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FIXED POINT THEOREMS FOR SPACES REPRESENTED BY PARTIALLY ORDERED SYSTEMS

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CHAPTER I

INTRODUCTION

A function f which maps a set into itself has a <u>fixed point</u> x if $f(x) = x$. A function F which maps a set into its collection of subsets has a <u>fixed point</u> x if $x \varepsilon F(x)$. A typical fixed point theorem then has the form: every function having properties P, defined on a set having properties C, has a (unique) fixed point.

Fixed point theorems may be usefully classified in at least two ways. First, the functions may be restrictive and the spaces rather general or vice versa. As instances one has the following results: Every monotone map on a complete lattice has a fixed point; every continuous map on an n-cube has a fixed point. In this dissertation most of the fixed point theorems will be of the second type, the functions will usually be required to satisfy only a mild continuity assumption, but the spaces will be restrictive.

It should be clear that the restrictions on the space cannot be strictly topological in nature, for one would have a good argument for the proposition that given almost any collection of topological properties, there exists a space having these properties and a continuous map on the space which has no fixed point. For example, the circle is a non-fixed point space which is compact, Hausdorff, connected, arcwise connected, completely regular, etc. We say almost any collection because it turns out that a compact, connected Hausdorff space with exactly two non-cutpoints is a fixed point space with respect to continuous functions. This is proved in Chapter III as a special case of a more general result. In this case there is a partially ordered system that represents the space, namely, the space itself with the cutpoint order. The proof uses both the topological properties of the space and the order properties of its representation.

The idea of studying the fixed point properties of a space which is simultaneously equipped with an order and a topology is due to Ward [20,21]. Others have also used this idea to obtain fixed point theorems for certain types of spaces [3,19]. In particular, Smithson has proved a fixed point theorem for trees in which the multi-valued functions under consideration need not be continuous [17].

A second way that fixed point theorems may be classified is with respect to their algebraic or topological nature. No attempt will be made to define precisely what is meant by this, but we point to the two results mentioned in the second paragraph as being of algebraic and topological nature respectively. The first is the prototype of fixed point theorems of an algebraic character; the second leads to the theory of retractions and topological fixed point theorems. Here, we will study topological spaces which are, in some sense, represented by partially ordered systems and the fixed point theorems will be of both topological and algebraic character.

Our main purpose, then, is to obtain fixed point theorems of the second type mentioned in paragraph two. The main theme is this: There are a number of ways that certain spaces may be represented by partially ordered systems; if the representation is good enough some information about the fixed point theory of the space will result.

For ease of reading, special results and definitions will be given when needed. However, the first part of Chapter II is devoted to some basic results and definitions which will be needed throughout this work. The last part of Chapter II is devoted to some rather general theorems for functions on compacta. These are needed in Chapter III, but some of them may be of independent interest.

In Chapter III fixed point theorems are obtained for spaces which are continua and have, at the same time, a lattice structure. Here the spaces are represented by a single partially ordered system and the fixed point theorems result. Also, in Chapter III, a structure theorem for trees is obtained and a fixed point theorem for trees results. In this case the space is represented by a collection of partially ordered systems.

If a space fails to be connected then it must fail to be a fixed point space with respect to continuous functions. Nevertheless, there is a pseudo-fixed point theory for spaces which are represented by a special class of lattices and a special class of functions. A few examples are presented in Chapter III.

Part of the motivation for Chapter IV is the theory of representation for rings [15]. In that theory maximal ideals, ring products, and the fundamental theorem of ring homeomorphisms play the vital roles. Since the counterpart of each of these is available in the topological case, it is natural to ask whether an analogous theory might be fruitful in topology. Unfortunately, the answer is no unless the spaces are totally disconnected, compact Hausdorff spaces. We answer this question in Chapter IV, and conclude that the intermingling of ring theory and topology probably stops with the relationship between Boolean rings and totally disconnected, compact Hausdorff spaces. We do, however, obtain necessary and sufficient conditions that a space be represented by a product space. This problem is closely related to the supply of partial orders that the space carries as is shown. A true fixed point theorem is not proved for spaces representable by products, but some responsibility for fixed points is delegated to the homeomorphisms that such a space may have.

De Marr [7] has given a representation for metric spaces, and has obtained the classical theorem on contraction maps on a complete metric space, in a new way, as a consequence. In Chapter V De Marr's representation is generalized to include Hausdorff uniform spaces, and a generalization of the contraction map theorem is proved. The author owes his interest in De Marr's paper to Davis, who has generalized the contraction map theorem in the setting of structures [5].

The problem of classifying those (complete) lattices for which a topology can be found which induces c convergence and only c convergence, for a prescribed lattice convergence c, is unsolved. De Marr [6] has recently solved the related problem for o-convergence. That is, which topological spaces (X,t) can be equipped with an order such that t-convergence and o-convergence coincide? In Chapter V we present De Marr's solution to the second problem, and a partial solution to the first.

Several unsolved problems appear throughout this work. Some of these are well known, others are due to the author. No attempt is made to trace these problems to their original sources.

CHAPTER II

BASIC DEFINITIONS AND THEOREMS

2.1 Preliminaries.

References for general topology are [4], [13], [14]. The reference for general lattice theory is [2]. Continuity for single valued maps of one topological space into another has its usual meaning. If F is a mapping of a topological space $(X, \mathcal{S})$ into its lattice $C(X)$ of $\mathcal{S}$-closed subsets, then, with Strother [18], F is continuous at $x \in X$ if and only if F is weakly and residually continuous at x and $F(x) \neq \emptyset$. This means that F satisfies the following conditions:

(W) If $F(x) \subset V$, and V is $\mathcal{S}$-open, then there exists a $\mathcal{S}$-open set G containing x, such that $z \in G$ implies that $F(z) \subset V$.

(R) If $F(x) \cap V \neq \emptyset$, and V is $\mathcal{S}$-open, then there exists a $\mathcal{S}$-open set G containing x, such that $z \in G$ implies that $F(z) \cap V \neq \emptyset$.

A compactum will consistently mean a compact, Hausdorff topological space. A continuum is a compactum which is also a connected topological space. A filter $\mathcal{F}$ on a set A is a collection of subsets of A such that

1. F_1, F_2 in $\mathcal{F}$ implies $F_1 \cap F_2 \in \mathcal{F}$,
2. $F \in \mathcal{F}$, $G \supset F$, implies $G \in \mathcal{F}$,
3. $\emptyset \notin \mathcal{F}$.

It will be necessary to consider uniform spaces in the sequel. A general reference is [4]. By a <u>uniform space</u> (E,J) is meant a set E and a filter J on $E\times E$ which has the following properties:

1. $\Delta \in J$,
2. $J \circ J = J$,
3. $J = J^{-1}$.

Here $J \circ J$ is the filter of all supersets of sets of the form $V \circ U$ with $V, U \in J$, and $\circ$ means relational composition. J^{-1} is the filter of all supersets of sets of the form V^{-1} with $V \in J$, and V^{-1} is the inverse of V. Δ is the set of all pairs (x,x), with $x \in E$.

If (E,J) is a uniform space, then t_J is the topology on E generated by taking the t_J-neighborhood system at $x \in E$ to be the filter generated by all sets of the form $V(x) = \{y \in E\colon (x,y) \in V\}$, where $V \in J$. It is known that if (E,J) is a uniform space, then its <u>associated topological space</u> (E,t_J) is completely regular. Also, if $(E,\mathcal{S})$ is a completely regular topological space, then for some filter J on $E\times E$, (E,J) is a uniform space and $\mathcal{S} = t_J$. In particular, this is true if $(E,\mathcal{S})$ is a compactum and this fact will be used later.

Now let (E,J) be a uniform space. The collection $C(E)$ of t_J-closed subsets of E is made into a uniform space $(C(E), 2^J)$ by defining the filter 2^J as follows (see [16]): for each $V \in J$ put

$$\mathcal{V}_V = \{(A,B) \in C(E) \times C(E) : A \subset V(B),\ B \subset V(A)\},$$

then the filter 2^J generated by $\{\mathcal{V}_V\colon V \in J\}$ has properties 1.,2.,3., and $(C(E), 2^J)$ is a uniform space.

Jacobs [12] has studied the continuity of functions defined on a topological space (X,t) and taking values in $(C(E), t_2J)$.

If $F\colon (X,t) \longrightarrow (C(E), t_{2^J})$, then with Jacobs and Berge, <u>F is u.s.c. at $a \in X$</u> if and only if

$$\bigcap_{G \in N(a)} \overline{\bigcup\{F(x) : x \in G\}}^{t_J} \subset F(a)$$

where $N(a)$ is the t-neighborhood system at $a \in X$. With similar notation <u>F is l.s.c. at $a \in X$</u> if and only if

$$\overline{\bigcup_{G \in N(a)} \bigcap\{F(X) : x \in G\}}^{t_J} \supset F(a)$$

The following is a result due to Jacobs [12] which greatly simplifies the study of spaces of closed subsets, and which will be used in Chapter III.

<u>Theorem 2.1.1.</u> Let (X,t) be a topological space and (E,J) a uniform space. If $F : X \longrightarrow C(E)$ has the property that $F(X) = \bigcup\{F(x): x \in X\}$ is compact in (E, t_J), then F is $t - t_{2^J}$ continuous at every point at which it is l.s.c. and u.s.c.

The next result will be referred to as the Fundamental Theorem because of its close resemblance to the algebraic theorem of the same name. It may easily be proved directly or derived from the results on quotient spaces in [13].

<u>Theorem 2.1.2.</u> Let X be a compact topological space and Y a Hausdorff space. If f is a continuous function which maps X onto Y, then there is a homeomorphism Φ such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ X/r_f \arrow[ur] & \end{tikzcd}$$

commutes. Here r_f is the equivalence relation on X induced by f and X/r_f has its quotient topology relative to the canonical projection π.

Now let (L, >) be a lattice. If there is only one order on L being considered we write L in place of (L, >). There are several natural ways of making L into a topological space. The <u>interval topology 1(L)</u> is obtained by taking all sets of the forms $\{x : x \geqslant a\}$, $\{x : x \leqslant b\}$ as a subbase for closed sets. The <u>complete topology K(L)</u> is obtained by taking the collection of all complete subsets of L as a subbase for closed sets. A net $(X_n : n \in D)$ in L <u>o-converges to $x \in L$</u> if and only if both lim inf X_n, lim sup X_n exist and x = lim inf X_n = lim sup X_n. The <u>order topology O(L)</u> has as a subbase for closed sets those sets F having the following property: if N is a net in F which o-converges to x, then $x \in F$.

<u>Remark</u>. In the definition of 1(L) one might be tempted to have, instead, a subbase of open sets of the forms $\{x : x > a\}$, $\{x : x < b\}$. But, if this were done, the product topology on [0,1] x [0,1] would not coincide with the internal topology on [0,1] x [0,1] with respect to the product order, as is easily seen. This is pointed out in [2]. With our present definition, arbitrary products of interval topologies are interval topologies on the product, a fact which will be proved later.

Frink [8], Insel [11], and Atsumi [1] have investigated the relations between these natural topologies. The following is a compilation of results due to these men. In particular, the important characterization of 1(L) convergence is due to Atsumi [1].

<u>Theorem 2.1.3.</u> For any lattice L, $1(L) \leqslant O(L)$ and $K(L) \leqslant O(L)$. If L is complete and Hausdorff in 1(L) then 1(L) is a compact topology and 1(L) = O(L). If L is complete, then a necessary and sufficient condition

that a net $(X_n : n \varepsilon D)$ 1(L) converge to $x \varepsilon L$ is that

$$\bigvee_C \bigwedge (X_n : n \varepsilon D) \leqslant x \leqslant \bigwedge_C \bigvee (X_n : n \varepsilon D),$$

where C is a cofinal subset of D.

A useful remark which is recorded for reference is

Theorem 2.1.4. In any lattice L, if a net o-converges to x, then it O(L) converges to x.

Proof: Suppose N is a net which o-converges to x but does not O(L) converge to x. Then N is not eventually in some O(L) - open set G containing x. So there is a subnet of N which is in G'. Since N o-converges to x, so does the subnet. But $x \notin G'$ which contradicts the fact that G' is O(L) closed.

