UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

SELF-DUAL REPRESENTATIONS WITH VECTORS FIXED UNDER AN IWAHORI SUBGROUP

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

Degree of

DOCTOR OF PHILOSOPHY

By

KUMAR BALASUBRAMANIAN
Norman, Oklahoma
2012

SELF-DUAL REPRESENTATIONS WITH VECTORS FIXED
UNDER AN IWAHORI SUBGROUP

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

BY

Dr. Alan Roche, Chair

Dr. Ralf Schmidt

Dr. Tomasz Przebinda

Dr. Jonathan Kujawa

Dr. Patrick Livingood

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my adviser Dr. Alan Roche, for his ever-present guidance, encouragement and help, without which this dissertation would never have been completed. I am forever grateful to him for having me as his student and for inspiring me with his tremendous knowledge and expertise in mathematics.

I would also like to thank Dr. Ralf Schmidt, Dr. Tomasz Przebinda, Dr. Jonathan Kujawa and Dr. Patrick Livingood for serving in my doctoral committee. My special thanks to Dr. Steven Spallone for his many valuable suggestions.

I would also like to thank the staff of the mathematics department for taking care of all the formalities and making my stay here a very enjoyable one. Lastly, I would like to thank my family and my friends especially Varadha, Kashyap, Houssein, Salam, Keshav, Tetsuya, Santosh, Esta, Yoe, and Thomas for their constant encouragement and support.

Contents

Abstract

Let G be the group of F-points of a split connected reductive F-group over a non-Archimedean local field F of characteristic 0. Let π be an irreducible smooth self-dual representation of G. The space W of π carries a non-degenerate G-invariant bilinear form $(\, , \,)$ which is unique up to scaling. The form is easily seen to be symmetric or skew-symmetric and we set $\varepsilon(\pi) = \pm 1$ accordingly. In this thesis, we show that $\varepsilon(\pi) = 1$ when π is a generic representation of G with non-zero vectors fixed under an Iwahori subgroup I.

Chapter 1

Introduction

Let G be a group and (π, V) be an irreducible representation of G such that $\pi \simeq \pi^\vee$ ($\pi^\vee$ denotes the dual or contragredient of π). This isomorphism gives rise to a non-degenerate G-invariant bilinear form on V which is unique up to scalars, and consequently is either symmetric or skew-symmetric. Accordingly, we set

$$\varepsilon(\pi) = \begin{cases} 1 & \text{if the form is symmetric} \\ -1 & \text{if the form is skew-symmetric} \end{cases}$$

and call it the sign of π. In this thesis, we study this sign for a certain class of representations of a reductive p-adic group G. To be more precise, we study the sign for representations with non-zero vectors fixed under an Iwahori subgroup in G. The structure of representations with Iwahori fixed vectors is well understood and exhibits many of the complications that occur when studying representations with fixed vectors under other compact open subgroups of G. This is one of the principal reasons we restrict our analysis to this particular class of representations.

1.1 Overview of the problem

Let G be a group and (π, V) be an irreducible self-dual complex representation of G. We say that the representation π is realizable over the real numbers if there exists a G-invariant real subspace W of V such that $V \simeq \mathbb{C} \otimes_{\mathbb{R}} W$ as representations of G. A classical problem in representation theory was to determine when such a W exists. This problem was settled for finite groups by Frobenius and Schur more than a century ago. They showed that π is realizable over the real numbers precisely when $\varepsilon(\pi) = 1$. They also gave a formula to compute $\varepsilon(\pi)$ in terms of the character χ_π of the representation π. They showed that

$$\varepsilon(\pi) = \frac{1}{|G|} \sum_{g \in G} \chi_\pi(g^2).$$

The sign $\varepsilon(\pi)$ has been fairly extensively studied for connected compact Lie groups and finite groups of Lie type. In this setting, the sign is sometimes referred to as the Frobenius-Schur indicator of π. There is a lot of literature available on computing these signs for such groups. For connected compact Lie groups the sign can be computed using the dominant weight attached to the representation π (see [2] pg. 261-264). In [7], Gow showed that for q a power of an odd prime and F_q the finite field with q elements, irreducible self-dual complex representations of $\mathrm{SO}(n, F_q)$ are always realizable over $\mathbb{R}$. He also showed that the same is true for any non-faithful representation of $\mathrm{Sp}(n, F_q)$. The proofs involve a detailed analysis of the conjugacy classes of these groups and are computationally quite complicated. In [10], Prasad introduced an elegant idea to compute the sign for a certain class of representations of finite groups of Lie type. These representations are called generic. He used this idea to determine the sign for many finite groups of Lie type, avoiding tedious conjugacy class computations. In a subsequent paper [11], he extended this idea to representations of a reductive p-adic group G and computed the sign for generic representations of certain classical groups in some cases. In [15], Vinroot used Prasad's idea along with other techniques to compute the sign for an irreducible self-dual representation of $\mathrm{GL}(n, F)$ where F is a p-adic field.

In this thesis, we determine the sign $\varepsilon(\pi)$ when π is a certain type of representation of an arbitrary connected reductive p-adic group G. Suppose K is the maximal compact open subgroup of G and π has non-zero vectors fixed under K. In this situation, it is easy to see that $\varepsilon(\pi)$ is always 1. A natural question is what is $\varepsilon(\pi)$ if π has non-zero Iwahori fixed vectors. There is enough evidence that the sign is one in the Iwahori fixed case. We have not been able to prove the result in complete generality. However, we do address a particular case of this problem. To be more precise, we prove the following theorem.

Theorem 1.1 (Main Theorem). *Let (π, W) be an irreducible smooth self-dual representation of G with non-zero vectors fixed under an Iwahori subgroup in G. Suppose that π is also generic. Then $\varepsilon(\pi) = 1$.*

1.2 Organization

In Chapter 2, we recall the basic definitions and theorems which we need throughout this report. In Chapter 3, we explain how the sign is attached to a self-dual representation π. In Chapter 4, we discuss representations of some classical groups. We use a theorem of Waldspurger to show that many representations of classical groups are self-dual. We have included this chapter just to motivate the problem of studying signs. In Chapter 5, we explain the important ideas of Rodier and Prasad which we use to study the sign. We also mention a few results about restricting an irreducible representation to a subgroup and recall an important characterization of representations with vectors fixed under an Iwahori subgroup. In Chapter 6, we prove the main theorem.

Chapter 2

Preliminaries

In this chapter, we recall the basic definitions and theorems which we need throughout this report and give some examples.

2.1 Valuations and local Fields

Let F be a field. A valuation on F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ such that, for $x, y \in F$, we have

$$|x| = 0 \Leftrightarrow x = 0$$
$$|xy| = |x||y|$$
$$|x+y| \leq |x| + |y|.$$

We say $|\cdot|$ is non-Archimedean if it satisfies

$$|x+y| \leq \max\{|x|, |y|\}, x, y \in F.$$

The valuation $|\cdot|$ defines a topology on F which has as a basis for the open sets all sets of the form $U(a, \varepsilon) = \{b \in F \mid |a-b| < \varepsilon\}$, $a \in F$, $\varepsilon \in \mathbb{R}_{>0}$. We call F a non-Archimedean local field, if it is locally compact and complete with respect to a non-trivial non-Archimedean valuation. Let $\mathfrak{O} = \{a \in F \mid |a| \leq 1\}$ and $\mathfrak{p} = \{a \in F \mid |a| < 1\}$. It is easy to see that $\mathfrak{O}$ is a ring and $\mathfrak{p}$ is a principal ideal in $\mathfrak{O}$. In fact, it can be shown that $\mathfrak{p}$ is the unique maximal ideal of $\mathfrak{O}$.

We fix a generator ϖ of the ideal $\mathfrak{p}$. We call $\mathfrak{O}$ the valuation ring (ring of integers) of the field F. The field $k = \mathfrak{O}/\mathfrak{p}$ is a finite field and we call it the residue field of F.

2.2 Representations of locally profinite groups

Let G be a topological group. G is called locally profinite if it is a Hausdorff topological space and every open neighbourhood of the identity element in G contains a compact open subgroup of G. Let V be a vector space over $\mathbb{C}$ (not necessarily finite dimensional) and $\mathrm{GL}(V)$ be the set of all invertible linear operators on V. A representation (π, V) of G in V is a map $\pi : G \to \mathrm{GL}(V)$ such that $\pi(gh) = \pi(g)\pi(h)$, $\forall g, h \in G$. Suppose W is a subspace of V which is invariant under G, i.e., $\pi(g)w \in W$, $\forall g \in G$, $w \in W$. Then restricting the operators $\pi(g)$ to W gives a representation of G in W. We call the invariant subspace W a sub-representation of V. The representation π is called irreducible if it has no proper invariant subspaces, i.e., $\{0\}$ and V are the only subspaces of V invariant under G. π is smooth if $\mathrm{Stab}_G(v) = \{g \in G \mid \pi(g)v = v\}$ is open for every $v \in V$ and admissible if $V^K = \{v \in V \mid \pi(k)v = v, \forall k \in K\}$ is a finite dimensional subspace of V for any compact open subgroup K of G. It is a well known fact that *an irreducible smooth representation of a locally profinite group is always admissible*. We will always assume this fact throughout this report. Given two representations (π_1, V_1) and (π_2, V_2) of G, a linear map T from V_1 to V_2 is called an intertwining map (intertwiner) if $\pi_2(g) \circ T = T \circ \pi_1(g)$, $\forall\, g \in G$. We call (π_1, V_1) and (π_2, V_2) isomorphic (equivalent) representations if there exists an intertwiner T which is an isomorphism. We will write $\mathrm{Hom}_G(\pi_1, \pi_2)$ or $\mathrm{Hom}_G(V_1, V_2)$ for the collection of all intertwining maps between V_1 and V_2. Given a representation (π, V) of G, we have a natural representation of G in the dual space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. For $\lambda \in V^*$ and $v \in V$, define $\pi^*(g)\lambda \in V^*$ by $\langle \pi^*(g)\lambda\,, v\rangle = \langle \lambda\,, \pi(g)^{-1}v\rangle, v \in V$, where $\langle\,,\,\rangle$ is the canonical pairing between V and V^* given by evaluation. This representation is not always smooth. Let $V^\vee = (V^*)^\infty = \{v \in V^* \mid \mathrm{Stab}_G(v) \text{ is open}\}$. It is easy to see that $V^\vee$ is a sub-representation of V and we get a smooth representation $(\pi^\vee, V^\vee)$ of G in $V^\vee$. The representation $(\pi^\vee, V^\vee)$ is called the smooth-dual or contragredient of (π, V). Given a subgroup H of G the restriction of π to H gives a representation of H in V. It is called the restriction of π denoted as $\pi|_H$. It is natural to ask whether one can construct a representation of G from a representation of the subgroup H. This process is called smooth induction and the representation of G so obtained is called the smoothly induced representation. We explain the construction below.

Let (ρ, W) be smooth a representation of H. Consider the space $\mathcal{W}$ of functions $f : G \to W$ which satisfy

(i) $f(hg) = \rho(h)f(g)$, $h \in H$, $g \in G$.

(ii) there is a compact open subgroup K of G (depending on f) such that $f(gk) = f(g)$, for $g \in G$, $k \in K$.

We define a homomorphism $\mathrm{Ind}_H^G \rho : G \to \mathrm{GL}(\mathcal{W})$ by

$$(\mathrm{Ind}_H^G \rho)(g)(f)(x) = f(xg), \quad g, x \in G.$$

The pair $(\mathrm{Ind}_H^G \rho, \mathcal{W})$ provides a smooth representation of G and is called the representation of G smoothly induced by ρ.

Theorem 2.1 (Schur's Lemma)**.** *Let (π_1, V_1) and (π_2, V_2) be irreducible representations of a group G and $T_1, T_2 : V_1 \to V_2$ be non-zero intertwining maps. Then, $T_1 = \lambda T_2$, for some $\lambda \in \mathbb{C}^\times$.*

Proof. See chapter 1, section 2.6 in [3]. $\square$

Remark 2.2. For Schur's Lemma to be valid in this setting, we need a technical restriction on the group G. *We assume that for any compact open subgroup K, the set G/K is countable.*

Theorem 2.3 (Frobenius Reciprocity)**.** *Let H be a closed subgroup of a locally profinite group G. For a smooth representation (ρ, W) of H and a smooth representation (π, V) of G, the canonical map*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \rho) \to \mathrm{Hom}_H(\pi|_H, \rho),$$
$$\phi \mapsto \alpha_\rho \circ \phi$$

is an isomorphism. Here $\alpha_\rho : \mathcal{W} \to W$ is the canonical H-homomorphism defined as $f \mapsto f(1)$.

Proof. See chapter 1, section 2.4 in [3]. $\square$

2.3 Hecke algebras

Throughout this section, we let G be a locally profinite group. For simplicity, we assume that G is unimodular and fix a Haar measure dg on G.

