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NON-UNIFORMLY SAMPLED RECURSIVE DIGITAL FILTERS

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ABSTRACT

Presented in this work is the analysis and design of recursive digital filters synchronously sampled nonuniformly, employing Z-transforms, Fourier transforms, and subsequently the state-space formulation. Suitable transfer sequence operators are developed for first and second order digital filters. However, much of the techniques developed could easily be adapted to higher order digital filters. Characteristic features of such transfer sequence operators are shown.

Statistical analysis of the quantization effects due to roundoffs are examined and some of the difficulties of analysis presented. A deterministic study of limit cycles due to roundoff of products in fixed-point arithmetic implementation is performed. Characteristic properties, such as stability constraints, magnitude and phase response of nonuniformly first and second order digital filters are presented and compared using computer simulations to those of the conventional digital filters.

Application of the developed nonuniform sampling decomposition methods, particularly to recursive variable digital filters and also to digital filtering in radar systems, is presented with examples.

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CHAPTER 1

INTRODUCTION

1.0 Introductory Remarks

Digital processing of time functions has been of considerable interest since the advent of the digital computer. The computational processes or algorithms may be that of low-pass filtering, band-pass filtering, interpolation, extrapolation or spectral decomposition [1]. Formally, a digital filter is a time sequential operator on a given discrete input time signal resulting in a discrete time signal as output. That is, the digital process transforms an input sequence $\{x_n\}$ into an output sequence $\{y_n\}$. For illustration, a common algorithm describing the transfer characteristics is a linear difference equation with constant coefficients of the form [3],

$$y_n = \sum_{i=o}^{N} a_i x_{n-i} - \sum_{j=1}^{M} b_j y_{n-j} \tag{1.1}$$

That is, at time n the output value can be computed from the M previous values of the output and N + 1 most recent values of the input. The set $\{a_i\}$ and $\{b_j\}$ are the filter parameters or "weights" that determine the nature of the filtering action on $\{x_n\}$.

Assuming uniform time sampling and taking the Z-transform of both sides of equation (1.1) term by term yields

$$Y(z)(1 + \sum_{i=1}^{M} b_i z^{-i}) = X(z) \sum_{i=o}^{N} a_i z^{-i} \qquad (1.2)$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{i=o}^{N} a_i z^{-i}}{1 + \sum_{i=1}^{M} b_i z^{-i}} \qquad (1.3)$$

$$= H(z)$$

where H(z) is the pulse transfer function of the filter.

The z-transform calculus, used to describe H(z), plays a useful role in the analysis and synthesis of linear discrete systems similar to the role of the Laplace transform calculus in the analysis and synthesis of linear continuous systems. No attempt has been made to discuss how the transfer function H(z) is obtained from the specification for a given digital filter. Techniques for determining H(z) from a given filter specification have received extensive coverage in the literature [1,3,4, 5,6]. For the present investigation, it is assumed that the transfer function satisfying the filter specification has been obtained and that a particular H(z) is therefore available.

1.1 Nonuniform Sampling

A discrete signal or sequence of numbers can be generated through the process of sampling a function of continuous time. That is, supposing the function is f(t), then the sequence $\{f_n\}$ may be used in representing f(t) where

$$f_n = f(nT) \qquad n = o \ldots. \infty.$$

Alternatively f(t) can be represented by the impulse f*(t) so that

$$f^*(t) = f(t)^* \sum_{n=o}^{\infty} \delta(t - nT)$$

which can also be written as

$$f^*(t) = \sum_{n=o}^{\infty} f(nT)\delta(t - nT) \tag{1.4}$$

The impulse train f*(t) as represented above is valid when samples are uniformly spaced. However, there are times when knowledge of only every other k-sample output is needed, and instances where it is too expensive to process an input that comes in every T seconds. As for example, in order to conserve energy consumption, the average number of samples per unit time is generally minimized in certain adaptive sampled data that are applicable in satellite control systems. In these cases, samples which are not uniformly spaced can be used. For nonuniform sampling, equation (1.4) is not necessarily valid. In general, equation (1.4) may be rewritten so that:

$$f^*(t) = f(t)* \sum_{k=o}^{\infty} \delta(t - t_k) \tag{1.5}$$

where t_j are separate but arbitrary spaced point in time. Thus, equation (1.4), applicable when samples are uniformly spaced, can be considered as a special case of equation (1.5) which is valid when samples are arbitrarily spaced.

Nonuniformly sampled data are quite often employed in radar tracking systems. For instance, McAulay [42] and others [40,41,43,44] have shown that the detection capabilities of radar systems for moving targets can be substantially extended by nonuniform sampling of the incoming bipolar video signals from the moving targets. Other viable applications for nonuniformly sampled data characteristics are in the design of variable digital filters and in the synthesis of mathematical models for certain biological systems in which samples are known to be nonuniformly spaced.

Some effort in the literature has been directed to the analysis of the difficulties that arise when data is processed at rates other than at uniform intervals. Yen [11] and Helms [12] have shown that a band-limited function is uniquely described, under certain conditions, by a knowledge of its values at nonuniformly spaced instants. Two theorems due to Yen [11] on nonuniform sampling are discussed briefly in Appendix A-1 of this dissertation.

Balakrishnan [18], Aggarwal and Kan [19] have considered error analysis of time jitters in sampling in which sampling is treated as a random phenomenon. This leads to the determination of signal-to-noise ratio where the noise is considered to be caused by the sampling. Brubaker [20] analyzed the effect of sampling interval drift in recursive digital filters designed using the impulse invariant approach. As shown in [20], the effect of sampling rate drift is to modify the critical frequencies of the filter by the ratio of the design sampling time and the actual sampling time.

Some aspects of nonuniform spacing of samples in discrete-data systems have been described in the literature. Jury [7,16] in discussing the z-transform approach to the analysis of discrete antenna arrays shows how the z-transform of the polynomial expression associated with the array can be expressed in terms of the z-transform of its known continuous nonuniform envelope function. Such an approach requires the existence of a known nonuniform envelope function continuous within a given interval range. Tou [14] and Ragazzini and Franklin [8], in their analysis of multirate sampled-data, derived the output sequences of a system with its input sampled at a slower rate than the output, and processed with a known transfer function H(s), where s is the Laplace transform operator. Ragazzini and Franklin [8] extended this analysis of multirate sampling to the case where the output of the system is sampled at a slower rate than the input. In their work, the input sequences are decomposed according to their location within a sampling period. These separate input sequences are appropriately weighted and processed with the system transfer function to yield the output sequences. This approach does not lend itself to the design of multirate discrete system, since the overall transfer function cannot be obtained by this approach. In addition, their decomposition approach is not applicable to the case where both the input and the output of such a discrete data system are sampled nonuniformly but synchronously.

Recently, Fjallbrant [45] has introduced a new type of recursive digital filter with multiple shifts in each sampling interval. Such multiple-shift digital filters are particularly suitable for the design of variable characteristic filters. Using linear difference equations and generating functions, Fjallbrant [45] derived the transfer function sequences of such filters. Jullien [15], using complex convolution integral, obtains the general transfer function for multiple shift sequences in digital filters. In addition, Atiya, Bilal and Abdou [46] have used state space techniques to analyze such multiple shifting sequences in digital signal processing. The approaches adopted in these works on multiple shift sequences in digital filters are not applicable to the analysis and the design of discrete systems with nonuniformly sampled inputs since during the shifting of signals within the filter, the input signals are set to zero.

1.2 Review of Research

In discrete-data systems where the input and the output are both subjected to nonuniform sampling but synchronized, analytic approaches advanced thus far are not directly applicable to the analysis of such systems. This dissertation will be directed toward developing a mathematical approach suitable for analysis and investigation of discrete-data systems with nonuniform samples. In particular, nonuniformly sampled recursive digital filters will be emphasized. The techniques developed in this work will provide a viable means of investigating design stability criteria, quantization error studies and limit cycle oscillations in recursive digital filters with nonuniform samples. Investigation of such problems in digital signal processing have been limited thus far to uniformly sampled digital filters. Application of the techniques of this work to other types of digital filtering will be presented.

Chapter 2 presents development of the mathematical decomposition technique applicable to discrete-data systems with nonuniform samples. Pulse transfer function sequences of these filters are derived as new results.

Chapter 3 extends the decomposition technique to analysis of design stability, quantization error effects and limit cycle oscillations in recursive digital filters in the presence of nonuniform sampling. Using computer simulation results, quantization errors and limit cycle oscillations in nonuniformly sampled digital filters are compared with those of filters with uniform sampling available from the works of Jackson [24,29] and others [5,6,27]. Chapter 3, also presents a new analytic approach of determining the frequency response of digital filters in the presence of nonuniform sampling. In particular, the frequency characteristics of first and second order recursive digital filters are shown. These characteristics are then compared with those of uniformly sampled digital filters available from the works of Kaiser [1,9] and others [2,3,4,5,13].

Chapter 4 presents applications of the decomposition techniques of Chapter 2 to the analysis of moving target indicator digital filters in radar systems. The mean square output function of a nonrecursive moving target indicator is derived and the results shown to agree with those of earlier work of Roy and Lowenschuss [40] and others [42,44] using a different approach. The simulation results of the mean square output of a recursive moving target indicator digital filter obtained by other works [43,44] are shown to compare favorably with the mean square output obtained analytically by employing the decomposition technique. Multiple-shift digital filter transfer function sequences are re-derived and shown to be a special case of the nonuniform sampling scheme of this work. Chapter 4 also discusses systems with inherent nonuniform samples. Signal-to-distortion ratio of such systems is derived for a special case of two intersamples per period using the Fourier transform approach.

Chapter 5 presents conclusions of this work and indication of those problems which remain subject to further research.

CHAPTER 2

TIME-DOMAIN SWITCH DECOMPOSITION FOR NON-UNIFORM SAMPLING

2.0 Introduction

This chapter presents the development of a new decomposition technique based on z-transforms and subsequent state space analysis of recursive digital filters, which are sampled nonuniformly, though periodic in the nonuniformity. This development is based on the concept of switch decomposition of time functions employed in multirate sampled data systems as analyzed by Ragazzini and Franklin [8], Tou [14] Wong and King [52] and quite recently by Boykin and Frazier [51]. The decomposition technique of this work is then applied in the analysis of fir t and second order recursive digital filters sampled nonuniformly. Employing the skip sampling theorem in the analysis, the pulse transfer function sequences of these filters are derived as new results. The z-transform of the output sequences of these filters are then formulated in terms of the pulse transfer function sequences and the advanced z-transform of the inputs. For reference purposes, nonuniform sampling, advanced z-transforms and the skip sampling theorems are discussed in Appendixes A-1, A-2 and A-3 respectively.

It should be noted that all the developments of this chapter assume infinite precision arithmetic so that all calculations are error free. Chapter 3 of this dissertation considers errors that result from such assumption.

2.1 Development of Decomposition Technique

The block diagram representation of discrete data system to be considered in this work is shown in Figure 2.1. It is assumed that the input to the system is appropriately band limited. Input signals appropriately band limited are discussed in Chapter 4. Appendix A-1 presents a discussion of the output recovery interpolation formula for nonuniform sampling.

In general, the discrete-time relation between the input and output may be expressed by the discrete convolution summation

$$y_n = \sum_{j=0}^{n} h_{n-j} x_j \tag{2.1}$$

where $y_n = y(t_n)$ and $h_j = 0;\ j < 0$.

It is desired to reorder the indices of (2.1) such that the event times (t_i) may be decomposed and regrouped into sets of periodic sequences consisting of n event times each. That is, if $t_n = T$, then the event times may be reorderd 0, t_1, $t_2, \ldots, t_{n-1}$, T, $T + t_1, \ldots,$ etc; provided that the periodicity exists for each point in the sequence. In other words if

$$\begin{aligned} t_i &= mt_n + t_k \\ &= mT + t_k \end{aligned} \tag{2.2}$$

then

$$t_i + T = (m+1)T + t_k$$

$$0 \le k \le n-1 \text{ for all } i. \tag{2.3}$$

This sequence arrangement is shown in Fig. 2.2. Accordingly, the input sequence may now be expressed

$$\begin{aligned} x_i &= x(t_i) \\ &= x(mT + t_k) \\ &= x_k(mT) \qquad 0 \le k \le n-1 \end{aligned} \tag{2.4}$$

The index k indicates the periodic time sequence with time shift t_k, $t_k < t_n$. These grouped sequences are shown in Fig. 2.3.

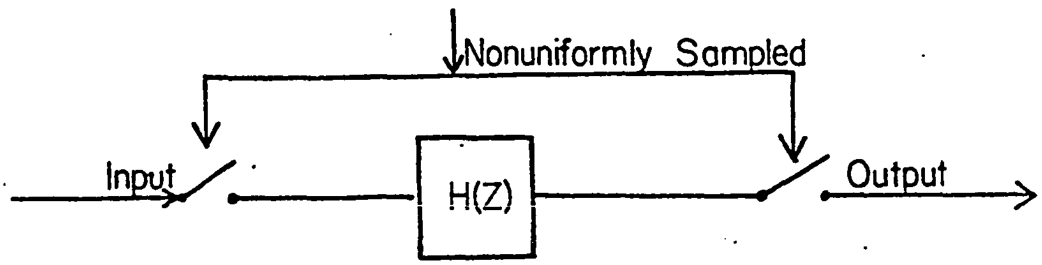


Fig. 2.1 Block diagram of nonuniformly sampled discrete system.

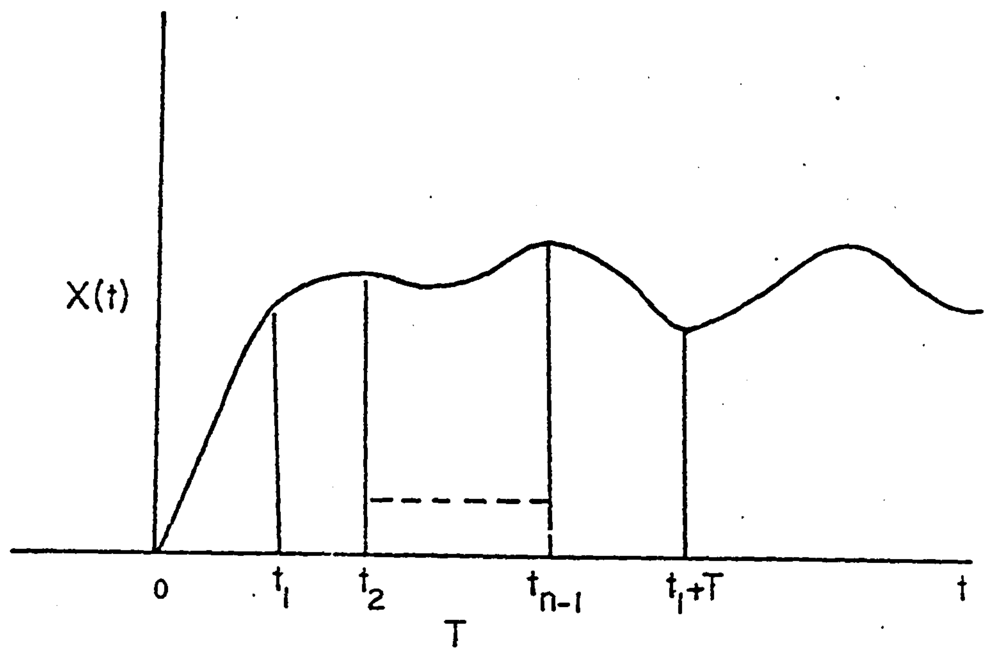


Fig. 2.2 Nonuniformly sampled function at 0, t_1, t_2 ... .

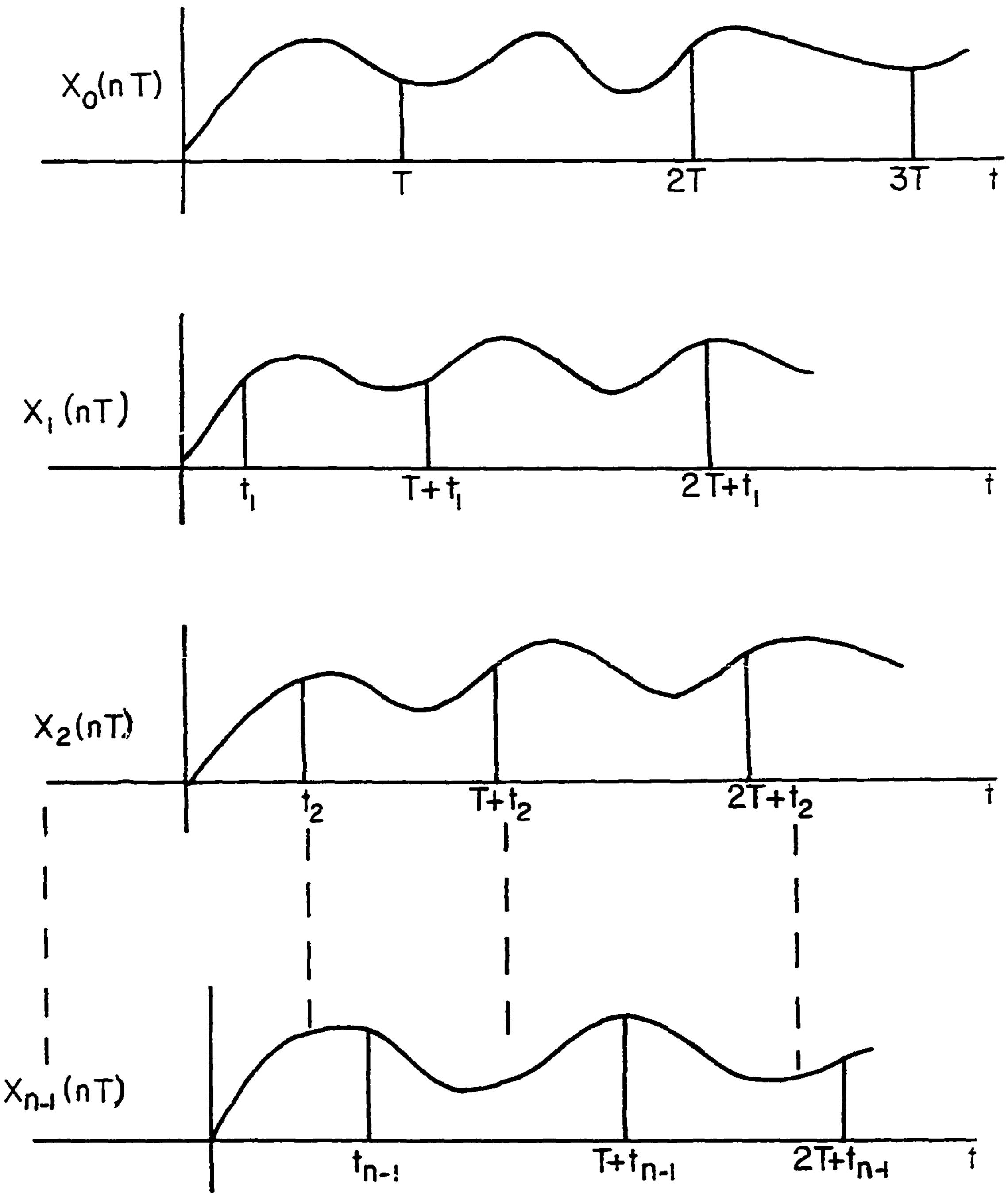


Fig. 2.3 Decomposed nonuniformly sampled function of Fig. 2.2.

The decomposition scheme of Ragazzini and Franklin [8] or Wong and King [47,52] is extended here to take into account the synchronized nonuniform sampling of the input and the output of the discrete system. For the entirely discrete transfer function H(z) that is available, the impulse sequence weightings that are applied to the respective input sequences are dependent not on the time that has elapsed since the application of the impulse, but only on the sequence index, k, of the sampling switch that has operated within T.

The decomposed set of uniform sequences may be used in the convolution summation (2.1) to determine the appropriately shifted output sequence. For the first output sequence $y_o(mT)$ and from (2.1) and (2.4)

$$
\begin{aligned}
y_o(mT) &= y_{m\cdot n} \\
&= \sum_{j=o}^{m\cdot n} h_{m\cdot n-j}x_j \\
&= \sum_{j=o}^{n} h_{m(n-j)}x_{mj} + \sum_{j=o}^{n} h_{m(n-j)-1}x_{m\cdot j+1} \\
&\quad + \dots + \sum_{j=o}^{n} h_{m(n-j)-(n-1)}x_{m\cdot j+(n-1)} \\
&= \sum_{j=o}^{n} h_o(nT-jT)x_o(jT) \\
&\quad + \sum_{j=o}^{n} h_{-1}(nT-jT)x_1(jT) \\
&\quad + \dots + \sum_{j=o}^{n} h_{-(n-1)}(nT-jT)x_{n-1}(jT) \qquad (2.5)
\end{aligned}
$$

In (2.5) the first output sequence has been decomposed into a set of convolution summations. The subscript on x directly indexes the input sequence advance position within T. The subscript on h should be taken to denote the delayed weighting sequence relative to the indicated sampling event within T.

In a similar fashion, $y_1(mT)$ may be determined, or

$$
\begin{aligned}
y_1(mT) &= y_{m\cdot n+1} \\
&= \sum_{j=o}^{n} h_{m(n-j)+1} x_{mj} \\
&\quad + \ldots + \sum_{j=o}^{n} h_{m(n-j)+1} x_{mj+(n-1)} \\
&= \sum_{j=o}^{n} h_1(nT-jT) x_o(jT) \\
&\quad + \ldots + \sum_{j=o}^{n} h_{-(n-1)+1}(nT-jT) x_{n-1}(jT) \qquad (2.6)
\end{aligned}
$$

To generalize,

$$
\begin{aligned}
y_k(mT) &= \sum_{j=o}^{n} h_k(nT-jT) x_k(jT) \\
&\quad + \ldots + \sum_{j=o}^{n} h_{-(n-1)}(nT-jT) x_{n-1}(jT) \\
&= \sum_{i=o}^{n-1} \left[\sum_{j=o}^{p} h_{-i+k}(nT-jT) x_i(jT) \right] \\
&\qquad 0 \le k \le n-1 \qquad (2.7)
\end{aligned}
$$

For the purpose of describing the Z-transform of nonuniformly sampled sequences, it is desirable to use precise notation to avoid confusion between the Z-transform of the function at various time intervals. The notation z_n should be taken to denote the number of samples n occurring in the overall sampling period T. In other words, z_n is a sequence operator. A delay of one sample is denoted by z_n^{-1} and has no unique time domain interpretation. However, the operator z_n^{-1} may only be interpreted in the time domain if the position of the sample sequence being delayed is known in time. The argument of the Z-transform taken with sample period T is denoted by (Z_n^n). This follows the notational convention as used by Rogazinni and Franklin [8].

Conventionally, if

$$y(nT) = \sum_{K=o}^{n} h(nT-kT)x(KT) \tag{2.8}$$

then by definition, the Z-transform of y(nT) becomes

$$\begin{aligned} Y(z) &= z\{y(nT)\} \\ &= H(z)X(z) \end{aligned} \tag{2.9}$$

Following the notation previously described, the Z-transform of the output sequence $y_0(mT)$ may be expressed directly from (2.8) and (2.9). Taking the Z-transform of (2.5),

$$\begin{aligned} z\{y_0(mT)\} &= Y_0(z_n^n) \\ &= H_0(z_n^n)X_0(z_n^n) + H_{-1}(z_n^n)X_1(z_n^n) \\ &\quad + \ldots + H_{-(n-1)}X_{n-1}(z_n^n) \\ &= \sum_{k=0}^{n-1} H_{-k}(z_n^n)X_k(z_n^n) \end{aligned} \tag{2.10}$$

A block diagram of the switch decomposition for the first output sequence, $Y_0(z_n^n)$ is shown in Figure 2.4. Note that $H_{-j}(z_n^n)$ should be taken to denote the weighting transform related to the jth previous sampling switch operation.

The second output sequence $y_1(mT)$ as expressed by (2.6) may be Z-transformed from (2.9) in the same manner as $y_0(mT)$, such that

$$Y_1(z_n^n) = \sum_{k=0}^{n-1} H_{-k+1}(z_n^n)X_k(z_n^n) \tag{2.11}$$

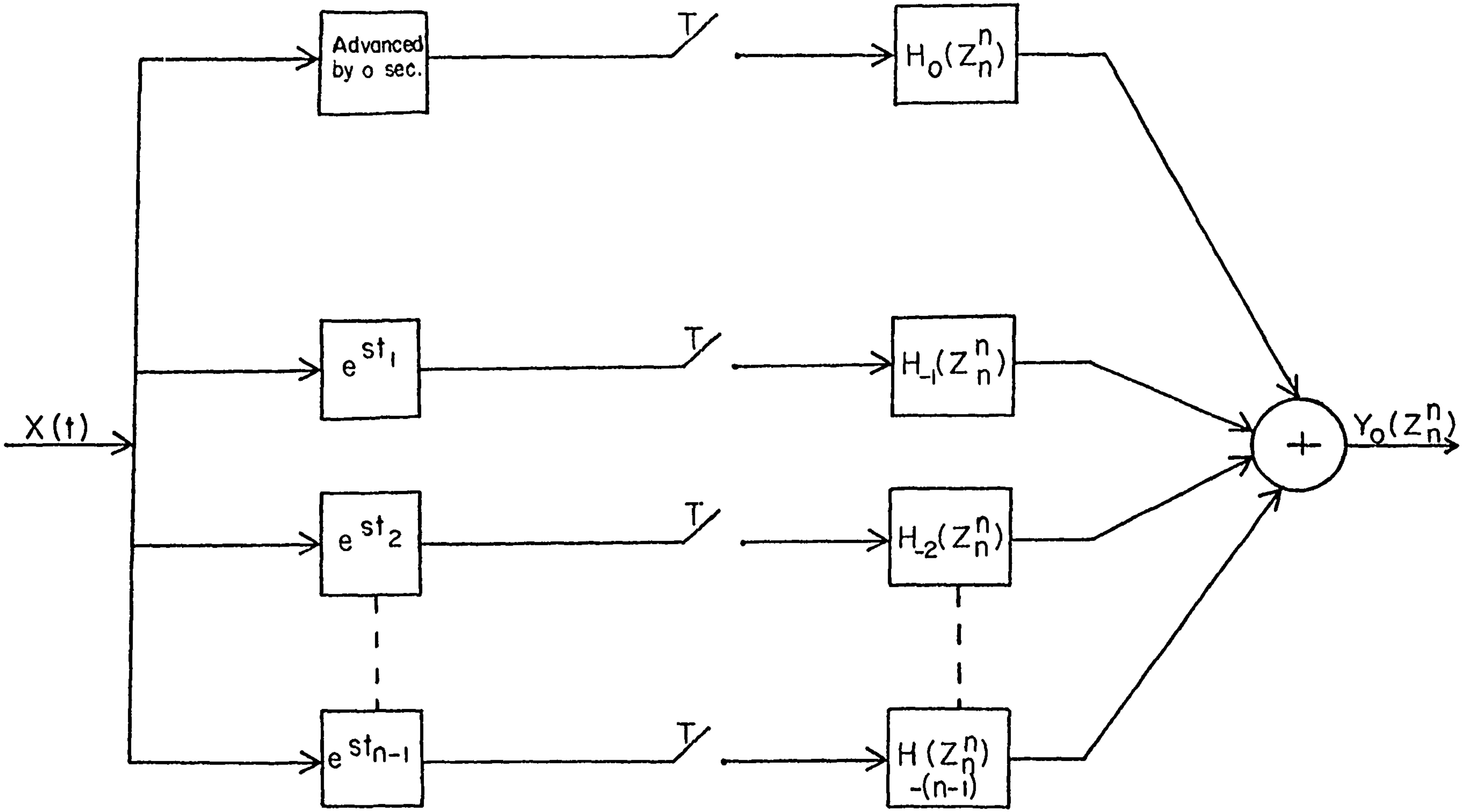


Fig. 2.4. Switch Decomposition block diagram for calculating the first output sequence.

Refer to Figure 2.5 for a block diagram representation. In general then, the transform of the jth output sequence can be expressed directly from (2.7) and (2.9). Thus

$$Y_j(z_n^n) = \sum_{k=0}^{n-1} H_{-k+j}(z_n^n)X_k(z_n^n) \quad 0 \le j \le n-1 \tag{2.12}$$

where $x_k(z_n^n)$, k = 0, +1, +2, +3 ..., +(n-1) are the advanced z-transform of all the input sequences taken with the overall period T units apart; and $H_k(z_n^n)$, k = 0, -1, -2, ..., -(n-1) are the delayed transfer function sequences of the filter. A block diagram representation of (2.12) is shown in Figure 2.6.

The overall output sequence transform $Y(z_n)$ can be obtained by convolution summation of all the output sequences $Y_k(z_n^n)$, appropriately located in the time sequence by multiplying by the delay operator Z_n^{-k}. That is,

$$\begin{aligned} y_n(n) = &\left[\sum_{j=0}^{n} h_0(nT - jT)x_0(jT) + \ldots \right.\\ &\left. + \sum_{j=0}^{n} h_{-(n-1)}(nT - jT)x_{n-1}(jT)\right]\\ &+ \left[\sum_{j=0}^{n} h_1(nT - jT - 1)x_0(jT) + \ldots \right.\\ &\left. + \sum_{j=0}^{n} h_{-(n-1)+1}(nT - jT - 1)x_{n-1}(jT)\right] + \ldots\\ &+ \left[\sum_{j=0}^{n} h_{(n-1)}(nT - jT - (n-1))x_0(jT) + \ldots \right.\\ &\left. + \sum_{j=0}^{n} h_{-(n-1)+n}(nT - jT - (n-1))x_n(jT)\right] \end{aligned} \tag{2.13}$$

Taking the z-transform of both sides of (2.13) yields

$$\begin{aligned} Y(Z_n) &= z\{y_n(n)\}\\ &= Y_0(z_n^n) + z_n^{-1}Y_1(z_n^n) + \ldots + Z_n^{-(n-1)}Y_{n-1}(z_n^n)\\ &= \sum_{k=0}^{n-1} Y_k(Z_n^n)Z_n^{-k} \end{aligned} \tag{2.14}$$

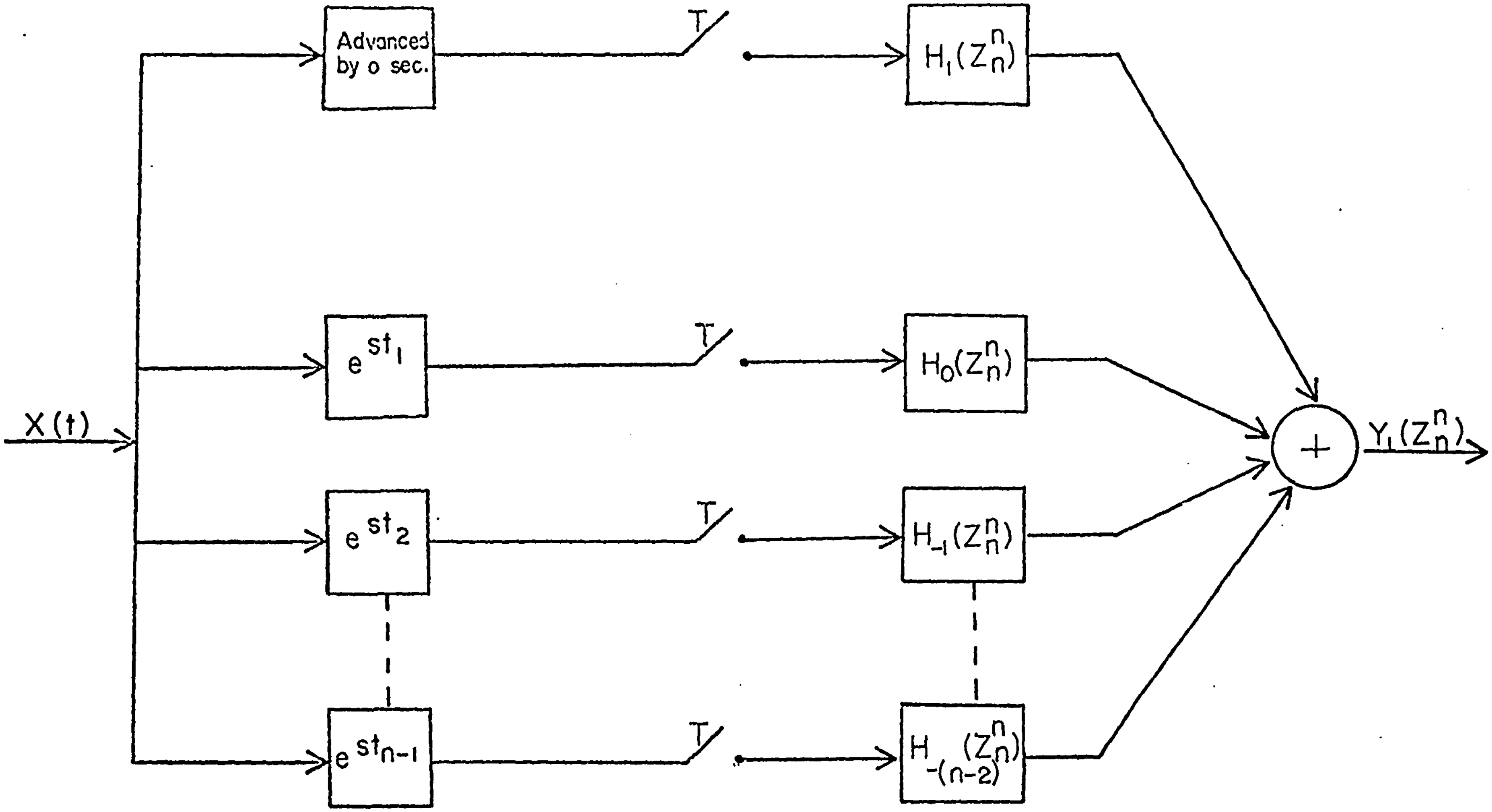


Fig. 2.5. Switch Decomposition block diagram for calculating the second output sequence.