2.2 Functions on compacta.

There are several fixed point theorems which result from assuming the existence of continuous surjections with right (left) inverses. For example, a homeomorphic image of a fixed point space is a fixed point space, and if X is a fixed point space and there exists a continuous surjection $f:X \longrightarrow Y$ with continuous right inverse, then Y is a fixed point space. Here, some fixed point theorems are obtained by assuming the existence of maps which need not have partial inverses. Our results will be applied in Chapter III.

A topological space X has the f.p.p. if and only if every continuous function from X into itself has a fixed point. X has the F.p.p. if and only if every continuous map from X into its lattice of closed subsets has a fixed point.

Theorem 2.2.1. Let E be a compactum and E^1 any Hausdorff space.

If there exists a continuous surjection f and a continuous map f^1, where $f, f^1: E \to E^1$, then the multivalued map $F: E \to C(E)$ defined by $F(x) = f^{-1}(f^1(x)) = \{y: f(y) = f^1(x)\}$ is weakly continuous on E.

Proof: Let V be an open set containing F(x). Suppose that for every $U \in N(x)$, N(x) the neighborhood system at x, there is an $x_u \in U$ such that $F(x_u) \not\subset V$. Then, for each $U \in N(x)$ there exists a $y_u \in F(x_u)$ such that $y_u \notin V$. The net $(x_u: u \in N(x))$ converges to x, and since E is compact, some subnet of $(y_u: u \in N(x))$ converges to a point s and $s \notin V$ for V is open. Hence there are nets (x_σ), (y_σ) such that $x_\sigma \to x, y_\sigma \to s$, $f(y_\sigma) = f^1(x_\sigma)$. By continuity $f(y_\sigma) \to f(s)$, $f^1(x_\sigma) \to f^1(x)$. Since E^1 is a Hausdorff space it must be that $f(s) = f^1(x)$. Thus $s \in F(x) \subset V$ which contradicts $s \notin V$. We conclude that there exists some $U_o \in N(x)$ such that $z \in U_o$ implies $F(z) \subset V$, and hence that F is weakly continuous at x. Since x was arbitrary, F is weakly continuous on E.

<u>Theorem 2.2.2.</u> Let f be a continuous, open surjection between compacta E, E^1. If g is any continuous map of E/r_f into itself, then the multivalued map $F: E \to C(E)$ defined by $F(x) = f^{-1}f^1(x)$, with $f^1 = \Phi g\pi$, is residually continuous on E. Here, Φ, π are the functions obtained from the Fundamental Theorem 2.1.2.

Proof: Suppose $F(x) \cap V \neq \emptyset$ with V open. Then $f(y) = f^1(x)$ and $y \in V$; hence, $f(y) = \Phi g \pi(x)$ and $\pi(y) \in \pi(V)$. From the Fundamental Theorem, $f = \Phi\pi$ and Φ is one-one so $\pi(y) = g\pi(x)$ with $\pi(y) \in \pi(V)$. Then, since f is open, π is open and $\tilde{V} = \pi(V)$ is a neighborhood of $y = g(x)$. Since g is continuous there is a neighborhood U of $\tilde{x}$ such that $g(U) \subset \tilde{V}$. Define $G = \pi^{-1}U$; then $G \in N(x)$ and $g(\pi G) = g(\pi\pi^{-1}U) \subset g(U) \subset \pi V$. It follows that $\Phi g\pi(G) \subset \Phi\pi(V) = f(V)$ so that $f^1(G) \subset f(V)$. Using this last relation,

$t \in G$ implies $f^1(t) = f(v)$ for some $v \in V$. That is, $t \in G$ implies $F(t) \cap V \neq \emptyset$, so F is residually continuous at x and hence on E since x was arbitrary.

Theorem 2.2.3. Let E be a compactum with the F.p.p. and let f be a continuous, open surjection from E to E^1 where E^1 is any Hausdorff space. Then E^1 has the f.p.p.

Proof: Since f is continuous and E is a compactum, E^1 is a compactum. Let π, Φ be the maps obtained from the Fundamental Theorem. Now let g be any continuous map from E/r_f into itself. We show that g has a fixed point. To this end put $f^1 = \Phi g\pi$. Then, the multivalued map $F(x) = f^{-1}(f^1(x))$ is continuous on E by Theorem 2.2.1 and Theorem 2.2.2. Since E has the F.p.p. there is a point $x_o \in E$ such that $x_o \in F(x_o)$. But, by definition of F, this means that $f(x_o) = f^1(x_o) = \Phi g\pi(x_o)$. Thus g has a fixed point as asserted. It has been shown that E/r_f has the f.p.p. Since E^1 and E/r_f are isomorphic, the same is true of E^1.

Unsolved Problem. Let E be a compactum which has the f.p.p. Does $E \times E$ have the f.p.p.? This is a well known unsolved problem.

Remark. In the next Chapter a result due to Strother [18] and Kakutani (independently) to the effect that any closed interval of real numbers has the F.p.p. will be obtained as a corollary to a more general result. Assuming this for the moment, the preceding theorem shows that a continuous, open, Hausdorff image of a closed interval has the f.p.p. If, in this case, the openness hypothesis in Theorem 2.2.3 could be removed a new proof of Brouwer's theorem would result, for there are continuous surjections $f:[a,b] \to [a,b]^n$. In the present case, however, it is readily seen that no open, continuous surjection $f:[a,b] \to [a,b]^n$ can exist, for such a map is automatically closed, so that the image of an open interval has void interior.

More generally, if E is a space with the F.p.p. then ExE has the f.p.p. provided there is a continuous, open surjection $f\colon E \to E\times E$. Thus, the preceding theorem might be relevant to the problem above for certain classes of spaces. It also gives motivation for a complete classification of the continuous, open, Hausdorff images of a space which is known to have the F.p.p.

Theorem 2.2.4. Let C be a class of continua. Suppose that C is closed under continuous images and countable intersections of decreasing chains. Then, if each member of C has the f.p.p. with respect to continuous surjections, each member of C has the f.p.p.

Proof: Let $f\colon X \to X$ be a continuous map on a member of C. Then $\bigcap_{n=1}^{\infty} f^n(X)$ is also a member of C by assumption. We claim that, if $A = \bigcap_{n=1}^{\infty} f^n(X)$, then f maps A onto itself. In order to show this let $y_o \in A$. Then there is a sequence of points $x_1, x_2, \cdots$ in X such that $y_o = f(x_1) = f^2(x_2) = \cdots$. So there are points y_n in X such that $y_o = f(y_n)$ and $y_n \in f^m(X)$ for all $m \leqslant n-1$.

Since X is compact, some subsequence (y_{n_k}) of (y_n) converges to a point s. If $s \notin f^{n_o}(X)$ for some n_o, it follows that $(f^{n_o}(X))^1$ contains y_{n_k} for n_k sufficiently large. But this contradicts the fact that $y_{n_k} \in f^m(X)$ for $n_k \leqslant m - 1$. Thus, $y_{n_k} \longrightarrow s \in A$. By continuity $f(y_{n_k}) \to f(s)$, and since $y_o = f(y_{n_k})$, $f(s) = y_o$. This shows that f maps A onto itself.

By assumption f fixes a point of A and so X has the f.p.p.

CHAPTER III

REPRESENTATION BY RECTANGULAR LATTICES

In this chapter spaces which are representable by a special type of lattice or families of such lattices are considered, and some fixed point theorems result. We develop the theory of these special lattices in section 1, deferring examples to section 2.

3.1 Fixed point theorems for rectangular lattices.

Let $(L, >)$ be a lattice and δ a topology on L. Then $>$ is continuous on L if, whenever $a < b$, there are δ-open neighborhoods G, H, of a,b, respectively, such that $x \in G$, $y \in H$, implies $x < y$. A rectangular lattice is a complete lattice $(L, >)$ such that $l(L)$ is Hausdorff and connected and $>$ is continuous with respect to $l(L)$.

Theorem 3.1.1. A rectangular lattice has the f.p.p. with respect to its interval topology.

Proof: If f is an $l(L)$-continuous map from a lattice L, which is Hausdorff in $l(L)$, into itself, then $\mathcal{J} = \{x : x \leqslant f(x)\}$ is $l(L)$-closed. To show this it suffices to show that $\mathcal{J}$ is $O(L)$-closed by Theorem 2.1.3. To this end let the net $(x_n : n \in D)$ o-converge to x, where $x_n \in \mathcal{J}$. Then,

(1) $$x = \liminf x_n.$$

Since $x_n \in \mathcal{L}$, $x_n \leqslant f(x_n)$ for all $n \in D$ and so

(2) $$\liminf x_n \leqslant \liminf f(x_n).$$

By Theorem 2.1.4 $(x_n: n \in D)$ O(L)-converges to x and by continuity $(f(x_n): n \in D)$ l(L) converges to f(x). As a consequence,

(3) $$\liminf f(x_n) \leqslant f(x).$$

For, if $\liminf f(x_n) \nleqslant f(x)$, then for some $n_o \in D$, $\inf \{f(x_n): n \geqslant n_o\}$ is not less than or equal to f(x). Hence, $\{s: s \ngtr \inf \{f(x_n): n > n_o\}\}$ is an l(L)-neighborhood of f(x). Since $f(x_n) \to f(x)$, $f(x_n) \ngtr \inf \{f(x_n): n \geqslant n_o\}$ for all n sufficiently large which is absurd. It has been shown that relation (3) holds. From relations (1), (2), (3), $x \in \mathcal{L}$. So every net in $\mathcal{L}$ which o-converges, o-converges to a point in $\mathcal{L}$; i.e., $\mathcal{L}$ is O(L) closed and hence l(L) closed as promised.

Suppose now that f is a continuous map on L which has no fixed point. Then $\mathcal{L} = \{x : x < f(x)\}$ and $\mathcal{L}$ is closed. $\mathcal{L}$ is not void $(o \in \mathcal{L})$ and $\mathcal{L}$ is not L$(1 \notin \mathcal{L})$ so $\mathcal{L}$ fails to be open since L is connected. So there exists a point $x_o \in \mathcal{L}$ and a net $(x_n: n \in D)$ outside of $\mathcal{L}$ such that $(x_n: n \in D)$ l(L)-converges to x_o. By continuity of > there are sets $H \in N(x_o)$, $G \in N(f(x_o))$, such that $t \in H$, $\mu \in G$, implies $t < \mu$. But $(x_n: n \in D)$ is eventually in H, and $(f(x_n): n \in D)$ is eventually in G, so $x_n < f(x_n)$ for n sufficiently large. But this contradicts the fact that $x_n \notin \mathcal{L}$. We conclude that f has a fixed point.

<u>Theorem 3.1.2.</u> If F is a non-void, l(L)-closed subset of a rectangular lattice L, then $\sup F \in F$, $\inf F \in F$.

Proof: Suppose that $\sup F \notin F$. Then, for each $x \in F$, $x < \sup F$. By continuity of > there are open sets Hx, Tx, such that $x \in Hx$, $\sup F \in Tx$ and $Hx \leqslant Tx$. Here we are using the notation $A \leqslant B$ to mean that $a \leqslant b$ for every $a \in A$ and every $b \in B$.

Now, $\{Hx : x \in F\}$ is an open cover of F and F is compact by Theorem 2.1.3. Hence $F \subset \bigcup_{i=1}^{n} Hx_i$ for some finite subset $x_1, \ldots, x_n$ of F. Then $y \in F$ implies $y \in Hx_{i_o} \leqslant Tx_{i_o}$ for some i_o. It follows that $\{y\} \leqslant \bigcap_{i=1}^{n} Tx_i$. Thus, there is an open set $T = \bigcap_{i=1}^{n} Tx_i$ containing sup F such that $y \in F$ implies $y \leqslant T$. Thence, $T \geqslant \{\sup F\}$ and

(1) $$\operatorname{Sup} F \in T \subset \{\mu : \mu \geqslant \sup F\} = A.$$

From relation (1), sup $F \in$ Int A; and if $\mu \in A$, $\mu > \sup F$, then $\mu \in$ Int A by the continuity of $>$. It follows from this that A is both open and closed. Since L is connected $A = \emptyset$ or $A = L$. The first alternative is impossible by (1). If $A = L$, then $\sup F = 0$ and $F = \{0\}$.

This completes the proof of the theorem for suprema; the proof for infima is similar.

A mild variation of the proof just given shows that $\inf S \in$ clos S, sup $S \in$ clos S, so that 1(L)-closed sets are complete. Since, by Theorem 2.1.3, $1(L) = O(L)$ for rectangular lattices, it follows that $1(L) = K(L)$. This proves the next result and shows why we ignore the canonical topologies K(L), O(L) when dealing with rectangular lattices.