Let $C_c^\infty(G)$ be the space of functions $f : G \to \mathbb{C}$ which are locally constant and compactly supported. For $f_1, f_2 \in C_c^\infty(G)$, we define

$$f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g) dg.$$

The pair $\mathcal{H}(G) = (C_c^\infty(G), *)$ is an associative $\mathbb{C}$-algebra called the Hecke algebra of G. Let $\mathcal{H}(G, K) = \{f \in \mathcal{H}(G) \mid f(k_1 g k_2) = f(g), \forall g \in G, k_1, k_2 \in K\}$. $\mathcal{H}(G, K)$ is a subalgebra of $\mathcal{H}(G)$ and is called the K-Hecke algebra.

Proposition 2.4. *Let (π, V) be an irreducible smooth representation of G and K be a compact open subgroup. The space V^K is either zero or a simple $\mathcal{H}(G, K)$-module.*

Proof. See chapter 1, section 4.3, proposition 1 in [3]. □

2.4 Preliminaries from Algebraic groups

Throughout this section, we let K be an algebraically closed field and k be a subfield of K. We write $K[t_1, \ldots, t_n]$ for the polynomial ring in n variables $t_1, \ldots, t_n$. For $S \subset K[t_1, \ldots, t_n]$ and $E \subset K^n$, we let $V(S) = \{x \in K^n \mid f(x) = 0 \ , \forall f \in S\}$ and $I(E) = \{f \in K[t_1, \ldots, t_n] \mid f(x) = 0 \ , \forall x \in E\}$.

A subset $X \subset K^n$ is called an algebraic set if $X = V(S)$ for some (finite) subset S of A_n. It is easy to see that the sets $V(S), S \subset K[t_1, \ldots, t_n]$, satisfy the axioms for closed sets in a topological space. The resulting topology on K^n is called the Zariski topology, and the induced topology on an algebraic set $X \subset K^n$ is the Zariski topology on X. Let $X \subset K^n$ be an algebraic set. The K-algebra $K[X] = K[t_1, \ldots, t_n]/I(X)$ is called the affine algebra (or coordinate ring) of X and the pair (X,K[X]) is called an affine algebraic variety. Suppose

X is an affine algebraic variety over K. If k is a subfield of K, we say that X is defined over k if the ideal $I(X)$ is generated by polynomials in $k[t_1, \dots, t_n]$. In this case, the set $X(k) := X \cap k^n$ is called the k-rational points of X. A map $f : K^n \to K^m$ is called a morphism if there exist polynomials $f_1, \dots, f_m$ in $K[t_1, \dots, t_n]$ such that $f(x) = (f_1(x), \dots, f_m(x))$ for all $x \in K^n$. If $X \subset K^n$ and $Y \subset K^m$ are affine algebraic varieties, a mapping $\phi : X \to Y$ is called a morphism if $\phi = f|X$ for some morphism $f : K^n \to K^m$. A morphism $\phi : X \to Y$ determines a unique K-algebra homomorphism $\phi^* : K[Y] \to K[X]$ given by $\phi^*(g) = g \circ \phi$. Suppose X and Y are defined over k. We say that a morphism $\phi : X \to Y$ is defined over k (or is a k-morphism) if there is a homomorphism $\phi_k^* : k[Y] \to k[X]$ such that the algebra homomorphism $\phi^* : K[Y] \to K[X]$ defined by ϕ is $\phi_k^* \otimes id_K$.

Definition 2.5. An linear algebraic group is (1) an affine algebraic variety and (2) a group, such that multiplication

$$\mu : G \times G \to G \quad (\mu(x, y) = xy)$$

and inversion

$$i : G \to G \quad (i(x) = x^{-1})$$

are morphisms of affine algebraic varieties.

Example 2.6. *We give some examples of algebraic groups.*

(i) $G = K$, *with* $\mu(x, y) = x + y$ *and* $i(x) = -x$. *We usually denote this group as* G_a.

(ii) Let n be a positive integer and let $\mathrm{M}(n,K)$ be the set of $n \times n$ matrices with entries in K. $G = \mathrm{GL}(n,K) = \{A \in \mathrm{M}(n,K) \mid detA \neq 0\}$ In the case when $n = 1$, the usual notation for $\mathrm{GL}(n,K)$ is G_m.

(iii) Let $A = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix}$ be the anti-diagonal matrix and let $J = \begin{bmatrix} & A \\ -A & \end{bmatrix}$. Let W be a vector space of dimension $2n$ over a field K and $\langle v, w\rangle = v^\top Jw$ be a non-degenerate skew-symmetric form on W. The symplectic group $\mathrm{Sp}(n,F) = \{g \in \mathrm{GL}(n,K) \mid \langle gv, gw\rangle = \langle v, w\rangle\}$.

Let G and G' be algebraic groups. A homomorphism of algebraic groups $\phi : G \to G'$ is a group homomorphism which is also a morphism of varieties. Given $x \in \mathrm{GL}(n,K)$, it is well known that there exists elements x_s and x_u in $\mathrm{GL}(n,K)$ such that x_s is semisimple, x_u is unipotent, and $x = x_s x_u = x_u x_s$. Furthermore, x_s and x_u are uniquely determined. Suppose that G is a linear algebraic group. It can be shown that there exists a positive integer n and an injective homomorphism $\varphi : G \to \mathrm{GL}(n,K)$ of algebraic groups. If $g \in G$, the semisimple and unipotent parts $\varphi(g)_s$ and $\varphi(g)_u$ of $\varphi(g)$ lie in $\varphi(G)$, and the elements g_s and g_u such that $\varphi(g_s) = \varphi(g)_s$ and $\varphi(g_u) = \varphi(g)_u$ depend only on g and not on the choice of φ (or n). The elements g_s and g_u are called the semisimple and unipotent part of g, respectively. An element $g \in G$ is semisimple if $g = g_s$ (and $g_u = 1$), and unipotent if $g = g_u$ (and $g_s = 1$). It can be shown that G contains a unique maximal connected normal solvable subgroup denoted $R(G)$. The set $R_u(G)$ of unipotent elements in $R(G)$ is called the unipotent radical of G. A torus T is a linear algebraic group which is isomorphic to the direct product $G_m^d = G_m \times \cdots \times G_m$, where d is a positive integer. If k is a subfield of K, we call T a *k-torus* if T is defined over k. We say that T

is k-split (or splits over k) whenever T is k-isomorphic to $G_m \times \cdots \times G_m$. For example, let T be the subgroup of $\mathrm{GL}(n, K)$ consisting of diagonal matrices in $\mathrm{GL}(n, K)$. Then T is a k-split k-torus for any subfield k of K. Let G be a connected reductive k-group (i.e., G is defined over k). Then it can be shown that G has a maximal torus T which is defined over k. We say that G is k-split (or splits over k) if T is k-split.

Remark 2.7. Let G be an algebraic group. Then the set of tori in G has maximal elements, relative to inclusion. Such maximal elements are called maximal tori of G.

Let G be a connected algebraic group. The set of connected closed solvable subgroups of G, ordered by inclusion, contains maximal elements. Such a maximal element is called a Borel subgroup of G. For example, if $G = \mathrm{GL}(n, K)$ and then the subgroup B of upper triangular matrices in G is a Borel subgroup.

2.4.1 Classification of split reductive groups

In this section, we explain the notion of a root datum and give examples of root data for some classical groups. The main result in this section is that the root datum determines G up to isomorphism (we state it without proof). Throughout this section, we let G be a connected reductive k-group and T be a k-split maximal torus in G.

Definition 2.8. A root datum $\mathfrak{R}$ is a quadruple $(X, \Phi, X^\vee, \Phi^\vee)$, where

(i) X and $X^\vee$ are free abelian groups of finite rank, in duality by a pairing $X \times X^\vee \to \mathbb{Z}$, denoted by $\langle\ ,\ \rangle$;

(ii) Φ and $\Phi^\vee$ are finite subsets of X and $X^\vee$, and we are given a bijection $\alpha \to \alpha^\vee$ of Φ on to $\Phi^\vee$.

For $\alpha \in \Phi$, we define endomorphisms s_α and $s_\alpha^\vee$ of X and $X^\vee$ by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \quad s_\alpha^\vee(y) = y - \langle \alpha, y \rangle \alpha^\vee \quad (x \in X, y \in X^\vee).$$

The following axioms are imposed

(1) If $\alpha \in \Phi$ then $\langle \alpha, \alpha^\vee \rangle = 2$.

(2) if $\alpha \in \Phi$ then $s_\alpha(\Phi) = \Phi$ and $s_\alpha^\vee(\Phi^\vee) = \Phi^\vee$.

The root datum $\mathfrak{R}$ is called reduced if

(3) $\alpha \in \Phi \Rightarrow 2\alpha \notin \Phi$.

We call Φ as the set of roots and $\Phi^\vee$ as the set of coroots.

Root datum attached to (G, T)

A character of a torus T is a homomorphism of algebraic groups $\chi : T \to G_m$. The set $X(T)$ of characters of T is in a natural way an abelian group $((\chi+\psi)(t) = \chi(t)\psi(t))$ and is called the character group of T. A cocharacter of a torus T is a homomorphism of algebraic groups $\lambda : G_m \to T$. The set $X^\vee(T)$ of cocharacters of T is again an abelian group $((\lambda+\mu)(x) = \lambda(x)\mu(x))$. There is a duality map $\langle\, , \rangle : X(T) \times X^\vee(T) \to \mathbb{Z}$ relating the character and cocharacter groups of T. Given $\chi \in X(T)$ and $\gamma \in X^\vee(T)$ we have $\chi \circ \gamma \in \mathrm{Hom}(k^\times, k^\times)$, thus $(\chi \circ \gamma)(\lambda) = \lambda^m$ for some $m \in \mathbb{Z}$. We write $\langle \chi, \gamma \rangle = m$. It induces isomorphisms

$$X(T) \simeq \mathrm{Hom}_{\mathbb{Z}}(X^\vee(T), \mathbb{Z}) \qquad X^\vee(T) \simeq \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z}).$$

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation of G. Since Ad_G is a homomorphism of algebraic groups, $\mathrm{Ad}_G(T)$ consists of commuting semisimple elements, hence is diagonalizable: this means that, relative to the action of T, $\mathfrak{g}$ decomposes as a direct sum

$$\mathfrak{g} = \bigoplus_{\alpha \in X(T)} \mathfrak{g}_\alpha$$

where for each character $\alpha \in X(T)$,

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{Ad}_G(t)X = \alpha(t)X, \forall t \in T\}.$$

The non-zero $\alpha \in X(T)$ such that $\mathfrak{g}_\alpha \neq 0$ are called the roots of G relative to T, and the set of roots is denoted by $\Phi(G, T)$. Let $\alpha \in \Phi(G, T)$ and let $T_\alpha = (\mathrm{Ker}\,\alpha)_0$, the connected component of the identity in the kernel of α; then T_α is a subtorus of T of codimension 1. Let G_α be the centralizer of T_α in G; it is easy to see that G_α is connected, and T is a maximal torus of G_α. The Weyl group $W(G_\alpha, T)$ has order 2, and embeds in $W(G, T)$. Let $s_\alpha \in W(G_\alpha, T)$ be the non-identity element of $W(G_\alpha, T)$; then s_α acts on $X(T)$ as follows:

$$s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha$$

for a unique $\alpha^\vee \in X^\vee(T)$. The unique element $\alpha^\vee \in X^\vee(T)$ is called the coroot of α, and the set of coroots is denoted by $\Phi^\vee(G, T)$.

It is clear that we can associate to any linear algebraic group G and a maximal torus T, a root datum $\mathfrak{R}(G, T) = (X(T), \Phi(G, T), X^\vee(T), \Phi^\vee(G, T))$.

