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"For today and its blessings, I owe the world an attitude of gratitude"

Clarence E. Hodges

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Chapter 0. Introduction

This chapter acquaints the reader with some basic problems of geometric control theory and introduces the notions of a control system as a system of vector fields and of an action on a differentiable manifold as formulated by Sontag [SO].

0.1. What is Geometric Control Theory?

Many problems of great importance in the contemporary world require that a dynamical system have the ability to be influenced so that it can behave in some desired fashion. For our purposes a control system is an underdetermined system of ordinary differential equations. In the most general autonomous form it is a system

$$\dot{x} = f(x, u) \tag{1}$$

where x is an element of a set M, called the state space, and u belongs to a set Ω, called the control space. We assume that the state space M is a finite dimensional differentiable manifold with the Euclidean space $\mathbb{R}^m$ being a prominent example. The control space is taken to be a subset of $\mathbb{R}^n$ or some differentiable manifold. The variable x represents the state of the system and u stands for an external influence on the system. It is the presence of the control variable that makes the system underdetermined. Under appropriate conditions, once the control is specified as a function of time $u(t)$, or a function of the state x and time, $u(x,t)$, and given an initial condition for x the system (1) will become completely determined. The unique solution of (1), whose local existence can be guaranteed by regularity of f and u, is called a trajectory. In this case u is referred to as an admissible control. The notion of an admissible control will be made more precise in sections (0.2) and (3.3).

We can also adopt a more geometric point of view in the study of control systems. For instance, (1) can be represented by a family of vector fields

$$\mathcal{F} = \{f_u\}_{u\in\Omega}, \text{ where } f_u(\cdot) = f(\cdot, u).$$

By a C^k vector field we mean a C^k mapping $X : M \to TM$ from M into the tangent bundle of M such that $\pi \circ X$ is the identity mapping on M. Here $\pi : TM \to M$ denotes the canonical projection.

One can also view a control system as a subset F of the tangent bundle TM, where

$$F = \bigcup_{x\in M} F(x),$$

so (1) can be written as

$$\dot{x} \in F(x), \text{ with } F(x) = f(x, \Omega).$$

The choice of u as a function of time $u = u(t)$ is called an open-loop control. Then given an initial condition $x(0) = x_0$, the resulting non-autonomous differential equation

$$\dot{x}(t) = f(x(t), u(t))$$

has a unique trajectory under certain regularity conditions (such as if f is Lipschitz with respect to x and continuous with respect to u).

Closed-loop control refers to the way of choosing the control parameter u as a function of the state, that is $u = u(x)$, or the state and time, i.e. $u = u(x, t)$. The terminology is suggested by the fact that in this case, unlike open-loop control, the information about the current state of the system is fed back into the system. Closed-loop controls are of interest due to robust behavior of the control law with respect to unknown disturbances and modeling errors.

An interesting problem in control theory is to find a stabilizing feedback control $u = u(x)$ such that the closed loop system $\dot{x} = f(x, u(x))$ becomes either locally or globally assymptotically stable. Such a problem is called the stabilization problem. The reader is referred to [AF] and [B] for more details on feedback stabilization.

For a control system (1) there is a family of trajectories starting from x_0 parametrized by all admissible controls. Points on a trajectory starting from x_0 are referred to as reachable points from x_0. The set of all reachable points from x_0 is called the reachable set from x_0 and is denoted by $\mathcal{R}(x_0)$.

Geometric control theory studies various qualitative properties of control systems, such as accessibility (the ability to reach a subset of the state space having nonempty interior from every initial state), complete controllability (the ability to reach the entire state space from every initial state), normal accessibility (the ability to reach a state with maximum rank from every initial state), local controllability (the initial state x_0 is in the interior of $\mathcal{R}(x_0)$), etc.

Optimal control problems are natural generalizations of problems from the calculus of variations. Given a control system (1), a class of admissible controls and an initial condition, the Lagrange form of the optimal control problem is to minimize a functional on the space of trajectories and controls given in the form

$$J = \int_0^T L(x(t), u(t))dt,$$

where T may or may not be fixed and an end-point condition for $x(T)$ may or may not be imposed. In the time-optimal problem one looks for a control that steers the system from an initial point x_0 to a target point x_1 in a minimal time. In this case $L \equiv 1$. In other problems T is fixed and one wants to minimize the functional J.

Here we mention two prominent theorems that are beyond the scope of the classical theory of the calculus of variations and that among other important results lie at the foundation of geometric control theory.

The study of optimal control problems led to the discovery of the Pontryagin's Maximum Principle (PMP), the more immediate of the two theorems [PBGM]. For the time-optimal problem one introduces a function $p = p(t)$ related to the state and control variables by means of the following system of differential equations

$$\dot{x} = f(x,u), \quad \dot{p} = -p\frac{\partial f}{\partial x}(x,u), \tag{2}$$

which is called the Hamiltonian lift of (1). Note that p here is an m-dimensional row vector. The function H that equals the inner product of p and $f(x,u)$, that is,

$$H(p,x,u) = pf(x,u),$$

is called the Hamiltonian of (1). Then the system (2) can be equivalently written in the form

$$\dot{x} = \frac{\partial H}{\partial p}(p,x,u), \; \dot{p} = -\frac{\partial H}{\partial x}(p,x,u). \tag{3}$$

Let $x_0 \in M$ and $x_1 \in M$ be an initial and a target point, respectively. Then the PMP for the time-optimal problem can be stated as follows [JR]:

Pontryagin's Maximum Principle If $\hat{u}(\cdot)$ is an admissible control $[0,T] \to U$, and if $\hat{x}(\cdot)$ is the corresponding trajectory joining x_0 with x_1 in a minimal time T, then there exists an absolutely continuous curve $\hat{p}(t) \neq 0$ on $[0,T]$, such that the triple $(\hat{p},\hat{x},\hat{u})(\cdot)$ satisfies (3) and the following maximum condition

$$H(\hat{p}(t),\hat{x}(t),\hat{u}(t)) = \max_{v\in U} H(\hat{p}(t),\hat{x}(t),v)$$

holds for almost all $t \in [0,T]$; i.e. $\hat{u}(t)$ maximizes the Hamiltonian with respect to the control variable at each $t \in [0,T]$.

For the Lagrange problem the Hamiltonian has the form

$$H = pf(x,u) + qL(x,u)$$

and one in addition claims that the constant $q \neq 0$, which is the case for so called normal optimal control problems.

The PMP relates optimal control with calculus of variations, but is more general since the set of control values U need not be open and optimal controls often take values at extreme points of U; these are so called "bang-bang" controls.

If we assume that U is open and we control the velocity of the system, i.e. $\dot{x} = u$, then Pontryagin's Maximum Principle yields the Euler-Lagrange equations of the calculus of variations.

It is interesting to note how the PMP relates with the accessibility property. One can show that for a control u^* to be optimal it is necessary that the response of a certain augmented control system be on the boundary of the reachable set. When the accessibility property fails, every reachable set has empty interior, and every control under consideration satisfies the PMP; therefore one cannot use the Pontryagin Maximum Principle to find an optimal control. However, if the accessibility property holds, the PMP might be useful in finding an optimal control.

Geometric control theory further generalizes the PMP forming a theoretical foundation for extensions of the maximum principle to optimal problems on arbitrary differentiable manifolds. There are important examples where the state space is naturally a differentiable manifold. In rigid body mechanics, for instance, a portion of the state lies in the matrix group $SO(3)$. Even for control systems whose state space is an open subset of $\mathbb{R}^n$ there are general structure theorems that tell us that the reachable sets lie on immersed submanifolds.

The other important theorem to be mentioned is **W. L. Chow's theorem** concerning the reachable sets for an arbitrary family of vector fields [C]. Specifically, let $\mathcal{F}$ be a family of local vector fields whose domains of definition collectively cover the whole manifold and let Lie($\mathcal{F}$) denote the family of all vector fields that are Lie brackets of elements of $\mathcal{F}$. Then Chow's theorem states that given $y \in M$, for y to be in the M interior of the $\mathcal{F}$ orbit of y, i.e. $y \in \mathrm{Int}_M \mathcal{O}(y)$, it is sufficient that Lie($\mathcal{F}$) contain a set of m linearly independent tangent vectors, where $m = \dim M$.

This theorem, published in 1939, and the related Hermann-Nagano theorem [J] concerning the existence of integral manifolds, lead to a distinguished class of control systems whose reachable sets can be described using Lie theoretical and algebraic criteria.

Geometric control theory is an exciting blend of differential topology, differential equations, and analysis and its interdisciplinary character places it at the crossroads of differential geometry, mechanics, and optimal control.

0.2. Control Systems as Systems of Vector Fields

In this section we introduce the notions of continuous and discrete-time control systems and provide the definitions used later in the thesis.

Let M be a C^r manifold of dimension m with $2 \le r \le \infty$ or $r = \omega$, let Ω be a separable metric space, let TM denote the tangent bundle of M, and let $\pi : TM \to M$ denote the canonical projection. We recall that TM is a C^{r-1} manifold. Throughout the dissertation we assume that manifolds are second countable and Hausdorff, hence metrizable, topological spaces.

Definition 0.2.1. A continuous-time control system on M with control space Ω

is a map

$$f : M \times \Omega \to TM \tag{4}$$

such that $(\pi \circ f)(x, \omega) = x \ \forall (x, \omega) \in M \times \Omega$, f is C^{r-1} in x, and f and $\frac{\partial f}{\partial x}$ are continuous jointly in (x, ω).

Associated to the control system is the family of initial value problems

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ x(t_0) &= x_0 \end{aligned} \tag{5}$$

where $u : \mathbb{R} \to \Omega$, the control, is a Lebesgue measurable mapping such that the initial value problem has a unique solution. We will call such a u admissible. The set of all admissible controls is denoted by $\mathcal{U}_{\text{meas}}(f)$.

Definition 0.2.2. If we denote the unique absolutely continuous and maximally defined solution of (5) by

$$t \mapsto \mu_f(t, t_0, x_0, u),$$

then the resulting map μ_f is referred to as the global flow of the control system (1).