Corollary. If L is a rectangular lattice, then $1(L) = O(L) = K(L)$.

Remark. The continuity of $>$ is essential in Theorem 3.1.2. For, if $[0,1] \times [0,1]$ is ordered with the product order, then the ordinary topology is the same as the interval topology on $[0,1] \times [0,1]$. Hence, a circular curve tangent to the four sides of $[0,1] \times [0,1]$ is closed in the interval topology. But this set contains neither its infimum nor its supremem.

Theorem 3.1.3. A rectangular lattice has the F.p.p. with respect to its interval topology.

Proof: Let $F:L \to C(L)$ be a continuous multi-valued map on L. By the preceding theorem sup $f(x) \in F(x)$ for each x. Define a function $g: L \to L$ by putting $g(x) = \sup F(x)$. It will be shown that g is continuous, and an appeal to Theorem 3.1.1 will complete the proof. To this end let $x_n \to x_o$. It must be shown that $(g(x_n))$ is eventually in any subbasic open set containing $g(x_o)$. Let V be such a set.

<u>Case 1.</u> $V = \{x : x \nleq a_o\}$.

Since $g(x_o) \in V$, $F(x_o) \cap V \neq \emptyset$. Since F is residually continuous on L, there is a $G \in N(x_o)$ such that $x \in G$ implies $F(x) \cap V \neq \emptyset$. Then, since (x_n) is eventually in G, there exist $y_n \in F(x_n)$ such that $y_n \nleq a_o$ for $n \geqslant n_o$. It follows that $g(x_n) \in V$, because $g(x_n) \leqslant a_o$, $g(x_n) = \sup F(x_n)$, implies $y_n \leqslant a_o$ contradicting the choice of the y_n.

<u>Case 2.</u> $V = \{x : x \ngeq a_o\}$.

Since $g(x_o) \in V$, $g(x_o) = \sup F(x_o)$, we have $t \in V$ for all $t \in F(x_o)$. That is $F(x_o) \subset V$. Since F is weakly continuous at x_o, there exists a set $G \in N(x_o)$ such that $x \in G$ implies $F(x) \subset V$. Thus, for n sufficiently large $g(x_n) \in V$. These cases show, then, that $x_n \to x$ implies $g(x_n) \to g(x)$ so that g is continuous.

<u>Corollary 1.</u> A continuous, open, Hausdorff image of a rectangular lattice has the f.p.p.

Proof: Theorem 2.2.3 and Theorem 3.1.3 together with the observation that a rectangular lattice is a compactum complete the proof.

<u>Corollary 2.</u> (Strother [18]) Any closed interval of real numbers has the F.p.p. (Moreover any continuous, open Hausdorff image of a closed interval has the f.p.p.)

Proof: Clearly a closed interval of real numbers is a rectangular lattice.

As remarked earlier, a product of rectangular lattices need not be rectangular. However we can prove

<u>Theorem 3.1.4.</u> Let $(L_\lambda)_{\lambda \varepsilon \Lambda}$ be a collection of rectangular lattices and let XL_λ be the topological product of the L_λ. If f is any continuous map from XL_λ into itself, then for each $\lambda \varepsilon \Lambda$ there is a $g^\lambda \varepsilon XL_\lambda$ such that $p_\lambda f(g^\lambda) = p_\lambda g^\lambda$, where the p_λ are the canonical projections.

Proof: A rectangular lattice has the f.p.p.

Anticipating the next theorem, we remark that if a space has the F.p.p. then so does any homeomorphic image of it. For let X have the F.p.p. and let $q:X \rightarrow Y$ be a homeomorphism. If F is a continuous multi-valued map on Y, then $F^1:X \rightarrow C(X)$ defined by $F^1(x) = q^{-1}F(q(x))$ is continuous on X and hence fixes some x_o. Then F fixes $q(x_o)$.

<u>Theorem 3.1.5.</u> Let L be a complete lattice which is a Hausdorff space and has continuous partial order with respect to some topology $\mathcal{S}$. Let L^1 be a complete lattice which is Hausdorff and connected in $l(L^1)$. If there exists an $l(L^1)$- $\mathcal{S}$ continuous surjection f from L^1 to L, then L has the F.p.p. relative to $\mathcal{S}$.

Proof: From the Fundamental Theorem one obtains the commutative diagram below.

$$\begin{tikzcd} L^1 \arrow[r, "f"] \arrow[d, "\pi"'] & L \\ L^1/r_f \arrow[ur, "\Phi"'] & \end{tikzcd}$$

It is asserted that there is an order on L^1/r_f such that the interval topology on L^1/r_f, with respect to this order, is precisely the quotient topology on L^1/r_f. In fact, define, for $\tilde{x}, \tilde{y} \in L^1/r_f$, $\tilde{x} \leqslant \tilde{y}$ if and only if $f(x) \leqslant f(y)$. Then L^1/r_f is a complete lattice, where $\bigwedge \tilde{x}_\lambda = \tilde{x}$, $f(x) = \bigwedge f(x_\lambda)$, and $V \tilde{x}_\lambda = \tilde{x}$, $f(x) = V f(x_\lambda)$.

Now π is $l(L^1) - l(L^1/r_f)$ continuous. To see this let $(x_n: n \in D)$ $l(L^1)$ converge to x. This means (see Theorem 2.1.3) $\underset{C}{V} \bigwedge(x_n: n \in C) = x = \underset{C}{\bigwedge} V(x_n: n \in C)$. Then, $\underset{C}{V} \bigwedge (\tilde{x}_n: n \in C) = \underset{C}{V} \tilde{\mu}_c$ where $f(\mu_c) = \bigwedge(f(x_n): n \in C)$, and $\underset{C}{V}\tilde{\mu}_c = \tilde{\mu}$ where $f(\mu) = \underset{C}{V} f(\mu_c)$. It follows that $f(\mu) = \underset{C}{V} \bigwedge (f(x_n): n \in C)$. imilarly, $f(\mu') = \underset{C}{\bigwedge} V(f(x_n): n \in C)$ for some $\mu' \in L^1$. But f is continuous and so $f(\mu) = f(\mu') = f(x)$. It follows that $\tilde{\mu} = \tilde{\mu}' = \tilde{x}$ and hence that $(\tilde{x}_n: n \in D)$ $l(L^1/r_f)$-converges to $\tilde{x}$. Thus, π is $l(L^1/r_f)$ continuous. By definition of the quotient topology q, $q \geqslant l(L^1/r_f)$. However, q is compact and $l(L^1/r_f)$ is Hausdorff so the topologies are identical. This proves our original assertion.

Since L has continuous partial order and is a homeomorphism it follows that L^1/r_f is a rectangular lattice and so has the F.p.p. By the remark preceding the theorem the same is true of L.

<u>Theorem 3.1.6.</u> Every rectangular lattice is topologically-isomorphic and order-isomorphic to a subspace of a space of closed subsets $(C(E), t_{2^J})$ of some uniform space (E,J).

Proof: Let L be a rectangular lattice. Since L is a compactum it is uniformizable so there is a uniformity J on LxL such that $t_J = l(L)$. It is asserted that the map $F:L \rightarrow C(L)$, defined by $F(x) = [0,x]$, embeds $(L, l(L))$ to topologically and algebraically in $(C(L), t_{2^J})$. To show that F is $l(L) - t_{2^J}$ continuous Theorem 2.1.1 will be employed. The overriding hypothesis of this theorem is satisfied because L is compact in $l(L)$.

First, F is u.s.c. on L. Let $a \varepsilon L$. It must be shown that

(1) $$\bigcap_{G \varepsilon N(a)} \overline{\bigcup \{F(x): x \varepsilon G\}}^{t_J} \subset F(a).$$

To this end let z be in the left hand member of (1). For each $V \varepsilon J$, $V(a)$ is in $N(a)$ and $V(z) \varepsilon N(z)$ so that $V(z) \cap F(x_v) \neq \emptyset$ for some $x_v \varepsilon V(a)$. Thus, for each $V \varepsilon J$, there are points $x_v \varepsilon V(a)$, $y_v \varepsilon V(z)$, such that $y_v \leqslant x_v$. The nets $(x_v: v \varepsilon J)$, $(y_v: v \varepsilon J)$ converge to a, z respectively so $a \geqslant \lim\inf x_v \geqslant \lim\inf y_v > V \wedge (y_v: v \varepsilon C) = z$. Thus $z \leqslant a$ and $z \varepsilon F(a)$. This shows that relation (1) is true and that F is u.s.c. on L.

Next, it will be shown that F is l.s.c. on L. For it is to be verified that

(2) $$\overline{\bigcup_{G \varepsilon N(a)} \bigcap \{F(x): x \varepsilon G\}}^{t_J} \supset F(a).$$

for each $a \varepsilon L$. Suppose that $z \varepsilon F(a)$, but that z is not in the right hand member of (2). Then for some closed neighborhood $V_o(z)$ of z, $V_o(z) \cap \left(\bigcap \{F(x): x \varepsilon V_o(a)\}\right) = \emptyset$.

Then $V \subset V_o$, $V \varepsilon J$, implies $V(z) \cap \left(\bigcap \{F(x): x \varepsilon V(a)\} \right) = \emptyset$. Since L is compact, and infima of closed sets F are members of F,

(3) $$V(z) \cap [0, t_v] = \emptyset, \; t_v \varepsilon V(a).$$

Then the net $(t_v: V \subset V_o)$ converges to a. Since $z \leqslant a$, either $z < a$ or $z = a$. But, if $z < a$ then by continuity of $>$ there are neighborhoods G, H of z, a, respectively, such that $G < H$. The t_v are in H for v sufficiently large, hence $z < t_v$ for v sufficiently large. However, this contradicts (3).

It must be, then, that $z = a$. But, in this case (3) is obviously false. We conclude that (2) holds, and that F is l.s.c. on L.

F is one-one and continuous; L is compact and $(C(E), t_{2^J})$ is Hausdorff. Hence, F is a homeomorphism. Since the order in C(E) is set inclusion, $x \leqslant y$ implies $F(x) \leqslant F(y)$ so F is order-preserving.

Problem. Can one exhibit a reasonable list of axioms, compatible with the above theorem such that if a subset $\mathcal{A}$ of a space $(C(E), t_{2^J})$ satisfies these axioms, then $\mathcal{A}$ is a rectangular lattice?

3.2. Some Examples.

Of course, any lattice, complete or not, can be equipped with its interval topology. Some of the more important lattices whose interval topology is of interest include:

(a) The n-cubes L^n, L a complete lattice;

(b) The complete lattice of subsystems of an algebraic system;

(c) The complete lattice of closed subsets of a topological space;

(d) The complete lattice of topologies (convergence schemes) on a set;

(e) A lattice of functions having some property with range contained in a complete lattice.

It is also true that there are well known spaces which have some interval topology. That is, $(X, \mathcal{S})$ is a lattice for some order $>$ and $\mathcal{S} = 1(X, >)$. Some illustrations follow; all spaces will be complete lattices.

Example 1. Let $(L_\lambda)_{\lambda \in \Lambda}$ be a set of complete lattices. If XL_λ is ordered with the product order, the interval topology on XL_λ

is the same as the product topology on XL_λ with respect to the topologies $l(L_\lambda)$. This follows from the order in XL_λ and Atsumi's characterization of $l(L)$-convergence.

Example 2. Let X be a circle in E^2 and order X by distinguishing two points 0, 1 on X. Then X has its interval topology; the order is not continuous. By the remarks in Example 1, the one-holed torus (the product of two circles) has its interval topology as does the unit cylinder.

Example 3. The spaces $[a, b]^n$ all have their interval topologies by Example 1. If $n = 1$, the order is continuous, and $[a,b]$ is a rectangular lattice. If $n > 1$, the order is not continuous. As a matter of fact, it is impossible to produce an order $>$ on $[a,b]^n$, $n \geqslant 2$, for which $>$ is continuous and $l([a,b]^n, >)$ is the usual topology. For Strother has shown that $[a,b]^n$, $n > 2$, does not have the F.p.p. while each rectangular lattice does.