Existence and Uniqueness Theorem

Theorem 2.9 (Uniqueness)**.** *Let G_1 and G_2 be connected reductive linear algebraic groups defined and split over k with split maximal tori T_1 and T_2. Then G_1 and G_2 are isomorphic as algebraic groups if and only if $\mathfrak{R}(G_1, T_1) \simeq \mathfrak{R}(G_2, T_2)$.*

Proof. See Theorem 9.6.2 in [14]. □

Theorem 2.10 (Existence)**.** *Let $\mathfrak{R}$ be a reduced root datum. Then there exists a connected reductive linear algebraic group G defined and split over k such that $\mathfrak{R} \simeq \mathfrak{R}(G, T)$.*

Proof. See Theorem 10.1.1 in [14]. □

Examples of Root Data

Example 2.11. $G = \mathrm{GL}(n, F)$, $n \geq 2$. *Let* $T = \Big\{t = diag(t_1, \dots, t_n),\ t_i \in F^{\times}\Big\}$. *It is easy to see that T is a maximal torus in G. We will now write the root datum $(X(T), \Phi(T), X^{\vee}(T), \Phi^{\vee}(T))$ corresponding to the pair (G, T).*

$$X(T) = \Big\{\chi : T \to F^{\times} \mid \chi(t) = t_1^{k_1} \cdots t_n^{k_n},\ k_1, \dots, k_n \in \mathbb{Z}\Big\}.$$

$$\Phi(T) = \Big\{\chi \in X(T) \mid \chi(t) = t_i t_j^{-1},\ 1 \leq i \neq j \leq n\Big\}.$$

$$X^{\vee}(T) = \Big\{\mu : F^{\times} \to T \mid \mu(\lambda) = diag(\lambda^{k_1}, \dots, \lambda^{k_n}),\ k_1, \dots, k_n \in \mathbb{Z}\Big\}.$$

$$\Phi^{\vee}(T) = \Big\{\mu \in X^{\vee}(T) \mid \mu(\lambda) = diag(1, \dots, \underbrace{\lambda}_{i}, \dots, \underbrace{\lambda^{-1}}_{j}, \dots, 1),\ 1 \leq i \neq j \leq n\Big\}.$$

Example 2.12. *Let* $G = \mathrm{Sp}(4, F)$. *Let* $T = \Big\{t = diag(a, b, b^{-1}, a^{-1}),\ a, b \in F^\times\Big\}$. *It is again easy to see that* T *is a maximal torus in* G. *We now describe the root datum* $(X(T), \Phi(T), X^\vee(T), \Phi^\vee(T))$ *attached to the pair* (G, T).

$$X(T) = \Big\{\chi_{(i,j)} : T \to F^\times \mid \chi_{(i,j)}(t) = a^i b^j,\ i, j \in \mathbb{Z}\Big\}.$$

Let $\alpha = \chi_{(1,-1)}$ *and* $\beta = \chi_{(0,2)}$. *Then we have*

$$\Phi(T) = \Big\{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\Big\}.$$

$$X^\vee(T) = \Big\{\mu_{(i,j)} : F^\times \to T \mid i, j \in \mathbb{Z}, \quad \mu_{(i,j)}(\lambda) = diag(\lambda^i, \lambda^j, \lambda^{-j}, \lambda^{-i})\Big\}.$$

Let $\alpha^\vee = \mu_{(1,-1)}$ *and* $\beta^\vee = \mu_{(0,1)}$. *Then we have*

$$\Phi^\vee(T) = \Big\{\pm\alpha^\vee, \pm\beta^\vee, \pm(\alpha^\vee+\beta^\vee), \pm(\alpha^\vee+2\beta^\vee)\Big\}.$$

2.5 Iwahori subgroups, Bruhat decomposition

In this section, we let F be a non-Archimedean local field. We write $\mathfrak{O}$ for the ring of integers in F, $\mathfrak{p}$ for the unique maximal ideal and k for the residue field. We also fix a maximal F-split torus T and write $T_\circ$ for the $\mathfrak{O}$-points of T. We write B for a Borel subgroup containing T and U for the unipotent radical of B as before.

Let B be a Borel subgroup in G. The Iwahori subgroup I is defined to be the inverse image of $B(k)$ (k-points of B) under the canonical map (reduction mod $\mathfrak{p}$) from $G(\mathfrak{O})$ to $G(k)$. For example, let $G = \mathrm{GL}(n, F)$. Take B to be the standard Borel subgroup (upper triangular matrices) and U to be the unipotent radical (unipotent matrices) of B in G. In this case the Iwahori subgroup is the collection of matrices of the following type.

$$I = \begin{bmatrix} \mathfrak{O}^\times & \mathfrak{O} & \cdots & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{O}^\times & \cdots & \mathfrak{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \cdots & \mathfrak{O}^\times \end{bmatrix}$$

Consider the group $\widetilde{W} := N_G(T)/T_\circ$. This group $\widetilde{W}$ is called the affine Weyl group. Given an Iwahori subgroup I in G, it can be shown that G is the disjoint union of the double cosets IwI, as w ranges over a set of representatives in $N_G(T)$ of the affine Weyl group $\widetilde{W}$, i.e.,

$$G = \bigsqcup_{w \in \widetilde{W}} IwI$$

The above decomposition of G is called the affine Bruhat Decomposition.

2.6 Non-degenerate characters, generic representations

In this section, we define the notion of a non-degenerate character and genereic representation. We continue with the notation of the previous section.

A (non-trivial) character ψ of U is non-degenerate (generic) if $\psi|_{U_\alpha} \neq 1$ for all simple roots α. For example, if $G = \mathrm{GL}(n, F)$ it can be shown that any non-degenerate character ψ of U is a character of the form

$$\psi(u) = \theta(\alpha_1 u_{12} + \alpha_2 u_{23} + \cdots + \alpha_{n-1} u_{n-1n})$$

where θ is a complex valued non-trivial additive character of F, $u = (u_{ij})$ is the collection of unipotent matrices and $\alpha_1, \ldots, \alpha_n \in F^\times$.

A representation π is called generic if there exists a non-degenerate character ψ of U such that $\mathrm{Hom}_U(\pi, \psi) \neq 0$.

2.7 Preliminaries from Group Cohomology

Let G be a group. An abelian group A on which G acts as automorphisms is called a G-module. If A is a G-module, let $A^G = \{a \in A \mid g.a = a \ \forall g \in G\}$ be the elements of A fixed by all the elements of G. Suppose G is a finite group and A is a G-module, define $C^\circ(G, A) = A$ and for $n \geq 1$ define $C^n(G, A)$ to be the collection of all maps from $G^n = G \times \cdots \times G$ (n copies) to A. The elements of $C^n(G, A)$ are called n-cochains (of G with values in A). For $n \geq 0$, we define the n^{th} coboundary homomorphism from $C^n(G, A)$ to $C^{n+1}(G, A)$ by

$$\begin{aligned} d_n(f)(g_1, \ldots, g_{n+1}) = {} & g_1.f(g_2, \ldots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \ldots, g_n) \end{aligned}$$

where the product $g_i g_{i+1}$ occupying the i^{th} position of f is taken in the group G. Let $Z^n(G, A) = \ker d_n$ for $n \geq 0$. The elements of $Z^n(G, A)$ are called n-cocycles. Let $B^n(G, A) = \mathrm{Im}\, d_{n-1}$ for $n \geq 1$ and $B^\circ(G, A) = 1$. The elements of $B^n(G, A)$ are called n-coboundaries. For any G-module A the quotient group $Z^n(G, A)/B^n(G, A)$ is called the n^{th} cohomology group of G with coefficients in A and is denoted by $H^n(G, A)$, $n \geq 0$.

If G is a profinite group then a discrete G-module A is a G-module A with the discrete topology such that the action of G on A is continuous, i.e., the map $(g, a) \mapsto g.a : G \times A \to A$ is continuous. If G is a profinite group and A is a discrete G-module, the cohomology groups $H^n(G, A)$ computed using continuous cochains (i.e., the continuous maps $f \in C^n(G, A)$) are called the profinite or continuous cohomology groups. When $G = \mathrm{Gal}(K/F)$ is the Galois group of a field extension K/F then the Galois cohomology groups $H^n(G, A)$ will always mean the cohomology groups computed using continuous cochains.

Theorem 2.13 (Long Exact Sequence in Group Cohomology)**.** *Suppose*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence of G-modules. Then there is a long exact sequence:

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \longrightarrow H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \longrightarrow \cdots$$
$$\cdots \longrightarrow H^n(G, A) \longrightarrow H^n(G, B) \longrightarrow H^n(G, C) \longrightarrow H^{n+1}(G, A) \longrightarrow \cdots$$

of abelian groups.

Proof. See chapter 17, theorem 21 in [5]. □

We now recall a key result which will be essential to a later argument.

Theorem 2.14 (Hilbert's Theorem 90)**.** *Let F be a field and let L/F be a Galois extension (not necessarily finite) of F with $\Gamma = \mathrm{Gal}(L/F)$. Then $H^1(\Gamma, L^\times) = \{1\}$.*

Chapter 3

Self-dual representations and signs

In this chapter, we introduce and briefly discuss the notion of signs associated to self-dual representations.

3.1 Sign of π

Let F be a non-Archimedean local field and G be the group of F-points of a connected reductive algebraic group. Let (π, W) be a smooth irreducible representation of G. We write $(\pi^\vee, W^\vee)$ for the smooth dual or contragredient of (π, W) and $\langle\ ,\ \rangle$ for the canonical non-degenerate G-invariant pairing on $W \times W^\vee$ (given by evaluation). Let $s : (\pi, W) \to (\pi^\vee, W^\vee)$ be an isomorphism. The map s can be used to define a bilinear form on W as follows

$$(w_1, w_2) = \langle w_1, s(w_2)\rangle, \quad \forall w_1, w_2 \in W.$$

It is easy to see that $(\ ,\ )$ is a non-degenerate G-invariant form on W, i.e., it satisfies,

$$(\pi(g)w_1, \pi(g)w_2) = (w_1, w_2), \quad \forall w_1, w_2 \in W.$$

Let $(\ ,\ )_*$ be a new bilinear form on W defined by

$$(w_1, w_2)_* = (w_2, w_1)$$

Clearly, this form is again non-degenerate and G-invariant. It follows from

Schur's Lemma (Theorem 2.1) that

$$(w_1, w_2)_* = c(w_1, w_2)$$

for some non-zero scalar c. A simple computation shows that $c \in \{\pm 1\}$. Indeed,

$$(w_1, w_2) = (w_2, w_1)_* = c(w_2, w_1) = c(w_1, w_2)_* = c^2(w_1, w_2).$$

We set $c = \varepsilon(\pi)$. It clearly depends only on the equivalence class of π. In sum, the form $(\ ,\ )$ is symmetric or skew-symmetric and the sign $\varepsilon(\pi)$ records its type.

Chapter 4

Representations of some classical groups

We use a theorem of Waldspurger to show that many representations of classical groups are self-dual. Throughout this chapter, we let F be a non-Archimedean local field of characteristic $\neq 2$ and W be a finite dimensional vector space over F. We write $\mathfrak{O}$ for the ring of integers in F, $\mathfrak{p}$ for the unique maximal ideal of $\mathfrak{O}$ and k for the residue field. We let $\langle\, ,\, \rangle$ to be a non-degenerate symmetric or skew-symmetric form on W. We take

$$G = \{g \in \mathrm{GL}(W) \mid \langle gw, gw' \rangle = \langle w, w' \rangle\}$$

For $x \in \mathrm{GL}(W)$ such that $xGx^{-1} = G$ and (π, V) a representation of G, we let π^x denote the representation of G defined by conjugation (i.e., $\pi^x(g) = \pi(xgx^{-1})$.

We recall the statement of Waldspurger's theorem and refer the reader to Chapter 4.II.1 in [8] for a proof.

Theorem 4.1 (Waldspurger)**.** *Let π be an irreducible admissible representation of G and $\pi^\vee$ be the smooth-dual or contragredient of π. Let $x \in \mathrm{GL}(W)$ be such that $\langle xw, xw' \rangle = \langle w', w \rangle$, $\forall\, w, w' \in W$. Then $\pi^x \simeq \pi^\vee$.*

Proof. See chapter 4.II.1 in [8]. $\square$

4.1 Orthogonal and special orthogonal groups

4.1.1 Orthogonal groups

Suppose the form $\langle\, ,\, \rangle$ is symmetric so that G is the orthgonal group $\mathrm{O}(W)$. Let π be any irreducible admissible representation of G. Then $x = 1 \in G$ satisfies $\langle xw, xw' \rangle = \langle w', w \rangle, \forall\, w, w' \in W$. Now using Waldspurger's Theorem, it follows that $\pi \simeq \pi^\vee$. So in the case of orthogonal groups every irreducible representation π is self-dual.

4.1.2 Special orthogonal groups

Suppose the dimension of W is odd. Take $G = \mathrm{SO}(W) = \mathrm{O}(W) \cap \mathrm{SL}(W)$ and π to be an irreducible admissible representation of G. Since $\mathrm{O}(W) \simeq \mathrm{SO}(W) \times \{\pm 1_W\}$, it follows that there exists an irreducible representation $\tilde{\pi}$ of $\mathrm{O}(W)$ such that $\tilde{\pi} \simeq \pi \otimes \chi$ (where χ is a character of $\{\pm 1_W\}$). Since $\chi = \chi^{-1}$ and $\tilde{\pi} \simeq \tilde{\pi}^\vee$ it follows that $\pi \simeq \pi^\vee$.

