JET SCHEME AND SIMPLE ISOLATED HYPERSURFACE SINGULARITIES

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Abstract. Let $m \geq 2$ be a positive integer. Let $(H(f), \mathfrak{o})$ be the analytic germ of hypersurface attached to a weighted-homogeneous polynomial $f \in \mathbf{C}[x_1, \ldots, x_m]$. In this article, we mainly characterize the simpleness of any isolated analytic quasi-homogeneous complex hypersurface singularities $(\mathscr{H}, \mathfrak{o})$ in $\mathbf{C}^m$ in terms of the associated jet schemes. We also compute the dimension of the jet scheme $\mathscr{L}_n(H)_{\mathfrak{o}}$, formed by the n-jets centered at $\mathfrak{o}$, for every positive integer $n >> 0$, which is the crucial technical part of our simpleness criterion.

1. Introduction

1.1. Let $m \geq 2$ be a positive integer. The *simple hypersurface singularities* form an important subclass of (reduced) quasi-homogeneous germs of analytic hypersurfaces ($F \in \mathbf{C}\{x_1, \ldots, x_m\}, \mathfrak{o} \in \mathbf{C}^m$) which has been deeply studied. Such an analytic singularity class is defined by the property to have, locally around the origin $\mathfrak{o}$ for the complex topology, a finite number of orbits for the right-equivalence (e.g., see [4, 5, 11, 13] for definitions and properties). Following the classical classification, an analytic isolated hypersurface singularity is simple if and only if it is equivalent (in a neighborhood of the origin $\mathfrak{o} \in \mathbf{C}^m$) to the hypersurface associated with one of the weighted-homogeneous polynomials $f(x_1, \ldots, x_m) = g(x_1, x_2) + x_3^2 + \ldots + x_m^2$, where g is a polynomial in the following list:

- (A_p) ($p \geq 1$ when $\mathfrak{o}$ is singular) $g(x_1, x_2) = x_1^{p+1} + x_2^2$.
- (D_p) ($p \geq 4$) $g(x_1, x_2) = x_1 x_2^2 + x_1^{p-1}$.
- (E_6) $g(x_1, x_2) = x_1^3 + x_2^4$.
- (E_7) $g(x_1, x_2) = x_1^3 + x_1 x_2^3$.
- (E_8) $g(x_1, x_2) = x_1^3 + x_2^5$.

In particular, let us stress that the origin $\mathfrak{o}$ is the unique singular point of $H(f)$; each polynomial g defines an affine plane curve with a single isolated singularity at the origin which is also simple. We call *A-D-E polynomials* the weighted-homogeneous polynomials g (and f) described in the list above.

1.2. Let k be a field. Let V be a k-variety, i.e., a k-scheme of finite type. Let us recall that one defines the *jet scheme* $\mathscr{L}_n(V)$ of level $n \in \mathbf{N}$ associated with V by the functorial property

$$\mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(A), \mathscr{L}_n(V)) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}(A[[T]]/\langle T^{n+1}\rangle), V) \tag{1.1}$$

for every k-algebra A. For every pair of integers $n' \geq n$, the natural bijection above induces the existence of a *truncation morphism* that we denote by $\pi_{n,n'} \colon \mathscr{L}_{n'}(V) \to \mathscr{L}_n(V)$. The morphisms $(\pi_{n,n'})_{n,n'}$ form a projective system in the category of k-schemes.

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1.3. Many characterizations for simpleness have been obtained, see, e.g., [4, 5, 11, 13]. One result of this article is to provide a dimensional criterion for the simpleness of the quasi-homogeneous isolated hypersurface singularities in terms of jet schemes. Precisely, we establish the following criterion, proved in section 4 (see subsection 2.2 for the definition of a weighted homogeneous polynomial and related terminology):

Theorem 1.4. *Let k be a field with characteristic zero embedded into the field of complex numbers. Let $m \geq 2$ be a positive integer. Let $f \in k[x_1, \ldots, x_m]$ be a weighted-homogeneous polynomial with* reduced *total weight $(w_1, \ldots, w_m; \delta)$ and associated hypersurface $H := V(f) \subset \mathbf{A}_k^m$. Let $\mathfrak{o} \in \mathbf{A}_k^m$ be the origin. We assume that the hypersurface H has a single isolated singularity at $\mathfrak{o}$. If $m \geq 3$, we assume that the hypersurface $H \otimes_k \mathbf{C}$ is integral. Then, the analytic hypersurface singularity $(H \otimes_k \mathbf{C}, \mathfrak{o} \in \mathbf{C}^m)$ is simple if and only if condition (A) or condition (B) is satisfied:*

(A) We have $m \geq 3$ and one of the following equivalent conditions is satisfied:
- *(a) For every integer $n \geq \delta$, we have $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) \leq 2|w| + (m-1)(n-\delta)$.*
- *(b) There exists an integer $n \geq \delta$ such that $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) \leq 2|w| + (m-1)(n-\delta)$.*
- *(c) We have $\dim(\pi_{0,\delta}^{-1}(\mathfrak{o})) \leq 2|w|$.*

(B) We have $m = 2$, and one of the following equivalent conditions is satisfied:
- *(i) For every integer $n \geq \delta$, we have $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) \leq \delta + |w| + \dim(\mathscr{L}_{n-\delta}(H)) - 1$.*
- *(ii) There exists an integer $n \geq \delta$ such that $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) \leq \delta + |w| + \dim(\mathscr{L}_{n-\delta}(H)) - 1$.*
- *(iii) We have $\dim(\pi_{0,\delta}^{-1}(\mathfrak{o})) \leq \delta + |w| + (m-2)$.*

The proof of theorem 1.4 is crucially based on dimensional computations linked to the dimensional estimate for the k-scheme $\pi_{0,n}^{-1}(\mathfrak{o})$ (e.g., see theorem 3.1). In particular, we introduce the *cyclicality default* $\mathrm{dft}_n(f)$ in subsection 3.7 which will play an important role in our criterion. Section 3, devoted to this study, can be considered independently from theorem 1.4. Let us point out that previous studies on the k-scheme $\pi_{0,n}^{-1}(\mathfrak{o})$ and its dimension have been conducted in [3, 8, 9], and specifically in the case of surface simple singularities, in [7].

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2. Conventions, notation

2.1. In this article, k is a field; a *k-variety* is a k-scheme of finite type. For every positive integer $m \geq 1$, every affine k-variety $V \subset \mathbf{A}_k^m = \mathrm{Spec}(k[x_1, \ldots, x_m])$ with $\mathcal{O}(V) = k[x_1, \ldots, x_m]/I$, and every integer $n \in \mathbf{N}$, a presentation of the k-scheme $\mathscr{L}_n(V)$ in $\mathscr{L}_n(\mathbf{A}_k^m) = \mathrm{Spec}(k[x_{i,j} : i \in \{1, \ldots, m\}, j \in \{0, \ldots, n\}])$ can be obtained as follows. Let us observe that, for every polynomial $f \in k[x_1, \ldots, x_m]$, there exists a unique family $(f_s)_{s \in \{0,\ldots,n\}}$ of polynomials in $k[x_{i,j} : i \in \{1, \ldots, m\}, j \in \{0, \ldots, n\}]$ such that the following equality holds:

$$f\left(\left(\sum_{j=0}^{n} x_{i,j} T^j\right)_{i \in \{1,\ldots,m\}}\right) = \sum_{s=0}^{n} f_s\left((x_{i,j})_{\substack{i \in \{1,\ldots,m\} \\ j \in \{0,\ldots,s\}}}\right) T^s \pmod{T^{n+1}}. \tag{2.1}$$

We define the ideal I_n of the ring $k[x_{i,j} : i \in \{1, \ldots, m\}, j \in \{0, \ldots, n\}]$ to be $I_n := \langle f_s : f \in I,\ s \in \{0, \ldots, n\} \rangle$. Then [1, Propositions 6.1.1 and 6.5.1] ensure that $\mathscr{L}_n(V) = \mathrm{Spec}(k[x_{i,j} : i \in \{1, \ldots, m\}, j \in \{0, \ldots, n\}]/I_n)$.