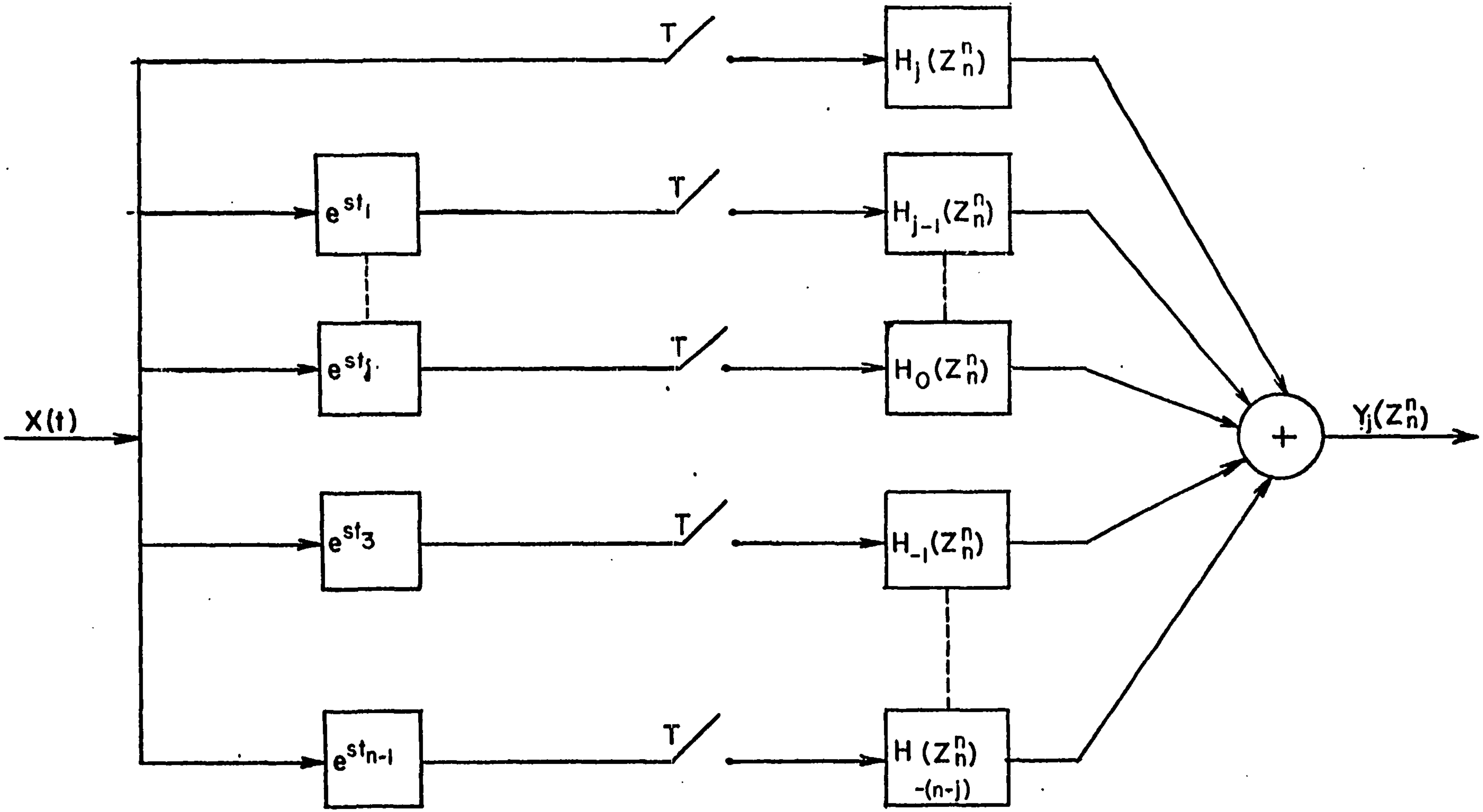


Fig. 2.6 Switch Decomposition block diagram for calculating the jth output sequence.

Thus, equation (2.12) shows that to determine the output sequences, the advanced Z-transform of the input sequences is weighted by the transfer function sequences. Then the output sequences thus obtained are correctly located in the overall output sequence by using equation (2.14).

2.2 Determination of the Pulse Transfer Sequences

The pulse transfer function sequences depend only on the number of times the sampling switch has operated for a digital transfer function. For this reason, the skip sampling theorem can be used to determine the pulse transfer function sequences.

Hence, invoking the skip sampling theorem [16] as discussed in Appendix A-3 yields

$$H_0(\cdot) = \frac{1}{2\pi j}\int_c \frac{H(z_n)}{(1 - z^{-1}z_n^n} \frac{dz_n}{z_n} \tag{2.15}$$

where $H_0(\cdot)$ is the transfer sequence operator for the overall period T. For the sake of notational convenience in later developments, (2.15) is put in the form

$$H_0(z_n^n) = \frac{1}{2\pi j}\int_c \frac{H(P_n)}{(1 - z_n^{-n}P_n^n)} \frac{dP_n}{P_n} \tag{2.16}$$

Similarly,

$$H_1(z_n^n) = \frac{1}{2\pi j}\int_c \frac{P_n H(P_n)}{(1 - z_n^{-n}P_n^n)} \frac{dP_n}{P_n} \tag{2.17}$$

and, in general,

$$H_k(z_n^n) = \frac{1}{2\pi j}\int_c \frac{P_n^k\, H(P_n)}{(1 - z_n^{-n}P_n^n)} \frac{dP_n}{P_n} \tag{2.18}$$

where k = 0, 1, 2 . . .

An approach other than the use of the skip sampling theorem can be used to determine the pulse transfer sequence operator $H(Z_n)$. Such an approach applies only when there are two non-uniform samples in a period

To formulate such an approach, let

$$H(Z_2) = \frac{N(Z_2)}{D(Z_2)} \tag{2.19}$$

Here $N(Z_2)$ and $D(Z_2)$ are polynomials in Z_n. Multiplying $H(Z_2)$ in (2.19) by another polynomial in Z_n yields

$$H(Z_2) = \frac{N(Z_2)}{D(Z_2)} \frac{B(Z_n)}{B(Z_n)} \tag{2.20}$$

The sequences $H(Z_n^{\ n})$ can now be determined. That is, $H_{\pm K}(Z_2^{\ 2})$ are obtained from (2.20) by multiplying it by $Z_n^{\pm K}$ and then selecting only those terms in $Z_n^{\ o}$, Z_n^{-n}, Z_n^{-2n} etc. Here K = 0, 1, ... The method by which $B(Z_n)$ is obtained is shown in Appendix A - 4 of this dissertation.

The following sections will discuss by example the transfer sequence operator for first and second order recursive digital filters.

2.3 First Order Non-uniformly Sampled Recursive Digital Filter

An all pole first order digital filter as shown in Figure 2.7 has a re-formulated pulse transfer function

$$H(Z) = \frac{1}{1-aZ^{-1}} \tag{2.21}$$

As shown by Kranc [21], $H(Z_n)$ is obtained by replacing every Z in (2.21) by Z_n. Hence

$$H(Z_n) = \frac{1}{1-a_1 Z_n^{-1}} \tag{2.22}$$

where $a_1 \neq a$.

For n = 2 and replacing Z_n by P_n for notational convenience, (2.22) can be rewritten as

$$H(P_2) = \frac{1}{1-a_1 P_2^{-1}} \tag{2.23}$$

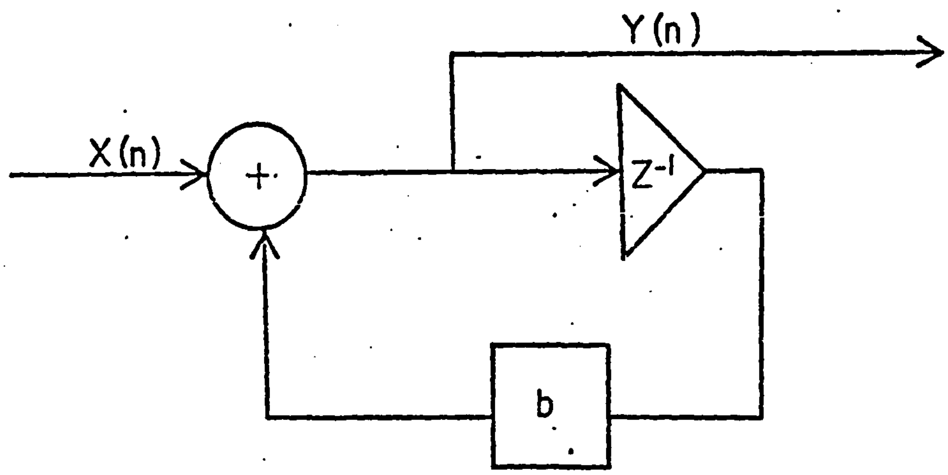


Fig. 2.7 A First Order Digital Filter with no Zeros.

$$H(Z) = \frac{1}{1 - bZ^{-1}}$$

The pulse transfer function sequences can be determined directly from (2.16), (2.17), or (2.18). Thus

$$H_o(Z_2^{\,2}) = \frac{1}{2\pi j} \int_c \frac{H(P_2)}{1-Z_2^{-2}P_2^{\,2}} \frac{dP_2}{P_2} \tag{2.24}$$

Substituting for $H(P_2)$ from (2.23) yields

$$H_o(Z_2^{\,2}) = \frac{1}{2\pi j} \int_c \frac{P_2}{(P_2 - a_1)(1 - Z_2^{-z}P_2^{\,2})} \frac{dP_2}{P_2} \tag{2.25}$$

Evaluating the above integral in the positive sense using the Cauchy residue theorem yields

$$H_o(Z_2^{\,2}) = \frac{1}{1 - a_1^{\,2}Z_2^{-2}} \tag{2.26}$$

Similarly,

$$H_1(Z_2^{\,2}) = \frac{1}{2\pi j} \int_c \frac{P_2 \cdot P_2}{(P_2 - a_1)(1 - Z_2^{-2}P_2^{\,2})} \frac{dP_2}{P_2}$$

$$= \frac{a_1}{1 - a_1^2\, Z_2^{-2}} \tag{2.27}$$

and

$$H_{-1}(Z_2^{\,2}) = \frac{1}{2\pi j} \int_c \frac{P_2 \cdot P_2^{-1}}{(P_2 - a_1)(1 - Z_2^{-2}P_2^{\,2})} \frac{dP_2}{P_2}$$

$$= \frac{1}{2\pi j} \int_c \frac{dP_2}{P_2(P_2 - a_1)(1 - Z_2^{-2}P_2^{\,2})}$$

$$= \frac{a_1 Z_2^{-2}}{1 - a_1^{\,2}Z_2^{-2}} \tag{2.28}$$

Alternatively, since $n = 2$, equation (A - 13) of Appendix A - 4 can be used in obtaining the transfer function sequences.

First $H(Z_2)$ is modified. That is,

$$H(Z_2) = \frac{1}{1 - a_1 Z_2^{-1}} \frac{B(Z_2)}{B(Z_2)}$$

where

$$B(Z_2) = \prod_{k=1}^{n-1} D(Z_2 e^{\frac{-j2\pi K}{n}})$$

and for $n = 2$

$$= \prod_{k=1}^{1} D(Z_2 e^{\frac{-j2\pi K}{n}}) = D(Z_2 e^{-j\pi}) \tag{2.29}$$

Hence

$$H(Z_2) = \frac{1}{1 - a_1 Z_2^{-1}} \frac{(1 - a_1 Z_2^{-1} e^{j\pi})}{(1 - a_1 Z_2^{-1} e^{j\pi})}$$

$$= \frac{1 + a_1\ Z_2^{-1}}{1 - a_1^{2} Z_2^{-2}} \tag{2.30}$$

Multiplying (2.30) by $Z_2^{\pm K}$, K = 0, 1, ... and selecting only those terms of the form Z_2^{0}, Z_2^{-n}, Z_2^{-2n} and so on, yields

$$H_o(Z_2^{\,2}) = \frac{1}{1 - a_1^{2} Z_2^{-2}} \tag{2.31}$$

$$H_1(Z_2^{\,2}) = Z_2 \left(\frac{1 + a_1 Z_2^{-1}}{1 - a_1^{2} Z_2^{-2}}\right) = \frac{Z_2 + a_1}{1 - a_1^{2} Z_2^{-2}}$$

$$= \frac{a_1}{1 - a_1^{2} Z_2^{-2}} \tag{2.32}$$

Similarly

$$H_{-1}(Z_2^{\,2}) = Z_2^{-1} \left(\frac{1 + a_1 Z_2^{-1}}{1 - a_1^{2} Z_2^{-2}}\right) = \frac{Z_2^{-1} + a_1 Z_2^{-2}}{1 - a_1^{2} Z_2^{-2}}$$

$$= \frac{a_1 Z_2^{-2}}{1 - a_1^{2} Z_2^{-2}} \tag{2.33}$$

Equations (2.31), (2.32) and (2.33) are seen to be the same as equations (2.26), (2.27) and (2.28) respectively obtained by use of the skip sampling theorem.

The transfer function sequences are calculated for cases where there are three and four nonuniform samples in a period. The results are given in Table 2.1.

Transfer function sequence	Two intersamples per sampling period. n = 2	Three intersamples per sampling period. n = 3	Four intersamples per sampling period. n = 4
$H_o(Z_n^n)$	$\frac{1}{1 - a_1^2 Z_2^{-2}}$	$\frac{1}{1 - a_1^3 Z_3^{-3}}$	$\frac{1}{1 - a_1^4 Z_4^{-4}}$
$H_1(Z_n^n)$	$\frac{a_1}{1 - a_1^2 Z_2^{-2}}$	$\frac{a_1}{1 - a_1^3 Z_3^{-3}}$	$\frac{a_1}{1 - a_1^4 Z_4^{-4}}$
$H_2(z_n^n)$	None	$\frac{a_1^2}{1 - a_1^3 Z_3^{-3}}$	$\frac{a_1^2}{1 - a_1^4 Z_4^{-4}}$
$H_3(Z_n^n)$	None	None	$\frac{a_1^3}{1 - a_1^4 Z_4^{-4}}$
$H_{-1}(Z_n^n)$	$\frac{a_1 Z_2^{-2}}{1 - a_1^2 Z_2^{-2}}$	$\frac{a_1^2 Z_3^{-3}}{1 - a_1^3 Z_3^{-3}}$	$\frac{a_1^3 Z_4^{-4}}{1 - a_1^4 Z_4^{-4}}$
$H_{-2}(Z_n^n)$	None	$\frac{a_1 Z_3^{-3}}{1 - a_1^3 Z_3^{-3}}$	$\frac{a_1^2 Z_4^{-4}}{1 - a_1^4 Z_4^{-4}}$
$H_{-3}(Z_n^n)$	None	None	$\frac{a_1 Z_4^{-4}}{1 - a_1^4 Z_4^{-4}}$

Table 2.1 Transfer function sequences for first order digital filter.

$$H(Z_2) = \frac{1}{1 - a_1 Z_2^{-1}}$$

The overall transfer function sequence can be determined from equations (2.26) and (2.27)

That is,

$$H(Z_2) = H_o(Z_2^{\,2}) + Z_2^{-1}H_1(Z_2^{\,2})$$

$$= \frac{1}{1 - a_1^{\,2}Z_2^{-2}} + \frac{a_1Z_2^{-1}}{1 - a_1^{\,2}Z_2^{-2}}$$

$$= \frac{1 + a_1Z_2^{-1}}{1 - a_1^{\,2}Z_2^{-2}} \tag{2.34}$$

Equation (2.34) is seen to be the same as equation (2.30) obtained by use of the method discussed in Appendix A - 4. This shows that the transfer function sequences are obtainable from the overall transfer function sequence and vice versa.

The output sequences of a first order filter sampled nonuniformly can be expressed in terms of the pulse transfer sequences and the input sequences. Using (2.26) and (2.28), equation (2.10) yields

$$Y_o(Z_2^{\,2}) = H_o(Z_2^{\,2})X_o(Z_2^{\,2}) + H_{-1}(Z_2^{\,2})X_1(Z_2^{\,2}) \tag{2.35}$$

Similarly, for the second output sequence, using (2.26) and (2.27), equation (2.11) yields

$$Y_1(Z_2^{\,2}) = H_1(Z_2^{\,2})X_0(Z_2^{\,2}) + H_o(Z_2^{\,2})X_1(Z_2^{\,2}) \tag{2.36}$$

Here $X_1(Z_2^{\,2})$ is the advanced Z transform of $x_1(nT)$. $H_{-1}(Z_2^{\,2})$ and $H_1(Z_2^{\,2})$ are the pulse transfer function respectively delayed and advanced by one sampling period.

Using (2.35) and (2.36) in (2.14), the overall output sequence is obtained as

$$Y(Z_2) = X_o(Z_2{}^2)[H_o(Z_2{}^2) + Z_2^{-1}H_1(Z_2{}^2)] +$$

$$X_1(Z_2{}^2)[H_{-1}(Z_2{}^2) + Z_2^{-1}H_o(Z_2{}^2)] \qquad (2.37)$$

Thus the overall output sequence is a function of both the pulse transfer function sequences and the input sequences.

2.4 Second Order Nonuniformly Sampled Recursive Digital Filter

The transfer function for an all pole second order filter is determined from Figure 2.8 by making $a_o = 1$, $a_1 = a_2 = 0$.

Hence

$$H(Z) = \frac{1}{1 + aZ^{-1} + bZ^{-2}} \qquad (2.38)$$

Following the procedure used for first order filter, the transfer function sequences are first obtained using the skip sampling theorem.

Hence

$$H_o(Z_2{}^2) = \frac{1}{2\pi j} \int_c \frac{H(P_2)}{(1 - Z_2^{-2}P_2{}^2)} \frac{dP_2}{P_2} \qquad (2.39)$$

$$H_1(Z_2{}^2) = \frac{1}{2\pi j} \int_c \frac{P_2^{+1}H(P_2)}{(1 - Z_2^{-2}P_2{}^2)} \frac{dP_2}{P_2} \qquad (2.40)$$

$$H_{-1}(Z_2{}^2) = \frac{1}{2\pi j} \int_c \frac{P_2^{-1}H(P_2)}{(1 - Z_2^{-2}P_2{}^2)} \frac{dP_2}{P_2} \qquad (2.41)$$

where

$$H(P_2) = \frac{1}{1 + a_1P_2^{-1} + b_1P_2^{-2}} \qquad (2.42)$$

a_1 and b_1 in equation (2.42) are not in general equal to a and b respectively in (2.38).

Evaluating the three integrals using the Cauchy residue theorem yields

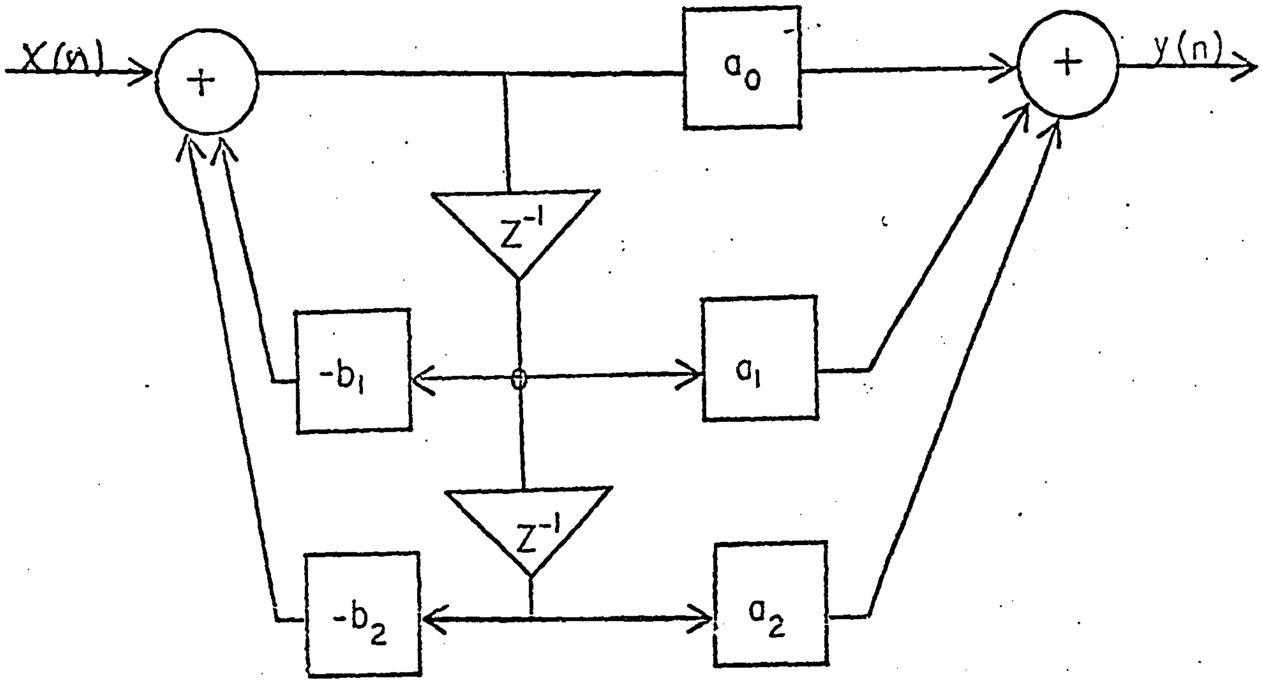


Fig. 2.8 A Second Order Digital Filter with Two Zeros in Canonic Form

$$H(Z) = \frac{a_0 + a_1Z^{-1} + a_2Z^{-2}}{1 + b_1Z^{-1} + b_2Z^{-2}} .$$

$$H_o(z_2^{\,2}) + \frac{1 + b_1 Z_2^{-2}}{1 + Z_2^{-2}(2b_1 - a_1^{\,2}) + b_1^{\,2} Z_2^{-4}} \tag{2.43}$$

$$H_1(Z_2^{\,2}) = \frac{-a_1}{1 + Z_2^{-2}(2b_1 - a_1^{\,2}) + b_1^{\,2} Z_2^{-4}} \tag{2.44}$$

$$H_{-1}(Z_2^{\,2}) = \frac{-a_1 Z_2^{-2}}{1 + Z_2^{-2}(2b_1 - a_1^{\,2}) + b_1^{\,2} Z_2^{-4}} \tag{2.45}$$

Alternatively, equation (A - 13) of Appendix A - 4 can be used. First $H(Z_2)$ is modified with $B(Z_2)$ to get

$$H(Z_2) = \frac{1}{1 + a_1 Z_2^{-1} + b_1 Z_2^{-2}} \, \frac{B(Z_2)}{B(Z_2)}$$

where

$$B(Z_2) = \prod_{L=1}^{1} D(Z_2 \exp(-j\tfrac{2\pi L}{n})) = D[Z_2 \exp(-j\pi)]$$

then

$$H(Z_2) = \frac{(1 + a_1 Z_2^{-1}\exp(j\pi) + b_1 Z_2^{-2}\exp(2j\pi))}{(1 + a_1 Z_2^{-1} + b_1 Z_2^{-2})(1 + a_1 Z_2^{-1}\exp(j\pi) + b_1 Z_2^{-2}\exp(2j\pi))} \tag{2.46}$$

Simplifying (2.46) yields

$$H(Z_2) = \frac{1 - a_1 Z_2^{-1} + b_1 Z_2^{-2}}{1 + Z_2^{-2}(2b_1 - a_1^{\,2}) + b_1^{\,2} Z_2^{-4}} \tag{2.47}$$

To obtain $H_o(Z_2^{-2})$ multiply (2.47) by $Z_2^{\,o}$ and select only those terms of the form $Z_2^{\,o}$, Z_2^{-n}, Z_2^{-2n}, etc., to get

$$H_o(Z_2^{\,2}) = \frac{1 + b_1 Z_2^{-2}}{1 + Z_2^{-2}(2b_1 - a_1^{\,2}) + b_1^{\,2} Z_2^{-4}} \tag{2.48}$$

Similarly, $H_1(Z_2^{\,2})$ is obtained from $Z_2^{\,1}H(Z_2)$ by selecting only terms of the form $Z_2^{\,o}$, Z_2^{-2}, Z_2^{-4}, etc. This yields

$$H_1(Z_2^{\,2}) = \frac{-a_1}{1 + Z_2^{-2}(2b_1 - a_1^{\,2}) + b_1^{\,2} Z_2^{-4}} \tag{2.49}$$

Similarly

$$H_{-1}(Z_2^{\ 2}) = \frac{-aZ_2^{-2}}{1 + Z_2^{-2}(2b_1 - a_1^{\ 2}) + b_1^{\ 2}Z_2^{-4}} \tag{2.50}$$

Equations (2.48), (2.49) and (2.50) are seen to be the same as equations (2.43), (2.44) and (2.45) respectively, obtained by use of the skip sampling theorem.

The overall transfer function sequences for two intersamples per sampling period can be obtained by using equations (2.43) and (2.44). That is,

$$\begin{aligned} H(Z_2) &= H_o(Z_2^{\ 2}) + Z_2^{-1}H_1(Z_2^{\ 2}) \\ &= \frac{1 + b_1 Z_2^{-2}}{1 + Z_2^{-2}(2b_1 - a_1^{\ 2}) + b_1^{\ 2}Z_2^{-4}} \\ &\quad - \frac{a_1 Z_2^{-1}}{1 + Z_2^{-2}(2b_1 - a_1^{\ 2}) + b_1^{\ 2}Z_2^{-4}} \end{aligned} \tag{2.51}$$

Simplifying (2.51) yields

$$H(Z_2) = \frac{1 - a_1 Z_2^{-1} + b_1 Z_2^{-2}}{1 + Z_2^{-2}(2b_1 - a_1^{\ 2}) + b_1^{\ 2}Z_2^{-4}} \tag{2.52}$$

Equation (2.52) agrees with (2.47) obtained by the method of Appendix A - 4. In general, the overall transfer sequence $H(Z_n)$ can be expressed as a function of the pulse transfer function sequences $H(Z_n^{\ n})$. That is,

$$H(Z_n) = \sum_{k=o}^{n-1} H_k(Z_n^{\ n})Z_n^{-k} \tag{2.53}$$

In a similar manner, the overall output sequence can be expressed, in general, as

$$Y(Z_n) = \sum_{k=0}^{n-1} Y_k(Z_n{}^n)Z_n^{-k} \tag{2.54}$$

Substituting (2.12) in (2.54) yields

$$Y(Z_n) = \sum_{k=0}^{n-1} \left(\left[\sum_{j=0}^{n-1} H_{-j+k}(Z_n{}^n)X_j(Z_n{}^n) \right] Z_n^{-k} \right) \tag{2.55}$$

where X_k, k = 0, 1, 2, ... are the input sequences and H_k, k = 0, $\pm$ 1, $\pm$ 2, ... are the pulse transfer function sequences. This result shows that the overall output sequence, which is a sequence operator, is available when the output sequences are determined.

The transfer function sequences for three and four non-uniform samples per period are given in Table 2.2.

Transfer function sequence	Two intersamples per sampling period. n = 2	Three intersamples per sampling period. n = 3	Four intersamples per sampling period. n = 4
$H_0(Z_n^{\ n})$	$\frac{1 + b_1 Z_2^{-2}}{D_1}$	$\frac{1 - a_1 b_1 Z_3^{-3}}{D_2}$	$\frac{1 + Z_4^{-4} b_1 (a_1^2 - b_1)}{D_3}$
$H_1(Z_n^{\ n})$	$\frac{-a_1}{D_1}$	$\frac{b_1^{\ 2} Z_3^{-3} - a_1}{D_2}$	$\frac{-a_1 - a_1 b_1^{\ 2} Z_4^{-4}}{D_3}$
$H_2(Z_n^{\ n})$	NONE	$\frac{a_1^{\ 2} - b_1}{D_2}$	$\frac{(a_1^2 - b_1) + b_1^3 Z_4^{-4}}{D_3}$
$H_3(Z_n^{\ n})$	NONE	NONE	$\frac{a_1(2b_1 - a_1^{\ 2})}{D_3}$
$H_{-1}(Z_n^{\ n})$	$\frac{-a_1 Z_2^{-2}}{D_1}$	$\frac{(a_1^{\ 2} - b_1) Z_3^{-3}}{D_2}$	$\frac{a_1(2b_1 - a_1^{\ 2}) Z_4^{-4}}{D_3}$
$H_{-2}(Z_n^{\ n})$	NONE	$\frac{(b_1 Z_3^{-3})^2 - a_1 Z_3^{-3}}{D_2}$	$\frac{Z_4^{-4}(a_1^{\ 2} - b_1) + b_1^3 Z_4^{-8}}{D_3}$
$H_{-3}(Z_n^{\ n})$	NONE	NONE	$\frac{-a_1(Z_4^{-4} - (b_1 Z_4^{-4})^2)}{D_3}$

Table 2.2. Transfer function sequences for second order digital filter.

$$H(Z_n) = 1/(1 + a_1 Z_n^{-1} + b_1 Z_n^{-2})$$

$$D_1 = 1 + Z_2^{-2}(2b_1 - a_1^{\ 2}) + b_1^{\ 2} Z_2^{-4}$$

$$D_2 = 1 + Z_3^{-3}(a_1^{\ 3} - 3a_1 b_1) + b_1^{\ 3} Z_3^{-6}$$

$$D_3 = 1 + Z_4^{-4}(4a_1^{\ 2} b_1 - 2b_1^{\ 2} - a_1^{\ 4}) + b_1^{\ 4} Z_4^{-8}$$

2.5 State-Space Description of Second-Order Digital Fitler Sampled Nonuniformly

In the z-transform analysis of digital systems with nonuniform sampling, a model with uniform sampling is first created. The state variable method may sometimes offer a unified approach to the analysis of digital filters with nonuniform sampling [17].

A second order recursive digital filter in time domain can be described by the following linear difference equation [27,32,46];

$$y_n = -ay_{n-1} - by_{n-z} + \mu_n \tag{2.56}$$

where $y_n = y(nT)$, $y_{n-1} = y(nT - T)$ and $y_{n-2} = y(nT - 2T)$ for uniform sampling.

Let

$$x_1(n) = y_{n-2} \text{ and } x_2(n) = y_{n-1} \tag{2.57}$$

Substituting (2.57) into (2.56) yields $x_1(n + 1) = y_{n-1} = x_2(n)$ $x_2(n + 1) = y_n = -ax_2(n) - bx_1(n) + \mu_n$ or in a more familiar notation

$$x_{n+1} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} x_n + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mu_n \tag{2.58}$$

A linear discrete system can be described by the following set of state equations [7,10,14,17];

$$x(n + 1) = Ax(n) + BU(n) \tag{2.59}$$

$$y(n) = cx(n) \text{ where}$$

$x(n) = n \times 1$ state vector, $\mu(n)$ and $y(n)$ are the 1×1 input and output, respectively. A, B and C are the state transition, the input, the output constant coefficient matrices, respectively, each with the appropraite dimension for the case of a linear, time-invariant system.

From (2.58) and (2.59) it is easily seen that

$$A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } C = [1, 0]$$

If the nonuniform samples have been rearranged so that the sampler samples at $t = nT$, $nT + T_1$, $nT + T_1 + T_2$, $(n + 1)T$, $(n + 1)T + T_1$, $(n + 1)T + T_1 + T_2 \ldots .T_k$ where $T = T_1 + T_2 + T_3 \ldots + T_k$, then the state transition equation can be determined [17]. Assuming in general that the transition of state of a digital system takes place from time zero to T_1, the state transition equation of the system at $t = kT + T_1$, using (2.59) can be expressed as

$$x(nT + T_1) = A_1x(nT) + B\,U(nT) \tag{2.60}$$

where A_1 is the transition matrix during the subinterval $0 < t - nT < T_1$ and B is assumed constant.

At $t = kT +_1 + T_2$, the state transition equation becomes

$$x(nT + T_1 + T_2) = A_2x(nT + T_1) + BU(nT + T_1) \tag{2.61}$$

At $t = nT + T_1 + T_2 + T_3$, the state transition equation becomes

$$x(nT + T_1 + T_2 + T_3) = A_3x(nT + T_1 + T_2) + BU(nT + T_1 + T_2) \tag{2.62}$$

and, in general,

$$\begin{aligned} x(nT + T_1 + \ldots + T_k) = A_kx(nT &+ T_1 + \ldots + T_{k-1}) \\ &+ BU(nT + T_1 + \ldots + T_{k-1}) \end{aligned} \tag{2.63}$$

Substituting (2.60) into (2.61) yields

$$\begin{aligned} x(nT + T_1 + T_2) &= A_2[A_1x(nT) + BU(nT)] \\ &\quad + BU(nT + T_1) \\ &= A_2A_1x(nT) + A_2BU(nT) + BU(nT + T_1) \end{aligned} \tag{2.64}$$

Substituting (2.64) into (2.62) yields

$$\begin{aligned} x(nT + T_1 + T_2 + T_3) &= A_3[A_2A_1x(nT) + A_2BU(nT) \\ &\quad + BU(nT + T_1)] + BU(nT + T_1 + T_2) \\ &= A_3A_2A_1x(nT) + A_3A_2BU(nT) + A_3BU(nT+T_1) \\ &\quad + BU(nT + T_1 + T_2) \end{aligned} \tag{2.65}$$

To generalize,

$$\begin{aligned} x(nT + T_1\ T_2 + \ldots + T_k) &= x(nT + T) \\ &= A_kA_{k-1}\ldots A_1x(nT) + A_kA_{k-2}\ldots A_{k-1}BU(nT) \\ &\quad + A_kBU(nT + T_1) \\ &\quad + BU(nT + T_1 + \ldots + T_{k-1}) \end{aligned} \tag{2.66}$$

where k is the number of intersamples in the overall sampling period.

2.6 Summary

The purpose of this chapter has been to develop, using the z-transform and subsequently the state space description, the mathematical technique suitable for analysis and design of discrete data systems with both the input and the output synchronously sampled nonuniformly. In particular, nonuniformly sampled recursive first and second order digital filters were considered. The approach adopted calls for decomposing the nonuniformly sampled input into a set of uniformly sampled input sequences. By noting that the pulse transfer function defined a time sequence of events and therefore the switch operates at discrete times, the input sequences can be appropriately weighted by the transfer function sequences to generate the output sequences.

Using the skip sampling theorem and derived alternate approach, the transfer function sequences of first and second order recursive digital filters are derived as new results. The output sequences of such filters are derived for a given set of input sequences, using the advanced z-transform and time-domain switch decomposition. The overall output sequences follow from the output sequences.

The following characteristic properties of the transfer function sequences and the output sequences of a digital filter sampled non-uniformly may be observed from the derived results of this work:

a) The order of the filter does not change as a result of nonuniform sampling. That is, a second order filter remains a second order. This is evidenced from the nature of the derived transfer function sequences shown in Table 2.2. The denominator of these transfer function sequences are of the 2nd order, though a function of the number of intersamples per sampling period.

b) The denominators are equal for all transfer function sequences belonging to a given filter with a given number of intersamples per sampling period. This is evident from the results tabulated in Table 2.1 and Table 2.2.

c) Some of the coefficients, which in conventional filters of discrete form, only affect the denominators of the transfer function of the filter, also appear in the numerators. Most of the multiplication constants are included in several of the coefficients of the polynomial terms. One significance of this is that even if there are no transmission zeros in the pulse transfer function of the filter with uniform samples, transmission zeros are introduced through nonuniform sampling. In other words, the filter constants are re-circulated in the registers during nonuniform sampling. Examples of this are evidenced in the results shown in Table 2.2.

d) Due to the form of the switch decomposition of this work, the number of the output sequences is exactly equal to the number of intersamples in one sampling period. This is evidenced from the results shown in equations (2.12), (2.35), and (2.36) of this work.

e) The number of the transfer function sequences is not necessarily equal to the number of intersamples in one sampling period. This can be observed from the results of Table 2.1 and Table 2.2.

f) The overall transfer function sequence is obtainable from the transfer function sequences or vice versa. Example is evidenced in equation (2.30) showing the overall transfer function from which equations (2.31), (2.32) and (2.33) are obtained as the transfer function sequences.

g) The advanced and the delayed pulse transfer function sequences can be related by the following equation

$$H_{-k}(Z_n^{\,n}) = Z_n^{-n} H_{n-k}(Z_n^{\,n}) \tag{2.67}$$

where n is the number of intersamples in one sampling period and k = 0, 1, 2 .... This is expected since the pulse transfer function sequences are obtained by using the skip sampling theorem. That this is so is observable from the results of Tables 2.1 and 2.2. This means that in calculating the pulse transfer function sequences, only the positively subscripted transfer function sequences need be derived and equation (2.67) then used to obtain the delayed pulse transfer function sequences.