4.2 Symplectic groups

Suppose that G is the symplectic group $Sp(W)$ or $Sp(n, F)$ (see example 2.6). We show that a certain class of representations of G is always self-dual. To be more precise, we prove,

Theorem 4.2. *Let (π, V) be an irreducible admissible representation of G with non-zero vectors fixed under an Iwahori subgroup I in G. Then $\pi \simeq \pi^\vee$.*

Consider $x = \left[\begin{smallmatrix} -\mathrm{I} & 0 \\ 0 & \mathrm{I} \end{smallmatrix}\right] \in \mathrm{GL}(W)$ (where I is the $n \times n$ identity matrix). It is easy to see that $\langle xw, xw' \rangle = \langle w', w \rangle$. By Theorem 4.1, $\pi^x \simeq \pi^\vee$. To prove $\pi \simeq \pi^\vee$, it suffices to show that $\pi \simeq \pi^x$. Observe that $xIx^{-1} = I$ and

$\pi^I = (\pi^x)^I$. Since $\pi^I = (\pi^x)^I \neq 0$, they can be realized as simple modules over $\mathcal{H}(G,I)$. Let $f \bullet v$ and $f \star v$ denote the action of $\mathcal{H}(G,I)$ on π^I and $(\pi^x)^I$ respectively. It will follow that $\pi \simeq \pi^x$ if π^I and $(\pi^x)^I$ are equivalent as $\mathcal{H}(G,I)$-modules. We establish this equivalence, by showing that the map $\phi = 1_V$ (identity map on V) defines an intertwining map between π^I and $(\pi^x)^I$.

Before we continue, we fix a collection of coset representatives for the affine Weyl group $\widetilde{\mathcal{W}}$ and record two lemmas we need.

Let $B_{i,i+1}, i = 1, 2, \ldots, n-1$, be the $n \times n$ matrix

$$B_{i,i+1} = \begin{array}{cc} \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} & \begin{array}{c} \\ \\ \left.\begin{array}{c} i \\ i+1 \end{array}\right\} \\ \\ \\ \end{array} \end{array}$$

and w_n and w_ℓ, $\ell \in \mathbb{Z}^n$, be the following $2n \times 2n$ matrices

$$w_n = \begin{array}{c} \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \begin{array}{c} n \\ n+1 \end{array} \\ \\ \\ \end{array},$$

$$w_\ell = \begin{bmatrix} \varpi^{l_1} & & & & & \\ & \ddots & & & & \\ & & \varpi^{l_n} & & & \\ & & & \varpi^{-l_n} & & \\ & & & & \ddots & \\ & & & & & \varpi^{-l_1} \end{bmatrix}, l_i \in \mathbb{Z}.$$

The group $\left\langle w_\ell, w_n, w_i = \left[\begin{smallmatrix} B_{i,i+1} & 0 \\ 0 & B_{i,i+1} \end{smallmatrix}\right] \;\middle|\; \ell \in \mathbb{Z}^n, i = 1, 2, \ldots, n-1 \right\rangle$ contains a collection of coset representatives for the affine Weyl group $\widetilde{\mathcal{W}}$.

Lemma 4.3. *For $f \in \mathcal{H}(G, I)$, let $f^x \in \mathcal{H}(G, I)$ be the function $f^x(g) = f(x^{-1}gx)$. The following statements are true.*

(i) $f \star v = f^x \bullet v$.

(ii) For $g = i_1 w i_2 \in G$ (Bruhat Decomposition), $f^x(g) = f(xwx^{-1})$ and $f(g) = f(w)$.

Proof. Clearly $f \star v = f^x \bullet v$. Indeed,

$$\begin{aligned} f \star v &= \int_G f(g)\pi^x(g)vdg \\ &= \int_G f(g)\pi(xgx^{-1})vdg \\ &= \int_G f(x^{-1}gx)\pi(g)vdg \\ &= f^x \bullet v. \end{aligned}$$

For (ii),

$$\begin{aligned} f^x(g) &= f(xi_1wi_2x^{-1}) \\ &= f(xi_1x^{-1}xwx^{-1}xi_2x^{-1}) \\ &= f(ixwx^{-1}i') \\ &= f(xwx^{-1}), \end{aligned}$$

and

$$\begin{aligned} f(g) &= f(i_1wi_2) \\ &= f(w). \end{aligned}$$

□

Lemma 4.4. *For $w \in \widetilde{W}$, $xwx^{-1} \in IwI$.*

Proof. If $w \in \widetilde{W}$, we can write $w = u_1u_2 \ldots u_n$ for $u_k \in \{w_\circ, w_i, w_\ell\}$. In this case we say w has length n and denote it as $l(w) = n$. We will use induction on the length $l(w)$ to show that $xwx^{-1} \in IwI$. Suppose that $l(w) = 1$. A simple computation shows that conjugation by the element x fixes the elements w_i,

w_ℓ and fixes w_n up to multiplication by elements in $T_\circ \subset I$, i.e., $xw_ix^{-1} = w_i$ for $i = 1, \ldots, n-1$, $xw_\ell x^{-1} = w_\ell$ for $\ell \in \mathbb{Z}^n$ and $xw_nx^{-1} = tw_nt^{-1}$ for some $t \in T_\circ \subset I$. Suppose $l(w) = 2$, i.e., $w = u_1u_2$ for $u_1, u_2 \in \{w_0, w_i, w_\ell\}$ such that $xu_1x^{-1} = t_1u_1t_1^{-1}$ and $xu_2x^{-1} = t_2u_2t_2^{-1}$, $t_1, t_2 \in T_\circ$. In this case, we have

$$\begin{aligned} xwx^{-1} &= xu_1x^{-1}xu_2x^{-1} \\ &= t_1u_1t_1^{-1}t_2u_2t_2^{-1} \\ &= t_1u_1u_2\,\underbrace{u_2^{-1}t_1^{-1}u_2}_{\in T_\circ}\,\underbrace{u_2^{-1}t_2u_2}_{\in T_\circ}\,t_2^{-1} \\ &= t_1u_1u_2t_1' \end{aligned}$$

where $t, t' \in T_\circ$.

Assume that the result is true for all words w such that $l(w) \leq n-1$. Suppose $w = u_1u_2\ldots u_n$. Now

$$\begin{aligned} xwx^{-1} &= xu_1u_2\ldots u_{n-1}x^{-1}xu_nx^{-1} \\ &= tu_1u_2\ldots u_{n-1}t^{-1}xu_nx^{-1} \\ &= twt' \quad \text{(by previous case)} \end{aligned}$$

where $t, t' \in T_\circ$.

$\square$

We are now ready to prove Theorem 4.2. To prove $\phi = 1_V$ is an intertwining map, we need to show that $f \bullet v = f \star v = f^x \bullet v$. By Lemma 4.3, it suffices to show that $f(xwx^{-1}) = f(w)$, for all $w \in \widetilde{\mathcal{W}}$. By Lemma 4.4, it follows that conjugation by the element x fixes every element in $\widetilde{W}$ (up to multiplication by elements in I). The result now follows.

Chapter 5

Results used in the main theorem

In this chapter, we recall the important results used in the proof of the main theorem.

5.1 Restriction of representations to subgroups

In this section, we recall some results about restricting an irreducible representation to a subgroup. These results hold when G is a locally compact totally disconnected group and H is an open normal subgroup of G such that G/H is finite abelian. For a more detailed account, we refer the reader to [6] (Lemma 2.1, 2.3).

Theorem 5.1 (Gelbart-Knapp)**.** *Let π be an irreducible admissible representation of G. Suppose that G/H is finite abelian. Then*

(i) $\pi|_H$ is a finite direct sum of irreducible admissible representations of H.

(ii) When the irreducible constituents of $\pi|_H$ are grouped according to their equivalence classes as

$$\pi|_H = \bigoplus_{i=1}^{M} m_i \pi_i$$

with the π_i irreducible and inequivalent, the integers m_i are equal.

Theorem 5.2 (Gelbart-Knapp)**.** *Let G be a locally compact totally disconnected group and H be an open normal subgroup of G such that G/H is finite abelian, and let π be an irreducible admissible representation of H. Then* *(i) There exists an irreducible admissible representation $\tilde{\pi}$ of G such that $\tilde{\pi}|_H$ contains π as a constituent.*

(ii) Suppose $\tilde{\pi}$ and $\tilde{\pi}'$ are irreducible admissible representations of G whose restrictions to H are multiplicity free and contain π. Then $\tilde{\pi}|_H$ and $\tilde{\pi}'|_H$ are equivalent and $\tilde{\pi}$ is equivalent with $\tilde{\pi}' \otimes \chi$ for some character χ of G that is trivial on H.

5.2 Unramified principal series and representations with Iwahori fixed vectors

In this section, we define the notion of an unramified principal series representation and state an important characterization of representations with non-zero vectors fixed under an Iwahori subgroup due to Borel and Casselman. We refer to ([1], [4]) for a proof.

Throughout this section, we let G be the group of F-points of a connected reductive algebraic group defined and split over F. We write T for a maximal F-split torus in G. We also fix a Borel subgroup B defined over F such that $B \supset T$ and write U for the unipotent radical of B.

Let (ρ, W) be a smooth representation of T. We can view ρ as a smooth representation of B which is trivial on U, and form the smooth induced representation $\mathrm{Ind}_B^G \rho$. This representation $\mathrm{Ind}_B^G \rho$ is called the representation parabolically induced from ρ. The representations $\mathrm{Ind}_B^G(\mu)$, where μ is an unramified character of T (i.e., $\mu|_{T_\circ} = 1$) are called the unramified principal series representations.

Theorem 5.3 (Borel-Casselman)**.** *Let (π, W) be any irreducible admissible representation of G. Then the following assertions are equivalent.*

(i) There are non-zero vectors in W invariant under I.

(ii) There exists some unramified character μ of T such that π imbeds as a sub-representation of $\operatorname{Ind}_B^G \mu$.

5.3 Prasad's idea for computing the sign

In [11], Prasad gives a criterion to compute the sign for an irreducible self-dual generic representation of a p-adic group G. He shows that for generic representations the sign is determined by the value of the central character ω_π at a special central element. We recall his result below and also sketch a proof.

Theorem 5.4 (Prasad)**.** *Let K be a compact open subgroup of G. Let s be an element of G which normalizes K and whose square belongs to the center of G. Let $\psi_K : K \to \mathbb{C}^\times$ be a one dimensional representation of K which is taken to its inverse by inner conjugation action of s on K. Let π be an irreducible representation of G in which the character ψ_K of K appears with multiplicity 1. Then if π is self-dual, $\varepsilon(\pi)$ is 1 if and only if the element s^2 belonging to the center of G operates by 1 on π.*

Proof. Let $(\, , \,)$ be a non-degenerate G-invariant form on the underlying space W of π. Let $w_0 \neq 0$ be a vector in W such that $\pi(h)w_0 = \psi_K(h)w_0$ for all $h \in K$. Since s normalizes K and takes ψ_K to its inverse, it follows that

$$\pi(h)\pi(s)w_0 = \psi_K^{-1}(h)\pi(s)w_0.$$

Assume $\psi_K^{-1} \neq \psi_K$. In this case, it is easy to see that w_0 and $\pi(s)w_0$ are linearly independent isotropic vectors which generate a two-dimensional non-degenerate subspace of W. The non-degenerate bilinear form $(\ ,\ )$ on W is symmetric if and only if its restriction to the two-dimensional subspace spanned by w_0 and $\pi(s)w_0$ is symmetric. Since w_0 and $\pi(s)w_0$ are isotropic vectors, $(w_0, \pi(s)w_0) \neq 0$. We have

$$(w_0, \pi(s)w_0) = (\pi(s)w_0, \pi(s^2)w_0) = \omega_\pi(s^2)(\pi(s)w_0, w_0).$$

This implies that $(\ ,\ )$ is symmetric if and only if s^2 acts by 1.

If the character $\psi_K = \psi_K^{-1}$, then the one-dimensional subspace on which K acts via ψ_K is a non-degenerate subspace of W, forcing the bilinear form $(\ ,\ )$ to be symmetric. □

5.4 Compact approximation of Whittaker models

In this section, we describe the idea of compact approximation and state an important result of Rodier which is used in the proof of the main theorem. We also give an example of compact approximation in the case of $\mathrm{GL}(2, F)$.

Let G be the group of F-points of a connected reductive algebraic group defined and split over F and π be an irreducible smooth generic representation of G. Let T be a maximal F-split torus in G. We let $X = X(T)$ and $X^\vee = X^\vee(T)$ denote the character and cocharacter groups of T and write Φ and $\Phi^\vee$ for the set of roots and coroots of T inside X and $X^\vee$ respectively. We also fix a minimal F-parabolic (Borel) subgroup B of G containing T. The group B corresponds to a positive system Φ^+ in Φ. We write Δ for the unique simple system contained in Φ^+. B has a Levi Decomposition $B = T \ltimes U$ where U is the unipotent radical of B. We write $\bar{B}$ for the opposite of B and $\bar{U}$ for the unipotent radical of $\bar{B}$. For each $\alpha \in \Phi$, we let U_α denote the root subgroup corresponding to α. We also fix an isomorphism $x_\alpha : F \to U_\alpha$ and a non-degenerate character ψ of U.