2.2. Let k be a field. Let $m \geq 2$ be a positive integer. A polynomial $f \in k[x_1, \dots, x_m]$ is said *weighted-homogeneous* if there exists an $(m+1)$-tuple of integers $(w_1, \dots, w_m; \delta)$ such that

$$f(t^{w_1}x_1, \dots, t^{w_m}x_m) = t^{\delta} f(x_1, \dots, x_m) \in k[t, x_1, \dots, x_m]. \tag{2.2}$$

The integers w_i are called *weights* and the $(m+1)$-tuple $(w_1, \dots, w_m; \delta)$ the *total weight* of f. Because of formula (2.2), every weighted-homogeneous polynomial vanishes at the origin $\mathfrak{o}$. If the origin $\mathfrak{o} \in \mathbf{A}_k^m$ is regular, then there exists an integer $i \in \{1, \dots, m\}$ such that $w_i/\delta = 1$; otherwise, one must have $\delta > w_i$ for every integer $i \in \{1, \dots, m\}$. We say that a total weight $(w_1, \dots, w_m; \delta)$ is *reduced* if $gcd(w_1, \dots, w_m) = 1$.

2.3. Let $(F \in \mathbf{C}\{x_1, \dots, x_m\}, \mathfrak{o} \in \mathbf{C}^m)$ be an analytic power series (at the origin) with $F(\mathfrak{o}) = 0$. Functorial property (1.1) can be easily extended for $(F, \mathfrak{o})$: if $H(F)$ is the analytic hypersurface associated with F, then the space $\mathscr{L}_n(H(F))_{\mathfrak{o}}$ whose A-points are the parametrizations $\varphi(T) \in (A[[T]]/\langle T^{n+1}\rangle)^m$ of F, with $\varphi_i(T) \in T(A[[T]]/\langle T^{n+1}\rangle)$ for every integer $i \in \{1, \dots, m\}$ and every $\mathbf{C}$-algebra A, is a k-scheme, by mimicking subsection 2.1, which takes the role of the fiber of $\pi_{0,n}$ over $\mathfrak{o}$. Recall that an analytic germ $(H(F), \mathfrak{o})$ is said *quasi-homogeneous* if there exists a weighted-homogeneous polynomial f with total weight $(w_1, \dots, w_m; \delta)$ such that the analytic hypersurfaces associated with F and f are (locally) analytically isomorphic. Because of the very definitions and functoriality, this isomorphism induces an isomorphism between the complex algebraic varieties $\mathscr{L}_n(H(F))_{\mathfrak{o}}$ and $\pi_{0,n}^{-1}(\mathfrak{o}) \subset \mathscr{L}_n(H(f))$ for every positive integer $n \geq 1$. If the analytic germ $(H(F), \mathfrak{o})$ is moreover assumed to have a simple singularity (at the origin), then one knows that $H(f)$ has a simple isolated singular point at the origin. Because of these observations, we easily deduce that the isolated quasi-homogeneous analytic hypersurface singularity $(H(F), \mathfrak{o})$ is simple if and only if $\dim(\mathscr{L}_n(H(F))_{\mathfrak{o}})$ is bounded by the corresponding upper bound provided by theorem 1.4 applied to $\pi_{0,n}^{-1}(H(f))$.

2.4. Let $m \geq 2$ be a positive integer. Let k a field and let $H \subset \mathbf{A}_k^m$ be an algebraic hypersurface with a single isolated singularity at the origin $\mathfrak{o}$. Let us note that the singular locus $\mathrm{Sing}(H)$ is then supported by the origin $\mathfrak{o}$ and the regular locus $\mathrm{Sm}(H)$ equals $H \setminus \mathfrak{o}$. The following observation will be used throughout the article, and it highlights the interest of studying the dimension of $\pi_{0,n}^{-1}(\mathfrak{o})$: for every integer $n \in \mathbf{N}$, the decomposition

$$\mathscr{L}_n(H) = \pi_{0,n}^{-1}(\mathfrak{o}) \cup \overline{\pi_{0,n}^{-1}(\mathrm{Sm}(H))}$$

implies that $\dim(\mathscr{L}_n(H)) = \max(\dim(\pi_{0,n}^{-1}(\mathfrak{o})), \dim(\pi_{0,n}^{-1}(\mathrm{Sm}(H))))$. Furthermore, one has

$$\dim(\pi_{0,n}^{-1}(\mathrm{Sm}(H))) = \mathscr{L}_n(\mathrm{Sm}(H)) = (m-1)(n+1).$$

2.5. Let us mention a consequence of [9] that we will use several times. By [9, Proposition 1.4, Theorem 3.2, Remark 2.2], one knows that the origin $\mathfrak{o}$ is a rational singularity if and only if one has $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) < \dim(\mathscr{L}_n(H))$, for every integer $n \geq \delta$, whenever H is assumed to be irreducible. In this case, by [9, Theorem 3.3], one knows that $\dim(\mathscr{L}_{n'}(H(f))) = (m-1)(n'+1)$ for every integer $n' \in \mathbf{N}$.

2.6. Lastly, recall that, if $m = 3$, every A-D-E polynomial f generates an algebraic hypersurface in $\mathbf{A}_{\mathbf{C}}^3$ which has a rational singularity at the origin $\mathfrak{o}$ (e.g., see [5, 10] for details and references).

3. The dimension of $\pi_{0,n}^{-1}(\mathfrak{o})$

The aim of this section consists in proving various results mainly collected in theorem 3.1 and linked to the computation, for every sufficiently large integer n, of the dimension of $\pi_{0,n}^{-1}(\mathfrak{o})$. Moreover, we also study the integer $\mathrm{dft}_n(f) := \dim(\pi_{0,n}^{-1}(\mathfrak{o})) - \dim(\pi_{\delta,n}^{-1}(C_w))$ (see subsection 3.7

for the precise definition) and prove that this integer, depending on $\dim(\pi_{0,n}^{-1}(\mathfrak{o}))$, $\dim(\mathscr{L}_{n-\delta}(H))$ and the total weight of f, can characterize the geometry of the germ $(H, \mathfrak{o})$.

Theorem 3.1. *Let k be a field with characteristic zero embedded into the field of complex numbers. Let $m \geq 2$ be a positive integer. Let $f \in k[x_1, \dots, x_m]$ be a weighted-homogeneous polynomial with* reduced *total weight $(w_1, \dots, w_m; \delta)$ and associated hypersurface $H := V(f) \subset \mathbf{A}_k^m$. Let $\mathfrak{o} \in \mathbf{A}_k^m$ be the origin. We assume that the hypersurface H has a* simple *single isolated singularity at $\mathfrak{o}$. Let $n \geq \delta$ be a positive integer.*

(1) If $m \geq 3$ then we have $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = 1 + (m-1)n$.
(2) If $m = 2$ then we have $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = 2\delta - |w| + \dim(\mathscr{L}_{n-\delta}(H))$.