The global flow is defined on an open subset of the product space $\mathbb{R} \times \mathbb{R} \times M \times \mathcal{U}_{\text{meas}}(f)$. Clearly

$$s \in \mathbb{R}, x_0 \in M, \text{ and } u \in \mathcal{U}_{\text{meas}}(f) \Rightarrow \mu_f(s, s, x_0, u) = x_0.$$

Moreover, as a consequence of uniqueness of solutions we have the transitivity property

$$\mu_f(t, s, \mu_f(s, r, x_0, u), u) = \mu_f(t, r, x_0, u)$$

for every $t, s, r \in \mathbb{R}$, $x_0 \in M$, and $u \in \mathcal{U}_{\text{meas}}(f)$. It follows that the map $x \mapsto \mu_f(t, s, x, u)$ is defined on an open (possibly proper or even empty) subset of M and is a C^1 diffeomorphism between open subsets of M with inverse $x \mapsto \mu_f(s, t, x, u)$.

Note that all piecewise constant maps of $\mathbb{R}$ into Ω are admissible controls. The family of piecewise constant maps of $\mathbb{R}$ into Ω will be denoted $\mathcal{U}_{\mathrm{pc}}$. Given a proper subset Δ of Ω, $\mathcal{U}_{\mathrm{pc}}^{\Delta}$ will denote the family of piecewise constant maps of $\mathbb{R}$ into Δ. Associated to the C^{r-1} control system

$$f : M \times \Omega \to TM$$

is the system of C^{r-1} vector fields

$$\{X^{\omega} \mid \omega \in \Omega\} \quad \text{where } X^{\omega}(x) = f(x, \omega).$$

If $u \in \mathcal{U}_{\mathrm{pc}}$ is defined on an interval $[0, T]$ $(T > 0)$ and has the form

$$u(t) = \begin{cases} \omega_1 & 0 = t_0 \leq t \leq t_1 \\ \omega_2 & t_1 < t \leq t_2 \\ \vdots & \\ \omega_k & t_{k-1} < t \leq t_k = T, \end{cases}$$

and if the trajectory $t \mapsto \mu_f(t, 0, x_0, u)$ is defined on $[0, T]$, then we have

$$\mu_f(t, 0, x_0, u) = \left(X^{\omega_k}_{T-t_{k-1}} \circ \cdots \circ X^{\omega_2}_{t_2 - t_1} \circ X^{\omega_1}_{t_1}\right)(x_0),$$

where $t \mapsto X^{\omega}_t(x)$ is the integral curve of the vector field $X^{\omega} : M \to TM$ that passes through x at time $t = 0$, that is

$$X^{\omega}_0(x) = x$$
$$\frac{d}{dt} X^{\omega}_t(x) = X^{\omega}(X^{\omega}_t(x))$$

for all t in the domain of the integral curve.

Definition 0.2.3. The forward reachable set $\mathcal{R}^+(x_0)$ of the control system (1) from x_0 with piecewise constant controls is defined as follows:

$$\mathcal{R}^+(x_0) = \{x \in M | \exists u \in \mathcal{U}_{\mathrm{pc}} \text{ and } t \geq 0 \text{ such that}$$

$$\mu_f(t, 0, x_0, u) \text{ is defined and equals } x\},$$

or equivalently

$$\mathcal{R}^+(x_0) = \{x \in M | \exists k \in \mathbb{N},\ (\omega_1, \ldots \omega_k) \in \Omega^k \text{ and } s_1, \ldots, s_k \geq 0 \text{ such that}$$

$$(X_{s_k}^{\omega_k} \circ \cdots \circ X_{s_1}^{\omega_1})(x_0) \text{ is defined and equals } x\}.$$

Definition 0.2.4. The (forward-backward) reachable set $\mathcal{R}(x_0)$ of the control system (1) from x_0 with piecewise constant controls is defined as follows:

$$\mathcal{R}(x_0) = \{x \in M | \exists k \in \mathbb{N},\ (\omega_1, \ldots \omega_k) \in \Omega^k \text{ and } s_1, \ldots, s_k \in \mathbb{R} \text{ such that}$$

$$(X_{s_k}^{\omega_k} \circ \cdots \circ X_{s_1}^{\omega_1})(x_0) \text{ is defined and equals } x\}$$

Since $(X_t)^{-1} = X_{-t}$ we can write

$$\mathcal{R}(x_0) = \{x \in M | \exists k \in \mathbb{N},\ (\omega_1, \ldots \omega_k) \in \Omega^k \text{ and } s_1, \ldots, s_k \geq 0 \text{ such that}$$

$$((X_{s_k}^{\omega_k})^{\pm} \circ \cdots \circ (X_{s_1}^{\omega_1})^{\pm})(x_0) \text{ is defined and equals } x\}$$

Here $(X_t^{\omega})^-$ is shorthand for X_{-t}^{ω}. If we want to stress that we are using the control system f and piecewise constant controls that take values in a specified subset $\Delta \subseteq \Omega$, we may write $\mathcal{R}_f^+(x_0|\mathcal{U}_{\mathrm{pc}}^{\Delta})$ or $\mathcal{R}_f(x_0|\mathcal{U}_{\mathrm{pc}}^{\Delta})$ for the forward and forward-backward reachable sets, respectively. One can similarly define $\mathcal{R}_f^+(x_0|\mathcal{U}_{\mathrm{meas}}(f))$, the forward reachable set of f from x_0 with the full family of controls $\mathcal{U}_{\mathrm{meas}}(f)$, as well as the forward-backward reachable set $\mathcal{R}_f(x_0|\mathcal{U}_{\mathrm{meas}}(f))$.

In computer science, logic, and operations research one often encounters control systems where the system state changes at discrete time steps rather than continuously. Automata and sequential machines are examples of such systems, which are collectively referred to as discrete-time control systems. More precisely we have the following definition:

Definition 0.2.5. A discrete-time control system is a map $h: M \times \Omega \to M$, whereby the evolution of the system is governed by the discrete-time equation

$$x(t+1) = h(x(t), u(t)), \; x(0) = x_0. \tag{6}$$

We assume that h is C^r in x and allow the possibility that the domain of h is a proper open subset of $M \times \Omega$.

If for each $\omega \in \Omega$ the partial map

$$h_\omega : \text{dom}(h_\omega) \to M, \text{ where } h_\omega(x) = h(x, \omega)$$

is a C^1 diffeomorphism of open subsets of M, then the discrete-time system (6) is called invertible.

For the discrete-time system (6) the forward reachable set from x_0 is defined as follows:

Definition 0.2.6. The forward reachable set, or positive orbit, of (6) from x_0 is defined as follows:

$$\mathcal{R}^+(x_0) = \{x \in M | x = x_0 \text{ or } \exists k \in \mathbb{N}, \text{ and } (\omega_1, \dots \omega_k) \in \Omega^k$$

$$\text{such that } (h_{\omega_k} \circ \cdots \circ h_{\omega_1})(x_0) \text{ is defined and equals } x\}.$$

For an invertible discrete-time system one introduces the concept of the (forward-backward) reachable set from x_0.

Definition 0.2.7. The forward-backward reachable set, or full orbit, of (6) from x_0 is given by

$$\mathcal{R}(x_0) = \{x \in M | x = x_0 \text{ or } \exists k \in \mathbb{N}, \text{ and } (\omega_1, \dots \omega_k) \in \Omega^k$$

$$\text{such that } (h^{\pm}_{\omega_k} \circ \cdots \circ h^{\pm}_{\omega_1})(x_0) \text{is defined and equals } x\}$$

where $h^+_\omega = h_\omega$ and h^-_ω denotes the inverse map

$$h^-_\omega = (h_\omega)^{-1}.$$

0.3. Actions on Differentiable Manifolds

Our interest in actions arises from the possibility of dealing with both continuous and discrete-time control systems under the framework of a single theory. Before the notion of an action was introduced by Sontag in [SO], the difficulties encountered in studying dicrete-time systems had made the general theory regarding controllability issues for discrete-time systems rather distinct from the more classical continuous-time case. Embracing both these types of control systems, Sontag's formalism of actions [SO] offers an effective tool of research in geometric control theory.

Let M and Λ be C^r differentiable manifolds with $\dim(M) = m \in \mathbb{N}$ and $\dim(\Lambda) = l \in \mathbb{N} \cup \{0\}$.

Definition 0.3.1. A local C^k diffeomorphism ϕ of M is a C^k diffeomorphism from an open subset $\mathrm{dom}(\phi)$ of M onto another open subset $\mathrm{im}(\phi)$ of M. $\mathrm{Diff}^k_{loc}(M)$ will denote the set of all local C^k diffeomorphisms on M.

Definition 0.3.2. A C^r action of Λ on M is a C^r map $\Gamma : M \times \Lambda \to M$ such that $\mathrm{dom}(\Gamma)$ is an open subset of $M \times \Lambda$ and $\lambda \in \Lambda \Rightarrow \Gamma_\lambda \in \mathrm{Diff}^1_{loc}(M)$, where Γ_λ is the partial map, $\Gamma_\lambda(x) = \Gamma(x, \lambda)$.

Given an action $\Gamma : M \times \Lambda \to M$ we define a map $\Gamma^- : M \times \Lambda \to M$ by

$$\Gamma^-(y, \lambda) = x \text{ if } \Gamma(x, \lambda) = y.$$

By the implicit function theorem, $\mathrm{dom}(\Gamma^-)$ is an open subset of $M \times \Lambda$, Γ^- is C^r on its domain of definition, and $\Gamma^- \in \mathrm{Diff}^1_{loc}(M)$ $\forall \lambda \in \Lambda$. It follows that Γ^- is also an action of Λ on M with the partial map $\Gamma^-_\lambda = (\Gamma_\lambda)^{-1}$. In the sequel the notations Γ^+_λ and Γ_λ will be used interchangeably.