Let $\mathfrak{O}$ be the ring of integers in F and θ be a non-trivial additive character of F with $\mathrm{Ker}(\theta) = \mathfrak{O}$. Let $\psi_\alpha = \psi \circ x_\alpha$. Clearly ψ_α is an additive character of F. Therefore there exists $a_\alpha \in F^\times$ such that $\psi_\alpha(\lambda) = \theta(a_\alpha \lambda)$, for $\lambda \in F$. Let $x'_\alpha : F \to U_\alpha$ be defined as

$$x'_\alpha(\lambda) = x_\alpha(a_\alpha^{-1}\lambda).$$

Clearly x'_α defines an isomorphism between F and U_α. Before we continue, we explore the relation between the characters ψ and θ and set up some more notation. For $u \in U$, we choose $\tau_\alpha \in F$ such that $u = \prod_{\alpha \in \Phi^+} x'_\alpha(\tau_\alpha)$. Now

$$\begin{aligned}\psi(u) = \psi\Big(\prod_{\alpha\in\Phi^+} x'_\alpha(\tau_\alpha)\Big) &= \psi\Big(\prod_{\alpha\in\Phi^+} x_\alpha(a_\alpha^{-1}\tau_\alpha)\Big) = \prod_{\alpha\in\Phi^+} \psi(x_\alpha(a_\alpha^{-1}\tau_\alpha)) \\ &= \prod_{\alpha\in\Delta} \psi_\alpha(a_\alpha^{-1}\tau_\alpha) = \prod_{\alpha\in\Delta} \theta(a_\alpha a_\alpha^{-1}\tau_\alpha) = \theta(\sum_{\alpha\in\Delta} \tau_\alpha).\end{aligned}$$

Let $G_m = G(\mathfrak{p}^m)$, $T_m = T(\mathfrak{p}^m)$, $U_m = U(\mathfrak{p}^m)$, $\bar{U}_m = \bar{U}(\mathfrak{p}^m)$. The group G_m has an Iwahori factorization, i.e., $G_m = \bar{U}_m T_m U_m$. We also choose $d \in T$ such that $\alpha(d) = \varpi^{-2}$ for all $\alpha \in \Delta$ and define $\theta_m : G_m \to \mathbb{C}^\times$ as

$$\theta_m(\bar{u}tu) = \theta_m(u) = \theta(\varpi^{-2m} \sum_{\alpha\in\Delta} \tau_\alpha), \quad \text{where as above } u = \prod_{\alpha\in\Phi^+} x'_\alpha(\tau_\alpha).$$

We can show that θ_m is a character. We refer to [12] for the details.

Let $K_m = d^m G_m d^{-m}$. Clearly K_m is a compact open subgroup and has an Iwahori factorization. Indeed,

$$\begin{aligned}
K_m &= d^m G_m d^{-m} \\
&= \underbrace{d^m \bar{U}_m d^{-m}}_{K_m^-} \underbrace{d^m T_m d^{-m}}_{K_m^\circ} \underbrace{d^m U_m d^{-m}}_{K_m^+} \\
&= K_m^- K_m^\circ K_m^+.
\end{aligned}$$

Let ψ_m be a character of K_m defined as

$$\psi_m(k) = \theta_m(d^{-m} k d^m).$$

For $k = k^- k^\circ k^+ \in K_m$, It is easy to see that $\psi_m(k) = \psi(k^+)$. Indeed,

$$\begin{aligned}
\psi_m(k) &= \theta_m(d^{-m} k d^m) \\
&= \theta_m(d^{-m} k^- d^m d^{-m} k^\circ d^m d^{-m} k^+ d^m) \\
&= \theta(\varpi^{-2m} \sum_{\alpha \in \Delta} \varpi^{2m} t_\alpha) \\
&= \psi(k^+), \quad \text{where } k^+ = \prod_{\alpha \in \Phi^+} x'_\alpha(t_\alpha).
\end{aligned}$$

From the above computations, it is clear that for m large enough, the pair (K_m, ψ_m) converges to (U, ψ). We fix an integer l large enough and call the pair (K_l, ψ_l) as the compact approximation of (U, ψ) in the above sense. From now on, to simplify notation we will write (K_l, ψ_l) as (K, ψ_K).

We now recall Rodier's result and refer the reader to [12] for further details.

Theorem 5.5 (Rodier)**.** *Let π be an irreducible admissible representation of G. There then exists a compact open subgroup $K = K_l$ and a character $\psi_K = \psi_{K_l}$*

of K such that $dim_{\mathbb{C}}\,\mathrm{Hom}_K(\pi, \psi_K) = dim_{\mathbb{C}}\,\mathrm{Hom}_U(\pi, \psi)$. Therefore, if π is generic, $dim_{\mathbb{C}}\,\mathrm{Hom}_K(\pi, \psi_K) = 1$.

5.4.1 An Example

We work out the details of Rodier's compact approximation in the case of $G = \mathrm{GL}(2, F)$. For $G = \mathrm{GL}(2, F)$, we have

$$G_m = \begin{bmatrix} 1+\mathfrak{p}^m & \mathfrak{p}^m \\ \mathfrak{p}^m & 1+\mathfrak{p}^m \end{bmatrix}, \qquad U_m = \begin{bmatrix} 1 & \mathfrak{p}^m \\ 0 & 1 \end{bmatrix},$$

$$T_m = \begin{bmatrix} 1+\mathfrak{p}^m & 0 \\ 0 & 1+\mathfrak{p}^m \end{bmatrix}, \qquad \bar{U}_m = \begin{bmatrix} 1 & 0 \\ \mathfrak{p}^m & 1 \end{bmatrix}$$

Let $\alpha \in X = X(T)$ be the simple root $\alpha(t) = ab^{-1}$ where $t = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $d = \begin{bmatrix} 1 & 0 \\ 0 & \varpi^2 \end{bmatrix}$. Clearly $\alpha(d) = \varpi^{-2}$ for the simple root α. We have

$$d^m \bar{U}_m d^{-m} = \begin{bmatrix} 1 & 0 \\ \mathfrak{p}^{3m} & 1 \end{bmatrix}, \qquad d^m U_m d^{-m} = \begin{bmatrix} 1 & \mathfrak{p}^{-m} \\ 0 & 1 \end{bmatrix},$$

$$K_m = \begin{bmatrix} 1+\mathfrak{p}^m & \mathfrak{p}^{-m} \\ \mathfrak{p}^{3m} & 1+\mathfrak{p}^m \end{bmatrix}$$

We see that as $m \to \infty$ the compact open subgroup $K_m \to U$. We will now show that the character ψ_m of K_m approximates the non-degenerate character ψ. For $k \in K_m$, we have $k = k^- k^\circ k^+$. Now $\psi_m(k) = \theta_m(d^{-m} k^+ d^m)$. Here

$k^+ \in d^m U_m d^{-m} \subset U$. Suppose that

$$k^+ = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in U.$$

Now

$$\begin{aligned} \psi_m(k^- k^\circ k^+) &= \theta_m(d^{-m} k^+ d^m)) \\ &= \theta_m\big(\left[\begin{smallmatrix} 1 & \varpi^{2m} x \\ 0 & 1 \end{smallmatrix}\right]\big) \\ &= \theta(\varpi^{-2m}(\varpi^{2m} x)) \\ &= \psi(k^+). \end{aligned}$$

Chapter 6

Main Theorem

In this chapter, we prove the main theorem. We recall the statement below.

Theorem 6.1 (Main Theorem)**.** *Let (π, W) be an irreducible smooth self-dual representation of G with non-zero vectors fixed under an Iwahori subgroup I in G. Suppose that π is also generic. Then $\varepsilon(\pi) = 1$.*

We first prove the result when G has connected center. In this case, we use a result of Rodier (Theorem 5.5) to get a compact open subgroup K and a character ψ_K of K which appears with multiplicity one in $\pi|_K$. We show that there exists an element $s \in T$ satisfying the hypotheses of Prasad's Theorem (Theorem 5.4). Finally we use the fact that π has non-zero Iwahori fixed vectors to show that $\varepsilon(\pi) = 1$.

When the center of G is not connected, we construct a split connected reductive F-group $\tilde{G}$ with a maximal F-split torus $\tilde{T}$. The group $\tilde{G}$ has a connected center $\tilde{Z}$ and contains G as a subgroup. We show that there exists an irreducible representation $\tilde{\pi}$ of $\tilde{G}$ that contains the representation π with multiplicity one on restriction to G and has non-zero vectors fixed under an Iwahori subgroup in $\tilde{G}$. The representation $\tilde{\pi}$ is not necessarily self-dual but is self-dual up to a twist by a character χ of $\tilde{G}$ which is trivial on G. We can still attach a sign $\varepsilon(\tilde{\pi})$ to $\tilde{\pi}$. Finally, we show that $\varepsilon(\tilde{\pi}) = \varepsilon(\pi)$ and $\varepsilon(\tilde{\pi}) = 1$.

Throughout this section, we let G be the group of F-points of a connected reductive algebraic group defined and split over F. We write T for a maximal F-split torus in G. We also fix a Borel subgroup B defined over F such that $B \supset T$. We write U for the unipotent radical of B (respectively $\bar{U}$ for the T-opposite of U) and fix a non-degenerate character ψ of U such that $\mathrm{Hom}_U(\pi, \psi) \neq 0$ (ψ exists since π is generic). We let X and $X^\vee$ be the character and cocharacter groups of T. We write Φ and $\Phi^\vee$ for the set of roots and coroots and Δ for the set of simple roots of T. Since T is F-split, we have unique subgroups $T_\circ$ and T_1 of T such that $T = T_\circ \times T_1$. To be more precise, the isomorphism $F^\times \simeq \mathfrak{O}^\times \times \mathbb{Z}$ given by $x\varpi^n \mapsto (x, n)$ induces the following isomorphism

$$T \simeq X^\vee \otimes F^\times \simeq X^\vee \otimes \mathfrak{O}^\times \oplus X^\vee \otimes \mathbb{Z}$$

and we take $T_\circ$ and T_1 to be the subgroups of T such that $T_\circ \simeq X^\vee \otimes \mathfrak{O}^\times$ ($\alpha^\vee \otimes y \to \alpha^\vee(y)$) and $T_1 \simeq X^\vee \otimes \mathbb{Z}$ ($\alpha^\vee \otimes n \to \alpha^\vee(\varpi^n)$). We have a similar decomposition for $\tilde{T}$ (i.e., $\tilde{T} = \tilde{T}_\circ \times \tilde{T}_1$). In what follows, we let $Z_\circ = Z \cap T_\circ$ and $Z_1 = Z \cap T_1$ (respectively $\tilde{Z}_\circ = \tilde{Z} \cap \tilde{T}_\circ$ and $\tilde{Z}_1 = \tilde{Z} \cap \tilde{T}_1$). We let $\omega_\circ = \omega_\pi|_{Z_\circ}$ and $\omega_1 = \omega_\pi|_{Z_1}$.

6.1 Center of G is connected

In this section, we show the existence of an element $s \in T$ satisfying the conditions of Prasad's theorem (Theorem 5.4) and use it to compute the sign $\varepsilon(\pi)$ when the center of G is connected.

Lemma 6.2. *Let $s \in T$ be such that $\alpha(s) = -1$ for all simple roots $\alpha \in \Delta$. The following are true.*

(i) For $u \in U$, we have $\psi(sus^{-1}) = \psi^{-1}(u)$.

(ii) The element s^2 belongs to the center of G.

(iii) The element s normalizes the compact open subgroups K_m and inner conjugation by s takes the characters ψ_m to its inverse.

Proof. Since U is generated by U_α, $\alpha \in \Phi$, it is enough to show that $\psi(sus^{-1}) = \psi^{-1}(u)$ for $u \in U_\alpha$. For $u = x_\alpha(\lambda) \in U_\alpha$ we have,

$$\begin{aligned}
\psi(sus^{-1}) &= \psi(sx_\alpha(\lambda)s^{-1}) \\
&= \psi(x_\alpha(\alpha(s)\lambda)) \\
&= \psi(x_\alpha(-\lambda)) \\
&= \psi^{-1}(x_\alpha(\lambda)) \\
&= \psi^{-1}(u).
\end{aligned}$$

For (ii), since $\alpha(s) = -1$ it is clear that $\alpha(s^2) = 1$ for all simple roots $\alpha \in \Delta$. Since $Z(G) = \bigcap_{\alpha \in \Delta} \mathrm{Ker}(\alpha)$, the result follows.

For (iii), It is easy to see that s normalizes U_m, $\bar{U}_m$. It follows that s normalizes G_m and hence K_m. Let $k = k^- k^\circ k^+ \in K_m$ (since $K_m = K_m^- K_m^\circ K_m^+$).