In particular, if $m \geq 3$ then the codimension of $\dim(\pi_{0,n}^{-1}(\mathfrak{o}))$ in $\mathscr{L}_n(\mathbf{A}_k^m)$ equals $n + m - 1$.

3.2. Let us note that, for every k-variety V and every integer $n \in \mathbf{N}$, one has $\mathscr{L}_n(V \otimes_k \mathbf{C}) \cong \mathscr{L}_n(V) \otimes_k \mathbf{C}$; hence, the involved dimensions in theorem 3.1 are not modified by this base-change. Then we may assume that $k = \mathbf{C}$ for proving theorem 3.1. By subsection 2.3, we may also assume that f is an A-D-E polynomial for computing $\dim(\pi_{0,n}^{-1}(\mathfrak{o}))$. (See also corollary 3.10 and propositions 3.12, 3.15.)

3.3. Let $m \geq 3$ be an integer. Let k be a field with *arbitrary* characteristic. Let $h \in k[x_1, \dots, x_{m-2}]$ be a non-constant polynomial and $f = h(x_1, \dots, x_{m-2}) + x_{m-1}x_m$. We show that, for every integer $n \geq 2$, we have $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = 1 + (m-1)n$. Let us set

$$D_{\mathfrak{o}}^1(x_{m,1}) := \mathrm{Spec}(\mathcal{O}(\pi_{0,n}^{-1}(\mathfrak{o}))[z]/\langle z x_{m,1} - 1\rangle) \text{ and } V_{\mathfrak{o}}^1(x_{m,1}) := \mathrm{Spec}(\mathcal{O}(\pi_{0,n}^{-1}(\mathfrak{o}))/\langle x_{m,1}\rangle).$$

Of course, we have $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = \max(\dim(D_{\mathfrak{o}}^1(x_{m,1})), \dim(V_{\mathfrak{o}}^1(x_{m,1})))$. Let us note that, with the notation of subsection 2.1, for every integer $s \in \mathbf{N}$, we have

$$f_s = h_s + \sum_{j_1 + j_2 = s} x_{m-1,j_1} x_{m,j_2}. \tag{3.1}$$

$\circ$ *We prove that* $\dim(D_{\mathfrak{o}}^1(x_{m,1})) = 1 + (m-1)n$. Recall that we have assumed $n \geq 2$. Because of the particular form of equation (3.1), we observe that the equation f_s, for every integer $s \geq 2$, becomes equivalent to

$$x_{m-1,s-1} = x_{m,1}^{-1}\left(f_s - h_s - \sum_{\substack{j_1+j_2=s \\ j_1 < s-1}} x_{m-1,j_1} x_{m,j_2}\right)$$

in the k-algebra $\mathcal{O}(D_{\mathfrak{o}}^1(x_{m,1}))$. Thus, we conclude that

$$\mathcal{O}(D_{\mathfrak{o}}^1(x_{m,1})) \cong k[x_{i,j}, x_{m-1,n}, z;\ i \in \{1, \dots, m\}, i \neq m-1,\ j \in \{1, \dots, n\}]/\langle x_{m,1} z - 1\rangle$$

and $\dim(D_{\mathfrak{o}}^1(x_{m,1})) = 1 + (m-1)n$.

$\circ$ *We prove that* $\dim(V_{\mathfrak{o}}^1(x_{m,1})) \leq 1 + (m-1)n$. If $n = 2$, then

$$\mathcal{O}(V_{\mathfrak{o}}^1(x_{m,1})) \cong k[x_{i,j};\ i \in \{1, \dots, m\},\ j \in \{1, 2\}]/\langle x_{m,1}, f_1, f_2\rangle$$

and $\dim(V_{\mathfrak{o}}^1(x_{m,1})) \leq 2m - 1$, which equals $1 + (m-1)n$ in this case. Now assume $n \geq 3$, then the proof is based on an iteration of the previous argument. Let us set

$$D_{\mathfrak{o}}^2(x_{m,2}) := \mathrm{Spec}(\mathcal{O}(\pi_{0,n}^{-1}(\mathfrak{o}))[z]/\langle x_{m,1}, z x_{m,2} - 1\rangle) \text{ and } V_{\mathfrak{o}}^2(x_{m,2}) := \mathrm{Spec}(\mathcal{O}(\pi_{0,n}^{-1}(\mathfrak{o}))/\langle x_{m,1}, x_{m,2}\rangle).$$

Using the same argument, we observe that the equation f_s, for every integer $s \geq 3$, becomes equivalent to

$$x_{m-1,s-2} = x_{m,2}^{-1}\left(f_s - h_s - \sum_{\substack{j_1+j_2=s\\ j_1<s-2}} x_{m-1,j_1}x_{m,j_2}\right)$$

in the k-algebra $\mathcal{O}(D_{\mathfrak{o}}^2(x_{m,2}))$. Then, for the same reason, we deduce that $\dim(D_{\mathfrak{o}}^2(x_{m,2})) \leq 1+(m-1)n$. (Let us stress that the equation f_2 becomes equivalent to $h_2 = 0$, and then could decrease our bound at most by one considering the Hauptidealsatz; this explains the existence of a simple inequality.) We are reduced to compute $\dim(V_{\mathfrak{o}}^2(x_{m,2}))$ and we iterate the process up to $x_{m,n-1}$ to conclude.

Corollary 3.4. *Let $m \geq 1$ be a positive integer. Let k be an algebraically closed field with characteristic zero. Let $h \in k[x_1,\dots,x_m]$ be an irreducible polynomial and $f = h + yz \in k[x_1,\dots,x_m,y,z]$. Let us assume that the hypersurface $H(h)$, attached to h, has a single singularity at the origin in $\mathbf{A}_k^m$. Then $H(f)$ has a rational singularity at the origin.*

Proof. We have shown that, for every integer $n \geq 2$, we have $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = 1+(m+1)n$ which is strictly smaller than $(m+1)(n+1) = \mathscr{L}_n(\mathrm{Sm}(H(f)))$. If $n = 1$ we check that $\pi_{0,1;f}^{-1}(\mathfrak{o}) \cong \mathbf{A}_k^{m+2}$, since $\mathfrak{o}$ is a singular point of $H(f)$; hence, we deduce that $\dim(\pi_{0,1;f}^{-1}(\mathfrak{o})) \leq m+2$, which is strictly smaller than $2(m+1) = \mathscr{L}_1(\mathrm{Sm}(H(f)))$ for $m \geq 1$. Then we conclude that H has rational singularities by subsection 2.5. □

3.5. Let $m \geq 2$ be a positive integer. Let k a field and $f \in k[x_1,\dots,x_m]$ be a weighted-homogeneous polynomial with associated hypersurface $H := H(f)$ and total weight $(w_1,\dots,w_m;\delta)$. Let C_w be the closed subset of $\mathscr{L}_\delta(H)$ defined by the datum of the ideal generated, for every $i \in \{1,\dots,m\}$, by the elements $x_{i,j}$ for every $j \in \{0,\dots,w_i-1\}$, in the ring $\mathcal{O}(\mathscr{L}_\delta(H))$. In particular, $C_w \subseteq \pi_{0,\delta}^{-1}(\mathfrak{o})$ and then $\pi_{\delta,n}^{-1}(C_w) \subseteq \pi_{0,n}^{-1}(\mathfrak{o}) = \pi_{\delta,n}^{-1}(\pi_{0,\delta}^{-1}(\mathfrak{o}))$ for every integer $n \geq \delta$. Recall that one has the following structure theorem:

Theorem 3.6 ([12])**.** *Let k be a field. Let $m \in \mathbf{N}$ be a positive integer. For every integer $n \geq \delta$, there is an isomorphism of k-schemes $\pi_{\delta,n}^{-1}(C_w) \to \mathscr{L}_{n-\delta}(H) \times_k \mathbf{A}_k^{m\delta-|w|}$.*

3.7. For every integer $n \geq \delta$, we introduce the following integer:

$$\mathrm{dft}_n(f) := \dim(\pi_{0,n}^{-1}(\mathfrak{o})) - \dim(\pi_{\delta,n}^{-1}(C_w)) \geq 0. \tag{3.2}$$

This integer, that we call *cyclicality default*, has the following properties:

Proposition 3.8. *Let k be a field with characteristic zero and let $m \geq 3$. Let $H \subset \mathbf{A}_k^m$ be an algebraic hypersurface with a single isolated singularity at the origin $\mathfrak{o}$. Then, for every integer $n \geq 1$,*

(1) we have $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) \geq n(m-1)+1$.

Let us assume in addition that H is associated with the datum of an irreducible weighted-homogeneous polynomial $f \in k[x_1,\dots,x_m]$, with reduced *total weight $(w_1,\dots,w_m;\delta)$. Then, for every integer $n \geq \delta$,*

(2) we have $\mathrm{dft}_n(f) \geq |w| - \delta - (m-2)$.

We assume that the field k is algebraically closed. Under the latter assumptions, for every integer $n \geq \delta$, we have the equality in (1) if and only if we have the equality in (2). Besides, if the equality holds, the origin $\mathfrak{o}$ is a rational singularity of H.

Proof. Since $\mathfrak{o} \in \mathrm{Sing}(H)$, we observe that $\pi_{0,1}^{-1}(\mathfrak{o}) = \mathbf{A}_k^m$. For $n \geq 2$, the k-scheme $\pi_{0,n}^{-1}(\mathfrak{o})$ is defined as the k-variety $\mathrm{Spec}(\mathcal{O}(\pi_{0,n-1}^{-1}(\mathfrak{o}))[x_{1,n}, \ldots, x_{m,n}]/\langle f_n \rangle)$. We conclude from the Hilbert Hauptidealsatz that, for every positive integer $n' \leq n$,

$$\dim(\pi_{0,n}^{-1}(\mathfrak{o})) \geq \dim(\pi_{0,n'}^{-1}(\mathfrak{o})) + (n-n')(m-1). \tag{3.3}$$

(Let us stress that this kind of elementary observations has already been given, e.g., in [9, Proposition 1.6]). Let us prove assertion (1). For $n' = 1$ formula (3.3) yields

$$\dim(\pi_{0,n}^{-1}(\mathfrak{o})) \geq \dim(\pi_{0,1}^{-1}(\mathfrak{o})) + (n-1)(m-1) = m + (n-1)(m-1) = n(m-1)+1.$$

Let us prove assertion (2). We assume that f is weighted-homogeneous and $n \geq \delta$. By the very definitions and theorem 3.6, we have

$$\begin{aligned} \mathrm{dft}_n(f) &= \dim(\pi_{0,n}^{-1}(\mathfrak{o})) - \dim(\pi_{\delta,n}^{-1}(C_w)) \\ &= \dim(\pi_{0,n}^{-1}(\mathfrak{o})) - m\delta + |w| - \dim(\mathscr{L}_{n-\delta}(H)). \end{aligned} \tag{3.4}$$

By subsection 2.4, we have two possibilities. If $\dim(\mathscr{L}_{n-\delta}(H)) = (m-1)(n-\delta+1)$, then part (1) and (3.4) imply

$$\begin{aligned} \mathrm{dft}_n(f) &= \dim(\pi_{0,n}^{-1}(\mathfrak{o})) - (m\delta - |w| + (m-1)(n-\delta+1)) \\ &\geq n(m-1) + 1 - (m\delta - |w| + (m-1)(n-\delta+1)) \\ &= |w| - \delta - (m-2). \end{aligned} \tag{3.5}$$

Otherwise, we must have $\dim(\mathscr{L}_{n-\delta}(H)) = \dim(\pi_{0,n-\delta}^{-1}(\mathfrak{o}))$. Let us observe that (3.3) with $n' = n-\delta$ is equivalent to $\dim(\pi_{0,n-\delta}^{-1}(\mathfrak{o})) \leq \dim(\pi_{0,n}^{-1}(\mathfrak{o})) - \delta(m-1)$. Using this fact, equation (3.4) implies

$$\begin{aligned} \mathrm{dft}_n(f) &= \dim(\pi_{0,n}^{-1}(\mathfrak{o})) - m\delta + |w| - \dim(\pi_{0,n-\delta}^{-1}(\mathfrak{o})) \\ &\geq \dim(\pi_{0,n}^{-1}(\mathfrak{o})) - m\delta + |w| - \dim(\pi_{0,n}^{-1}(\mathfrak{o})) + \delta(m-1) \\ &= |w| - \delta \\ &> |w| - \delta - (m-2). \end{aligned}$$

This proves assertion (2). Let us stress that we have in particular shown that, if $\dim(\mathscr{L}_{n-\delta}(H)) = \dim(\pi_{0,n-\delta}^{-1}(\mathfrak{o}))$ then $\mathrm{dft}_n(f) > |w| - \delta - (m-2)$. We assume now that the field k is algebraically closed; let us prove the end of the proposition. If $\mathrm{dft}_n(f) = |w| - \delta - (m-2)$, then we must have $\dim(\mathscr{L}_{n-\delta}(H)) = (m-1)(n-\delta+1) > \dim(\pi_{0,n-\delta}^{-1}(\mathfrak{o}))$ by the former remark. Then in (3.5) the equality holds, which implies that $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = n(m-1)+1$. Conversely, we assume $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = n(m-1)+1$. Since $m \geq 3$, this integer is strictly smaller than $(m-1)(n+1)$. Then, by subsection 2.5, we deduce that the origin $\mathfrak{o}$ is a rational singularity of H. Then $\dim(\mathscr{L}_{n-\delta}(H)) = (m-1)(n-\delta+1)$; we conclude as in (3.5), where, in this case, we only have equalities instead of inequalities. This completes the proof. ☐

3.9. The former results imply assertion (1) of theorem 3.1 when $m \geq 4$ or $m = 3$ and the germ $(H, \mathfrak{o})$ is the (A_p)-hypersurface singularity (with $p \geq 1$). Indeed, in these cases we may assume that the hypersurface H in the statement of the theorem is defined by an A-D-E polynomial which, by subsection 1.1, can be written in the form $h(x_1, \ldots, x_{m-2}) + x_{m-1}^2 + x_m^2$ (up to reordering the variables). Then, after a linear change of variables on the last two variables, we may assume that H is defined by a polynomial f in the aforementioned form $f = h(x_1, \ldots, x_{m-2}) + x_{m-1}x_m$. Thus we have obtained in this case:

Corollary 3.10. *We adopt the notation and assumptions of theorem 3.1. Let $n \geq 2$. If $m \geq 4$, or if $m = 3$ and the germ $(H, \mathfrak{o})$ is the (A_p)-hypersurface singularity (with $p \geq 1$), we have*

$\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = 1 + (m-1)n$. *Besides, we have* $\mathrm{dft}_n(f) = |w| - \delta - (m-2)$ *and, if the field k is assumed to be algebraically closed, the origin is a rational singularity for H.*

3.11. We prove now assertion (1) in theorem 3.1 for the remaining cases, that is, $m = 3$ and the germ of hypersurface $(H(f), \mathfrak{o})$ is *not* the (A_p)-hypersurface singularity. Actually, we will prove the following stronger proposition:

Proposition 3.12. *We adopt the notation of theorem 3.1. We assume that $m = 3$. Let $n \geq \delta$ be a positive integer. Then, we have*

$$\dim(\mathscr{L}_n(H)) - 1 = \dim(\pi_{0,n}^{-1}(\mathfrak{o})) = 2n + 1.$$

In particular, we have $\mathrm{dft}_n(f) = |w| - \delta - (m-2)$.

Let us stress that the equality $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = 2n + 1$ in case $m = 3$ can already be found in [7], which provides other proof arguments.

Remark 3.13. Let us observe that, if $f \in \mathbf{C}[x_1, x_2, x_3]$ is an A-D-E polynomial such that $(H(f), \mathfrak{o})$ is a simple isolated singularity different from the (A_p)-singularity with even p, then $\mathrm{dft}_n(f) = 0 = |w| - \delta - (m-2)$. In particular, it implies that the upper bound for $\dim(\pi_{0,n}^{-1}(\mathfrak{o}))$ in theorem 1.4 coincides with the value of the bound for $m = 2$ in this case. If $(H(f), \mathfrak{o})$ is the (A_p)-singularity with even p, then $\mathrm{dft}_n(f) = 1$.

Proof. Let us assume that f is an A-D-E polynomial. One knows (e.g., see [5, 10] for details and references) that the origin is a rational singularity, then as explained in subsection 2.5 we have $\dim(\mathscr{L}_n(H)) = (n+1)(m-1) = 2(n+1)$. If $(H(f), \mathfrak{o})$ is the (A_p)-singularity, then corollary 3.10 proves the result. Assume that $(H(f), \mathfrak{o})$ is not the (A_p)-singularity. In each case of the classification recalled in subsection 1.1, we check that $\dim(\mathscr{L}_n(H)) = \dim(\pi_{\delta,n}^{-1}(C_w)) + 1$ using theorem 3.6; since $\dim(\mathscr{L}_n(H)) > \dim(\pi_{0,n}^{-1}(\mathfrak{o}))$ and $\pi_{0,n}^{-1}(\mathfrak{o}) \supset \pi_{\delta,n}^{-1}(C_w)$, we conclude the proof.

- (D_p) We assume that $p \geq 4$ is a positive integer. The total weight equals $(2, p-2, p-1; 2p-2)$ and $\dim(\pi_{\delta,n}^{-1}(C_w)) = (6p-6) - (2p-1) + 2(n - 2p + 3) = 2n + 1$.
- (E_6) The total weight equals $(4, 3, 6; 12)$ and $\dim(\pi_{\delta,n}^{-1}(C_w)) = 36 - 13 + 2(n - 11) = 2n + 1$.
- (E_7) Let $n \geq \delta$. The total weight equals $(6, 4, 9; 18)$ and $\dim(\pi_{\delta,n}^{-1}(C_w)) = 54 - 19 + 2(n - 17) = 2n + 1$.
- (E_8) Let $n \geq \delta$. The total weight equals $(10, 6, 15; 30)$ and $\dim(C_w) = 90 - 31 + 2(n - 29) = 2n + 1$.

Let us consider an arbitrary simple hypersurface singularity in $\mathbf{A}^3_{\mathbf{C}}$. By subsection 2.3 and the previous cases, we deduce that $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = 2n+1$. We observe that, in this case, it means that $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = n(m-1) + 1$. Thus, by proposition 3.8, we have $\mathrm{dft}_n(f) = |w| - \delta - (m-2)$. $\square$

3.14. Finally, we prove assertion (2) in theorem 3.1. Precisely we prove the following result:

Proposition 3.15. *We adopt the notation of theorem 3.1. We assume that $m = 2$. Let $n \geq \delta$ be a positive integer. Then, we have*

$$\dim(\pi_{\delta,n}^{-1}(C_w)) = \dim(\mathscr{L}_n(H(f))) = \dim(\pi_{0,n}^{-1}(\mathfrak{o})) = 2\delta - |w| + \dim(\mathscr{L}_{n-\delta}(H)).$$

In particular, we have $\mathrm{dft}_n(f) = 0$.

Proof. By subsection 3.2, we may assume that $k = \mathbf{C}$. By [2], one knows that, up to re-ordering the variables, the polynomial f can be uniquely written under the form $f(x_1, x_2) = x_1^a x_2^b \prod_{i=1}^{c}(x_1^{w_2} + \lambda_i x_2^{w_1})$. In this way, we conclude that the triple $(w_1, w_2; c)$ (hence the reduced total-weight $(w_1, w_2; \delta)$) is an analytic invariant of the germ $(H, \mathfrak{o})$. If $H(f)$ is the union of two or three lines, then it is easy to check that $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = n+1 = \mathscr{L}_n(H(f))$ for every integer $n \geq \delta$ (e.g., use an induction with theorem 3.6). Otherwise, since we have assumed that $\mathfrak{o}$ is a simple singularity, the polynomial f has an irreducible factor f_0 whose associated hypersurface $H(f_0)$ has a single singularity at the origin. In that case too, we have $\dim(\mathscr{L}_n(H(f))) = \dim(\pi_{0,n}^{-1}(\mathfrak{o}))$. Indeed, if H is assumed to be irreducible, then we conclude by [9, Proposition 1.4, Proposition 1.7]; otherwise, assume that $\dim(\mathscr{L}_n(H(f))) = n+1 > \dim(\pi_{0,n}^{-1}(\mathfrak{o}))$. Then, we also have $\dim(\mathscr{L}_n(H(f_0))) = n+1 > \dim(\pi_{0,n;f_0}^{-1}(\mathfrak{o}))$, which implies $\mathscr{L}_n(H(f_0))$ is irreducible; hence, we conclude by *loc. cit.* that $H(f_0)$ is normal, which is a contradiction.

From now on, we assume that $k = \mathbf{C}$ *and* that the polynomial f is A-D-E by subsection 3.2. We need to study different cases.

$\circ$ *Plane curve singularities with* $\pi_{0,\delta}^{-1}(\mathfrak{o}) = C_w$. In this case, we have $\pi_{\delta,n}^{-1}(C_w) = \pi_{\delta,n}^{-1}(\pi_{0,\delta}^{-1}(\mathfrak{o})) = \pi_{0,n}^{-1}(\mathfrak{o})$. Every homogeneous polynomial satisfies this property. Then, we have $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = \dim(\pi_{\delta,n}^{-1}(C_w))$ for the (A_1)-plane curve singularity and for the (D_4)-plane curve singularity. By proposition 3.16, we also assert that it holds true for (A_p)-plane curve singularity (with $p \geq 2$), (E_6)-plane curve singularity and (E_8)-plane curve singularity.