Example 4. Let $(X, \mathcal{S})$ be a continuum with exactly two non-cutpoints a,b. We show that X is a rectangular lattice. If the cutpoint order $<$ is placed on X, ($x < y$ if $x = a$ or x separates b and y) then the following properties hold:

1. for each distinct x,y, $x < y$ or $x > y$;
2. if $x < y$, then $x > y$ is false;
3. if $x < y$ and $y < z$, then $x < z$;
4. the topology on X which has sets of the forms $\{S : \{S\} > x\}$, $\{S : S < y\}$ as a subbase for open sets is the same as $\mathcal{S}$. For details see [9].

Now define $x \leqslant y$ if $x = y$ or $x < y$. Then $(X, >)$ is a chain; $l(X, >) = \mathcal{S}$; and since $\mathcal{S}$ is compact and Hausdorff $(X, >)$ is a complete lattice. Also, $>$ is continuous, for, if $x < y$, the sets $A_x = \{s : s \not> y\} = \{s : s < y\}$, $A_y = \{s : s \not< x\} = \{s : s > x\}$, are $l(X, >)$ open and $A_x < A_y$.

Thus, $(X, >)$ is a rectangular lattice. Every continuum with exactly two non-cutpoints is a rectangular lattice and so has the F.p.p.

Example 5. As a particular instance of the type of space mentioned in Example 4 which is not a metric space take the long line of Bing with an additional point ∞ added [9].

Example 6. Homeomorphic images of complete lattices with interval topologies, or rectangular lattices, are spaces of the same type, for, let $f\colon L \longrightarrow (X, \mathcal{S})$ be an $l(L)$- $\mathcal{S}$ homeomorphism. Define $>$ on X by $x \leqslant y$ if and only if $f^{-1}(x) \leqslant f^{-1}(y)$. This order is continuous if the order in L is continuous and it makes X into a complete lattice with $\inf x_\alpha = x$ if $f^{-1}(x) = \inf f^{-1}(x_\alpha)$, $\sup x_\alpha = y$ if $f^{-1}(y) = \sup f^{-1}(x_\alpha)$.

Finally, $(x_n : n \in D) \xrightarrow[\mathcal{S}]{} x$ iff $(f^{-1}(x_n) : n \in D) \xrightarrow[l(L)]{} f^{-1}(x)$ iff $\bigvee_C \bigwedge (f^{-1}(x_n) : n \in C) = f^{-1}(x) = \bigwedge_C \bigvee (f^{-1}(x_n) : n \in C)$ iff $\bigvee_C \bigwedge (x_n : n \in C) = x = \bigwedge_C \bigvee (x_n : n \in C)$ iff $(x_n : n \in D) \xrightarrow[l(X)]{} x$.

Hence, $\mathcal{S} = l(X)$ and the conclusion follows. In particular every arc has the F.p.p., and every homeomorphic image of an arbitrary product of copies of intervals is a complete lattice with interval topology.

Problem. For examples (b), (c), (d) determine algebraic conditions that insure connectedness, the Hausdorff property, etc. in the interval (complete) (order) topology. This is an innocent sounding but non-trivial problem. In this connection see [22].

3.3. Topological unions of rectangular lattices.

There are spaces which are not rectangular lattices but which are made up of such lattices. For example, $[0,1] \times [0,1]$ fails to have a continuous partial order, but it is made up of straight lines through the origin.

More precisely, we say that a space (X, δ) is the <u>topological union</u> of spaces $(Y_\lambda, \delta_\lambda)$ if $X = \bigcup Y_\lambda$ and $\delta_\lambda = \delta \restriction Y_\lambda$ for each λ. Here, $\delta \restriction Y$ is the restriction of δ to Y.

Thus, for example, all simplexes, the cubes $[a,b]^n$, the real line, or any disconnected combination of these is a topological union of rectangular lattices.

As another illustration it will be shown that a tree is a topological union of rectangular lattices.

A <u>tree</u> T is a continuum such that every two distinct points x,y can be separated in the sense that there is a third point z and sets, A,B for which

(1) $$A \cap \bar{B} = \bar{A} \cap B = \emptyset,$$

(2) $$T - \{z\} = A \cup B,$$

(3) $$x \in A,\ y \in B.$$

The <u>associated order</u> $>$ for a tree is defined by selecting a fixed $e \in T$ and putting $x \leqslant y$ if and only if $x = e$, $x = y$, or x separates e and y.

Every compact, connected, chain in T has a maximal and minimal element with respect to its associated order, and every interval $[a,b]$ is a compact, connected chain.

To fill in the details of this discussion see [17].

<u>Theorem 3.3.1.</u> Every tree is the topological union of rectangular lattices.

Proof: Define a <u>branch</u> of a tree $(T, \mathcal{S})$ to be a maximal compact, connected chain. Notice that if B is a branch then it contains maximal and minimal elements, and so must be of the form $[e,x]$, e the least element of T. Moreover each member y_o of T is contained in a branch for the collection $N = \{[e,x] : y_o \in [e, x]\}$ has a maximal element which is a branch. Then $T = \bigcup B$, the union taken over all branches of T. It remains to show that any branch B is a rectangular lattice, and that $\mathcal{S} | B = l(B)$. The order on B, of course, is just the order in T restricted to B. Now each subbasic $l(B)$-closed set $\{x : x \leqslant (\geqslant) a\} \cap B$ is $\mathcal{S} | B$ closed so that $l(B) \leqslant \mathcal{S} | B$. Then $l(B)$ is a compact, connected topology because $\mathcal{S} | B$ is. Also $l(B)$ is Hausdorff and $\geqslant | B$ is continuous. For if $x \neq y$, $x, y \in B$, then, since B is a chain and $>$ is order dense, the sets

$A_x = \{t : t \ngeq z\} \cap B$,

$A_y = \{t : t \nleq z\} \cap B$,

are $l(B)$-open, and $A_x \quad A_y = \emptyset$, $A_x < A_y$, where z is any element of T such that $x < z < y$.

We now have $l(B) \leqslant \mathcal{S} | B$ with $\mathcal{S} | B$ compact and $l(B)$ Hausdorff. Thus, $\mathcal{S} | B = l(B)$.

This theorem, along with our previous results, makes the following two results clear.

<u>Corollary 1.</u> If $(T, \mathcal{S})$ is a tree, every $\mathcal{S}$- open set is a union of intervals, each of which is open in some branch. The converse is false.

Proof: If V is $\mathcal{S}$-open, $V = \bigcup_B (V \cap B)$; $V \cap B$ is $\mathcal{S} | B = l(B)$ open; thus $V \cap B$ is a union of intervals and so is V. That the converse is false may be seen by considering two rays which meet at one point.

<u>Corollary 2.</u> (Ward [21]) A closed, non-void subset of a tree

has maximal and minimal elements in itself.

Proof: If C is closed and non void so is $C \cap B$ for some branch B. Then $C \cap B$ is $t \setminus B = l(B)$ closed, and by Theorem 3.1.2 $\inf C \cap B$, $\sup C \cap B$ are in $C \cap B$. It follows that these are minimal (maximal) in C.

It is known that every tree has the f.p.p. We illustrate our results by proving this for the case when the number of branches is finite.

<u>Theorem 3.3.2.</u> A tree with finitely many branches has the f.p.p.

Proof: Let $f: T \longrightarrow T$ be continuous and $A = \{x : x \leqslant f(x)\}$. Then $A = \bigcup_B (A \cap B)$. Since the union is finite and each $A \cap B$ is closed so is A. If f has no fixed point then $A = \{x : x < f(x)\}$. By Corollary 2, A has a maximal element x_o. Pick μ, $x_o < \mu < f(x_o)$. The set $\{s : s > \mu\} \epsilon N(f(x_o))$, so by Corollary 1 f maps an interval containing x_o into $\{s : s > \mu\}$. Choose y to be a member of this interval such that $x_o < y < \mu$. Then $f(y) > \mu > y$ so $y \epsilon A$ and $x_o < y$. But this contradicts the maximal character of x_o. We conclude that f has a fixed point.

It is false that the topological union of rectangular lattices has the f.p.p. even when the number of components is finite. For a triangle is the topological union of its edges in an obvious way but fails to have the f.p.p.

<u>Problem</u>. A space X is <u>unicoherent</u> if it is not the union of two continua whose intersection is disconnected. Does every unicoherent topological union of rectangular lattices have the f.p.p.? We conjecture that it does.

<u>3.4. Pseudo fixed point theory</u>.

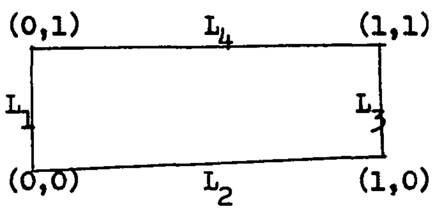
We turn now to spaces, which, for one reason or another, are not fixed point spaces. We show that a bit of information may be obtained about functions on certain of these spaces.

Let $(X, \mathcal{S})$ be a topological union of spaces $(Y_\lambda)_{\lambda \in \Lambda}$. A class of maps $\{f_{\lambda B} : (\lambda, B) \in \Lambda \times \Lambda\}$, where $f_{\lambda B}: Y_\lambda \to Y_B$, is said to be a <u>globally continuous class for X</u> provided that the following condition holds:

(GC) Given any net $(x_n : n \in D)$ which $\mathcal{S}$-converges to x, and given $\lambda(n) \in \Lambda$, λ, $B \in \Lambda$ with $x_n \in Y_{\lambda(n)}$, $x \in Y_\lambda$, the net $(f_{\lambda(n),B}(x_n) : n \in D)$ $\mathcal{S}$-converges to $f_{\lambda B}(x)$.

Although the definition is rather complicated notationally, it is a natural thing to require for topological unions of rectangular lattices. In fact, globally continuous classes are not hard to obtain for such spaces.

<u>Example 1.</u> Consider the space X in E^2 as shown below. X is the topological union of L_1, L_2, L_3, L_4.



The class of maps $f_{\ B}$ is globally continuous, where $f_{11} = $ id., $f_{22} = $ id., $f_{33} = $ id., $f_{44} = $ id.,

$f_{12}: (0,y) \to (y,0)$,

$f_{13}: (0,y) \to (1,1-y)$,

$f_{14}: (0,y) \to (1-y,1)$,

$f_{23}: (y,0) \to (1,1-y)$,

$f_{24}: (y,0) \to (1-y,1)$,

$f_{34}: (1,y) \to (y,1)$,

$f_{B\lambda} = f^{-1}_{\lambda B}$.

<u>Example 2.</u> Let X be the unit disk. Then X is the topological union of the rectangular lattices $(L_\theta)_{0 \leq \theta \leq 2\pi}$, where $L_\theta = \{(r \cos\theta, r \sin\theta), 0 \leq r \leq 1\}$. Take $f_{\theta\theta}$ to be the identity map on L_θ, and define $f_{\theta\delta} : L_\theta \to L_\delta$ by putting $f_{\theta\delta}(r \cos\theta, r \sin\theta) = (r \cos\delta, r \sin\delta)$.

Now let $(x_n,y_n) \to (x,y)$ and suppose $(x_n,y_n) \in L_{\theta_n}$, $(x,y) \in L_\theta$. If $0 \leq \delta \leq 2\pi$, then $f_{\theta_n,\delta}(x_n,y_n) = f_{\theta_n,\delta}(r_n \cos\theta_n, r_n \sin\theta_n) = (r_n \cos\delta, r_n \sin\delta)$, and $f_{\theta,\delta}(x,y) = f_{\theta,\delta}(r_o \cos\theta, r_o \sin\theta) = (r_o \cos\delta, r_o \sin\delta)$.

Since $(x_n,y_n) \to (x,y)$, $r_n \cos\theta_n \quad r_o \cos\theta$, $r_n \sin\theta_n \to r_o \sin\theta$, and so $r_n^2 \to r_o^2$. Since r_n, r_o are non negative $r_n \to r_o$. Then $(r_n \cos\delta, r_n \sin\delta) \to (r_o \cos\delta, r_o \sin\delta)$. Hence, $\{f_{\theta\delta} : 0 \leq \delta, \theta \leq 2\pi\}$ is a globally continuous class for X.

<u>Example 3.</u> Let $X = [1, \infty]$, then X is the topological union of the lattices $L_n = [n, n+1]$, $n = 1,2,\ldots$. Define $f_{n,n} = \text{id}$, and

$f_{n,n+1}(x) = x + 1$,

$f_{n,n-1}(x) = x - 1$,

$f_{n,m}(x) = x + m + n + 1$ if $m - n$ is even and not zero,

$f_{n,m}(x) = x + m - n$ if $m - n$ is odd and not 1.