We have

$$\begin{aligned}\psi_m(sks^{-1}) &= \psi_m(sk^-s^{-1}sk^\circ s^{-1}sk^+s^{-1}) \\ &= \psi(sk^+s^{-1}) \\ &= \psi^{-1}(k^+) \\ &= \psi_m^{-1}(k).\end{aligned}$$

□

Theorem 6.3. *Let (π, W) be an irreducible smooth self-dual generic representation of G with non-zero vectors fixed under an Iwahori subgroup I in G. Suppose there exists an element $s \in T_\circ$ such that $\alpha(s) = -1$ for all simple roots α. Then $\varepsilon(\pi) = 1$.*

Proof. By Theorem 5.4, it is enough to show that $\omega_\pi(s^2) = 1$ (ω_π is the central character). Let $v \neq 0 \in \pi^I$. We have $v = \pi(s^2)v = \omega_\pi(s^2)v$. From this it follows that $\omega_\pi(s^2) = 1$. □

Theorem 6.4. *There exists $s \in T_\circ$ such that $\alpha(s) = -1$ for all the simple roots α.*

Proof. We know that $X^\vee \otimes F^\times \simeq T$, via $y \otimes \lambda \mapsto y(\lambda)$. Since $F^\times \simeq \mathfrak{O}^\times \rtimes \mathbb{Z}$, we see that $T \simeq X^\vee \otimes \mathfrak{O}^\times \oplus X^\vee \otimes \mathbb{Z}$. Now define $f : T \longrightarrow \prod_{\alpha_i \in \Delta} F^\times$ by

$$f(y \otimes \lambda) = (\lambda^{\langle \alpha_1, y\rangle}, \ldots, \lambda^{\langle \alpha_k, y\rangle}).$$

We will show that there exists $y \in X^\vee$ such that $\langle \alpha_i, y\rangle$ is an odd integer for every simple root $\alpha_i, i = 1, \ldots, k$. Since Z is connected $X/\mathbb{Z}\Phi$ is torsion free.

Since Δ spans Φ we see that $\mathbb{Z}\Phi = \mathbb{Z}\Delta$. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z}\Delta \longrightarrow X \longrightarrow X/\mathbb{Z}\Delta \longrightarrow 0. \tag{6.1}$$

Since (6.1) is an exact sequence of finitely generated free abelian groups, it is split. i.e., $X = \mathbb{Z}\Delta \oplus L$, where $L \simeq X/\mathbb{Z}\Delta$. Let $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z})$. Clearly, g extends to an element of $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ (say trivial on L). Since $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}) \simeq X^{\vee}$, there exists $y \in X^{\vee}$ such that $g = \langle -, y\rangle$. We now choose $h \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}\Delta, \mathbb{Z})$ such that $h(\alpha_i)$ is odd for every $\alpha_i, i = 1, 2, \ldots, k$. Then $h(\alpha_i) = \langle \alpha_i, y\rangle$ is an odd integer. Now consider the element $y \otimes -1 \in X^{\vee} \otimes \mathfrak{O}^{\times}$. Let $s = y(-1)$. Then $s \in T_{\circ}$ clearly acts by -1 on all the simple root subgroups U_{α} of U, i.e.,

$$\begin{aligned} s x_{\alpha_i}(\mu) s^{-1} &= x_{\alpha_i}(\alpha_i(s)\mu) \\ &= x_{\alpha_i}(\alpha_i(y(-1))\mu) \\ &= x_{\alpha_i}((-1)^{\langle \alpha_i, y\rangle}\mu) \\ &= x_{\alpha_i}(-\mu). \end{aligned}$$

□

6.2 Center of G is not connected

Construction of $(\tilde{G}, \tilde{T})$

Let $q : X \to X/\mathbb{Z}\Phi$ be the canonical quotient map. Choose a free abelian group L of finite rank such that there exists a surjective map $p : L \to X/\mathbb{Z}\Phi$. Let p_1 and p_2 be the projection maps from $X \times L$ onto X and L respectively.

Let

$$\tilde{X} = \{(x, l) \in X \times L \mid q(x) = p(l)\}.$$

Clearly $\tilde{X}$ is a free abelian group of finite rank. Let $\tilde{\Phi} = \{(\alpha, 0) \mid \alpha \in \Phi\}$. The map $\alpha \mapsto (\alpha, 0)$ induces an injection $\mathbb{Z}\Phi \hookrightarrow \tilde{X}$ and we identify its image under the map with $\mathbb{Z}\tilde{\Phi}$. Let $\tilde{X}^\vee = \mathrm{Hom}_{\mathbb{Z}}(\tilde{X}, \mathbb{Z})$. Given $\tilde{\alpha} \in \tilde{\Phi}$ we want to describe $\tilde{\alpha}^\vee \in \tilde{\Phi}^\vee \subset \tilde{X}^\vee$. Now $\tilde{\alpha} = (\alpha, 0)$ for some $\alpha \in \Phi$. For this α, there exists $\alpha^\vee \in \Phi^\vee \subset X^\vee \simeq \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$. Let $\tilde{x} = (x, 0) \in \tilde{X}$. Define $\tilde{\alpha}^\vee(\tilde{x}) = \tilde{\alpha}^\vee((x, 0)) = \alpha^\vee(p_1(\tilde{x}))$. Clearly $\tilde{\alpha}^\vee \in \mathrm{Hom}_{\mathbb{Z}}(\tilde{X}, \mathbb{Z})$. It is easy to see that $(\tilde{X}, \tilde{\Phi}, \tilde{X}^\vee, \tilde{\Phi}^\vee)$ is a root datum. By Theorem 2.10, the existence of $(\tilde{G}, \tilde{T})$ follows. Since $\tilde{X}/\mathbb{Z}\tilde{\Phi} \hookrightarrow L$, it follows that it is torsion free and the center $\tilde{Z}$ of $\tilde{G}$ is connected.

Extension of the central character

In this section, we show that there exists a character ν of $\tilde{Z}$ which extends the central character ω_π and satisfies $\nu^2 = 1$.

Lemma 6.5. *There exists an unramified character $\mu : T \to \mathbb{C}^\times$ such that $\mu|_Z = \omega_\pi$.*

Proof. Since π has Iwahori fixed vectors, there exists an unramified character μ of T such that $\pi \hookrightarrow \mathrm{Ind}_B^G \mu$. Let (ρ, E) be an irreducible sub-representation of $\mathrm{Ind}_B^G \mu$ that is isomorphic to π. Let $x \in Z, f \in E, g \in G$. Clearly,

$$(\rho(x)f)(g) = f(gx) = f(xg) = \mu(x)f(g) \tag{6.2}$$

On the other hand,

$$(\rho(x)f)(g) = \omega_\rho(x)f(g) \tag{6.3}$$

From (6.2) and (6.3) it follows that $\omega_\rho(x) = \mu(x) = \omega_\pi(x)$. $\square$

Since μ is unramified it follows that $\mu|_Z = \omega_1$ (since $\mu|_{Z_\circ} = 1$). If we can extend ω_1 to a self-dual character $\tilde{\omega}_1$ of $\tilde{Z}_1$ then we get a self-dual character ν of $\tilde{Z}$ extending the central character ω_π. We record the result in a lemma below.

Lemma 6.6. *Suppose that $\tilde{\omega}_1$ is an extension of ω_1 to $\tilde{Z}_1$. Then there exists $\nu : \tilde{Z} \to \{\pm 1\}$ such that ν extends ω_π.*

Proof. For $\tilde{z} = \tilde{z}_0\tilde{z}_1 \in \tilde{Z}$, $\tilde{z}_0 \in \tilde{Z}_\circ, \tilde{z}_1 \in \tilde{Z}_1$ define $\nu(\tilde{z}) = \tilde{\omega}_1(\tilde{z}_1)$. Clearly ν is a well-defined character of $\tilde{Z}$ and $\nu|_Z = \tilde{\omega}_1|_Z = \omega_1 = \mu|_Z = \omega_\pi$. $\square$

From Lemma 6.6, it follows that we can extend the central character ω_π to a self-dual character ν of $\tilde{Z}$ if there exists an extension $\tilde{\omega}_1$ of ω_1. Consider the map $\omega_1' : Z_1/Z_1^2 \to \{\pm 1\}$ defined by $\omega_1'(aZ_1^2) = \omega_1(a)$. Since Z_1/Z_1^2 is an elementary abelian 2-group, ω_1' can be thought of as a $\mathbb{Z}_2$-linear map. If the natural map from Z_1/Z_1^2 to $\tilde{Z}_1/\tilde{Z}_1^2$ is an embedding, then we can extend the $\mathbb{Z}_2$-linear map ω_1' to a $\mathbb{Z}_2$-linear map $\tilde{\omega}_1'$ of $\tilde{Z}_1/\tilde{Z}_1^2$. Now defining $\tilde{\omega}_1(a) = \tilde{\omega}_1'(a\tilde{Z}_2)$ gives us an extension of ω_1. The natural map is an embedding precisely when $\tilde{Z}_1^2 \cap Z_1 \subset Z_1^2$. We record the result in the following lemma.

Lemma 6.7. *The natural map Z_1/Z_1^2 to $\tilde{Z}_1/\tilde{Z}_1^2$ is an embedding and ω_1 extends to a character $\tilde{\omega}_1$ of $\tilde{Z}_1$.*

Proof. It is enough to show that $\tilde{Z}_1^2 \cap Z_1 \subset Z_1^2$. Consider $z_1 \in \tilde{Z}_1^2 \cap Z_1$. Clearly $z_1 = \tilde{z}_1^2$ for some $\tilde{z}_1 \in \tilde{Z}_1$. It is enough to show that $\tilde{z}_1 \in T$ (since $\tilde{z}_1 \in T$ implies $\tilde{z}_1 \in \tilde{Z}_1 \cap T = Z_1$ and $\tilde{z}_1^2 \in Z_1$). Since $T \hookrightarrow \tilde{T}$ there exists a sub-torus

S such that $\tilde{T} = \tilde{T}_\circ \times \tilde{T}_1 = T \times S$. Clearly, $\tilde{T}_1 = T_1 \times S_1$. Indeed,

$$\begin{aligned}\tilde{T} &= \tilde{T}_\circ \times \tilde{T}_1 \\ &= T_\circ \times T_1 \times S_\circ \times S_1 \\ &= T_\circ \times S_\circ \times T_1 \times S_1.\end{aligned}$$

Now $\tilde{z}_1 \in \tilde{Z}_1 \subset \tilde{T}_1 = T_1 \times S_1$. Therefore $\tilde{z}_1 = t_1 s_1$ for $t_1 \in T_1$, $s_1 \in S_1$. Also $z_1 = \tilde{z}_1^2 = t_1^2 s_1^2 \in Z_1 \subset T_1$. We see that $s_1^2 = 1$. Since $\tilde{T}_1$ is torsion free it follows that $s_1 = 1$ and $\tilde{z}_1 \in T$. □

Irreducible representation $\tilde{\pi}$ of $\tilde{G}$

In this section, we show that there exists an irreducible representation $\tilde{\pi}$ of $\tilde{G}$ which contains π with multiplicity one on restriction to G.

The main idea behind the proof is Theorem 5.2. We first extend the representation π to an irreducible representation $\pi\nu$ of $\tilde{Z}G$ and show that the group $\tilde{G}/\tilde{Z}G$ is finite abelian. Before we continue, we recall a result of Serre which we use in proving the finiteness of $\tilde{G}/\tilde{Z}G$.

Theorem 6.8 (Serre)**.** *If A is a finite Γ module, $H^n(\Gamma, A)$ is finite for every n.*

Proof. See Proposition 14, Sec. 5.1 in [13]. □

Let (π, W) be an irreducible representation of G and ν be a self-dual character of $\tilde{Z}$ extending the central character ω_π. Let $\pi\nu : \tilde{Z}G \to \mathrm{GL}(W)$ be defined as $(\pi\nu)(\tilde{z}g) = \nu(\tilde{z})\pi(g)$. Clearly, $\pi\nu$ is a well-defined irreducible representation of $\tilde{Z}G$. The irreducibility of $\pi\nu$ is trivial. To see $\pi\nu$ is well-defined, suppose $z_1 g_1 = z_2 g_2$. Then

$$\begin{aligned}
(\pi\nu)(z_1 g_1) &= (\pi\nu)(z_1 z_2^{-1} z_2 g_1) \\
&= \nu(z_1 z_2^{-1})\nu(z_2)\pi(g_1) \\
&= \omega_\pi(z_1 z_2^{-1})\nu(z_2)\pi(g_1) \quad (\text{since } z_1 z_2^{-1} \in \tilde{Z} \cap G) \\
&= \omega_\pi(z_1 z_2^{-1})\nu(z_2)\pi(z_1^{-1} z_2 g_2) \\
&= \omega_\pi(z_1 z_2^{-1})\nu(z_2)\omega_\pi(z_1^{-1} z_2)\pi(g_2) \\
&= (\pi\nu)(z_2 g_2).
\end{aligned}$$

We now prove the finiteness of $\tilde{G}/\tilde{Z}G$.

