \\ \frac{L-2aR}{aR^2} & \frac{L}{aR} - 1 \end{bmatrix} \tag{4.7}$$

Note that in the Euclidean case where the indicatrix is a unit circle, ($a = 1$ and $b = 0$), Then $D_P\mathcal{T}$ reduces to a more familiar tangent matrix $D_P\mathcal{T} = \begin{bmatrix} \frac{L}{R} - 1 & L \\ \frac{L}{R^2} - \frac{2}{R} & \frac{L}{R} - 1 \end{bmatrix}$, see [22]. Because $\mathcal{T}$ is symplectic, the eigenvalues λ of $D_p\mathcal{T}$ satisfy $\lambda^2 - \mathrm{Tr}(D_P\mathcal{T})\lambda + 1 = 0$, where $\mathrm{Tr}(D_P\mathcal{T}) = \frac{2L}{aR} - 2$. Thus we can define $\lambda = \tilde{a}_{10} - i\sqrt{-\tilde{a}_{01}\tilde{b}_{10}}$ and $\overline{\lambda} = \frac{1}{\lambda}$. It follows that $D_P T$ is

(1) parabolic if $L = 2aR$,

(2) hyperbolic if $L > 2aR$,

(3) elliptic if $0 < L < 2aR$.

Going forward, we will assume $D_P\mathcal{T}$ is elliptic, and we will also assume the non-resonance condition needed to calculate the first twist coefficient τ_1:

(A1) $\qquad \lambda^4 \neq 1$ or equivalently, $L \in (0, 2aR) \setminus \{aR\}$

We emphasize that the dynamics pertaining to the resonance condition are not necessarily significant. The resonances are just an unfortunate byproduct of the coordinate changes needed for the Birkhoff transformation.

Theorem 4.1. *Let $Q(L,\alpha(t),\beta(t))$ be given such that $R_0 = R_1 = R$, and assume the condition given in (A1) is satisfied. Then the first twist coefficient of the one-step billiard map $\mathcal{T} : (s,u) \mapsto (s_1,u_1)$ is given by*

$$\tau_1(\mathcal{T},P) = \frac{a^2LRR'' + (4-3a)aL + 2(2a-9)aR + 12R}{8a^2R(L-2aR)}. \tag{4.8}$$

As done in [22], the functions $\alpha(t)$ and $\beta(t)$ assumed to be even polynomials in order to simplify the computation of τ_1. Without these conditions, the expression for τ_1 would be significantly more complex as it would also depend on R'. Letting $a = 1$ and $b = 0$ into the equation above, we recover the first twist coefficient for the periodic orbit of period 2 for Euclidean billiards: $\tau_1 = \frac{1}{8}\left(\frac{1}{R} - \frac{LR''}{2R-L}\right)$. The proof of Theorem 4.1 is given over the subsequent sections and follows closely the approaches given in [22] and [30].

4.3 Proof of Theorem

4.3.1 The Taylor Polynomial of the One-Step Billiard Map

To begin, we first compute the Taylor expansion of the billiard map $\mathcal{T}(s,u) = (s_1,u_1)$ at $P = (0,0)$. We already computed the first order partial derivatives of s_1 and u_1 at P which are given in equations (4.3) - (4.6). To compute the higher order derivatives, we can apply the chain rule to equations (3.19)-(3.22), or better yet, use mathematical software to aid in this task. [2] Note that $R(s)$ has a local minimum at $\gamma_0(0)$ and $\gamma_1(0)$. Thus $R'(0) = 0$, and because of this, further calculations show that

[2] The Mathematica code can be found in the appendix.

$\frac{\partial^{j+k}s_1}{\partial s^j \partial u^k}(0,0) = \frac{\partial^{j+k}u_1}{\partial s^j \partial u^k}(0,0) = 0$ for $j+k=2$. Denote $R^{(j)}(0) = R^{(j)}$, we list the third order partial derivatives of s_1 and u_1 needed for the calculation of τ_1:

$$\begin{aligned}\frac{\partial^3 s_1}{\partial s^3}(0,0) = \frac{1}{a^4R^5}\Big[&2a^4LR^2 - 3a^3R\left(L^2 - LR + 2R^2\right) - a^3LR^3R''\\ &+ a^2\left(L^3 + 3LR^2\right) - 3aLR(L-3R) - 6LR^2\Big],\end{aligned} \tag{4.9}$$

$$\begin{aligned}\frac{\partial^3 s_1}{\partial s^2\partial u}(0,0) = \frac{1}{a^4R^4}\Big[&2a^4LR^2 - 2a^3R\left(L^2 + R^2\right) + a^2\left(L^3 + 2LR^2\right)\\ &- 3aLR(L-3R) - 6LR^2\Big],\end{aligned} \tag{4.10}$$

$$\frac{\partial^3 s_1}{\partial s\partial u^2}(0,0) = \frac{L\left(-a^3LR + a^2\left(L^2+R^2\right) - 3aR(L-3R) - 6R^2\right)}{a^4R^3}, \tag{4.11}$$

$$\frac{\partial^3 s_1}{\partial u^3}(0,0) = \frac{L\left(a^2L^2 - 3aR(L-3R) - 6R^2\right)}{a^4R^2}, \tag{4.12}$$

$$\begin{aligned}\frac{\partial^3 u_1}{\partial s^3}(0,0) = \frac{1}{a^4R^6}\Big[&6LR^2 - 8a^5R^3 - 12a^3LR(L+R) + 3a^2L^2(L+3R)\\ &+ 2a^4R^2(8L+3R) - 3aL(L^2 - LR + 3R^2)\\ &+ aR(L^3 - 3aL^2R + 4a^2LR^2 - 2a^3R^3)R''\Big],\end{aligned} \tag{4.13}$$

$$\begin{aligned}\frac{\partial^3 u_1}{\partial s^2\partial u}(0,0) = \frac{-1}{a^4R^5}\Big[&6a^4LR^2 - a^3R\left(9L^2 + 3LR + 2R^2\right)\\ &+ a^2L\left(3L^2 + 6LR + R^2\right) - 3aL\left(L^2 - LR + 3R^2\right)\\ &+ aLRR''(L-aR)^2 + 6LR^2\Big],\end{aligned} \tag{4.14}$$

$$\begin{aligned}\frac{\partial^3 u_1}{\partial s\partial u^2}(0,0) = \frac{-1}{a^4R^4}\Big[&6LR^2 + 2a^4LR^2 + a^2L(3L^2 + 3LR + 2R^2)\\ &- 3aL(L^2 - LR + 3R^2) - 2a^3(3L^2R + R^3)\\ &+ aL^2R(L-aR)R''\Big],\end{aligned} \tag{4.15}$$

$$\frac{\partial^3 u_1}{\partial u^3}(0,0) = \frac{3(a-1)L\left(a^2LR - a\left(L^2 - LR + R^2\right) + 2R^2\right) - aL^3RR''}{a^4R^3}. \tag{4.16}$$

The Taylor polynomial of $\mathcal{T}(s,u) = (s_1, u_1)$ at $P = (0,0)$ is thus given by

$$s_1(s,u) = \tilde{a}_{10}s + \tilde{a}_{01}u + \sum_{j+k=3} \tilde{a}_{jk}s^k u^k + \text{h.o.t} \tag{4.17}$$

$$u_1(s,u) = \tilde{b}_{10}s + \tilde{b}_{01}u + \sum_{j+k=3} \tilde{b}_{jk}s^k u^k + \text{h.o.t} \tag{4.18}$$

where $\tilde{a}_{jk} = \frac{1}{j!k!}\frac{\partial^{j+k}s_1}{\partial s^j \partial u^k}(0,0)$, $\tilde{b}_{jk} = \frac{1}{j!k!}\frac{\partial^{j+k}u_1}{\partial s^j \partial u^k}(0,0)$, and h.o.t stands for higher order terms.

4.3.2 The First Transformation: A Rigid Rotation

Having computed the third order Taylor polynomial of the billiard map $\mathcal{T}$, we may now initiate the construction of a symplectic coordinate transformation for $\mathcal{T}$ is in Birkhoff normal form. The first step consists of finding a coordinate transformation for which $D_p\mathcal{T}$ acts as a rigid rotation.

Given $D_P\mathcal{T}$ as in equation (4.7), $\lambda = \tilde{a}_{10} - i\sqrt{-\tilde{a}_{01}\tilde{b}_{10}}$ is an eigenvalue of $D_P\mathcal{T}$ with corresponding eigenvector $v_\lambda = \begin{bmatrix} i\eta \\ \eta^{-1} \end{bmatrix}$, where $\eta = \left(-\tilde{a}_{01}/\tilde{b}_{10}\right)^{1/4} > 0$. It follows that $D_P\mathcal{T}$ is conjugate to a rotation matrix R_θ where $\theta \in (0, 2\pi)$ is the argument of λ. That is, we can write $D_P\mathcal{T} = BR_\theta B^{-1}$ where

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad B = \begin{bmatrix} \eta & 0 \\ 0 & \eta^{-1} \end{bmatrix}. \tag{4.19}$$

Define the coordinate transformation $(s,u) \mapsto (x,y)$ given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = B^{-1} \begin{bmatrix} s \\ u \end{bmatrix} = \begin{bmatrix} s/\eta \\ \eta u \end{bmatrix}. \tag{4.20}$$

We abuse notation by adopting the convention that $\mathcal{T}$ will always denote the billiard map regardless of the coordinate system. This will hold true for later transformations as well. Therefore we denote $\mathcal{T}(x,y) = (x_1, y_1) = (s_1/\eta, \eta u_1)$, and it follows form equations (4.17) and (4.18) that

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = B^{-1} \begin{bmatrix} \tilde{a}_{10}s + \tilde{a}_{01}u + \sum\limits_{j+k=3} \tilde{a}_{jk}s^j u^k \\ \tilde{b}_{10}s + \tilde{b}_{01}u + \sum\limits_{j+k=3} \tilde{b}_{jk}s^j u^k \end{bmatrix} + \text{ h.o.t} \tag{4.21}$$

$$= R_\theta \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \sum\limits_{j+k=3} a_{jk}x^j y^k \\ \sum\limits_{j+k=3} b_{jk}x^j y^k \end{bmatrix} + \text{ h.o.t} \tag{4.22}$$

where we obtained (4.22) from (4.21) by using the relation $\begin{bmatrix} s \\ u \end{bmatrix} = B \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \eta x \\ y/\eta \end{bmatrix}$. The coefficients a_{jk} and b_{jk} of the billiard map $\mathcal{T}$ in the new coordinates (x,y) are given by

$$a_{jk} = \eta^{j-k-1}\tilde{a}_{jk} \qquad \text{and} \qquad b_{jk} = \eta^{j-k+1}\tilde{b}_{jk} \tag{4.23}$$

for $j + k = 1, 3$. Finally, note that $\mathcal{T} : (x,y) \mapsto (x_1, y_1)$ is symplectic since both B and $T : (s,u) \mapsto (s_1, u_1)$ are symplectic.

$$\begin{tikzcd}
(s,u) \arrow[r, "B^{-1}"] \arrow[d, "\mathcal{T}"'] & (x,y) \arrow[d, "\mathcal{T}"'] \\
(s_1,u_1) \arrow[r, "B^{-1}"'] & (x_1,y_1)
\end{tikzcd}$$

Figure 4.3: First Coordinate Transformation

4.3.3 The Second Transformation: Diagonalizing the Linear Terms

Next, we now need a second coordinate transformation in which R_θ becomes the diagonal matrix $D = \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}$. To do so, we embed $\mathbb{R}^2 \subset \mathbb{C}^2$ and consider $\mathcal{T} : (x, y) \to (x_1, y_1)$ to be a complex symplectic function on a small neighborhood $(0,0) \in \mathbb{C}^2$. Define new complex coordinates z and w as

$$z = x + iy, \qquad w = x - iy, \tag{4.24}$$

and let $\mathcal{T}(z, w) = (z_1, w_1) = (x_1 + iy_1, x_1 - iy_1)$. Note that under this coordinate transformation, the subspace $\mathbb{R}^2$ of $\mathbb{C}^2$ is mapped to $\{(z, w) \in \mathbb{C}^2 : w = \overline{z}\}$ which may also be regarded as a real subspace. Observe further that $\det D_P\mathcal{T} \neq 0$. Thus by inverse function theorem, there exists a neighborhood about P on which $\mathcal{T} : (z, w) \mapsto (z_1, w_1)$ is a diffeomorphism so that $w_1 = \overline{z}_1$ if and only if $w = \overline{z}$.

To compute the coefficients of the billiard map $\mathcal{T} : (z, w) \mapsto (z_1, w_1)$, we plug the Taylor expansions of x_1 and y_1 given in equation (4.22) into the definitions for z_1 and w_1 to get

$$\begin{aligned} z_1 &= x_1 + iy_1 && (4.25) \\ &= x\cos\theta - y\sin\theta + \sum_{j+k=3} a_{jk}x^j y^k \\ &\qquad + i\left(x\sin\theta + y\cos\theta + \sum_{j+k=3} b_{jk}x^j y^k\right) + \text{h.o.t.} && (4.26) \\ &= \lambda\left(z + \sum_{j+k=3} c_{jk}z^j w^k\right) + \text{h.o.t.}, && (4.27) \end{aligned}$$

and similarly,

$$w_1 = \overline{\lambda}\left(w + \sum_{j+k=3} \overline{c}_{jk} w^j z^k\right) + \text{h.o.t.}. \tag{4.28}$$

The c_{jk} in equations (4.27) and (4.28) are given by

$$c_{30} = 2^{-3}\overline{\lambda}\left(a_{30} + ib_{30} - ia_{21} + b_{21} - a_{12} - ib_{12} + ia_{03} - b_{03}\right), \tag{4.29}$$

$$c_{21} = 2^{-3}\overline{\lambda}\left(3a_{30} + 3ib_{30} - ia_{21} + b_{21} + a_{12} + ib_{12} - 3ia_{03} + 3b_{03}\right), \tag{4.30}$$

$$c_{12} = 2^{-3}\overline{\lambda}\left(3a_{30} + 3ib_{30} + ia_{21} - b_{21} + a_{12} + ib_{12} + 3ia_{03} - 3b_{03}\right), \tag{4.31}$$

$$c_{03} = 2^{-3}\overline{\lambda}\left(a_{30} + ib_{30} + ia_{21} - b_{21} - a_{12} - ib_{12} - ia_{03} + b_{03}\right). \tag{4.32}$$

$$\begin{tikzcd}
(s,u) \arrow[r, "B^{-1}"] \arrow[d, "\mathcal{T}"'] & (x,y) \arrow[r] \arrow[d, "\mathcal{T}"'] & (z,w) \arrow[d, "\mathcal{T}"'] \\
(s_1,u_1) \arrow[r, "B^{-1}"'] & (x_1,y_1) \arrow[r] & (z_1,w_1)
\end{tikzcd}$$

Figure 4.4: Second Coordinate Transformation

Since $\mathcal{T} : (x,y) \mapsto (x_1,y_1)$ is symplectic, then $D_P\mathcal{T} = 1$ in the (x,y) coordinates. Further, the transformation $(x,y) \mapsto (z,w)$ has constant Jacobian. It follows that $\frac{\partial(z_1,w_1)}{(z,w)} = 1 + (3c_{30} + \overline{c}_{12})z^2 + (2c_{21} + 2\overline{c}_{21})zw + (c_{12} + 3\overline{c}_{30})w^2 + \text{h.o.t.}$ is identically equal to one so that all the non-constant terms vanish. We thus get the following relations:

$$c_{12} = -3\overline{c}_{30}, \qquad c_{21} = \overline{c}_{21}. \tag{4.33}$$

In particular, c_{21} is purely imaginary, and we can thus write $c_{21} = i\tau_1$ where $\tau_1 \in \mathbb{R}$ is given by:

$$\tau_1 = \frac{a^2 LRR'' + (4-3a)aL + 2(2a-9)aR + 12R}{8a^2R(L-2aR)}. \tag{4.34}$$

Shortly we will show that τ_1 is in fact, the first twist coefficient of $\mathcal{T}$ at P.

4.3.4 The Third Transformation: Killing Most Third Order Terms

Now we need a transformation $(z,w) \mapsto (z',w')$ such that most third order terms cancel. Let this transformation be given by

$$z' := z + p_3(z,w) = z + \sum_{j+k=3} d_{jk} z^j w'^k, \tag{4.35}$$

$$w' := w + \overline{p}_3(w,z) = w + \sum_{j+k=3} \overline{d}_{jk} w^j z^k. \tag{4.36}$$

Then z_1' satisfies,

$$z_1' = z_1 + \sum_{j+k=3} d_{jk} z_1^j w_1^k \tag{4.37}$$

$$= \lambda \left(z + \sum_{j+k=3} c_{jk} z^j w^k \right) + \sum_{j+k=3} d_{jk} \lambda^{j-k} z^j w^k + \text{ h.o.t} \tag{4.38}$$

$$= \lambda \left[z' + \sum_{j+k=3} \left(-d_{jk} + c_{jk} + d_{jk} \lambda^{j-k-1} \right) (z')^j (w')^k \right] + \text{ h.o.t..} \tag{4.39}$$

Observe how in equation (4.39) the d_{21} terms always cancel. Therefore without loss of generality, we can set $d_{21} = 0$. By setting $-d_{jk} + c_{jk} + d_{jk}\lambda^{j-k-1} = 0$ for $(j,k) \in \{(3,0),(1,2),(2,1)\}$ we obtain

$$d_{30} = \frac{c_{30}}{1-\lambda^2}, \qquad d_{12} = \frac{c_{12}}{1-\overline{\lambda}^2} = -3\overline{d}_{30}, \qquad d_{03} = \frac{c_{03}}{1-\overline{\lambda}^4}, \tag{4.40}$$

so that equation (4.39) can be rewritten as $z_1' = \lambda\big(z' + c_{21}(z')^2(w')\big) + \text{h.o.t.}$ The ellipticity of $D_P\mathcal{T}$ implies $\lambda \neq \pm 1$, ensuring that the expressions for d_{30} and d_{12} do not involve division by zero. Similarly, assumption (A1) does the same for the expression of d_{03}.

Considering we have $c_{21} = i\tau$ and $\lambda = e^{i\theta}$, we can further rewrite z_1' if we restrict to the subspace where $w = \overline{z}$:

$$z_1' = e^{i\theta}\left(z' + i\tau_1|z'|^2 z'\right) + \text{h.o.t.} = e^{i\left(\theta+\tau_1|z'|^2\right)} z' + h.o.t. \tag{4.41}$$

This shows how the linear portion of billiard map $\mathcal{T} : (z', w') \mapsto (z_1', w_1')$ acts as a rotation about the elliptic fixed point. However, in the (z', w') coordinates, $\mathcal{T}$ is not necessarily symplectic.

$$\begin{tikzcd}
(s,u) \arrow[r, "B^{-1}"] \arrow[d, "\mathcal{T}"'] & (x,y) \arrow[r] \arrow[d, "\mathcal{T}"'] & (z,w) \arrow[r] \arrow[d, "\mathcal{T}"'] & (z',w') \arrow[r] \arrow[d, "\mathcal{T}"'] & \cdots \\
(s_1,u_1) \arrow[r, "B^{-1}"'] & (x_1,y_1) \arrow[r] & (z_1,w_1) \arrow[r] & (z_1',w_1') \arrow[r] & \cdots
\end{tikzcd}$$

Figure 4.5: Third coordinate transformation

4.3.5 The Fourth Transformation: A Symplectic Transformation

As already mentioned, the billiard map $\mathcal{T} : (z', w') \mapsto (z_1', w_1')$ given in the previous section is not necessarily symplectic. This is an obstacle to using Moser's twist mapping theorem to prove the nonlinear stability of periodic billiard orbits. In this section, we will use the previous transformation $(z, w) \mapsto (z', w')$ to compute a real symplectic transformation $h_N : (x, y) \mapsto (X, Y)$ such that

$$h_N^{-1} \circ T \circ h_N \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \cos\Theta(r) & -\sin\Theta(r) \\ \sin\Theta(r) & \cos\Theta(r) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \text{h.o.t.} \tag{4.42}$$

where $r^2 = x^2 + y^2$ and $\Theta(r) = \theta + \tau_1 r^2 + \tau_2 r^4 + \cdots + \tau_{N-1} r^{2(N-1)}$.