Then $f_{n,m}$ is a globally continuous class for X.

<u>Example 4.</u> Let X be the subspace of E^2 consisting of all line segments connecting $(n,0)$ and $(n,1)$ n an integer. A globally continuous class for X consists of all the translation homeomorphisms of these spaces in addition to all the identity maps.

The following result gives information about pseudo fixed points on each rectangular component of a space equipped with a globally continuous class.

Theorem 3.4.1. Let X be the topological union of rectangular lattices L_λ, $\lambda \varepsilon \Lambda$. Let $\{f_{\lambda B}\}$ be any globally continuous class for X and let f: $X \rightarrow X$ be continuous. Then, for each $\lambda \varepsilon \Lambda$ there is a $B \varepsilon \lambda$ and an $x_o \varepsilon L$ such that $f_{B\lambda} f(x_o) = x_o$. Both B, x_o depend on λ.

Proof: For each $x \varepsilon X$ consider the class of all L_λ which contain f(x). From the axiom of choice there is a function $T: x \rightarrow L_{T(x)}$ with $f(x) \varepsilon L_{T(x)}$.

Now define $\Psi : XL_\lambda \rightarrow XL_\lambda$ in the following way: if $g \varepsilon XL_\lambda$, then $g(\lambda) \varepsilon L_\lambda$ for each λ, and $fg(\lambda) \varepsilon L_{T(g(\lambda))}$.

Then $f_{T(g\lambda)),\lambda} fg(\lambda) \varepsilon L_\lambda$, and $\Psi(g)$ is defined by putting $\Psi(g)(\lambda) = f_{T(g(\lambda)),\lambda} fg(\lambda)$ for each $\lambda \varepsilon \Lambda$.

The continuity of Ψ follows from the continuity of f and the global continuity of $\{f_{\lambda B}\}$.

By Theorem 3.1.4, if $\lambda \varepsilon \Lambda$ is given, there is an $f^1 \varepsilon XL_\lambda$ such that $p_\lambda \Psi f^1 = p_\lambda f^1$. Hence $f_{T(f^1(\lambda)),\lambda} ff^1(\lambda) = f^1(\lambda)$. The theorem now follows with $x_o = f^1(\lambda)$ and $B = T(f^1(\lambda))$.

Notice that the theorem applies to spaces which need not be compact or connected or have the f.p.p. The spaces in Examples 1,3,4 are not fixed point spaces. In Example 1, the map f which pushes each point one half unit in the direction indicated in the figure is continuous. As is guaranteed by the theorem, $(0,1/4) \xrightarrow{f} (1/4, 0) \xrightarrow{f_{21}} (0,1/4)$.

In Example 3, if f is any continuous map on X the theorem shows that for each m there is an $x_m \varepsilon [m,m+1]$ and an integer n_m such that one of the following equations holds:

$f(x_m) = x_m$,

$f(x_m) + 1 = x_m$,

$f(x_m) - 1 = x_m$,

$f(x_m) + m + n_m + 1 = x_m$,

$f(x_m) + m - n_m + 1 = x_m$.

In Example 4, let any continuous function f on X be given. For each L_n, f either fixes a point in L_n, or else moves some point in L_n along a line parallel to the x-axis to some L_m.

Problem. Does every topological union of rectangular lattices have a globally continuous class such that the $f_{\lambda B}$ are not constant maps?

Problem. It has recently been announced that the statement every two commuting functions on [0,1] have a common fixed point is false. However, many special results have been obtained over the years that this was still a conjecture. The problem makes sense with [0,1] replaced by any space X but it is doubtful that many results go through unless X has some order properties. How many results may be obtained with [0,1] replaced by rectangular lattice ?

Is there a complete characterization of rectangular lattices?

CHAPTER IV

REPRESENTATION BY PRODUCTS

As promised in the Introduction, representation of spaces by products will be studied in this chapter. In analogy with the ring case, Theorem 2.1.2, topological products, continuous maps, equivalence relations, play the role of Fundamental Theorem, ring products, homomorphisms, ideals, respectively.

In Section 1 necessary and sufficient conditions that a compactum be a subdirect product of subspaces is obtained.

In Section 2 this result is translated to one involving the partial orders on a certain class of spaces. Finally, it is shown that spaces representable by products will have the f.p.p. if there are enough homeomorphisms.

4.1 Subdirect products of compacta.

Let us begin with a definition. A topological space $(X, \mathcal{S})$ has a representation as a subdirect product of topological spaces $(X_\lambda, \mathcal{S}_\lambda)$ if X is isomorphic to a subspace S of XX_λ, and the projection maps ρ_λ map S onto X.

Theorem 4.1.1. A compact space X has a representation as a subdirect product of Hausdorff spaces $(X_\lambda)_{\lambda \varepsilon \Lambda}$ if and only if for each

$\lambda \varepsilon \Lambda$ there is a continuous map Φ_λ of X onto X_λ such that if $x \neq y$, there is at least one $\lambda_0 \varepsilon \Lambda$ for which $\Phi_{\lambda_0}(x) \neq \Phi_{\lambda_0}(y)$.

Proof: If $\Phi : X \rightarrow S \subset XX_\lambda$ is an homeomorphism define $\Phi_\lambda = \rho_\lambda \Phi$. Then the properties required of Φ_λ are clear.

Suppose now that $\Phi_\lambda: X \rightarrow X_\lambda$ is a continuous map for each $\lambda \varepsilon \Lambda$. Define $f: X \rightarrow XX_\lambda$ by putting $\rho_\lambda f(x) = \Phi_\lambda(x)$. Then f is continuous, and by assumption is one-one. Since X is compact and XX_λ is Hausdorff, f is a homeomorphism onto $S = f(X)$. Also, $\rho_\lambda(f(X)) = X_\lambda$ for each $\lambda \varepsilon \Lambda$ so X has a representation as a subdirect product of the X_λ.

<u>Theorem 4.1.2.</u> A compact space X has a representation as a subdirect product of Hausdorff spaces $(X_\lambda)_{\lambda \varepsilon \Lambda}$ if and only if there exists a family of compact equivalence relations r_λ ($\lambda \varepsilon \Lambda$) in X x X such that X/r_λ is isomorphic to X_λ, and $\bigcap r_\lambda = \Delta$, where Δ is the diagonal in XxX.

Proof: If X has a representation by the X_λ, then, by Theorem 4.1.1, there are continuous maps $\Phi_\lambda: X \rightarrow X_\lambda$ such that $x \neq y$ implies there exists $\lambda_0 \varepsilon \Lambda$ for which $\Phi_{\lambda_0}(x) \neq \Phi_{\lambda_0}(y)$. Then X/r_{Φ_λ} is isomorphic to X_λ by the Fundamental Theorem. Set $r_\lambda = r_{\Phi_\lambda}$.

Each r_λ is compact, for it is the inverse image of the diagonal in X under the continuous map $(x,y) \rightarrow (\Phi_\lambda(x), \Phi_\lambda(y)$; and the diagonal in X_λ is compact, for it is the image of X under the composition of the maps $x \rightarrow (x,x)$, Φ_λ. Also, each r_λ is an equivalence relation. Finally, $\bigcap r_\lambda = \Delta$. For, if $(x,y) \varepsilon r_\lambda$ for all $\lambda \varepsilon \Lambda$ and $x \neq y$, then $\Phi_\lambda(x) = \Phi_\lambda(y)$ for all λ which is a contradiction. Thus, the r_λ, as defined, satisfy the conditions of the theorem.

Conversely, suppose equivalence relations satisfying the hypotheses of the theorem exist. Define $\Phi_\lambda: X \to X_\lambda$ by $\Phi_\lambda = h_\lambda \pi_\lambda$, where the π_λ are the canonical maps $\pi_\lambda : X \to X/r_\lambda$, and the h_λ are the given homeomorphisms $h_\lambda : X/r_\lambda \to X$. Since $\bigcap r_\lambda = \Delta$, it follows from Theorem 4.1.1 that X has a representation by the X_λ.

The following result amounts to saying that almost every compact space has at least a countable number of compact equivalence relations which are properly decreasing.

Theorem 4.1.3. If X is a compact space which satisfies the descending chain condition for compact equivalence relations, then every continuous, one-one, map $f: X \longrightarrow X$ is a surjection.

Proof: Let $f: X \longrightarrow X$ be continuous and one-one and assume $f(X) \neq X$. Then $f \times f: X \times X \longrightarrow X \times X$ defined by $f \times f(x,y) = (f(x), f(y))$ has the same properties with respect to $X \times X$.

All inclusions in the chain $X \times X \supset f \times f(X \times X) \supset (f \times f)^2(X \times X)\ldots$ are proper. For the statement $(f \times f)^n(X \times X) = (f \times f)^{n+1}(X \times X)$ leads to $(f \times f)(X \times X) = X \times X$.

Now $(f \times f)^m(X \times X)$ is compact and the continuity of $x \longrightarrow (x,x)$ shows that the diagonal Δ (in X) is compact. Thus, $Y_n = (f \times f)^n(X \times X) \cup \Delta$ is a compact equivalence relation for each n. It is asserted that $Y_1 \supsetneq Y_2 \supsetneq Y_3 \ldots .$

To see this suppose $Y_n = Y_{n+1}$. Let $x \in X$ and select $y \neq x$. Then $(f \times f)^n(x,y) \in Y_n = Y_{n+1}$ but $(f \times f)^n(x,y) \notin \Delta$. Hence $(f \times f)^n(x,y) = (f \times f)^{n+1}(\xi, \eta)$ for some $(\xi, \eta) \in X \times X$. From this $x = f(\xi)$ so that f maps X onto itself contrary to assumption. This proves the assertion; but the assertion contradicts the fact that X satisfies the descending chain condition. Therefore, f is a surjection and the proof is complete.

The next theorem shows that maximal compact equivalence relations cannot play the role of maximal ideals, but a few interesting results will follow.

Theorem 4.1.4. If X is a compactum and r is a maximal compact equivalence relation in X, then X/r consists of exactly two points.

Proof: If R is a compact equivalence relation in X/r, then the same is true of $(\pi \times \pi)^{-1}R$ in X because a compact set in the compact Hausdorff space $X/r \times X/r$ is closed and $\pi \times \pi$ is continuous. Since $(\pi \times \pi)^{-1}R \supset r$ and r is maximal, it follows that $(\pi \times \pi)^{-1}R = r$ or $(\pi \times \pi)^{-1}R = X \times X$. Then $R = \Delta$ or $R = X/r \times X/r$.

Now since $r \neq X \times X$, there are at least two distinct members of X/r say $\tilde{x}$, $\tilde{y}$. From the remarks above, $\Delta \cup \{(\tilde{x},\tilde{y}), (\tilde{y},\tilde{x})\} = X/r \times X/r$ and hence $\tilde{x}$, $\tilde{y}$ are the only points in X/r.

Theorem 4.1.5. A compactum X is connected if and only if there is no maximal compact equivalence relation in $X \times X$.

Proof: If there is a compact equivalence relation r which is maximal then the previous result shows that there are only two points in X/r. Since X/r is Hausdorff, these points are open and then the inverse images of these points under $\pi: X \rightarrow X/r$ gives a disconnection of X.

Conversely, if X is disconnected, there is a proper subset A of X which is open and closed; Then, $r_o = AxA \cup A'xA'$ is an equivalence relation which is compact, because AxA, $A'xA'$, are closed and hence compact in $X \times X$. It will be shown that r_o is maximal.

If this is not the case, $r_o \subsetneq r$ with r a compact equivalence relation. Let $(x,y) \in X \times X$. If $(x,y) \notin r$, then (say) $x \in A$, $y \in A'$. There is an $(a,b) \in r$ such that $(a,b) \notin r_o$. Thus, there is an $(a,b) \in r$

such that (say) $a \in A$, $b \in A'$. It follows that $(x,a) \in r$, $(y,\bar{b}) \in r$, $(a,b) \in r$, and therefore $(x,y) \in r \circ r \circ r \subset r$. This shows that $r = X \times X$ and so r_o is maximal.

Suppose now that X is a totally disconnected compact Hausdorff space. Then there is an open base $(B_\lambda)_{\lambda \in \Lambda}$ for X such that each B_λ is also closed. By the proof of the preceding theorem $r_\lambda = B_\lambda \times B_\lambda \cup B_\lambda' \times B_\lambda'$ is a maximal compact equivalence relation on X; since X is Hausdorff $\bigcap r_\lambda = \Delta$. Hence, as an application of Theorem 4.1.2, the following old result is proved:

<u>Corollary 1.</u> Any totally disconnected compact Hausdorff space is homeomorphic to a closed subspace of a direct product of discrete, two point spaces.