CHAPTER 3

STABILITY, FREQUENCY, AND QUANTIZATION CHARACTERISTICS

3.0 Introduction

The filter methods described in chapter two assumes infinite precision arithmetic. Thus, all computations are error free. This chapter formulates the stability constraints and the quantization effects due to finite arithmetic operations in digital filters sampled nonuniformly. The analysis of such effects thus far in the literature has been limited to digital filters sampled uniformly or slight modifications thereof. The ability to predict accurately these quantization effects on the overall digital processing scheme is essential in determining how one can reduce the amount of hardware and processing time for particular applications.

The developments of Chapter 2, along with Jury's stability test [25] will be employed in the design stability criteria development. Applying the stability criteria to first and second order recursive filters, conclusions generated are compared with those obtained by other work for uniformly sampled filters.

Two effects of quantization are considered. First, statistical analysis of quantization errors is presented. The approach developed in Chapter 2 will be used along with standard techniques for analyzing such errors. Computer simulations are then used in comparing the quantization errors of nonuniformly sampled filters to those of uniformly sampled filters presented in other work [3,24].

The second quantization effect to be analyzed is that of limit cycle oscillations. The results of Chapter 2 along with well known techniques will form the basis for deriving the amplitude bounds of such limit cycles. Computer simulation using the developed approach will provide an avenue for comparison of limit cycle oscillations in nonuniformly sampled digital filters with those in uniformly sampled filters as found in the literature.

Finally, the decomposition technique is applied to the frequency response aspects in terms of magnitude and phase characteristics in digital filters sampled nonuniformly for demonstration. Magnitude plots of the derived expressions are presented, thereby enabling comparison to be made between frequency responses of uniformly and nonuniformly sampled digital filters.

3.1 Stability Considerations

Finite register or word lengths result in either rounding or truncation of the computed (ideal) filter coefficient. This altering of the coefficient values may manifest itself in not only altered response characteristics but possibly the stability of the filter algorithms.

Two types of stability to be considered are design stability and numerical stability. Design instability results from inherent instability of the mathematical design of system parameters. Numerical instability, on the other hand, is dependent on the structure or the type of arithmetic employed in the computations. This type of instability generally leads to limit cycle oscillations which will be considered.

Jury [25] and others [10, 17, 22] have shown that only the poles of the transfer function of the filter, H(Z), determine its stability.

Thus the poles of the filter must lie within the unit circle in the Z-plane. The zeros lying inside the unit circle in the Z-plane ensure that there is no excess phase characteristics.

First consider a first order recursive filter sampled nonuniformly with two intersamples per sampling period. Using equation (2.34),

$$H(Z_2) = \frac{1 + a_1 Z_2^{-1}}{1 - a_1^2 Z_2^{-2}} = \frac{Z_2^2 + a_1 Z_2}{Z_2^2 - a_1^2} \qquad (3.1)$$

In order for the filter to be stable, the parameter a_1 must satisfy the condition $-1 < a_1 < 1$. Otnes [26] and others [23,24] have shown that the condition for stability for the recursive first order filter sampled uniformly is that the pole must lie between minus one and plus one. This means that the condition for stability for first order recursive digital filter is the same for both uniform and nonuniform sampling. This is expected since the order of the denominator of the transfer function for uniform sampling is not altered by nonuniform sampling.

The overall transfer function sequence for a second order digital filter with two intersamples per sampling period is given in 52) as

$$H(Z_2) = \frac{1 - a_1 Z_2^{-1} + b_1 Z_2^{-2}}{1 + Z_2^{-2} (2b_1 - a_1^2) + b_1^2 Z_2^{-4}} \qquad (3.2)$$

The constraints for stability can be determined by examining the poles of the denominator of (3.2). Let

$$D(Z_2^2) = (Z_2^2)^2 + \beta_1 Z_2^2 + \beta_2 \qquad (3.3)$$

where

$$\beta_1 = 2b_1 - a_1^2 \qquad \text{and} \qquad \beta_2 = b_1^2$$

To determine the constraints on β_1 and β_2 for stability, Jury's stability test [25] for linear discrete systems is applicable.

Applying the test yields

$$1 + \beta_2 > |\beta_1| \quad \text{and}$$

$$\beta_2 < 1 \tag{3.4}$$

or

$$b_1^2 < 1 \text{ and}$$

$$|2b_1 - a_1^2| < 1 + b_1^2 < 2 \tag{3.5}$$

The constraints on the filter parameters for stability is determined from (3.5) as

$$|b_1| < 1 \text{ and}$$

$$|a_1| < |1 + b_1| \leq |b_1| + 1 \tag{3.6}$$

When there are four intersamples per sampling period, then

$$\beta_1 = 4a_1^2 b_1 - 2b_1^2 - a_1^4 \quad \text{and}$$

$$\beta_2 = b_1^4 \tag{3.7}$$

Using (3.4) and (3.7) yield

$$|4a_1^2 b_1 - 2b_1^2 - a_1^4| < 1 + b_1^4$$

which reduces to

$$-(b_1^2 - 1)^2 < (a_1 - 2b_1)^2 < (b_1^2 + 1)^2$$

$$|a_1^2 - 2b_1| < |b_1^2 + 1|$$

$$|a_1| < |b_1 + 1| \leq |b_1| + 1 \tag{3.8}$$

From (3.8), it is seen that (3.7) reduces to (3.6). Hence, the design parameter constraints for stability are in general given by equation (3.6).

Otnes [26] and others [5,22,23,50] have derived design coefficients constraints for stability in uniformly sampled filters. Their results and the results of this dissertation are exhibited in Table 3.1.

It should be noted that both results do agree since, as noted in chapter two, nonuniform sampling does not alter the order of the denominator polynomial of the uniformly sampled pulse transfer function. The results in Table 3.1 can be viewed as a one-to-one mapping of the a-b coefficient plane of the uniformly sampled filter into the a_1-b_1 coefficient plane of the nonuniformly sampled filter.

Type of filter	Transfer function sequence	Design Parameter Constraints
Uniformly sampled first order 26,5	$\frac{1}{1 - aZ^{-1}}$	$-1 < a < 1$
Nonuniformly sampled first order	$\frac{1 + a_1 Z_2^{-1}}{1 - a_1^2 Z_2^{-2}}$	$-1 < a_1 < 1$
Uniformly sampled second order 26,5	$\frac{Z^2}{Z^2 + aZ + b}$	$b < 1$ $\|a\| < \|1 + b\|$
Nonuniformly sampled second order--two intersamples per period	$\frac{1 - a_1 Z_2^{-1} + b_1 Z_2^{-2}}{1 + Z_2^{-2}(2b_1 - a_1^2) + b_1^2 Z_2^4}$	$b_1 < 1$ $\|a_1\| < \|1+b_1\|$

TABLE 3.1
Design coefficient constraints for stability in digital filters sampled uniformly and nonuniformly.

Stable second order filters fall into two categories, depending on the position of the roots of the characteristic equation of the filter. From equation (3.3) the characteristic equation of a second order filter sampled nonuniformly is given as

$$(Z_2^2)^2 + \beta_1 Z_2^2 + \beta_2 = 0$$

If $\beta_1^2 - 4\beta_2 > 0$, then the roots are real and unequal. If $\beta_1^2 - 4\ _2 < 0$, then the roots are complex conjugate of each other.

The real and imaginary boundary of the filter parameters can be determined. That is, for two intersamples per sampling period, this boundary can be given as

$$b_1^2 = \frac{(2b_1 - a_1^2)^2}{4} \quad \text{which reduces to}$$

$$b_1 = \frac{a_1^2}{4} \tag{3.9}$$

For four intersamples per sampling period, the boundary can be expressed as

$$b_1^4 = \frac{(4a_1^2 b_1 - 2b_1^2 - a_1^4)^2}{4} \quad \text{which reduces to}$$

$$-2b_1^2 = 4a_1^2 b_1 - 2b_1^2 - a_1^4 \quad \text{and simplifying the above yields}$$

$$b_1 = \frac{a_1^2}{4} \quad \text{which is the same as equation (3.9).}$$

Figure 3.1 shows the familiar bounded stability region [26,29] in the parameter (a_1, b_1) plane. Values outside result in unstable filters, while those on the borders result in marginally stable filters. Also the figure shows the real and the imaginary boundary of equation (3.9).

3.2 Statistical Analysis of quantization errors.

The quantization effects due to round-offs in nonuniformly sampled filters is analyzed from statistical point of view. To do this, two assumptions are necessary. The first assumption is that each of the noise samples is uncorrelated and independent of the signal input. This assumption is not necessarily valid for constant inputs. However, Jackson [24] and others [3,33,35] have shown experimentally that the assumption is valid in discrete-data systems for a broad class of inputs including random signals.

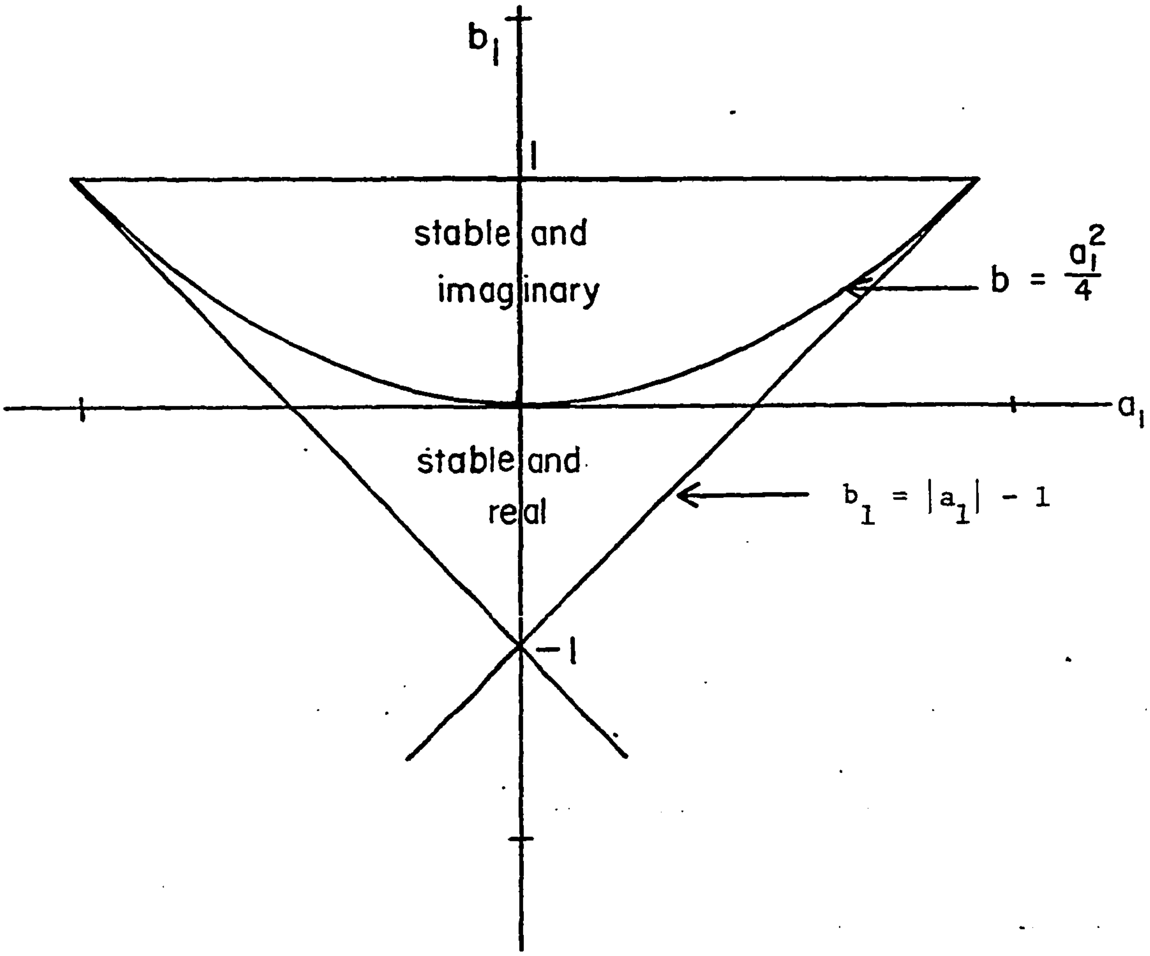


Figure 3.1

Design coefficient stability region in terms of a_1 and b_1.

The second assumption is that the probability density of the quantization noise is uniform as shown in Figure 2.3a and is stationary in the wide sense. Hence if e_n is the error due to rounding then

$$E[e_n] = 0$$

where $E[\cdot]$

is the expected value. The variance σ^2 may be computed.

$$\sigma^2 = E[e_n^2] = \int_{-\infty}^{\infty} e_n^2 p(e_n) de_n$$

$$= \int_{-2^{-b-1}}^{2^{-b-1}} e_n^2 (2^b) de_n$$

$$= 2^b \left. \frac{e_n^2}{3} \right]_{-2^{-b-1}}^{2^{-b-1}}$$

$$= \frac{2^{-2b}}{12}$$

$$= \frac{q^2}{12} \tag{3.10}$$

where

$$q = 2^{-b}$$

is the quantization step. The error e_n can be bound so that $|e_n| < 2^{-b}$ for all n, where b is the number of places to the right of the binary point after rounding.

The quantization error of the input signal can be represented in a block diagram as shown in Fig. 3.2b. Ignoring all other errors in the filter, Crold and Rader [3], Mitra, Hirano and Sakaguchi [34], have shown that the steady-state value of the output noise variance, using Fig 3.2b, is given by

$$\sigma_o^2 = \frac{q^2}{12} \frac{1}{2\pi j} \int_c H(Z)\ H(Z^{-1})\ Z^{-1}\ dZ \tag{3.11}$$

where H(Z) is the pulse transfer function of the discrete system

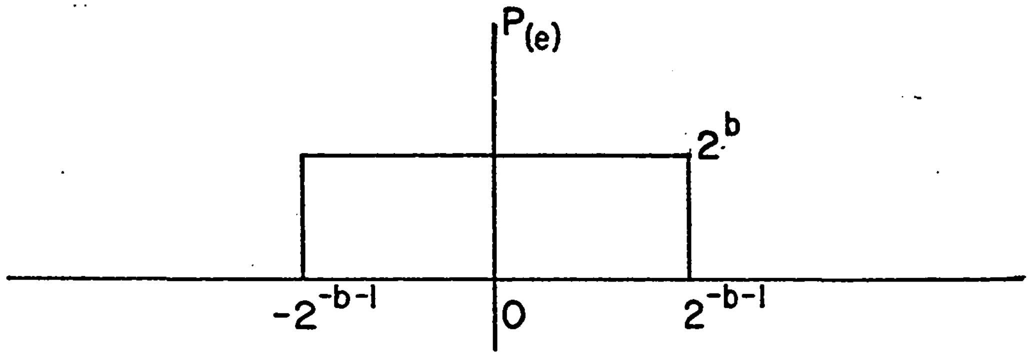


Figure 3.2a

Probability density for round-off error.

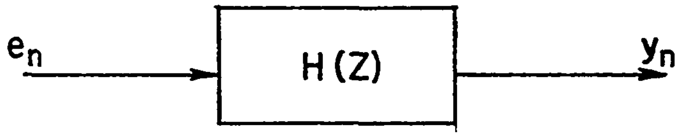


Figure 3.2b

Input signal error block diagram.

and C is the unit circle. Equation (3.11) also gives the output noise variance due to internal product roundoffs caused by coefficient multiplication in the discrete system. However, H(Z) is now taken as the pulse transfer function from the output of the multipliers to the discrete system output, as appropriate.

When the coefficient errors are assumed to be random variables, Avenhaus [36] has shown that the noise variance can be described by

$$\sigma^2(w) = \frac{q^2}{12} \sum_{i=1}^{NX} \left[\frac{\partial P(w)}{\partial x_i}\right]^2 \tag{3.12}$$

where

$$P(w) = H(w)H^*(w)$$

is the power transfer function of the discrete system and NX is the total number of erroreous coefficients. The alternative statistical estimate of the noise variance in (3.12) can be used in determining the word length requirements of a discrete system and sometimes can be very useful since it is in terms of frequency.

As derived in Chapter 2 of this work, the overall transfer function sequence of nonuniformly sampled filter is, in general, not regular in the time domain. For this reason, it is necessary to use each pulse transfer function sequences in determining the steady-state output noise variance due to roundoff.

Consider, for example, a second order digital filter sampled nonuniformly with two intersamples per period. The steady-state value of the output noise variance can be evaluated from equation (3.11) to yield

$$\sigma_o^2 = \frac{q^2}{12} \frac{1}{2\pi_j} \int_c H_o(Z_2^2)H_o(Z_2^{-2})Z_2^{-2}dZ_2^2 \tag{3.13}$$

where $H_o(\cdot)$ is the first pulse transfer function sequences as derived in Chapter 2. The noise spectral density can be normalized with respect to $q^2/12$ to give $\sigma_0^2/(q^2/12)$

Equation (3.13) then becomes

$$\sigma_{No}^2 = \frac{1}{2\pi_j} \int_c H_o(Z_2^2)H(Z_2^{-2})Z_2^{-2}dZ_2^2 \tag{3.14}$$

For the second pulse transfer function sequences:

$$\sigma_{N1}^2 = \frac{1}{2\pi_j} \int_c H_1(Z_2^2)H_1(Z_2^{-2})Z_2^{-2}dZ_2^2 \tag{3.15}$$

A computer program written in Fortran IV and based on the recursive algorithm of Astrom, Jury, and Agniel [38] has been used in evaluating the integrals of equations (3.14) and (3.15). The program has been run on the IBM S/370-158 of the University of Oklahoma Computing Center and is listed in Appendix C-1.

A wide variety of values for the filter coefficients have been used for computer runs with the analysis program. In this analysis program, equation (3.11) as derived by Gold and Rader [3] for uniformly sampled filter is evaluated along with equations (3.14) and (3.15) of this work. For comparison purposes, the eigenvalues of the uniformly sampled filter is set to be the same as those of nonuniformly sampled filters.

As an example, consider a nonuniformly sampled filter with two intersamples per period such that the denominator parameters satisfy the following relations:

$$2b_1 - a_1^2 = a = -1.7 \text{ and}$$

$$b_1^2 = b = 0.8 \tag{3.16}$$

where a and b are the filter coefficients for the uniformly sampled filters. From equation (3.16), one obtains

$$a_1 = 1.867847525 \text{ and}$$

$$b_1 = 0.89442719 \tag{3.17}$$

The values of a_1 and b_1 in (3.17) are used in evaluating the integral of equations (3.14) and (3.15), where

$$H_o(Z_2^{-2}) = \frac{1 + b_1 Z_2^{-2}}{1 + Z_2^{-2}(2b_1 - a_1^2) + Z_2^{-4} b_1^2}$$

and

$$H_1(Z_2^{-2}) = \frac{-a_1}{1 + Z_2^{-2}(2b_1 - a_1^2) + Z_2^{-4} b_1^2}.$$

The result of the variance computation for each sequence is displayed in Table 3.2. Additional result for filter coefficients b = 0.950 and 0.945 respectively are shown in Table 3.3a and Table 3.3b. Note that the total noise variance in the overall output sequence is the average of the steady-state noise variances in the output sequences. Table 3.4 shows the variance computation when there are four intersamples in the overall sampling period.

The graphical displays of the normalized noise variance for some representative values of filter coefficients are shown in Figure 3.3 and Figure 3.4. In Figure 3.5, the noise variance for filters with four intersamples per sampling period is exhibited. In all these plots, one filter coefficient is held fixed while the other is allowed to vary as long as the filter stability is maintained. Fixing one filter parameter makes the magnitude of the pole constant and changing the pole location causes damping changes.

From these graphical displays, some observations are noted. The output noise variance is about the same for each pulse transfer function sequences for a given number of intersamples. For two intersamples per sampling period, nonuniformly sampled filters have the same noise characteristics as uniformly sampled filters, and tend to have somewhat better noise variance characteristics for positive

EROR1 IS STEADY-STATE VALUE OUTPUT NOISE VARIANCE FOR UNIFORMLY SAMPLED

EROR2 IS 1ST OUTPUT SEQUENCE NOISE VARIANCE

EROR3 IS FOR 2ND OUTPUT SEQUENCE IN NONUNIFORMLY SAMPLED DIGITAL FILTERS

b = 0.800000

a	EROR1	EROR2	EROR3
-1.7000	0.2571404E 02	0.8972841E 02	0.8971249E 02
-1.6500	0.1739124E 02	0.5982260E 02	0.5980646E 02
-1.6000	0.1323523E 02	0.4486894E 02	0.4485255E 02
-1.5500	0.1074623E 02	0.3589648E 02	0.3587985E 02
-1.5000	0.9090863E 01	0.2991539E 02	0.2989850E 02
-1.4500	0.7912052E 01	0.2564322E 02	0.2562608E 02
-1.4000	0.7031232E 01	0.2243896E 02	0.2242155E 02
-1.3500	0.6349188E 01	0.1994684E 02	0.1992915E 02
-1.3000	0.5806423E 01	0.1795316E 02	0.1793518E 02
-1.2500	0.5365112E 01	0.1632199E 02	0.1630371E 02
-1.2000	0.4999976E 01	0.1496276E 02	0.1494418E 02
-1.1500	0.4693596E 01	0.1381263E 02	0.1379374E 02
-1.1000	0.4433473E 01	0.1282686E 02	0.1280764E 02
-1.0500	0.4210498E 01	0.1197252E 02	0.1195297E 02
-1.0000	0.4017838E 01	0.1122502E 02	0.1120512E 02
-0.9500	0.3850252E 01	0.1056550E 02	0.1054524E 02
-0.9000	0.3703687E 01	0.9979300E 01	0.9958660E 01
-0.8500	0.3574960E 01	0.9454791E 01	0.9433763E 01
-0.8000	0.3461521E 01	0.8982791E 01	0.8961357E 01
-0.7500	0.3361333E 01	0.8555758E 01	0.8533905E 01
-0.7000	0.3272713E 01	0.8167581E 01	0.8145290E 01
-0.6500	0.3194312E 01	0.7813176E 01	0.7790432E 01
-0.6000	0.3124987E 01	0.7488342E 01	0.7465122E 01
-0.5500	0.3063820E 01	0.7189531E 01	0.7165818E 01
-0.5000	0.3010024E 01	0.6913722E 01	0.6889493E 01
-0.4500	0.2962952E 01	0.6658370E 01	0.6633603E 01
-0.4000	0.2922071E 01	0.6421289E 01	0.6395961E 01
-0.3500	0.2886919E 01	0.6200603E 01	0.6174685E 01
-0.3000	0.2857137E 01	0.5994668E 01	0.5968132E 01
-0.2500	0.2832405E 01	0.5802033E 01	0.5774849E 01
-0.2000	0.2812492E 01	0.5621489E 01	0.5593625E 01
-0.1500	0.2797195E 01	0.5451924E 01	0.5423346E 01
-0.1000	0.2786372E 01	0.5292359E 01	0.5263030E 01
-0.0500	0.2779917E 01	0.5141970E 01	0.5111847E 01

Table 3.2

Normalized steady-state output noise

Variance for filter parameter b = 0.8.

EROR1 IS STEADY-STATE VALUE OUTPUT NOISE VARIANCE FOR UNIFORMLY SAMPLED

EROR2 IS 1ST OUTPUT SEQUENCE NOISE VARIANCE

EROR3 IS FOR 2ND OUTPUT SEQUENCE IN NONUNIFORMLY SAMPLED DIGITAL FILTERS

b = 0.950000

a	EROR1	EROR2	EROR3
1.8000	0.6933379E 02	0.1044118E 02	0.1035561E 02
1.7500	0.5270311E 02	0.1057102E 02	0.1050684E 02
1.7000	0.4273996E 02	0.1070895E 02	0.1065760E 02
1.6500	0.3611116E 02	0.1085301E 02	0.1081021E 02
1.6000	0.3138621E 02	0.1100246E 02	0.1096577E 02
1.5500	0.2785687E 02	0.1115717E 02	0.1112506E 02
1.5000	0.2512067E 02	0.1131675E 02	0.1128822E 02
1.4500	0.2294116E 02	0.1148157E 02	0.1145590E 02
1.4000	0.2116690E 02	0.1165159E 02	0.1162825E 02
1.3500	0.1969696E 02	0.1182698E 02	0.1180558E 02
1.3000	0.1846150E 02	0.1200793E 02	0.1198817E 02
1.2500	0.1741063E 02	0.1219466E 02	0.1217632E 02
1.2000	0.1650793E 02	0.1238752E 02	0.1237040E 02
1.1500	0.1572573E 02	0.1258659E 02	0.1257054E 02
1.1000	0.1504335E 02	0.1279235E 02	0.1277724E 02
1.0500	0.1444442E 02	0.1300502E 02	0.1299075E 02
1.0000	0.1391602E 02	0.1322485E 02	0.1321133E 02
0.9500	0.1344814E 02	0.1345239E 02	0.1343955E 02
0.9000	0.1303251E 02	0.1368804E 02	0.1367581E 02
0.8500	0.1266223E 02	0.1393205E 02	0.1392038E 02
0.8000	0.1233195E 02	0.1418502E 02	0.1417385E 02
0.7500	0.1203692E 02	0.1444740E 02	0.1443670E 02
0.7000	0.1177347E 02	0.1471959E 02	0.1470932E 02
0.6500	0.1153835E 02	0.1500245E 02	0.1499257E 02
0.6000	0.1132892E 02	0.1529631E 02	0.1528680E 02
0.5500	0.1114278E 02	0.1560195E 02	0.1559278E 02
0.5000	0.1097814E 02	0.1592013E 02	0.1591128E 02
0.4500	0.1083326E 02	0.1625151E 02	0.1624297E 02
0.4000	0.1070687E 02	0.1659709E 02	0.1658881E 02
0.3500	0.1059777E 02	0.1695767E 02	0.1694966E 02
0.3000	0.1050499E 02	0.1733429E 02	0.1732652E 02
0.2500	0.1042774E 02	0.1772803E 02	0.1772047E 02
0.2000	0.1036541E 02	0.1814015E 02	0.1813281E 02
0.1500	0.1031740E 02	0.1857185E 02	0.1856473E 02
0.1000	0.1028341E 02	0.1902466E 02	0.1901772E 02
0.0500	0.1026313E 02	0.1950012E 02	0.1949336E 02

Table 3.3a

Normalized steady-state output noise variance for filter parameter b = 0.950.

EROR1 IS STEADY-STATE VALUE OUTPUT NOISE VARIANCE FOR UNIFORMLY SAMPLED

EROR2 IS 1ST OUTPUT SEQUENCE NOISE VARIANCE

EROR3 IS FOR 2ND OUTPUT SEQUENCE IN NONUNIFORMLY SAMPLED DIGITAL FILTERS

b = 0.945000

a	EROR1	EROR2	EROR3
1.8000	0.6512373E 02	0.9489798E 01	0.9392224E 01
1.7500	0.4908023E 02	0.9604971E 01	0.9532432E 01
1.7000	0.3960001E 02	0.9728923E 01	0.9671180E 01
1.6500	0.3334554E 02	0.9858948E 01	0.9810983E 01
1.6000	0.2891484E 02	0.9994130E 01	0.9953127E 01
1.5500	0.2561613E 02	0.1013427E 02	0.1009845E 02
1.5000	0.2306790E 02	0.1027907E 02	0.1024728E 02
1.4500	0.2104314E 02	0.1042857E 02	0.1039999E 02
1.4000	0.1939830E 02	0.1058296E 02	0.1055700E 02
1.3500	0.1803780E 02	0.1074225E 02	0.1071847E 02
1.3000	0.1689583E 02	0.1090672E 02	0.1088478E 02
1.2500	0.1592572E 02	0.1107636E 02	0.1105600E 02
1.2000	0.1509306E 02	0.1125168E 02	0.1123269E 02
1.1500	0.1437235E 02	0.1143266E 02	0.1141486E 02
1.1000	0.1374396E 02	0.1161972E 02	0.1160297E 02
1.0500	0.1319275E 02	0.1181308E 02	0.1179727E 02
1.0000	0.1270680E 02	0.1201304E 02	0.1199807E 02
0.9500	0.1227670E 02	0.1222001E 02	0.1220580E 02
0.9000	0.1189472E 02	0.1243435E 02	0.1242080E 02
0.8500	0.1155467E 02	0.1265629E 02	0.1264337E 02
0.8000	0.1125137E 02	0.1288644E 02	0.1287408E 02
0.7500	0.1098065E 02	0.1312517E 02	0.1311333E 02
0.7000	0.1073885E 02	0.1337295E 02	0.1336158E 02
0.6500	0.1052320E 02	0.1363028E 02	0.1361935E 02
0.6000	0.1033103E 02	0.1389773E 02	0.1388721E 02
0.5500	0.1016037E 02	0.1417594E 02	0.1416580E 02
0.5000	0.1000937E 02	0.1446555E 02	0.1445576E 02

Table 3.3b

Normalized steady-state output noise variance for b = 0.945

ERORI IS STEADY-STATE VALUE OUTPUT NOISE VARIANCE FOR UNIFORMLY SAMPLED

EROR2 IS 1ST OUTPUT SEQUENCE NOISE VARIANCE

EROR3 IS FOR 2ND OUTPUT SEQUENCE IN NONUNIFORMLY SAMPLED FILTERS WITH 4 INTERSAMPLES PER PERIOD

b = 0.835000

a	EROR1	EROR2	EROR3
1.8000	0.8741304E 02	0.4035618E 01	0.8663804E 01
1.7500	0.3649518E 02	0.5172359E 01	0.8399973E 01
1.7000	0.2330365E 02	0.5821754E 01	0.8436002E 01
1.6500	0.1724539E 02	0.6308205E 01	0.8562682E 01
1.6000	0.1377699E 02	0.6717596E 01	0.8728967E 01
1.5500	0.1152778E 02	0.7083990E 01	0.8917099E 01
1.5000	0.9954247E 01	0.7424136E 01	0.9119283E 01
1.4500	0.8793357E 01	0.7747655E 01	0.9321982E 01
1.4000	0.7902905E 01	0.8060524E 01	0.9553261E 01
1.3500	0.7199437E 01	0.8368875E 01	0.9782309E 01
1.3000	0.6630698E 01	0.8669562E 01	0.1001858E 02
1.2500	0.6162244E 01	0.8970840E 01	0.1026201E 02
1.2000	0.5770561E 01	0.9272424E 01	0.1051260E 02
1.1500	0.5438946E 01	0.9575815E 01	0.1077060E 02
1.1000	0.5155307E 01	0.9882156E 01	0.1103621E 02
1.0500	0.4910589E 01	0.1019260E 02	0.1130983E 02
1.0000	0.4697965E 01	0.1050813E 02	0.1159184E 02
0.9500	0.4512122E 01	0.1082963E 02	0.1188269E 02
0.9000	0.4348519E 01	0.1115804E 02	0.1218290E 02
0.8500	0.4205035E 01	0.1149416E 02	0.1249298E 02
0.8000	0.4077828E 01	0.1183888E 02	0.1281356E 02
0.7500	0.3965197E 01	0.1219306E 02	0.1314527E 02
0.7000	0.3865240E 01	0.1255758E 02	0.1348879E 02
0.6500	0.3776644E 01	0.1293336E 02	0.1384491E 02
0.6000	0.3698149E 01	0.1332134E 02	0.1421442E 02
0.5500	0.3628767E 01	0.1372251E 02	0.1459821E 02
0.5000	0.3567654E 01	0.1413795E 02	0.1499724E 02
0.4500	0.3514108E 01	0.1456879E 02	0.1541257E 02
0.4000	0.3467536E 01	0.1501621E 02	0.1584528E 02
0.3500	0.3427462E 01	0.1548157E 02	0.1629668E 02
0.3000	0.3393477E 01	0.1596622E 02	0.1676807E 02
0.2500	0.3365238E 01	0.1647174E 02	0.1726093E 02
0.2000	0.3342481E 01	0.1699979E 02	0.1777690E 02
0.1500	0.3324991E 01	0.1755214E 02	0.1831773E 02
0.1000	0.3312615E 01	0.1813091E 02	0.1888547E 02
0.0500	0.3305229E 01	0.1873822E 02	0.1948221E 02

Table 3.4

Normalized steady-state output noise variance for filter parameter b = 0.835.