$\circ$ (E_7)-*plane curve singularity.* We assume that $f = x_1^3 + x_1 x_2^3 = x_1(x_1^2 + x_2^3)$ with total weight $(3,2;9)$. Then, we have $\delta + |w| = 14$ (recall that $m = 2$ here). We set $D_{\mathfrak{o}} = \operatorname{Spec}(\mathcal{O}(\pi_{0,n}^{-1}(\mathfrak{o}))[z]/\langle x_{2,1}z - 1\rangle)_{\text{red}}$; set-theoretically we have:

$$\pi_{0,n}^{-1}(\mathfrak{o}) = D_{\mathfrak{o}} \sqcup \pi_{\delta,n}^{-1}(C_w).$$

Indeed, we observe that, in the k-algebra $B_n := \mathcal{O}(\pi_{0,n}^{-1}(\mathfrak{o}))_{\text{red}}$ the regular functions f_s, with $s \geq 3$, are equivalent to:

- $s = 3$: the function $x_{1,1}^3$ (hence, $x_{1,1}$ vanishes in B_n).
- $s = 4$: the function $0 \pmod{\langle x_{1,1}\rangle}$.
- $s = 5$: the function $x_{2,1}^3 x_{1,2} \pmod{\langle x_{1,1}\rangle}$.
- $s = 6$: the function $x_{1,3}x_{2,1}^3 + 3x_{1,2}x_{2,1}^2x_{2,2} + x_{1,2}^3 \pmod{\langle x_{1,1}\rangle}$.
- $s = 7$: the function $x_{2,1}^3x_{1,4} + 3x_{1,2}^2x_{1,3} + 3x_{2,1}^2x_{2,2}x_{1,3} + 3x_{2,1}^2x_{2,3}x_{1,2} + 3x_{2,1}x_{2,2}^2x_{1,2} \pmod{\langle x_{1,1}\rangle}$.
- $s = 8$: the function $x_{1,5}x_{2,1}^3 + 3x_{1,4}x_{2,1}^2x_{2,2} + 3x_{1,3}x_{2,1}x_{2,2}^2 + x_{1,2}x_{2,2}^3 + 3x_{1,3}x_{2,1}^2x_{2,3} + 6x_{1,2}x_{2,1}x_{2,2}x_{2,3} + 3x_{1,2}x_{2,1}^2x_{2,4} + 3x_{1,2}x_{1,3}^2 + 3x_{1,2}^2x_{1,4} \pmod{\langle x_{1,1}\rangle}$.
- $s = 9$: the function $x_{1,6}x_{2,1}^3 + 3x_{1,5}x_{2,1}^2x_{2,2} + 3x_{1,4}x_{2,1}x_{2,2}^2 + x_{1,3}x_{2,2}^3 + 3x_{1,4}x_{2,1}^2x_{2,3} + 6x_{1,3}x_{2,1}x_{2,2}x_{2,3} + 3x_{1,2}x_{2,2}^2x_{2,3} + 3x_{1,2}x_{2,1}x_{2,3}^2 + 3x_{1,3}x_{2,1}^2x_{2,4} + 6x_{1,2}x_{2,1}x_{2,2}x_{2,4} + 3x_{1,2}x_{2,1}^2x_{2,5} + x_{1,3}^3 + 6x_{1,2}x_{1,3}x_{1,4} + 3x_{1,2}^2x_{1,5} \pmod{\langle x_{1,1}\rangle}$.

From this computation, we deduce, first, that $(C_w)_{\text{red}} = \operatorname{Spec}(\mathcal{O}(\pi_{0,\delta}^{-1}(\mathfrak{o}))/\langle x_{2,1}\rangle)_{\text{red}}$ and that $D_{\mathfrak{o}}$ is an affine space over $\mathbf{G}_{m,k}$, and has dimension $n+3$. Moreover, by [9, Proposition 1.6] applied to $Z = H$, we deduce that $\dim(\mathscr{L}_{n-9}(H)) \geq n - 8$. Since $2\delta - |w| = 13$, we conclude that

$$\dim(D_{\mathfrak{o}}) = n + 3 \leq n + 5 = 13 + n - 8 \leq 2\delta - |w| + \dim(\mathscr{L}_{n-\delta}(H)) = \dim(\pi_{\delta,n}^{-1}(C_w)).$$

$\circ$ (D_p)-*plane curve singularity.* Let $p \geq 5$ be a positive integer and $f = x_1x_2^2 + x_1^{p-1}$. We set, for every integer $r \in \{2, \ldots, n\}$,

$$D_{\mathfrak{o}}^1 = \operatorname{Spec}(\mathcal{O}(\pi_{0,n}^{-1}(\mathfrak{o}))[z]/\langle x_{2,1}z - 1\rangle)_{\text{red}},\ D_{\mathfrak{o}}^r = \operatorname{Spec}(\mathcal{O}(\pi_{0,n}^{-1}(\mathfrak{o}))[z]/\langle x_{2,1}, \ldots, x_{2,r-1}x_{2,r}z - 1\rangle)_{\text{red}}.$$

Let $r \in \{1, \ldots, n\}$. We assume that $2r + j \neq (p-1)j$ for every integer $j \leq n - 2r$. We prove that $\dim(D_{\mathfrak{o}}^r) = n + r + 1$ using arguments similar to the case of the (E_7)-singularity. Let $j \in \{1, \ldots, n-2r\}$. Let us assume that $x_{1,j'}$ vanishes in $\mathcal{O}(D_{\mathfrak{o}}^r)$ for every $j' < j$ (which is true for $j = 1$). Then f_s vanishes in $\mathcal{O}(D_{\mathfrak{o}}^r)$ for $0 \leq s < \min(2r + j, (p-1)j)$; if $2r + j < (p-1)j$ then f_{2r+j} can be reduced to $x_{2,r}^2 x_{1,j}$, and if $2r + j > (p-1)j$ then $f_{(p-1)j}$ can be reduced to $x_{1,j}^{p-1}$

(all the other terms contain some $x_{2,r'}$ with $r' < r$, or some $x_{1,j'}$ with $j' < j$). In both cases, since $x_{2,r}^2$ is a unit in this ring, which is reduced by construction, we deduce that $x_{1,j}$ vanishes. Hence

$$\mathcal{O}(D_{\mathfrak{o}}^r) \cong (k[x_{2,r}, z]/\langle x_{2,1}z - 1\rangle)\,[x_{1,n-2r+1}, \ldots, x_{1,n}, x_{2,r+1} \ldots, x_{2,n}].$$

Thus, we deduce that:

$$\dim(D_{\mathfrak{o}}^r) = n + r + 1 \tag{3.6}$$

Let us assume that p is even, and let $q \geq 3$ be such that $p = 2q$. In this case, the polynomial f has total weight $(1, q-1; 2q-1 =: \delta)$. We easily observe that the very definitions of $D_{\mathfrak{o}}^r$ and C_w imply that, set-theoretically, we have

$$\pi_{0,n}^{-1}(\mathfrak{o}) = \left(\bigsqcup_{r=1}^{q-2} D_{\mathfrak{o}}^r\right) \sqcup \pi_{\delta,n}^{-1}(C_w).$$

Let $r \in \{1, \ldots, q-2\}$. Then $r \leq q-2$ is equivalent to $2r + j \leq p - 4 + j$, and the latter inequality implies $2r + j < (p-1)j$ for every $j \geq 1$. Then, by formula (3.6), we deduce $\max_{1 \leq r \leq q-2}(\dim(D_{\mathfrak{o}}^r)) = \dim(D_{\mathfrak{o}}^{q-2}) = n + q - 1$. On the other hand, we obtain from theorem 3.6 and [9, Proposition 1.6] that $\dim(\pi_{\delta,n}^{-1}(C_w)) = 2(2q-1) - q + \dim(\mathscr{L}_{n-(2q-1)}(H)) \geq 3q - 2 + (n - 2q + 1 + 1) = n + q$. Hence, we conclude that $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = \dim(\pi_{\delta,n}^{-1}(C_w))$.