First, notice that $p_3(z,w)$ satisfies $\frac{\partial p_3}{\partial z} = -\frac{\partial \overline{p}_3}{\partial w}$. Hence there exists a polynomial $s_4(z,w)$ satisfying $\frac{\partial s_4}{\partial w}(z,w) = p_3(z,w)$ and $\frac{\partial s_4}{\partial z}(z,w) = -\overline{p}_3(w,z)$, given by

$$s_4(s,w) = \frac{1}{4}\frac{\overline{c}_{03}}{\lambda^4 - 1}z^4 - \frac{1}{4}\frac{c_{03}}{\overline{\lambda}^4 - 1}w^4 + \frac{\overline{c}_{30}}{\overline{\lambda}^2 - 1}zw^3 - \frac{c_{30}}{\lambda^2 - 1}z^3 w \tag{4.43}$$

that is purely imaginary whenever $w = \overline{z}$. Hence we can thus define a real valued function $g_4(x+iy, x-iy) = \frac{i}{2}s_4(x+iy, x-iy)$. Next, consider the generating function $G(x,Y) := xY + g_4(x+iY, x-iY)$ of the symplectic transformation $(x,y) \mapsto (X,Y)$ where $X = X(x,y)$ and $Y = Y(x,y)$. Then

$$\begin{aligned}
X := \frac{\partial G}{\partial Y} &= x + \frac{i}{2}\big[-\overline{p}_3(x-iY, x+iY)(i) + p_3(x+iY, x-iY)(-i)\big] \\
&= x + \Re\left(p_3(x+iY, x-iY)\right) \\
&= x + \Re\left(\sum_{j+k=3} d_{jk}(x+iY)^j(x-iY)^k\right) \\
&=: x + \sum_{j+k=3} p_{jk}x^jY^k \qquad (4.44) \\
y := \frac{\partial G}{\partial x} &= Y + \frac{i}{2}\Big[-\overline{p}_3(x-iY, x+iY) + p_3(x+iY, x-iY)\Big] \\
&= Y - \Im\left(p_3(x+iY, x-iY)\right) \\
&= Y - \Im\left(\sum_{j+k=3} d_{jk}(x+iY)^j(x-iY)^k\right) \\
&=: Y - \sum_{j+k=3} q_{jk}x^jY^k. \qquad (4.45)
\end{aligned}$$

It follows from equations (4.44) and (4.45), that on a small neighborhood about $(x,y) = (0,0)$, we have

$$X = x + \sum_{j+k=3} p_{jk}x^jy^k + \text{h.o.t.} \tag{4.46}$$

$$Y = y + \sum_{j+k=3} q_{jk}x^jy^k + \text{h.o.t.} \tag{4.47}$$

Letting, $Z = X + iY$ and $W = X - iY$ gives

$$\begin{aligned} Z &= x + iy + \Re\left(p_3(x+iy, x-iy)\right) + i\Im\left(p_3(x+iy, x-iy)\right) + \text{h.o.t.} \\ &= z + p_3(z, w) + \text{h.o.t.} \end{aligned} \tag{4.48}$$

Similarly,

$$W = w + \overline{p}_3(w, z) + \text{h.o.t.}, \tag{4.49}$$

so that the billiard map $\mathcal{T} : (Z, W) \mapsto (Z_1, W_1)$ is given by

$$Z_1 = e^{i(\theta + \tau_1 |Z|)} Z + \text{h.o.t.} \tag{4.50}$$

$$W_1 = e^{-i(\theta + \tau_1 |Z|)} W + \text{h.o.t.} \tag{4.51}$$

Letting $\Theta = \theta + \tau_1 |Z|^2$ and converting back to to X and Y in equations (4.50) and (4.51) yields the desired Birkhoff normal form, showing τ_1 is the first twist coefficient of $\mathcal{T}$.

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta \\ \sin\Theta & \cos\Theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \text{h.o.t.} \tag{4.52}$$

$$\begin{tikzcd} (s,u) \arrow[r, "B^{-1}"] \arrow[d, "\mathcal{T}"'] & (x,y) \arrow[r] \arrow[d, "\mathcal{T}"'] & (X,Y) \arrow[r] \arrow[d, "\mathcal{T}"'] & (Z,W) \arrow[d, "\mathcal{T}"'] \\ (s_1,u_1) \arrow[r, "B^{-1}"'] & (x_1,y_1) \arrow[r] & (X_1,Y_1) \arrow[r] & (Z_1,W_1) \end{tikzcd}$$

Figure 4.6: Fourth coordinate transformation

4.4 The Fist Twist Coefficient for Asymmetric Billiards

In this section we study the general case where the (Euclidean) radius of curvature functions $R_j(s)$, have different values at $s = 0$. That is, $R_0(0) = R_0$ and $R_1(0) = R_1$ are not equal. For the class of billiard tables $Q(L, \alpha(t), \beta(t))$, this will occur whenever the even polynomials $\alpha(t) = \sum_{n\geq 1} \alpha_{2n} t^{2n}$ and $\beta(t) = \sum_{n\geq 1} \beta_{2n} t^{2n}$ have different values for α_2 and β_2. Going forward we will assume that $\alpha_2 > \beta_2$ so that $R_0 \leq R_1$. As before, we will consider the periodic 2-orbit $\mathcal{O}_2 = \{P, T(P)\}$.

The billiard map $\mathcal{T}^2$ is given by the composition $(s, u) \xrightarrow{\mathcal{T}} (s_1, u_1) \xrightarrow{\mathcal{T}} (s_2, u_2) = (s_2(s_1, u_1), u_2(s_1, u_1))$ where as before $s_1 = s_1(s, u)$ and $u_1 = (s, u)$. The partial derivatives $\frac{\partial^{j+k} s_2}{\partial s^j \partial u^k}$ and $\frac{\partial^{j+k} u_2}{\partial s^j \partial u^k}$ are straightforward albeit tedious applications of the chain rule. To illustrate it, we give expressions for $\frac{\partial s_2}{\partial s}$ and $\frac{\partial^2 s_2}{\partial s^2}$:

$$\frac{\partial s_2}{\partial s} = \frac{\partial s_2}{\partial s_1}\frac{\partial s_1}{\partial s} + \frac{\partial s_2}{\partial u_1}\frac{\partial u_1}{\partial s}, \tag{4.53}$$

$$\frac{\partial^2 s_2}{\partial s^2} = \frac{\partial^2 s_2}{\partial s_1^2}\left(\frac{\partial s_1}{\partial s}\right)^2 + 2\frac{\partial^2 s_2}{\partial s_1 \partial u_1}\frac{\partial u_1}{\partial s}\frac{\partial s_1}{\partial s} + \frac{\partial s_2}{\partial s_1}\frac{\partial^2 s_1}{\partial s^2} + \frac{\partial^2 s_2}{\partial u_1^2}\left(\frac{\partial u_1}{\partial s}\right)^2 + \frac{\partial s_2}{\partial u_1}\frac{\partial^2 u_1}{\partial s^2}. \tag{4.54}$$

Denote $\tilde{a}_{jk} = \frac{1}{j!k!}\frac{\partial^{j+k} s_2}{\partial s^j u^k}(0,0)$ and $\tilde{b}_{jk} = \frac{1}{j!k!}\frac{\partial^{j+k} u_2}{\partial s^j u^k}(0,0)$. Calculating the remaining first order partial derivatives and evaluating at $\mathcal{O}_2$ shows that $D_P\mathcal{T}^2$ is given by

$$D_P\mathcal{T}^2 = \begin{bmatrix} \tilde{a}_{10} & \tilde{a}_{01} \\ \tilde{b}_{10} & \tilde{b}_{01} \end{bmatrix} = \begin{bmatrix} \frac{L}{aR_1} - 1 & \frac{L}{a} \\ \frac{L - a(R_0+R_1)}{aR_0R_1} & \frac{L}{aR_0} - 1 \end{bmatrix} \begin{bmatrix} \frac{L}{aR_0} - 1 & \frac{L}{a} \\ \frac{L - a(R_0+R_1)}{aR_0R_1} & \frac{L}{aR_1} - 1 \end{bmatrix}, \tag{4.55}$$

where

$$\tilde{a}_{10} = \tilde{b}_{01} = \frac{aR_0\left(aR_1 - 2L\right) + 2L\left(L - aR_1\right)}{a^2R_0R_1}, \tag{4.56}$$

$$\tilde{a}_{01} = \frac{2L}{a}\left(\frac{L}{aR_1} - 1\right), \tag{4.57}$$

$$\tilde{b}_{10} = \frac{2\left(L - aR_0\right)\left(L - a\left(R_0 + R_1\right)\right)}{a^2 R_0^2 R_1}. \tag{4.58}$$

It follows that the orbit $\mathcal{O}_2$ is parabolic for $L \in \{aR_0, aR_1, a(R_0 + R_1)\}$, hyperbolic for $L \in (aR_0, aR_1) \cup (a(R_0 + R_1), \infty)$, and elliptic for $L \in (0, aR_0) \cup (aR_1, a(R_0 + R_1))$. Additionally, we assume the following non-resonance condition:

(A2) $\lambda^4 \neq 1$ or equivalently, $L \neq \frac{a}{2}\left[R_0 + R_1 \pm \sqrt{R_0^2 + R_1^2}\right]$.

Theorem 4.2. *Let $F(v) = a(v_1^2+v_2^2)^{1/2}+b(v_1^4+b_2^4)^{1/4}$ with $a+b = 1$, and $Q(\alpha(t), \beta(t), L)$ be given. Additionally, assume that the resonance condition (A2) is satisfied. Then the first twist coefficient of the two-step billiard map $\mathcal{T}^2$ is given by*

$$\tau_1 = \frac{\Delta}{8aR_0R_1(L - aR_0)(L - aR_1)\left(L - a(R_0 + R_1)\right)}, \tag{4.59}$$

where

$$\begin{aligned}
\Delta = {} & a^3(4 - 3a)LR_0^3 + a^3(2a - 9)R_0R_1^3 + a^3(2a - 9)R_0^3R_1 + 2a^2(3a - 4)L^2R_0^2 \\
& - 4a^2(2a - 9)LR_0R_1^2 + 6a^2R_0R_1^3 + 6a^2R_0^3R_1 + a(4 - 3a)L^3R_0 \\
& + 4(a(2a - 9) + 6)L^2R_0R_1 - 24aLR_0R_1^2 - a(3a - 4)LR_1\left(L - aR_1\right)^2 \\
& + 2a(a(2a - 9) + 6)R_0^2R_1\left(aR_1 - 2L\right) \\
& + R_0''\left(a^4LR_0R_1^3 - 2a^3L^2R_0R_1^2 + a^2L^3R_0R_1\right) \\
& + R_1''\left(a^4LR_0^3R_1 - 2a^3L^2R_0^2R_1 + a^2L^3R_0R_1\right).
\end{aligned} \tag{4.60}$$

Proof. After computing the $\tilde{a}_{jk}$'s and $\tilde{b}_{jk}$'s (which we list in the appendix), we find that the $\tilde{a}_{jk}$'s and $\tilde{b}_{jk}$'s for $j + k = 2$ are identically zero. Hence the calculation for τ_1 is exactly the same as in the case where $R_0 = R_1$. $\square$

As in the symmetric case, the first twist coefficient in the Euclidean case is recovered by setting $a = 1$:

$$\tau_1 = \frac{1}{8}\left(\frac{R_0+R_1}{R_0R_1} - \frac{L}{R_0+R_1-L}\left(\frac{R_1-L}{R_0-L}R_0'' + \frac{R_0-L}{R_1-L}R_1''\right)\right). \tag{4.61}$$

4.5 Applications of the First Twist Coefficient

In this section we use the first twist coefficient given in Theorem 4.8 to prove the nonlinear stability of the periodic orbit $\mathcal{O}_2 = \{P, \mathcal{T}(P)\}$ on various symmetric billiard tables. Recall that the orbit $\mathcal{O}_2$ is

(1) parabolic if $L = 2aR$,

(2) hyperbolic if $L > 2aR$, or

(3) elliptic if $0 < L < 2aR$.

4.5.1 Symmetric Lemon Billiards

Consider the symmetric lemon billiard table $Q(L)$ discussed in chapter 2, where γ_0 and γ_1 are circular arcs. In this case, $R = 1, R'' = 0$, and $L \in (0, 2)$. From the stability conditions, the orbit $\mathcal{O}_2(a, L) = \{P, \mathcal{T}(P)\}$ is elliptic whenever $L < 2a$ with nonresonance condition $L \neq a$. The first twist coefficient is given by

$$\tau_1(\mathcal{T}, P) = \frac{a^2(3L-4) + a(18-4L) - 12}{8a^2(2a-L)}, \tag{4.62}$$

and $\tau_1(\mathcal{T}, P) = 0$ whenever $L = \frac{2\left(2a^2-9a+6\right)}{a(3a-4)}$.

Corollary 4.3. *Let $0 < a < 1$ and $\mathcal{O}_2(a, L)$ be the periodic orbit on the symmetric billiard table $Q(L)$ running along the x-axis. The orbit $\mathcal{O}_2(a, L)$ is nonlinearly stable when $L \in (0, 2a) \backslash \left\{a, \frac{4a^2-18a+12}{a(3a-4)}\right\}$.*

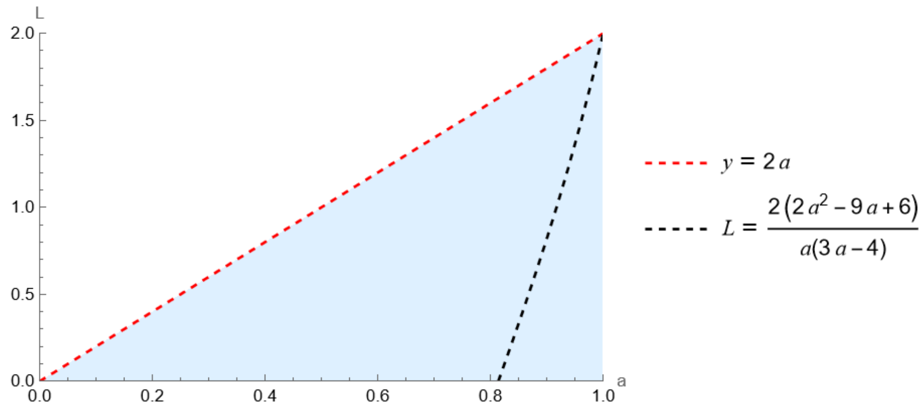
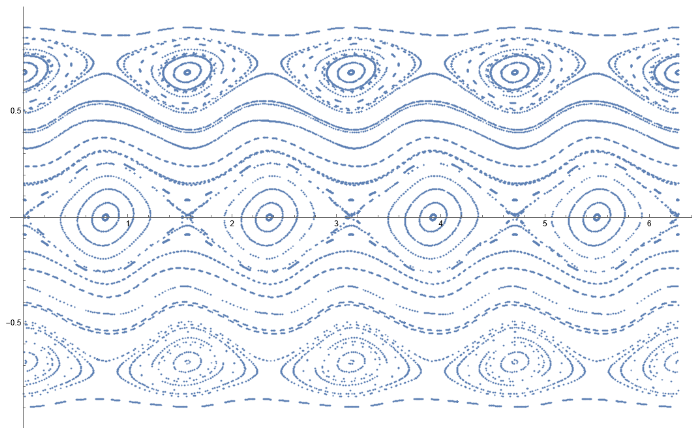


Figure 4.7: Stability graph for $\mathcal{O}_2(a, L)$ on a symmetric billiard table.

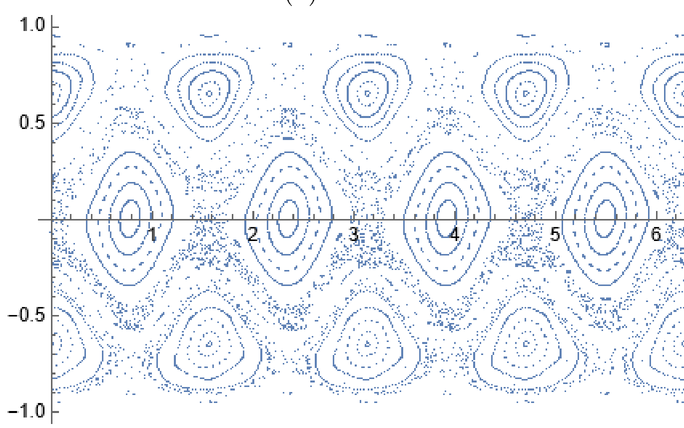
4.5.2 Circular Billiards

Now consider the Euclidean circular billiard table Q_r bounded by the circle $x^2 + y^2 = r^2$. Then the orbit $\mathcal{O}_2(a, 1) = \{P, \mathcal{T}(P)\}$ bouncing back and forth along the $y - axis$ is a 2-periodic orbit. In this case the $R = r$ and $L = 2r$. From the stability conditions above, it follows that $\mathcal{O}_2(a, 1)$ is hyperbolic for $0 < a < 1$ and parabolic only when $a = 1$, corresponding to the Euclidean norm $F(v) = \|v\|_2$. In particular such an orbit cannot be elliptic. Moreover, because of the symmetry of the table and indicatrix, the orbit traveling along the x-axis is also hyperbolic

Below we present some numerical results for $(a, b) \in \{(0.8, 0.2), (0.5, 0.5)\}$. For the case $a = 0.8$ and $b = 0.2$, a small perturbation of the hyperbolic orbit $\mathcal{O}_2(0.8, 1)$ discussed above can be seen in Figure 4.9(a) and is represented in the phase space (see Figure 4.8(a)) by the central ∞-curve. This curve surrounds the elliptic islands of the periodic 2-orbits $\left\{(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\right\}$ and $\left\{(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\right\}$. A small perturbation of the first is shown in Figure 4.9(b). Although the twist coefficient given in equation (4.8) does not determine the nonlinear stability of these 2-orbits, we can obtain the appropriate first twist coefficient in a similar way. However, equations (3.19)-(3.22) would not simplify as nicely as they do in equations (4.3)-(4.6). Near the top of the phase space, we encounter the elliptic islands for the periodic 4-orbit $\{(1,0),(0,1),(-1,0),(0,-1)\}$. A small perturbation of this orbit is given in Figure 4.9(d). Note how the invariant curves between the elliptical islands disappear as a decreases.