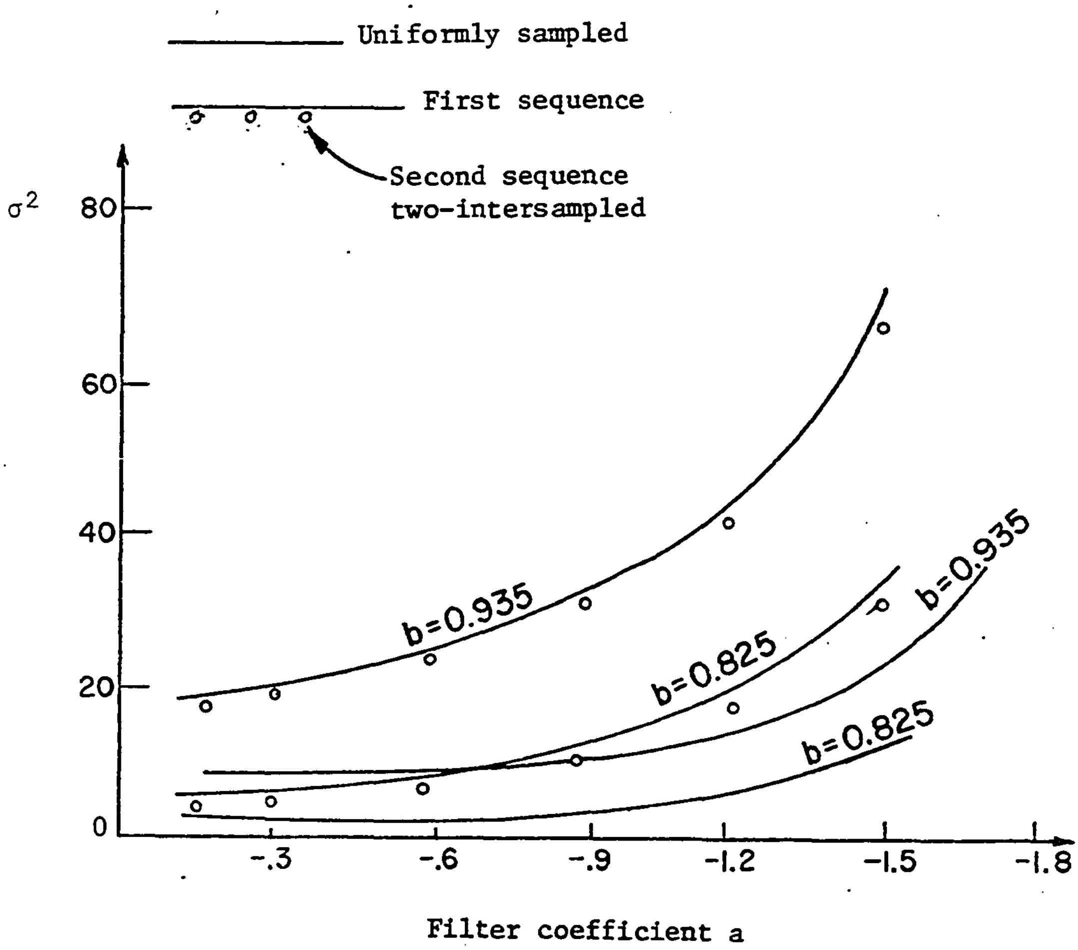


Figure 3.3

Normalized output noise variance

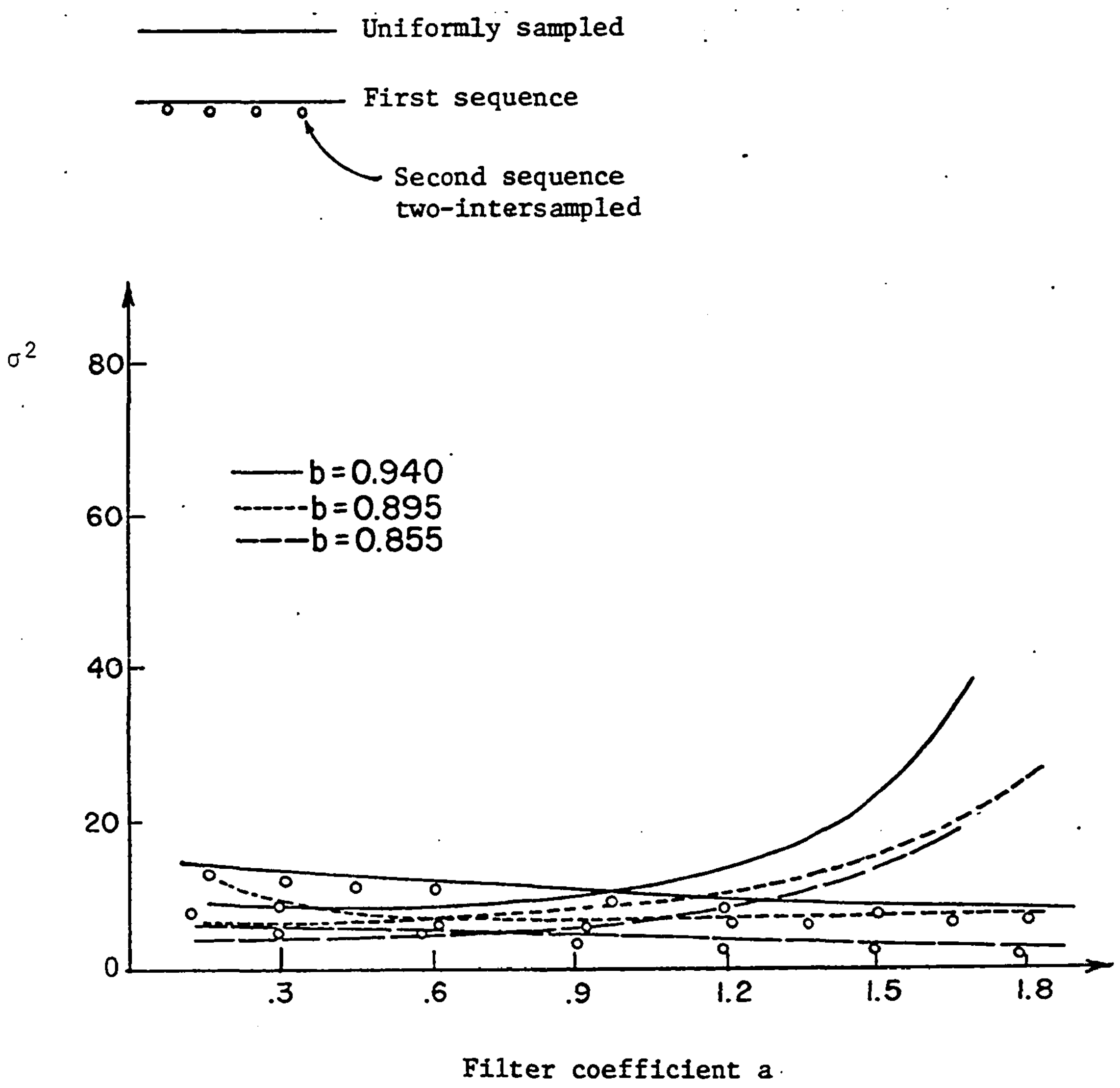


Figure 3.4

Normalized output noise variance

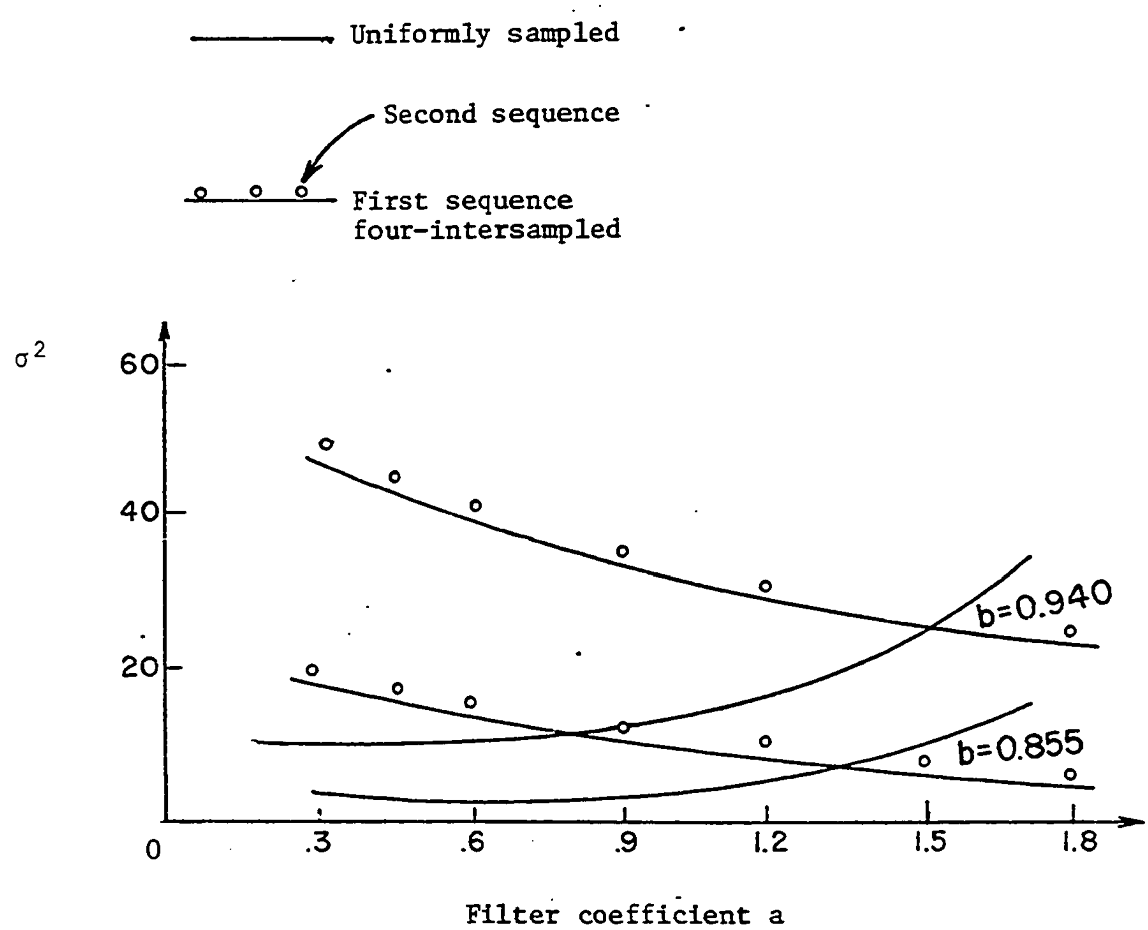


Figure 3.5

Normalized output noise variance

values of the filter parameter a. These remarks are evidenced in Figure 3.3 and Figure 3.4. For four intersamples per sampling period, nonuniformly sampled filters tend to have lower noise variance for high Q filters but higher noise variance for low Q filters when compared with uniformly sampled filter with the same eigenvalues. This remark is evidenced in Figure 3.5.

The purpose of this study has been to investigate quantization errors in nonuniformly sampled filters using the results developed earlier in this dissertation along with established techniques for analyzing such errors. The results generated in this section are then compared with those for uniformly sampled filters obtained by others. From the graphical displays of the results, some observations are made. It is observed that all the transfer function sequences for a given number of intersamples generate about the same level of output noise variance. For two intersamples, it is noted that nonuniformly sampled filter has the same noise characteristics when compared to uniformly sampled filter having the same eigenvalues. They also tend to have somewhat better noise variance characteristics for positive values of filter coefficient a. For four intersamples, the nonuniformly sampled filter is observed to have lower noise variance for high Q filters with positive value of filter coefficient "a," and higher noise variance for low Q filters when compared to uniformly sampled filter having about the same eigenvalues. These observations are only restricted to second order nonuniformly sampled filters.

3.3 Zero-input limit cycle oscillations.

Jackson [29] and others [27,28,30,31,32] have considered the zero-input limit cycle, which is the periodic output of the filter in case the input is zero. Parker and Hess [27] have shown that limit cycles under forced response conditions can be readily reduced to zero-input conditions. The work on limit cycles has been restricted to uniformly sampled filters.

In this research, the decomposition technique is extended to limit cycle analysis of nonuniformly sampled filters. For simplicity, a zero-input filter model with poles and no zeros is used. This simple model is employed because it allows the study of the natural response of the filter without the distracting influence of zeros in the transfer function sequences. Also, zeros in the transfer function may change only the magnitude of the limit cycle but not the the nature. As shown by Jatnieks and Shenoi [32] and others [27,33], limit cycle oscillations are completely defined by a knowledge of the poles or eigenvalues of the discrete data system.

Jackson [29] and others [27,32] have shown that the limit cycle amplitude bounds on the output y(n) in a first order filter sampled uniformly can be evaluated as

$$|y(n)| \le \frac{0.5}{1 - |a|} \tag{3.18}$$

Equation (3.18) assumes that the quantization step has been normalized to one.

Using the results derived in chapter 2 along with standard techniques, the amplitude bounds on limit cycles in nonuniformly sampled filters can be evaluated.

From equation (2.34), the overall pulse transfer function sequence is given as

$$H(Z_2) = \frac{1 + a_1 Z_2^{-1}}{1 - a_1^2 Z_2^{-2}} \tag{3.19}$$

Since the zeros of $H(Z_2)$ have no influence on the nature of limit cycles, the amplitude bound can be evaluated as

$$|y(n)| \le \frac{0.5}{1 - a_1^2} \tag{3.20}$$

where $|a|$ in (3.18) has been replaced by a_1^2 for filter with nonuniform sampling.

There is a large variety of modes of limit cycle behavior in second order filters. Parker and Hess [27] and others [29,31,32] have shown that the amplitude bounds that apply to two modes of limit cycles in uniformly sampled second order filters can be given as

$$|y(n)| \le \frac{0.5}{1 - |b|}$$

for oscillating limit cycles and

$$|y(n)| \le \frac{1}{1 - |a| + b} \tag{3.21}$$

for constant magnitude limit cycles. The quantization step is assumed to be one in equation (3.21).

A second order digital filter with two nonuniform samples per period has the overall transfer function sequence which is given from (2.52) as

$$H(Z_2) = \frac{1 - C_1 Z_2^{-1} + C_2 Z_2^{-2}}{1 + \beta_1 Z_2^{-2} + \beta_2 Z_2^{-4}} \tag{3.22}$$

where

$$\beta_1 = 2b_1 - a_1^2 \text{ and } \beta_2 = b_1^2$$

The values for C_1 and C_2 can be expressed in terms of β_1 and β_2 as

$$C_1 = (2C_2 - \beta_1)^{\frac{1}{2}}$$

$$C_2 = (\beta_2)^{\frac{1}{2}} \qquad (3.23)$$

Equation (3.22) represents a mapping of the $a_1 - b_1$ parameter plane of the filter with nonuniform samples into $\beta_1 - \beta_2$ coefficient plane of filter with uniform samples when the zeros of equation (3.22) are ignored. Since the feedforward coefficients of equation (3.22) do not contribute to the nature of the limit cycle, the bounds of (3.21) are applicable.

The amplitude bounds on limit cycles in filters with nonuniform samples with two intersamples per period for the case of roundoff quantization after multiplication are thus given as:

$$Bd_1 = \frac{0.5}{1 - |\beta_2|} = \frac{0.5}{1 - b_1^2} \qquad (3.24)$$

for oscillating limit cycles, and

$$Bd_2 = \frac{1}{1 - |\beta_1| + \beta_2} = \frac{1}{1 - |2b_1 - a_1^2| + b_1^2} \qquad (3.25)$$

for constant magnitude limit cycles.

Similarly, the amplitude bounds where there are four intersamples per samplig period are

$$Bd_1 = \frac{0.5}{1 - b_1^4} \quad \text{and}$$

$$Bd_2 = \frac{1}{1 - |4a_1^2 b_1 - 2b_1^2 - a_1^4| + b_1^4} \qquad (3.26)$$

The limit cycle behavior of a second order filter may be viewed on a phase plane plot which displays the successive states of a second order discrete system on a cartesian plane with y(n) plotted on the x-axis and y(n-1) on the y-axis. Once a state point is chosen in the phase plane plot for a given $H(Z)$, the resulting state trajectory is uniquely specified. A limit cycle on the phase plane plot is recognized as a repeating sequence of state points.

In order to display the limit cycle in a second order filter due to roundoff errors, a computer program in PL/1 language is written. This program has been run on the IBM S/370-158 computer and is listed in Appendix C-2.

A PL/1 programming language has been chosen in order to minimize errors which occur when decimal constants are converted into binary format. A PL/1 language allows for exact decimal representation of the filter coefficients since the language internal coded arithmetic form is decimal. This is discussed in the IBM S/360 PL/1 operating manual [37].

During the simulation, several initial conditions are employed in a systematic manner until a specific representative range of initial conditions has been established. Then using a variety of values for the filter coefficient parameters, all possible limit cycles in a specified area of search for a given set of filter coefficients are enumerated. As for example, the computer results for $a = -1.4$ and $b = 0.8$ for uniformly sampled second order recursive digital filter are displayed in Table 3.5 and the phase plane plot is shown in Figure 3.6. Additional results for $a = -1.4$ and $b = 0.94$ for uniformly sampled second order filter are displayed in Table 3.6 and the phase plane plot is shown in Figure 3.7. These results agree with those of Hess and Parker [27] in comparison.

```
a=-1.400000000 , b=0.800000000
THE CALCULATED BOUNDS ARE BND1=  2.5000,BND2=  2.5000


LIMIT CYCLE #  1   IS
   -2  -1   1   2   2   1  -1  -2

LIMIT CYCLE #  2   IS
   -1   0   1   1   0  -1

LIMIT CYCLE #  3   IS   0


LIMIT CYCLE #  4   IS   2
```

Table 3.5

Limit cycle oscillations in second order

filter with uniform sampling

LIMIT CYCLES ARRANGED IN PHASE PLANE, X(N) VERSUS X(N-1)

a= -1.400000 b= 0.800000

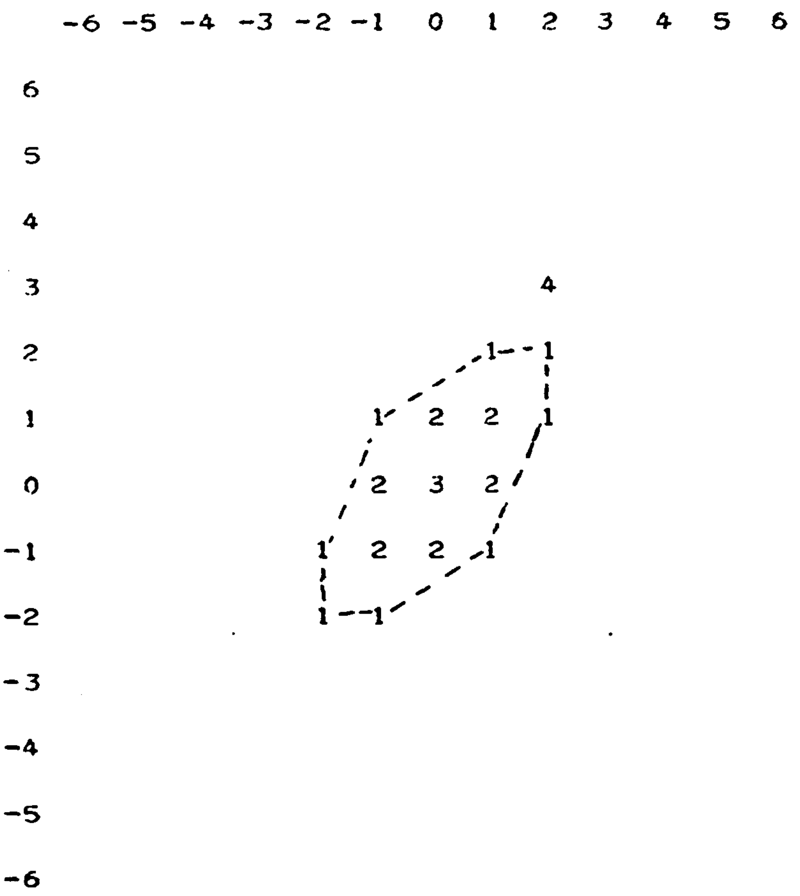


Figure 3.6

Phase-plane plot of Table 3.5

```
a= -1.400000000 , b=0.940000000
THE CALCULATED BOUNDS ARE BND1=  8.3333,BND2=  1.8519


LIMIT CYCLE #  1    IS
   -7  -3   3   7   7   3  -3  -7

LIMIT CYCLE #  2    IS
   -6  -1   5   8   6   0  -6  -8  -5   1   6   7   4  -1  -5  -6
   -3   2   6   6   2  -3  -6  -5  -1   4   7   6   1  -5  -8  -6
    0   6   8   5  -1  -6  -7  -4   1   5   6   3  -2  -6  -6  -2
    3   6   5   1  -4  -7

LIMIT CYCLE #  3    IS
   -5   0   5   7   5   0  -5  -7

LIMIT CYCLE #  4    IS
   -2   4   8   7   2  -4  -8  -7

LIMIT CYCLE #  5    IS
   -4   0   4   6   4   0  -4  -6

LIMIT CYCLE #  6    IS
   -5  -2   2   5   5   2  -2  -5

LIMIT CYCLE #  7    IS
   -4  -1   3   5   4   1  -3  -5

LIMIT CYCLE #  8    IS
   -3   1   4   5   3  -1  -4  -5
```

Table 3.6

Limit cycle oscillations in second order filter

with uniform sampling.

LIMIT CYCLES ARRANGED IN PHASE PLANE, X(N) VERSUS X(N-1)
a=-1.400000 b=0.940000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8																2	2	4			
7													4	1	2	3	2	1			
6											2	2	2	2	5	2	2	2	2		
5										2	3	2	6	8	7	6	2	3	2		
4										2	5	7				8	5	2	4		
3								1	2	8						7	2	1			
2							4	2	6							6	2				
1						2	2	7							8	2	2				
0					2	3	5								5	3	2				
-1					2	2	8							7	2	2					
-2					2	6							6	2	4						
-3				1	2	7						8	2	1							
-4			4	2	5	8				7	5	2									
-5			2	3	2	6	7	8	6	2	3	2									
-6			2	2	2	2	5	2	2	2	2										
-7				1	2	3	2	1	4												
-8				4	2	2															
-9																					
-10																					

Figure 3.7

Phase-plane plot of Table 3.6.

The filter parameter for nonuniformly two-intersampled digital filter has been chosen such that its eigenvalues are the same as those of uniformly sampled filter with a = -1.4 and b = 0.8. Thus a_1 = 1.7857364 and b_1 = 0.89442719. The computer results for the limit cycle in nonuniformly sampled filter with its coefficient values thus chosen as above is displayed in Table 3.7 and the phase plane plot in Figure 3.8.

Similarly, the filter parameter for non-uniformly four-intersampled filter has been chosen such that its eigenvalues are the same as those of uniformly sampled filter with a = -1.4 and b = 0.94. Hence a_1 = 1.94848996 and b_1 = 0.98465018. The corresponding limit cycles and phase plane plots are displayed respectively in Table 3.8 and Figure 3.9.

In both tables 3.7 and 3.8, most of the limit cycle oscillations in the area of search are enumerated. These limit cycles are designated by an identification number and the values of the limit cycles point are printed. Furthermore, the amplitude bounds of equations (3.24), (3.25) and (3.26) are stated. In Table 3.7, all the limit cycles in the area of search stay below the bound calculated. However, in Table 3.8, limit cycles #6 exceeds the calculated bound by as much as three quantization steps. This means that the bounds are not exact bounds but can only give approximate results. Parker and Hess [27] arrived at the same conclusion for their bound on limit cycles in uniformly sampled filters.

The corresponding state trajectories for Tables 3.7 and 3.8 are shown respectively in Figures 3.8 and 3.9. A trajectory in the phase-plane is made up of those point values bearing the same number.

```
a=-1.4C0000000 , b=0.8C0000000
THE CALCULATED BOUNDS ARE BND1=  2.5000,BND2=  2.5000


LIMIT CYCLE #  1   IS
   -2  -1   1   2   2   1  -1  -2

LIMIT CYCLE #  2   IS
   -1   0   1   1   0  -1

LIMIT CYCLE #  3   IS   0


LIMIT CYCLE #  4   IS   2
```

Table 3.7

Limit cycle oscillations in second order

filter with nonuniform sampling

LIMIT CYCLES ARRANGED IN PHASE PLANE, X(N) VERSUS X(N-1)

a=-1.400000 b=0.800000

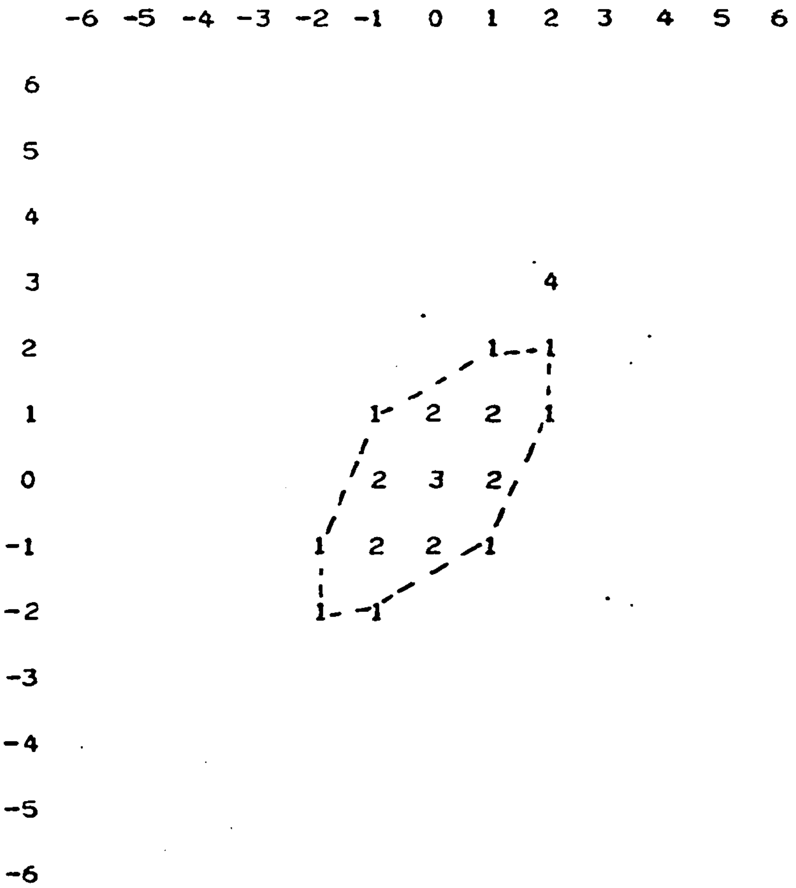


Figure 3.8

Phase-plane plot of Table 3.7.

a=-1.40000000 , b=0.940000000
THE CALCULATED BOUNDS ARE BND1= 8.3333,BND2= 1.8519

LIMIT CYCLE # 1 IS

-7	-3	3	7	7	3	-3	-7								

LIMIT CYCLE # 2 IS

-7	-4	1	5	6	3	-2	-6	-6	-2	3	6	5	1	-4	-7
-6	-1	5	8	6	0	-6	-8	-5	1	6	7	4	-1	-5	-6
-3	2	6	6	2	-3	-6	-5	-1	4	7	6	1	-5	-8	-6
0	6	8	5	-1	-6										

LIMIT CYCLE # 3 IS

-7	-5	0	5	7	5	0	-5

LIMIT CYCLE # 4 IS

-7	-8	-4	2	7	8	4	-2

LIMIT CYCLE # 5 IS

2	-4	-8	-7	-2	4	8	7

LIMIT CYCLE # 6 IS

-7	-11	-8	-1	7	11	8	1

LIMIT CYCLE # 7 IS

-6	-4	0	4	6	4	0	-4

LIMIT CYCLE # 8 IS

-5	-2	2	5	5	2	-2	-5

LIMIT CYCLE # 9 IS

-5	-3	1	4	5	3	-1	-4

LIMIT CYCLE # 10 IS

-5	-4	-1	3	5	4	1	-3

Table 3.8

Limit cycle oscillations in second order

filter with non-uniform sampling

LIMIT CYCLES ARRANGED IN PHASE PLANE. X(N) VERSUS X(N-1)
a=-1.400000 b=0.940000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8												6			4	2	2	5			
7													5	1	2	3	2	1	4		
6											2	2	2	2	7	2	2	2	2		
5										2	3	2	8	9	10	8	2	3	2		
4									4	2	7	10				9	7	2	5		
3								1	2	9						10	2	1			
2							5	2	8							8	2	4			
1				6		2	2	10							9	2	2				
0					2	3	7								7	3	2				
-1					2	2	9							10	2	2		6			
-2				4	2	8							8	2	5						
-3				1	2	10						9	2	1							
-4			5	2	7	9				10	7	2	4								
-5			2	3	2	8	10	9	8	2	3	2									
-6			2	2	2	2	7	2	2	2	2										
-7			4	1	2	3	2	1	5												
-8				5	2	2	4			6											
-9																					
-10																					

Figure 3.9

Phase-plane plot of Table 3.8

These numbers are identical to the limit cycle number appearing in Tables 3.7 and 3.8. For example, consider limit cycle #1 of Table 3.7, its state trajectory is made up of those state points designated by the number "1" in Figure 3.8. These include (1,2), (2,2), (2,1), (1,-1), (-1,-2), (-2,-2), (-2,-1) and (-1,1). The trajectory is very close to an ellipse, which indicates that this limit cycle is nearly sinusoidal in nature. It can be verified that there is no difference between the nature of the limit cycle oscillations in uniformly sampled filter of Figure 3.6 and that of nonuniformly sampled filter of Figure 3.8, for two intersamples per period. It can also be verified from Tables 3.6 and 3.8 that in the area of search, the limit cycles in nonuniformly four-intersampled filter are not exactly the same as those in uniformly sampled filter with equivalent eigenvalues. As can be verified from Figure 3.9, the pattern of limit cycle oscillations in nonuniformly sampled filter with four-intersamples is somewhat slightly distorted.

The purpose of this section has been to extend the decomposition method to the analysis of limit cycle oscillations of nonuniformly sampled filters. In comparison with well-known techniques, the amplitude bounds and the general nature of limit cycles in nonuniformly sampled filters are shown. The conclusions arrived at fairly agree with those established for uniformly sampled filters.

3.4 Frequency and Phase characteristics of recursive digital filters sampled nonuniformly.

In applications, it may be necessary to have the filter realization match some desired characteristics. For a stable system, this would mean that the system function $H(\cdot)$ would be interpretable as a frequency response function.

The derivation of the frequency characteristics of uniformly sampled filter is fairly straight-forward. However, for nonuniformly sampled filters, the derivation is more complicated due to the fact that the pulse transfer function sequence is not regular in the Z-domain.

For a nonuniformly sampled filter, assume that the filter with transfer function of equation (2.38) is subjected to nonuniform sampling. Then the kth output sequences can be written, using equations (2.12) and (2.14) as

$$Y(Z_n) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} Z_n^{-k} X_j(Z_n^n) H_{k-j}(Z_n^n) \tag{3.27}$$

where Z_n^{-k} are the sample delays that place each periodic output sequence correctly in time and $X_j(Z_n^n)$, $j = 1, 2, \ldots, n-1$, are the advanced Z-transforms of the input sequences.

Rewriting (3.27) to take into consideration the advanced Z-transform of the given input sequences yields

$$Y(Z_n) = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} e^{jwt_j} X_0(Z_n^n) H_{k-j}(Z_n^n) Z_n^{-k} \tag{3.28}$$

Define the gain of a filter with the overall output sequence of (3.28) as

$$\frac{Y(Z_n)}{X_o(Z_n^n)} = \bar{H}(jw) \tag{3.29}$$

Then, (3,28) can be rewritten as

$$\bar{H}(jw) = \frac{1}{T} \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} e^{jwt_i} e^{-jwt_k} H_{k-i}(e^{jwT}) \qquad (3.30)$$

where e^{jwT} has been substituted for Z_n^n.

The summation in (3.30) may be rearranged to yield

$$\frac{1}{T} \sum_{i=0}^{n-1} \{ \sum_{k=0}^{i-1} e^{-jw(t_k - t_i)} H_{k-i}(e^{jwT}) + \sum_{k=i}^{n-1} e^{-jw(t_k - t_i)} H_{k-i}(e^{jwT}) \} \qquad (3.31)$$

Letting r=k-i, then (3.31) becomes

$$\frac{1}{T} \sum_{i=0}^{n-1} \{ \sum_{r=-i}^{-1} e^{-jw(t_{r+i} - t_i)} H_r(e^{jwT}) + \sum_{r=0}^{n-i-1} e^{-jw(t_{r+i} - t_i)} H_r(e^{jwT}) \} \qquad (3.32)$$

In (3.32) $t_{r+i} - t_i$ may be conveniently written as $\sum_{v=k} t_v$ so that (3.32) becomes

$$\frac{1}{T} \sum_{i=0}^{n-1} \{ \sum_{r=-i}^{-1} \exp(-jw \sum_{v=k} t_v) H_r(e^{jwT}) + \sum_{r=0}^{n-i-1} \exp(-jw \sum_{v=k} t_v) H_r(e^{jwT}) \} \qquad (3.33)$$

Equation (3.30) can now be written as

$$\bar{H}(jw) = \bar{H}_1(jw) + \bar{H}_2(jw) \qquad (3.34)$$

where

$$\bar{H}_1(jw) = \frac{1}{T} \sum_{i=0}^{n-1} \sum_{r=-i}^{-1} \exp(-jw \sum_{v=k} t_v) H_r(e^{jwT}) \qquad (3.35)$$

$$\bar{H}_2(jw) = \frac{1}{T} \sum_{i=0}^{n-1} \sum_{r=0}^{n-i-1} \exp(-jw \sum_{v=k} t_v) H_r(e^{jwT}) \qquad (3.36)$$

Now consider equation (3.36) which has positive values for subscript r. Equation (3.35) with negative values for subscript r can be similarly handled as (3.36) which thus can be rewritten for convenience as

$$\bar{H}_2(jw) = \frac{1}{T}\sum_{i=0}^{n-1} H_i(e^{jwT}) \sum_{k=1}^{n} \exp(-jw \sum_{v=k}^{k-1+i} t_v) \qquad (3.37)$$

Expanding (3.37) yeilds

$$\bar{H}_2(jw) = \frac{1}{T}\{(H_o(e^{jwT}) \sum_{k=1}^{n} \exp(-jw \sum_{v=k}^{k-1} t_v)) +$$

$$(H_1(e^{jwT}) \sum_{k=1}^{n} \exp(-jw \sum_{v=k}^{k} t_v)) +$$

$$(H_2(e^{jwT}) \sum_{k=1}^{n} \exp(-jw \sum_{v=k}^{k+1} t_v)) + \ldots$$

$$+ (H_{n-1}(e^{jwT}) \sum_{k=1}^{n} \exp(-jw \sum_{v=k}^{k+n-2} t_v))\} \qquad (3.38)$$

Simplifying futher yields

$$\bar{H}_2(jw) = \frac{1}{T}\{(nH_o(e^{jwT})) + (e^{-jwt_1} + e^{-jwt_2} +$$

$$e^{-jwt_3} \ldots + e^{-jwt_n})H_1(e^{jwT}) +$$

$$(e^{-jw(t_1+t_2)} + e^{-jw(t_2+t_3)} + e^{-jw(t_3+t_4)} + \ldots$$

$$+ (e^{-jw(t_{n-1}+t_n)}H_2(e^{jwT}) + \ldots$$

$$+ (e^{-jw(t_1+\ldots+t_{n-1})} + e^{-jw(t_1+t_2 \ldots t_n)} + \ldots$$

$$+ e^{-jw(t_n+t_1 \ldots +t_{n-2})})H_{n-1}(e^{jwT})\} \qquad (3.39)$$

The overall transfer function sequences for nonuniformly sampled filter can be written from equation (2.53) as

$$H(Z_n) = \sum_{k=0}^{n-1} H_k(Z_n^n)Z_n^{-k} \qquad (3.40)$$

Expanding (3.40) yields

$$H(Z_n) = H_o(Z_n^n) + Z_n^{-1}H_1(Z_n^n) + Z_n^{-2}H_2(Z_n^n)$$

$$+ \ldots + Z_n^{-(n-1)}H_{n-1}(Z_n^n) \qquad (3.41)$$

Comparing equations (3.39) and (3.41) one finds that as long as the denominator parameters of the overall pulse transfer function sequence have Z_n^n as multipliers, then the following correspondence can be established:

$$\frac{1}{n}\{e^{-jwt_1} + e^{-jwt_2} + \ldots + e^{-jwt_n}\} \Longleftrightarrow Z_n^{-1},$$

$$\frac{1}{n}\{e^{-jw(t_1+t_2)} + e^{-jw(t_2+t_3)} + \ldots + e^{-jw(t_{n-1}+t_n)}\} \Longleftrightarrow Z_n^{-2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{1}{n}\{e^{-jw(t_1+t_2+\ \ldots\ t_{n-1})} + e^{-jw(t_1+t_2+\ \ldots\ +\ t_n)} + \ldots$$

$$+ e^{-jw(t_n+t_1+\ldots+\ t_{n-2})}\} \Longleftrightarrow Z_n^{-(n-1)} \tag{3.42}$$

Equation (3.42) is significant since it provides a means of describing the magnitude and phase characteristics of a digital filter sampled nonuniformly from its overall transfer function sequence $H(Z_n)$. The result shown in (3.42) is in contrast with the operator $H(e^{jwT})$ which describes the magnitude and phase response of a uniformly sampled digital filter having a transfer function $H(Z)$.