<u>4.2 Representation and partial orders.</u>

Our main interest lies with compacta, and the reader is reminded that a compactum is uniformizable. By a <u>continuous curve</u> is meant a continuous image of [0,1]. A compactum $(X, \mathcal{J})$ is <u>admissable for partial orders</u> if it has a uniform structure J, with $t_J = \mathcal{J}$, such that if $V \in J$ and V excludes (x,y), (y,x), then a continuous curve containing (x,y), (y,x) meets V.

If X is a given space, pick any point $p = (x_o, y_o)$ not on the diagonal $\Delta \subset X \times X$. If $(x,y) \in X \times X$, then (x,y) is on the <u>p-side of Δ</u> if either $(x,y) \in \Delta$ or $(x,y) \notin \Delta$ but there exists a continuous curve f beginning at (x,y) and ending at p which fails to meet Δ. (i.e., $f(0) = (x,y)$, $f(1) = p$.)

If $S \subset X \times X$, we define $p(S)$ to be that part of S which lies on the p-side of Δ. A partial order $\rho \subset X \times X$ is <u>compact</u> if it is compact in the product topology. Notice that ρ is compact in a compactum if and only if ρ is closed, and this is true if and only if $(X_n, Y_n) \to (X,Y)$, $(X_n, Y_n) \in \rho$, implies $(X,Y) \in \rho$. This last is true if and only if

$$\begin{pmatrix} X_n \to X, \\ Y_n \to Y, \\ X_n \rho Y_n, \end{pmatrix} \quad \text{implies } X \rho Y.$$

Thus, for example, the order > for rectangular lattices is a compact partial order.

<u>Lemma 4.2.1.</u> If X is admissable for partial orders, then a continuous curve f, which begins at (y,x) and ends at (x,y), must meet Δ.

Proof: It may be assumed that $y \neq x$. Since the space (X, t_J) is Hausdorff if and only if $\bigcap \{V : V \in J\} = \Delta$, there is a $V_o \in J$ which excludes (y,x) and (x,y). For each $V \in J$, V V_o, there is an $X_v \in \text{imf} \cap V$. The net $(X_v : V \subset V_o)$ has a convergent subnet by the compactness of X. This subnet must converge to a point in im $f \wedge \Delta$, and this completes the proof.

<u>Remark.</u> That the preceding lemma fails to hold for uniform spaces with strong topological properties is seen by considering the (one-holed) torus.

<u>Lemma 4.2.2.</u> If X is admissable for partial orders and locally arcwise connected, if there are continuous curves beginning at (x,y), (y,z) and ending at $p = (x_o, y_o)$ which do not meet Δ, then there is a continuous curve beginning at (x,z) and ending at p which does not meet Δ or else $(x,z) \in \Delta$.

Proof: Let f, g: $[0,1] \to X \times X$ be the given maps with $f(0) = (x,y)$, $f(1) = p$, $g(0) = (y,z)$, $g(1) = p$.

If $z = y$ the conclusion follows for then $(x,y) = (x,z)$. Otherwise, by the arcwise connectedness of X, there is a homeomorphism $h: [0,1] \to X$, $h(0) = z$, $h(1) = y$.

Now, if $H: t \to (x,h(t))$ fails to meet Δ, then the curve from (x,z) to p by way of (x,y) fails to meet Δ. More precisely, put

$$H^1(t) = H(2t) \text{ for } 0 \leqslant t \leqslant 1/2,$$

$$= f(2t-1) \text{ for } 1/2 \leqslant t \leqslant 1.$$

Then H^1 is continuous and $H^1(0) = H(0) = (x,z)$, $H^1(1) = f(1) = p$.

Hence, H^1 is a continuous curve beginning at (x,z) and ending at p. Assume that H meets Δ. That is, $h(t_1) = x$ for $t_1 \varepsilon [0,1]$. Now $t_1 \neq 0$, $t_1 \neq 1$; for then, $(x,z) \varepsilon \Delta$ or $(x,y) \varepsilon \Delta$. In the first case the conclusion of the theorem follows. In the second a contradiction to the fact that f doesn't meet Δ obtains. Thus, if H meets Δ, $h(t_1) = x$ for $t_1 \varepsilon (0,1)$.

Define, in this case, $T : [0,1] \to [0,t_1]$ by $T(t) = t_1 t$. Then $M: t \to (hT(t),y)$ is a continuous curve between $M(0) = (z,y)$ and $M(1) = (x,y)$. Since h is one-one and takes value y at $t = 1$, M fails to meet Δ. Then, the curve from (z,y) to (y,z) by way of (x,y) and p fails to meet Δ. More precisely, define

$$F(t) = M(3t) \text{ if } 0 \leqslant t < 1/3,$$

$$= f(3t-1) \text{ if } 1/3 < t \leqslant 2/3,$$

$$= g(1-(3t-2)) \text{ if } 2/3 \leqslant t \leqslant 1.$$

Then F is continuous, F fails to meet Δ since M, f, g fail to do so, and $F(0) = (z,y)$, $F(1) = (y,z)$.

But this contradicts lemma 4.2.1 unless $z = y$. Then $(x,y) = (x,z)$ and the conclusion follows.

<u>Lemma 4.2.3.</u> Let X be admissable for partial orders and locally arcwise connected. If r is a compact equivalence relation in X x X, then p(r) is compact.

Proof: If (x_n, y_n) is a net in p(r) which converges to (x,y), then $(x,y) \varepsilon r$. If $x = y$, then $(x,y) \varepsilon p(r)$ by definition. Otherwise, there is an open set V containing (x,y) which excludes Δ. For some n_o, $(x_{n_o}, y_{n_o}) \varepsilon V$ and hence there is a continuous curve f which connects (x_{n_o}, y_{n_o}), (x,y) and fails to meet Δ.

By assumption there is a continuous curve g which connects (x_{n_o}, y_{n_o}) and p which does not meet Δ. Putting these together as in lemma 4.2.3, it is seen that (x,y) is on the p-side of Δ.

<u>Theorem 4.2.4.</u> If X is admissable for partial orders and locally arcwise connected, then X has a representation as a subdirect product of Hausdorff spaces if and only if there exist compact partial orders ρ_λ, on the p-side of Δ, such that inf ρ_λ is the trivial order.

Proof: If X has such a representation, then there are compact equivalence relations r_λ such that $\bigcap r_\lambda = \Delta$. For each λ, $p(r_\lambda)$ is a partial order. For,

(1) $(x,x) \varepsilon p(r_\lambda)$ by definition,

(2) if (x,y), $(y,x) \varepsilon p(r_\lambda)$, then $x = y$ by lemma 4.2.1,

(3) if (x,y), $(y,z) \varepsilon p(r_\lambda)$, then $(x,z) \varepsilon p(r_\lambda)$ by lemma 4.2.2.

By lemma 4.2.3, $p(r_\lambda)$ is compact if r_λ is, and clearly $\bigcap p(r_\lambda) = \Delta$. Conversely, suppose $(\rho_\lambda)_{\lambda \varepsilon \Lambda}$ is a collection of partial orders such that $\bigcap \rho_\lambda = \Delta$. Then $Y_\lambda = \rho_\lambda \cup \text{dual}\rho_\lambda \cup \Delta$ is a compact equivalence relation by the continuity of $(x,y) \rightarrow (y,x)$, and, also,

$\bigcap \gamma_\lambda = \Delta$. Then X has a representation by Hausdorff spaces by Theorem 4.1.2.

If we say that a representation by spaces X/r_λ is <u>non-trivial</u> if no $r_\lambda = \Delta$, then, using Theorem 4.1.2, the following result obtains:

<u>Theorem 4.2.5.</u> If X is admissable for partial orders and locally arcwise connected, then X has a non-trivial representation as a sub direct product of Hausdorff spaces if and only if there exist partial orders ρ_λ , on the p-side of Δ , none of which is the trivial order with $\inf \rho_\lambda$ the trivial order.

<u>Theorem 4.2.6.</u> Let f be a continuous map on a space X into itself, and suppose there is a representation $\Phi : X \rightarrow XX_v$ of X onto a product of fixed point spaces such that $p_v \Phi(x) = p_v \Phi(y)$ implies $p_v \Phi f(x) = p_v \Phi f(y)$ for each projection p_v. Then f has at least one fixed point.

Proof: From the diagram below, where the π_v are constructed from the Fundamental Theorem, we construct maps $g_v : X/r_v \rightarrow X/r_v$ by putting $g_v(\pi v(x)) = \pi v f(x)$.

$$\begin{tikzcd}
X \arrow[r, "\Phi"] \arrow[d, "\pi v"'] & XXv \arrow[r, "p_v"] & Xv \\
X/r_v \arrow[urr, "\phi_v"'] & &
\end{tikzcd}$$

Notice that g_v is well defined for

$\pi v(x) = \pi v(y)$ implies $(x,y) \in r_v$

implies $p_v \Phi(x) = p_v \Phi(y)$

implies $p_v \Phi f(x) = p_v \Phi f(y)$

implies $(f(x), f(y)) \in \gamma_v$

implies $\pi_v f(x) = \pi_v f(y)$.

Next, g_v is continuous for

$\pi v(x_n) \longrightarrow \pi v(x)$ implies $\phi_v\ \pi_v(x_n) \rightarrow \phi_v\ \pi_v(x)$

implies $p_v\ \Phi\ (x_n) \rightarrow p_v\ \Phi\ (x)$

implies $p_v\ \Phi f(x_n) \rightarrow p_v\ \Phi\ f(x)$

implies $\phi_v\ \pi_v\ f(x_n) \rightarrow \phi_v\ \pi_v\ f(x)$

implies $\pi_v\ f(x_n) \rightarrow \pi_v f(x)$

implies $g_v(x_n) \longrightarrow g_v(x)$.

Now g_v induces a continuous map $\phi_v\ g_v\ \phi_v^{-1}$: $X_v \longrightarrow X_v$, and since X_v is a fixed point space there is an $x^v \in X_v$ such that

(1) $\phi_v g_v \phi_v^{-1}(x^v) = x^v$.

Also, by commutativity of the diagram above,

(2) $p_v\ \Phi\ = \phi_v \pi_v$.

Select the x such that $p_v\ \Phi\ (x) = x^v$ for each v. Then,

$\phi_v\ \pi_v(x) = x^v$ by (2),

$\pi_v(x) = \phi_v^{-1}(x^v)$ for ϕ_v is one-one,

$g_v \pi_v(x) = g_v \phi_v^{-1}(x^v)$,

$\pi_v f(x) = g_v \phi_v^{-1}(x^v)$,

$\pi_v f(x) = \phi_v^{-1}(x^v)$ by (1),

$\phi_v \pi_v f(x) = x^v$,

$p_v\ \Phi f(x) = x^v$ by (2).

Thus, $p_v\ \Phi\ f(x) = p_v\ \Phi\ (x)$ for all v and hence $\Phi f(x) = \Phi x$. Since is one-one, $f(x) = x$ and this completes the proof.

CHAPTER V

REPRESENTATION FOR UNIFORM SPACES; LATTICE CONVERGENCE

De Marr [7] has represented metric spaces by a partially ordered system, and obtained the fixed point theorem for contraction maps in a new way by using this representation. In Section 1 of this chapter De Marr's principal results are generalized to include representation for Hausdorff uniform spaces. In particular, the contraction map theorem is generalized to complete, Hausdorff uniform spaces.

In Section 2 the problem of representing a lattice convergence by a topology is taken up, and some partial results are obtained.

5.1 Fixed point theorem for complete contractions on a complete, Hausdorff uniform space.

We remind the reader of some results from the theory of uniform spaces. (See section 2.1, [4], [14])

If (E,J) is a uniform space, let S(J) be the class of all pseudo metrics ρ on E whose natural uniform structure J_ρ is coarser than J. Then, it is known that $\sup_{\rho \in S(J)} J_\rho = J$. It is in this sense that a uniform space is generated by a class of pseudo metrics. In fact, one defining class of pseudo metrics for (E,J) is obtained as follows: for each U in a base for J, put $C_u(x,y) = \inf \{ 2^{-n} : (x,y) \varepsilon U^n \}$, and define $\rho_u(x,y) = \inf \sum_C C_u(x_{i-1},x_i)$, where the infirmum is taken over all chains $x = x_0, x_1, \ldots, x_n = y$ connecting x and y. Then $\{\rho_u\}$ is a defining class of pseudo metrics for (E,J) [4].