(a) $a = 0.8$

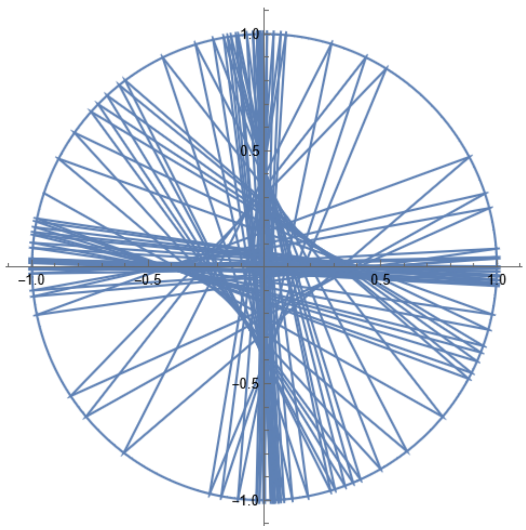


(b) $a = 0.5$

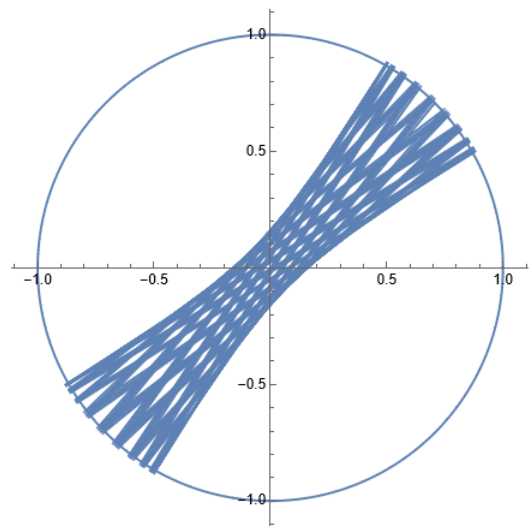
Figure 4.8: Phase spaces on Circular billiards

4.5.3 Elliptical Billiards

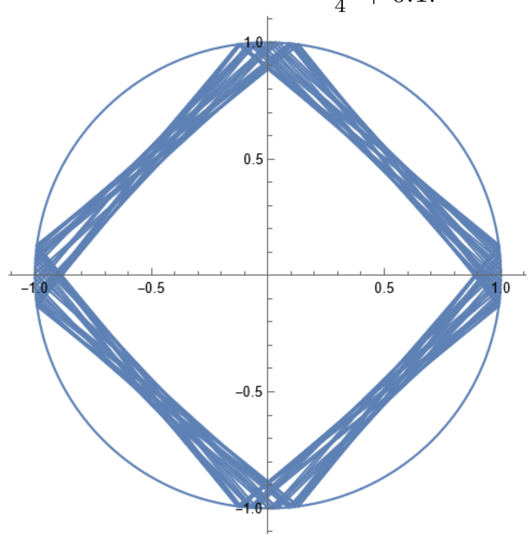
Let $\delta \in (0,1)$, and $E(\delta)$ be the billiard table bounded by the Euclidean ellipse $x^2 + \frac{y^2}{\delta} = 1$. It is easy to see that there is a periodic orbit $\mathcal{O}_2(a,\delta) = \{P, \mathcal{T}(P)\}$ of period 2 running along the minor axis of the ellipse. Then $L = 2\delta$ and $R = \frac{1}{\delta}$, so that $\frac{L}{R} = 2\delta^2$.



(a) 100 reflections of the billiard orbit with initial point $(0, -1)$ and direction $\frac{\pi}{2} + 0.01$.



(b) 100 reflections of the billiard orbit with initial point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and direction $\frac{5\pi}{4} + 0.1$.



(c) 100 reflections of the billiard orbit with initial point $(0, -1)$ and direction $\frac{3\pi}{4} - 0.05$.

Figure 4.9: Dynamics on the circular billiard table with $a = 0.8$.

It follows that $\mathcal{O}_2(a,\delta)$ is elliptic for $\delta < \sqrt{a}$, parabolic for $\delta = \sqrt{a}$, and hyperbolic for $\delta > \sqrt{a}$. Additionally, the non-resonance condition is satisfied for $\delta \neq \frac{\sqrt{a}}{2}$. A short calculation shows $R'' = \frac{3(\delta^2-1)}{\delta}$ so that the first twist coefficient is given by

$$\tau_1(P,\mathcal{T}) = \frac{\delta\left(-6 + a(9 + a - 4\delta^2)\right)}{8a^2(a-\delta^2)}. \tag{4.63}$$

Note that $\tau_1(P,\mathcal{T})$ is equal to zero whenever $\delta = \frac{\sqrt{a^2+9a-6}}{2\sqrt{a}}$. This value of δ only makes sense when it is real and less than $\sqrt{a}$ which occurs whenever $a \in \left(\frac{\sqrt{105}-9}{2}, 1\right)$.

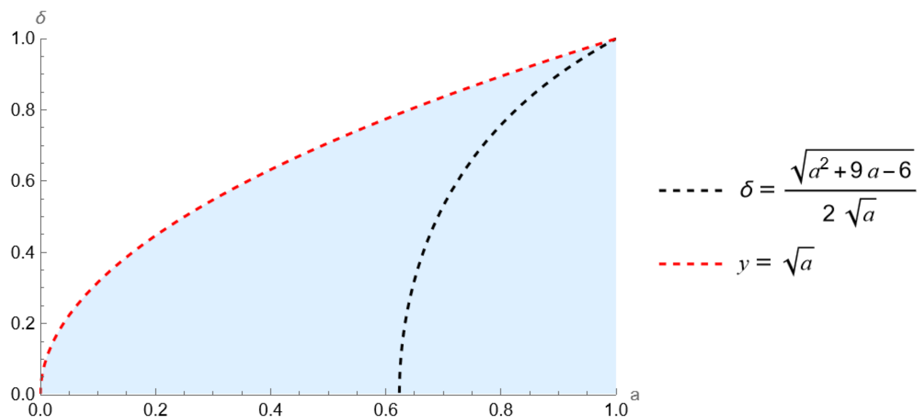
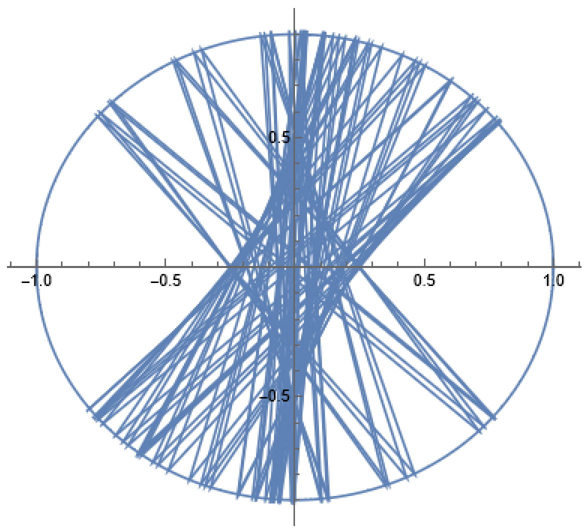


Figure 4.10: Stability graph for $\mathcal{O}_2(a,\delta)$ on an elliptic billiard table

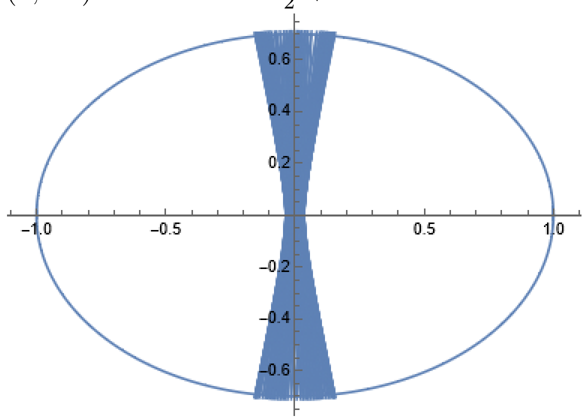
Corollary 4.4. *On the elliptical billiard table $E(\delta)$, the periodic orbit $\mathcal{O}_2(a,\delta)$ running along the minor axis is nonlinearly stable whenever $\delta < \sqrt{a}$ and $\delta \neq \frac{\sqrt{a^2+9a-6}}{2\sqrt{a}}$ and $\delta \neq \frac{\sqrt{a}}{2}$.*

Some numerical results for $(a,b) \in \{(0.8, 0.2), (0.5, 0.5)\}$ on elliptical billiards are given below. Note that $\sqrt{0.8} \approx 0.894$ and $\sqrt{0.5} \approx 0.707$ so that the orbit $\mathcal{O}_2(a, 0.9)$ is hyperbolic for both $a = 0.8$ and $a = 0.5$, see figure 4.11(a) for the latter case. We can see this reflected in their respective phase space diagrams shown in Figures 4.12(a) and 4.12(b), as we can see the characteristic ∞-shaped curves - albeit very small in Figure 4.12(a). The remaining phase space diagrams correspond to $\delta = 0.7$

and $\delta = 0.5$ which are both less than $\sqrt{0.8}$ and $\sqrt{0.5}$. As stated in Corollary 4.4, the orbits $\mathcal{O}_2(a, \delta)$ are nonlinearly stable in these cases resulting in the presence of elliptic islands on the center of the diagrams.

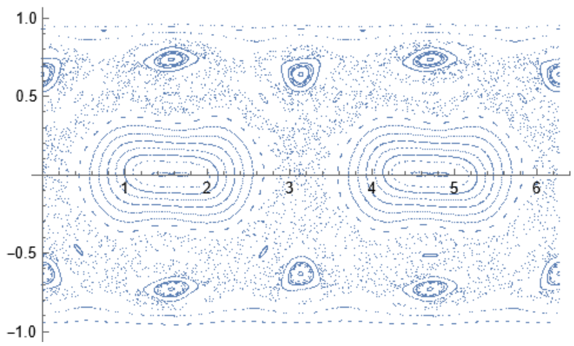


(a) 100 reflections on $E(0.9)$ with initial point $(0, -1)$ and direction $\frac{\pi}{2} + 0.05$.

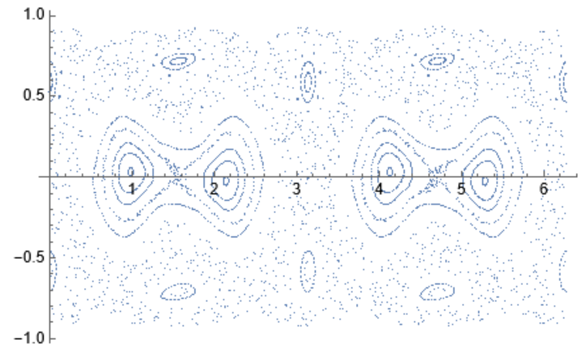


(b) 100 reflections on $E(0.7)$ with initial point $(0, -1)$ and direction $\frac{\pi}{2} + 0.01$.

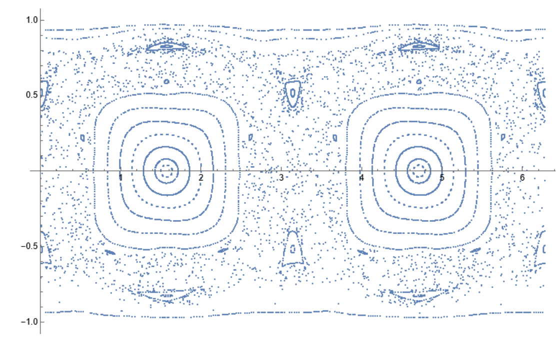
Figure 4.11: Dynamics on elliptical billiard tables with $a = 0.5$



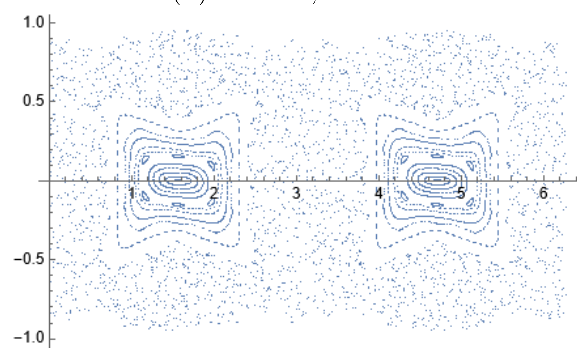
(a) $a = 0.8$, $\delta = 0.9$



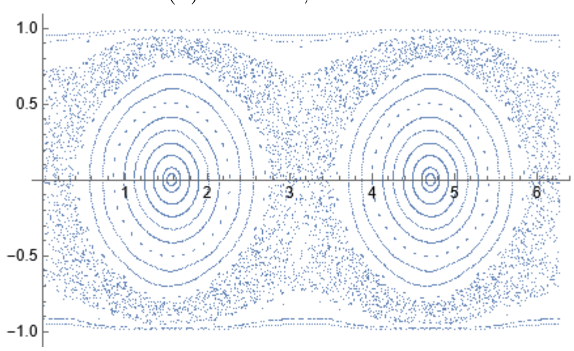
(b) $a = 0.5$, $\delta = 0.9$



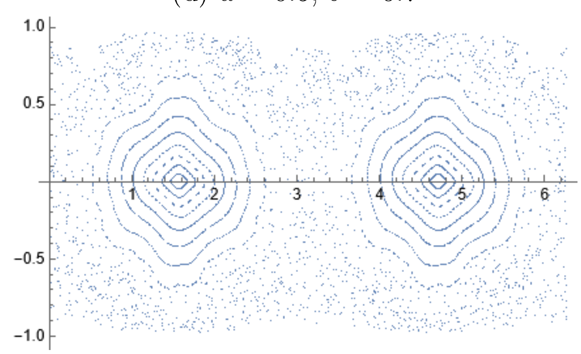
(c) $a = 0.8$, $\delta = 0.7$



(d) $a = 0.5$, $\delta = 0.7$



(e) $a = 0.8$, $\delta = 0.5$



(f) $a = 0.5$, $\delta = 0.5$

Figure 4.12: Phase spaces for elliptical billiards

4.5.4 General Symmetric Billiards

For the class of billiard tables $Q(\alpha, \beta, L)$ where $\alpha(t) = \beta(t)$, $R(s)$ satisfies $R = \frac{1}{2\alpha_2}$ and $R'' = 6\alpha_2 - \frac{6\alpha_4}{\alpha_2^2}$ at both $\gamma_0(0)$ and $\gamma_1(0)$. Hence the orbit $\mathcal{O}_2(a, \alpha) = \{P, \mathcal{T}(P)\}$ is elliptic whenever $L \in \left(0, \frac{a}{\alpha_2}\right)$. Further, the non-resonance condition is satisfied whenever $L \neq \frac{a}{2\alpha_2}$. From equation (4.8), we have that $\tau_1(\mathcal{T}, P) = 0$ whenever

$$R'' = -\frac{-3a^2L + 4a^2R + 4aL - 18aR + 12R}{a^2LR}, \tag{4.64}$$

or equivalently, when

$$\alpha_4 = -\frac{-2a^2\alpha_2^2 + 9a\alpha_2^2 - 4a\alpha_2^3L - 6\alpha_2^2}{3a^2L}. \tag{4.65}$$

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Appendix A

Mathematica Code Symmetric Twist Coefficient

Part 1: Symmetric Twist Coefficient

1. One-step DpT

In[31]:=
```
(* Define p - Has to be an even number *)
w = 2;
p = 2 w;
$Assumptions = L > 0;
```

1.1 Derivatives of γ

1.1.1 Minkowski curvature

In[34]:=
```
(* s1 = C/2  half the circumference of billiard table *)
s1[0, u_, a_, b_] = s₁;

(* Euclidean curvature *)
ke[s_] := 1 / R[s];

(* Minkowski curvature *)
ρ[θ_, a_, b_] := 1 / (a + b (Cos[θ] ^p + Sin[θ] ^p) ^ (1 / p));
(* Euclidean radius of Minkowski circle *)
Derivative[1][θ][s_] := ke[s] ρ[θ[s], a, b];
km[s_, a_, b_] := ke[s] ρ[θ[s], a, b] ^3
```

1.1.2 Normal field $n_\gamma(s)$

In[39]:=
```
dir[θ_] := {Cos[θ], Sin[θ]}
n[s_] := 1 / (ρ[θ[s], a, b] ^2) * D[ρ[θ[s], a, b], {θ[s], 1}] × dir[θ[s]] +
  1 / ρ[θ[s], a, b] * D[dir[θ[s]], {θ[s], 1}]
```

1.1.3 Derivatives of γ

```
In[41]:= Derivative[1, 0, 0][γ][s_, a_, b_] := ρ[θ[s], a, b] dir[θ[s]]
Derivative[2, 0, 0][γ][s_, a_, b_] := km[s, a, b] × n[s]
(* First order derivatives *)
dg1[s_, a_, b_] := Derivative[1, 0, 0][γ][s, a, b][[1]]
dg2[s_, a_, b_] := Derivative[1, 0, 0][γ][s, a, b][[2]]

(* Second order derivatives *)
ddg1[s_, a_, b_] := Derivative[2, 0, 0][γ][s, a, b][[1]]
ddg2[s_, a_, b_] := Derivative[2, 0, 0][γ][s, a, b][[2]]

dg1[s1[s, u, a, b], 1, 0];
dg2[s1[s, u, a, b], 1, 0];
```

1.2 Derivatives of v_1 and v_2

```
In[49]:= (* Define partial derivatives of v1 =
 gamma1[s1]- gamma1[s] and v2 = gamma2[s1]- gamma2[s] *)
Derivative[m_, n_][v1][s_, u_] := D[γ[s1[s, u, a, b], a, b] - γ[s, a, b], {s, m}, {u, n}][[1]]
Derivative[m_, n_][v2][s_, u_] := D[γ[s1[s, u, a, b], a, b] - γ[s, a, b], {s, m}, {u, n}][[2]]
```

1.3 Metric F and it's partial derivatives

```
In[51]:= (* Define Finsler metric F *)
F[a_, b_, v1_, v2_] := a * Sqrt[v1^2 + v2^2] + b (v1^(p) + v2^(p))^(1 / (p));

(* Define Partial Derivatives of F wrt v1 and v2 *)
Fderi[a_, b_, v1_, v2_, j_, k_] :=
  Fderi[a, b, v1, v2, j, k] = D[F[a, b, v1, v2], {v1, j}, {v2, k}];
```

1.4 Define 1st partial derivatives of s_1 and u_1

```
In[53]:= (* ds1du *)
ds1ds[s_, u_, a_, b_, v1_, v2_] :=
 (Fderi[a, b, v1, v2, 2, 0] × dg1[s, a, b]^2 + 2 * Fderi[a, b, v1, v2, 1, 1] ×
     dg1[s, a, b] × dg2[s, a, b] + Fderi[a, b, v1, v2, 0, 2] × dg2[s, a, b]^2 -
    Fderi[a, b, v1, v2, 1, 0] × ddg1[s, a, b] - Fderi[a, b, v1, v2, 0, 1] × ddg2[s, a, b]) /
  (Fderi[a, b, v1, v2, 2, 0] × dg1[s, a, b] × dg1[s1[s, u, a, b], a, b] +
    Fderi[a, b, v1, v2, 1, 1] × dg1[s, a, b] × dg2[s1[s, u, a, b], a, b] +
    Fderi[a, b, v1, v2, 1, 1] × dg1[s1[s, u, a, b], a, b] × dg2[s, a, b] +
    Fderi[a, b, v1, v2, 0, 2] × dg2[s, a, b] × dg2[s1[s, u, a, b], a, b])

Derivative[1, 0, 0, 0][s1][s_, u_, a_, b_] := ds1ds[s, u, a, b, v1[s, u], v2[s, u]]
```

In[55]:=
```
(* ds1du *)
ds1du[s_, u_, a_, b_, v1_, v2_] :=
 -1 / (Fderi[a, b, v1, v2, 2, 0] × dg1[s, a, b] × dg1[s1[s, u, a, b], a, b] +
    Fderi[a, b, v1, v2, 1, 1] × dg1[s, a, b] × dg2[s1[s, u, a, b], a, b] +
    Fderi[a, b, v1, v2, 1, 1] × dg1[s1[s, u, a, b], a, b] × dg2[s, a, b] +
    Fderi[a, b, v1, v2, 0, 2] × dg2[s, a, b] × dg2[s1[s, u, a, b], a, b])

Derivative[0, 1, 0, 0][s1][s_, u_, a_, b_] := ds1du[s, u, a, b, v1[s, u], v2[s, u]]
```