Definition 0.3.3. The positive orbit of an action $\Gamma : M \times \Lambda \to M$ of Λ on M from x_0 is given by

$$\mathcal{O}^+_\Gamma(x_0) = \{x \in M | \; x = x_0 \text{ or } \exists k \in \mathbb{N},\ (\lambda_1, \ldots \lambda_k) \in \Omega^k$$

$$\text{such that } (\Gamma^+_{\lambda_k} \circ \cdots \circ \Gamma^+_{\lambda_1})(x_0) \text{ is defined and equals } x\}.$$

Definition 0.3.4. The full orbit of Γ from x_0 is the set

$$\mathcal{O}_\Gamma(x_0) = \{x \in M | \; x = x_0 \text{ or } \exists k \in \mathbb{N},\ (\lambda_1, \ldots \lambda_k) \in \Omega^k$$

$$\text{such that } (\Gamma^\pm_{\lambda_k} \circ \cdots \circ \Gamma^\pm_{\lambda_1})(x_0) \text{ is defined and equals } x\}.$$

Given a continuous-time control system $f : \; M \times \Omega \to TM$ let $\{X^\omega | \; \omega \in \Omega\}$ be the associated family of C^1 vector fields, that is $X^\omega(x) = f(x, \omega)$. Consider a set Λ formed as a disjoint union

$$\Lambda = \bigcup_{\omega \in \Omega} (0, \infty) \times \{\omega\}$$

Note that Ω has the structure of a one-dimensional differentiable manifold, but if Ω has an uncountable number of points, then the set Λ is not second countable. The map $\Gamma: M \times \Lambda \to M$ given by $\Gamma(x,(t,\omega)) = X_t^\omega(x)$ is an action whose orbits coincide with the orbits of the control system if one uses piecewise constant controls, i.e.

$$\mathcal{O}^+(x_0) = \mathcal{R}_f^+(x_0|\mathcal{U}_{\mathrm{pc}}) \text{ and } \mathcal{O}(x_0) = \mathcal{R}_f(x_0|\mathcal{U}_{\mathrm{pc}}).$$

If the discrete-time system (0.4) is invertible with $h: M \times \Omega \to M$ being C^r and if Ω is a differentiable manifold, then clearly h is a C^r action of Ω on M and the orbits of h are the same whether h is viewed as an action or a control system.

0.4. Outline of Results

In this thesis we deal with a refinement of the accessibility property known as the normal accessibility property (see definitions 1.1.1, 1.1.2, and 1.1.3). The normal accessibility property is easily seen to imply the accessibility property, but it has the additional advantage of being demonstrably robust with respect to perturbations in system parameters. Furthermore, in many useful situations the normal accessibility property has been shown to be equivalent to the accessibility property. This equivalence was established for systems of vector fields (that is, control systems with piecewise constant controls) by H. Sussmann [SU] and K. Grasse [G2], and later for control systems with arbitrary measurable controls by K. Grasse and H. Sussmann [GS]. The results of this thesis will further extend the equivalence of the accessibility and normal accessibility properties to actions, as defined in Section 0.3, and to an even more general formulation of control systems involving pseudo semi-groups of local diffeomorphisms (see Section 3.2). (As an additional source about pseudo semi-groups the reader may refer to [KS]). The latter notion is sufficiently general to encompass control systems with arbitrary measurable controls, but also holds the promise of facilitating the study of normal accessibility in other types of "systems" not specifically addressed here, such as control systems governed by functional differential equations, or control systems with impulsive inputs.

In Chapter 1 we review the results obtained by K. Grasse and H. Sussmann that are closely related with the thesis and introduce the concept of a pseudo semi-group of local diffeomorphisms on a differentiable manifold. In Chapter 2 we establish a sufficient condition for the normal accessibility of an action and thereby extend known results for control systems [GS]. Finally, Chapter 3 addresses the problem of the normal accessibility property of pseudo semi-groups. Here, using the formalism of an action and the concept of a pseudo semi-group generated by the action, we derive the vector field case as a particular case providing a different proof of the results established in [GS].

Chapter 1. Preliminary Results

1.1. Accessibility and Normal Accessibility in Systems of Vector Fields

The maximal rank of normal reachablility implies that reachablility of one state from another is stable , in some sense, under perturbations of the system and the control. This property signifies the importance of studying the concept of normal reachability and accessibility in geometric control theory.

Let M be an m-dimensional C^1 differentiable manifold and let Ω be a separable metric space. Consider a C^1 control system $f: M \times \Omega \to TM$ with global flow μ_f.

Definition 1.1.1. Given a nonempty subset Δ of Ω, $x_0, y \in M$ and $k \in \mathbb{N}$ with $1 \leq k \leq m$ we say that y is normally k-reachable from x_0 via f with controls in $\mathcal{U}_{\mathrm{pc}}^{\Delta}$ if there exist $q \in \mathbb{N}$, $\omega_1, \ldots, \omega_q \in \Delta$ and positive real numbers $s_1, \ldots, s_q$ such that

$$\mu_{s_q}^{\omega_q} \circ \cdots \circ \mu_{s_1}^{\omega_1}(x_0) = y,$$

and the map

$$(t_1, \ldots, t_q) \mapsto \mu_{t_q}^{\omega_q} \circ \cdots \circ \mu_{t_1}^{\omega_1}(x_0),$$

which is defined and C^1 on an open neighborhood of $(s_1, \ldots, s_q)$ in $\mathbb{R}^q$, has rank k at $(s_1, \ldots, s_q)$. We denote by $AN_k^{\Delta}(x_0)$ the set of all points normally k-reachable from a given $x_0 \in M$ by controls in $\mathcal{U}_{\mathrm{pc}}^{\Delta}$.

Definition 1.1.2. Let S be a family of C^1 vector fields let $\mathcal{R}_S^+(x)$ denote the reachable set of S from x. The property that $\mathrm{Int}\mathcal{R}_S^+(x) \neq \emptyset$ for every $x \in M$, is called the accessibility property of S. We say that the C^1 control system $f : M \times \Omega \to TM$ has the accessibility property (Δ) if $\mathrm{Int}\mathcal{R}_f(x|\mathcal{U}^{\Delta}_{\mathrm{pc}}) \neq \emptyset \ \forall x \in M$. When $\Delta = \Omega$ we simply say that f has the accessibility property.

Definition 1.1.3. We say that the C^1 control system $f : M \times \Omega \to TM$ has the normal accessibility property (Δ) if $AN^{\Delta}_m(x_0) \neq \emptyset \ \forall x_0 \in M$. When $\Delta = \Omega$ we simply say that f has the normal accessibility property.

As a direct consequence of the surjective mapping theorem [B, p.378] we state the following proposition:

Proposition 1.1.4. If $y \in M$ is normally m-reachable from $x \in M$ via f with controls in $\mathcal{U}^{\Delta}_{\mathrm{pc}}$, then

$$y \in \mathrm{Int}(\mathcal{R}^+_f(x|\mathcal{U}^{\Delta}_{\mathrm{pc}})),$$

where $\mathcal{R}^+_f(x|\mathcal{U}^{\Delta}_{\mathrm{pc}})$ is the reachable set from x via f with controls in $\mathcal{U}^{\Delta}_{\mathrm{pc}}$.

The following three statements are easily verified:

(i) If $x \in \mathcal{R}^+_f(w|\mathcal{U}^{\Delta}_{\mathrm{pc}})$, y is normally m-reachable from x with controls in $\mathcal{U}^{\Delta}_{\mathrm{pc}}$ and $z \in \mathcal{R}^+_f(y|\mathcal{U}^{\Delta}_{\mathrm{pc}})$, then z is normally m-reachable from w via f with controls in $\mathcal{U}^{\Delta}_{\mathrm{pc}}$.

(ii) The set $AN^{\Delta}_m(x)$ is an open, possibly empty, subset of M for every $x \in M$. This follows from the surjective mapping theorem and the fact that the rank of a C^1 mapping is locally non-decreasing.

(iii) The normal accessibility property Δ of f implies the accessibility property of f relative to $\mathcal{U}^{\Delta}_{\mathrm{pc}}$.

The converse of Proposition 1.1.4 does not hold unless the control system is real analytic. Next we recall an example from [G1] of a smooth control system for which there exist $x, y \in M$ such that $y \in \mathrm{Int}(\mathcal{R}_f^+(x|\mathcal{U}_{\mathrm{pc}}^{\Delta}))$, but y is not normally m-reachable from x.

Let $M = \mathbb{R}^2$ and denote the coordinates on $\mathbb{R}^2$ by (p, q). Choose a C^∞ function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(p) = 0 \ \forall p \leq 0$ and $\phi(p) > 0$ if $p > 0$. Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a nonnegative-valued C^∞ function with $\psi^{-1}(0,0) = \{(p,0)|p \geq 0\}$. Consider the system of C^∞ vector fields on M given by

$$\{\frac{\partial}{\partial p}, -\psi(p,q)\frac{\partial}{\partial p}, \pm\phi(p)\frac{\partial}{\partial q}\}. \tag{7}$$

One can readily verify that for this system of vector fields the reachable set is given by

$$\mathcal{R}^+((p,q)) = \begin{cases} \mathbb{R}^2, & p < 0, \ q = 0 \\ \mathbb{R}^2 \setminus \{(p,q)|p < 0, \ q = 0\}, & p = 0, \ q = 0 \\ \mathbb{R}^2 \setminus \{(p,q)|p \leq 0, \ q = 0\}, & (p,q) \notin \{(p,q)|p \leq 0, \ q = 0\}. \end{cases}$$

We see that $(p,q) \in \mathrm{Int}\mathcal{R}^+((p,q))$ when $p < 0$ and $q = 0$. Nevertheless such a (p,q) is not normally 2-reachable from itself via (7).

Finally, we list a few very useful and interesting results concerning control systems, their reachable sets and normal accessibility. Our references here are [G3], [G2] and [GS].

Theorem 1.1.5. Given a C^1 control system $f : M \times \Omega \to TM$, a countable dense subset Δ of Ω, and $x \in M$ with

$$x \in \mathrm{Int}(\mathcal{R}_f^+(x|\mathcal{U}_{\mathrm{meas}}(f))),$$

we have

$$\mathcal{R}_f^+(x|\mathcal{U}_{\mathrm{meas}}(f)) = \mathrm{Int}(\mathcal{R}_f^+(x|\mathcal{U}_{\mathrm{meas}}(f))) = \mathcal{R}_f^+(x|\mathcal{U}_{\mathrm{pc}}^{\Delta})$$

Moreover, for a smooth control system the assumptions imply that $AN_m^{\Delta}(x)$ is an open dense subset of $\mathcal{R}_f^+(x|\mathcal{U}_{\mathrm{meas}}(f))$.