Using the relationship in (3.42), the frequency and phase characteristics of first and second order digital filters with nonuniform sampling can be derived.

For a first order filter, the overall pulse transfer function as given in equation (2.34) is

$$H(Z_2) = \frac{1 + a_1 Z_2^{-1}}{1 - a_1^2 Z_2^{-2}} = \frac{1 + aZ_2^{-1}}{1 - a^2 Z_2^{-2}} \tag{3.43}$$

Then using (3.42), (3.43) yields:

$$H(jw) = \frac{1}{T}\left\{\frac{2 + a_1 \exp(-jwt_1) + a_1 \exp(-jwt_2)}{1 - a_1^2 \exp(-jwT)}\right\} \tag{3.44}$$

where t_1 and t_2 are the two nonuniform intersamples. Simplifying (3.44) yields

$$H(jw) = \frac{2}{T}\ \{1 + \frac{a}{2}(\cos wt_1 + \cos wt_2) - j\frac{a}{2}(\sin wt_1 + \sin wt_2)\}/$$

$$(1 - a^2 \cos wT + ja^2 \sin wT) \tag{3.45}$$

where T is the overall sampling interval, that is $T = T_1 + T_2$.

$$\theta(jw) = \text{angle of } H(jw) \tag{3.46}$$

A first order filter with four intersamples per sampling period is similarly considered. The overall pulse transfer function is given as

$$H(Z_4) = H_o(Z_4^4) + Z_2^{-1} H(Z_4^4) + Z_2^{-2}H(Z_4^4) + Z_3^{-3}H_3(Z_4^4) \tag{3.47}$$

Substituting from Table 2.1 and simplifying yields

$$H(Z_4) = \frac{1 + aZ_2^{-1} + a^2Z_2^{-2} + a^3Z_3^{-3}}{1 - a^4Z_4^{-4}} \tag{3.48}$$

Using (3.42), (3.48) simplifies to

$$H(jw) = \frac{1}{T}\{4 + aA + a^2B + a^3c\}|D \tag{3.49}$$

where

$$A = e^{-jwt_1} + e^{-jwt_2} + e^{-jwt_3} + e^{-jwt_4}$$

$$B = e^{-jw(t_1+j_2)} + e^{-jw(t_2+t_3)} + e^{-jw(t_3+t_4)} + e^{-jw(t_4+t_1)}$$

$$C = e^{-jw(t_1+t_2+t_3)} + e^{-jw(t_2+t_3+t_4)} + e^{-jw(t_3+t_4+t_1)} + e^{-jw(t_4+t_1+t_2)}$$

$$D = 1 - a^4 \cos wT + ja^4 \sin wT$$

The overall pulse transfer function sequence for a second order filter with two intersamples per sampling period is given as of equation (2.52)

$$H(Z_2) = \frac{1 - a_1Z_2^{-1} + b_1Z_2^{-2}}{1 + Z_2^{-2}(2b_1 - a_1^2) + b_1^2Z_2^{-4}}$$

$$= \frac{1 - aZ_2^{-1} + bZ_2^{-2}}{1 + Z_2^{-2}(2b - a^2) + b^2Z_2^{-4}} \tag{3.50}$$

where the subscripts on a and b have been dropped for convenience.

From (3.50) and using (3.42)

$$H(jw) = \frac{2}{T} \frac{\{1 - \frac{a(e^{-jwt_1} + e^{-jwt_2})}{2} + be^{-jwT}\}}{1 + e^{-jwT}(2b - a^2) + b^2e^{-j2wT}} \tag{3.51}$$

Hence

$$|H(jw)|^2 = \frac{4}{T^2} \{\frac{A^2 + B^2}{C^2 + D^2}\} \tag{3.52}$$

$$\theta(jw) = \text{angle of } H(jw)$$

$$= \arctan(B/A) + \arctan(D/C) \tag{3.53}$$

where

$$A = 1 - \frac{a_1}{2} (\cos wt_1 + \cos wt_2) + b_1 \cos wT$$

$$B = \frac{a_1}{2} (\sin wt_1 + \sin wt_2) - b_1 \sin wT$$

$$C = 1 + (2b_1 - a_1^2) \cos wT + b_1^2 \cos 2wT$$

$$D = (2b_1 - a_1^2) \sin wT + b_1^2 \sin 2wT$$

A second order filter with four intersamples would have the following overall pulse transfer function sequences:

$$1 - aZ_4^{-1} + (a^2-b)Z_4^{-2} + (2ab-a^3)Z_4^{-3} + (a^2b-b^2)Z_4^{-4} - ab^2Z_4^{-5}$$
$$+ b^3Z_4^{-6} / (1 - Z_4^{-4}(4a^2b-2b^2-a^4) + b^4Z_4^{-8}) \tag{3.54}$$

Hence

$$H(jw) = \frac{1}{T} \{4 - aA + (a^2-b)B + (2ab-a^3)C + (a^2b-b^2)D$$
$$- ab^2E + b^3F/(1 + Ge^{-jwT} + b^4e^{-j2wT})\} \tag{3.55}$$

where

$$D = e^{-jwT}$$

$$E = e^{-jw(t_1+T)} + e^{-jw(t_2+T)} + e^{-jw(t_3+T)} + e^{-jw(t_4+T)}$$

$$F = e^{-jw(t_3+t_4+T)} + e^{-jw(t_2+t_3T)} + e^{-jw(t_3+t_4+T)}$$
$$+ e^{-jw(t_4+t_1+T)}$$

$$G = (4a^2b - 2b^2 - a^4)$$

and A,B and C are as defined in equation (3.49) It should be noted that when using equation (3.42), whenever $v > n$ then $t_v = t_{v-n}$.

The frequency characteristic plots for first order digital filter with nonuniform sampling is exhibited in Figure 3.10. These plots are obtained by using equation (3.44) for the case of two intersamples per period and equation(3.49) for the case of four intersamples per period. Also for comparison, the frequency response, of first order filter with uniform sampling is included. For the filter with nonuniform samples, the two intersamples are in the ratio of 3:7, and the four intersamples are in the ratios of 2:1:4:3. For the uniformly sampled filter, the normalized sampling frequency $fs/f = 0.5$ and the filter parameter $a = 0.8$.

From Figure 3.10, it can be observed that nonuniformly sampled filter tend to have a higher gain than the uniformly sampled filter of comparable eigenvalue. For example, the frequency response magnitude in the passband for uniformly sampled filter is about 7.3 db. For two intersampled filter, it is about 11.7 db and for the four intersampled it is about 12.6 db, using Figure 3.10. This is expected since nonuniform sampling tend to introduce transmission zeros into the pulse transfer function sequences. Also from Figure 3.10, it can be observed that the critical frequencies are the same for the same filter eigenvalues.

The phase characteristics for the first order filter is shown in Figure 3.11. The plot shows that the phase characteristics are nonlinear for both uniformly and nonuniformly sampled filter having the same eigenvalues.

In Figure 3.12 and Figure 3.13 are shown respectively, the frequency and the phase characteristics for a second order filter. These plots are obtained from equations (3.52) and (3.53) for the

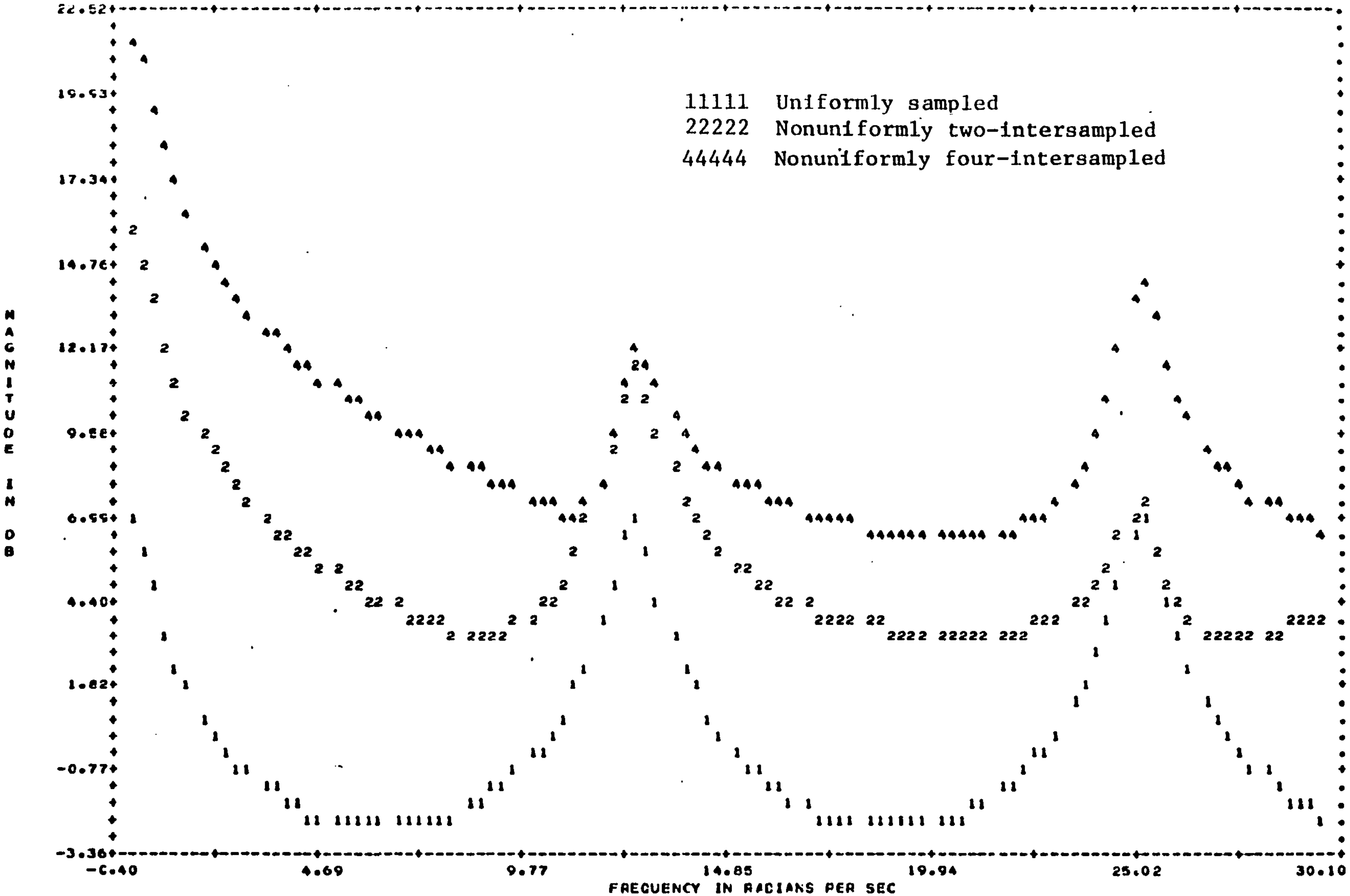


Figure 3.10

First order digital filter frequency response.

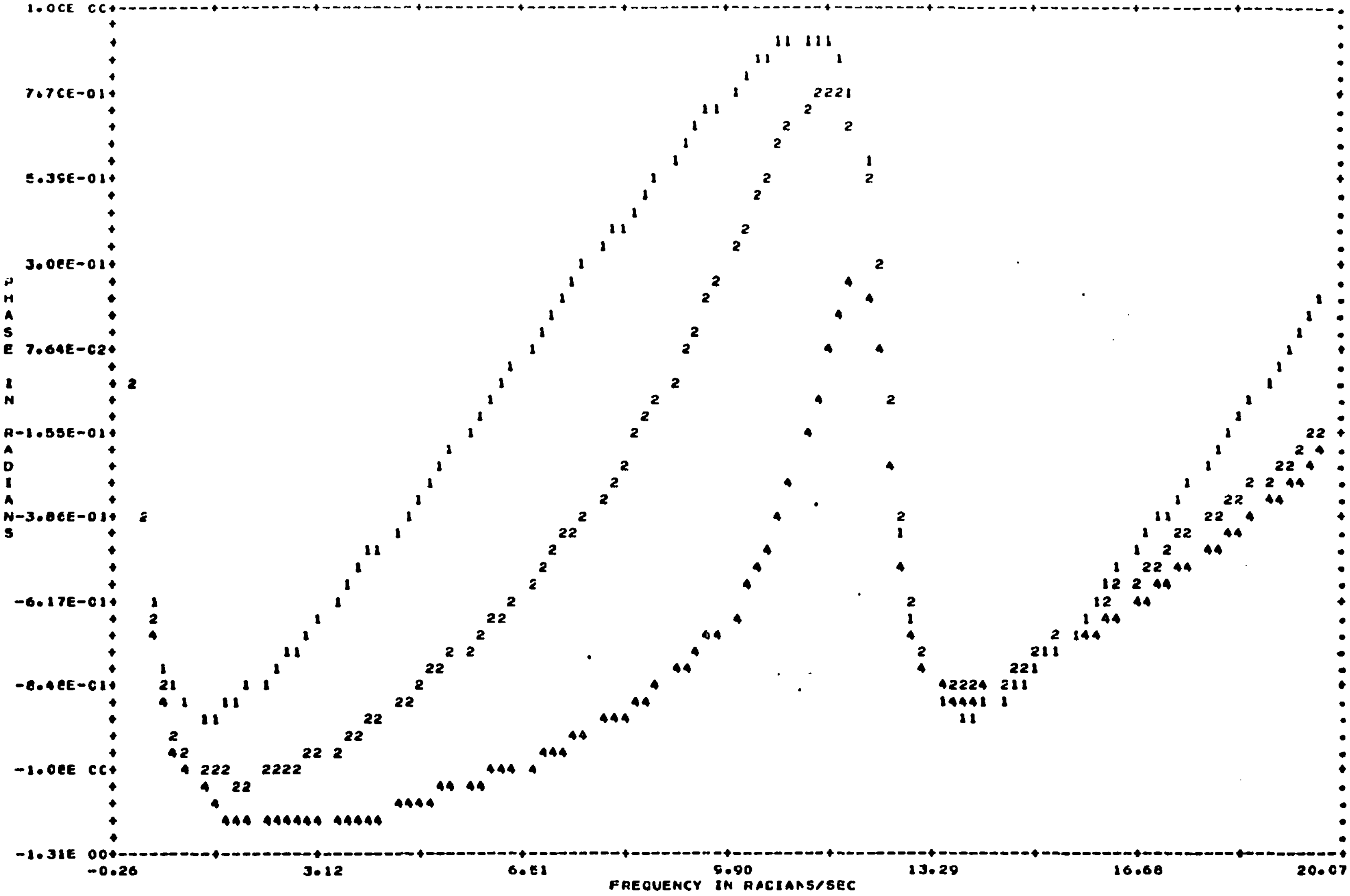


Figure 3.11

First order digital filter phase characteristics.

two intersampled filter and from equation (3.55) for the four intersampled filter. Also shown, for comparison, is the frequency response for uniformly sampled filter with the filter parameter a = -1.8 and b = 0.937 at a normalized sampling frequency fs/f = 0.9. For the nonuniformly sampled case, the two intersampling intervals are in the ratio of 3:7 and the four intersampling intervals are in the ratios 2:1:3:4.

From Figure 3.12, it can be observed that the gain at the passband for uniformly sampled is 13.7 db, for nonuniformly sampled with two intersamples per period, the gain is 23.2 db and for four intersampling, the gain is about 24.1 db. The same observation noted earlier holds. Also from Figure 3.12, it can be observed that a second order nonuniformly sampled filter exhibits some ripples in the stopband. The phase characteristic of Figure 3.13 points to the fact that the phase characteristics for the four intersampled filter exhibit more nonlinear characteristics than the two intersampled filter with approximate eigenvalues. However, recursive digitial filters, in general, exhibit some nonlinear phase characteristics [13]. This is also evidenced in Figure 3.11 and Figure 3.13 of this work for nonuniformly sampled filters.

The purpose of this section has been to demonstrate the application of the decomposition method to determining the frequency characterisitcs of recursive digital filter sampled nonuniformly. This new approach has been applied to first and second order recursive filters. From the graphical displays of frequency response function, it is observed that at critical frequencies, the gain of nonuniformly sampled filter tend to be higher than that of uniformly sampled filter

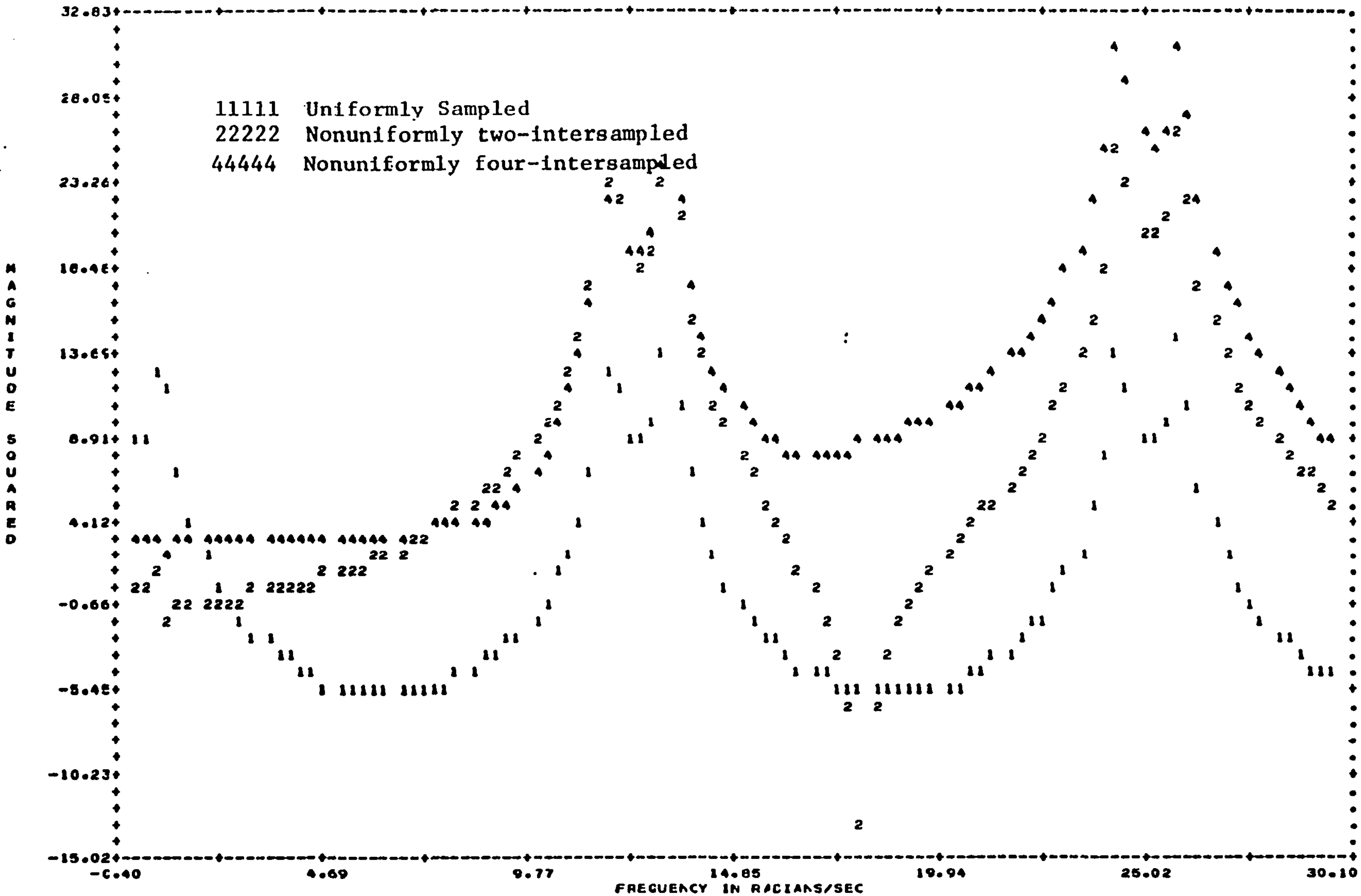


Figure 3.12

Second order digital filter frequency respone

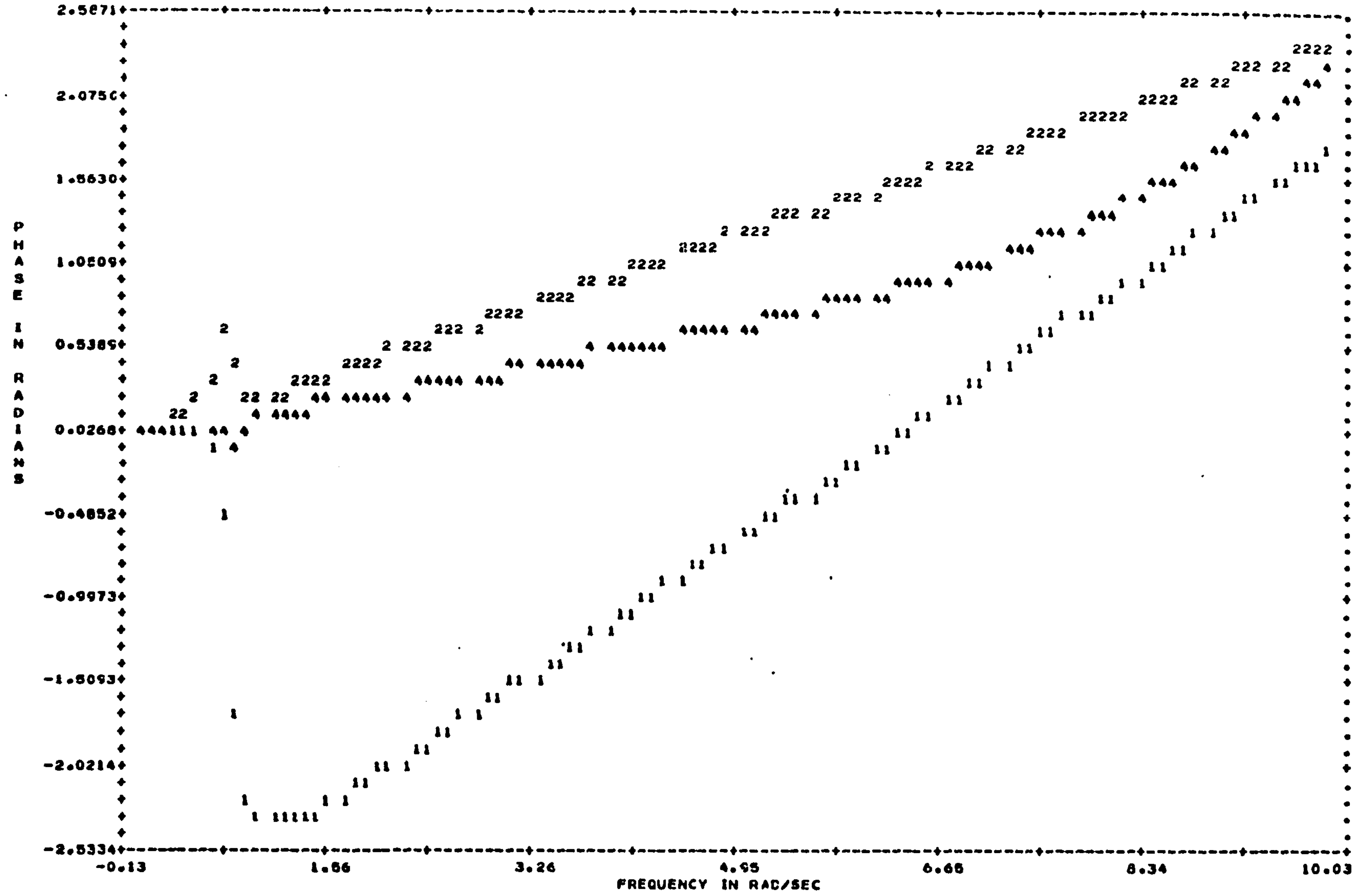


Figure 3.13

Second order digital filter phase characteristics.

of approximate eigenvalues due to transmission zeros introduced into the transfer function sequences. It is also observed that nonuniform sampling does not improve the phase characteristics of recursive digital filters which, in general, tend to be nonlinear.

The frequency response characteristics as derived in this section is important in that it enables one to match a filter response characteristics to a design filter response.

Chapter 4

APPLICATION OF DECOMPOSITION TECHNIQUE.

4.0 Introduction

In this chapter, the decomposition approach is employed in the analysis of moving target indicator (MTI) digital filters in radar systems. The characteristics of such filters when nonuniform sampling is used are re-derived and compared with the results of earlier investigators. In extension, the decomposition approach will be applied to the analysis of recursive moving target indicator filters. The effect of nonuniform samples in moving target indicator filters will then be shown. Thus far, the effect of nonuniform samples in recursive moving target indicator filters has been limited to direct simulation of these filters.

Secondly, multiple shift sequences in digital filters is considered. It will be shown that multiple shifting in digital filters can be regarded as a special case of nonuniform sampling. Filters with multiple shift sequences have been shown to be particularly suitable for the realization of variable filters.

Thirdly in this chapter, system with inherent nonuniform samples will be investigated. It will be shown that distortion results when nonuniform samples are processed as uniform samples.

4.1 Moving Target Indicator (MTI) digital filter in Radar Systems.

Moving target indicator filters [39,41] are used in radar systems to reject signals from fixed unwanted targets, such as buildings, trees, and hills, and to retain for detection or display the signals from moving targets such as aircrafts. Echo signals from fixed targets are normally not shifted in frequency, but the echo from a moving target is invariably shifted in frequency. Hence, in MTI radars, the doppler shift in frequency is used to discern moving targets even when the echo signal from fixed targets is orders of magnitude greater. Zverev [39] has shown the advantages of digital MTI filters over their analog counterparts.

The frequency response characteristic of MTI digital filters is given in [40,41] and shown in Figure 4.1a. This frequency characteristic indicates that, for uniform samples, the response of a MTI filters is zero whenever

$$f = \frac{n}{T} = n_{fs} \text{ where } n = 0,1,2.$$

and fs is the sampling frequency. Thus the uniformly sampled MTI filter not only eliminates the d.c. component caused by clutter targets, but unfortunately it also rejects any moving target whose doppler frequency happens to be $f = \frac{n}{T}$ or multiples thereof.

Those relative target velocities which result in zero MTI filter response are called blind velocities [39,40,41]. The presence of these blind velocities within the doppler frequency band reduces the detection capabilities of the radar system.

One solution to the problem caused by the uniform sampling is to nonuniformly sample the incoming bipolar video signal from the targets and then to repeat this nonuniform sampling pattern. The net result of such a sampling scheme can be construed as a superposition of a set of uniformly sampled radar pulse bursts, each set of bursts having a different sampling frequency. Roy and Lowenschuss [40], and

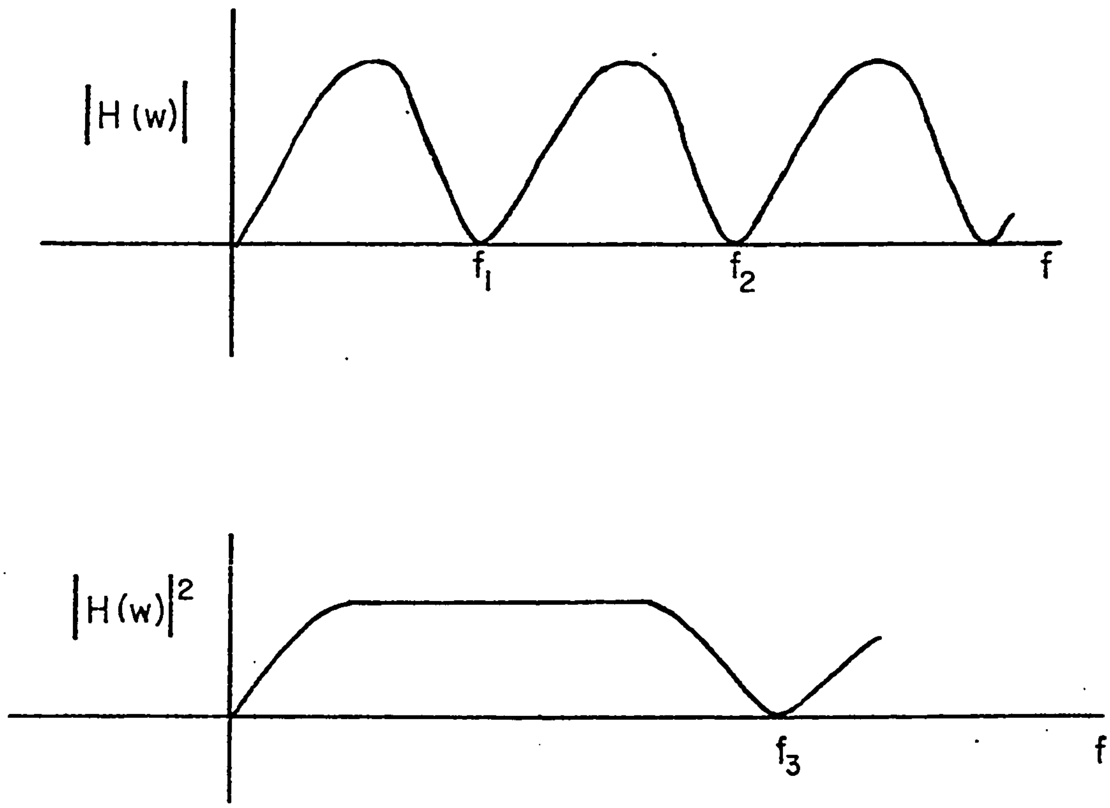


Figure 4.1

Frequency respone of a MTI filter.

(a) Uniformly sampled MTI filter response.

(b) Desired frequency response.

McAulay [42] have shown that the usable frequency range of a nonrecursive MTI filter can be extended so that the frequency characteristic will ideally approach that of Figure 4.1b by operating the filter with nonuniform sample periods. Prinsen [44], using computer search techniques in the time domain, considered the elimination of the blind velocities of nonrecursive MTI radar filter by optimal choice of the nonuniform sampling period and the optimal choice of the weighting of the filter coefficients. Haykin and Boulter [43] have considered the response of a recursive digital MTI filter and arrived at the conclusion, by direct simulation of the filter, that both the width of the transition from the passband to the stopband and the passband ripple of the filter can be substantially reduced by nonuniformly sampling the input signal and proper choice of the feedback multiplier coefficient of the filter.

The most common type of MTI digital filter is shown in Figure 4.2. The transfer function of such a filter can be given as

$$H(Z) = 1 - Z^{-1} \tag{4.1}$$

Roy and Lowenschuss [40] and McAulay [42] investigated the effect of nonuniform samples on MTI digital filter whose transfer function is as shown in (4.1). Since the overall purpose of the MTI filter is to detect the presence or absence of a target at any particular range, the transfer function of a nonuniformly sampled MTI digital filter is generally defined [40,41,42] as the ratio of the mean-squared output to the mean-square sinusoidal input. For $x(t_m) = \sin(wt_m+\theta)$, where θ is the phase angle uniformly distributed between $-\Pi$ and Π, then the output of such a filter is given by the difference between the present sample and the previous sample as

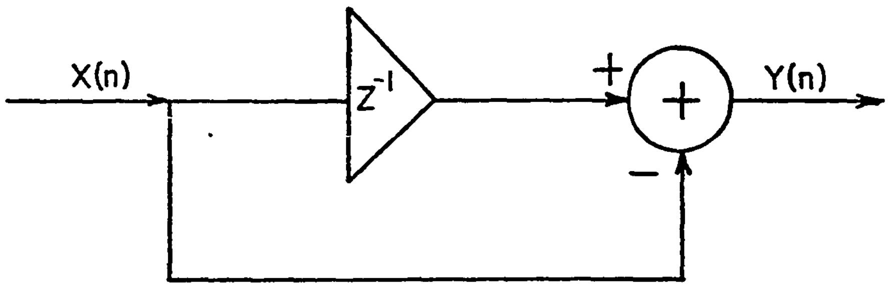


Figure 4.2

Nonrecursive MTI digital filter.

defined by (4.1). Assuming the nonuniform samples occur at periods $T_i(i=1,2.)$, Roy and Lowenschuss [40], using trigonometric approach showed that the Kth output of the filter is given as

$$y_K = \sin[\theta + w\sum_{i=1}^{K} T_i] - \sin[\theta + w\sum_{i=1}^{k-1} T_i] \quad (4.2)$$

The squared value of (4.2) is then reduced, after much algebra, to

$$y_k^2 = 1 - \cos wT_K - \frac{1}{2}\cos 2[\theta + w\sum_{i=1}^{K} T_i] -$$

$$\frac{1}{2}\cos 2[\ \theta + w\sum_{i=1}^{K-1} T_i] + \cos[2\theta + 2w\sum_{i=1}^{K-1} T_i + wT_K] \quad (4.3)$$

By summing all of the output pulses and averaging over all the pulses and all possible starting phase angles, Roy and Lowenschuss [40] showed that the mean square output using (4.3) is given as

$$E[y_K^2] = 1 - \frac{1}{N} \sum_{i=1}^{N} \cos wT_i \quad (4.4)$$

The terms associated with 2θ in (4.3) are the envelopes from which the maximum deviation from the mean square output can be calculated. Since the sinusoidal input has a mean-squared value of $\frac{1}{2}$, the unnormalized transfer function of the MTI filter with nonuniform samples is given [40] as

$$|H(w)|^2 = 2(1 - \frac{1}{N} \sum_{i=1}^{N} \cos wT_i) \quad (4.5)$$

It will now be shown that the result given in (4.5) can alternatively be obtained by using the decomposition approach developed in Chapter 2.

In general, the Kth output sequence for a filter with nonuniform samples can be written, using equation (2.12), as

$$Y_K(Z_n^n) = \sum_{j=0}^{n-1} X_j(Z_n^n)H_{K-j}(Z_n^n) \quad (4.6)$$

where k = 0, 1, 2, . . .