In[57]:=
```
(* duds *)
duds[s_, u_, a_, b_, v1_, v2_] :=
 Fderi[a, b, v1, v2, 2, 0] × dg1[s, a, b] × dg1[s1[s, u, a, b], a, b] +
  Fderi[a, b, v1, v2, 1, 1] × dg1[s, a, b] × dg2[s1[s, u, a, b], a, b] +
  Fderi[a, b, v1, v2, 1, 1] * dg1[s1[s, u, a, b], a, b] × dg2[s, a, b] +
  Fderi[a, b, v1, v2, 0, 2] × dg2[s, a, b] × dg2[s1[s, u, a, b], a, b] -
  ds1ds[s, u, a, b, v1, v2] * (Fderi[a, b, v1, v2, 2, 0] × dg1[s1[s, u, a, b], a, b]^2 +
     2 * Fderi[a, b, v1, v2, 1, 1] × dg1[s1[s, u, a, b], a, b] × dg2[s1[s, u, a, b], a, b] +
     Fderi[a, b, v1, v2, 0, 2] * dg2[s1[s, u, a, b], a, b]^2 + Fderi[a, b, v1, v2, 1, 0] ×
      ddg1[s1[s, u, a, b], a, b] + Fderi[a, b, v1, v2, 0, 1] × ddg2[s1[s, u, a, b], a, b])

Derivative[1, 0, 0, 0][u1][s_, u_, a_, b_] := duds[s, u, a, b, v1[s, u], v2[s, u]]
```

In[59]:=
```
(* dudu *)
dudu[s_, u_, a_, b_, v1_, v2_] :=
 (Fderi[a, b, v1, v2, 2, 0] × dg1[s1[s, u, a, b], a, b]^2 + 2 * Fderi[a, b, v1, v2, 1, 1] ×
     dg1[s1[s, u, a, b], a, b] × dg2[s1[s, u, a, b], a, b] + Fderi[a, b, v1, v2, 0, 2] ×
     dg2[s1[s, u, a, b], a, b]^2 + Fderi[a, b, v1, v2, 1, 0] * ddg1[s1[s, u, a, b], a, b] +
    Fderi[a, b, v1, v2, 0, 1] * ddg2[s1[s, u, a, b], a, b]) /
  (Fderi[a, b, v1, v2, 2, 0] × dg1[s, a, b] × dg1[s1[s, u, a, b], a, b] +
    Fderi[a, b, v1, v2, 1, 1] × dg1[s, a, b] × dg2[s1[s, u, a, b], a, b] +
    Fderi[a, b, v1, v2, 1, 1] × dg1[s1[s, u, a, b], a, b] × dg2[s, a, b] +
    Fderi[a, b, v1, v2, 0, 2] × dg2[s, a, b] × dg2[s1[s, u, a, b], a, b])

Derivative[0, 1, 0, 0][u1][s_, u_, a_, b_] := dudu[s, u, a, b, v1[s, u], v2[s, u]]
```

2. Higher Order Derivatives of s_1 and u_1

2.1 Billiard table conditions

2.1.1 Symmetric Billiard Table

In[61]:=
```
(* Fordward map (s₀,u₀)→ (s₁, u₁) *)
Symm = {
   s → 0,
   v1[s, u] → L, v2[s, u] → 0,
   θ[s] → -Pi / 2, θ[s1[s, u, a, b]] → Pi / 2,

   R[0] :→ R,
   Derivative[1][R][s] → 0,
   Derivative[2][R][0] → R'',

   R[s₁] :→ R,
   Derivative[1][R][s₁] → 0,
   Derivative[2][R][s₁] → R''
  };
```

2.2 Higher order derivatives

In[62]:=
```
Clear[sderi, uderi, srderi, urderi]
sderi[j_, k_, A_, B_, l_] := sderi[j, k, A, B, l] =
  Simplify[D[s1[s, u, a, b], {s, j}, {u, k}] //. Symm /. {a → A, b → B, L → l}]
uderi[j_, k_, A_, B_, l_] := uderi[j, k, A, B, l] =
  Simplify[D[u1[s, u, a, b], {s, j}, {u, k}] //. Symm /. {a → A, b → B, L → l}]
```

2.3 Some partial derivatives

In[65]:=
```
Simplify[sderi[1, 0, a, 1 - a, L]]
```

Out[65]= $-1 + \frac{L}{a R}$

In[66]:=
```
Simplify[sderi[0, 1, a, 1 - a, L]]
```

Out[66]= $\frac{L}{a}$

In[67]:=
```
Simplify[uderi[1, 0, a, 1 - a, L]]
```

Out[67]= $\frac{L - 2 a R}{a R^2}$

In[68]:= `Simplify[uderi[0, 1, a, 1 - a, L]]`

Out[68]= $-1+\frac{L}{a\,R}$

In[69]:= `Simplify[sderi[2, 0, a, 1 - a, L]]`

Out[69]= 0

In[70]:= `Simplify[sderi[1, 1, a, 1 - a, L]]`

Out[70]= 0

In[71]:= `Simplify[sderi[0, 2, a, 1 - a, L]]`

Out[71]= 0

In[72]:= `Simplify[uderi[2, 0, a, 1 - a, L]]`

Out[72]= 0

In[73]:= `Simplify[uderi[1, 1, a, 1 - a, L]]`

Out[73]= 0

In[74]:= `Simplify[uderi[0, 2, a, 1 - a, L]]`

Out[74]= 0

In[75]:= `Simplify[sderi[3, 0, a, 1 - a, L]]`

Out[75]= $$\frac{-3\,a\,L\,(L-3\,R)\,R-6\,L\,R^2+2\,a^4\,L\,R^2-3\,a^3\,R\,\left(L^2-L\,R+2\,R^2\right)+a^2\,\left(L^3+3\,L\,R^2\right)-a^3\,L\,R^3\,R''}{a^4\,R^5}$$

In[76]:= `Simplify[sderi[2, 1, a, 1 - a, L]]`

Out[76]= $$\frac{-3\,a\,L\,(L-3\,R)\,R-6\,L\,R^2+2\,a^4\,L\,R^2-2\,a^3\,R\,\left(L^2+R^2\right)+a^2\,\left(L^3+2\,L\,R^2\right)}{a^4\,R^4}$$

In[77]:= `Simplify[sderi[1, 2, a, 1 - a, L]]`

Out[77]= $$\frac{L\,\left(-a^3\,L\,R-3\,a\,(L-3\,R)\,R-6\,R^2+a^2\,\left(L^2+R^2\right)\right)}{a^4\,R^3}$$

In[78]:= `Simplify[sderi[0, 3, a, 1 - a, L]]`

Out[78]= $$\frac{L\,\left(a^2\,L^2-3\,a\,(L-3\,R)\,R-6\,R^2\right)}{a^4\,R^2}$$

In[79]:= `Simplify[uderi[3, 0, a, 1 - a, L]]`

Out[79]= $$-\frac{1}{a^4\,R^6}\Big(6\,L\,R^2-8\,a^5\,R^3-12\,a^3\,L\,R\,(L+R)+3\,a^2\,L^2\,(L+3\,R)+ 2\,a^4\,R^2\,(8\,L+3\,R)-3\,a\,L\,\left(L^2-L\,R+3\,R^2\right)+a\,R\,\left(L^3-3\,a\,L^2\,R+4\,a^2\,L\,R^2-2\,a^3\,R^3\right)\,R''\Big)$$

In[80]:=
```
Simplify[uderi[2, 1, a, 1 - a, L]]
```

Out[80]= $$-\frac{1}{a^4 R^5}\left(6 L R^2+6 a^4 L R^2+a^2 L\left(3 L^2+6 L R+R^2\right)-a^3 R\left(9 L^2+3 L R+2 R^2\right)-3 a L\left(L^2-L R+3 R^2\right)+a L R(L-a R)^2 R''\right)$$

In[81]:=
```
Simplify[uderi[1, 2, a, 1 - a, L]]
```

Out[81]= $$-\frac{6 L R^2+2 a^4 L R^2+a^2 L\left(3 L^2+3 L R+2 R^2\right)-3 a L\left(L^2-L R+3 R^2\right)-2 a^3\left(3 L^2 R+R^3\right)+a L^2 R(L-a R) R''}{a^4 R^4}$$

In[82]:=
```
Simplify[uderi[0, 3, a, 1 - a, L]]
```

Out[82]= $$\frac{3(-1+a) L\left(a^2 L R+2 R^2-a\left(L^2-L R+R^2\right)\right)-a L^3 R R''}{a^4 R^3}$$

3. Taylor expansion of the billiard map

3.1 Taylor Expansion of the one-step billiard map

In[83]:=
```
ta10[a_, e_, L_] := sderi[1, 0, a, e, L]
ta01[a_, e_, L_] := sderi[0, 1, a, e, L]
ta30[a_, e_, L_] := sderi[3, 0, a, e, L] / 6
ta21[a_, e_, L_] := sderi[2, 1, a, e, L] / 2
ta12[a_, e_, L_] := sderi[1, 2, a, e, L] / 2
ta03[a_, e_, L_] := sderi[0, 3, a, e, L] / 6

tb10[a_, e_, L_] := uderi[1, 0, a, e, L]
tb01[a_, e_, L_] := uderi[0, 1, a, e, L]
tb30[a_, e_, L_] := uderi[3, 0, a, e, L] / 6
tb21[a_, e_, L_] := uderi[2, 1, a, e, L] / 2
tb12[a_, e_, L_] := uderi[1, 2, a, e, L] / 2
tb03[a_, e_, L_] := uderi[0, 3, a, e, L] / 6
```

3.1 Check that $D_P\, T$ is symplectic :

In[95]:=
```
Simplify[ta10[a, b, L] * tb01[a, b, L] - ta01[a, b, L] * tb10[a, b, L]]
```

Out[95]= 1

4. Coordinate transformation $B^{-1}:(s, u) \mapsto (x, y)$

4.1 Calculate the coefficients of the billiard map $T:(x, y) \mapsto (x_1, y_1)$ where

$x_1 = \sum a_{j\,k}\, x^j\, y^k$ and $y_1 = \sum b_{j\,k}\, x^j\, y^k$.

In[

```
(* Define coordinate transformation *)
Binv = {{1 / η, 0}, {0, η}};
B = {{η, 0}, {0, 1 / η}} ;

(* Compute s and u in terms of x and y *)
s = (η * x);
u = (y / η);
vect1 = Expand[ta10 s + ta0.1 u + ta30 s^3 + ta21 s^2 * u + ta12 s * u^2 + ta0.3 u^3];
vect2 = Expand[tb10 s + tb0.1 u + tb30 s^3 + tb21 s^2 * u + tb12 s * u^2 + tb0.3 u^3];

BV = Binv.{{vect1}, {vect2}};
x1 = Expand[BV[[1]][[1]]]
y1 = Expand[BV[[2]][[1]]]
```

Out[

$$\frac{y\,ta_{0.1}}{\eta^2} + \frac{y^3\,ta_{0.3}}{\eta^4} + x\,ta_{10} + \frac{x\,y^2\,ta_{12}}{\eta^2} + x^2\,y\,ta_{21} + x^3\,\eta^2\,ta_{30}$$

Out[

$$y\,tb_{0.1} + \frac{y^3\,tb_{0.3}}{\eta^2} + x\,\eta^2\,tb_{10} + x\,y^2\,tb_{12} + x^2\,y\,\eta^2\,tb_{21} + x^3\,\eta^4\,tb_{30}$$

4.2 Define the a_{jk} and b_{jk}

In[96]:=

```
μ[a_, b_, L_] := Sqrt[-ta01[a, b, L] × tb10[a, b, L]]
λ[a_, b_, L_] := ta10[a, b, L] - ⅈ * μ[a, b, L]
η[a_, b_, L_] := (-ta01[a, b, L] / tb10[a, b, L]) ^ (1 / 4)

a10[a_, b_, L_] := ta10[a, b, L]
a01[a_, b_, L_] := ta01[a, b, L] * η[a, b, L] ^ (-2)
a30[a_, b_, L_] := ta30[a, b, L] * η[a, b, L] ^2
a21[a_, b_, L_] := ta21[a, b, L]
a12[a_, b_, L_] := ta12[a, b, L] * η[a, b, L] ^ (-2)
a03[a_, b_, L_] := ta03[a, b, L] * η[a, b, L] ^ (-4)

b10[a_, b_, L_] := tb10[a, b, L] * η[a, b, L] ^ (2)
b01[a_, b_, L_] := tb01[a, b, L]
b30[a_, b_, L_] := tb30[a, b, L] * η[a, b, L] ^4
b21[a_, b_, L_] := tb21[a, b, L] * η[a, b, L] ^2
b12[a_, b_, L_] := tb12[a, b, L]
b03[a_, b_, L_] := tb03[a, b, L] * η[a, b, L] ^ (-2)
```

5. Coordinate transformation $(x, y) \mapsto (z, w)$

5.1 Calculate the coefficients of $z_1 = \lambda\left(z + \sum c_{j\,k}\, z^j\, w^k\right)$ and $w_1 = \overline{\lambda}\left(w + \sum \overline{c}_{j\,k}\, w^j\, z^k\right)$.

In[

```
(* Define z & w in terms of x and y *)
z = x + i * y;
w = x - i * y;

(* Define z1 in terms of x1 and y1 and then collect terms *)
z1 = Expand[λ (z + c30 * z^3 + c21 * z^2 * w + c12 * z * w^2 + c0.3 * w^3)];
Collect[z1, {x^3, x^2 * y, x * y^2, y^3, x, y}]

(* Solve for c30, c21, c12, and c03 *)
Solve[{λ (c30 + c21 + c12 + c0.3) == a30 + i * b30,
   λ (3 i * c30 + i * c21 - i * c12 - 3 i * c0.3) == a21 + i * b21,
   λ (-3 * c30 + c21 + c12 - 3 c0.3) == a12 + i * b12,
   λ (-i * c30 + i * c21 - i * c12 + i * c0.3) == a0.3 + i * b0.3},
 {c30, c21, c12, c0.3}]
```

Out[

$$x\,\lambda + i\,y\,\lambda + x\,y^2\,(-3\,\lambda\,c_{0.3} + \lambda\,c_{12} + \lambda\,c_{21} - 3\,\lambda\,c_{30}) + y^3\,(i\,\lambda\,c_{0.3} - i\,\lambda\,c_{12} + i\,\lambda\,c_{21} - i\,\lambda\,c_{30}) + x^2\,y\,(-3\,i\,\lambda\,c_{0.3} - i\,\lambda\,c_{12} + i\,\lambda\,c_{21} + 3\,i\,\lambda\,c_{30}) + x^3\,(\lambda\,c_{0.3} + \lambda\,c_{12} + \lambda\,c_{21} + \lambda\,c_{30})$$

Out[

$$\Big\{\Big\{c_{30} \to \frac{i\,(a_{0.3} + i\,a_{12} - a_{21} - i\,a_{30} + i\,b_{0.3} - b_{12} - i\,b_{21} + b_{30})}{8\,\lambda},$$
$$c_{21} \to -\frac{i\,(3\,a_{0.3} + i\,a_{12} + a_{21} + 3\,i\,a_{30} + 3\,i\,b_{0.3} - b_{12} + i\,b_{21} - 3\,b_{30})}{8\,\lambda},$$
$$c_{12} \to -\frac{-3\,i\,a_{0.3} - a_{12} - i\,a_{21} - 3\,a_{30} + 3\,b_{0.3} - i\,b_{12} + b_{21} - 3\,i\,b_{30}}{8\,\lambda},$$
$$c_{0.3} \to -\frac{i\,a_{0.3} + a_{12} - i\,a_{21} - a_{30} - b_{0.3} + i\,b_{12} + b_{21} - i\,b_{30}}{8\,\lambda}\Big\}\Big\}$$

5.2 Define the c_{jk}'s

In[111]:=

```
c30[a_, b_, L_] :=
  2^(-3) * 1 / λ[a, b, L] * (a30[a, b, L] + i * b30[a, b, L] - i * a21[a, b, L] +
     b21[a, b, L] - a12[a, b, L] - i * b12[a, b, L] + i * a03[a, b, L] - b03[a, b, L]);
c21[a_, b_, L_] :=
  2^(-3) * 1 / λ[a, b, L] * (3 * a30[a, b, L] + 3 * i * b30[a, b, L] - i * a21[a, b, L] +
     b21[a, b, L] + a12[a, b, L] + i * b12[a, b, L] - 3 * i * a03[a, b, L] + 3 * b03[a, b, L]);
c12[a_, b_, L_] :=
  2^(-3) * 1 / λ[a, b, L] * (3 * a30[a, b, L] + 3 i * b30[a, b, L] + i * a21[a, b, L] -
     b21[a, b, L] + a12[a, b, L] + i * b12[a, b, L] + 3 i * a03[a, b, L] - 3 * b03[a, b, L]);
c03[a_, b_, L_] :=
  2^(-3) * 1 / λ[a, b, L] * (a30[a, b, L] + i * b30[a, b, L] + i * a21[a, b, L] -
     b21[a, b, L] - a12[a, b, L] - i * b12[a, b, L] - i * a03[a, b, L] + b03[a, b, L]);
```

5.3 Calculate $\dfrac{\partial(z_1, w_1)}{\partial(z, w)}$

In[

```
Clear[z, w]

(* Calculate partial derivatives *)
dz1dz = λ (1 + 3 c30 z^2 + 2 c21 z * w + c12 w^2);
dz1dw = λ ( c21 z^2 + 2 c12 z * w + 3 c0.3 w^2);
dw1dz = 1 / λ (OverBar[c]21 w^2 + 2 OverBar[c]12 w * z + 3 OverBar[c]0.3 z^2);
dw1dw = 1 / λ (1 + 3 * OverBar[c]30 w^2 + 2 * OverBar[c]21 w * z + OverBar[c]12 * z^2);

(* Calculate Jacobian *)
CoefficientList[Expand[dz1dz * dw1dw - dz1dw * dw1dz], {z, w}] // MatrixForm
```

Out[

$$\begin{pmatrix} & 0 & c_{12} + 3\,\bar{c}_{30} & 0 \\ & 2\,c_{21} + 2\,\bar{c}_{21} & 0 & -6\,c_{0.3}\,\bar{c}_{12} + 6\,c_{21} \\ 3\,c_{30} + \bar{c}_{12} & & -9\,c_{0.3}\,\bar{c}_{0.3} - 3\,c_{12}\,\bar{c}_{12} + 3\,c_{21}\,\bar{c}_{21} + 9\,c_{30}\,\bar{c}_{30} & 0 \\ & -6\,c_{12}\,\bar{c}_{0.3} + 6\,c_{30}\,\bar{c}_{21} & 0 & 0 \\ -3\,c_{21}\,\bar{c}_{0.3} + 3\,c_{30}\,\bar{c}_{12} & 0 & 0 & 0 \end{pmatrix}$$

---

6. Coordinate transformation $(z, w) \mapsto (z', w')$

6.1 Define the d_{jk}'s

In[115]:=

```
d30[a_, b_, L_] := c30[a, b, L] / (1 - λ[a, b, L])
d21[a_, b_, L_] := c21[a, b, L]
d12[a_, b_, L_] := c12[a, b, L] / (1 - λ[a, b, L] ^ (-2))
d03[a_, b_, L_] := c03[a, b, L] / (1 - λ[a, b, L] ^ (-4))
```