Theorem 6.9. *$\tilde{G}/\tilde{Z}G$ is a finite abelian group.*

Proof. Clearly, $\tilde{G} = \tilde{T}G$. Now

$$\begin{aligned}
\tilde{G}/\tilde{Z}G &= \tilde{T}G/\tilde{Z}G \\
&= (\tilde{T}\tilde{Z}G)/\tilde{Z}G \\
&= \tilde{T}/(\tilde{T} \cap \tilde{Z}G) \\
&= \tilde{T}/\tilde{Z}T.
\end{aligned}$$

It follows that $\tilde{G}/\tilde{Z}G$ is abelian. Let $\bar{F}$ be the algebraic closure of F and $\Gamma = \mathrm{Gal}(\bar{F}/F)$. Let $m : T(\bar{F}) \times \tilde{Z}(\bar{F}) \to \tilde{T}(\bar{F})$ be the multiplication map. This map is surjective with $\mathrm{Ker}(m) = \{(z, z^{-1}) \mid z \in Z(\bar{F})\}$ (follows by considering the dimensions). Considering $Z(\bar{F})$ embedded diagonally in $T(\bar{F}) \times \tilde{Z}(\bar{F})$, we get the following exact sequence of abelian groups

$$1 \longrightarrow Z(\bar{F}) \longrightarrow T(\bar{F}) \times \tilde{Z}(\bar{F}) \xrightarrow{m} \tilde{T}(\bar{F}) \longrightarrow 1.$$

Γ clearly acts on these groups and applying Galois cohomology, we get a long exact sequence of cohomology groups

$$\begin{gathered} 1 \longrightarrow Z(\bar{F})^{\Gamma} \longrightarrow T(\bar{F})^{\Gamma} \times \tilde{Z}(\bar{F})^{\Gamma} \longrightarrow \tilde{T}(\bar{F})^{\Gamma} \longrightarrow H^1(\Gamma, Z(\bar{F})) \\ \longrightarrow H^1(\Gamma, T(\bar{F}) \times \tilde{Z}(\bar{F})) \longrightarrow H^1(\Gamma, \tilde{Z}(\bar{F})) \longrightarrow \cdots \end{gathered}$$

We note that $H^1(\Gamma, T(\bar{F}) \times \tilde{Z}(\bar{F})) = 1$ (Theorem 2.14) and we get the short exact sequence

$$1 \longrightarrow Z \longrightarrow T \times \tilde{Z} \xrightarrow{m} \tilde{T} \xrightarrow{\varphi} H^1(\Gamma, Z(\bar{F})) \longrightarrow 1. \tag{6.4}$$

From (6.4) it follows that φ is surjective, $\mathrm{Im}(m) = T\tilde{Z} = \mathrm{Ker}(\varphi)$, and $\tilde{T}/\tilde{Z}T \simeq H^1(\Gamma, Z(\bar{F}))$. It is enough to show that $H^1(\Gamma, Z(\bar{F}))$ is finite. Let Z° be the identity component of the algebraic group Z. Consider the short exact sequence

$$1 \longrightarrow Z^{\circ}(\bar{F}) \longrightarrow Z(\bar{F}) \longrightarrow Z(\bar{F})/Z^{\circ}(\bar{F}) \longrightarrow 1. \tag{6.5}$$

Applying Galois cohomology again to (6.5), we get the sequence

$$\begin{gathered} 1 \longrightarrow Z^{\circ} \longrightarrow Z \longrightarrow Z/Z^{\circ} \longrightarrow H^1(\Gamma, Z^{\circ}(\bar{F})) \longrightarrow H^1(\Gamma, Z(\bar{F})) \\ \longrightarrow H^1(\Gamma, Z(\bar{F})/Z^{\circ}(\bar{F})) \longrightarrow \cdots \end{gathered}$$

Since $Z^{\circ}(\bar{F})$ is connected, we have $H^1(\Gamma, Z^{\circ}(\bar{F})) = 1$ and it follows that $H^1(\Gamma, Z(\bar{F})) \hookrightarrow H^1(\Gamma, Z(\bar{F})/Z^{\circ}(\bar{F}))$. Since F is a local field of char 0 and

$Z(\bar{F})/Z^\circ(\bar{F})$ is a finite abelian group, $H^1(\Gamma, Z(\bar{F})/Z^\circ(\bar{F}))$ is finite (Theorem 6.8). Hence the result follows. $\square$

By Theorem 5.1 and Theorem 5.2, we get an irreducible representation $(\tilde{\pi}, V)$ of $\tilde{G}$ which breaks up as a finite direct sum of distinct irreducible representations $\pi_1, \ldots, \pi_k$ each occurring with the same multiplicity m on restriction to $\tilde{Z}G$ and contains $\pi\nu$ as a constituent. Without loss of generality, we assume that $\pi_1 \simeq \pi\nu$. To simplify notation, we again denote the restriction of π_i's to G by π_i so that $\tilde{\pi}|_G = m\pi_1 \oplus m\pi_2 \oplus \ldots \oplus m\pi_k$ and $\pi \simeq \pi_1$. We now show that each π_i occurs with multiplicity one in $\tilde{\pi}|_G$.

Lemma 6.10. *The representation $(\tilde{\pi}, V)$ of $\tilde{G}$ is generic and each irreducible representation π_i occurs with multiplicity one.*

Proof. Since (π, W) is generic, there exists a non-degenerate character ψ of U such that $\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G \psi) \neq 0$. It is enough to show that $\mathrm{Hom}_{\tilde{G}}(\tilde{\pi}, \mathrm{Ind}_{\tilde{U}}^{\tilde{G}} \psi) \neq 0$. Observe that $\tilde{U} = U \subset G$. Consider the restriction $\tilde{\pi}|_U$ of $\tilde{\pi}$. Since π is generic, $\mathrm{Hom}_U(\pi|_U, \psi) \neq 0$. It follows that

$$\mathrm{Hom}_U(\tilde{\pi}|_U, \psi) \simeq \mathrm{Hom}_{\tilde{G}}(\tilde{\pi}, \mathrm{Ind}_{\tilde{U}}^{\tilde{G}} \psi) \neq 0.$$

Indeed,

$$\begin{aligned}
\mathrm{Hom}_U(\tilde{\pi}|_U, \psi) &= \mathrm{Hom}_U((\tilde{\pi}|_G)|_U, \psi) \\
&= \mathrm{Hom}_U((m\pi_1 \oplus m\pi_2 \oplus \cdots \oplus m\pi_k)|_U, \psi) \\
&= m \bigoplus_{i=1}^{k} \mathrm{Hom}_U(\pi_i|_U, \psi) \\
&\neq 0.
\end{aligned}$$

As $\tilde{\pi}$ is generic it follows that $dim(\mathrm{Hom}_{\tilde{G}}(\tilde{\pi}, \mathrm{Ind}_U^{\tilde{G}} \psi)) = 1$. Thus by Frobenius Reciprocity, $m = 1$. □

Choosing $\tilde{\pi}$ with non-zero $\tilde{I}$ fixed vectors

In this section, we show that the representation $\tilde{\pi}$ can be modified in such a way that it has non-zero vectors fixed under an Iwahori subgroup $\tilde{I}$ in $\tilde{G}$.

Lemma 6.11. *Suppose that τ_1 is a linear character of $\tilde{I}$ which is trivial on I. Then τ_1 extends to a linear character $\tilde{\tau}$ of $\tilde{G}$ which is trivial on G.*

Proof. Let $I^- = I \cap \bar{U}$ and $I^+ = I \cap U$. We know that $I = I^- T_\circ I^+$. Since $U = \tilde{U}$ we have $\tilde{I} = I^- \tilde{T}_\circ I^+$. Now

$$\begin{aligned}\tilde{I}/I &= \tilde{T}_\circ I/I \ (\tilde{T}_\circ \,\text{normalizes}\, I^-,\, I^+) \\ &= \tilde{T}_\circ/\tilde{T}_\circ \cap I \\ &= \tilde{T}_\circ/T_\circ.\end{aligned}$$

It follows that we can consider τ_1 as a linear character of $\tilde{T}_\circ$ which is trivial on $T_\circ$. We first extend τ_1 to a character $\tilde{\tau}_1$ of $\tilde{T}$ by making it trivial on $\tilde{T}_1$, i.e., $\tilde{\tau}_1(\tilde{t}_\circ \tilde{t}_1) = \tau_1(\tilde{t}_\circ)$. Now define an extension $\tilde{\tau}$ of $\tilde{\tau}_1$ to $\tilde{G}$ as $\tilde{\tau}(\tilde{t}g) = \tilde{\tau}_1(\tilde{t}), \tilde{t} \in \tilde{T}, g \in G$ (this is possible since $\tilde{G} = \tilde{T}G$). Using

$$\tilde{G}/G = \tilde{T}G/T = \tilde{T}/T = \tilde{T}_\circ/T_\circ \times \tilde{T}_1/T_1$$

it follows that $\tilde{\tau}$ is well-defined and a character of $\tilde{G}$. □

Theorem 6.12. *The representation $(\tilde{\pi}\tau^{-1}, V)$ of $\tilde{G}$ has non-zero $\tilde{I}$ fixed vectors.*

Proof. Let $v \neq 0 \in V$ be such that $\tilde{\pi}(i)v = v$, $\forall i \in I$ (v exists since $\tilde{\pi}|_G \supset \pi$ and π has non-zero vectors fixed under I). Let $V_0 = \mathrm{Span}_{\mathbb{C}}\{\tilde{\pi}(k)v \mid k \in \tilde{I}\}$. Clearly, V_0 is an invariant subspace for $\tilde{I}$ and thus we get a representation (ρ, V_0) of $\tilde{I}$. Suppose $\rho = \tau_1 \oplus \cdots \oplus \tau_k$, where each τ_i is an irreducible representation of $\tilde{I}$. We know that $1_I \subset \rho$. Pick an irreducible component, say τ_1, that contains 1_I. By Clifford's theorem, $I \leq \mathrm{Ker}(\tau_1)$. Since $\tilde{I}/I$ is a compact abelian group, it follows that τ_1 is a linear character of $\tilde{I}$ which is trivial on I. By Lemma 6.11, τ_1 extends to a linear character $\tilde{\tau}$ of $\tilde{G}$ trivial on G. Consider the irreducible representation $\tilde{\pi}\tau^{-1}$. Clearly it has an $\tilde{I}$ fixed vector. Indeed for w in the space of τ_1 and $k \in \tilde{I}$, we have

$$\begin{aligned}
(\tilde{\pi}\tau^{-1})(k)w &= \tau^{-1}(k)\tilde{\pi}(k)w \\
&= \tau^{-1}(k)\tau_1(k)w \\
&= w \ (\,\mathrm{since}\, \tau|_{\tilde{I}} = \tau_1\,).
\end{aligned}$$

□

It is easy to see that $\tilde{\pi}\tau^{-1}|_G$ contains the representation π with multiplicity one, in addition to having non-zero $\tilde{I}$ fixed vectors. To simplify notation, we will denote the representation $\tilde{\pi}\tau^{-1}$ as $\tilde{\pi}$.

Sign of $\tilde{\pi}$

In this section, we attach a sign $\varepsilon(\tilde{\pi})$ to the representation $\tilde{\pi}$. We also give a formula to compute $\varepsilon(\tilde{\pi})$ and show that $\varepsilon(\tilde{\pi}) = \varepsilon(\pi)$. Finally we show that $\varepsilon(\tilde{\pi}) = 1$ to complete the proof of the main theorem.

Consider the representation $\tilde{\pi}^{\vee}$. This is again an irreducible representation of $\tilde{G}$ which on restriction to $\tilde{Z}G$ and contains the representation $\pi\nu$ with multiplicity 1 (since $\nu = \nu^{-1}$ and $\pi \simeq \pi^\vee$). By Theorem 5.2, there's a linear character χ of $\tilde{G}$ trivial on $\tilde{Z}G$ such that $\tilde{\pi}^\vee \simeq \tilde{\pi} \otimes \chi$. We use this isomorphism to define a non-degenerate bilinear form $[\, , \,]$ on V.

Lemma 6.13. *There exists a non-degenerate form* $[\, , \,] : V \times V \to \mathbb{C}$ *satisfying* $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] = \chi^{-1}(g)[v_1, v_2]$.