Theorem 1.1.6. Given a C^1 control system $f: M \times \Omega \to TM$ and a countable dense subset Δ of Ω, then the accessibility property of f implies the normal accessibility property Δ of f, that is

$$\mathrm{Int}(\mathcal{R}_f^+(x|\mathcal{U}_{\text{meas}}(f))) \neq \emptyset \ \forall x \in M \Rightarrow AN_m^\Delta(x) \neq \emptyset \ \forall x \in M.$$

The proof of Theorem 1.1.6 relies on the following interesting result:

Theorem 1.1.7. Given a C^1 control system $f: M \times \Omega \to TM$, a countable dense subset Δ of Ω, and $x_0 \in M$, let

$$k = \max\{\ell \in \{1, \ldots, m\} | AN_\ell^\Delta(x_0) \neq \emptyset\} < m.$$

Then the set $AN_k^\Delta(x_0)$ is first category in M and is invariant under the flow of f, that is

$$x \in AN_k^\Delta(x_0), \ u \in \mathcal{U}_{\text{meas}}(f)) \text{ and } T > 0 \Rightarrow \mu_f(T, 0, x, u) \in AN_k^\Delta(x_0).$$

In Chapter 2 we will show that the conclusion of the preceding theorem holds for a more general geometric control theoretic object such as an action.

1.2. Pseudo Semi-Groups of Local Diffeomorphisms

The definition of an action $\Gamma: M \times \Lambda \to M$ requires that the control manifold Λ be finite dimensional. Because of this restriction, the framework of actions can only readily encompass control systems with piecewise constant controls. If one wants an abstract framework that encompasses control systems with arbitrary measurable controls and also provides the possibility of encompassing other types of control systems not addressed here (such as systems with impulsive inputs, or systems governed by functional differential equations), then it is necessary to move to the formalism of pseudo semi-groups.

For differential geometric definitions and elementary results our main references for this section are [SS] and [KS]. In Section 0.3 we introduced the notion of a local diffeomorphism. The following are elementary concepts about local diffeomorphisms:

(a) Restriction of ϕ: if $U \subseteq \mathrm{dom}(\phi)$ is open, the restriction of ϕ to U is a local diffeomorphism denoted by $\phi|U$.

(b) Composition: if ϕ and ψ are two local C^k diffeomorphisms with $\mathrm{dom}(\phi) = \mathrm{im}(\psi)$, then the composition $\phi \circ \psi$ is a local C^k diffeomorphism.

(c) Inverse: if ϕ is a local C^k diffeomorphism, the mapping $\phi^{-1} : \mathrm{im}(\phi) \to \mathrm{dom}(\phi)$ satisfying $(\phi^{-1} \circ \phi)(x) = x$, $\forall x \in \mathrm{dom}(\phi)$ is also a local C^k diffeomorphism.

(d) Units: if U is an open subset of M, then the canonical injection $i_U : U \to M$ is a local diffeomorphism with $\mathrm{dom}(i_U) = \mathrm{im}(i_U) = U$.

Definition 1.2.1. A subset $\mathcal{S}$ of $\mathrm{Diff}^1_{loc}(M)$ is called a C^k-pseudo semi-group (PSG), if:

(i) $\mathcal{S}$ contains all the units i_U, where $U \subseteq M$ is any open set;

(ii) $\mathcal{S}$ is closed under restriction to open subsets, composition, whenever defined, and

(iii) Pasting: if $\{U_j | j \in J\}$ is a family of open subsets of M, ϕ is a local C^k diffeomorphism with $\mathrm{dom}(\phi) = \bigcup_{j \in J} U_j$ and $\phi|U_j \in \mathcal{S}$ for all $j \in J$, then $\phi \in \mathcal{S}$.

Definition 1.2.2. A subset $\mathcal{S}$ of $\mathrm{Diff}^1_{loc}(M)$ is called a C^k pseudo group (PG), if it is a pseudo semi-group and is closed under inversion.

We note that $\mathrm{Diff}^1_{loc}(M)$ itself is a C^1 PG.

Definition 1.2.3. The orbit of $x \in M$ under a PSG $\mathcal{S}$, denoted by $\mathcal{O}_{\mathcal{S}}(x)$, is the subset of M given by

$$\mathcal{O}_{\mathcal{S}}(x) = \{y \in M | \ \exists \phi \in \mathcal{S} \text{ with } x \in \mathrm{dom}(\phi) \text{ and } y = \phi(x)\}$$

Remark 1.2.4. Every action Γ generates a PSG that we denote by $\mathcal{S}_\Gamma$, and we have

$$\mathcal{O}_\Gamma(x) = \mathcal{O}_{\mathcal{S}_\Gamma}(x).$$

Later we will see in Section 3.3. how control systems with measurable controls can be formulated as PSG's.

Chapter 2. On the Normal Accessibility of Actions on Differentiable Manifolds

In [G2, theorem 3.12] K. A. Grasse proved that for a C^1 family S of vector fields on a differentiable manifold M the accessibility property of S is sufficient for normal accessibility of S. The aim of this chapter is to prove a variant of this result and show that for an action the accessibility property implies the normal accessibility property. First we introduce the following definitions.

Definition 2.1.1. Given an action $\Gamma : M \times \Lambda \to M$ and $x_0 \in M$ the set of normally k-reachable points from x_0 via Γ, $AN_k^\Lambda(x_0)$, is defined as follows:

$$\begin{aligned} AN_k^\Lambda(x_0) = \{x \in M | \exists n \in \mathbb{N} \text{ and } (\alpha_1, \dots, \alpha_n) \in \Lambda^n \text{ such that} \\ x = (\Gamma_{\alpha_n} \circ \cdots \circ \Gamma_{\alpha_1})(x_0) \text{ and the } \mathrm{C}^1 \text{ map} \\ f_n : (\lambda_1, \dots, \lambda_n) \mapsto (\Gamma_{\lambda_n} \circ \cdots \circ \Gamma_{\lambda_1})(x_0) \text{ has} \\ \text{rank } k \text{ at } (\alpha_1, \dots, \alpha_n)\}. \end{aligned}$$

Definition 2.1.2. We say that an action Γ has the normal accessibility property if $\forall x_0 \in M$ the set $AN_m^\Lambda(x_0) \neq \emptyset$, where $m = \dim M$.

Now we state the principal result of this chapter:

Theorem 2.1.3. Let M and Λ be finite dimensional C^1 differentiable manifolds with $m = \dim M$ and Λ being second countable. If the action $\Gamma : M \times \Lambda \to M$ of Λ on M has the accessibility property, then it has the normal accessibility property.

Proof: Since the action Γ has the accessibility property, for any $x_0 \in M$ the Γ-orbit $\mathcal{O}_\Gamma^+(x_0)$ of x_0 has nonempty interior. We want to show that for any $x_0 \in M$ $\mathcal{O}_\Gamma^+(x_0)$ contains a point

$$y = (\Gamma_{\nu_p} \circ \cdots \circ \Gamma_{\nu_1})(x_0) \text{ for some } p \in \mathbb{N} \text{ and } (\nu_1, \dots \nu_p) \in \Lambda^p$$

for which the map

$$(\lambda_1, \ldots, \lambda_p) \to (\Gamma_{\lambda_p} \circ \cdots \circ \Gamma_{\lambda_1})(x_0)$$

has full rank $m = \dim M$ at $(\nu_1, \ldots \nu_p)$. We argue by contradiction: suppose Γ does not have the normal accessibility property. Then there exists $x_0 \in M$ such that for any $z \in \mathcal{O}_\Gamma^+(x_0)$ of the form

$$z = \Gamma_{\mu_q} \circ \cdots \circ \Gamma_{\mu_1}(x_0)$$

the rank of the map

$$(\lambda_1, \ldots, \lambda_q) \to (\Gamma_{\lambda_q} \circ \cdots \circ \Gamma_{\lambda_1})(x_0)$$

is less than m at $(\mu_1, \ldots, \mu_q)$. For the rest of this section we assume that x_0 is chosen with this property. Let $k < m$ be the maximum positive integer such that $AN_k^\Lambda(x_0) \neq \emptyset$. Note that the positive integer k depends on $x_0 \in M$ and we write $k(x_0)$ for k when we want to emphasize this fact.

To prove the theorem we will employ the following two assertions:

Claim 1. The set $AN_k^\Lambda(x_0)$ is of the first category in M.

Proof of claim 1: let $\{U_i\}_{i=1}^\infty$ be a countable base of the second countable space Λ. Choose $y \in AN_k^\Lambda(x_0)$. Then there exist $p \in \mathbb{N}$ and $(\beta_1, \ldots, \beta_p) \in \Lambda^p$ such that $y = (\Gamma_{\beta_p} \circ \cdots \circ \Gamma_{\beta_1})(x_0)$ and the C^1 map

$$f_p : \ (\lambda_1, \ldots, \lambda_p) \longmapsto (\Gamma_{\lambda_p} \circ \cdots \circ \Gamma_{\lambda_1})(x_0)$$

has rank k at $(\beta_1, \ldots, \beta_p)$. In addition, by the maximality property of k and the fact that the rank of a C^1 mapping is locally nondecreasing it follows that there exists an open neighborhood $V_0 \subseteq \Lambda^p$ of $(\beta_1, \ldots, \beta_p)$, so that the above map has rank k at each point of V_0. The rank theorem implies that we can shrink V_0 to another open neighborhood $V_1 \subseteq V_0$ of $(\beta_1, \ldots, \beta_p)$ if necessary such that for every open subset V of V_1 the set

$$\{\Gamma_{\lambda_p} \circ \cdots \circ \Gamma_{\lambda_1}(x_0) |\ (\lambda_1, \ldots, \lambda_p) \in V\}$$

is a C^1 imbedded submanifold of M. Pick elements $U_{i_1}, \ldots, U_{i_p}$ of $\{U_i\}_{i=1}^{\infty}$ with $U_{i_1} \times \cdots \times U_{i_p} \subseteq V$. Thus f_p has rank k at every point of $U_{i_1} \times \cdots \times U_{i_p}$ and the set

$$N(x_0) = \{\Gamma_{\lambda_p} \circ \cdots \circ \Gamma_{\lambda_1}(x_0) |\ \lambda_j \in U_{i_j},\ j = 1, \ldots, p\}$$

is a C^1 imbedded k-dimensional submanifold of M. Because $k < m$ this submanifold is nowhere dense. Since each $y \in AN_k^{\Lambda}(x_0)$ is contained in one of the countably many of these sets

$$N(x_0) \subseteq AN_k^{\Lambda}(x_0),$$

we have that $AN_k^{\Lambda}(x_0)$ is the union of a countable number of nowhere dense subsets of M, and hence $AN_k^{\Lambda}(x_0)$ is of the first category in M. This proves claim 1.

Note: in the sequel we often employ the notation $N_j(x_0)$ for a representative of this family of countably many k-dimensional imbedded submanifold of M.