---

7. Twist Coefficient

7.1 Simplify d_{21}

```
d21[a, e, L];
```

In[126]:=

```
FullSimplify[d21[a, 1 - a, L]]
```

Out[126]=

$$-\frac{\sqrt{-\frac{R^2}{L-2\,a\,R}}\,\left(i\,\sqrt{L}\,R - L\,\sqrt{-\frac{R^2}{L-2\,a\,R}} + a\,R\,\sqrt{-\frac{R^2}{L-2\,a\,R}}\right)\,\left((4-3\,a)\,a\,L + 12\,R + 2\,a\,(-9+2\,a)\,R + a^2\,L\,R\,R''\right)}{8\,a^2\,R^3\,\left(i\,L + a\,R\,\left(-i + \sqrt{\frac{L\,(-L+2\,a\,R)}{a^2\,R^2}}\right)\right)}$$

7.2 Define Symmetric Twist Coefficient

In[127]:=
```
(* After a quick simplification by hand we get: *)

twistsymm[a_, L_, r_, r2_] :=
```

$$\frac{i\left((4-3a)\,a\,L+12\,R+2\,a\,(-9+2\,a)\,R+a^2\,L\,R\,R''\right)}{8\,a^2\,R\,(L-2\,a\,R)} \text{ /. }\{R\to r,\ R''\to r2\}$$

In[128]:=
```
twistsymm[a, L, R, R'']
```

Out[128]=

$$\frac{i\left((4-3a)\,a\,L+12\,R+2\,a\,(-9+2\,a)\,R+a^2\,L\,R\,R''\right)}{8\,a^2\,R\,(L-2\,a\,R)}$$

8. Nonlinear Stability

8.1 Elliptical Billiards

Nonlinear Stability

In[
```
twistsymm[a, 1 - a, 2 δ, 1 / δ, 3 (δ^2 - 1) / δ]
```

Out[

$$\frac{\delta\left(-6+a\left(9+a-4\,\delta^2\right)\right)}{8\,a^2\left(a-\delta^2\right)}$$

In[
```
Solve[twistsymm[a, 1 - a, 2 δ, 1 / δ, 3 (δ^2 - 1) / δ] == 0, δ]
```

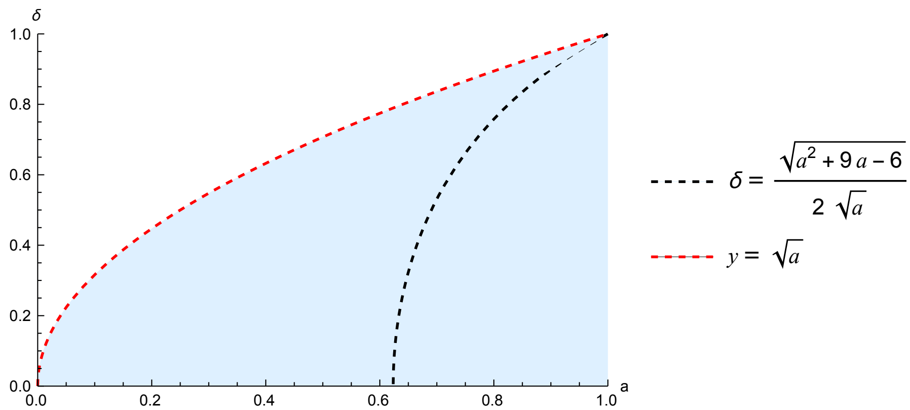
Out[

$$\left\{\{\delta\to 0\},\ \left\{\delta\to -\frac{\sqrt{-6+9\,a+a^2}}{2\,\sqrt{a}}\right\},\ \left\{\delta\to \frac{\sqrt{-6+9\,a+a^2}}{2\,\sqrt{a}}\right\}\right\}$$

In[

```
Plot[{Sqrt[-6 + 9 a + a^2] / (2 * Sqrt[a]), Sqrt[a]}, {a, 0, 1},
 PlotRange → {{0, 1}, {0, 1}},
 PlotStyle → {{Dashed, Black}, {Dashed, Red}},
 Filling → {2 → {Bottom, LightBlue}},
 AxesLabel → {"a", "δ"},
 PlotLegends → {"δ = Sqrt[a^2 + 9 a - 6] / (2 Sqrt[a])", "y = Sqrt[a]"}]
```

Out[



8.2 Symmetric Lemon Billiards

In[

```
twistsymm[a, 1 - a, L, 1, 0]
```

Out[

$$\frac{-12 + a\,(18 - 4\,L + a\,(-4 + 3\,L))}{8\,a^2\,(2\,a - L)}$$

In[

```
Solve[twistsymm[a, 1 - a, L, 1, 0] == 0, L]
```

Out[

$$\left\{\left\{L \to \frac{2\,(6 - 9\,a + 2\,a^2)}{a\,(-4 + 3\,a)}\right\}\right\}$$

In[

```
Reduce[2 (6 - 9 a + 2 a^2) / (a (-4 + 3 a)) > 0, a]
```

Out[

$$a < 0 \;||\; \frac{1}{4}\left(9 - \sqrt{33}\right) < a < \frac{4}{3} \;||\; a > \frac{1}{4}\left(9 + \sqrt{33}\right)$$

In[

```
Solve[2 (6 - 9 a + 2 a^2) / (a (-4 + 3 a)) == 0, a]
```

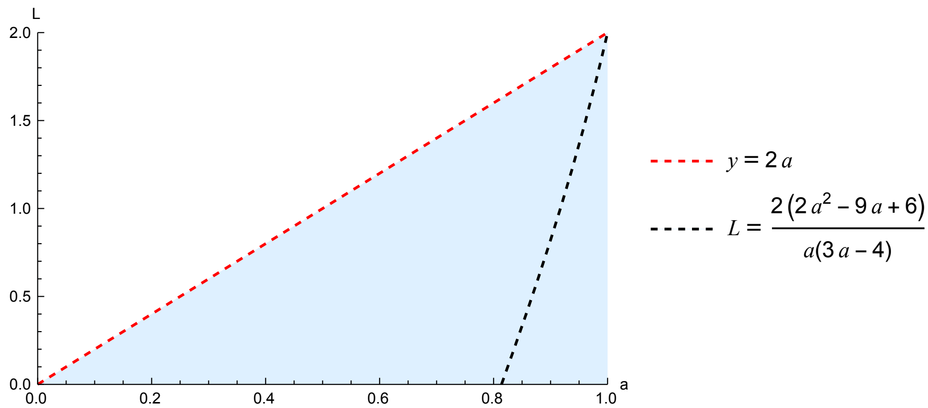
Out[

$$\left\{\left\{a \to \frac{1}{4}\left(9 - \sqrt{33}\right)\right\}, \left\{a \to \frac{1}{4}\left(9 + \sqrt{33}\right)\right\}\right\}$$

In[

```
Plot[{2 a, (2 (6 - 9 a + 2 a^2)) / (a (-4 + 3 a))}, {a, 0, 1},
 PlotStyle → {{Dashed, Red}, {Dashed, Black}},
 Filling → {1 → {Bottom, LightBlue}},
 AxesLabel → {"a", "L"},
 PlotRange → {{0, 1}, {0, 2}},
 PlotLegends → {"y = 2 a", "L = 2 (2 a^2 - 9 a + 6) / (a (3 a - 4))"}]
```

Out[



8.3 Even Polynomial

In[

```
twistsymm[a, 1 - a, L, 1 / (2 * α2), 6 * α2 - 6 α4 / (α2^2)]
```

Out[

$$\frac{-\alpha_2^2\,(6 + a\,(-9 + 2\,a)) + 4\,a\,L\,\alpha_2) + 3\,a^2\,L\,\alpha_4}{4\,a^2\,\alpha_2\,(a - L\,\alpha_2)}$$

In[

```
Solve[twistsymm[a, 1 - a, L, R, R''] == 0, R'']
```

Out[

$$\left\{\left\{R'' \to -\frac{4\,a\,L - 3\,a^2\,L + 12\,R - 18\,a\,R + 4\,a^2\,R}{a^2\,L\,R}\right\}\right\}$$

In[

```
FullSimplify[-(4 a L - 3 a^2 L + 12 R - 18 a R + 4 a^2 R) / (a^2 L R) /. {R → 1 / (2 * α2)}]
```

Out[

$$\frac{2\,(-6 + (9 - 2\,a)\,a + a\,(-4 + 3\,a)\,L\,\alpha_2)}{a^2\,L}$$

In[

```
Solve[6 * α2 - 6 α4 / (α2^2) == 2 (-6 + (9 - 2 a) a + a (-4 + 3 a) L α2) / (a^2 L), α4]
```

Out[

$$\left\{\left\{\alpha_4 \to -\frac{-6\,\alpha_2^2 + 9\,a\,\alpha_2^2 - 2\,a^2\,\alpha_2^2 - 4\,a\,L\,\alpha_2^3}{3\,a^2\,L}\right\}\right\}$$

In[

```
FullSimplify[-(-6 α₂^2 + 9 a α₂^2 - 2 a^2 α₂^2 - 4 a L α₂^3)/(3 a^2 L) /. {α₂ → x, L → 1}]
```

$$\mathtt{FullSimplify}\left[-\frac{-6\,\alpha_2^2+9\,a\,\alpha_2^2-2\,a^2\,\alpha_2^2-4\,a\,L\,\alpha_2^3}{3\,a^2\,L}\ /.\ \{\alpha_2\to x,\ L\to 1\}\right]$$

Out[

$$\frac{x^2\,(6+a\,(-9+2\,a+4\,x))}{3\,a^2}$$

Appendix B

Mathematica Code Asymmetric Twist Coefficient

Part 2: Asymmetric Twist Coefficient

1. One-step DpT

In[30]:=
```
(* Define p - Has to be an even number *)
w = 2;
p = 2 w;
$Assumptions = L > 0;
```

1.1 Derivatives of γ

1.1.1 Minkowski curvature

In[33]:=
```
(* s1 = C/2  half the circumference of billiard table *)
s1[0, u_, a_, b_] = s1;

(* Euclidean curvature *)
ke[s_] := 1 / R[s];

(* Minkowski curvature *)
ρ[θ_, a_, b_] := 1 / (a + b (Cos[θ]^p + Sin[θ]^p)^(1 / p));
(* Euclidean radius of Minkowski circle *)
Derivative[1][θ][s_] := ke[s] ρ[θ[s], a, b];
km[s_, a_, b_] := ke[s] ρ[θ[s], a, b]^3
```

1.1.2 Normal field $n_\gamma(s)$

In[38]:=
```
dir[θ_] := {Cos[θ], Sin[θ]}
n[s_] := 1 / (ρ[θ[s], a, b]^2) * D[ρ[θ[s], a, b], {θ[s], 1}] × dir[θ[s]] +
  1 / ρ[θ[s], a, b] * D[dir[θ[s]], {θ[s], 1}]
```

1.1.3 Derivatives of γ

In[40]:=
```
Derivative[1, 0, 0][γ][s_, a_, b_] := ρ[θ[s], a, b] dir[θ[s]]
Derivative[2, 0, 0][γ][s_, a_, b_] := km[s, a, b] × n[s]
(* First order derivatives *)
dg1[s_, a_, b_] := Derivative[1, 0, 0][γ][s, a, b][[1]]
dg2[s_, a_, b_] := Derivative[1, 0, 0][γ][s, a, b][[2]]

(* Second order derivatives *)
ddg1[s_, a_, b_] := Derivative[2, 0, 0][γ][s, a, b][[1]]
ddg2[s_, a_, b_] := Derivative[2, 0, 0][γ][s, a, b][[2]]
```

1.2 Derivatives of v_1 and v_2

In[46]:=
```
(* Define partial derivatives of v1 =
 gamma1[s1]- gamma1[s] and v2 = gamma2[s1]- gamma2[s] *)
Derivative[m_, n_][v1][s_, u_] := D[γ[s1[s, u, a, b], a, b] - γ[s, a, b], {s, m}, {u, n}][[1]]
Derivative[m_, n_][v2][s_, u_] := D[γ[s1[s, u, a, b], a, b] - γ[s, a, b], {s, m}, {u, n}][[2]]
```

1.3 Metric F and it's partial derivatives

In[48]:=
```
(* Define Finsler metric F *)
F[a_, b_, v1_, v2_] := a * Sqrt[v1^2 + v2^2] + b (v1^(p) + v2^(p))^(1 / (p));

(* Define Partial Derivatives of F wrt v1 and v2 *)
Fderi[a_, b_, v1_, v2_, j_, k_] :=
  Fderi[a, b, v1, v2, j, k] = D[F[a, b, v1, v2], {v1, j}, {v2, k}];
```

1.4 Define 1st partial derivatives of s_1 and u_1

```
(* ds1du *)
ds1ds[s_, u_, a_, b_, v1_, v2_] :=
 (Fderi[a, b, v1, v2, 2, 0] × dg1[s, a, b]^2 + 2 * Fderi[a, b, v1, v2, 1, 1] ×
     dg1[s, a, b] × dg2[s, a, b] + Fderi[a, b, v1, v2, 0, 2] × dg2[s, a, b]^2 -
    Fderi[a, b, v1, v2, 1, 0] × ddg1[s, a, b] - Fderi[a, b, v1, v2, 0, 1] × ddg2[s, a, b]) /
  (Fderi[a, b, v1, v2, 2, 0] × dg1[s, a, b] × dg1[s1[s, u, a, b], a, b] +
    Fderi[a, b, v1, v2, 1, 1] × dg1[s, a, b] × dg2[s1[s, u, a, b], a, b] +
    Fderi[a, b, v1, v2, 1, 1] × dg1[s1[s, u, a, b], a, b] × dg2[s, a, b] +
    Fderi[a, b, v1, v2, 0, 2] × dg2[s, a, b] × dg2[s1[s, u, a, b], a, b])

Derivative[1, 0, 0, 0][s1][s_, u_, a_, b_] := ds1ds[s, u, a, b, v1[s, u], v2[s, u]]
```

```
(* ds1du *)
ds1du[s_, u_, a_, b_, v1_, v2_] :=
 -1 / (Fderi[a, b, v1, v2, 2, 0] × dg1[s, a, b] × dg1[s1[s, u, a, b], a, b] +
     Fderi[a, b, v1, v2, 1, 1] × dg1[s, a, b] × dg2[s1[s, u, a, b], a, b] +
     Fderi[a, b, v1, v2, 1, 1] × dg1[s1[s, u, a, b], a, b] × dg2[s, a, b] +
     Fderi[a, b, v1, v2, 0, 2] × dg2[s, a, b] × dg2[s1[s, u, a, b], a, b])

Derivative[0, 1, 0, 0][s1][s_, u_, a_, b_] := ds1du[s, u, a, b, v1[s, u], v2[s, u]]

(* duds *)
duds[s_, u_, a_, b_, v1_, v2_] :=
 Fderi[a, b, v1, v2, 2, 0] × dg1[s, a, b] × dg1[s1[s, u, a, b], a, b] +
  Fderi[a, b, v1, v2, 1, 1] × dg1[s, a, b] × dg2[s1[s, u, a, b], a, b] +
  Fderi[a, b, v1, v2, 1, 1] * dg1[s1[s, u, a, b], a, b] × dg2[s, a, b] +
  Fderi[a, b, v1, v2, 0, 2] × dg2[s, a, b] × dg2[s1[s, u, a, b], a, b] -
  ds1ds[s, u, a, b, v1, v2] * (Fderi[a, b, v1, v2, 2, 0] × dg1[s1[s, u, a, b], a, b]^2 +
     2 * Fderi[a, b, v1, v2, 1, 1] × dg1[s1[s, u, a, b], a, b] × dg2[s1[s, u, a, b], a, b] +
     Fderi[a, b, v1, v2, 0, 2] * dg2[s1[s, u, a, b], a, b]^2 + Fderi[a, b, v1, v2, 1, 0] ×
      ddg1[s1[s, u, a, b], a, b] + Fderi[a, b, v1, v2, 0, 1] × ddg2[s1[s, u, a, b], a, b])

Derivative[1, 0, 0, 0][u1][s_, u_, a_, b_] := duds[s, u, a, b, v1[s, u], v2[s, u]]

(* dudu *)
dudu[s_, u_, a_, b_, v1_, v2_] :=
 (Fderi[a, b, v1, v2, 2, 0] × dg1[s1[s, u, a, b], a, b]^2 + 2 * Fderi[a, b, v1, v2, 1, 1] ×
      dg1[s1[s, u, a, b], a, b] × dg2[s1[s, u, a, b], a, b] + Fderi[a, b, v1, v2, 0, 2] ×
      dg2[s1[s, u, a, b], a, b]^2 + Fderi[a, b, v1, v2, 1, 0] * ddg1[s1[s, u, a, b], a, b] +
     Fderi[a, b, v1, v2, 0, 1] * ddg2[s1[s, u, a, b], a, b]) /
   (Fderi[a, b, v1, v2, 2, 0] × dg1[s, a, b] × dg1[s1[s, u, a, b], a, b] +
     Fderi[a, b, v1, v2, 1, 1] × dg1[s, a, b] × dg2[s1[s, u, a, b], a, b] +
     Fderi[a, b, v1, v2, 1, 1] × dg1[s1[s, u, a, b], a, b] × dg2[s, a, b] +
     Fderi[a, b, v1, v2, 0, 2] × dg2[s, a, b] × dg2[s1[s, u, a, b], a, b])

Derivative[0, 1, 0, 0][u1][s_, u_, a_, b_] := dudu[s, u, a, b, v1[s, u], v2[s, u]]
```

2. Higher Order Derivatives of s_1 and u_1

2.1 Billiard table conditions

2.1.1 Symmetric Billiard Table

In[58]:=
```
(* Fordward map (s₀,u₀)→ (s₁, u₁) *)
Symm = {
   s → 0,
   v1[s, u] → L, v2[s, u] → 0,
   θ[s] → -Pi / 2, θ[s1[s, u, a, b]] → Pi / 2,

   R[0] :→ R₀,
   Derivative[1][R][s] → 0,
   Derivative[2][R][0] → R₀'',

   R[s₁] :→ R₁,
   Derivative[1][R][s₁] → 0,
   Derivative[2][R][s₁] → R₁''
  };

(* Reverse map (s₁,u₁) → (s₀,u₀) *)
SymmReverse = {
   s → 0,
   v1[s, u] → -L, v2[s, u] → 0,
   θ[s] → Pi / 2, θ[s1[s, u, a, b]] → -Pi / 2,
   R[0] :→ R₁,
   Derivative[1][R][s] → 0,
   Derivative[2][R][0] → R₁'',
   R[s₁] :→ R₀,
   Derivative[1][R][s₁] → 0,
   Derivative[2][R][s₁] → R₀''
  };
```

2.2 Higher order derivatives

```
In[60]:= Clear[sderi, uderi, srderi, urderi]
sderi[j_, k_, A_, B_, l_] := sderi[j, k, A, B, l] =
  Simplify[D[s1[s, u, a, b], {s, j}, {u, k}] //. Symm /. {a → A, b → B, L → l}]
uderi[j_, k_, A_, B_, l_] := uderi[j, k, A, B, l] =
  Simplify[D[u1[s, u, a, b], {s, j}, {u, k}] //. Symm /. {a → A, b → B, L → l}]

srderi[j_, k_, A_, B_, l_] := srderi[j, k, A, B, l] =
  Simplify[D[s1[s, u, a, b], {s, j}, {u, k}] //. SymmReverse /. {a → A, b → B, L → l}]
urderi[j_, k_, A_, B_, l_] := urderi[j, k, A, B, l] =
  Simplify[D[u1[s, u, a, b], {s, j}, {u, k}] //. SymmReverse /. {a → A, b → B, L → l}]
```