When the input to a nonuniformly sampled filter is specified, the periodic steady-state overall output sequence can be determined from (4.6). Thus for $x(t) = e^{+jwt}$ as the input, the advanced Z-transforms of the input sequences are

$$X_K(Z_n^n) = \frac{e^{jwt_k}}{1 - e^{jwT}Z_n^{-n}} \tag{4.7}$$

$k = 1, 2, \ldots, n-1.$

The periodic output sequences can be obtained by substituting (4.7) into (4.6) to yield

$$\begin{aligned} Y_K(Z_n^n) &= \sum_{m=0}^{n-1} X_m(Z_n^n)H_{K-m}(Z_n^n) \\ &= \sum_{m=1}^{n} \frac{e^{jwt_m}}{1 - e^{jwT}Z_n^{-n}} H_{K-m}(Z_n^n) \end{aligned} \tag{4.8}$$

where T is the overall sampling interval and t_m represent the nonuniform sampling instants. Inverting (4.8) yields the steady-state output sequence. Thus

$$Y_K(LT) = \frac{1}{2\pi j} \int_c \sum_{m=1}^{n} \frac{Z_n^n e^{jwtm}}{Z_n^n - e^{jwT}} H_{k-m}(Z_n^n)Z_n^{n(L-1)} dZ_n^n \tag{4.9}$$

Assuming all transients have died out, (4.9) can be evaluated by summation of the residues at the poles of the input function expression. Thus

$$\begin{aligned} Y_K(1T) &= \sum_{m=1}^{n} e^{jwt_m} H_{k-m}(e^{jwT}) e^{jwLT} \\ &= y_K(e^{jwT}) e^{jw1T} \end{aligned} \tag{4.10a}$$

where

$$y_K(e^{jwT}) = \sum_{m=1}^{n} e^{jwt_m} H_{k-m}(e^{jwT}) \tag{4.10b}$$

For a MTI with a sinusoidal input $x(t) = \sin(wt \pm \theta_K)$, the envelope of each periodic output sequence can be written, using (4.10b) as

$$\left| y_K(e^{jwT}) \right| \sin(wt + \theta_k) \tag{4.11}$$

where θ_k, k=0, 1, 2, . . ., denotes the various starting phase angles. The mean square value of all the output sequences over one complete cycle is

$$\frac{1}{2}\pi \int_0^{2\pi} \sum_{j=0}^{n-1} |y_K(e^{jwT})|^2 \sin^2(wt + \theta_K)d(wt)$$

$$= \sum_{j=0}^{n-1} |y_K(e^{jwT})|^2 \frac{1}{2\pi} \int_0^{2\pi} \sin^2(wt + \theta_K)\, d(wt)$$

$$= \frac{1}{2} \sum_{j=0}^{n-1} |y_K(e^{jwT})|^2 \tag{4.12}$$

The transfer function of a MTI digital filter with uniform samples is given, from (4.1) as

$$H(Z) = 1 - Z^{-1}$$

Hence $$H(Z_n) = 1 - Z_n^{-1} \tag{4.13}$$

Using the method presented in Appendix A-4 which is also applicable to nonrecursive transfer function, the transfer function sequences can be determined to yield

$$H_j(Z_n^n) = \begin{cases} 1 & j = 0 \\ -1 & j = 1 \\ 0 & j = -1, -2 \ . \ . \ .-(n-2), 2, 3, \ldots \\ -Z_n^{-n} & j = -(n-1) \end{cases} \tag{4.14}$$

The output sequences can then be determined from (4.6) and (4.14) as follows:

$$Y_o = X_oH_o + X_1H_{-1} + X_2H_{-2} + \ . \ . \ . + X_{n-1}H_{-(n-1)}$$

Simplifying yields

$$Y_o = X_oH_o + 0 + 0 + \ldots + X_{n-1}H_{-(n-1)} \tag{4.15}$$

Similarly

$$Y_1 = X_0H_1 + X_1H_0 + 0 + 0 + - - + 0.$$

$$Y_2 = 0 + X_1H_1 + X_2H_0 + 0 + 0 + - - + 0.$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$Y_{K-1} = 0 + 0 + - - + X_{n-2}H_1 + X_{n-1}H_0. \tag{4.16}$$

In (4.15) and (4.16), the suffix (Z_n^n) has been omitted for convenience.

Rewriting (4.15) and (4.16) in the time domain and normalizing the input to one yield:

$$\begin{aligned} y_0(e^{jwT}) &= 1 - e^{jwT_{n-1}}\, e^{-jwT} \\ &= 1 - e^{jw(T_{n-1} - T)} \\ &= 1 - e^{jw(T_{n-1} - T_n)} \end{aligned} \tag{4.17}$$

To simplify calculations, equations (2.2) and (2.3) are used in rewriting (4.17) to yield

$$y_0(e^{jwT}) = -(e^{-jwt_n} - 1) \tag{4.18}$$

Similarly

$$\begin{aligned} y_1(e^{jwT}) &= -1 + e^{jwT_1} = (e^{jwt_1} - 1) \\ y_2(e^{jwT}) &= -e^{jwT_1} + e^{jwT_2} \\ &= (e^{jw(T_2 - T_1)} - 1)e^{jwT_1} \\ &= (e^{jwt_2} - 1)e^{jwT_1} \\ &\vdots \\ y_{K-1}(e^{jwT}) &= e^{jwT_{n-1}} - e^{jwT_{n-2}} \\ &= (e^{jw(T_{n-1} - T_{n-2})} - 1)e^{jwT_{n-2}} \\ &= (e^{jwt_{n-1}} - 1)e^{jwT_{n-2}} \end{aligned} \tag{4.19}$$

where t_j, j=1,2... are the actual nonuniform sampling instants.

Since the output sequences $Y_K(Z_n^n)$ do not exist simultaneously, then the total mean square value of the output is the mean of all $\bar{y}_K^2$ in (4.18) and (4.19).

Using (4.12) along with (4.18) and (4.19), the mean square value of the output sequence (MSVO) will be given as:

$$\frac{1}{2}\sum_{k=0}^{n-1}|y_K(e^{jwT})|^2 = \frac{1}{2}\ \{|y_1(e^{jwT})|^2 + |y_2(e^{jwT})|^2 + \ldots + |y_{K-1}(e^{jwT})|^2\}$$

$$= \frac{1}{2}\sum_{k=0}^{n-1}|(e^{jwt_K} - 1)e^{jwT_{K-1}}|^2$$

$$= \frac{1}{2}\sum_{k=0}^{n-1}|(e^{jwt_K} - 1)|^2$$

$$= \frac{1}{2}\sum_{k=0}^{n-1}|(\cos wt_K - 1) + j\sin wt_K)|^2$$

$$= \frac{1}{2}\sum_{k=0}^{n-1}((2 - 2\cos wt_K)^{\frac{1}{2}})^2$$

$$\mathrm{MSVO} = \sum_{k=0}^{n-1}(1 - \cos wt_K) \qquad (4.20)$$

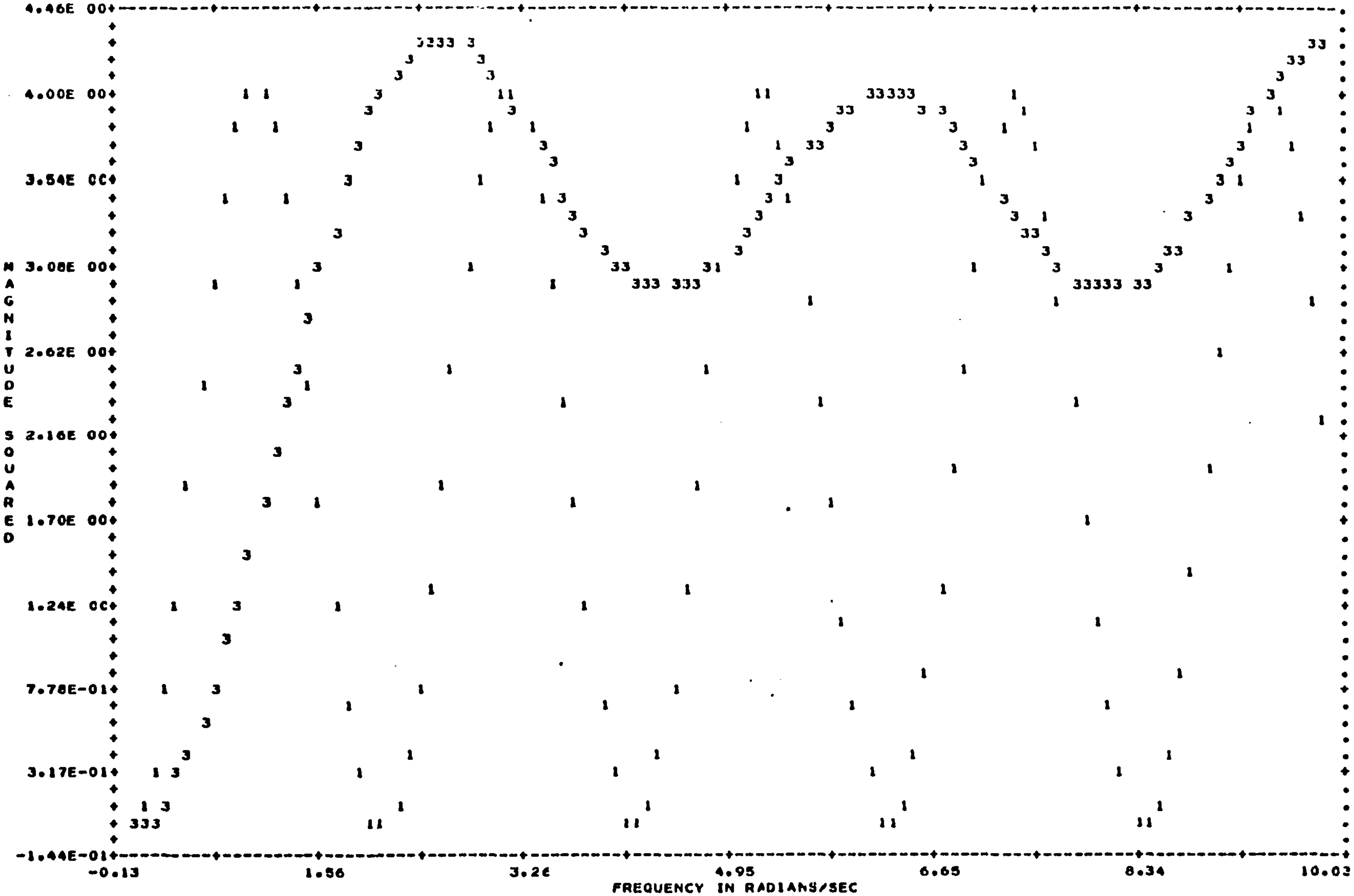
Equation (4.20) applies to all radar signal bursts to the MTI filter.

Since the mean squared of the sinusoidal input is $\frac{1}{2}$, then the unnormalized transfer function to MTI digital filter with nonuniform samples is

$$|H(e^{jwT})|^2 = 2(\sum_{k=0}^{n-1}(1 - \cos wt_K)) \qquad (4.21)$$

The result shown in equation (4.21) agrees with that obtained by Roy and Lowenschuss [40] using a different approach.

A computer plot of (4.21) for three intersamples per sampling period in the ratios 1:2:3 is shown in Figure 4.3 along with a plot for MTI filter uniformly sampled at f/fs = 3. Both filters have the same overall sampling frequency. From Figure 4.3, it is observed that at those frequencies for which the response of uniformly sampled MTI is zero, the response of the nonuniformly sampled MTI is not. Thus



1111 Uniformly Sampled

3333 Nonuniformly Sampled

Figure 4.3

Derived frequency response of a nonrecursive MTI filter.

the usable frequency range of the filter has been extended. This agrees with earlier observations by McAulay [42] and others [40,41, 43,44].

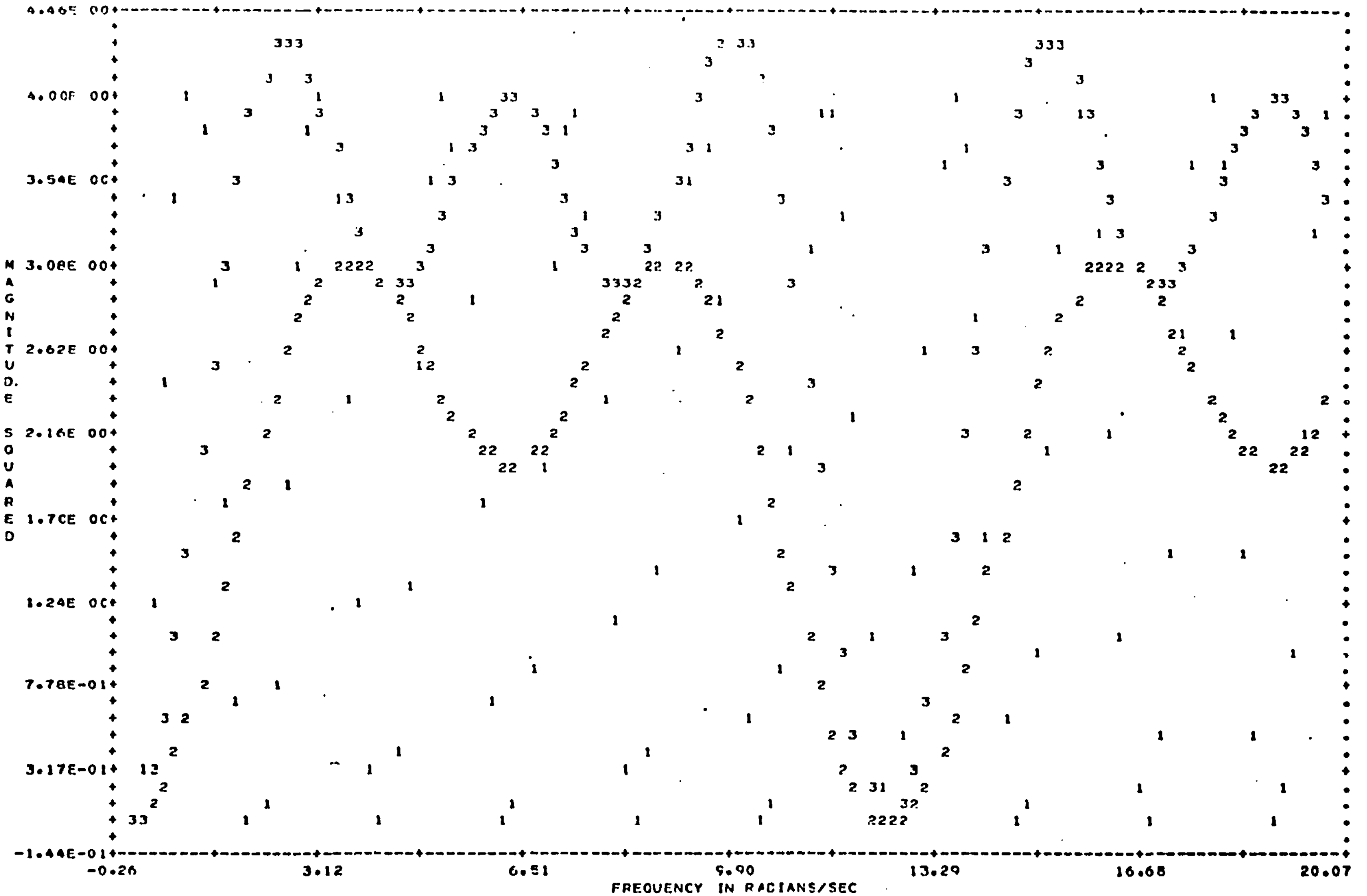
Figure 4.4 shows the plot of uniformly sampled MTI filter with sampling period $T_s/T = 3$ along with a plot for two intersamples per period in the ratio of 1:2 and three intersamples per period in the ratios of 1:2:3. In this figure, it is observed that for the nonuniformly sampled MTI, the number of ripples in the passband corresponds to the number of intersamples employed per overall sampling period. Also in the figure, the effect of starting transient is noticeable in the transition region of the response curve for the nonuniformly sampled MTI filter. The transient effects can be removed by ignoring the response to the first few samples. The plots in Figure 4.3 and Figure 4.4 agree with the results of earlier investigators [40,41, 42, 44].

4.2 Recursive MTI Digital Filter.

The recursive moving target indicator radar filter is as shown in Figure 4.5. The transfer function of such a filter can be shown as

$$H(Z) = \frac{1 - Z^{-1}}{1 - bZ^{-1}} \qquad (4.22)$$

The effect of nonuniform sampling in recursive MTI filters has always been considered by direct simulation of the filter since the manipulation of trigonometric approach becomes intractable at best. The decomposition approach is now shown to be applicable to the analysis of such filters.



1111 Uniformly Sampled
2222 Two intersampling
3333 Three intersampling

Figure 4.4

Frequency response for two and three-intersampled nonrecursive MTI filter.

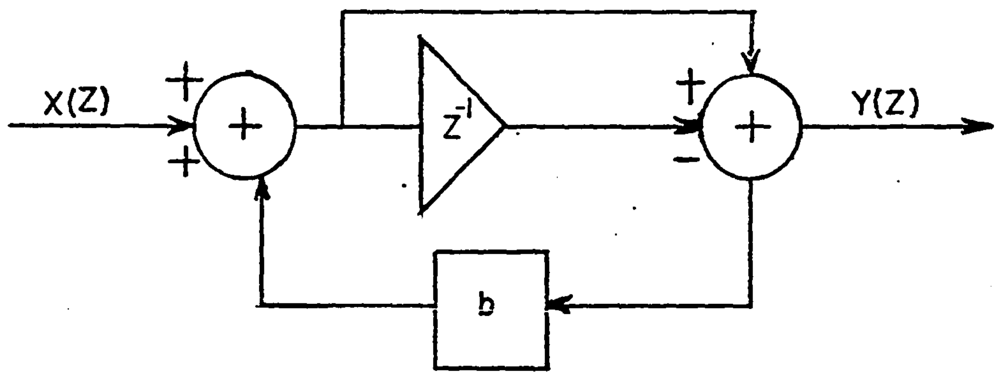


Figure 4.5

Recursive MTI digital filter

Let the transfer function sequences be obtained from (4.22) by use of the techniques of Chapter 2. For two intersamples per sampling period, the set of transfer function sequences can be shown to be

$$H_{-1}(Z_2^2) = \frac{Z_2^{-2}(b - 1)}{1 - b^2 Z_2^{-2}} \tag{4.23a}$$

$$H_o(Z_2^2) = \frac{1 - bZ_2^{-2}}{1 - b^2 Z_2^{-2}} \tag{4.23b}$$

$$H_1(Z_2^2) = \frac{(b - 1)}{1 - b^2 Z_2^{-2}} \tag{4.23c}$$

Using (4.6) and the set of equations in (4.23), the output sequences are

$$Y_o(Z_2^2) = X_o(Z_2^2)H_o(Z_2^2) + X_1(Z_2^2)H_{-1}(Z_2^2) \tag{4.24a}$$

$$Y_1(Z_2^2) = X_o(Z_2^2)H_1(Z_2^2) + X_1(Z_2^2)H_o(Z_2^2) \tag{4.24b}$$

Normalizing the input to one, the set of equations in (4.24) becomes

$$Y_o(Z_2^2) = H_o(Z_2^2) + e^{jwt_1}H_{-1}(Z_2^2) \tag{4.25a}$$

$$Y_1(Z_2^2) = H_1(Z_2^2) + e^{jwt_1}\, H_o(Z_2^2) \tag{4.25b}$$

Substituting the set of equations of (4.23) in (4.25) yields

$$Y_o(Z_2^2) = \frac{1 - bZ_2^{-2} + e^{jwt_1}Z_2^{-2}(b-1)}{1 - b^2 Z_2^{-2}} \tag{4.26a}$$

$$Y_1(Z_2^2) = \frac{e^{jwt_1}\,(1 - bZ_2^2) + (b - 1)}{1 - b^2 Z_2^{-2}} \tag{4.26b}$$

The output mean square value (OMV), using (4.12) becomes

$$\begin{aligned} \text{OMV} &= \tfrac{1}{2}\{\,|\,Y_1(e^{jwT})\,|^2 + |\,Y_2(e^{jwT})\,|^2\} \\ &= \tfrac{1}{2}\{\,|\,1 - be^{-jwT} + e^{jwt_1}\, e^{-jwT}\,(b-1)|^2 \\ &\quad + |\,(b-1) + e^{jwt_1}\,(1 - be^{-jwT})|^2\,\}\,/ \\ &\quad (|1 - b^2 e^{-jwT}|^2) \end{aligned}$$

$$= \frac{\frac{1}{2}\{|1-be^{-jwT}+(b-1)e^{-jw(T-t_1)}|^2 + |(b-1)+e^{jwt_1}-be^{-jw(T-t_1)}|^2\}}{|1 - b^2e^{-jwT}|^2}$$

$$= \frac{\frac{1}{2}\{|1-be^{-jwT}+(b-1)e^{-jwt_2}|^2 + |(b-1)+e^{jwt_1}-be^{-jwt_2}|^2\}}{|1 - b^2e^{-jwT}|^2} \qquad (4.27)$$

If the mean squared value of the sinusoidal input is normalized to one, then the transfer function of a recursive MTI filter with two intersamples per sampling period is given by equation (4.27).

Figure 4.6 shows a computer plot of equation (4.27) with intersampling ratio $\frac{t_1}{t_2} = 0.5$ and the filter coefficient $b = 0.8$, along with a recursive MTI filters having uniform samples spaced at $T = .12$ seconds.

Comparing Figure 4.6 with Figure 4.4 for nonrecursive MTI with two intersamples in the ratio 1:2, it is observed that:

a) the frequency response is smoother inside the passband

b) the low frequency transient has been reduced drastically without having to resort to ignoring the effect of the first few samples, and

c) the width of the passband ripples has been decreased.

These observations point to the fact that the proper choice of feedback coefficient in recursive MTI filter in conjunction with nonuniform sampling of the input bipolar video signal can result in significant improvement in its frequency response compared to nonrecursive MTI filter. Haykin and Boulter [43] arrived at similar conclusion by direct simulation of recursive MTI digital filter in a CDC 1700 digital computer. Also in Figure 4.6, it can be observed that the useful frequency range is extended before a blind speed is encountered by nonuniformly sampling scheme.

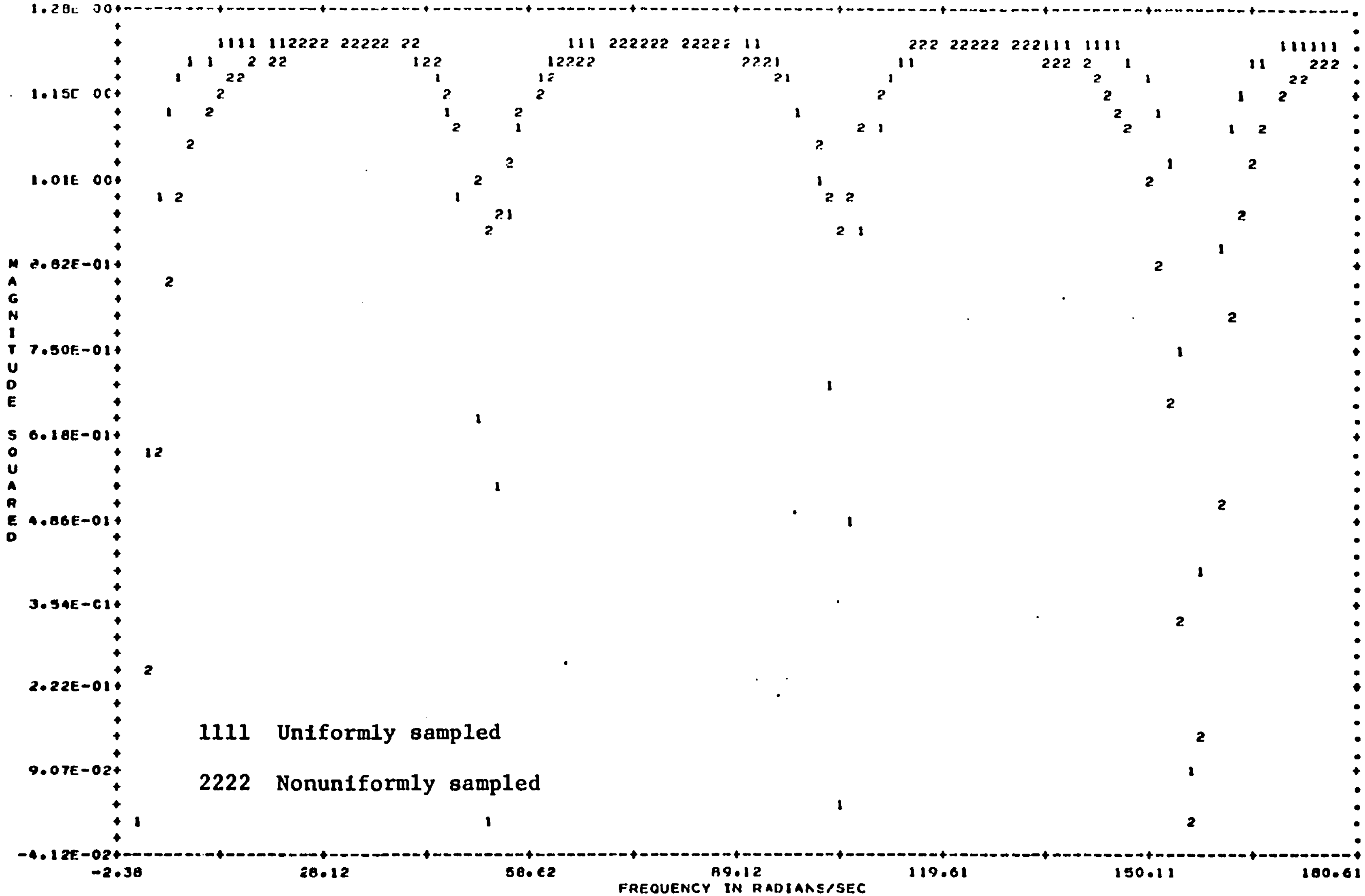


Figure 4.6

Derived frequency response of recursive MTI filter

4.3 Multiple-Shift Sequences in digital filters.

In conventional digital filters, data is shifted from register to register at a rate equal to the sampling frequency. However, the time between consecutive samples can be used to further process the stored information without need for additional memory. This process is inherent in multiple-shift digital filters. In multi-shift filters, the shifting of data is continued several times during each sampling interval at whch time signals stored in registers re-circulate in the internal paths of the filter. A block diagram of a multi-shift digital filter is shown in Figure 4.7.

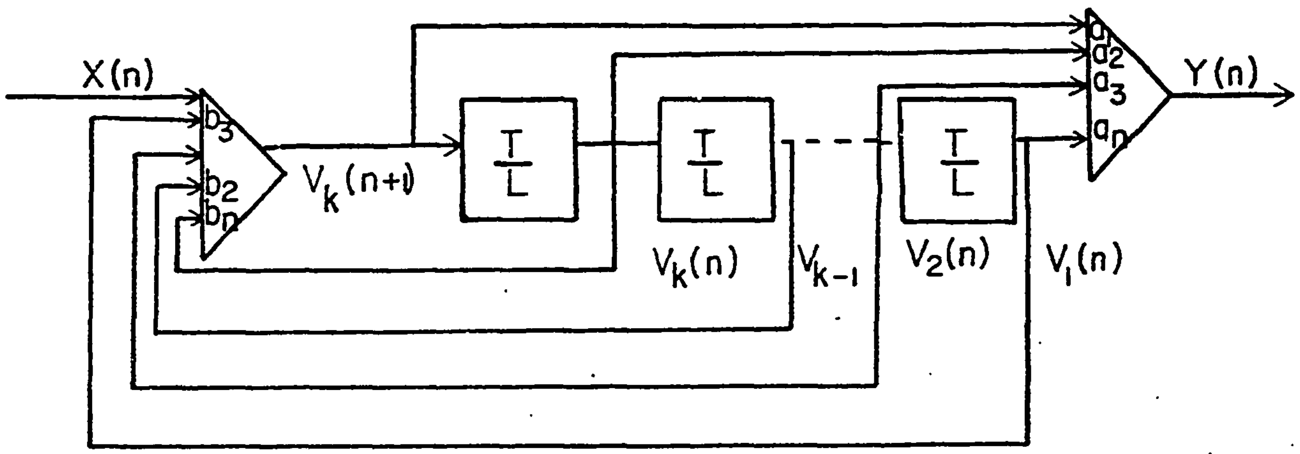
Fjallbrant [45] has derived the transfer function sequences of the filter as shown in Figure 4.7, using difference equations and the Z-transform. Atiya, Bilal and Abdou [46], Wong and King [47] have generalized the approach of obtaining the transfer function sequences of such filters by the use of state space techniques. When shifting occurs twice in each sampling interval, then the block diagram representation is as shown in Figure 4.7b. Fjallbrant [45] has shown that for two shifts per sampling interval, the following set of four euqations can be written:

$$Y(Z^{-1}) = V(Z^{-1}) - aY_1(Z^{-1})Z^{-\frac{1}{2}} - bY_1(Z^{-1})Z^{-1}$$

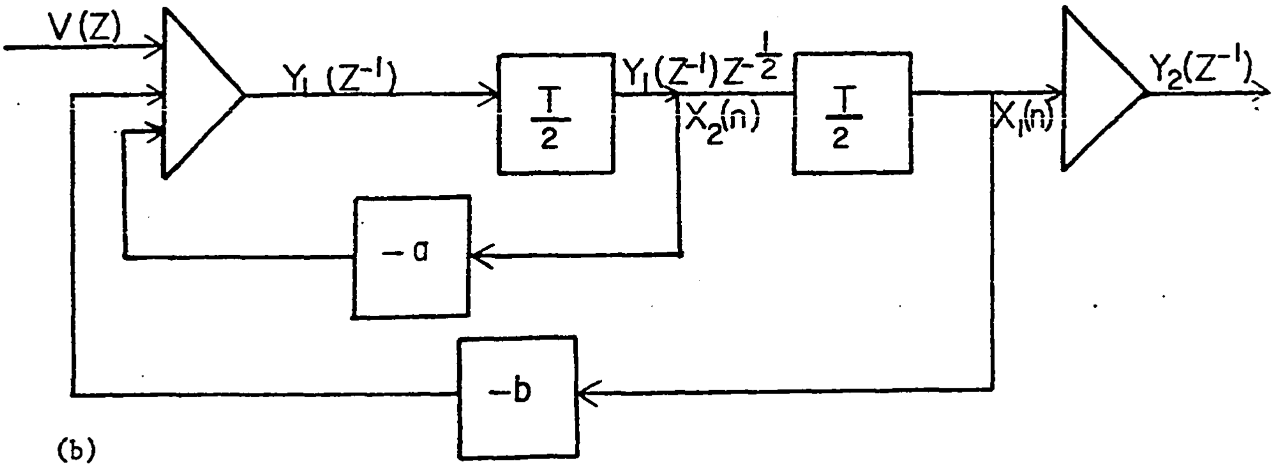
$$Y_2(Z^{-1}) = Y_1(Z^{-1})Z^{-1}$$

$$Y_1(Z^{-1})Z^{-\frac{1}{2}} = -aY_1(Z^{-1})Z^{-1} - bY_1(Z^{-1})Z^{-3/2}$$

$$Y_2(Z^{-1})Z^{-\frac{1}{2}} = Y_1(Z^{-1})Z^{-3/2} \qquad (4.28)$$



(a)



(b)

Figure 4.7

Multiple-shift digital filter.

(a) An nth order multiple-shift filter with L-shifts per period.

(b) Second order multi-shift filter with two shifts per period for a conventional filter with $H(Z) = Z^2/(Z^2 + aZ + b.)$

Solving the set of equation in (4.28) for the ratios $Y_2(Z^{-1})/V(Z^{-1})$ and $Y_2(Z^{-1})Z^{-\frac{1}{2}}/V(Z^{-1})$ Fjallbrant shows, after much algebra, that the applicable transfer function sequences can be written as

$$H_1(Z) = \frac{Y_2(Z^{-1})}{V(Z^{-1})} = \frac{Z + b}{Z^2 + Z(2b-a^2) + b^2} \tag{4.29}$$

$$H_2(Z) = \frac{Y_2(Z^{-1})Z^{-\frac{1}{2}}}{V(Z^{-1})} = \frac{-a}{Z^2 + Z(2b-a^2) + b^2} \tag{4.30}$$

Equations (4.29) and (4.30) hold when the filter coefficients do not vary with each shift sequence. When the filter coefficient vary with each shift sequence, it can be shown [45] that the applicable transfer function sequences can be written as

$$H_1(Z) = \frac{Z + b_2}{Z^2 + Z(b_1+b_2+a_1a_2) + b_1b_2} \tag{4.31}$$

$$H_2(Z) = \frac{-a_2}{Z^2 + Z(b_1+b_2 + a_1a_2) + b_1b_2} \tag{4.32}$$

Here a_1 and b_1 denote the values of the filter coefficients during the first sequence while a_2 and b_2 denote the values of the filter coefficients during the second shift, assuming that shifting is performed twice during each sampling interval.

It will now be shown that the decomposition approach can be successfully used in identifying the transfer function sequences in a multi-shift digital filter. Multiple-shifting in digital filters can be considered as a special case of nonuniform sampling.

In a nonuniformly sampled filter, the output sequences are obtainable from equation (4.6). Assuming two shifts per sampling period which correspond to two intersamples per period, equation (4.6) yields

$$Y_o(Z_2^2) = X_o(Z_2^2)H_o(Z_2^2) + X_1(Z_2^2)H_{-1}(Z_2^2) \text{ and}$$

$$Y_1(Z_2^2) = X_o(Z_2^2)H_1(Z_2^2) + X_1(Z_2^2)H_o(Z_2^2) \tag{4.33}$$

But in multi-shift digital filter, there are no inputs during the second and subsequent shifting of data. This means that $X_1(Z_2^2) = 0$. Thus for the case when filter coefficients do not vary with each shifting operation, equation (4.33) reduces to

$$Y_o(Z_2^2) = X_o(Z_2^2)H_o(Z_2^2) \text{ and}$$

$$Y_1(Z_2^2) = X_o(Z_2^2)H_1(Z_2^2) \tag{4.34}$$

The desired transfer function sequences are obtainable from (4.34) as

$$H_1(Z) = \frac{Y_o(Z)}{X_o(Z)} = Z^{-1}H_o(Z) \tag{4.35}$$

$$H_2(Z) = \frac{Y_1(Z)}{X_o(Z)} = Z^{-2}H_1(Z) \tag{4.36}$$

In equations (4.35) and (4.36), Z replaces Z_2^2 in (4.34) and the operator Z^{-1} locates the output sequences correctly in the overall time sequence.

For two intersamples per period, $H_o(Z_2^2)$ and $H_1(Z_2^2)$ are obtainable from Table 2.2 as

$$H_o(Z_2^2) = \frac{1 + bZ_2^{-2}}{1 + Z_2^{-2}(2b-a^2) + b^2Z_2^{-4}} \tag{4.37}$$

$$H_1(Z_2^2) = \frac{-a}{1 + Z_2^{-2}(2b-a^2) + b^2Z_2^{-4}} \tag{4.38}$$

Substituting the above in equations (4.35) and (4.36) after replacing Z_2^2 by Z, one obtains after simplification

$$H_1(Z) = \frac{Z + b}{Z^2 + Z(2b-a^2) + b} \tag{4.39}$$

$$H_2(Z) = \frac{-a}{Z^2 + Z(2b-a^2) + b} \tag{4.40}$$

Equations (4.39) and (4.40) are seen to be the same as equations (4.29) and (4.30) respectively. Equations (4.29) and (4.30) are obtained by Fjallbrant, using finite difference equation approach.