Claim 2. If $z \in \mathcal{O}_{\Gamma}^{+}(y)$ and $y \in AN_k^{\Lambda}(x_0)$ then $z \in AN_k^{\Lambda}(x_0)$; that is, the set $AN_k^{\Lambda}(x_0)$ is invariant under the action Γ.

Proof of claim 2: if $z = y$ the statement is trivial. Assume that $z \in \mathcal{O}_{\Gamma}^{+}(y) \setminus \{y\}$. Then there exists $l \in \mathbb{N}$ and $(\gamma_1, \cdots, \gamma_l) \in \Lambda^l$ with

$$z = \Gamma_{\gamma_l} \circ \cdots \circ \Gamma_{\gamma_1}(y) = \Gamma_{\gamma_l} \circ \cdots \circ \Gamma_{\gamma_1} \circ \Gamma_{\beta_p} \circ \cdots \circ \Gamma_{\beta_1}(x_0),$$

where $y = \Gamma_{\beta_p} \circ \cdots \circ \Gamma_{\beta_1}(x_0)$. The partial map $\Gamma_{\lambda} \subseteq \mathrm{Diff}^{\,1}_{loc}(M)$ maps open sets onto open sets and carries C^1 imbedded submanifolds onto C^1 imbedded submanifolds thereby preserving their dimensions. Therefore the map

$$f_{p+l}: \ (\lambda_1, \dots, \lambda_{p+l}) \longmapsto \Gamma_{\lambda_{p+l}} \circ \cdots \circ \Gamma_{\lambda_1}(x_0)$$

is defined on an open neighborhood of $(\beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_l) \in \Lambda^{p+l}$ and by the maximality property of k has rank k at this point. Thus

$$z = f_{p+l}(\beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_l) \in AN_k^{\Lambda}(x_0),$$

and since $z \in \mathcal{O}_\Gamma^+(y)$ was arbitrary we obtain $\mathcal{O}_\Gamma^+(y) \subseteq AN_k^{\Lambda}(x_0)$ for every $y \in AN_k^{\Lambda}(x_0)$. The proof of claim 2 is now complete.

By the accessibility property $\mathrm{Int}\mathcal{O}_\Gamma^+(y) \neq \emptyset$, so $\mathrm{Int}AN_k^{\Lambda}(x_0) \neq \emptyset$. However M is locally compact, and therefore it is a Baire space. In a Baire space no set of the first category has nonempty interior. In claim 1 we showed that $AN_k^{\Lambda}(x_0)$ is of the first category. The resulting contradiction completes the proof of the theorem.■

Chapter 3. A Sufficient Condition for the Normal Accessibility of Actions in Terms of Pseudo Semi-Groups

Our aim in this chapter is to embed control systems in the more general formalism of pseudo semi-groups, prove that under reasonable assumptions the accessibility property of a certain PSG implies the normal accessibility property of an underlying action, and, finally, to show the effectiveness of our new approach in the case of a control system with arbitrary measurable controls.

3.1. Preliminary Definitions and Results

Let M be a differentiable manifold of dimension $m < \infty$ and $\mathcal{S}$ be a pseudo semi-group of local diffeomorphisms on M. We equip $\mathcal{S}$ with the compact-open topology (COT), whose definition we recall below:

Let X be a locally compact Hausdorff space, and Y any Hausdorff space. Denote by Y^X the set of continuous functions $X \to Y$.

Definition 3.1.1. The compact-open topology (COT) on Y^X is generated by the sets

$$S(C,U) = \{\phi \in Y^X | \phi(C) \subseteq U\},$$

where $C \subseteq X$ is compact and $U \subseteq Y$ is open. In other words, these sets form a subbasis for the COT.

Throughout this chapter we will assume that the PSG $\mathcal{S}$ contains an action $\Gamma : \; M \times \Lambda \to M$ in the sense that $\forall\, \lambda \in \Lambda \; \Gamma_\lambda \in \mathcal{S}$; this in particular implies that

$$\forall x \in M \quad \mathcal{O}^+_\Gamma(x) \subseteq \mathcal{O}^+_\mathcal{S}(x)\,,$$

where $\mathcal{O}_\Gamma^+(x)$ is the positive orbit of Γ as defined in (0.3.3). Consider the countable family of maps

$$\{F_p : \Lambda^p \to \mathcal{S}_\Gamma \subseteq \mathcal{S}, p \in \mathbb{N}\}$$

defined as follows:

$$F_p(\lambda_1, \lambda_2, ..., \lambda_p) = \Gamma_{\lambda_p} \circ \cdots \circ \Gamma_{\lambda_1}.$$

The domain of F_p is the open subset of Λ^p for which the map $\Gamma_{\lambda_p} \circ \cdots \circ \Gamma_{\lambda_1}$ has a nonempty domain of definition. Denote $\mathcal{S}^{-1} = \{\phi^{-1} | \phi \in \mathcal{S}\}$. We also assume that $\mathcal{S}^{-1}$ is topologized with the COT. Because the manifold is metrizable, on $\mathcal{S}$ the COT and the topology of compact convergence coincide [M, p.286]. Hence we have the following standard result:

Proposition 3.1.2. Given $\phi \in \mathcal{S}$, a compact set $K \subseteq \text{dom}(\phi)$, and $\epsilon > 0$ there exists an open neighborhood B of ϕ in the COT on $\mathcal{S}$ such that

$$\psi \in B \Rightarrow K \subseteq \text{dom}\psi \text{ and } \forall x \in K \ \text{dist}(\phi(\text{x}), \psi(\text{x})) < \epsilon.$$

The COT is a natural choice for a topology on $\mathcal{S}$ because the group operations are rendered continuous as we show in the next proposition.

Proposition 3.1.3. The maps $\Phi : \mathcal{S} \to \mathcal{S}^{-1}$ and $\Delta : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ given by $\Phi(\phi) = \phi^{-1}$ and $\Delta(\phi, \psi) = \phi \circ \psi$ are continuous.

Proof: Note that the composition map Φ is defined only on an appropriate subset of $\mathcal{S} \times \mathcal{S}$.

The continuity of Δ can be established easily. Indeed, let $\phi, \psi \in \mathcal{S}$ and $\phi \circ \psi \in S(C, U)$, where $S(C, U)$ is a subbasic open set in the COT on $\mathcal{S}$, so C is compact and U is open in M. Then $\phi\circ\psi(C)\subseteq U$ and the set $\phi\circ\psi(C)$ is compact. We can show readily that there exists

$$V\subseteq U \text{ open in } M \text{ with } \phi\circ\psi(C)\subset V\subset\overline{V}\subset U$$

and $\overline{V}$ being compact. Indeed, cover the compact set $\phi\circ\psi(C)$ with a finite family of open sets $W_1,\dots,W_n$ such that $\overline{W}_i\subseteq U$ and $\overline{W}_i$ is compact for every $i=1,\dots,n$. This is possible by the local compactness of M. Then if V is the open set given by

$$V=\bigcup_{i=1}^{n} W_i,$$

we see that

$$\overline{V}=\overline{\bigcup_{i=1}^{n} W_i}=\bigcup_{i=1}^{n}\overline{W}_i,$$

so $\overline{V}$ is compact. In addition

$$\phi\circ\psi(C)\subseteq V\subseteq\overline{V}\subseteq U.$$

Next, $\phi^{-1}(V)$ is open and $\psi(C)\subset\phi^{-1}(V)$. It follows that $\psi\in S(C,\phi^{-1}(V))$. Clearly $\phi\in S(\phi^{-1}(\overline{V}),U)$.

Let (ϕ',ψ') be an element of $S(\phi^{-1}(\overline{V}),U)\times S(C,\phi^{-1}(V))$. It follows that

$$\phi'\circ\psi'(C)\subset\phi'(\phi^{-1}(V))\subset\phi'(\phi^{-1}(\overline{V}))\subset U$$

Thus $\phi'\circ\psi'\in S(C,U)$, so the map Δ is continuous.

Next we establish the continuity of Φ. Let ϕ^{-1} belong to the subbasic open set $S(C,U)\subseteq\mathcal{S}^{-1}$, where again C is compact and U is open in M. We need to find an open neighborhood B of ϕ in $\mathcal{S}$ such that $\psi\in B$ implies that $\psi^{-1}\in S(C,U)$.

Without loss of generality we may assume that the compact set $C\subseteq M$ represents a single closed topological ball. Indeed, choose closed balls $D_1,D_2,\cdots,D_n$

with $C \subseteq \bigcup_{i=1}^{n} D_i$. This is possible by the compactness of C. Further, we may assume that C is of the form $\bigcup_{i=1}^{n} D_i$, because if we find an open neighborhood B of ϕ with $\psi^{-1} \in S(\bigcup_{i=1}^{n} D_i, U)$ for every $\psi \in B$, then $\psi^{-1} \in S(C, U)$ for every $\psi \in B$. Moreover, if we find a B_i such that

$$\psi \in B_i \Rightarrow \psi^{-1} \in S(D_i, U)$$

for every $i = 1, 2, ..., n$, then

$$\psi \in \bigcap_{i=1}^{n} B_i \Rightarrow \psi^{-1} \in S(\bigcup_{i=1}^{n} D_i, U).$$

Thus, we will complete the proof with $C = D$, where D is a closed topological ball. Choose $x_0 \in \mathrm{Int}\phi^{-1}(D)$ and a closed ball $D' \supset D$ with $\phi^{-1}(D') \subset U$. Let $\epsilon > 0$ be such that

$$\epsilon < \min\{\mathrm{dist}(D, \partial D'), \mathrm{dist}(\phi(x_0), \partial D)\},$$

where $\partial D = \mathrm{boundary}(D) = \overline{D} \cap \overline{M \setminus D}$. From Proposition (3.1.2) it follows that there exists an open neighborhood B of ϕ such that

$$\psi \in B \Rightarrow \mathrm{dist}(\phi(x), \psi(x)) < \epsilon \quad \forall x \in \phi^{-1}(D'). \tag{8}$$

Then we claim that for every $\psi \in B$ we have $\psi(\phi^{-1}(\partial D')) \cap D = \emptyset$. To see this, argue by contradiction and assume that

$$\exists z \in \psi(\phi^{-1}(\partial D')) \cap D.$$

Then $z = \psi(z')$ for some $z' \in \phi^{-1}(\partial D')$. It follows from (8) that

$$\epsilon > \mathrm{dist}(\phi(z'), \psi(z')) = \mathrm{dist}(z, z''),$$

where $z'' \in \partial D'$ and $z' = \phi^{-1}(z'')$. On the other hand, because $z \in$ D and $z'' \in \partial D'$, by our choice of ϵ we have

$$0 < \epsilon < \min\{\mathrm{dist}(D, \partial D'), \mathrm{dist}(\phi(x_0), \partial D)\}.$$