A uniform space (E,J) is Hausdorff if and only if $\cap \{V : V \varepsilon J\} = \Delta$, a net (filter) on E is a t_J-Cauchy net (filter) if and only if it is a t_J Cauchy net (filter) for each $\rho \varepsilon S(J)$.

Remark. In what follows if (E,d) is a metric space, then the class of pseudo metrics which consists precisely of d is a defining class of pseudo metrics for (E,d). Thus, our results are true generalizations of those of De Marr [7]; in particular, the fixed point theorem to be obtained is a true generalization of the classical contraction map theorem.

Definition 5.1.1. Let (E,J) be a Hausdorff uniform space and let S(J) be a defining class of pseudo metrics. Let R be the real numbers and put $X = E \times R$. If $x = (p, \lambda)$, $y = (q,\mu)$ are in X, define $x \leqslant y$ if and only if $\rho(p,q) \leqslant \mu - \lambda$ for all $\rho \varepsilon S(J)$.

Lemma 5.1.1. The set X in the previous definition is partially ordered by $<$.

Proof: Suppose $x \leqslant y$ and $y \leqslant x$. Then $\rho(p,q) \leqslant \mu - \lambda$ and $\rho(q,p) \leqslant \lambda - \mu$ for each $\rho \varepsilon S(J)$. Thus, $\rho(p,q) = 0$ for each $\rho \varepsilon S(J)$ and $\mu = \lambda$. Since (E,J) is Hausdorff, $\cap \{V : V \varepsilon J\} = \Delta$. Hence, V ε J implies

$$\begin{aligned} V = \bigcap_\rho V \varepsilon_\rho \quad & \text{implies } (p,q) \varepsilon V \\ & \text{implies } (p,q) \varepsilon \cap \{V : V \varepsilon J\} \\ & \text{implies } (p,q) \varepsilon \Delta \\ & \text{implies } p = q. \end{aligned}$$

It has been shown that $x \leqslant y$, $y \leqslant x$, implies $x = y$. The other two properties required of a partial order follow from the non-negativity property and the triangle inequality which hold for each $\rho \in S(J)$.

Definition 5.1.2. Let $(S, >)$ be a partially ordered set. If for every increasing [decreasing] net N bounded above [below] sup N [inf N] exists, then $(S, >)$ is said to be upper [lower] net-complete.

Theorem 5.1.2. Let (E,J) be a complete, Hausdorff uniform space and let $(X, >)$ be the partially ordered set of Lemma 5.1.1. Then $(X, >)$ is upper and lower net complete.

Proof: Let $(y_n : n \in D)$ be a decreasing net in X such that $y_n \geqslant y$ for all $n \in D$. Put $y_n = (p_n, \lambda_n)$ and $y = (p, \lambda)$.

It is asserted that $(\lambda_n : n \in D)$ is a decreasing net of real numbers bounded below by λ. To see this notice that if $n \geqslant m$, then $y_n \leqslant y_m \leqslant y$ so that

$$\rho(p_n, p_m) \leqslant \lambda_m - \lambda_n,$$

$$\rho(p, p_m) \leqslant \lambda_m - \lambda, \text{ for all } \lambda \in S(J).$$

Then, since ρ is non-negative, $\lambda n \leqslant \lambda m$, $\lambda m \geqslant \lambda$. This proves our assertion; hence $\lambda_n \longrightarrow \lambda_o = \inf \lambda_n$.

Now, since $\rho(p_n, p_m) \leqslant |\lambda_m - \lambda_n|$ it follows that the net $(p_n : n \in D)$ is at t_J Cauchy net for each $\rho \in S(J)$. Hence $(p_n : n \in D)$ is a t_{J_ρ} Cauchy net and, since (E,J) is complete, $p_n \xrightarrow[t_J]{} x$ for some $x \in E$. Put $t = (x, \lambda_o)$. It is asserted that $t = \inf y_n$. To show this it must be demonstrated that

(1) $$t \leqslant y_n \text{ for all } n \in D,$$

(2) if $t' = (x', \lambda')$ y_n for all $n \in D$, then $t' \leqslant t$.

<u>Proof of 1.</u> We must show that for all $n \in D$ and all $\rho \in S(J)$ $\rho(x,p_n) \leq \lambda_n - \lambda_o$. Suppose contrariwise that for some $\rho_o \in S(J)$, $n_o \in D$, $\rho_o(x,p_{n_o}) > \lambda_{n_o} - \lambda_o$. Then define $\epsilon = \rho_o(x,p_{n_o}) - (\lambda_{n_o} - \lambda_o) > 0$. There exists $n_1 \in D$ such that if $n \geqslant n_1$, $\rho_o(x,p_n) < \rho_o(x,p_{n_o}) - (\lambda_{n_o} - \lambda_o)$. Then, for $n \geqslant n_1$, $\rho_o(x,p_n) < \rho_o(x,p_n) + \rho_o(p_n,p_{n_o}) - (\lambda_{n_o} - \lambda_o)$, or $\rho_o(p_n,p_{n_o}) > \lambda_{n_o} - \lambda_o$. However $\rho_o(p_n,p_{n_o}) \leq |\lambda_{n_o} - \lambda_n|$. Putting these last two statements together the relation

(a) $\rho_o(p_n,p_{n_o}) \leq |\lambda_{n_o} - \lambda_n|$ for $n > n_1$, obtains.

Select $n \in D$ such that $n \geqslant n_o$, $n \geqslant n_1$. Then $\lambda_{n_o} - \lambda_o < \lambda_{n_o} - \lambda_n$ from (a) and so $\lambda_o > \lambda_n$ which contradicts $\lambda_o = \inf \lambda_n$. This contradiction shows that 1. holds.

<u>Proof of 2.</u> If $t' = (x', \lambda') \leq y_n$ for all $n \in D$, then for all $\rho \in S(J)$ and all $n \in D$, $\rho(x',p_n) < \lambda_n - \lambda'$. It follows that $\rho(x', x) \leq \lambda_o - \lambda'$ and hence that $(x', \lambda') \leq (x, \lambda_o)$.

The proof that $(X, >)$ is lower net complete is complete. The proof that $(X, >)$ is upper net complete is similar.

<u>Corollary 1.</u> If (E,J) is a complete, Hausdorff uniform space and C is a chain in $(X, >)$ which is bounded above [below], then sup C [inf C] exists.

Proof: If $C = \{(p_\lambda, \mu_\lambda), \lambda \in \Lambda\}$ with $(p_\lambda, \mu_\lambda) \leq (p_o,\mu_o)$ for all $\lambda \in \Lambda$, then $\mu_\lambda \leq \mu_o$ for all $\lambda \in \Lambda$. So there is an increasing net $(\mu_n : n \in D)$, $D \subset \Lambda$, such that $\mu_n \longrightarrow \mu = \sup \{\mu_\lambda : \lambda \in \Lambda\}$. Then the net $((p_n,\mu_n): n \in D)$ is increasing in X; for if $n \geqslant m$ and $(p_n,\mu_n) < (p_m,\mu_m)$, then, for $\rho \in S(J)$, $\rho(p_n,p_m) < \mu_m - \mu_n < 0$ a contradiction.

The existence of sup C now follows from the preceeding theorem, and the proof for the existence of inf C is similar.

Our definition of contraction is as follows.

Definition 5.1.3. Let (E,J) be a uniform space, $S(J)$ a defining class of pseudo metrics. A map $f:E \to E$ is a complete contraction on E if there is an $\alpha \in R$, $\alpha < 1$, such that for all $p,q \in E$ and all $\rho \in S(J)$, $\rho(f(p), f(q)) \leq \alpha\rho(p,q)$.

The proof of the following result is essentially the same as that given in [6].

Theorem 5.1.3. Let (E,J) be a complete, Hausdorff uniform space and $(X, >)$ the partially ordered space of Theorem 5.1.2. If $F: X \to X$ is isotone ($x \leq y$ implies $F(x) \leq F(y)$) and there exists $x_0, x_1 \in X$ such that $x_0 \leq F(x_0) \leq F(x_1) \leq x_1$, then F has a fixed point.

This section is concluded with the main result.

Theorem 5.1.4. A complete contraction on a complete, Hausdorff uniform space has a unique fixed point.

Proof: Let $f:E \to E$ be the complete contraction. Define $F:X \to X$ by putting $F(p,\mu) = (f(p), \lambda_0\mu)$, where λ_0 is the contraction coefficient.

Then F is isotone because if $\rho \in S(J)$ and $(p,\mu) \geq (p',\mu')$, then $\rho(p,p') \leq \mu - \mu'$ so $\rho(f(p), f(p')) < \lambda_0(\mu - \mu')$. This last means $f(p,\mu) \geq F(p',\mu')$.

Now there is no loss of generality in assuming that the class $S(J)$ is uniformly bounded. For each $\rho \in S(J)$ can be replaced by $\rho/1+\rho$. For each $\rho \in S(J)$ define λ_ρ by the equation $\rho(p_0, f(p_0)) = (1 - \lambda_0)\lambda_\rho$, where p_0 is any point in E. Put $\gamma_0 = \sup \lambda_\rho$, and $x_0 = (p_0, -\gamma_0)$, $x_1 = (p_0, \gamma_0)$. An easy computation shows that $x_0 \leq F(x_0) \leq F(x_1) \leq x_1$, hence, by Theorem 5.1.4, F has at least one fixed point y_0. But

$F(y_0) = y_0$ means $F(y_0) \leqslant y_0$ and $F(y_0) \geqslant y_0$ so that $\rho(f(y_0), y_0) = 0$ for all $\rho \in S(J)$. As in Lemma 5.1.1, this means $(f(y_0), y_0) \in \bigcap \{V : V \in J\} = \triangle$ and so $f(y_0) = y_0$. If y_1 is another fixed point one easily obtains $\rho(y_0, y_1) = 0$ for all $\rho \in S(J)$ and hence $y_0 = y_1$.

5.2. The general problem in lattice convergence.

Given a lattice L there are a number of types of convergence on L which may be considered, for example:

(a) 1(L) convergence,

(b) K(L) convergence,

(c) O(L) convergence,

(d) O-convergence,

(e) ^{1} convergence, $x_n \xrightarrow[1]{} x$ if and only if every subnet of (x_n) has an o-convergent subnet which o-converges to x.

A lattice equipped with a convergence cannot very well be studied until succinct necessary and sufficient conditions for convergence are established.

For instance, although the interval topology is rather old, having been studied by Birkoff, it was only recently that Atsumi showed that $(x_n : n \in D)$ 1(L)-converges to x if and only if $\bigvee_C \bigwedge (x_n : n \in C) \leq x \leq \bigwedge_C \bigvee (x_n : n \in C)$. This fact was used repeatedly in Chapter III. We remark that K(L), O(L) convergence has been studied by Insel [10], [11].

Unsolved Problems. Given a lattice L and a convergence c. When can a topology $\mathcal{T}$ be defined on L such that a net c-converges to x if and only if it $\mathcal{T}$-converges to x?

Atsumi's result may be interpreted in the following way: the lattice convergence $x_n \xrightarrow[c]{} x$ if and only if $\bigvee_C \bigwedge (x_n : n \in C) \leqslant x \leqslant \bigwedge_C \bigvee (x_n : n \in C)$ is definable in terms of a topology for complete lattices, and, in fact, the defining topology is l(L).

<u>Unsolved Problems</u>. Given a space (X, δ) and a lattice convergence c, when does there exist a partial order $>$ on X such that X is a lattice and a net c converges to x if and only if it δ-converges to x?

Let (X, δ) be an arc, torus, continuum with exactly two non-cutpoints, or arbitrary product of intervals. Let c be the lattice convergence defined by $x_n \xrightarrow[c]{} x$ if and only if $\bigvee_C \bigwedge (x_n : n \in C) \leqslant x \leqslant \bigwedge_C \bigvee (x_n : n \in C)$. Then, the results of Chapter III show that X admits an order $>$ such that a net δ-converges to x if and only if it c-converges to x.

In the remainder of this chapter we will examine the problems above in the case of o-convergence.

In this case DeMarr [6] has solved the second problem. DeMarr's result is briefly described below.