When the filter coefficients are allowed to change with each shift sequence, then equations (4.37) and (4.38) respectively become

$$H_o(Z_2^2) = \frac{1 + b_2 Z_2^{-2}}{1 + Z_2^{-2}(b_1+b_2-a_1a_2) + b_1b_2Z_2^{-4}}$$

$$H_1(Z_2^2) = \frac{-a_2}{1 + Z_2^{-2}(b_1+b_2-a_1a_2) + b_1b_2Z_2^{-4}}$$

Substituting these equations in (4.35) and (4.36) after replacing Z_2^2 by Z, the pulse transfer functions become

$$H_1(Z) = \frac{Z + b_2}{Z^2 + Z(b_1+b_2-a_1a_2) + b_1b_2Z^{-2}} \tag{4.41}$$

$$H_2(Z) = \frac{-a_2}{Z^2 + Z(b_1+b_2-a_1a_2) + b_1b_2Z^{-2}} \tag{4.42}$$

Equations (4.41) and (4.42) are respectively the same as (4.31) and (4.32) obtained by another approach.

When the number of shiftings per period is large, that is for N large, the difference equation method becomes complicated. To obtain a generalized solution, state space techniques as discussed in Chapter two of this dissertation can be used.

Thus equation (2.60) is

$$X(nT + T_1) = A_1X(nT) + BU(nT) \tag{4.43}$$

Since no signals enter the filter during the second and subsequent shiftings, equations (2.61) and (2.62) are respectively rewritten as

$$X(nT + T_1 + T_2) = A_2 X (nT + T_1) \tag{4.44}$$

$$X(nT + T_1 + T_2 + T_3) = A_3 X (nT + T_1 + T_2) \tag{4.45}$$

However, multi-shift filter can be regarded as a filter sampled nonuniformly at times $T_1 = T_2 = T_k = \frac{T}{N}$ where N is the number of shiftings per period. Hence all the transition matrices are equal. Thus $A_1 = A_2 = A_k = A$. Equations (4.44) and (4.45) can now be rewritten as

$$X(nT + T_1 + T_2) = A X (nT + T_1) \tag{4.46}$$

$$X(nT + T_1 + T_2 + T_3) = A X (nT + T_1 + T_2) \tag{4.47}$$

and in general

$$X(nT + NT_1) = A X (nT + (N-1)T_1) \tag{4.48}$$

Substituting (4.43) into (4.46) yields

$$\begin{aligned} X(nT + T_1 + T_2) &= A[A X (nT) + BU(nT)] \\ &= A^2 X (nT) + ABU(nT) \end{aligned} \tag{4.49}$$

Substituting (4.49) into (4.47) and so on yield, in general

$$X(nT + T) = A^N X (nT) + A^{N-1} BU (nT) \tag{4.50}$$

Taking the Z-transform of (4.50) and simplifying yields

$$X(Z) = [ZI - A^N]^{-1} A^{N-1} BU(Z) \tag{4.51}$$

Using the output equation of (2.59) yields

$$H(Z) = C[ZI - A^N]^{-1} A^{N-1} B \tag{4.52}$$

where I is the identity matrix.

Equation (4.52) agrees with the relation obtained by Atiya, Bilal and Abdon [46], Wong and King [47] for the multi-shift digital filters. Thus, the decomposition approach has been shown to be a viable alternative to the analysis of multiple-shift filters.

4.4 Signal-to-distortion ratio for system with inherent nonuniform samples.

The decomposition of nonuniform sampling into a superposition of uniform sampling is particularly useful when the spectrum of nonuniformly sampled signal is being sought. Consider, then a signal X(t) bandlimited to 1/2T Hertz which has a Fourier transform X(w). If this signal is nonuniformly impulse sampled, then the sampled signal can be written [10] as

$$X^*(t) = \sum_{n=-\infty}^{\infty} \sum_{i=1}^{K} X(t)\,\delta\,(t-(b_i+nK)T) \qquad (4.53)$$

where X*(t) is the sampled values for the given condition such that

$$0 \leq b_1 < b_2 < \ldots b_K < K.$$

The signal thus sampled as shown in (4.53) can be thought of as a superposition of uniformly sampled signal as given by

$$X^*(t) = \sum_{i=1}^{K} X_i^*(t)$$

$$X_i(t) = \sum_{n=-\infty}^{\infty} X(t)\,\delta\,(t-(b_i+nk)T) \qquad (4.54)$$

If the Fourier transform of $X_i^*(t)$ is taken to be $X_i^*(w)$ and X(w), the Fourier transform of X(t) is bandlimited, then the Fourier transform $X_i^*(w)$ of $X_i(t)$ can be found[I]. Assuming K is even, then

$$X_i^*(w) = \frac{1}{KT} \sum_{n=-\frac{K}{2}}^{K/2} \exp\frac{(-j2\pi b_i n)}{K}\; X\left(w-\frac{2\pi n}{KT}\right) \qquad (4.55)$$

where w is bandlimited to $\left(\frac{2\pi m}{KT}\ \frac{2(m+1)\pi}{KT}\right)$ m = -1 0 . . .

Using (4.55), then

$$X^*(w) = \sum_{i=1}^{K} X_i^*(w) = \frac{1}{T}\,X\,(w) + \frac{1}{KT}\sum_{\substack{n=-K/2 \\ n\neq 0}}^{K/2}\left(\sum_{i=1}^{K}\exp\frac{(-j2\pi b_i n)}{K}\; X\left(w-\frac{2\pi n}{KT}\right)\right) \qquad (4.56)$$

[I] See, for example, Freeman [10], p. 78.

The term under the summation sign causes the occurrence of aliases of X(w) and because the sampling rate is $\frac{1}{K}$ times the Nygnist rate these aliases overlap in frequency, giving rise to distortion. Consequently, a digital filter operated with nonuniform sample periods will be restricted to the input frequency range fo/n where fo is the maximum frequency that could be handled by the same filter operating with a uniform sample period T/n where n is the number of intersamples per sampling interval.

Typical system where nonuniform sampling of (4.53) are often encountered include the code format conversion system as described by Goodman [48], and systems that include the pulse code modulated channel banks. For example, available pulse code modulated channel banks for telephone communication systems which has been designed with an 8-kHZ sampling rate that can accomodate about 3.5kHZ bandwidth of voice channels, may be needed for the transmission of signals of greater bandwidth such as television audio, or studio-to-transmitter links. Such channels generally require bandwidths of from 5 to 15 kHZ as shown by Flood and Hoskins [49]. In such a case, the existing channel would have to be modified to allow the transmission of the signals of greater bandwidth.

A 24-channel bank is illustrated in Figure 4.8a [49]. Each channel of the 24-channel bank has 8 bits and the 24-channel bank is followed by a single framing bit for a total of 193 bits in a certain time frame. Increasing the sampling rate by a factor of K, where K divides 24 evenly, can be achieved by feeding the signal in parallel into K channels symmetrically located in the frame. The case of K=2 is illustrated in Figure 4.8b.

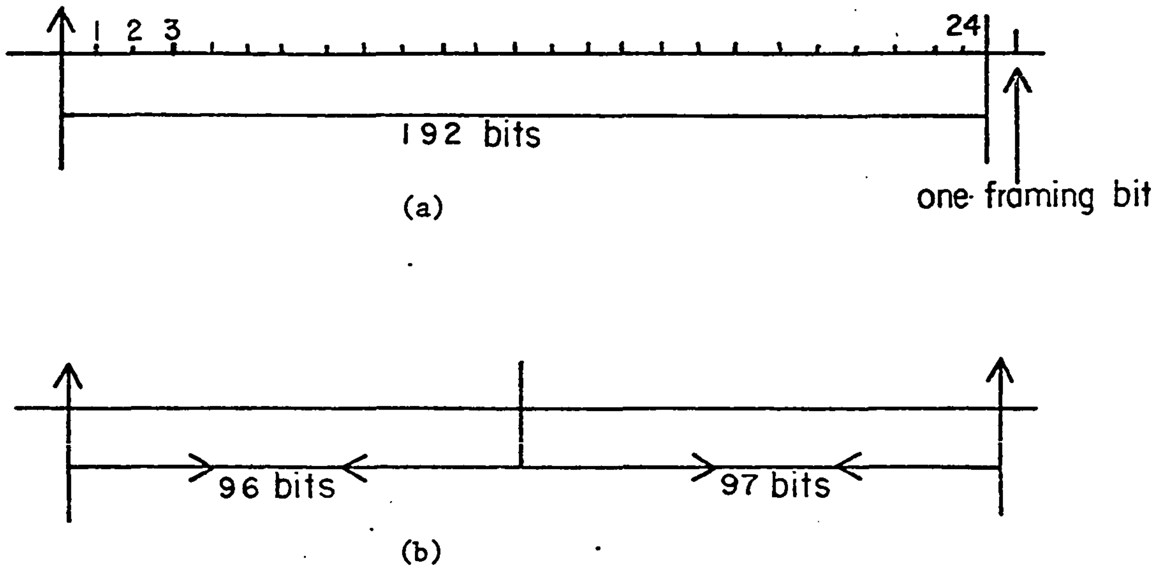


Figure 4.8

A 24-channel Bank.

(a) Frame format for 192 bits plus 1 bit and 8 bits per channel. Sampling rate is 8KHZ.

(b) Double sampling rate (K=2) sampling rate is 16KHZ.

Because of the extra framing bit, the sampling is no longer uniform since the two samples containing the framing bit are separated by more time frame than other adjacent samples. If for example, the 193 bits are in a 120μ sec frame, then the two sampling containing the framing bit are separated by 120/193 = 0.622μ more than the other adjacent samples since KT = 120μ sec. If the signal is reconstructed as if it were uniformly sampled, then some distortion results.

The distortion resulting from passing the nonuniformly sampled signal through a filter, as if it were uniformly sampled can be considered in general terms for the case when the distortion is considered small. The error spectrum resulting when the signal sampled as in (4.53) is applied to a given filter can be written, using (4.56) as

$$TX^*(w) - X(w) = \{\frac{1}{K}\sum_{\substack{n=-K/2 \\ n\neq 0}}^{K/2}\sum_{i=1}^{K}\exp(-j2\pi b_i n/K)\; X\,(w-\frac{2\pi n}{KT})\} \qquad (4.57)$$

Let the signal-to-distortion ratio (S/D) be defined as

$$S/D = \frac{\frac{1}{2\pi}\int_0^{2\pi/T}|X(w)|^2\,dw}{\frac{1}{2\pi}\int_0^{2\pi/T}|TX^*(w)-X(w)|^2\,dw} \qquad (4.58)$$

It is shown in Appendix B that for any arbitrary input, equation (4.58) yields for K = 2

$$S/D = \frac{4}{\sum_{j=1}^{2}\sum_{j=1}^{2}\exp(-j\pi(bj-bj))} \qquad (4.59)$$

Equation (4.59) is seen to be a function of the nonuniform sampling instants only but not of the input signal. However, for K > 2 but even, the signal-to-distortion ratio, as can be seen from equation (4.57), becomes a function of both the input signal and nonuniform sampling instants. Evaluation is then dependent on the type of input sequence.

Equation (4.59) is plotted in Figure 4.9 by fixing b_1 and varying b_2, the other nonuniform instant. From this plot, it is seen that more distortion results as the separation between b_1 and b_2 is increased for the system nonuniformly sampled as in equation (4.53). This result is intuitively expected.

4.5 Summary

Applications have been presented of the nonuniform sampling decomposition technique as developed. First, the effect of nonuniform samples in nonrecursive moving target indicator filters in radar systems may be appropriately investigated using the approach presented. It is shown that the transfer function of such filters can alternatively be derived by the techniques of this work.

Next, the decomposition approach is extended to the analysis of the recursive MTI filter. This work has provided an analytical tool for investigation of the effect of nonuniform samples and the feedback coefficients which has been previously limited to simulation. The plot from an example using the derived result demonstrates that recursive MTI filter has significant frequency response improvement when compared to the nonrecursive MTI filter. Haykin and Boulter [43] arrived at similar result by direct simulation of the filter.

Secondly, digital filters with multi-shift sequences are considered. By using the results from Chapter 2, it is shown that multiple-shifting in digital filter may be treated as a special case of nonuniform sampling. The transfer function sequences are re-derived with results in agreement with other works [45,46,47]. These transfer

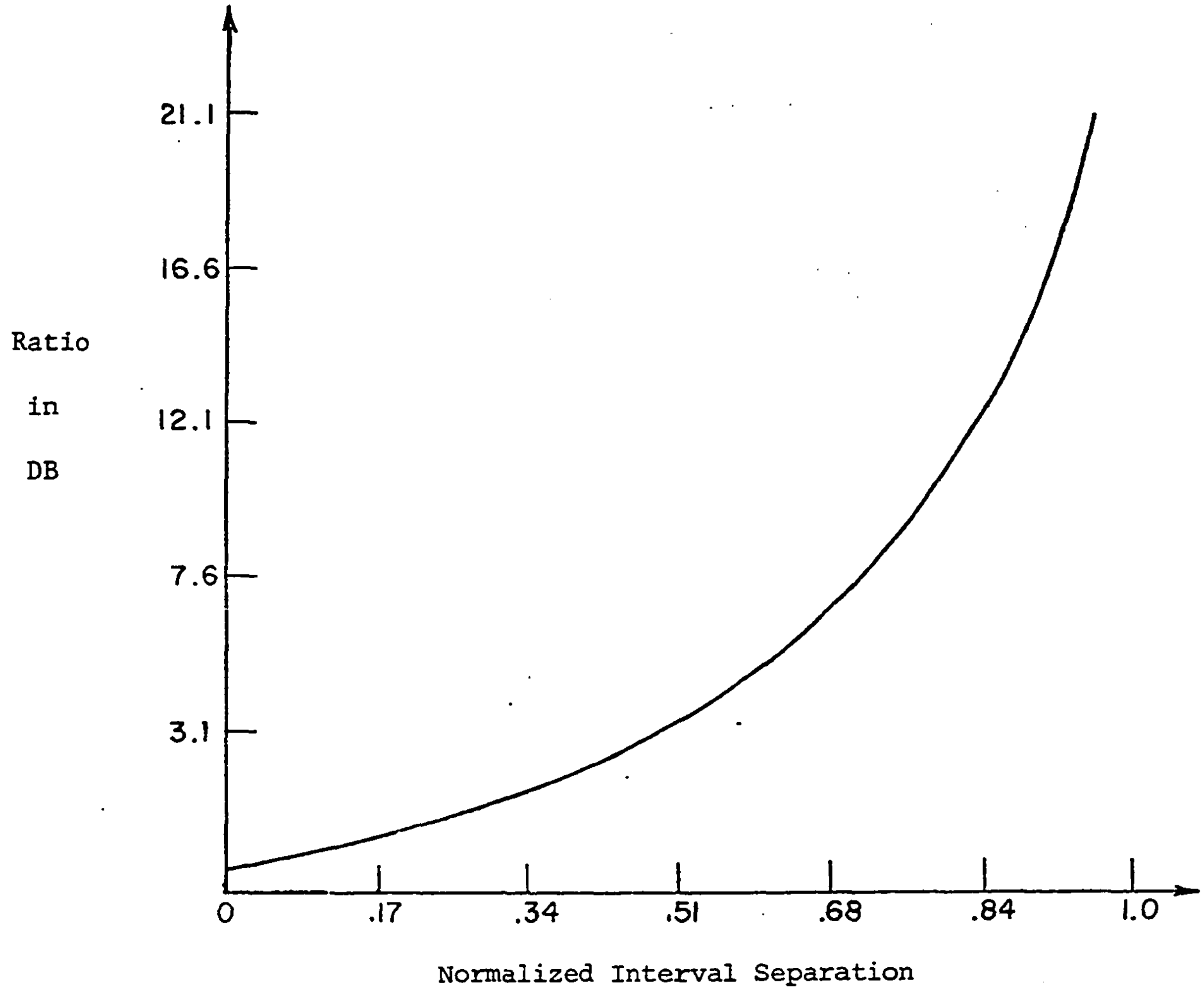


Figure 4.9

Signal-to-distortion Ratio.

function sequences are suitable for the realization of variable digital filters and for the reduction of coefficient accuracy requirements in the second and higher order filters [45,47].

Thirdly, Fourier transform of a signal sampled nonuniformly is used in the investigation of the distortion that arises as a result of processing a nonuniformly sampled signal as if it were uniformly sampled. An example of a 24-channel bank [49] is used in showning that for two intersamples per period, more distortion results as the separation between the nonuniform samples is increased within the system considered. Also it is shown that a digital filter operated with nonuniform sample periods will be restricted, to avoid aliasing effects, to a much smaller frequency range when compared to the frequency range in the same filter with uniform samples.

Chapter 5

CONCLUSIONS

This work has detailed a new analytical approach for the analysis of a nonuniformly sampled recursive digital filter, based on $\mathcal{Z}$-transforms along with the time-domain switch decomposition, the Fourier transform and subsequently the state-space technique. First and second order filters have been emphasized. Consequently the transfer function sequence operator of such filters have been derived as new results. The characteristic properties of the derived transfer sequence operator were presented. It was demonstrated that the order of the pulse transfer function of a conventional filter subjected to nonuniform sampling is not changed, but rather that additional transmission zeroes were introduced as a result of the nonuniform sampling.

Using the derived transfer sequence operator, the stability constraints, quantization effects and frequency characterists of nonuniformly sampled first and second order filters were investigated. The analysis of such filter characteristics thus far in the literature has been limited to filters sampled uniformly or slight modifications thereof. In considering the quantization effects due to uncorrelated noise, the steady-state values of the output noise variance normalized with respect to the input noise variance were computed using an analysis computer program. It was shown through graphical displays of the computer results that the output noise variance is approximately the same for each transfer sequence operator for a given number of intersamples. It was also shown that for two intersamples per period, second order nonuniformly sampled filters have the same general type of noise characteristic as second order uniformly sampled filters having comparable eigenvalues. It was further shown that the input quantization noise of high Q nonuniformly sampled filter is significantly higher than that of uniformly sampled filter having approximate eigenvalues.

Limit cycle oscillations result in filters when quantizations errors are due to correlated noise spectrum. The decomposition technique developed in Chapter 2 has been extended to the analysis of limit cycles of nonuniformly sampled recursive filters. Such limit cycles in nonuniformly sampled filters were simulated in the IBM S/370-158 digital computer. By neglecting the influence of zeroes in the transfer sequence operator, it was shown from the simulation results that limit cycle oscillations in uniformly and nonuniformly sampled filters have identical properties such as symmetry. However, the influence of zeroes will most likely result in a reduction of the limit cycle as has been suggested by Parker and Hess [27] in their analysis of limit cycles in uniformly sampled filters.

Since in applications, it may be necessary to have the filter realization match certain desired design characteristics, a new analytic approach has been presented for determining the frequency and phase characteristics of nonuniformly sampled filters. The approach provides a means of describing the magnitude and phase response of nonuniformly discrete-systems from the pulse transfer sequence operator. From the graphical displays of the derived magnitude response, it was demonstrated that uniformly and nonuniformly sampled recursive filters with approximate eigenvalues exhibit the same critical frequencies. It was also shown that in general, nonuniformly sampled filters exhibit higher gain at critical frequencies than uniformly sampled filters with approximate eigenvalues. However, this is expected since nonuniform sampling generally introduces zeroes to the pulse transfer function sequences. It was further shown that the phase characteristics of recursive filters were not improved by nonuniform sampling and that the phase response tend to be nonlinear. The same statement can be made for uniformly sampled filters as has been shown by Kaiser [1] and others [3, 6, 9]. The results thus advanced provide venues of comparison between uniformly and nonuniformly sampled recursive filters. It should be noted that even though fixed point arithmetic computation has been assumed in describing stability and quantization characteristics of filters sampled nonuniformly, much of the analysis could easily be adapted to floating-point filter algorithm.

Application of the nonuniform sampling decomposition technique as developed in this work has been extended to the analysis of radar filters (MTI) employed for detection of moving target from the clutter background. Such filters, for greater detectability and efficiency, use nonuniform sampling. It has been shown that the output of nonuniformly sampled MTI filters could be obtained directly from the pulse transfer functions of such filters, regardless of whether the transfer functions describe recursive or nonrecursive MTI filters. This then has provided greater flexibility in the analysis and design of nonuniformly sampled MTI filters since thus far in the literature the analysis of nonuniformly sampled recursive MTI filters has been by direct simulation. In addition, it has been demonstrated that multiple-shifting in digital filters could be treated as a special case of nonuniform sampling. Multi-shift sequences in digital filters have the form of transfer function sequences that are suitable for the implementation of variable digital filters and for the reduction of the requirements for filter coefficient accuracy [45, 47].

Employing the Fourier transform formulation, it has been shown in this work that for a system with inherent nonuniform samples, distortion often results when nonuniformly sampled signal is processed as it were uniformly sampled. From the design concept and from the discussion on the signal-to-distortion ratio in a system with inherent nonuniform samples, it has been demonstrated that a limitation of nonuniform sampling in digital filter designed to operate on uniform samples is that the effective Nyquist frequency range is reduced. However, within this limited frequency range, the introduction of zeroes into the transfer function operator by nonuniform sampling can provide additional control of the frequency characteristics. The introduction of additional zeroes may not necessarily require additional multiplier since all the numerator coefficients also are present in the denominator of the transfer function operator.

The limitations and the applications of the analytic techniques employed in this and other works are summarized in Table 5.1. This table exhibits the scope of this work in relation to other works in the literature on digital filters with nonuniform or multirate samples. Each work listed contains a summary of the analytical techniques employed, constraints on the analytic approach, the sampling scheme considered, the specific application and problems considered in the specific application.

There remain some unanswered problems which have arisen as a result of this work and should be noted for further investigation.

	Limitations		Filter Application				
Work	Analytical Tools & Sampling Scheme Used	Other	Non-recursive MTI filter analysis	Recursive MTI filter analysis	Multi-shift sequences filter	Quantization effects	Sensitivity and frequency analysis
Atiya et al [46,47,52]	State space Z-transforms multirate sampling	state transition matrix available. No signal input after first sampled value.	-	-	Yes	Yes	Sensitivity only
Fjallbrant [45]	Z-transform finite difference equation multirate sampling	no signal input in second and subsequent processing	-	-	Yes variable filter considered	-	-
Ragazzini and Franklin [8] Tou [14]	Z-transform switch decomposition multirate scheme	system function H(s) available	-	-	-	For multirate system	-
Haykin and Boulter [43]	Nonuniform sampling in time domain	Analogue-digital filter transformation	by simulation	by simulation	-	-	frequency by simulation
Jury [7,16]	Z-transform multirate nonuniform scheme	nonuniform envelope of function available	-	-	-	Limit cycles in closed-loop systems	Frequency in multirate system
McAulay [42,44]	Nonuniform sampling Fourier Transform	Fourier transform of input signal available	Yes	-	-	-	Frequency only
Roy and Lowenschuss [40]	Nonuniform sampling difference equation with trignometric identity	Finite difference equation available	Yes	-	-	-	Frequency only
This work	Nonuniform sampling switch decomposition Z-transform state space, Fourier transform	System H(Z) available input and output nonuniformly sampled synchronously	Yes	Yes	Yes	Statistical analysis and limit cycle oscillations	Frequency only

Table 5.1.

Summary of Scope and Limitations of This and Other Works in the Literature.

Among them are the following:

a) Optimization of both the intersample periods and the feedback coefficients in recursive MTI digital filters.

b) An investigation of instability thresholds in terms of the bandwidth of digital filters sampled nonuniformly, using different arithmetic algorithms.

c) Generalization of the steady-state noise errors encompassing higher number of intersamples per period along with sensitivity considerations in filters sampled nonuniformly.

d) An exact analytical expression for the frequency of the limit cycle generated in the presence of a driving function.

e) Optimum choice of intersamples to achieve a given design criteria.

f) A general relationship between the overall sampling period T and the number of significant digits required to represent data if the data is to approximate a continuous signal in some specified sense.

Some of these problems are problems yet to be solved even for filters subjected to uniform samples.

APPENDIX A-1

Nonuniform Sampling Theorems. [10, 11, 12].

Suppose a bandlimtied function f(t) is uniformly sampled so that a finite set of samples f(kT) is generated, where k ranges over all integers. If a total of 1 samples are eliminated and replaced by an arbitrarily spaced samples f(tp) where $^{t}P/T$ is not an integer, then the following theorem due to Yen [11] is applicable.

Theorem A-1.1: A function {f(t)}, which has a Fourier spectrum F(jw) such that F(jw) = 0 for $|w| > \frac{W}{2}$, is uniquely described by a knowledge of its values at nonuniformly spaced instants but satisfy the condition that there be precisely K distinct samples to every interval of length KT where K is some finitie integer and T is the sampling interval.

This theorem is proved by Yen [11], Helms [12] and Freeman [10]. The theorem, however is not valid when the number of shifted uniform sample values increases without bound. For this case, a meaningful interpolation function can be developed for reconstruction if it is possible to shift half the uniform sample values by an equal distance with respect to the rest of the sample values.

Another type of nonuniform sampling process is the recurrent nonuniform sampling. In this case, a bandlimited function {f(t)} has its values given at $t = t_{nk} = nT + t_k$ where $k = 0,1,\ldots,N$ and n is a finite integer. The spacing between samples for which maximum $(t_k) < T$ is thus nonuniform but with a pattern periodic with period

T. For periodic nonuniform sampling, the following theorem due to Yen [11] is valid.

Theorem A-1.2: A function $\{f(t)\}$, which has a Fourier spectrum $F(jw)$ such that $F(jw) = 0$ for $|w| > \frac{W}{2}$, is uniquely described by its values at a set of periodic nonuniform sample points $t_{nk} = nT + t_k$ where $k = 1, 2, \ldots, N$ and $n = -1, 0, 2, \ldots$. All sampling instants t_k are distinct and (maximum t_k) $< T$.

This theorem has been proved by Freeman [10] and Yen [11]. The interpolation formula which generates the original f(t) from its nonuniform samples is given by [10, 11] as

$$f(t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{N} f(t_{nk}) \frac{\prod_{i=0}^{N} \sin \frac{W(t-nT-t_i)}{2}}{\frac{W(t-nT-t_k)}{2} \prod_{\substack{i=0 \\ i \neq k}}^{N} \frac{\sin W(t_k-t_i)}{2}} \qquad \text{(A.1)}$$

As can be readily appreciated from this interpolation formula, the generation of f(t) from its periodic nonuniformly sampled sequence is a fairly complex task and is reserved mainly for theoretical considerations.

APPENDIX A-2

Advanced Z-transform [7, 8, 16]

Advanced Z-transform gives the Z-transform of pulse sequences which are derived from time functions delayed by nonintegral multiplies of the sampling frequency.

If x(t) is a continuous function and is delayed or advanced by a time mT, where m is nonintegral, then x(t) is sampled at times (n+m)T, so that

$$X^*(t,m) = \sum_{n=0}^{\infty} x(nT + mT)\ \delta\ (t - nT) \qquad \text{(A.2a)}$$

Using the definition of the Z-transform as given by Derusso, Roy and Close [50], the expression for the Z-transform of equation (A·2a) can be stated as

$$X(Z,\ m) = \sum_{n=0}^{\infty} X\ (nT + mT)Z^{-n} \qquad \text{(A·2b)}$$

Many advanced Z-transforms of time sequences can be evaluated by direct use of (A·2b). It should be noted that if m is an integer, the result of the Z-transform of (A·2a) is trivial since the Z-transform of the resulting function is simply $Z^{m}X(Z)$, assuming zero initial conditions.

APPENDIX A-3

Skip Sampling Theorem [8,16]

This theorem shows the relationship between the Z-transform of a time function sampled at T/n units apart and the same time function sampled at T units apart. To show this relationship, let the Z-transform of a time function, X(t), sampled at T/n units apart be denoted by $X(Z_n)$ where n is any integer greater than one. Also denote the Z-transform of X(t) sampled at T units apart by X(Z). Then by the definition of the Z-transform,

$$X(Z) = \sum_{N=0}^{\infty} X\,(NT)Z^{-N} \tag{A.3}$$

Similarly

$$X(Z_n) = \sum_{k=0}^{\infty} X\,(k\,\frac{T}{n})Z_n^{-k} \tag{A.4}$$

Applying the Z-transform inversion theorem to (A.4) yields

$$X(k\,\frac{T}{n}) = \frac{1}{2\pi j} \int_c X\,(Z_n)Z_n^{k-1}dZ_n \tag{A.5}$$

Substituting (A.5) into (A.3) with k = nN yields

$$X(Z) = \sum_{N=0}^{\infty} \left[\frac{1}{2\pi j}\int_c X\,(Z_n)Z_n^{nN-1}dZ_n\right]Z^{-N} \tag{A.6}$$

Summation and integration signs in (A.6) can be interchanged if $\sum_{N=0}^{\infty} (Z_n^{nN-1}Z^{-N})$ is absolutely convergent and provided that $|Z_n^nZ^{-1}| < 1$. Equation (A.6) then yields

$$X(Z) = \frac{1}{2\pi j} \int_c X\ (Z_n)[\sum_{N=0}^{\infty} Z_n^{Nn-1} Z^{-N}]dZ_n$$

$$= \frac{1}{2\pi j} \int_c X\ (Z_n) \frac{Z_n^{-1}}{1 - Z_n^n Z^{-1}} dZ_n \tag{A.7}$$

The integral in (A.7) can be evaluated by obtaining the residues at the singularities of $X(Z_n)Z_n^{-1}$ or by evaluating the residues at the singularities of $(1 - Z_n^n Z^{-1})$ which are outside the contour C.

APPENDIX A-4

Modification of Overall Transfer Sequence Operator

The pulse transfer function sequences $H(Z_n^n)$ can be obtained from the overall transfer sequence $H(Z_n)$ when there are only two nonuniform samples in a period. Let the overall transfer function

$$H(Z_2) = \frac{N(Z_2)}{D(Z_2)}\,\frac{B(Z_2)}{B(Z_2)} = \frac{N(Z_2)B(Z_2)}{F(Z_2^2)}$$

where $F(Z_2^2) = D(Z_2)B(Z_2)$. Each pulse transfer sequence $H_K(Z_2^2)$, $K=0, \pm 1, \pm 2, \ldots$ will then have $F(Z_2^2)$ as a common denominator. $F(Z_2^2)$ written in factored form becomes

$$F(Z_2^2) = \prod_{K=1}^{m} (Z_n^n - C_K^n) \tag{A.8a}$$

where C_K and m are constants to be determined. The factors $(Z_n^n - C_K^n)$ can also be written as follows,

$$Z_n = C_K \exp(\frac{j2\pi l}{n}),\quad l = 0, 1, \ldots (n-1) \tag{A.8b}$$

Using (A.8b), (A.8a) becomes

$$F(Z_2^2) = \prod_{K=1}^{m} \prod_{l=0}^{n-1} (Z_n - C_K \exp(j\frac{2\pi l}{n}))$$

$$= \prod_{K=1}^{m} \prod_{l=0}^{n-1} [\exp(j\frac{2\pi l}{n})\{Z_n \exp(-j\frac{2\pi l}{n}) - C_K\}]$$

$$= \prod_{K=1}^{m} \prod_{l=0}^{n-1} [Z_n \exp(-j\frac{2\pi l}{n}) - C_K] \tag{A.9}$$

Also $D(Z_n)$ can be factored to yield

$$D(Z_n) = \prod_{K=1}^{M} (Z_n - b_K) \tag{A.10}$$

Now since $F(Z_2^2) = B(Z_2)D(Z_2)$ then

$$B(Z_2) = \frac{F(Z_2^2)}{D(Z_2)}$$

Using (A.9) and (A.10)

$$B(Z_2) = \frac{\prod_{K=1}^{m} \prod_{1=0}^{n-1} [Z_n \exp(-j\frac{2\pi 1}{n}) - C_K]}{\prod_{k=1}^{M} (Z_n - b_K)} \tag{A.11}$$

Equation (A.11) can be furhter simplified if M=m and $C_K=b_K$. Then (A.11) is rewritten as

$$B(Z_2) = \frac{\prod_{k=1}^{m} \prod_{1=1}^{n-1} [Z_n \exp(-j\frac{2\pi 1}{n}) - C_K] \prod_{K=1}^{M} (Z_n - C_K)}{\prod_{K=1}^{M} (Z_n - C_K)}$$

$$= \prod_{K=1}^{m} \prod_{1=1}^{n-1} [Z_n \exp(-j\frac{2\pi 1}{n}) - C_K] \tag{A.12}$$

Using (A.10), equation (A.12) becomes

$$B(Z_2) = \prod_{1=1}^{n-1} D[Z_n \exp(-j\frac{2\pi 1}{n})] \tag{A.13}$$

where

1=0, 1, ..., (n-1).

APPENDIX B

Derivation of Signal-to-Distortion Energy Function

Using equation (4.57), the total distortion energy spectrum can be given as

$$\frac{1}{2\pi}\int_0^{\frac{2\pi}{T}} |TX^*(w) - X(W)|^2\, dw = \frac{1}{\Pi}\int_{\frac{-2\pi}{KT}}^{\frac{2\pi}{KT}} \left|\frac{1}{K}\sum_{\substack{n=-K/2\\ n\neq 0}}^{K/2}\left(\sum_{i=1}^{K}\exp\left(-j\frac{2\pi n b_i}{K}\right)\right) X\left(w-\frac{2\pi n}{KT}\right)\right|^2 dw$$

$$= \frac{1}{K^2\Pi}\int_{\frac{-2\Pi}{KT}}^{\frac{2\Pi}{KT}} \left(X\left(w-\frac{2\pi n}{KT}\right)\overline{X\left(w-\frac{2\pi r}{KT}\right)}\right)dw$$

$$\left[\sum_{\substack{n=-K/2\\ n\neq 0}}^{K/2}\ \sum_{\substack{m=-K/2\\ m\neq 0}}^{K/2}\left(\sum_{i=1}^{K}\ \sum_{i=1}^{K}\exp\left(-j2\pi\left(\frac{nb_i-mb_i}{K}\right)\right)\right)\right] \tag{B.1}$$

For a special case of K=2, then (B.1) becomes

$$\frac{1}{2\pi}\int_0^{\frac{2\pi}{T}} |Tx^*(w) - x(w)|^2\, dw = \frac{1}{4\pi}\int_{\frac{-\pi}{T}}^{\frac{\pi}{T}} \left|X\left(w-\frac{\pi}{T}\right)\right|^2\left(\sum_{i=1}^{2}\ \sum_{i=1}^{2}\exp(-j\,\Pi(b_i-b_i))\right)dw \tag{B.2}$$

But from (4.58)

$$S/D \equiv \frac{\frac{1}{2\pi}\int_0^{\frac{2\pi}{T}} |X(w)|^2\, dw}{\frac{1}{2\pi}\int_0^{\frac{2\pi}{T}} |TX^*(w)-X(w)|^2 dw}$$

Substituting (B.2) in the above definition gives

$$S/D = \frac{\frac{1}{2\pi}\int_0^{\frac{2\pi}{T}} |X(w)|^2 dw}{\frac{1}{4\pi}\int_{-\pi/T}^{\pi/T} |X(W - \frac{\pi}{T})|^2 (\sum_{i=1}^{2}\sum_{i=1}^{2} \exp(-j\pi(b_i - b_i))dw} \quad (B.3)$$

The equation (B.3) gives the S/D ratio in terms of any arbitrary input sequence which has a Fourier transform.