Therefore

$$\epsilon < \mathrm{dist}(D, \partial D') \leq \mathrm{dist}(z'', z)$$

and the contradiction is now obvious. Thus we must have

$$\psi(\phi^{-1}(\partial D')) \cap D = \emptyset. \tag{9}$$

¿From (9) the Jordan separation theorem [D, p. 358] implies that D lies in one of the complementary components of $\psi(\phi^{-1}(\partial D'))$. From (8) we obtain

$$\mathrm{dist}(\phi(x_0), \psi(x_0)) < \epsilon,$$

and therefore $\psi(x_0) \in D$. From this we infer that

$$D \subseteq \psi(\phi^{-1}(D')),$$

whence $\psi^{-1}(D) \subseteq \phi^{-1}(D') \subset U$. But this shows that $\psi^{-1} \in S(D, U)$ for every $\psi \in B$.■

We recall from Claim 1 of Theorem 2.1.3 that for a given $x_0 \in M$ the set $AN_k^{\Lambda}(x_0)$ is a countable union of $k = k(x_0)$-dimensional imbedded submanifolds of M

$$AN_k^{\Lambda}(x_0) = \bigcup_{j=1}^{\infty} N_j(x_0).$$

Definition 3.1.4. The pseudo semi-group of local diffeomorphisms $\mathcal{S} = (\mathcal{S}, \mathcal{P})$ is said to be parametrized if there exists a family of continuous maps

$$\mathcal{P} = \{\rho_\alpha : E_\alpha \to \mathcal{S} | \alpha \in \mathcal{A}\},$$

where $\mathcal{A}$ is some index set, each E_α is a topological space and

$$\mathcal{S} = \bigcup_{\alpha \in \mathcal{A}} \rho_\alpha(E_\alpha).$$

Definition 3.1.5. Let $S = (\mathcal{S}, \mathcal{P})$ be a parametrized pseudo semi-group of local diffeomorphisms. A k-dimensional imbedded submanifold $N \subseteq M$ is called stable with respect to perturbations induced by $\mathcal{P}$, or $(\mathcal{S}, \mathcal{P})$-stable, if $\forall \phi_0 \in \mathcal{S}$, $\forall x_1 \in N$ with $x_1 \in \mathrm{dom}(\phi_0)$, $\forall \alpha \in \mathcal{A}$ with $\phi_0 \in \rho_\alpha(E_\alpha)$, and $\forall \xi_0 \in E_\alpha$ with $\rho_\alpha(\xi_0) = \phi_0$ there exist an open neighborhood V_1 of x_1 relative to N and an open neighborhood W_0 of ξ_0 in E_α such that $\mathrm{Cl}_M(V_1) \subseteq N \cap \mathrm{dom}(\phi_0)$ and

$$\xi \in W_0 \Rightarrow \rho_\alpha(\xi)(\mathrm{Cl}_M(V_1)) \subseteq \phi_0(N \cap \mathrm{dom}(\phi_0)). \tag{10}$$

Note that every pseudo semi-group $\mathcal{S}$ with the COT has a trivial parametrization induced by the identity map $id_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$.

The next remark gives one simple class of submanifolds for which stability is assured.

Remark 3.1.6. Let $m = \dim M$. Then any m-dimensional imbedded submanifold of M is $(\mathcal{S}, id)$-stable.

Proof: Let $M_0 \subseteq M$ be an m-dimensional imbedded submanifold of M. Pick $x_1 \in M_0$ and $\phi_0 \in \mathcal{S}$ with $x_1 \in \mathrm{dom}(\phi_0)$. The open mapping theorem readily implies that M_0 is open in M, as is the set $\phi_0(M_0 \bigcap \mathrm{dom}(\phi_0)) = M_1$. Choose an open $W_1 \subset M_1$ so that $\phi(x_0) \in W_1$ and $\overline{W}_1$ is compact and contained in M_1. Set $V_1 = \phi_0^{-1}(W_1)$. Because ϕ_0 is a local diffeomorphism $\overline{V}_1 = \phi_0^{-1}(\overline{W}_1)$ is compact, $\overline{V}_1 \subseteq M_0$, and $x_0 \in V_1$. The set $S(\overline{V}_1, M_1)$ is a subbasic open set in the COT on $\mathcal{S}$ and $\phi_0 \in S(\overline{V}_1, M_1)$. Setting $\mathcal{N}_0 = S(\overline{V}_1, M_1)$ we see that

$$\phi \in \mathcal{N}_0 \Rightarrow \phi(\overline{V}_1) \subseteq \phi_0(M_0 \cap \mathrm{dom}(\phi_0)).$$

This proves that M_0 is $(\mathcal{S}, id)$-stable.■

In the following statement $\mathcal{S}$ is a parametrized pseudo semi-group of local diffeomorphisms as given in definition (3.1.5).

Proposition 3.1.7. A k-dimensional submanifold N of M is stable with respect to perturbations induced by $\mathcal{P}$ if and only if $\forall x_1 \in N$, $\forall \phi_0 \in \mathcal{S}$ with $x_1 \in \mathrm{dom}(\phi_0)$, $\forall \alpha \in \mathcal{A}$ with $\phi_0 \in \rho_\alpha(E_\alpha)$, and $\forall \xi_0 \in E_\alpha$ with $\rho_\alpha(\xi_0) = \phi_0$ there exists an open neighborhood V_1 of x_1 relative to N and an open neighborhood W_0 of ξ_0 in E_α such that $\mathrm{Cl}_M(V_1) = \overline{V}_1 \subseteq N$ and

$$\xi_1, \xi_2 \in W_0 \Rightarrow \rho_\alpha(\xi_2)^{-1} \circ \rho_\alpha(\xi_1)(\overline{V}_1) \subseteq N.$$

Proof: Sufficiency: set $\xi_2 = \xi_0 \in W_0$ to get

$$\xi_1 \in W_0 \Rightarrow \rho_\alpha(\xi_0)^{-1} \circ \rho_\alpha(\xi_1)(\overline{V}_1) = \phi_0^{-1} \circ \rho_\alpha(\xi_1)(\overline{V}_1) \subseteq N$$

$$\Rightarrow \rho_\alpha(\xi_1)(\overline{V}_1) \subseteq \rho_\alpha(\xi_0)(N) = \phi_0(N). \tag{11}$$

Necessity: let $x_1 \in N$ and $\phi_0 \in \mathcal{S}$ with $x_1 \in \mathrm{dom}(\phi_0)$. Let V_1 be an open neighborhood of x_1 relative to N with $\overline{V}_1 \subseteq N$, $\alpha \in \mathcal{A}$, $\xi_0 \in E_\alpha$ and an open neighborhood W_0' of ξ_0 be chosen so that $\phi_0 = \rho_\alpha(\xi_0)$ and

$$\xi \in W_0' \Rightarrow \rho_\alpha(\xi)(\overline{V}_1) \subseteq \phi_0(N).$$

Without loss of generality we may assume that $\overline{V}_1$ is a compact subset of $\mathrm{dom}(\phi_0)$. The map $\phi_0 : \overline{V}_1 \to \phi_0(\overline{V}_1)$ is a homeomorphism, hence ϕ_0 has a continuous right inverse at every point of $\phi_0(\overline{V}_1)$, in particular at every $z \in \phi_0(\overline{V}_2)$, where V_2 is an open neighborhood of x_1 with $\overline{V}_2 \subseteq V_1$. Clearly $\overline{V}_2$ is compact and $\phi_0(\overline{V}_2) \subseteq \phi_0(\overline{V}_1)$. Since M is metrizable the COT coincides with the topology of compact convergence on $\mathcal{S}$. Let M be equipped with a metric which induces its topology as a differentiable manifold. Then Theorem 3.11 of [G2] yields a basic open neighborhood

$$B_{\overline{V}_1}(\phi_0, \epsilon) = \{\phi | \max_{x \in \overline{V}_1} \mathrm{dist}_M[\phi(x), \phi_0(x)] < \epsilon\}$$

of ϕ_0 in the COT on $\mathcal{S}$ such that

$$\psi \in B_{\overline{V}_1}(\phi_0, \epsilon) \Rightarrow \phi_0(\overline{V}_2) \subseteq \psi(\overline{V}_1).$$

For every $y \in \overline{V}_1$ there exists an open neighborhood V_y relative to N with

$$\phi_0(\overline{V}_y) \subseteq B\left(\phi_0(y), \frac{\epsilon}{2}\right) \cap \phi_0(N),$$

where W_y is an open neighborhood of $\phi_0(y)$ relative to $\phi_0(N)$ and $B\left(\phi_0(y), \frac{\epsilon}{2}\right)$ denotes the open $\epsilon/2$-ball in M centered at $\phi_0(y)$. Cover the compact set $\overline{V}_1$ with finitely many such V_y, say for $y = y_1, \ldots, y_n$, and set $K_{y_i} = \overline{V}_1 \cap \overline{V}_{y_i}$. Then each K_{y_i} is compact. The set $\bigcap_{i=1}^n S\left(K_{y_i}, B\left(\phi_0(y_i), \frac{\epsilon}{2}\right)\right)$ is a basic open set in the COT on $\mathcal{S}$ containing ϕ_0. In addition,

$$\bigcap_{i=1}^n S\left(K_{y_i}, B\left(\phi_0(y_i), \frac{\epsilon}{2}\right)\right) \subseteq B_{\overline{V}_1}(\phi_0, \epsilon),$$

and by the choice of $\epsilon > 0$

$$\psi \in \bigcap_{i=1}^n S\left(K_{y_i}, B\left(\phi_0(y_i), \frac{\epsilon}{2}\right)\right) \Rightarrow \phi_0(\overline{V}_2) \subseteq \psi(\overline{V}_1).$$