3. Taylor expansion of the billiard map

3.1 Taylor Expansion of the two-step billiard map

```
In[166]:= ta10[a_, b_, L_] :=
  sderi[1, 0, a, b, L] × srderi[1, 0, a, b, L] + srderi[0, 1, a, b, L] × uderi[1, 0, a, b, L];
ta01[a_, b_, L_] :=
  sderi[0, 1, a, b, L] × srderi[1, 0, a, b, L] + srderi[0, 1, a, b, L] × uderi[0, 1, a, b, L];
ta20[a_, b_, L_] :=
  (sderi[2, 0, a, b, L] × srderi[1, 0, a, b, L] + sderi[1, 0, a, b, L]^2 × srderi[2, 0, a, b, L] +
     2 sderi[1, 0, a, b, L] × srderi[1, 1, a, b, L] × uderi[1, 0, a, b, L] +
     srderi[0, 2, a, b, L] × uderi[1, 0, a, b, L]^2 +
     srderi[0, 1, a, b, L] × uderi[2, 0, a, b, L]) / 2;
ta11[a_, b_, L_] := sderi[1, 1, a, b, L] × srderi[1, 0, a, b, L] +
   sderi[0, 1, a, b, L] × sderi[1, 0, a, b, L] × srderi[2, 0, a, b, L] +
   sderi[1, 0, a, b, L] × srderi[1, 1, a, b, L] × uderi[0, 1, a, b, L] +
   sderi[0, 1, a, b, L] × srderi[1, 1, a, b, L] × uderi[1, 0, a, b, L] +
   srderi[0, 2, a, b, L] × uderi[0, 1, a, b, L] × uderi[1, 0, a, b, L] +
   srderi[0, 1, a, b, L] × uderi[1, 1, a, b, L];
ta02[a_, b_, L_] :=
  (sderi[0, 2, a, b, L] × srderi[1, 0, a, b, L] + sderi[0, 1, a, b, L]^2 × srderi[2, 0, a, b, L] +
     2 sderi[0, 1, a, b, L] × srderi[1, 1, a, b, L] × uderi[0, 1, a, b, L] +
     srderi[0, 2, a, b, L] × uderi[0, 1, a, b, L]^2 +
     srderi[0, 1, a, b, L] × uderi[0, 2, a, b, L]) / 2;
ta30[a_, b_, L_] :=
  (sderi[3, 0, a, b, L] × srderi[1, 0, a, b, L] + 3 sderi[1, 0, a, b, L] × sderi[2, 0, a, b, L] ×
      srderi[2, 0, a, b, L] + sderi[1, 0, a, b, L]^3 × srderi[3, 0, a, b, L] +
     3 sderi[2, 0, a, b, L] × srderi[1, 1, a, b, L] × uderi[1, 0, a, b, L] +
     3 sderi[1, 0, a, b, L]^2 × srderi[2, 1, a, b, L] × uderi[1, 0, a, b, L] +
     3 sderi[1, 0, a, b, L] × srderi[1, 2, a, b, L] × uderi[1, 0, a, b, L]^2 +
```

```
    srderi[0, 3, a, b, L] × uderi[1, 0, a, b, L]^3 +
    3 sderi[1, 0, a, b, L] × srderi[1, 1, a, b, L] × uderi[2, 0, a, b, L] +
    3 srderi[0, 2, a, b, L] × uderi[1, 0, a, b, L] × uderi[2, 0, a, b, L] +
    srderi[0, 1, a, b, L] × uderi[3, 0, a, b, L]) / 6;
ta21[a_, b_, L_] :=
  (sderi[2, 1, a, b, L] × srderi[1, 0, a, b, L] + 2 sderi[1, 0, a, b, L] × sderi[1, 1, a, b, L] ×
     srderi[2, 0, a, b, L] + sderi[0, 1, a, b, L] × sderi[2, 0, a, b, L] × srderi[2, 0, a, b, L] +
    sderi[0, 1, a, b, L] × sderi[1, 0, a, b, L]^2 × srderi[3, 0, a, b, L] +
    sderi[2, 0, a, b, L] × srderi[1, 1, a, b, L] × uderi[0, 1, a, b, L] +
    sderi[1, 0, a, b, L]^2 × srderi[2, 1, a, b, L] × uderi[0, 1, a, b, L] +
    2 sderi[1, 1, a, b, L] × srderi[1, 1, a, b, L] × uderi[1, 0, a, b, L] +
    2 sderi[0, 1, a, b, L] × sderi[1, 0, a, b, L] × srderi[2, 1, a, b, L] × uderi[1, 0, a, b, L] +
    2 sderi[1, 0, a, b, L] × srderi[1, 2, a, b, L] × uderi[0, 1, a, b, L] × uderi[1, 0, a, b, L] +
    sderi[0, 1, a, b, L] × srderi[1, 2, a, b, L] × uderi[1, 0, a, b, L]^2 +
    srderi[0, 3, a, b, L] × uderi[0, 1, a, b, L] × uderi[1, 0, a, b, L]^2 +
    2 sderi[1, 0, a, b, L] × srderi[1, 1, a, b, L] × uderi[1, 1, a, b, L] +
    2 srderi[0, 2, a, b, L] × uderi[1, 0, a, b, L] × uderi[1, 1, a, b, L] +
    sderi[0, 1, a, b, L] × srderi[1, 1, a, b, L] × uderi[2, 0, a, b, L] +
    srderi[0, 2, a, b, L] × uderi[0, 1, a, b, L] × uderi[2, 0, a, b, L] +
    srderi[0, 1, a, b, L] × uderi[2, 1, a, b, L]) / 2;
ta12[a_, b_, L_] := (sderi[1, 2, a, b, L] × srderi[1, 0, a, b, L] +
    sderi[0, 2, a, b, L] × sderi[1, 0, a, b, L] × srderi[2, 0, a, b, L] +
    2 sderi[0, 1, a, b, L] × sderi[1, 1, a, b, L] × srderi[2, 0, a, b, L] +
    sderi[0, 1, a, b, L]^2 × sderi[1, 0, a, b, L] × srderi[3, 0, a, b, L] +
    2 sderi[1, 1, a, b, L] × srderi[1, 1, a, b, L] × uderi[0, 1, a, b, L] +
    2 sderi[0, 1, a, b, L] × sderi[1, 0, a, b, L] × srderi[2, 1, a, b, L] × uderi[0, 1, a, b, L] +
    sderi[1, 0, a, b, L] × srderi[1, 2, a, b, L] × uderi[0, 1, a, b, L]^2 +
    sderi[1, 0, a, b, L] × srderi[1, 1, a, b, L] × uderi[0, 2, a, b, L] +
    sderi[0, 2, a, b, L] × srderi[1, 1, a, b, L] × uderi[1, 0, a, b, L] +
    sderi[0, 1, a, b, L]^2 × srderi[2, 1, a, b, L] × uderi[1, 0, a, b, L] +
    2 sderi[0, 1, a, b, L] × srderi[1, 2, a, b, L] × uderi[0, 1, a, b, L] × uderi[1, 0, a, b, L] +
    srderi[0, 3, a, b, L] × uderi[0, 1, a, b, L]^2 × uderi[1, 0, a, b, L] +
    srderi[0, 2, a, b, L] × uderi[0, 2, a, b, L] × uderi[1, 0, a, b, L] +
    2 sderi[0, 1, a, b, L] × srderi[1, 1, a, b, L] × uderi[1, 1, a, b, L] +
    2 srderi[0, 2, a, b, L] × uderi[0, 1, a, b, L] × uderi[1, 1, a, b, L] +
    srderi[0, 1, a, b, L] × uderi[1, 2, a, b, L]) / 2;
ta03[a_, b_, L_] :=
  (sderi[0, 3, a, b, L] × srderi[1, 0, a, b, L] + 3 sderi[0, 1, a, b, L] × sderi[0, 2, a, b, L] ×
     srderi[2, 0, a, b, L] + sderi[0, 1, a, b, L]^3 × srderi[3, 0, a, b, L] +
    3 sderi[0, 2, a, b, L] × srderi[1, 1, a, b, L] × uderi[0, 1, a, b, L] +
    3 sderi[0, 1, a, b, L]^2 × srderi[2, 1, a, b, L] × uderi[0, 1, a, b, L] +
    3 sderi[0, 1, a, b, L] × srderi[1, 2, a, b, L] × uderi[0, 1, a, b, L]^2 +
    srderi[0, 3, a, b, L] × uderi[0, 1, a, b, L]^3 +
    3 sderi[0, 1, a, b, L] × srderi[1, 1, a, b, L] × uderi[0, 2, a, b, L] +
    3 srderi[0, 2, a, b, L] × uderi[0, 1, a, b, L] × uderi[0, 2, a, b, L] +
    srderi[0, 1, a, b, L] × uderi[0, 3, a, b, L]) / 6;
```

```
tb10[a_, b_, L_] :=
  uderi[1, 0, a, b, L] × urderi[0, 1, a, b, L] + sderi[1, 0, a, b, L] × urderi[1, 0, a, b, L];
tb01[a_, b_, L_] :=
  uderi[0, 1, a, b, L] × urderi[0, 1, a, b, L] + sderi[0, 1, a, b, L] × urderi[1, 0, a, b, L];
tb20[a_, b_, L_] :=
  (uderi[2, 0, a, b, L] × urderi[0, 1, a, b, L] + uderi[1, 0, a, b, L]^2 × urderi[0, 2, a, b, L] +
     sderi[2, 0, a, b, L] × urderi[1, 0, a, b, L] + 2 sderi[1, 0, a, b, L] × uderi[1, 0, a, b, L] ×
      urderi[1, 1, a, b, L] + sderi[1, 0, a, b, L]^2 × urderi[2, 0, a, b, L]) / 2;
tb11[a_, b_, L_] := uderi[1, 1, a, b, L] × urderi[0, 1, a, b, L] +
   uderi[0, 1, a, b, L] × uderi[1, 0, a, b, L] × urderi[0, 2, a, b, L] +
   sderi[1, 1, a, b, L] × urderi[1, 0, a, b, L] +
   sderi[1, 0, a, b, L] × uderi[0, 1, a, b, L] × urderi[1, 1, a, b, L] +
   sderi[0, 1, a, b, L] × uderi[1, 0, a, b, L] × urderi[1, 1, a, b, L] +
   sderi[0, 1, a, b, L] × sderi[1, 0, a, b, L] × urderi[2, 0, a, b, L];
tb02[a_, b_, L_] :=
  (uderi[0, 2, a, b, L] × urderi[0, 1, a, b, L] + uderi[0, 1, a, b, L]^2 × urderi[0, 2, a, b, L] +
     sderi[0, 2, a, b, L] × urderi[1, 0, a, b, L] + 2 sderi[0, 1, a, b, L] × uderi[0, 1, a, b, L] ×
      urderi[1, 1, a, b, L] + sderi[0, 1, a, b, L]^2 × urderi[2, 0, a, b, L]) / 2;
tb30[a_, b_, L_] :=
  (uderi[3, 0, a, b, L] × urderi[0, 1, a, b, L] + 3 uderi[1, 0, a, b, L] × uderi[2, 0, a, b, L] ×
      urderi[0, 2, a, b, L] + uderi[1, 0, a, b, L]^3 × urderi[0, 3, a, b, L] +
     sderi[3, 0, a, b, L] × urderi[1, 0, a, b, L] +
     3 sderi[2, 0, a, b, L] × uderi[1, 0, a, b, L] × urderi[1, 1, a, b, L] +
     3 sderi[1, 0, a, b, L] × uderi[2, 0, a, b, L] × urderi[1, 1, a, b, L] +
     3 sderi[1, 0, a, b, L] × uderi[1, 0, a, b, L]^2 × urderi[1, 2, a, b, L] +
     3 sderi[1, 0, a, b, L] × sderi[2, 0, a, b, L] × urderi[2, 0, a, b, L] +
     3 sderi[1, 0, a, b, L]^2 × uderi[1, 0, a, b, L] × urderi[2, 1, a, b, L] +
     sderi[1, 0, a, b, L]^3 × urderi[3, 0, a, b, L]) / 6;
tb21[a_, b_, L_] :=
  (uderi[2, 1, a, b, L] × urderi[0, 1, a, b, L] + 2 uderi[1, 0, a, b, L] × uderi[1, 1, a, b, L] ×
      urderi[0, 2, a, b, L] + uderi[0, 1, a, b, L] × uderi[2, 0, a, b, L] × urderi[0, 2, a, b, L] +
     uderi[0, 1, a, b, L] × uderi[1, 0, a, b, L]^2 × urderi[0, 3, a, b, L] +
     sderi[2, 1, a, b, L] × urderi[1, 0, a, b, L] +
     sderi[2, 0, a, b, L] × uderi[0, 1, a, b, L] × urderi[1, 1, a, b, L] +
     2 sderi[1, 1, a, b, L] × uderi[1, 0, a, b, L] × urderi[1, 1, a, b, L] +
     2 sderi[1, 0, a, b, L] × uderi[1, 1, a, b, L] × urderi[1, 1, a, b, L] +
     sderi[0, 1, a, b, L] × uderi[2, 0, a, b, L] × urderi[1, 1, a, b, L] +
     2 sderi[1, 0, a, b, L] × uderi[0, 1, a, b, L] × uderi[1, 0, a, b, L] × urderi[1, 2, a, b, L] +
     sderi[0, 1, a, b, L] × uderi[1, 0, a, b, L]^2 × urderi[1, 2, a, b, L] +
     2 sderi[1, 0, a, b, L] × sderi[1, 1, a, b, L] × urderi[2, 0, a, b, L] +
     sderi[0, 1, a, b, L] × sderi[2, 0, a, b, L] × urderi[2, 0, a, b, L] +
     sderi[1, 0, a, b, L]^2 × uderi[0, 1, a, b, L] × urderi[2, 1, a, b, L] +
     2 sderi[0, 1, a, b, L] × sderi[1, 0, a, b, L] × uderi[1, 0, a, b, L] × urderi[2, 1, a, b, L] +
     sderi[0, 1, a, b, L] × sderi[1, 0, a, b, L]^2 × urderi[3, 0, a, b, L]) / 2;
tb12[a_, b_, L_] := (uderi[1, 2, a, b, L] × urderi[0, 1, a, b, L] +
     uderi[0, 2, a, b, L] × uderi[1, 0, a, b, L] × urderi[0, 2, a, b, L] +
     2 uderi[0, 1, a, b, L] × uderi[1, 1, a, b, L] × urderi[0, 2, a, b, L] +
```

```
    uderi[0, 1, a, b, L]^2 × uderi[1, 0, a, b, L] × urderi[0, 3, a, b, L] +
    sderi[1, 2, a, b, L] × urderi[1, 0, a, b, L] +
    2 sderi[1, 1, a, b, L] × uderi[0, 1, a, b, L] × urderi[1, 1, a, b, L] +
    sderi[1, 0, a, b, L] × uderi[0, 2, a, b, L] × urderi[1, 1, a, b, L] +
    sderi[0, 2, a, b, L] × uderi[1, 0, a, b, L] × urderi[1, 1, a, b, L] +
    2 sderi[0, 1, a, b, L] × uderi[1, 1, a, b, L] × urderi[1, 1, a, b, L] +
    sderi[1, 0, a, b, L] × uderi[0, 1, a, b, L]^2 × urderi[1, 2, a, b, L] +
    2 sderi[0, 1, a, b, L] × uderi[0, 1, a, b, L] × uderi[1, 0, a, b, L] × urderi[1, 2, a, b, L] +
    sderi[0, 2, a, b, L] × sderi[1, 0, a, b, L] × urderi[2, 0, a, b, L] +
    2 sderi[0, 1, a, b, L] × sderi[1, 1, a, b, L] × urderi[2, 0, a, b, L] +
    2 sderi[0, 1, a, b, L] × sderi[1, 0, a, b, L] × uderi[0, 1, a, b, L] × urderi[2, 1, a, b, L] +
    sderi[0, 1, a, b, L]^2 × uderi[1, 0, a, b, L] × urderi[2, 1, a, b, L] +
    sderi[0, 1, a, b, L]^2 × sderi[1, 0, a, b, L] × urderi[3, 0, a, b, L]) / 2;
tb03[a_, b_, L_] :=
  (uderi[0, 3, a, b, L] × urderi[0, 1, a, b, L] + 3 uderi[0, 1, a, b, L] × uderi[0, 2, a, b, L] ×
     urderi[0, 2, a, b, L] + uderi[0, 1, a, b, L]^3 × urderi[0, 3, a, b, L] +
    sderi[0, 3, a, b, L] × urderi[1, 0, a, b, L] +
    3 sderi[0, 2, a, b, L] × uderi[0, 1, a, b, L] × urderi[1, 1, a, b, L] +
    3 sderi[0, 1, a, b, L] × uderi[0, 2, a, b, L] × urderi[1, 1, a, b, L] +
    3 sderi[0, 1, a, b, L] × uderi[0, 1, a, b, L]^2 × urderi[1, 2, a, b, L] +
    3 sderi[0, 1, a, b, L] × sderi[0, 2, a, b, L] × urderi[2, 0, a, b, L] +
    3 sderi[0, 1, a, b, L]^2 × uderi[0, 1, a, b, L] × urderi[2, 1, a, b, L] +
    sderi[0, 1, a, b, L]^3 × urderi[3, 0, a, b, L]) / 6;
```

3.2 Some higher order terms

In[161]:= `Simplify[ta10[a, 1 - a, L]]`

Out[161]= $$\frac{2 L (L - a R_1) + a R_0 (-2 L + a R_1)}{a^2 R_0 R_1}$$

In[160]:= `Simplify[ta01[a, 1 - a, L]]`

Out[160]= $$\frac{2 L \left(-1 + \frac{L}{a R_1}\right)}{a}$$

In[159]:= `Simplify[tb10[a, 1 - a, L]]`

Out[159]= $(2 (-L + a R_0) (-L + a R_0 + a R_1)) \Big/ \left(a^2 R_0^2 R_1\right)$

In[158]:= `Simplify[tb01[a, 1 - a, L]]`

Out[158]= $$\frac{2 L (L - a R_1) + a R_0 (-2 L + a R_1)}{a^2 R_0 R_1}$$

In[162]:= `Simplify[ta20[a, 1 - a, L]]`

Out[162]= 0

In[163]:= **Simplify[ta11[a, 1 - a, L]]**

Out[163]= 0

In[164]:= **Simplify[ta02[a, 1 - a, L]]**

Out[164]= 0

In[184]:= **Simplify[tb20[a, 1 - a, L]]**

Out[184]= 0

In[185]:= **Simplify[tb11[a, 1 - a, L]]**

Out[185]= 0

In[186]:= **Simplify[tb02[a, 1 - a, L]]**

Out[186]= 0

In[187]:= **Simplify[ta30[a, 1 - a, L]]**

Out[187]=
$$\frac{1}{3\, a^7\, R_0^5\, R_1^3}\Big(4\, a^2\, L^3\, (L - a\, R_1)^3 + 6\, a\, \big(1 + a^2\big)\, L^2\, R_0\, (L - a\, R_1)^2\, (-2\, L + a\, R_1) +$$
$$R_0^5\, \big(-3\, a^3\, \big(-8 + 12\, a - 2\, a^2 + a^3\big)\, L + a^6\, R_1\, (3 + L\, R_1'')\big) +$$
$$a\, R_0^3\, \Big(-\big(\big(-72 + 108\, a + 18\, a^2 + 9\, a^3 + 4\, a^4\big)\, L^3\big) + a^5\, R_1^3\, (3 + L\, R_0'') -$$
$$a^2\, L\, R_1^2\, \big(-18 + 27\, a + 27\, a^2 + 2\, a^4 + a^2\, L\, R_0''\big) + 3\, a\, L^2\, R_1\, \big(-24 + 36\, a + 19\, a^2 + 2\, a^4 + a^2\, L\, R_1''\big)\Big) +$$
$$3\, a^2\, R_0^4\, \Big(\big(-24 + 36\, a - 2\, a^2 + 3\, a^3\big)\, L^2 + 2\, a^4\, R_1^2 - a\, L\, R_1\, \big(6\, \big(-2 + 3\, a + a^2\big) + a^2\, L\, R_1''\big)\Big) +$$
$$L\, R_0^2\, \Big(3\, \big(-8 + 12\, a + 10\, a^2 + a^3 + 4\, a^4\big)\, L^3 + a^2\, \big(-24 + 36\, a + 39\, a^2 + 3\, a^3 + 14\, a^4\big)\, L\, R_1^2 -$$
$$a^3\, \big(-6 + 9\, a + 3\, a^2 + 3\, a^3 + 2\, a^4\big)\, R_1^3 - a\, L^2\, R_1\, \big(6\, \big(-6 + 9\, a + 12\, a^2 + 4\, a^4\big) + a^2\, L\, R_1''\big)\Big)\Big)$$