For $w \in [-\frac{\Pi}{T}, \frac{\Pi}{T}]$, (B.3) can be rewritten as

$$S/D = \frac{\frac{1}{\Pi}\int_{-\pi/T}^{\Pi/T} |X(w)|^2 dw}{\frac{1}{4\Pi}\int_{-\Pi/T}^{\Pi/T} |X(w - \frac{\Pi}{T})|^2 (\sum_{i=1}^{2}\sum_{i=1}^{2} \exp(-j\pi(b_i - b_i))dw}$$

$$= \frac{4}{\sum_{i=1}^{2}\sum_{i=1}^{2} \exp(-j\pi(b_i - b_i)} \quad (B.4)$$

APPENDIX C-1

Analysis Computer program for the steady-state output noise variance

```
C THIS IS THE SOURCE DECK.
C CALCULATION OF QUANTIZATION ERRORS.
C VARIABLES THAT CAN BE CHANGED ARE A & B.
      DIMENSION A(3),B(3),C(3),D(3)
      WRITE(6,102)
  102 FORMAT(///18X,'NORMALIZED QUANTIZATION ERRORS IN'//
     118X,'UNIFORMLY AND NONUNIFORMLY SAMPLED DIGITAL FILTERS'//)
      WRITE(6,130)
  130 FORMAT(18X,'EROR1 IS STEADY-STATE VALUE OUTPUT NOISE VARIANCE
     1FOR UNIFORMLY SAMPLED'//18X,'EROR2 IS 1ST OUTPUT SEQUENCE NOISE
     2VARIANCE'//18X,'EROR3 IS FOR 2ND OUTPUT SEQUENCE IN NONUNIFORMLY
     3SAMPLED DIGITAL FILTERS'//)
      READ(5,10) A,B
   10 FORMAT(6F10.4)
  201 B2=B(3)
       WRITE(6,160) B2
  160 FORMAT(13X,'B2= ', F10.6///)
      WRITE(6,120)
  120 FORMAT(13X,'B1', 23X, 'EROR1', 26X, 'EROR2', 26X, 'EROR3'//)
C CHECK THE STABILITY OF THE FILTER
    6 IF(B(3) .LT. 1.0 .AND. ABS(B(2)) .LT. (1.0+B(3))) GO TO 12
   15 IF(B(3).LT. 1.0 .AND. ABS(B(2)) .GE.(1.0+B(3))) GO TO 200
C  REPLACE A VECTOR WITH WORKING VECTOR C
C  REPLACE B VECTOR WITH WORKING VECTOR D
   12 DO 110 K=1,3
       C(K)=A(K)
      D(K)=B(K)
  110 CONTINUE
C DETERMINE ERROR IN UNIFORMLY SAMPLED DIGITAL FILTER
      CALL SUB(C,D,ENTGL)
      EROR1=ENTGL
      T=SQRT(B(3))
      T1=-SQRT(2*T-B(2))
C DETERMINE ERROR IN THE FIRST OUTPUT SEQUENCE
      C(1)=1.0
      C(2)=T
      C(3)=0.0
      D(1)=1.0
      D(2)=2*T-T1*T1
      D(3)=T*T
      CALL SUB(C,D,ENTGL)
      EROR2=ENTGL
C DETERMINE THE ERROR IN THE SECOND OUTPUT SEQUENCE
      C(1)=-T1
      C(2)=0.0
      C(3)=0.0
      CALL SUB(C,D,ENTGL)
      EROR3=ENTGL
      WRITE(6,72) B(2),EROR1,EROR2,EROR3
   72 FORMAT(8X,F10.4,12X,E19.7,10X,E19.7,10X,E19.7)
      B(2)=B(2)-0.05
      IF(ABS(B(2)) .GE. 0.01) GO TO 6
  200  STOP
       END
```

```
C THE CALLING SUBROUTINE NOW FOLLOWS:
      SUBROUTINE SUB(C,D,ENTGL)
      DIMENSION C(3),D(3)
      X1=(C(1)*C(1))+(C(2)*C(2))+(C(3)*C(3))
      X2=(C(1)*C(2)+C(2)*C(3))*2.0
      X3=C(1)*C(3)*2.0
      Y=D(1)+D(3)
      Y1=(D(2)*D(2)*X3)-Y*X3*D(3)+(D(1)*Y*X1)-(D(1)*D(2)*X2)
      Y2=D(2)*D(2)*D(3)-D(1)*D(2)*D(2)-D(3)*D(3)*Y+D(1)*D(1)*Y
      Y2=Y2*D(1)
      ENTGL=Y1/Y2
      RETURN
      END
```

APPENDIX C-2

Computer simulation program for limit cycles in nonuniformly sampled second order filter with roundoff errors.

```
 1            CC1: PROCEDURE OPTIONS(MAIN);
              /* THIS IS THE SOURCE DECK */
              /* PROGRAM TO STUDY LIMIT CYCLE OSCILLATIONS IN SECOND ORDER FILTERS*/
 2    1       DECLARE FOF ENTRY (DECIMAL FIXED(6),DECIMAL FIXED(6),
              DECIMAL FIXED(6)) RETURNS(DECIMAL FIXED(6));
 3    1       OPEN FILE(SYSPRINT) PAGESIZE(75);
 4    1       DECLARE (A,B) DECIMAL FIXED(9,6);
 5    1       A=-1.4;
 6    1       B=0.8;
              /* INVESTIGATE VALUES TO SUPPRESS LIMIT CYCLES */
 7    1       TRIAL: DO WHILE ( C1 < C2);
 8    1    1  C1=0.40;
 9    1    1  C2=B/C1;
10    1    1  C1=C1+0.1;
11    1    1  END TRIAL;
12    1       DECLARE (D1,D2) DECIMAL FIXED (9,6);
13    1       D1=2.0*C1-C2;
14    1       D2=C1*C1;
15    1       ONE: BND1=0.5/(1.-ABS(B));
16    1       BND2=1.0/(1.0-(ABS(A))+B);
17    1       IF (B*4.0-A*A)<0
18    1       THEN DO;
19    1    1  C1=C1+0.01;
20    1    1  GO TO TWO;
21    1    1  END;
22    1       TWO: PUT EDIT
              ('LIMIT CYCLE OSCILLATIONS IN SECOND ORDER DIGITAL FILTER WITH
              NONUNIFORM SAMPLES')( PAGE,LINE(5),COLUMN(20),A);
23    1       PUT EDIT('A=',A,',B=',
              B,'THE CALCULATED BOUNDS ARE BND1=',BND1,',BND2=' ,
              BND2)(LINE(8),COLUMN(20),A,F(12,9),X(1),A,F(12,9),
              LINE(9),COLUMN(20),A,F(8,4),A,F(8,4));
24    1       PUT SKIP(3);
25    1       BND1=BND1+3.0;
26    1       BND2=BND2+5.0;
27    1       IF BND1>=BND2
28    1       THEN NN=BND1;
29    1       ELSE NN=BND2;
30    1       THREE: BEGIN;
              /* TRY ALL POSSIBLE INITIAL CONDITIONS FOR THE STATES*/
31    2       DECLARE MATRIX(-NN:NN,-NN:NN) BINARY FIXED(6);
32    2       DECLARE R DECIMAL FIXED(4);
33    2       R=0;
34    2         M=1;
35    2       MM=10*BND1;
36    2       DO I=-NN TO NN;
37    2    1  DO J= -NN TO NN;
38    2    2  MATRIX(I,J)=-1;
39    2    2    END;
40    2    1    END;
```

```
41      2          DO I=1 TO -7 BY -1;
42      2    1     DO J=1 TO -7 BY -1;
43      2    2     IFLAG=0;
44      2    2     ISIGN=0;
45      2    2     N=1;
                   /* CALCULATE TRANSIENT RESPONSE FOR FILTER USING INITIAL CONDITIONS*/
46      2    2     FOUR: BEGIN;
47      3    2     DECLARE LIMIT(MM) BINARY FIXED(15), (X1,X2) DECIMAL FIXED(6);
48      3    2     IF MATRIX(I,J)>0
49      3    2     THEN GO TO EIGHT;
50      3    2     MATRIX(I,J)=0;
51      3    2     X1=I;
52      3    2     X2=J;
53      3    2     FIVE:   K=FCF(X1,X2,L);
54      3    2     IF ABS(K)>NN | ABS(X1)> NN
55      3    2     THEN GOTO SEVEN;
56      3    2     IF MATRIX(K,X1)=-1
57      3    2     THEN DO;
58      3    3     MATRIX(K,X1)=0;
59      3    3     SEVEN: X2=X1;
60      3    3     X1=K;
61      3    3     GOTO FIVE;
62      3    3     END;
63      3    2     IF MATRIX(K,X1)=0
                   /* CHECK FOR NEW LIMIT CYCLE */
64      3    2     THEN DO;
65      3    3     IFLAG=1;
66      3    3     MATRIX(K,X1)=M;
67      3    3     LIMIT(N)=K;
68      3    3     N=N+1;
69      3    3     IF SIGN(K)=SIGN(X1)
70      3    3     THEN GOTO SEVEN;
71      3    3     ELSE IF SIGN(SIGN(K)+SIGN(X1))=1
72      3    3     THEN GOTO SEVEN;
73      3    3     ELSE DO;
                   /* CHECK FOR SIGN CHANGE*/
74      3    4     ISIGN=ISIGN+1;
75      3    4     GOTO SEVEN;
76      3    4     END;
77      3    3     END;
78      3    2     DO I2=- NN TO NN;
                   /* SET MATRIX TO ZERO */
79      3    3     DO J2=- NN TO NN;
80      3    4     IF MATRIX(I2,J2)=0
81      3    4     THEN MATRIX(I2,J2)=-1;
82      3    4     END;
83      3    3     END;
84      3    2     IF IFLAG=0
85      3    2     THEN GO TO EIGHT;
```

```
 86   3   2   PUT SKIP(2);
 87   3   2   PUT EDIT('LIMIT CYCLE #',M,' IS')(COLUMN(20),A,F(3),X(2),A,X(3));
 88   3   2   N1=N1+1;
 89   3   2   IF N1=16 THEN DO;
 91   3   3   PUT EDIT('   ')(SKIP,COLUMN(19),A);
 92   3   3   N1=0;
 93   3   3   END;
 94   3   2   M=M+1;
 95   3   2   END FOUR;
 96   2   2   EIGHT: END;
 97   2   1   NINE: /* SET MATRIX FOR THE PHASE PLANE PLOT */
              BEGIN;
 98   3   1   PUT EDIT('LIMIT CYCLES ARRANGED IN PHASE PLANE, X(N) VERSUS X(N-1)')
              (PAGE, SKIP(10), COLUMN(20),A);
 99   3   1   PUT EDIT ('A= ',A, 'B= ',B)
              (SKIP(1),COLUMN(20),A,F(12,6),X(1),A,F(12,6));
100   3   1   PUT SKIP(5);
101   3   1   IF NN< 10
102   3   1   THEN I3=NN;
103   3   1   ELSE I3=10;
104   3   1   DO I2=-I3 TO I3;
105   3   2   IF I2=-I3
106   3   2   THEN PUT EDIT(I2)(COLUMN(20),F(3));
107   3   2   ELSE PUT EDIT(I2)(F(3));
108   3   2   END;
109   3   1   DO J2=I3 TO -I3 BY -1;
110   3   2   PUT EDIT(J2)(COLUMN(17),F(3));
111   3   2   DO I2=-I3 TO I3;
112   3   3   IF MATRIX(I2,J2)>0
113   3   3   THEN PUT EDIT(MATRIX(I2,J2))(F(3));
114   3   3   ELSE PUT EDIT('   ')(A(3));
115   3   3   END;
116   3   2   END NINE;
118   2   1   END THREE;
120   1       FOF:PROCEDURE(X1,X2,R) RETURNS(DECIMAL FIXED(4));
121   2       DECLARE (X,X1,X2,R) DECIMAL FIXED(4);
122   2       X=ROUND(-D1*X1,0)+ROUND(-D2*X2,0)+R;
123   2       RETURN(X);
124   2       END FOF;
125   1       END CO1;
```

APPENDIX C-3

Analysis Computer Program for recursive MTI filter.

```
C THIS IS THE SOURCE DECK.
C DETERMINATION OF RECURSIVE  MTI FREQUENCY CHARACTERISTICS
      DIMENSION X(200),Y(200),DUMY(200)
      COMPLEX R3,R7,D,C7,C3,CMPLX
      X(1)=0.0
      DO 1 I=1,99
    1 X(I+1)=X(I)+1.0
C CALCULATE MAGNITUDE FOR ORDINARY MTI DIGITAL FILTER
      B=0.8
      T1=.04
      T2=.08
      T=T1+T2
      DO 2 I=1,100
      R1=1.0-(COS(T*X(I)))
      R2=SIN(T*X(I))
      R3=CMPLX(R1,R2)
      R4=CABS(R3)
      R4=R4*R4
      R5=1.0-(B*COS(T*X(I)))
      R6=B*R2
      R7=CMPLX(R5,R6)
      R8=CABS(R7)
      R8=R8*R8
      Y(I)=R4/R8
    2 CONTINUE
      DO 3 I=1,100
    3 X(I+100)=X(I)
C CALCULATE MAGNITUDE FOR 2 INERSAMPLES PER PERIOD MTI
      DO 4 I=1,100
      A1=1.0-(B*COS(T*X(I)))+(B*COS(T2*X(I)))-(COS(T2*X(I)))
      A2=(B*SIN(T*X(I)))-(B*SIN(T2*X(I)))+(SIN(T2*X(I)))
      D=CMPLX(A1,A2)
      D1=CABS(D)
      D1=D1*D1
      C1=B-1.0+(COS(T1*X(I)))-(B*COS(T2*X(I)))
      C2=(SIN(T1*X(I)))+(B*SIN(T2*X(I)))
      C3=CMPLX(C1,C2)
      C4=CABS(C3)
      C4=C4*C4
      C5=1.0-(B*B*COS(T*X(I)))
      C6=(B*B*SIN(T*X(I)))
      C7=CMPLX(C5,C6)
      C8=CABS(C7)
      C8=C8*C8
      Y(I+100)=(D1+C4)/(2.0*C8)
    4 CONTINUE
      CALL PLOT(X,C,Y,0,DUMY,0,200,2,1,2,2,0,1)
      STOP
      END
```

```
C THE CALLING SUBROUTINE NOW FOLLOWS: .
      SUBROUTINE PLOT(X,LX,Y,LY,Z,LZ,NPT,NPLOT,NCOPY,NCD,NDIM,NXTREM,
     *NPAGE)
      REAL N
      DIMENSION X(1),Y(1),Z(1),SX(7),TITLE(20),L(134),NCH(41),NH(41)
     *,MOP(42),FM1(6),FM2(6),FM3(6),FM4(8),FM5(8),FM6(8),FMT(6),FPT(8)
     1,SCL(3)
      DATA FM1/'(1H ','','A1,','F9.2',',121','A1) ',','     '/
      DATA FM2/'(1H ','','A1,','F9.4',',121','A1) ',','     '/
      DATA FM3/'(1H ','','A1,','1PE9','.2,1','21A1',')'    '/
      DATA FM4/'(1X,','1PE1','6.2,',',SE20','.2,E','15.2',')    ',
     1'    '/
      DATA FM6/'(5X,','F9.2',',5(1','1X,F','9.2)',',9X,',
     1'F9.2',')    '/
      DATA FM5/'(5X,','F9.4',',5(1','1X,F','9.4)',',9X,','F9.4',
     1')   '/
      DATA NCH/1H0,1H1,1H2,1H3,1H4,1H5,1H6,1H7,1H8,1H9,1HA,1HB,1HC,1HD,
     *1HE,1HF,1HG,1HH,1HI,1HJ,1HK,1HL,1HM,1HN,1HO,1HP,1HQ,1HR,1HS,1HT,1
     *HU,1HV,1HW,1HX,1HY,1HO,1H*,1H$,1H=,1H+,1H-/
      DATA ND,NP,NM,NB/1H.,1H+,1H-,1H /
      DATA SCL/1HX,1HY,1HZ/
    1 FORMAT(20A4)
    2 FORMAT(52A1,7A4)
    3 FORMAT(1H ,26X,20A4)
    5 FORMAT(132A1)
    6 FORMAT(5H ALL ,A1,57H VALUES ARE NEGATIVE.  LOG SCALE MAY NOT BE S
     *PECIFIED.                  )
    7 FORMAT(1PE17.2,E115.2)
    8 FORMAT(1PE17.2,E61.2,E54.2)
    9 FORMAT(1PE17.2,2E40.2,E35.2)
   10 FORMAT(1PE17.2,3E30.2,E25.2)
   11 FORMAT(1PE17.2,4E24.2,E19.2)
   12 FORMAT(1HK,59X,7A4)
   74 FORMAT(6E12.2)
   76 FORMAT(1H1)
   80 FORMAT(70H NPT(NO. OF PTS.) MUST BE EVENLY DIVISIBLE BY NPLOT(NO.
     *OF CURVES.)                )
   84 FORMAT(1X,36HONE DIMENSIONAL PLOT NOT ALLOWED.       )
   94 FORMAT(42A1,10X,7A4)
      DO95I=1,41
   95 NH(I)=NCH(I)
      N=50+66*(NPAGE-1)
      LLX=LX+1
      NDD=NCD+1
        GOTO(15,13,14,13),NDD
   13   READ(5,1)(TITLE(I),I=1,20)
   14 IF(NDD.LT.3)GOTO15
      IF(NDIM.NE.3)GOTO93
      READ(5,94)(MOP(I),I=1,42),TAB1,TAB2,TAB3,TAB4,TAB5,TAB6,TAB7
      GO TO 81
   93 READ(5,2)(MOP(I),I=1,42),( NH(I),I=1,10),TAB1,TAB2,TAB3,TAB4,TAB5,
     1TAB6,TAB7
```

```
      GOTO81
   15 DO82I=1,42
   82 MOP(I)=NB
   81 NCH(41)=NB
      IF(FLOAT(NPT/NPLOT)-FLOAT(NPT)/FLOAT(NPLOT))101,85,101
  85  IF(NXTREM.EQ.1)GOTO72
      YMIN=1.00E 75
      XMIN=YMIN
      ZMIN=YMIN
      YMAX=1.00E-75
      XMAX=YMAX
      ZMAX=YMAX
      DO51I=1,NPT
      GOTO(103,52,53),NDIM
   53 IF(Z(I).GE.ZMAX)ZMAX=Z(I)
      IF(Z(I).LE.ZMIN)ZMIN=Z(I)
   52 IF(Y(I).GE.YMAX)YMAX=Y(I)
      IF(Y(I).LE.YMIN)YMIN=Y(I)
      IF(X(I).GE.XMAX)XMAX=X(I)
      IF(X(I).LE.XMIN)XMIN=X(I)
   51 CONTINUE
      IF(LX.GT.0)GOTO900
      XMIN=XMIN-(XMAX-XMIN)/75.0
      XMAX=XMAX+(XMAX-XMIN)/75.0
  900 IF(LY.GT.0)GOTO73
      YMIN=YMIN-(YMAX-YMIN)/30.0
      YMAX=YMAX+(YMAX-YMIN)/30.0
      GOTO73
   72 READ(5,74)XMIN,XMAX,YMIN,YMAX,ZMIN,ZMAX
   73 IF(YMAX.GT.10000.0.OR.YMIN.LT.(-10000.0))GOTO60
      IF (YMAX-YMIN.LT.5.0)GOTO60
      IF(YMAX.GT.10.0.OR.YMIN.LT.(-10.0))GOTO62
      IF(YMAX.LT.0.01.OR.ABS(YMIN).LT.0.01)GOTO60
      DO59I=1,6
   59 FMT(I)=FM2(I)
      GOTO65
   60 DO61I=1,6
   61 FMT(I)=FM3(I)
      GOTO65
   62 DO63I=1,6
   63 FMT(I)=FM1(I)
   65 IF(XMAX.GT.10000.0.OR.XMIN.LT.(-10000.0))GOTO67
      IF(XMAX-XMIN.LT.5.0)GOTO67
      IF(XMAX.GT.10.0.OR.XMIN.LT.(-10.0))GOTO69
      DO66 I=1,8
      IF(XMAX.LT.0.01.OR.ABS(XMIN).LT.0.01)GO TO 67
   66 FRT(I)=FM5(I)
      GOTO70
   67 DO68I=1,8
   68 FRT(I)=FM4(I)
      GOTO70
   69 DO71I=1,8
```

```
   71 FRT(I)=FM6(I)
   70 IF(LX.GT.0)GOTO17
      CX=120./(XMAX-XMIN)
      SX(1)=XMIN
      SX(7)=XMAX
      U=XMIN
      DO16K=2,6
      U=(XMAX-XMIN)/6.+U
   16 SX(K)=U
      GOTO19
   17 XLX=LX
      MS=1
      IF(XMAX.LE.0.0)GOTO100
      XMIN=10.0**(ALOG10(XMAX*9.999999)-FLOAT(LX))
      CX=120./XLX
      NX=ALOG10(XMIN)
      IF(XMIN.LT.1.0)NX=ALOG10(XMIN)-0.9999999
      XMIN=10.0**NX
      DO18K=1,LLX
   18 SX(K)=10.**(NX+K-1)
   19 CALLPOT(X,XMIN,LX,NPT,0,120.,CX)
      IF(LY.GT.0)GOTO20
      CY=N/(YMAX-YMIN)
      GOTO21
   20 YLY=LY
      MS=2
      IF(YMAX.LE.0.0)GOTO100
      YMIN=10.0**(ALOG10(YMAX*9.999999)-FLOAT(LY))
      CY=N/YLY
      KY=CY
      NY=ALOG10(YMIN)
      IF(YMIN.LT.1.0)NY=ALOG10(YMIN)-0.9999999
      YMIN=10.0**NY
   21 CALLPOT(Y,YMIN,LY,NPT,1,N,CY)
      IF(NDIM.LT.3)GOTO24
      FPLOT=NPLOT
      IF(LZ.GT.0)GOTO22
      CZ=FPLOT/(ZMAX-ZMIN)
      GOTO23
   22 ZLZ=LZ
      MS=3
      IF(ZMAX.LE.0.0)GOTO100
      ZMIN=10.0**(ALOG10(ZMAX*9.999999)-ZLZ)
      CZ=FPLOT/ZLZ
      IF(ZMIN.GE.1.0)GO TO 23
      IA=ALOG10(ZMIN)-.9999999
      ZMIN=10.0**IA
   23 CALL POT(Z,ZMIN,LZ,NPT,0,FPLOT,CZ)
      DO77I=1,NPT
      IZ=Z(I)+1.0
      Z(I)=NCH(IZ)
   77 CONTINUE
```

```
      GOTO99
   24 NPN=NPT/NPLOT
      DO 78I=1,NPLOT
      JJ=(I-1)*NPN+1
      JK=I*NPN
      DO78J=JJ,JK
      Z(J)=NH(I)
   78 CONTINUE
   99 DO102I=1,NPT
      DO102J=I,NPT
      IF(Y(I).GE.Y(J))GOTO102
      A=Y(I)
      B=X(I)
      Y(I)=Y(J)
      X(I)=X(J)
      Y(J)=A
      X(J)=B
      C=Z(I)
      Z(I)=Z(J)
      Z(J)=C
  102 CONTINUE
      DO56NC=1,NCOPY
      M1=1
      M2=1
      LL=0
      LYY=LY
      TT=N
      WRITE(6,76)
      IF(NDD.EQ.1.OR.NDD.EQ.3)GO TO 97
      WRITE(6,3)(TITLE(I),I=1,20)
   97 NN=N+1.0
      DO43KK=1,NN
      M=1
      NNN=NPN
      JED=1
      T=INT(N)-KK+1
      DO25J=1,133
   25 L(J)=NB
      L(133)=ND
      IF(LY.GT.0)GOTO26
      L(13)=NP
      IF(T.GT.TT)GOTO30
      SCALE=T/CY+YMIN
      L(133)=NP
      M=0
      TT=TT-5.
      IF(T.LT.0.)SCALE=YMIN
      GOTO30
   26 GOTO(27,27,28,28,27,28),LY
   27 SS=KY*LYY
      GOTO29
   28 SS=KY*LYY+1
```

```
   29 L(13)=ND
      IF(T.GT.SS)GOTO30
      SCALE=10.**(NY+LYY)
      M=0
      LYY=LYY-1
      L(13)=NP
      L(133)=NP
   30 IF(N.EQ.T)GOTO31
      IF(0..NE.T)GOTO37
   31 DO32J=14,133
   32 L(J)=NM
      IF(LX.GT.0)GOTO34
      DO33J=13,133,10
   33 L(J)=NP
      GOTO36
   34 KX=120/LX
      DO35J=13,133,KX
   35 L(J)=NP
   36 IF(N.EQ.T)L(133)=ND
   37 LL=LL+1
      IF(LL.GT.NPT)GOTO79
      IF(Y(LL)-T)79,92,37
   92 J=X(LL)
      L(J+13)=Z(LL)
      GOTO37
   79 LL=LL-1
      IF(M1.GT.4+( INT(N)-50)/2   .AND.M1.LT.47+( INT(N)-50)/2   )GOTO54
      L(2)=NB
      GOTO55
   54 L(2)=NOP(M2)
      M2=M2+1
   55 M1=M1+1
   41 IF(M.EQ.1)GOTO42
      WRITE(6,FMT)L(2),SCALE,(L(J),J=13,133)
      GOTO43
42    WRITE(6,5)(L(J),J=1,11),(L(J),J=13,133)
   43 CONTINUE
      GOTO(44,45,46,47,48,49,44),LLX
   44 WRITE(6,FRT)(SX(K),K=1,7)
      GOTO50
   45 WRITE(6,7)(SX(K),K=1,LLX)
      GOTO50
   46 WRITE(6,8)(SX(K),K=1,LLX)
      GOTO50
   47 WRITE(6,9)(SX(K),K=1,LLX)
      GOTO50
   48 WRITE(6,10)(SX(K),K=1,LLX)
      GOTO50
   49 WRITE(6,11)(SX(K),K=1,LLX)
   50 IF(NCD.LT.3)GOTO56
      WRITE(6,12)TAB1,TAB2,TAB3,TAB4,TAB5,TAB6,TAB7
   56 CONTINUE
```

```
   83 RETURN
  100 WRITE(6,6)SCL(MS)
      RETURN
  101 WRITE(6,80)
      RETURN
  103 WRITE(6,84)
      RETURN
      END
```

REFERENCES

[1] Kaiser, J. F., "Digital Filters," in System Analysis by Digital Computer, edited by F. F. Kuo and J. F. Kaiser, Wiley, 1966.

[2] G-AE Concepts Subcommittee, On Digital Filtering, IEEE Trans. on Audio and Electroacoustics, V. AU-16, No. 3: 303-314, Sept. 1968.

[3] Gold, B. and Rader, C. M., Digital Processing of Signals, McGraw-Hill Book Co., 1969.

[4] Golden, R. M., and J.F. Kaiser, "Design of Wideband sampled-data filters," Bell Systems Technical Journal No. 43, Part 2, July 1964.

[5] Jackson, L. B., J. F. Kaiser, and H. S. McDonald, "Implementation of digital filters," 1968 IEEE International Convention Digest, IEEE, 1968.

[6] Rader, C. M. and B. Gold, "Digital filter design techniques in the frequency domain," Proc IEEE, Vol. 55, February 1967.

[7] Jury, E. I., Sampled-Data Control Systems, New York: John Wiley & Sons, 1958.

[8] Ragazzini, J. R. and Franklin, G. F., Sampled-Data Control Systems, New York: McGraw-Hill Book Company, Inc., 1958.

[9] Kaiser, J.F., "Some practical considerations in the realiztion of linear digital filters," Papers on Digital Signal Processing, edited by Rabiner, L. R. and Rader, C. M., IEEE Press, New York, 1972.

[10] Freeman, H., Discrete-time Systems, New York, John Wiley & Sons, Inc., 1965, Chapter 4.

[11] Yen, J. L., "On nonuniform Sampling of Bandwidth-Limited Signals," IRE Trans. Circuit Theory, Vol. CT-3, December 1956.

[12] Helms, H. D., Generalizations of the Sampling theorem and error Calculations, Ph.D. dissertation, Princeton University, Princeton, N.J., 1961.

[13] Bogner, R. E., "Time Domain Filter Methods," Proceedings of the NATO Advanced Study Institute on Signal Processing, Loughborough, England, Academic Press, New York, 1973.

[14] Tou, J. T., Digital and Sampled-Data Control Systems, New York, McGraw-Hill Book Company, Inc., 1959.

[15] Jullien, G. A., "A general Analysis for the Time-invariant multirate digital filter," IEEE Transactions on Circuit Theory, Vol. CT-20, No. 4, July 1973.

[16] Jury, E. L., Theory and application of the Z-transform method, New York, John Wiley & Sons, Inc., 1964.

[17] Kuo, B. C., Discrete-data Control Systems, Englewood Cliffs, N. J., Prentice-Hall, Inc., 1970.

[18] Balakrishnan, A. U., "On the Problem of Time Jitter in Sampling," IRE Trans. on Information Theory, Vol. IT-18, April 1962.

[19] Aggarwal, J. K. and Kan, P. E., "Randomly Sampled digital filters," IEEE Trans. on Audio and Electroacoustics, Vol. AU-20, No. 1, March 1972.

[20] Brubaker, T. A. and Risley, W. B., "Sampling interval drift in digital filters," International J. Electronics, Vol. 35, No. 4, October 1973.

[21] Kranc, G. M., "Compensating Error-Sampled System by Multirate Controller," Trans. AIEE (Applications and Industry), 76, Part 2, July 1957.

[22] Meron, P., Sekey, A. A., and Zeheb, E., "Design method for stable second-order digital filters," IEEE Trans. on Acoustics, Speech, and Signal Processing, Vol. ASSP-22, No. 3, June 1974.

[23] Otnes, R. K., and McNamee, L. P., "Instability Thresholds in digital filters due to coefficient rounding," IEEE Trans. on Audio and Electroacoustics, Vol. AU-18, December 1970.

[24] Jackson, L. B., "Roundoff-noise analysis for fixed point digital filters realized in cascade or parallel form," IEEE Trans. on Audio and Electroacoustics, V. AU-18, No. 2, June 1970.

[25] Jury, E. I., "A stability Test for Linear discrete systems using a simple division," Proceedings IRE, Vol. 49, No. 12, December 1961.

[26] Otnes, R. K., Instability Thresholds in digital filters, Ph.D. dissertation, University of California, Los Angeles, California, 1970.

[27] Parker, S. R. and Hess, S. F., "Limit-cycle Oscillations in digital filters," IEEE Trans. on circuit theory, Vo. CT-18, No. 6, November 1971.

[28] Ebert, P. M., Mazo, J. E. and Taylor, M. G., "Overflow oscillations in digital filters," Bell Systems Technical Journal, November 1969.

[29] Jackson, L. B., "Analysis of Limit Cycles due to multiplication rounding in recursive digital filters," Papers on Digital Signal Processing, edited by Rabiner, L. R. and Rader, C. M., IEEE Press, New York, 1972.

[30] Sandberg, J. W., "Floating-point-roundoff Accumulation in digital filter realizations," Bell Systems Technical Journal, V. 46, No. 8, October 1967.

[31] Claasen, T., Mecklenbräuker, W. F. G. and Peek, J. B. H., "Frequency domain criteria for the absence of zero-input limit cycles in nonlinear discrete-time systems, with applications to digital filters," IEEE Trans. on Circuits and Systems, Vol. CAS-22, No. 3, March 1975.

[32] Jatnieks, G. U. and Shenoi, B. A., "Zero-input Limit Cycles in Coupled digital filters," IEEE Trans. on Acoustics, speech, and signal processing, Vol. ASSP-22, No. 2, April 1974.

[33] Jackson, L. B., "On the interaction of roundoff noise and dynamic range in digital filters," Bell Systems Technical Journal, Vol. 49, February 1970.

[34] Mitra, S. K., Hirano, K., and Sakaguchi, H., "A simple method of computing the input quantization and multiplication roundoff errors in a digital filter," IEEE Transactions on Acoustics, speech and signal processing, Vol. ASSP-22, No. 5, October 1974.

[35] Knowles, J. B., and Olcayto, E. M., "Coefficient Accuracy and digital filter response," IEEE Transactions on Circuit Theory, Vol. CT-15, No. 1, March 1968.

[36] Avenhaus, E., "On the design of digital filters with coefficients of limited word length," IEEE Transactions on Audio and electroacoustics, Vol. AU-20, No. 3, August 1972.

[37] IBM System/360 Operating System: PL/I(F) Language Reference Manual, GC-28-8201-2, 3rd Edition, pages 27-28, October 1969.

[38] Åström, K. J., Jury, E. I., and Agniel, R. G., "A numerical method for the evaluation of complex integrals," IEEE Transactions on Automatic Control, Vol. AC-15, August 1970.

[39] Zverev, A. I., "Digital MTI Radar Filters," IEEE Transactions on Audio and Electoracoustics, Vol. AU-16, No. 3, September 1968.

[40] Roy, R., and Lowenschuss, O., "Design of MTI Detection filters with nonuniform interpulse periods," IEEE on Circuit Theory, Vol. CT-17, No. 4, November 1970.

[41] Skolnik, M. I., Introduction to Radar Systems, McGraw-Hill Book Co Company, Inc., New York 1962.

[42] McAulay, R. J., "The effect of Staggered PRF's on MTI signal detection," IEEE Transactions on Aerospace and Electronic Systems, Vol. AES-9, No. 4, July 1973.

[43] Haykin, S., and Boulter, J. F., "Response of a recursive M.T.I. Filter to inputs with Staggered P.R.F.," Electronics Letters, Vol. 7, No. 14, July 1971.

[44] Prinsen, P. J. A., "Elimination of blind velocities of MTI Radar by modulating the interpulse period," IEEE Transactions on Aerospace and Electronic Systems, Vol. AES-9, No. 5, September 1973.

[45] Fjallbrant, T., "Digital filters with a number of shift sequences in each pulse repetition interval," IEEE Transactions on Circuit Theory, Vol. CT-17, No. 3, August 1970.

[46] Atiya, F. S., Bilal, A. Y., and Abdou, I. E., "Multiple-Shift sequences in digital Filter Design," IEEE Transactions on Circuits and Systems, Vol. CAS-21, No. 1, January 1974.

[47] Wong, K. M. and King, R. A., "Pole sensitivity of a digital filter with multishift sequences in each sampling interval," Electronics Letters, Vol. 9, No. 21, October 1973.

[48] Goodman, D. J., "Digital Filters for code format Conversion," Electornics Letters, Vol. 11, No. 4, February 1975.

[49] Flood, J. E., and Hoskins, R. F., "TDM transmission of program channels," Proceeding IEEE, Vol. 112, No. 8, August 1965.

[50] Derusso, P. M., Roy, R. J., and Close, C. M., State Variables for Engineers, New York: John Wiley & Sons, Inc., 1967.

[51] Boykin, W. H., and Frazier, B. D., "Multirate Sampled-data systems Analysis via vector Operators," IEEE Transactions on Automatic Control, Vol. AC-20, No. 4, August 1975.

[52] Wong, K. M., and King, R. A., "State space description of multirate digital filter," Electronics Letters, Vol. 9, No. 2, January 1973.