Now let V be an open neighborhood of x_1 relative to N with $\overline{V} \subseteq V_2$. Because ϕ_0 is a local diffeomorphism the set $\phi_0(V_2)$ is open relative to $\phi_0(N)$, so we can write $\phi_0(V_2) = H \cap \phi_0(N)$, where H is open in M. Then because $\phi_0(\overline{V}) \subseteq \phi_0(V_2) \subseteq H$ we have that $\phi_0 \in S(\overline{V}, H)$. It follows that

$$\phi \in S(\overline{V}, H), \psi \in \bigcap_{i=1}^n S\left(K_{y_i}, B\left(\phi_0(y_i), \frac{\epsilon}{2}\right)\right) \Rightarrow$$

$$\phi(\overline{V}) \subseteq \phi_0(V_2) \subseteq \phi_0(\overline{V}_2) \subseteq \psi(\overline{V}_1) \subseteq \psi(N \cap \mathrm{dom}(\psi)) \Rightarrow$$

$$\psi^{-1}(\phi(\overline{V})) \subseteq N. \tag{12}$$

Set

$$\mathcal{N}' = S(\overline{V}, H) \cap \left(\bigcap_{i=1}^{n} S\left(K_{y_i}, B\left(\phi_0(y_i), \frac{\epsilon}{2}\right)\right)\right).$$

Then $\mathcal{N}'$ is an open neighborhood of ϕ_0 in the COT. Choose an open neighborhood $W_0 \subseteq W_0'$ of ξ_0 in E_α with $\rho_\alpha(W_0) \subseteq \mathcal{N}'$. This is possible by the continuity of ρ_α. Then if $\xi_1 \in W_0$, setting $\phi = \rho_\alpha(\xi_1)$ yields

$$\phi(\overline{V}_1) \subseteq \phi_0(N) \Rightarrow \phi(K_{y_i}) \subseteq \phi_0(N) \text{ and } \phi(\overline{V}) \subseteq \phi_0(N).$$

If $\xi_2 \in W_0$ then for $\psi = \rho_\alpha(\xi_2)$ we have $\psi \in \mathrm{S}\left(K_{y_i}, B\left(\phi_0(y_i), \frac{\epsilon}{2}\right)\right)$. To complete the proof we refer back to Definition (3.1.5.) and notice that

$$\phi \in S(\overline{V}, H) \Rightarrow \phi(\overline{V}) \subseteq \phi_0(N) \cap H = \phi_0(V_2),$$

so $\phi \in S(\overline{V}, H)$.■

Consider the countable family of maps

$$\mathcal{P} = \{F_p : \Lambda^p \to \mathcal{S}_\Gamma,\ p \in \mathbb{N}\}$$

defined as follows

$$F_p(\lambda_1, \lambda_2, ..., \lambda_p) = \Gamma_{\lambda_p} \circ \cdots \circ \Gamma_{\lambda_1}. \tag{13}$$

Then we have the following:

Lemma 3.1.8. For every $p \in \mathbb{N}$ the map F_p defined in (13) is continuous.

Proof: Let $\phi = \Gamma_{\bar{\lambda}_p} \circ \cdots \circ \Gamma_{\bar{\lambda}_a} \in \mathcal{S}_\Gamma$ for some $(\bar{\lambda}_p, \ldots, \bar{\lambda}_1) \in \Lambda^p$ and let $S(C, V)$ be a subbasic open neighborhood of ϕ in $\mathcal{S}_\Gamma$ in the COT. We want to find an open neighborhood U of $(\bar{\lambda}_1, \ldots, \bar{\lambda}_p)$ in Λ^p such that

$$(\lambda_1, \lambda_2, \ldots, \lambda_p) \in U \Rightarrow F_p(\lambda_1, \lambda_2, \ldots, \lambda_p) \in S(C, V).$$

Clearly $\Gamma_{\bar{\lambda}_p} \circ \cdots \circ \Gamma_{\bar{\lambda}_1}(C) \subseteq V$. Because the action $\Gamma: M \times \Lambda \to M$ is C^1, for each $x \in C \subseteq M$ there exist an open neighborhood $W_x \subseteq M$ of x and an open neighborhood U_x of $(\bar{\lambda}_1, \ldots, \bar{\lambda}_p)$ in Λ^p such that $\Gamma(W_x \times U_x) \subseteq V$. Here, by abuse of notation

$$\Gamma(W_x \times U_x) = \bigcup_{(\lambda_p, \ldots, \lambda_1) \in U_x} \Gamma_{\lambda_p} \circ \cdots \circ \Gamma_{\lambda_1}(W_x).$$

Consider an open cover in M of the compact set C with the family $\{W_x | x \in C\}$ and extract a finite subcover $\{W_{x_j}\}_{j=1}^n$. Thus we have $\Gamma(W_{x_j} \times U_{x_j}) \subseteq V$ for every $j = 1, 2, \ldots, n$. Let $U = \bigcap_{j=1}^n U_{x_j}$. Then $\Gamma(C \times U) \subseteq V$. It follows that

$$(\lambda_1, \lambda_2, \ldots, \lambda_p) \in U \Rightarrow F_p(\lambda_1, \lambda_2, \ldots, \lambda_p)(C) \subseteq V,$$

and we infer that $F_p(\lambda_1, \lambda_2, \ldots, \lambda_p) \in S(C, V)$.■

In the following proposition we assume $\mathcal{S} = \mathcal{S}_\Gamma$ is the pseudo semi-group generated by an action $\Gamma: M \times \Lambda \to M$ and $\mathcal{P}$ is the family of parametrization maps defined by (13). This proposition gives another general situation in which the existence of stable submanifolds is assured.

Proposition 3.1.9. For every $x_0 \in M$ the $k(x_0)$- dimensional imbedded submanifolds $N_j(x_0)$ of M defined in Claim 1 of Theorem 2.1.3 are stable with respect to perturbations induced by $\mathcal{P}$.

Proof: Let $\phi_0 = \Gamma_{\bar{\mu}_q} \circ, \cdots, \circ \Gamma_{\bar{\mu}_1} \in \mathcal{S}_\Gamma$, where $(\bar{\mu}_1, \ldots \bar{\mu}_q) \in \Lambda^q$, and let $x \in N_j(x_0)$ with $x \in \text{dom}(\phi_0)$ be given. Then $x = \Gamma_{\bar{\lambda}_p} \circ, \cdots, \circ \Gamma_{\bar{\lambda}_1}(x_0)$ for some $(\bar{\lambda}_1, \ldots \bar{\lambda}_p) \in \Lambda^p$ and the map

$$(\lambda_1, \lambda_2, ..., \lambda_p) \longmapsto \Gamma_{\lambda_p} \circ \cdots \circ \Gamma_{\lambda_1}(x_0)$$

has rank $k(x_0)$ at $(\bar{\lambda}_1, \cdots \bar{\lambda}_p)$. The following two maps

$$g_1: (\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_q) \longmapsto \Gamma_{\mu_q} \circ \cdots \circ \Gamma_{\mu_1} \circ \Gamma_{\lambda_p} \circ \cdots \circ \Gamma_{\lambda_1}(x_0)$$

and

$$g_2 : (\lambda_1, \ldots, \lambda_p) \longmapsto \Gamma_{\bar{\mu}_q} \circ \cdots \circ \Gamma_{\bar{\mu}_1} \circ \Gamma_{\lambda_p} \circ \cdots \circ \Gamma_{\lambda_1}(x_0)$$
$$= \Gamma_{\bar{\mu}_q} \circ \cdots \circ \Gamma_{\bar{\mu}_1} \circ f_p(\lambda_1, \ldots, \lambda_p)$$

also have rank $k(x_0)$ at $(\bar{\lambda}_1, \ldots \bar{\lambda}_p, \bar{\mu}_1, \ldots \bar{\mu}_q)$ and $(\bar{\lambda}_1, \cdots, \bar{\lambda}_p)$, respectively.

Clearly image(g_2) $\subseteq$ image(g_1). By the maximality of $k(x_0)$ there exist open neighborhoods $U_p \subseteq \Lambda^p$ of $(\bar{\lambda}_1, \ldots, \bar{\lambda}_p)$ and $U_q \subseteq \Lambda^q$ of $(\bar{\mu}_1, \cdots \bar{\mu}_q)$ such that f_p and g_1 still have rank $k(x_0)$ at every $(\lambda_1, \ldots, \lambda_p) \in U_p \subseteq \Lambda^p$ and $(\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_q) \in U_p \times U_q \subseteq \Lambda^{p+q}$ respectively. By the rank theorem the sets U_p and U_q can be chosen so that $g_1(U_p \times U_q)$ and $g_2(U_p)$ are $k(x_0)$-dimensional imbedded submanifolds of M, both of which contain $y = \Gamma_{\bar{\mu}_q} \circ, \cdots, \circ \Gamma_{\bar{\mu}_1}(x)$. Thus $g_2(U_p)$ is open in $g_1(U_p \times U_q)$. Therefore we can further shrink the sets U_p and U_q to open neighborhoods $\widetilde{U}_p \subseteq \Lambda^p$ of $(\bar{\lambda}_1, \cdots, \bar{\lambda}_p)$ and $\widetilde{U}_q \subseteq \Lambda^q$ of $(\bar{\mu}_1, \cdots \bar{\mu}_q)$ with

$$g_1(\widetilde{U}_p \times \widetilde{U}_q) \subseteq g_2(U_p) \subseteq \phi_0(N_j(x_0) \cap \mathrm{dom}(\phi_0)).$$

By the local compactness of Λ^P the point $(\bar{\lambda}_1, \cdots, \bar{\lambda}_p) \in \Lambda^p$ has an open neighborhood $\widetilde{U}'_p$ of in Λ^p such that $\mathrm{Cl}_{\Lambda^p}(\widetilde{U}'_p)$ is a compact subset of Λ^p and

$$\widetilde{U}'_p \subseteq \mathrm{Cl}_{\Lambda^p}(\widetilde{U}'_p) \subseteq \widetilde{U}_p.$$

Let $V_1 = f_p(\widetilde{U}'_p)$, so we have $V_1 \subseteq f_p(\widetilde{U}_p)$. The map f_p has rank k at every point of $\widetilde{U}_p$ and by the rank theorem both V_1 and $f_p(\widetilde{U}_p)$ are open subsets of $N_j(x_0)$. Moreover, since $f_p : \Lambda^p \to M$ is continuous $f_p(\mathrm{Cl}_{\Lambda^p}(\widetilde{U}'_p)$ is a compact, hence closed, subset of M, so $\mathrm{Cl}_M(V_1) = f_p(\mathrm{Cl}_{\Lambda^p}(\widetilde{U}'_p)$. It follows that $\mathrm{Cl}_M(V_1) \subseteq N_j(x_0)$. Then for $W_0 = \widetilde{U}_q$, which is an open neighborhood of $(\bar{\mu}_1, \cdots \bar{\mu}_q)$, we see that

$$(\mu_1, \ldots, \mu_q) \in W_0 \Rightarrow \Gamma_{\mu_q} \circ \cdots \circ \Gamma_{\mu_1}(\mathrm{Cl}_M(V_1)) \subseteq \phi_0(N_j(x_0) \cap \mathrm{dom}(\phi_0)).$$

Referring back to the notations in definition (3.1.5.) we conclude that $N_j(x_0)$ is $(\mathcal{S}, \mathcal{P})$-stable.■

3.2. The Main Theorem

Our primary goal in this section is to provide a sufficient condition for an action to have the normal accessibility property when its ambient pseudo semi-group of local diffeomorphisms has the accessibility property.