Proof. Since $\tilde{\pi}^\vee \simeq \tilde{\pi} \otimes \chi$, there exists a non-zero map $\phi : V \to V^\vee$ such that $\tilde{\pi}^\vee(g)(\phi(v)) = \phi((\tilde{\pi} \otimes \chi)(g)v)$. Let $\langle\, , \,\rangle : V \times V^\vee \to \mathbb{C}$ be the canonical $\tilde{G}$ invariant pairing. We define $[\, , \,] : V \times V \to \mathbb{C}$ as $[v_1, v_2] = \langle v_1, \phi(v_2)\rangle$. Clearly this form is non-degenerate and satisfies $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] = \chi^{-1}(g)[v_1, v_2]$. Indeed,

$$\begin{aligned}
[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] &= \langle \tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2 \rangle \\
&= \langle \tilde{\pi}(g)v_1, \chi^{-1}(g)\tilde{\pi}^\vee(g)(\phi(v_2)) \rangle \\
&= \chi^{-1}(g)\langle \tilde{\pi}(g)v_1, \tilde{\pi}^\vee(g)(\phi(v_2)) \rangle \\
&= \chi^{-1}(g)\langle v_1, \phi(v_2) \rangle \\
&= \chi^{-1}(g)[v_1, v_2].
\end{aligned}$$

□

The form $[\, , \,]$ is unique up to scalars and is easily seen to be symmetric or skew-symmetric as before, i.e.,

$$[v_1, v_2] = \varepsilon(\tilde{\pi})[v_2, v_1].$$

where $\varepsilon(\tilde{\pi}) \in \{\pm 1\}$. We call $\varepsilon(\tilde{\pi})$ the sign of $\tilde{\pi}$.

Let $[\,,\,] : V \times V \longrightarrow \mathbb{C}$ be the non-degenerate bilinear form on V(obtained above). Suppose that $[\,,\,]\Big|_{W_1 \times W_j} = 0, \forall j = 2, 3, \cdots, k$, then it is easy to see that $[\,,\,]\Big|_{W_1 \times W_1}$ is non-degenerate. We now show that $[\,,\,]\Big|_{W_1 \times W_j} = 0$ for $j = 2, 3, \cdots, k$.

Lemma 6.14. $[\,,\,]\Big|_{W_1 \times W_j} = 0, \ \forall j = 2, 3, \cdots, k$.

Proof. Suppose $[\,,\,]\Big|_{W_1 \times W_j} \neq 0$. Let $v \in W_1$ and $u \in W_j$ be such that $[v, u] \neq 0$. Let $\phi(w) = \phi_w$ be defined as $\phi_w(v) = [v, w]$. Then clearly ϕ is a non-zero intertwining map between $\pi_1^\vee$ and π_j. Indeed,

$$\begin{aligned}
(\pi_1^\vee(g) \circ \phi)(w_j)(w_1) &= \pi_1^\vee(g)(\phi(w_j)(w_1) \\
&= \phi(w_j)(\pi_1(g^{-1})w_1) \\
&= [\pi_1(g^{-1})w_1, w_j] \\
&= [w_1, \pi_j(g)w_j] \\
&= \phi(\pi_j(g)w_j)(v_1).
\end{aligned}$$

Since $\pi_1^\vee \simeq \pi_1$ and the representations π_i are distinct (up to isomorphism), the lemma follows. □

We know that $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] = \chi^{-1}(g)[v_1, v_2], \forall g \in \tilde{G}, v_1, v_2 \in V$. Now if $g \in G$ then $\chi(g) = 1$ and we have $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_2] = [v_1, v_2]$. In particular if $v_1 \in W_1$ and $v_j \in W_j$, then $[\tilde{\pi}(g)v_1, \tilde{\pi}(g)v_j] = [v_1, v_j]$. Since $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k, \tilde{\pi}(g)v_1 = \pi_1(g)v_1, \tilde{\pi}(g)v_j = \pi_j(g)v_j$.

Lemma 6.15. *With notation as above,* $\varepsilon(\tilde{\pi}) = \varepsilon(\pi)$.

Proof. Since $\chi(g) = 1$ for $g \in G$, we have

$$\begin{aligned}[\tilde{\pi}(g)w_1, \tilde{\pi}(g)\dot{w}_1] &= [\pi_1(g)w_1, \pi_1(g)\dot{w}_1] \\ &= [w_1, \dot{w}_1].\end{aligned}$$

Since $\pi_1 \simeq \pi$, we see that $[\,,\,]\big|_{W_1 \times W_1}$ is G-invariant. Therefore $[w_1, \dot{w}_1] = \varepsilon(\pi)(w_1, \dot{w}_1)$. But we also know that $[w_1, \dot{w}_1] = \varepsilon(\tilde{\pi})[\dot{w}_1, w_1]$. $\square$

By Lemma 6.15, it follows that the sign $\varepsilon(\pi)$ is completely determined by the sign $\varepsilon(\tilde{\pi})$. Since $\tilde{Z}$ is connected, applying Theorem 6.4 we get an element $s \in \tilde{T}_\circ$ such that $\alpha(s) = -1$ for all simple roots α of $\tilde{T}$. We will show that the sign $\varepsilon(\tilde{\pi})$ is controlled by the central character $\omega_{\tilde{\pi}}$ and the character χ. Before we proceed, we prove a lemma we need.

Lemma 6.16. *Let W_1, W_2 be irreducible K-invariant subspaces of V. Let $\rho_1 = \tilde{\pi}|_{W_1}$ and $\rho_2 = \tilde{\pi}|_{W_2}$. Let $b : W_2 \to W_1^\vee$ be the map $w_2 \mapsto [-, w_2]$. If $b \neq 0$, then $\rho_2 \simeq \rho_1^\vee \chi$.*

Proof. We first show that b defines an intertwining map between ρ_2 and $\rho_1^\vee \chi$. Indeed, for $w_1 \in W_1, w_2 \in W_2$ and $k \in K$, we have

$$\begin{aligned}b(\rho_2(k)(w_2))(w_1) &= [w_1, \rho_2(k)w_2] \\ &= \chi(k)[\rho_1(k^{-1})(w_1), w_2] \\ &= \chi(k)b(w_2)(\rho_1(k^{-1})w_1) \\ &= \chi(k)\rho_1^\vee(k)(b(w_2))(w_1).\end{aligned}$$

Since $b \neq 0$, Schur's Lemma (Theorem 2.1) applies and the result follows. $\square$

Let V_0 be the space of $\psi_{\tilde{K}}$ and $v_0 \in V_0$. Since $[\,,\,]$ is non-degenerate, $b(v_0)(v_1) = [v_1, v_0] \neq 0$ for some $v_1 \in V_1$ where V_1 is an irreducible $\tilde{K}$-invariant subspace of V. We denote ρ for the restriction of $\tilde{\pi}|_{V_1}$. By Lemma 6.16, it follows that $\rho \simeq \psi_{\tilde{K}}^{-1}\chi$. Since χ is smooth, we can in fact choose $\tilde{K}$ such that χ is trivial on $\tilde{K}$. It follows that any vector $v_0 \in V_0$ has to pair non-trivially with some vector in the space of $\psi_{\tilde{K}}^{-1}$, i.e., $[\tilde{\pi}(s)v_0, v_0] \neq 0$. We will use this in the following theorem.

Theorem 6.17. *Let $(\tilde{\pi}, V)$ be the irreducible representation of $\tilde{G}$ obtained above. Then $\varepsilon(\tilde{\pi}) = \omega_{\tilde{\pi}}(s^2)\chi(s)$.*

Proof. Clearly $s^2 \in \tilde{Z}$. Since $\tilde{\pi}$ is generic it follows by Theorem 5.5 that there exists a compact open subgroup $\tilde{K}$ and a character $\psi_{\tilde{K}}$ of $\tilde{K}$ such that $\psi_{\tilde{K}}$ occurs with multiplicity one in $\tilde{\pi}|_{\tilde{K}}$. Let V_0 be the space of $\psi_{\tilde{K}}$ and $0 \neq v_0 \in V_0$. Now

$$\begin{aligned} [\tilde{\pi}(s)v_0, \tilde{\pi}(s^2)v_0] &= \omega_{\tilde{\pi}}(s^2)[\tilde{\pi}(s)v_0, v_0] \quad \text{(since } s^2 \in \tilde{Z}\text{)} \\ &= \chi^{-1}(s)[v_0, \tilde{\pi}(s)v_0] \quad \text{(invariance of the form)} \end{aligned}$$

It follows that $\varepsilon(\tilde{\pi}) = \omega_{\tilde{\pi}}(s^2)\chi(s)$. $\square$

Using $\tilde{\pi}$ has Iwahori fixed vectors and $s^2 \in \tilde{T}_\circ$, we see that $\omega_{\tilde{\pi}}(s^2) = 1$. It will follow that $\varepsilon(\pi) = 1$ once we show that $\chi(s) = 1$. We do this by showing that χ is an unramified character. Before we continue, we recall a result about intertwining maps which we need in the proof.

For K a compact open subgroup of $\tilde{G}$ and ρ an irreducible representation of K, we let $\hat{K}$ denote the set of equivalence classes of irreducible smooth representations of K, $K^g = g^{-1}Kg$, $g \in \tilde{G}$ and ρ^g the irreducible representation of K^g defined as $x \to \rho(gxg^{-1})$.

Proposition 6.18. *For $i = 1, 2$, let K_i be a compact open subgroup of $\tilde{G}$ and let $\rho_i \in \hat{K}_i$. Let (Π, V) be an irreducible representation of $\tilde{G}$ which contains both ρ_1 and ρ_2. There then exists $g \in \tilde{G}$ such that* $\mathrm{Hom}_{K_1^g \cap K_2}(\rho_1^g, \rho_2) \neq 0$.

Proof. We refer the reader to [3] (Chapter 3, Section 11, Proposition 1) for a proof of the above proposition. In [3], the authors prove the result for $\mathrm{GL}(2, F)$. The same proof works in the case of any connected reductive group. □

Theorem 6.19. *The character χ is an unramified character. In particular $\chi(s) = 1$.*

Proof. We know that $\tilde{\pi}^\vee \simeq \tilde{\pi} \otimes \chi$. Since $\tilde{\pi}$ has non-trivial $\tilde{I}$ fixed vectors it follows that $\tilde{\pi}^\vee$ and hence $\tilde{\pi} \otimes \chi$ has non-trivial $\tilde{I}$ fixed vectors. Therefore $(\tilde{\pi} \otimes \chi)\Big|_{\tilde{I}} \supset 1$ and $(\tilde{\pi} \otimes \chi)\Big|_{\tilde{I}} \supset \chi$. By Proposition 6.18 there exists $g \in \tilde{G}$ such that $\mathrm{Hom}_{\tilde{I}^g \cap \tilde{I}}(1^g, \chi) \neq 0$. Since $\tilde{G} = \coprod_{w \in \tilde{W}} \tilde{I} w \tilde{I}$ we see that $\mathrm{Hom}_{\tilde{I}^w \cap \tilde{I}}(1^w, \chi) \neq 0$ when $g \in \tilde{I} w \tilde{I}$. From this it follows that $\chi(h) = 1, \forall h \in \tilde{I}^w \cap \tilde{I}$. Since $\tilde{T}_\circ \subset \tilde{I}^w \cap \tilde{I}$ it follows that χ is unramified. □

Bibliography

[1] Armand Borel. Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup. *Invent. Math.*, 35:233–259, 1976.

[2] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Translated from the German manuscript, Corrected reprint of the 1985 translation.

[3] C. J Bushnell and G. Henniart. *The Local Langlands Conjecture for $GL(2)$*. Grundlehren der Mathematischen Wissenschaften, 335 Springer - Verlag, Berlin, 2006.

[4] W. Casselman. The unramified principal series of $\mathfrak{p}$-adic groups. I. The spherical function. *Compositio Math.*, 40(3):387–406, 1980.

[5] David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2004.

[6] S. S. Gelbart and A. W. Knapp. L-indistinguishability and R groups for the special linear group. *Adv. in Math.*, 43(2):101–121, 1982.

[7] R. Gow. Real representations of the finite orthogonal and symplectic groups of odd characteristic. *J. Algebra*, 96(1):249–274, 1985.

[8] Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger. *Correspondances de Howe sur un corps p-adique*, volume 1291 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[9] Fiona Murnaghan. Linear algebraic groups. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 379–391. Amer. Math. Soc., Providence, RI, 2005.

[10] Dipendra Prasad. On the self-dual representations of finite groups of Lie type. *J. Algebra*, 210(1):298–310, 1998.

[11] Dipendra Prasad. On the self-dual representations of a p-adic group. *Internat. Math. Res. Notices*, (8):443–452, 1999.

[12] F. Rodier. Modèle de Whittaker et caractères de représentations. In *Noncommutative harmonic analysis (Actes Colloq., Marseille-Luminy, 1974)*, pages 151–171. Lecture Notes in Math., Vol. 466. Springer, Berlin, 1975.

[13] Jean-Pierre Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.

[14] T. A. Springer. *Linear algebraic groups*, volume 9 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1998.

[15] C. Ryan Vinroot. Involutions acting on representations. *J. Algebra*, 297(1):50–61, 2006.