Let us assume that p is odd. In this case, the polynomial f has total weight $(2, p-2; 2p-2 =: \delta)$. Again, by the very definitions of $D_{\mathfrak{o}}^r$ and C_w, we have the set-theoretical equality

$$\pi_{0,n}^{-1}(\mathfrak{o}) = \left(\bigsqcup_{r=1}^{p-3} D_{\mathfrak{o}}^r\right) \sqcup \pi_{\delta,n}^{-1}(C_w).$$

Let $r \in \{1, \ldots, p-3\}$. Then $r \leq p-3$ is equivalent to $2r + j \leq 2p - 6 + j$, and the latter inequality implies $2r + j < (p-1)j$ for every $j \geq 2$. If $j = 1$, then $2r + 1$ is odd, whereas $p - 1$ is even, then they cannot be equal. Then, by the preliminary computation (3.6), we deduce $\max_{1 \leq r \leq p-3}(\dim(D_{\mathfrak{o}}^r)) = \dim(D_{\mathfrak{o}}^{p-3}) = n + p - 2$. On the other hand, $\dim(\pi_{\delta,n}^{-1}(C_w)) = 2(2p-2) - p + \dim(\mathscr{L}_{n-(2p-2)}(H)) \geq 3p - 4 + (n - 2p + 2 + 1) = n + p - 1$. Hence, we have $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = \dim(\pi_{\delta,n}^{-1}(C_w))$. □

The following result can be deduced from [6, Proposition 4.1]. We include here a proof for the convenience of the reader.

Proposition 3.16. *Let $p, q > 0$ be two positive integers, and $f = x^p + y^q \in k[x, y]$ be a* reduced *weighted-homogeneous polynomial with total weight $(w_1, w_2; \delta)$. Then, the associated plane curve $H(f)$ satisfies the condition $C_w = \pi_{0,\delta}^{-1}(\mathfrak{o})$.*

Proof. Keeping the notation of subsection 2.1, in this particular case the polynomial f_s can be written in the form

$$f_s = \sum_{\substack{(i_a)_{1 \leq a \leq p} \\ \sum i_a = s}} \prod x_{i_a} + \sum_{\substack{(j_b)_{1 \leq b \leq q} \\ \sum j_b = s}} \prod y_{j_b}$$

for every integer s. Let us note that in the above sum there may be many occurrences of the same monomial. Let $r \in \mathbf{N}$. Let us observe the following facts:

(1) the polynomial f_{rp} (resp. f_{rq}) contains the monomial x_r^p (resp. y_r^q);
(2) any other monomial in f_{rp} (resp. f_{rq}) is either a monomial in the variables $(y_j)_{0 \leq j \leq rp}$ (resp. $(x_i)_{0 \leq i \leq rq}$), or it involves a variable $x_{r'}$ (resp. $y_{r'}$) with $r' < r$ (if there were a monomial $\prod x_{i_a}$ with $i_a \geq r$ for every $1 \leq a \leq p$, then $\sum_{a=1}^p i_a \geq rp$, and the equality only holds for $i_a = r$ for every a; this is a contradiction).

After this preliminary observations, we are now ready to prove that, for every integers $r \in \{0, \ldots, w_1 - 1\}$, $u \in \{0, \ldots, w_2 - 1\}$, we have $x_r, y_u \in \sqrt{I_\delta + \langle x_0, y_0\rangle}$. It is clear that $x_0, y_0 \in \sqrt{I_\delta + \langle x_0, y_0\rangle}$. Let $r \in \{0, \ldots, w_1 - 1\}$, $u \in \{0, \ldots, w_2 - 1\}$. We assume that $x_{r'}, y_{u'} \in \sqrt{I_\delta + \langle x_0, y_0\rangle}$ for every integer $r' \in \{0, \ldots, r - 1\}$ and $u' \in \{0, \ldots, u - 1\}$. We also assume that $rp < uq$. Since $rp < \delta = w_1 p$, we observe that the polynomial f_{rp} belongs to I_δ. By the former remark, the monomials in f_{rp} are x_r^p, monomials involving $x_{r'}$ for $r' < r$ (and thus belonging to $\sqrt{I_\delta + \langle x_0, y_0\rangle}$ by our assumption), or monomials in the variables $(y_j)_{0\leq j\leq rp}$. Indeed, each of the monomials of the latter sort $\prod y_{j_b}$ involves a variable $y_{j'}$ with $j' < u$: otherwise, $\sum_{b=1}^{q} j_b \geq uq \geq rp$, and the equality $uq = rp$ would imply that f is weighted homogeneous with total degree $(r, u; rp)$, which contradicts the reducedness of the total weight. Then the latter monomials also belong to $\sqrt{I_\delta + \langle x_0, y_0\rangle}$. We deduce that $x_r \in \sqrt{I_\delta + \langle x_0, y_0\rangle}$. In case $uq < rp$, a similar argument shows that $y_u \in \sqrt{I_\delta + \langle x_0, y_0\rangle}$. Then, we have the following equality of ideals in $k[x_i, y_j; 0 \leq i, j \leq \delta]$

$$\sqrt{I_\delta + \langle x_0, y_0\rangle} = \sqrt{I_\delta + \langle x_i, y_j \ : \ 0 \leq i < w_1, 0 \leq j < w_2\rangle},$$

which corresponds to the sought-for equality of sets $C_w = \pi_{0,\delta}^{-1}(\mathfrak{o})$ in $H(f)$. □

4. The end of the proof of theorem 1.4

Let $m \geq 2$ be a positive integer. In this section, we prove theorem 1.4. We show how simpleness criteria $(A)-(a) \Leftrightarrow (A)-(b) \Leftrightarrow (A)-(c)$ and $(B)-(a') \Leftrightarrow (B)-(b') \Leftrightarrow (B)-(c')$ are linked to another criterion $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ thanks to corollary 3.10 and propositions 3.12 and 3.15. The characterization $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ is crucially based on the geometric structure of the jet schemes associated with quasi-homogeneous hypersurfaces (see theorem 3.6) and can be understood as a formulation, in terms of jet schemes, of Saito's result as stated in [5, Characterization B.9].