<u>Definition 5.2.1.</u> A topological space (X, δ) is an O space if and only if it is topologically homeomorphic to a subset Λ_o of a complete lattice Λ, and a net in Λ_o converges to a point in Λ_o if and only if it o-converges to this point in the lattice.

<u>Theorem 5.2.1.</u> (DeMarr [6]) (X, δ) is an O space if and only if it is Hausdorff and regular.

We remark that if L is a complete lattice and δ is a topology on L for which (L, δ) is an O space it need not follow that δ-convergence and o-convergence coincide. That is, the order placed on L as a consequence of Theorem 5.2.1 has nothing to do with the order already there even when the order and the topology are natural.

<u>Example</u>. Let (E,J) be the real numbers with metric uniform structure. Then $(C(E), t_{2^J})$ is regular and Hausdorff. The first assertion is true in any uniform space, and the second is true for any space $(C(E), t_{2^J})$, regardless of whether (E,J) is Hausdorff. Thus, $(C(E), t_{2^J})$ is an O space and a complete lattice with its natural order. But o-convergence and t_{2^J}-convergence do not coincide.

For let $x_1, x_2, \ldots$ be a fixed sequentialization of the rational reals and define $A_n = \{x_1, \ldots x_n\}$. Then the net $(A_n : n \in Z^t)$ o-converges to E because

$$\overline{\bigcup_{n_o} \bigcap_{n \geqslant n_o} A_n} = \lim\inf A_n = \bigcap_{n_o} \overline{\bigcup_{n \geqslant n_o} A_n}$$

$$= \lim\sup A_n = E.$$

But $(A_n : n \in Z^t)$ fails to t_{2^J}-converge to E. For this would require that if $\epsilon > o$ is given, there exists an $n_\epsilon \in Z^t$ such that if $n \geqslant n_\epsilon$ then $E \subset \{x : \text{there exists } x_n \in A_n \text{ such that } |x - x_n| < \epsilon\}$. This, however, is not possible.

A partial solution to the first problem for o-convergence will now be given. The problem will be attacked by means of convergence schemes which we briefly describe. The reference for convergence schemes is [4].

Let E be a set, F(E) the class of filters on E. If $\delta : E \to F(E)$ is a function from E into F(E), write $\mathcal{F} \xrightarrow[\delta]{} x$ if and only if $\mathcal{F} \in \delta(x)$.

A function $\delta : E \to F(E)$ is called a <u>convergence scheme</u> on E. The usual notation for filters will be employed. $\mathcal{F} \geqslant g$ means every $G \in g$ is in $\mathcal{F}$. An <u>ultrafilter</u> $\mathcal{U}$ is a filter which is maximal with respect to the order just defined. Every filter is contained in an ultrafilter. A collection B of subsets of E is a <u>filter base</u> on E if it is closed under finite intersections of its members and if $\emptyset \notin B$. Then [B], the <u>filter generated by B</u>, consists of B and all supersets of members of B. If δ is a convergence scheme on E, some of the following are usually required of δ.

1. $\dot{x} \to x$, $\dot{x} = [\{x\}]$.
2. $\mathcal{F} \to x$, $\mathcal{G} \geqslant \mathcal{F}$ implies $\mathcal{G} \to x$.
3. $\mathcal{F} \to x$, $\mathcal{G} \to x$ implies $\mathcal{F} \wedge \mathcal{G} \to x$.
4. $\mathcal{F} \not\to x$ implies there exists $\mathcal{G} \geqslant \mathcal{F}$, such that if $\mathcal{H} \geqslant \mathcal{G}$ then $\mathcal{H} \to x$.
5. $\mathcal{F} \not\to x$ implies there exists $V \subset E$ with $x \in V$, $x \notin \mathcal{F}$, such that if $y \in V$ and $\mathcal{G} \to y$, then $V \in \mathcal{G}$.

If δ is a convergence scheme that satisfies 1., 5., then there exists a topology δ^1 on E such that filters δ-converge if and only if they δ^1 converge [4].

It is well known that filters and nets give the same convergence theory in the following sense [4]:

(a) Given a net N, the filter $\mathcal{F}^N$ of final sections of N converges to precisely the same points as does N.

(b) Given a filter $\mathcal{F}$, the net $N^{\mathcal{F}}$ defined on $\{(F,n,x) : x \in F,\ F \in \mathcal{F},\ n \in Z^t\}$

with dictionary order by $N^{\mathcal{F}}(F,n,x) = x$ converges to exactly the same points as does $\mathcal{F}$.

Lemma 5.2.2. If, for a complete lattice E and $\mathcal{F} \in F(E)$ we define $\liminf \mathcal{F} = \inf_{F \in \mathcal{F}} \sup F$ and $\limsup \mathcal{F} = \sup_{F \in \mathcal{F}} \inf F$, then a net N [filter $\mathcal{F}$] o-converges to y if and only if its associated filter, $\mathcal{F}^N$ [net $N^{\mathcal{F}}$] o-converges to y.

Proof: If, for a filter $\mathcal{F}$, $\liminf \mathcal{F} = y = \limsup \mathcal{F}$, let F_o be arbitrary in $\mathcal{F}$ and x_o be arbitrary in F_o and let $n_o \in Z^t$. Put $S_{(F_o,n_o,x_o)} = \{N^{\mathcal{F}}(F,n,x) : (F,n,x) \geqslant (F_o,n_o,x_o)\}$.
Then $S_{(F_o,n_o,x_o)} \subset F_o$ so that $\liminf N^{\mathcal{F}} \geqslant \inf S_{(F_o,n_o,x_o)} \geqslant \inf F_o$. Since F_o was arbitrary in $\mathcal{F}$ it follows that $\liminf N^{\mathcal{F}} \geqslant \liminf \mathcal{F} = y$. Similarly, $\limsup N^{\mathcal{F}} \leqslant y$, so $\liminf N^{\mathcal{F}} \geqslant y \geqslant \limsup N^{\mathcal{F}}$. It follows that $N^{\mathcal{F}}$ o-converges to y. The proof for nets is similar.

Now define, for filters $\mathcal{F}$ on E, $\mathcal{F} \xrightarrow[o\downarrow]{} x$ if and only if $\liminf \mathcal{F} \geqslant x$, $\mathcal{F} \xrightarrow[o\uparrow]{} x$ if and only if $\limsup \mathcal{F} \leqslant x$.

Lemma 5.2.3. Suppose that the convergence schemes $o\downarrow$, $o\uparrow$ satisfy 1.,5., for some complete lattice E. Then there exists a topology $\mathcal{S}$ on E such that nets o-converge if and only if they $\mathcal{S}$-converge.

Proof: By assumption there are topologies $\mathcal{S}(o\uparrow)$, $\mathcal{S}(o\downarrow)$ on E such that filters $(o\uparrow)$ [$(o\downarrow)$] converge if and only if they $o\uparrow$ [$o\downarrow$] converge. Put $\mathcal{S} = \mathcal{S}(o\uparrow) \wedge (o\downarrow)$. Then, using Lemma 5.2.2, one sees that a net o-converges to x if and only if it $\mathcal{S}$-converges to x.

Definition 5.2.2. A complete lattice is an O lattice if, for each $a \in L$, $\sup\{\mu : \mu < a\} = a$, and $\inf\{\mu : \mu > a\} = a$, provided that the sets under consideration are not empty.

Theorem 5.2.4. Let L be an O lattice. If there fails to be a topology δ on L such that o-convergence and δ-convergence coincide, then there is an ultrafilter $\mathcal{U}$ on L and a point $u_o \in L$ such that $\{u : u, u_o \text{ are not comparable}\}$ is in $\mathcal{U}$, and either $\lim\inf \mathcal{U} > u_o$ or $\lim\sup \mathcal{U} < u_o$.

Proof: Since the schemes $o\downarrow$, $o\uparrow$ satisfy 1., it must be that (say) $o\downarrow$ fails to satisfy 5. if the hypotheses of this theorem hold. Suppose $\mathcal{F} \not\xrightarrow[o\downarrow]{} x$. Then $\lim\inf \mathcal{F} \not\geqslant x$. Since L is an O lattice, there is a point $u_o \in L$ such that $u_o < x$ and $\lim\inf \mathcal{F} \not\geqslant u_o$.

Now define $V = \{\mu : \mu > u_o,\ \mu \not\leqslant \lim\inf \mathcal{F}\}$. Notice that $\inf V = u_o$ and $x \in V$. Also, $V \notin \mathcal{F}$, for then $\lim\inf \mathcal{F} \geqslant \inf V = u_o$ which is a contradiction. But since 5. fails to be satisfied for the scheme $o\downarrow$, it must be that there is a filter $\mathcal{W}$ such that $\mathcal{W} \xrightarrow[o\downarrow]{} y$, $y \in V$, but $V \notin \mathcal{W}$.

Then $\{W \cap V' : W \in \mathcal{W}\}$ is a filter base for a filter $\mathcal{B}$ having the property $\mathcal{B} \geqslant \mathcal{W}$. Let $\mathcal{U}$ be an ultrafilter such that $\mathcal{U} \geqslant \mathcal{B} \geqslant \mathcal{W}$. $\mathcal{U}$ contains $V' = \{u : u \not> u_o\} \cup \{u : u \leqslant \lim\inf \mathcal{F}\}$, hence it contains one of the sets in the right hand member. It cannot contain $\{u : u \leqslant \lim\inf \mathcal{F}\}$ for then $\lim\inf \mathcal{F} \geqslant \lim\sup \mathcal{U} \geqslant \lim\inf \mathcal{U} \geqslant \lim\inf \mathcal{W} \geqslant y > u_o$, which contradicts $\lim\inf \mathcal{F} \not\geqslant u_o$.

Thus $\{u : u \not> u_o\} \in \mathcal{U}$, and by using the ultrafilter property again with $\{u : u \not> u_o\} = \{u : u \leqslant u_o\} \cup T$, T the set of elements not comparable to u_o, we find $T \in \mathcal{U}$. $\mathcal{U}$ then satisfies the conditions of the theorem. A similar result obtains if $o\uparrow$ fails to satisfy 5.

With respect to nets, this theorem takes the following form:

<u>Corollary</u>. Let L be an O lattice and suppose no topology δ on L exists such that nets o-converge if and only if they δ-converge. Then, there is a universal net $(x_v : v \in V)$ and a point $u_o \in L$ such that no x_v is comparable to u_o and either $\liminf x_v > u_o$ or $\limsup x_v < u_o$.

Almost every infinite, complete lattice is an O lattice, even, for example, the lattice of closed subsets of a Hausdorff space in which points are not open. Hence one would not expect that being an O lattice is necessary and sufficient to insure the existence of a topology δ for which o and δ convergence coincide. In fact, it is not necessary even for chains; put $L = [0,1] \cup [2,3]$ with the usual order, then L is not an O space but o-convergence and topological convergence coincide. It is sufficient for chains; this is clear from Theorem 5.2.4. From these remarks we see that connectedness is essential in

<u>Theorem 5.2.5.</u> Let L be a complete chain; in order that there exist a connected topology δ on L such that δ-convergence and o-convergence coincide, it is necessary and sufficient that L be an O lattice.

Proof: If L is an O lattice and the conclusion fails to hold, then, since L is a chain, Theorem 5.2.4 shows that $\emptyset$ is a member of a filter which is a contradiction.

Conversely, suppose δ is a connected topology on L such that o and convergence coincide. Since o-convergent nets in $\{u : u \leq a\}$ converge to points in this set, $\{u : u > a\}$ is δ-open. If $\inf\{u : u > a\} > a$ then no $x \in L$ can satisfy $\inf\{u : u > a\} > x > a$. Then L is disconnected by $\{u : u < \inf\{u : u > a\}\}$ and $\{u : u > a\}$.

This is a contradiction and the proof is completed with a similar argument for suprema.

The difficulty of solving the first problem of section 2 just for chains indicates, as Insel has mentioned, that a general solution, if possible at all, would be very complex. As illustrated here, an attack by means of convergence schemes might yield results in special cases.

Problem. The contraction map theorem has been generalized by the author, Davis [5] and others. To carry this one step further can it be generalized to the setting of convergence schemes? This would entail finding the right definition of contraction map, etc. for convergence schemes. That is, the properties required would need to be phrased in terms of convergent filters.

More generally, given a fixed point theorem for topological spaces, is there a generalized theorem for convergence schemes?

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