In[188]:= **Simplify[ta21[a, 1 - a, L]]**

Out[188]=
$$\frac{1}{a^7\, R_0^4\, R_1^3}\Big(4\, a^2\, L^3\, (L - a\, R_1)^3 + 2\, a\, \big(3 + 2\, a^2\big)\, L^2\, R_0\, (L - a\, R_1)^2\, (-2\, L + a\, R_1) + a\, R_0^3$$
$$\Big(-6\, \big(-8 + 12\, a + a^3\big)\, L^3 - 2\, a^2\, \big(-6 + 9\, a + 5\, a^2\big)\, L\, R_1^2 + a^5\, R_1^3 + 2\, a\, L^2\, R_1\, \big(-24 + 36\, a + 10\, a^2 + a^2\, L\, R_1''\big)\Big) +$$
$$L\, R_0^2\, \Big(\big(-24 + 36\, a + 18\, a^2 + 3\, a^3 + 4\, a^4\big)\, L^3 + 2\, a^2\, \big(-12 + 18\, a + 13\, a^2 + 3\, a^4\big)\, L\, R_1^2 -$$
$$a^3\, \big(-6 + 9\, a + 2\, a^2 + 2\, a^4\big)\, R_1^3 - a\, L^2\, R_1\, \big(-36 + 54\, a + 48\, a^2 + 8\, a^4 + a^2\, L\, R_1''\big)\Big) +$$
$$a^2\, R_0^4\, \Big(3\, \big(-8 + 12\, a - 2\, a^2 + a^3\big)\, L^2 + a^4\, R_1^2 - a\, L\, R_1\, \big(2\, \big(-6 + 9\, a + a^2\big) + a^2\, L\, R_1''\big)\Big)\Big)$$

In[189]:= **Simplify[ta12[a, 1 - a, L]]**

Out[189]=
$$\frac{1}{a^7\, R_0^3\, R_1^3}L\, \Big(4\, a^2\, L^2\, (L - a\, R_1)^3 + 2\, a\, \big(3 + a^2\big)\, L\, R_0\, (L - a\, R_1)^2\, (-2\, L + a\, R_1) +$$
$$a\, R_0^3\, \Big(-3\, \big(-8 + 12\, a - 2\, a^2 + a^3\big)\, L^2 - a^2\, \big(-6 + 9\, a + a^2\big)\, R_1^2 + a\, L\, R_1\, \big(-24 + 36\, a + a^2 + a^2\, L\, R_1''\big)\Big) -$$
$$R_0^2\, \Big(-3\, \big(-8 + 12\, a + 2\, a^2 + a^3\big)\, L^3 + a^2\, \big(24 - 36\, a - 13\, a^2\big)\, L\, R_1^2 +$$
$$a^3\, \big(-6 + 9\, a + a^2\big)\, R_1^3 + a\, L^2\, R_1\, \big(6\, \big(-6 + 9\, a + 4\, a^2\big) + a^2\, L\, R_1''\big)\Big)\Big)$$

In[190]:= `Simplify[ta03[a, 1 - a, L]]`

Out[190]=

$$
\begin{aligned}
&\frac{1}{3\, a^7\, R_0^2\, R_1^3}\\
&L\, \Big(4\, a^2\, L^2\, (L - a\, R_1)^3 + 6\, a\, L\, R_0\, (L - a\, R_1)^2\, (-2\, L + a\, R_1) - R_0^2\, \Big(-3\, \big(-8 + 12\, a - 2\, a^2 + a^3\big)\, L^3 + 12\, (2 - 3\, a)\\
&\qquad a^2\, L\, R_1^2 + 3\, a^3\, (-2 + 3\, a)\, R_1^3 + a\, L^2\, R_1\, \big(-36 + 54\, a + a^2\, L\, R_1''\big)\Big)\Big)
\end{aligned}
$$

In[191]:= `Simplify[tb30[a, 1 - a, L]]`

Out[191]=

$$
\begin{aligned}
&\frac{1}{3\, a^7\, R_0^6\, R_1^3}\Big(12\, (-1 + a)\, a\, L^3\, (-L + a\, R_1)^3 -\\
&\quad 2\, a\, L^2\, R_0\, (L - a\, R_1)^2\, \big(a\, R_1\, \big(-3 - 9\, a + 12\, a^2 - 2\, L\, R_0''\big) + 2\, L\, \big(3 + 9\, a - 12\, a^2 + L\, R_0''\big)\big) -\\
&\quad a^6\, R_0^6\, (6 - 3\, a + a\, R_1\, R_1'') + 2\, R_0^5\, \big(-6\, a^3\, \big(-2 + 3\, a - 3\, a^2 + a^3\big)\, L + a^6\, R_1\, (-3 + 2\, L\, R_1'')\big) +\\
&\quad L\, R_0^2\, \Big(a^3\, R_1^3\, \big(6 - 9\, a - 12\, a^3 + 16\, a^4 - 6\, a^2\, L\, R_0''\big) - 4\, a^2\, L\, R_1^2\, \big(6 - 9\, a - 6\, a^2 - 12\, a^3 + 22\, a^4 - 6\, a^2\, L\, R_0''\big) +\\
&\qquad 3\, L^3\, \big(-8 + 12\, a + 6\, a^2 + 13\, a^3 - 24\, a^4 + 4\, a^2\, L\, R_0''\big) +\\
&\qquad a\, L^2\, R_1\, \big(6\, \big(6 - 9\, a - 8\, a^2 - 12\, a^3 + 24\, a^4\big) - 30\, a^2\, L\, R_0'' - a^2\, L\, R_1''\big)\Big) -\\
&\quad a^2\, R_0^4\, \Big(a^5\, R_1^3\, R_0'' + 2\, L^2\, \big(36 - 54\, a + 30\, a^2 - 9\, a^3 + 6\, a^4 - 2\, a^2\, L\, R_0''\big) + 4\, R_1^2\, \big(a^6 - a^4\, L\, R_0''\big) -\\
&\qquad 6\, a\, L\, R_1\, \big(6 - 9\, a + 4\, a^2 + 2\, a^4 - a^2\, L\, R_0'' - a^2\, L\, R_1''\big)\Big) +\\
&\quad a\, R_0^3\, \Big(a^5\, R_1^3\, ((3 - 4\, a)\, a + 4\, L\, R_0'') + 4\, a^2\, L\, R_1^2\, \big(6 - 9\, a - 3\, a^3 + 8\, a^4 - 4\, a^2\, L\, R_0''\big) -\\
&\qquad 12\, L^3\, \big(-6 + 9\, a - 2\, a^2 + 2\, a^3 - 4\, a^4 + a^2\, L\, R_0''\big) +\\
&\qquad 2\, a\, L^2\, R_1\, \big(9\, \big(-4 + 6\, a + a^3 - 4\, a^4\big) + 12\, a^2\, L\, R_0'' + 2\, a^2\, L\, R_1''\big)\Big)\Big)
\end{aligned}
$$

In[192]:= `Simplify[tb21[a, 1 - a, L]]`

Out[192]=

$$
\begin{aligned}
&\frac{1}{a^7\, R_0^5\, R_1^3}\Big(12\, (-1 + a)\, a\, L^3\, (-L + a\, R_1)^3 -\\
&\quad 2\, a\, L^2\, R_0\, (L - a\, R_1)^2\, \big(a\, R_1\, \big(-3 - 6\, a + 9\, a^2 - 2\, L\, R_0''\big) + 2\, L\, \big(3 + 6\, a - 9\, a^2 + L\, R_0''\big)\big) +\\
&\quad a^5\, R_0^5\, (-3\, (-2 + a)\, L + a\, R_1\, (-1 + L\, R_1'')) +\\
&\quad a^2\, R_0^4\, \big(3\, \big(-8 + 12\, a - 10\, a^2 + 3\, a^3\big)\, L^2 - 2\, a^4\, R_1^2 + a\, L\, R_1\, \big(2\, \big(6 - 9\, a + 7\, a^2\big) - 3\, a^2\, L\, R_1''\big)\big) +\\
&\quad L\, R_0^2\, \Big(a^3\, R_1^3\, \big(6 - 9\, a + a^2 - 3\, a^3 + 6\, a^4 - 4\, a^2\, L\, R_0''\big) + L^3\, \big(-24 + 36\, a + 6\, a^2 + 15\, a^3 - 36\, a^4 + 8\, a^2\, L\, R_0''\big) +\\
&\qquad a^2\, L\, R_1^2\, \big(-24 + 36\, a + 11\, a^2 + 15\, a^3 - 42\, a^4 + 16\, a^2\, L\, R_0''\big) +\\
&\qquad a\, L^2\, R_1\, \big(6\, \big(6 - 9\, a - 4\, a^2 - 4\, a^3 + 12\, a^4\big) - 20\, a^2\, L\, R_0'' - a^2\, L\, R_1''\big)\Big) +\\
&\quad a\, R_0^3\, \Big(a^5\, R_1^3\, (-1 + L\, R_0'') + a^2\, L\, R_1^2\, \big(18 - 27\, a + 9\, a^2 + 6\, a^4 - 5\, a^2\, L\, R_0''\big) +\\
&\qquad L^3\, \big(48 - 72\, a + 30\, a^2 - 9\, a^3 + 12\, a^4 - 4\, a^2\, L\, R_0''\big) +\\
&\qquad a\, L^2\, R_1\, \big(-48 + 72\, a - 19\, a^2 - 18\, a^4 + 8\, a^2\, L\, R_0'' + 3\, a^2\, L\, R_1''\big)\Big)\Big)
\end{aligned}
$$

In[193]:= **Simplify[tb12[a, 1 - a, L]]**

Out[193]= $$\frac{1}{a^7 R_0^4 R_1^3}\Big(12\,(-1+a)\,a\,L^3\,(-L+a\,R_1)^3 - 2\,a\,L^2\,R_0\,(L-a\,R_1)^2\,\big(a\,R_1\,(-3-3\,a+6\,a^2-2\,L\,R_0'')+2\,L\,(3+3\,a-6\,a^2+L\,R_0'')\big) - a^4\,R_0^4\,\big(-3\,(-2+a)\,L^2+a^2\,R_1^2+a\,L\,R_1\,(-2+L\,R_1'')\big) + L\,R_0^2\,\big(-2\,a^2\,L\,R_1^2\,(12-18\,a+a^2+7\,a^4-4\,a^2\,L\,R_0'')+a^3\,R_1^3\,(6-9\,a+2\,a^2+2\,a^4-2\,a^2\,L\,R_0'') + L^3\,(3\,(-8+12\,a-2\,a^2+a^3-4\,a^4)+4\,a^2\,L\,R_0'')+a\,L^2\,R_1\,(36-54\,a+24\,a^4-10\,a^2\,L\,R_0''-a^2\,L\,R_1'')\big) + a\,R_0^3\,\big(-6\,(-4+6\,a-4\,a^2+a^3)\,L^3+2\,a^2\,(6-9\,a+5\,a^2)\,L\,R_1^2-a^5\,R_1^3 + 2\,a\,L^2\,R_1\,(-2\,(6-9\,a+5\,a^2)+a^2\,L\,R_1'')\big)\Big)$$

In[194]:= **Simplify[tb03[a, 1 - a, L]]**

Out[194]= $$\frac{1}{3\,a^7\,R_0^3\,R_1^3}$$

$$L\,\Big(12\,(-1+a)\,a\,L^2\,(-L+a\,R_1)^3-2\,a\,L\,R_0\,(L-a\,R_1)^2\,\big(a\,R_1\,(-3+3\,a^2-2\,L\,R_0'')+2\,L\,(3-3\,a^2+L\,R_0'')\big) + a^3\,R_0^3\,\big(-3\,(-2+a)\,L^2+3\,(2-3\,a+a^2)\,R_1^2+a\,L\,R_1\,(-3+L\,R_1'')\big)+R_0^2\,\big(3\,(-2+a)^3\,L^3 - 3\,a^2\,(8-12\,a+5\,a^2)\,L\,R_1^2+3\,a^3\,(2-3\,a+a^2)\,R_1^3+a\,L^2\,R_1\,(6\,(6-9\,a+4\,a^2)-a^2\,L\,R_1'')\big)\Big)$$

3.3 Check that $D_P\,T$ is symplectic :

In[89]:= **Simplify[ta10[a, b, L] * tb01[a, b, L] - ta01[a, b, L] * tb10[a, b, L]]**

Out[89]= 1

4. Coordinate transformation $B^{-1}: (s, u) \mapsto (x, y)$

4.1 Define the a_{jk} and b_{jk}

In[94]:=
```
μ[a_, b_, L_] :=  Sqrt[-ta01[a, b, L] × tb10[a, b, L]]
λ[a_, b_, L_] := ta10[a, b, L] - ⅈ * μ[a, b, L]
η[a_, b_, L_] := (-ta01[a, b, L] / tb10[a, b, L]) ^ (1 / 4)

a10[a_, b_, L_] := ta10[a, b, L]
a01[a_, b_, L_] := ta01[a, b, L] * η[a, b, L] ^ (-2)
a30[a_, b_, L_] := ta30[a, b, L] * η[a, b, L] ^2
a21[a_, b_, L_] := ta21[a, b, L]
a12[a_, b_, L_] := ta12[a, b, L] * η[a, b, L] ^ (-2)
a03[a_, b_, L_] := ta03[a, b, L] * η[a, b, L] ^ (-4)

b10[a_, b_, L_] := tb10[a, b, L] * η[a, b, L] ^ (2)
b01[a_, b_, L_] := tb01[a, b, L]
b30[a_, b_, L_] := tb30[a, b, L] * η[a, b, L] ^4
b21[a_, b_, L_] := tb21[a, b, L] * η[a, b, L] ^2
b12[a_, b_, L_] := tb12[a, b, L]
b03[a_, b_, L_] := tb03[a, b, L] * η[a, b, L] ^ (-2)
```

5. Coordinate transformation $(x, y) \mapsto (z, w)$

5.1 Define the c_{jk}'s

In[109]:=
```
c30[a_, b_, L_] :=
  2^ (-3) * 1 / λ[a, b, L] * (a30[a, b, L] + ⅈ * b30[a, b, L] - ⅈ * a21[a, b, L] +
     b21[a, b, L] - a12[a, b, L] - ⅈ * b12[a, b, L] + ⅈ * a03[a, b, L] - b03[a, b, L]);
c21[a_, b_, L_] :=
  2^ (-3) * 1 / λ[a, b, L] * (3 * a30[a, b, L] + 3 * ⅈ * b30[a, b, L]  - ⅈ * a21[a, b, L] +
     b21[a, b, L] + a12[a, b, L] + ⅈ * b12[a, b, L] - 3 * ⅈ * a03[a, b, L] + 3 * b03[a, b, L]);
c12[a_, b_, L_] :=
  2^ (-3) * 1 / λ[a, b, L] * (3 * a30[a, b, L] + 3 ⅈ * b30[a, b, L] + ⅈ * a21[a, b, L] -
     b21[a, b, L] + a12[a, b, L] + ⅈ * b12[a, b, L] + 3 ⅈ * a03[a, b, L] - 3 * b03[a, b, L]);
c03[a_, b_, L_] :=
  2^ (-3)  * 1 / λ[a, b, L] * (a30[a, b, L] + ⅈ * b30[a, b, L] + ⅈ * a21[a, b, L] -
     b21[a, b, L] - a12[a, b, L] - ⅈ * b12[a, b, L] - ⅈ * a03[a, b, L] + b03[a, b, L]);
```

6. Coordinate transformation $(z, w) \mapsto (z', w')$

6.1 Define the d_{jk}' s

In[113]:=
```
d30[a_, b_, L_] := c30[a, b, L] / (1 - λ[a, b, L])
d21[a_, b_, L_] := c21[a, b, L]
d12[a_, b_, L_] := c12[a, b, L] / (1 - λ[a, b, L] ^ (-2))
d03[a_, b_, L_] := c03[a, b, L] / (1 - λ[a, b, L] ^ (-4))
```

7. Asymmetric Twist Coefficient

7.1 Expression for d_{21}

In[121]:=
```
Simplify[Simplify[d21[a, b, L]] /. {b → 1 - a}]
```

Out[121]=

$$\Bigg(\sqrt{-\frac{L R_0^2 (L - a R_1)}{(L - a R_0)(L - a R_0 - a R_1)}} \Bigg(-2\, i\, L (L - a R_1) \sqrt{-\frac{L R_0^2 (L - a R_1)}{(L - a R_0)(L - a R_0 - a R_1)}} + R_0 \Bigg(-2 L \Bigg(L - i\, a \sqrt{-\frac{L R_0^2 (L - a R_1)}{(L - a R_0)(L - a R_0 - a R_1)}}\Bigg) + a R_1 \Bigg(2 L - i\, a \sqrt{\frac{L R_0^2 (-L + a R_1)}{(L - a R_0)(L - a R_0 - a R_1)}}\Bigg)\Bigg)\Bigg)$$

$$\Big(-a(-4 + 3a) L R_1 (L - a R_1)^2 + a^2 R_0^3 \big((4 - 3a) a L + R_1 (6 - 9a + 2a^2 + a^2 L R_1'')\big) +$$
$$2 a R_0^2 \big(a(-4 + 3a) L^2 + a (6 - 9a + 2a^2) R_1^2 - L R_1 (12 - 18a + 4a^2 + a^2 L R_1'')\big) +$$
$$R_0 \big((4 - 3a) a L^3 + a^2 R_1^3 (6 - 9a + 2a^2 + a^2 L R_0'') - 2 a L R_1^2 (12 - 18a + 4a^2 + a^2 L R_0'') +$$
$$L^2 R_1 (24 - 36a + 8a^2 + a^2 L R_0'' + a^2 L R_1'')\big)\Big)\Bigg) \Bigg/ \Bigg(8 a^2 L R_0^3 R_1 (L - a R_1)^2$$

$$\Bigg(2 L (L - a R_1) + a R_0 \Bigg(-2 L + R_1 \Bigg(a - 2\, i\, a \sqrt{\frac{L(-L + a R_0)(-L + a R_1)(-L + a R_0 + a R_1)}{a^4 R_0^2 R_1^2}}\Bigg)\Bigg)\Bigg)\Bigg)$$