4.1. We introduce a dimensional characterization of simpleness in terms of the variety C_w.

Proposition 4.2. *We adopt the notation of theorem 3.1. Let $n \geq \delta$ be a positive integer. Then, the analytic hypersurface singularity $(H \otimes_k \mathbf{C}, \mathfrak{o} \in \mathbf{C}^m)$ is simple if and only if one the following (equivalent) assertions is satisfied.*

1. *For every integer $n \geq \delta$, we have $\dim(\pi_{\delta,n}^{-1}(C_w)) \leq \delta + |w| + \dim(\mathscr{L}_{n-\delta}(H)) - 1$.*
2. *There exists an integer $n \geq \delta$ such that $\dim(\pi_{\delta,n}^{-1}(C_w)) \leq \delta + |w| + \dim(\mathscr{L}_{n-\delta}(H)) - 1$.*
3. *We have $\dim(C_w) \leq \delta + |w| + (m - 2)$.*

Proof. Let $n \geq \delta$ be a positive integer. By theorem 3.6, we have:

$$\dim(\pi_{\delta,n}^{-1}(C_w)) = m\delta - |w| + \dim(\mathscr{L}_{n-\delta}(H))$$

Then, the inequality $\dim(\pi_{\delta,n}^{-1}(C_w)) \leq \delta + |w| + \dim(\mathscr{L}_{n-\delta}(H)) - 1$ is equivalent to $m\delta - |w| \leq \delta + |w| - 1$, which can rewritten as $(m - 1)\delta < 2|w|$. This is equivalent to the origin being a simple singularity by [11] (see also [5, B9]). This concludes the proof of the equivalence and also shows that assertions (1), (2) and (3) are equivalent. □

4.3. Let us prove the simpleness criterion $(A)-(a) \Leftrightarrow (A)-(b) \Leftrightarrow (A)-(c)$. Let us assume that $m \geq 3$. It is clear that $(a) \Rightarrow (c) \Rightarrow (b)$. Let us assume that $(H \otimes_k \mathbf{C}, \mathfrak{o})$ is a simple singularity. By corollary 3.10 and proposition 3.12, we know that $\mathrm{dft}_n(f) = |w| - \delta - (m - 2)$. Besides, we have

$$\begin{aligned}
\dim(\pi_{0,n}^{-1}(\mathfrak{o})) &= \dim(\pi_{\delta,n}^{-1}(C_w)) + \mathrm{dft}_n(f) \\
&\leq \delta + |w| + \dim(\mathscr{L}_{n-\delta}(H)) - 1 + |w| - \delta - (m - 2) \\
&= 2|w| + (m - 1)(n - \delta + 1) - (m - 1) \\
&= 2|w| + (m - 1)(n - \delta).
\end{aligned}$$

This proves $(A)-(a)$.

Let us assume that assertion $(A)-(b)$ holds true. Then, by the very definitions and proposition 3.8, we have:

$$\begin{aligned}\dim(\pi_{\delta,n}^{-1}(C_w)) &= \dim(\pi_{0,n}^{-1}(\mathfrak{o})) - \mathrm{dft}_n(f)\\ &\leq 2|w| + (m-1)(n-\delta) - \mathrm{dft}_n(f)\\ &\leq 2|w| + (m-1)(n-\delta) - |w| + \delta + (m-2)\\ &= \delta + |w| + (m-1)(n-\delta+1) - 1\\ &\leq \delta + |w| + \dim(\mathscr{L}_{n-\delta}(H)) - 1.\end{aligned}$$

The last inequality comes from $\dim(\mathscr{L}_{n-\delta}(H)) \geq \dim(\mathscr{L}_{n-\delta}(\mathrm{Sm}(H)))$. We conclude by proposition 4.2 that $(H \otimes_k \mathbf{C}, \mathfrak{o})$ is a simple singularity.

Remark 4.4. Let $m \geq 3$ be a positive integer. Let $f \in k[x_1, \ldots, x_m]$ be a homogeneous polynomial with an isolated singularity at the origin $\mathfrak{o} \in \mathbf{A}_k^m$; hence, its (total) degree is $d \geq 2$. Then, the associated hypersurface obviously satisfies $\pi_{0,d}^{-1}(\mathfrak{o}) = C_w$ and, by theorem 3.6, we have $\dim(\pi_{0,d}^{-1}(\mathfrak{o})) = md - m + (m-1) = md - 1$. We see that $md - 1 \leq 2m$ means $m(d-2) \leq 1$. We conclude that $(H(f), \mathfrak{o})$ is a simple singularity if and only if $d = 2$. By a classical result in the theory of quadratic forms, this means that $(H(f), \mathfrak{o})$ is the (A_1)-hypersurface singularity.

4.5. Let us show the simpleness criterion $(B)-(i) \Leftrightarrow (B)-(ii) \Leftrightarrow (B)-(iii)$. Let us note that implication $(i) \Rightarrow (iii) \Rightarrow (ii)$ is formal. Let us assume that the germ $(H, \mathfrak{o})$ is a simple hypersurface singularity. Then, by proposition 3.15, we know that, in each case, we have $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) = \dim(\pi_{\delta,n}^{-1}(C_w))$ for every integer $n \geq \delta$; hence, assertion $(B)-(i)$ is a direct consequence of proposition 4.2. Conversely, let us assume that there exists an integer $n \geq \delta$ such that $\dim(\pi_{0,n}^{-1}(\mathfrak{o})) \leq \delta + |w| + \dim(\mathscr{L}_{n-\delta}(H)) - 1$. It follows that $\dim(C_w) \leq \delta + |w| + \dim(\mathscr{L}_{n-\delta}(H)) - 1$ and we conclude the proof by assertion (2) of proposition 4.2.

Remark 4.6. Let $m = 2$ and $f \in k[x_1, x_2]$ be a homogeneous polynomial with an isolated singularity at the origin $\mathfrak{o} \in \mathbf{A}_k^2$; again, its (total) degree is $d \geq 2$. Here the condition provided by theorem 1.4 is $2d - 1 \leq d + 2$ which equivalent to $d \leq 3$. If $d = 2$, we re-find the (A_1)-hypersurface singularity. If $d = 3$, by considering the polynomial $f(1, T) \in \mathbf{C}[T]$ (by assumption the expression of f contains both variables x_1, x_2), we deduce, up to a linear change of variables, a factorization of f under the form $x_1x_2(\alpha x_1 + \beta x_2)$, with $\alpha, \beta \in \mathbf{C}$, which is analytically (actually algebraically) equivalent to the (D_4)-singularity.

Example 4.7. Let us consider the polynomial $f = x_1^2(x_1^2 + x_2^3) \in \mathbf{C}[x_1, x_2]$. It is weighted-homogeneous with reduced total weight $(3, 2; 12)$. The associated plane curve $\mathscr{C}$ has a single singular point at the origin, but $(\mathscr{C}, \mathfrak{o})$ does not define a simple hypersurface singularity since, *e.g.*, the multiplicity at the origin is greater than 3. We check that theorem 1.4 fails in this case by analyzing the equations f_s for every integer $s \in \{1, \ldots, 12\}$ in the complex algebra $\mathcal{O}(\pi_{0,12}^{-1}(\mathfrak{o}))_{\mathrm{red}}$ similarly to the arguments used in the proof of proposition 3.15. We obtain $\pi_{0,12}^{-1}(\mathfrak{o})_{\mathrm{red}} = C_w \sqcup D_w$ with $D_w := \{x_{1,0} = x_{2,0} = x_{1,1} = 0\} \cap \{x_{2,1} \neq 0\}$. Besides, we find $\dim(D_w) = 12 + 8 = 20$ and also $\dim(C_w) = 2 \cdot 12 - 5 + 1 = 20$ whereas our upper bound is given here by $12 + 3 + 2 = 17$ (recall that $m = 2$ here). Let us also stress that $\pi_{0,12}^{-1}(\mathfrak{o})_{\mathrm{red}}$ differs here from C_w and is not irreducible.

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