7.2 Simplify numerator of d_{21}

7.2.1 Simplify First Set of terms in numerator of

In[135]:=
$$\text{numerator} = \text{FullSimplify}\Big[\sqrt{\Big(-\big(\big(L R_0^2 (L - a R_1)\big) / \big((L - a R_0)(L - a R_0 - a R_1)\big)\big)\Big)}\ \Big(-2 i L (L - a R_1) \sqrt{\Big(-\big(\big(L R_0^2 (L - a R_1)\big) / \big((L - a R_0)(L - a R_0 - a R_1)\big)\big)\Big)} + R_0 \Big(-2 L \Big(L - i a \sqrt{\Big(-\big(\big(L R_0^2 (L - a R_1)\big) / \big((L - a R_0)(L - a R_0 - a R_1)\big)\big)\Big)}\Big) + a R_1 \Big(2 L - i a \sqrt{\big(\big(L R_0^2 (-L + a R_1)\big) / \big((L - a R_0)(L - a R_0 - a R_1)\big)\big)}\Big)\Big)\Big)\Big]$$

Out[135]=
$$\frac{L R_0^2 \left(2 i L (L - a R_1)^2 - i a R_0 (-2 L + a R_1)(-L + a R_1) + \frac{2 L R_0 (L - a R_1)^2}{\sqrt{\frac{L R_0^2 (-L + a R_1)}{(L - a R_0)(L - a (R_0 + R_1))}}}\right)}{(L - a R_0)(L - a (R_0 + R_1))}$$

7.2.2 Simplify Second Set of terms in numerator

In[134]:=
$$\text{delta} = \text{FullSimplify}\Big[\big(-a (-4 + 3 a) L R_1 (L - a R_1)^2 + a^2 R_0^3 \big((4 - 3 a) a L + R_1 \big(6 - 9 a + 2 a^2 + a^2 L R_1''\big)\big) + 2 a R_0^2 \big(a (-4 + 3 a) L^2 + a \big(6 - 9 a + 2 a^2\big) R_1^2 - L R_1 \big(12 - 18 a + 4 a^2 + a^2 L R_1''\big)\big) + R_0 \big((4 - 3 a) a L^3 + a^2 R_1^3 \big(6 - 9 a + 2 a^2 + a^2 L R_0''\big) - 2 a L R_1^2 \big(12 - 18 a + 4 a^2 + a^2 L R_0''\big) + L^2 R_1 \big(24 - 36 a + 8 a^2 + a^2 L R_0'' + a^2 L R_1''\big)\big)\big)\Big]$$

Out[134]=
$$-a (-4 + 3 a) L R_1 (L - a R_1)^2 + a^2 R_0^3 \big((4 - 3 a) a L + R_1 \big(6 + a (-9 + 2 a) + a^2 L R_1''\big)\big) + 2 a R_0^2 \big(a (-4 + 3 a) L^2 + R_1 \big((6 + a (-9 + 2 a))(-2 L + a R_1) - a^2 L^2 R_1''\big)\big) + R_0 \Big((4 - 3 a) a L^3 + R_1 \Big(4 (6 + a (-9 + 2 a)) L^2 + a \big(a R_1^2 \big(6 + a (-9 + 2 a) + a^2 L R_0''\big) - 2 L R_1 \big(12 + 2 a (-9 + 2 a) + a^2 L R_0''\big) + a L^3 (R_0'' + R_1'')\big)\Big)\Big)$$

7.3 Simplify denominator of d_{21}

In[128]:=
$$\text{Factor}\left[\left(8 a^2 L R_0^3 R_1 (L - a R_1)^2 \left(2 L (L - a R_1) + a R_0 \left(-2 L + R_1 \left(a - 2 i a \sqrt{\frac{L (-L + a R_0)(-L + a R_1)(-L + a R_0 + a R_1)}{a^4 R_0^2 R_1^2}}\right)\right)\right)\right)\right]$$

Out[128]=
$$8 a^2 L R_0^3 R_1 (L - a R_1)^2 \left(2 L^2 - 2 a L R_0 - 2 a L R_1 + a^2 R_0 R_1 - 2 i a^2 R_0 R_1 \sqrt{-\frac{L (L - a R_1)\left(L^2 - 2 a L R_0 + a^2 R_0^2 - a L R_1 + a^2 R_0 R_1\right)}{a^4 R_0^2 R_1^2}}\right)$$

In[129]:=
$$\text{Simplify}\left[2 L^2 - 2 a L R_0 - 2 a L R_1 + a^2 R_0 R_1\right]$$

Out[129]=
$$2 L (L - a R_1) + a R_0 (-2 L + a R_1)$$

In[130]:= `Simplify[L^2 - 2 a L R_0 + a^2 R_0^2 - a L R_1 + a^2 R_0 R_1]`

Out[130]= $(L - a R_0)(L - a R_0 - a R_1)$

In[132]:= `denominator = 8 a^2 L R_0^3 R_1 (L - a R_1)^2 (2 L (L - a R_1) + a R_0 (-2 L + a R_1) - 2 I Sqrt[-L (L - a R_1) (L - a R_0) (L - a R_0 - a R_1)]);`

7.4 Twist

In[136]:= `twist = (numerator / denominator) * delta`

Out[136]=
$$\left(\left(2 i L (L - a R_1)^2 - i a R_0 (-2 L + a R_1)(-L + a R_1) + \frac{2 L R_0 (L - a R_1)^2}{\sqrt{\frac{L R_0^2 (-L + a R_1)}{(L - a R_0)(L - a (R_0 + R_1))}}}\right)\right.$$
$$\Big(-a(-4+3a) L R_1 (L - a R_1)^2 + a^2 R_0^3 \left((4-3a) a L + R_1 \left(6 + a(-9+2a) + a^2 L R_1''\right)\right) +$$
$$2 a R_0^2 \left(a(-4+3a) L^2 + R_1 \left((6 + a(-9+2a))(-2L + a R_1) - a^2 L^2 R_1''\right)\right) +$$
$$R_0 \Big((4-3a) a L^3 + R_1 \Big(4(6 + a(-9+2a)) L^2 + a \Big(a R_1^2 \left(6 + a(-9+2a) + a^2 L R_0''\right) -$$
$$\left.2 L R_1 \left(12 + 2a(-9+2a) + a^2 L R_0''\right) + a L^3 (R_0'' + R_1'')\Big)\Big)\Big)\Big)\right) \Big/$$
$$\left(8 a^2 R_0 (L - a R_0) R_1 (L - a R_1)^2 (L - a(R_0 + R_1)) \left(2 L (L - a R_1) - 2 i \sqrt{-L (L - a R_0)(L - a R_1)(L - a R_0 - a R_1)} + a R_0 (-2L + a R_1)\right)\right)$$

```
(* After a long a tendious calculation by hand,
(numerator / denominator) simplifies to i/(8a^2R0R1(L-aR0)(L-aR1)(L-a(R0+R1))) *)
```

In[140]:= `delta`

Out[140]=
$$-a(-4+3a) L R_1 (L - a R_1)^2 + a^2 R_0^3 \left((4-3a) a L + R_1 \left(6 + a(-9+2a) + a^2 L R_1''\right)\right) +$$
$$2 a R_0^2 \left(a(-4+3a) L^2 + R_1 \left((6 + a(-9+2a))(-2L + a R_1) - a^2 L^2 R_1''\right)\right) +$$
$$R_0 \Big((4-3a) a L^3 + R_1 \Big(4(6 + a(-9+2a)) L^2 +$$
$$a \left(a R_1^2 \left(6 + a(-9+2a) + a^2 L R_0''\right) - 2 L R_1 \left(12 + 2a(-9+2a) + a^2 L R_0''\right) + a L^3 (R_0'' + R_1'')\right)\Big)\Big)$$

In[149]:= `Collect[delta, {R_0'', R_1''}]`

Out[149]=
$$(4-3a) a L^3 R_0 + 2 a^2 (-4+3a) L^2 R_0^2 + (4-3a) a^3 L R_0^3 + 4(6 + a(-9+2a)) L^2 R_0 R_1 +$$
$$6 a^2 R_0^3 R_1 + a^3 (-9+2a) R_0^3 R_1 - 24 a L R_0 R_1^2 - 4 a^2 (-9+2a) L R_0 R_1^2 + 6 a^2 R_0 R_1^3 +$$
$$a^3 (-9+2a) R_0 R_1^3 - a(-4+3a) L R_1 (L - a R_1)^2 + 2a(6 + a(-9+2a)) R_0^2 R_1 (-2L + a R_1) +$$
$$\left(a^2 L^3 R_0 R_1 - 2 a^3 L^2 R_0 R_1^2 + a^4 L R_0 R_1^3\right) R_0'' + \left(a^2 L^3 R_0 R_1 - 2 a^3 L^2 R_0^2 R_1 + a^4 L R_0^3 R_1\right) R_1''$$

Appendix C

Mathematica Code Minkowski Reflections

Part 3: Billiard Trajectories

3.1 Calculate new parameters

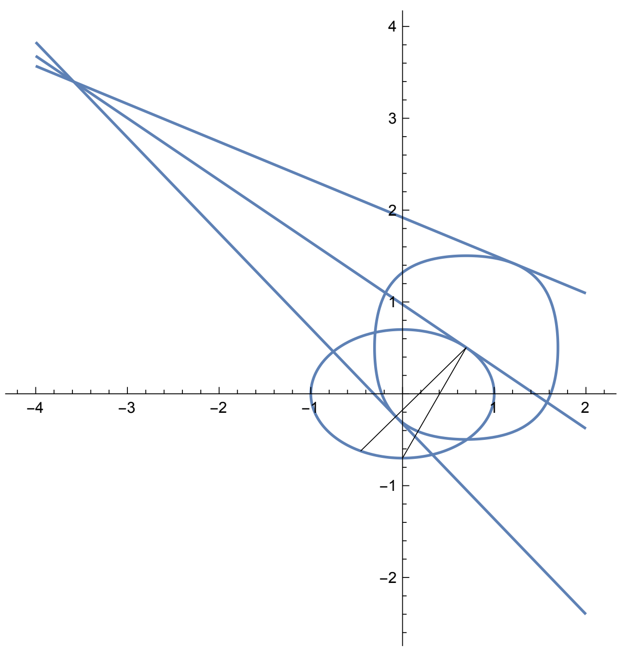
```
Clear[a, b, A, B, posi, r, Idx, u, tau, newPosi, newPara, ellislope,
 TpM, Islope, TvI, qxsolver, qx, qy, thetasolver, thetalist, newTheta,
 iteTheta, itePara, itePosi, Finsnorm, Finstang, F1, F2, U, newU, iteU]

(* Fixed parameters *)
a = 0.5; (* Euclidean part of Finsler metric *)
b = 1 - a; (* Deformation parameter of Finsler metric *)
A = 1; (* Ellipse semi-major axis *)
B = 0.7; (* Ellipse semi-minor axis *)

(* Position on ellipse *)
posi[ϕ_] := {A * Cos[ϕ], B * Sin[ϕ]};

(* Define the function r(theta) that determines indicatrix I *)
r[θ_] := 1 / (a + b (Cos[θ] ^4 + Sin[θ] ^4)) ^ (1 / 4)

(* Define the parametric coordinates for I at (x_k,y_k) *)
Idx[θ_, ϕ_] := {r[θ] * Cos[θ] + posi[ϕ][[1]] , r[θ] * Sin[θ] + posi[ϕ][[2]]}

(* Direction vector u in I*)
u[θ_] := {r[θ] * Cos[θ], r[θ] * Sin[θ]}

(* Determine Euclidean length between (x_k,y_k) and (x_k+1 ,y_k+1) *)
tau[θ_, ϕ_] := -2 (u[θ][[1]] * posi[ϕ][[1]] / A^2 + u[θ][[2]] * posi[ϕ][[2]] / B^2) /
   (u[θ][[1]]^2 / A^2 + u[θ][[2]]^2 / B^2)

(* Determine new point (x_k+1,y_k+1) and phi values *)
newPosi[θ_, ϕ_] := posi[ϕ] + tau[θ, ϕ] * u[θ]
newPara[θ_, ϕ_] :=
 If[newPosi[θ, ϕ][[2]] ≥ 0, ArcCos[newPosi[θ, ϕ][[1]] / A], 2 Pi - ArcCos[newPosi[θ, ϕ][[1]] / A]]

(* Calculate tangent line TpM to ellipse at (x_k+1,y_k+1) *)
ellislope[θ_, ϕ_] := -B * Cos[newPara[θ, ϕ]] / (A * Sin[newPara[θ, ϕ]])
TpM[θ_, ϕ_] := Simplify[ellislope[θ, ϕ] * (x - newPosi[θ, ϕ][[1]]) + newPosi[θ, ϕ][[2]]]

(* Calculate the tangent line TvI of I at v (here v is an arbitrary vector)*)
Islope[θ_, ϕ_] := D[Idx[z, ϕ][[2]], z] / D[Idx[z, ϕ][[1]], z] /. {z → θ}
TvI[θ_, ϕ_] :=
 Simplify[Islope[θ, ϕ] * (x - Idx[θ, newPara[θ, ϕ]][[1]]) + Idx[θ, newPara[θ, ϕ]][[2]]]
```

```
(* Calculate the point q where TpM and TvI intersect *)
qxsolver[θ_, ϕ_] := Solve[TpM[θ, ϕ] == TvI[θ, ϕ], x]
qx[θ_, ϕ_] := x /. qxsolver[θ, ϕ][[1]]
qy[θ_, ϕ_] := TpM[θ, ϕ] /. {x → qx[θ, ϕ]}

(* Calculate direction of reflected vector i.e. newtheta *)
(* newtheta is a solution of Iy(@) - qy = Islope(@)[Ix(@) - qx] *)
(* That is, solve for the theta value
 such that Islope goes through the point (qx,qy) *)
thetasolver[θ_, ϕ_] := NSolve[Idx[z, newPara[θ, ϕ]][[2]] - qy[θ, ϕ] ==
    Islope[z, newPara[θ, ϕ]] * (Idx[z, newPara[θ, ϕ]][[1]] - qx[θ, ϕ]) && 0 < z < 2 Pi, z]
thetalist[θ_, ϕ_] := {z /. thetasolver[θ, ϕ][[1]], z /. thetasolver[θ, ϕ][[2]]}
newTheta[θ_, ϕ_] :=
 If[thetalist[θ, ϕ][[1]] == N[θ], thetalist[θ, ϕ][[2]], thetalist[θ, ϕ][[1]]]



(* Iterate *)
iteTheta[θ_, ϕ_, 0] := θ;
itePara[θ_, ϕ_, 0] := ϕ;

iteTheta[θ_, ϕ_, n_] :=
  iteTheta[θ, ϕ, n] = newTheta[iteTheta[θ, ϕ, n - 1], itePara[θ, ϕ, n - 1]];
itePara[θ_, ϕ_, n_] :=
  itePara[θ, ϕ, n] = newPara[iteTheta[θ, ϕ, n - 1], itePara[θ, ϕ, n - 1]];

itePosi[θ_, ϕ_, n_] := itePosi[θ, ϕ, n] = posi[itePara[θ, ϕ, n]]

(* Finsler Parameters *)
Finsnorm[v__] := a * Norm[v] + b * (v[[1]]^4 + v[[2]]^4)^(1 / 4)
Finstang[ϕ_] := {-A * Sin[ϕ], B * Cos[ϕ]} / Finsnorm[{-A * Sin[ϕ], B * Cos[ϕ]}]

F1[v__] := a * v[[1]] / Sqrt[v[[1]]^2 + v[[2]]^2] + b * v[[1]]^3 (v[[1]]^4 + v[[2]]^4)^(-3 / 4)
F2[v__] := a * v[[2]] / Sqrt[v[[1]]^2 + v[[2]]^2] + b * v[[2]]^3 (v[[1]]^4 + v[[2]]^4)^(-3 / 4)

U[θ_, ϕ_] := -F1[newPosi[θ, ϕ] - posi[ϕ]] * Finstang[ϕ][[1]] -
  F2[newPosi[θ, ϕ] - posi[ϕ]] * Finstang[ϕ][[2]]
newU[θ_, ϕ_] := -F1[newPosi[θ, ϕ] - posi[ϕ]] * Finstang[newPara[θ, ϕ]][[1]] -
  F2[newPosi[θ, ϕ] - posi[ϕ]] * Finstang[newPara[θ, ϕ]][[2]]
iteU[θ_, ϕ_, n_] := iteU[θ, ϕ, n] = U[iteTheta[θ, ϕ, n], itePara[θ, ϕ, n]]
```

3.2 Check code

In[671]:=
```
ellipse = ParametricPlot[{A * Cos[θ], B * Sin[θ]}, {θ, 0, 2 Pi}, PlotRange → All];
I1 = ParametricPlot[{Idx[θ, newPara[Pi / 3, 3 Pi / 2]][[1]],
    Idx[θ, newPara[Pi / 3, 3 Pi / 2]][[2]]}, {θ, 0, 2 Pi}, PlotRange → All];
TvI1 = Plot[TvI[Pi / 3, 3 Pi / 2], {x, -4, 2}];
TpM1 = Plot[TpM[Pi / 3, 3 Pi / 2], {x, -4, 2}];

(* Equation of TvI where v reflected ray *)
x2 = qx[Pi / 3, 3 Pi / 2];
y2 = qy[Pi / 3, 3 Pi / 2];
m2 = Islope[newTheta[Pi / 3, 3 Pi / 2], newPara[Pi / 3, 3 Pi / 2]];
TvI2 = Plot[m2 * (x - x2) + y2, {x, -4, 2}];

(* Incident and reflected rays *)
start1 = posi[3 Pi / 2];
stop1 = newPosi[Pi / 3, 3 Pi / 2];
start2 = newPosi[Pi / 3, 3 Pi / 2];
stop2 = newPosi[newTheta[Pi / 3, 3 Pi / 2], newPara[Pi / 3, 3 Pi / 2]];

l1 = Graphics[Line[{start1, stop1}]];
l2 = Graphics[Line[{start2, stop2}]];

Show[ellipse, I1, TvI1, TvI2, TpM1, l1, l2]
```

Out[685]=

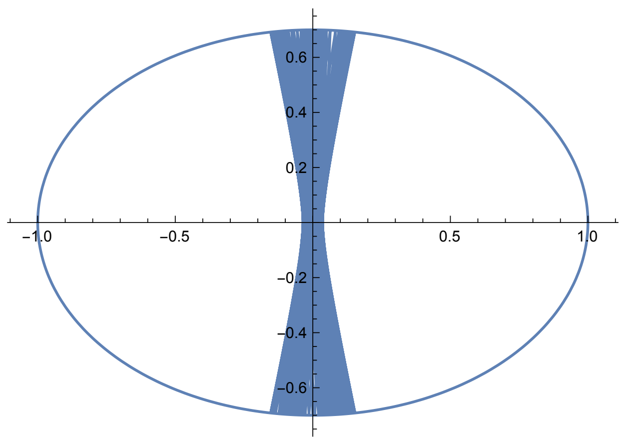


3.3 Billiard Orbits

In[72]:=
```
Clear[ellipse, billiards, elliFinPhase]

ellipse = ParametricPlot[{A * Cos[θ], B * Sin[θ]}, {θ, 0, 2 Pi}, PlotRange → All];
billiards[θ_, ϕ_, m_] :=
 ListLinePlot[Table[itePosi[θ, ϕ, n], {n, 0, m}], PlotStyle → PointSize[0.01]]
elliFinPhase[θ_, ϕ_, m_] :=
 ListPlot[ParallelTable[{itePara[θ, ϕ, n], iteU[θ, ϕ, n]}, {n, 0, m}],
  PlotStyle → PointSize[0.01], PlotRange → All]
PhaseList[θ_, ϕ_, m_] := Table[{itePara[θ, ϕ, n], iteU[θ, ϕ, n]}, {n, 0, m}]
```

In[688]:=
```
Show[ellipse, billiards[Pi / 2 + .05, 3 Pi / 2, 100]]
ListPlot[PhaseList[Pi / 2 + .05, 3 Pi / 2, 100]]
```

Out[688]=



Out[689]=