Definition 3.2.1. Let $\mathcal{S}$ be a parametrized pseudo semi-group of local diffeomorphisms in $\mathrm{Diff}^1_{loc}(M)$ with the COT, let $\mathcal{P} = \{\rho_\alpha : E_\alpha \to \mathcal{S} | \alpha \in \mathcal{A}\}$ be a parametrization of $\mathcal{S}$, and let $\Gamma : \ M \times \Lambda \to M$ be an action on M. Then we say that the action Γ is dense relative to the parametrization $\mathcal{P}$ if:

(a) $\lambda \in \Lambda \Rightarrow \Gamma_\lambda \in \mathcal{S}$, so $\mathcal{S}$ contains the PSG $\mathcal{S}_\Gamma$ generated by Γ;

(b) $\forall \alpha \in \mathcal{A}$, $\forall$ open set $W \subseteq E_\alpha$ $\exists \phi \in \mathcal{S}_\Gamma$ such that $\phi \in \rho_\alpha(W)$.

Theorem 3.2.2. Let $\mathcal{S}$, $\mathcal{P}$, and Γ be as in Definition 3.2.1, and assume additionally that Λ is second countable. If for any $x_0 \in M$ the $k(x_0)$-dimensional imbedded submanifolds $N_j(x_0)$ (as defined in Claim 1 of Theorem 2.1.3) are $(\mathcal{S}, \mathcal{P})$-stable and the action Γ is dense relative to the parametrization $\mathcal{P}$, then the accessibility property of $\mathcal{S}$ implies the normal accessibility property of Γ.

Proof: Assume for contradiction that $\mathcal{S}$ has the accessibility property but Γ does not have the normal accessibility property. Then there exists $x_0 \in M$ such that $k = k(x_0) < m = \dim(M)$, where $k(x_0)$ is defined as in the proof of Theorem 2.1.3. Pick $x_1 \in N_j(x_0) \subseteq AN_k^\Lambda(x_0)$. Let $\phi_0 \in \mathcal{S}$ be any element with $x_1 \in \mathrm{dom}(\phi_0)$ and select $\alpha \in \mathcal{A}$ and $\xi_0 \in E_\alpha$ such that $\phi_0 = \rho_\alpha(\xi_0)$. Because $N_j(x_0)$ is $(\mathcal{S}, \mathcal{P})$-stable Proposition 3.1.7 implies that there exists an open neighborhood V_1 of x_1 relative to $N_j(x_0)$ and an open neighborhood W_0 of ξ_0 in E_α such that $\overline{V}_1 \subsetneq N_j(x_0)$ and

$$\xi_1, \xi_2 \in W_0 \Rightarrow \rho_\alpha(\xi_2)^{-1} \circ \rho_\alpha(\xi_1)(\overline{V_1}) \subsetneq N_j(x_0).$$

Since Γ is dense relative to the parametrization $\mathcal{P}$ there exists $\psi \in \mathcal{S}_\Gamma$ such that $\psi = \rho_\alpha(\zeta)$ for some $\zeta \in W_0$. Setting $\xi_1 = \xi_0$, so $\rho_\alpha(\xi_0) = \phi_0$ and $\xi_2 = \zeta$ we obtain

$$\rho_\alpha(\zeta)^{-1} \circ \rho_\alpha(\xi_0)(\overline{V}_1) \subseteq N_j(x_0)$$

$$\Rightarrow (\psi^{-1} \circ \phi)(\overline{V}_1) \subseteq N_j(x_0)$$

$$\Rightarrow \phi_0(\overline{V}_1) \subseteq \psi(N_j(x_0) \cap \mathrm{dom}(\psi)).$$

In particular,

$$\phi_0(x_1) \subseteq \psi(N_j(x_0) \cap \mathrm{dom}(\psi)).$$

From Claim 2 of Theorem 2.1.3 we recall that by the maximality property of $k = k(x_0)$ the set $AN_k^\Lambda(x_0)$ is invariant under the action Γ. Therefore

$$\phi_0(x_1) \in AN_k^\Lambda(x_0)$$

Since ϕ_0 was an arbitrary element of $\mathcal{S}$ with $x_1 \in \mathrm{dom}(\phi_0)$ we conclude that

$$\mathcal{O}_\mathcal{S}^+(x_1) \subseteq AN_k^\Lambda(x_0).$$

By the accessibility property of $\mathcal{S}$ $\mathrm{Int}\mathcal{O}_\mathcal{S}^+(x_1) \neq \emptyset$, so $\mathrm{Int}AN_k^\Lambda(x_0) \neq \emptyset$. In Claim 1 of Theorem 2.1.3 we showed that when $k(x_0) < m$ the set $AN_k^\Lambda(x_0)$ is of the first category and as such cannot have nonempty interior since M is a Baire space. We have arrived at a contradiction, and therefore $k = m$. Thus Γ must have the normal accessibility property.■

3.3. The Vector Field Case under the Framework of Actions

Having proved Theorem 3.2.2 one may expect that the well-known result providing a sufficient condition for the normal accessibility property of a family of vector fields [G2] can be derived as a particular case. This is the issue we are going to address in this final part of the thesis. First we introduce the necessary definitions and prove two lemmas that, along with the results obtained in Section 3.2, shall pave the way for the principal result of this section.

Given the control system $f: \ M \times \Omega \to TM$, recall from Section 0.2 that we have the associated global flow, or response, of the control system for the initial condition (t_0, x_0)

$$\mu_f : t \to \mu_f(t, \ t_0, \ x_0, \ u),$$

where u is an admissible control. By admissibility of a control we mean that the map $(t, x) \longmapsto f(x, u(t))$ satisfies local C^1 Carathéodory conditions in its local representation with respect to every coordinate chart of M. The pertinent definitions are recalled here.

Definition 3.3.1. Let $V \subseteq \mathbb{R}^m$ be open. A map $h : V \times \mathbb{R} \mapsto \mathbb{R}^m$ is said to satisfy the C^1 Carathéodory conditions if:

(i) for every $t \in \mathbb{R}$ the map $y \mapsto h(y, t)$ is C^1;

(ii) for every $y \in \mathbb{R}^m$ the maps $t \mapsto h(y, t), \ t \mapsto D_1 h(y, t)$ are measurable;

(iii) for every $(y_0, t_0) \in V \times \mathbb{R}^m$ there exist $\delta > 0$ and $\lambda \in L^1[t_0 - \delta, t_0 + \delta]$ such that $\| h(y, t) \| + \| D_1 h(y, t) \| \leq \lambda(t)$ (i.e. h and $D_1 h$ are locally L^1 bounded on $V \times \mathbb{R}$).

The set of all admissible controls for f is denoted by $\mathcal{U}_{\text{meas}}(f)$. Let $\mathcal{U}_{\text{pc}}$ and $\mathcal{U}_{\text{meas}}$ denote the set of all piecewise constant and Lebesgue-measurable maps

$\mathbb{R} \to \Omega$ respectively. Then $\mathcal{U}_{\text{pc}} \subseteq \mathcal{U}_{\text{meas}}(f) \subseteq \mathcal{U}_{\text{meas}}$. Following [GS] we define a useful topology on $\mathcal{U}_{\text{meas}}(f)$, called the f-topology, as follows:

(i) Let $V \subseteq \mathbb{R}^m$ be an open and let $h : V \times \mathbb{R} \to \mathbb{R}^m$ satisfy C^1 Carathéodory conditions. For a compact subset $Q \subseteq V$ the map

$$t \mapsto \sup\{\| h(y,t) \| + \| D_1 h(y,t) \| : y \in Q\}$$

is measurable and locally L^1 bounded. Define

$$\| h \|_{Q,[a,b]} = \int_a^b \sup\{\| h(y,t) \| + \| D_1 h(y,t) \| : y \in Q\} dt$$

for $[a,b] \subset \mathbb{R}$.

(ii) For a C^1 control system $f : M \times \Omega \to TM$, an admissible control $u \in \mathcal{U}_{\text{meas}}(f)$, and a chart $\phi : \text{dom}\phi \to \mathbb{R}^m$ of M, we let $f_\phi^u : \text{dom}\phi \times \mathbb{R} \to \mathbb{R}^m$ denote the map

$$f_\phi^u(y,t) = f_\phi(y, u(t)),$$

where

$$f_\phi(y,u) = d\phi_{\phi^{-1}(y)} f(\phi^{-1}(y), u)$$

is the local representative of f with respect to ϕ. Note that f_ϕ^u satisfies C^1 Carathéodory conditions.

(iii) For f and ϕ as in (ii), for $K \subseteq \text{dom}\phi$ compact, for $l \in \mathbb{N}$ and for $u, v \in \mathcal{U}_{\text{meas}}(f)$, we let

$$\rho_{\phi,K,l}(u,v) = \| f_\phi^u - f_\phi^v \|^1_{\phi(K),[-l,l]},$$

Then $\rho_{\phi,K,l}$ is a pseudo metric on $\mathcal{U}_{\text{meas}}(f)$.

Definition 3.3.2. The topology on $\mathcal{U}_{\text{meas}}(f)$ generated by the family of pseudo metrics

$$\{\rho_{\phi,K,l} | \phi : \text{dom}\phi \to \mathbb{R}^m \text{ a chart of } M,\ K \subseteq \text{dom}\phi \text{ compact},\ l \in \mathbb{N}\}$$

is called the f-topology on $\mathcal{U}_{\text{meas}}(f)$. It is known from [GS] that the f-topology is pseudometrizable and $\mathcal{U}_{\text{pc}}$ is a dense subset of $\mathcal{U}_{\text{meas}}(f)$ in the f-topology. Another nice feature of the f-topology is that if $u_k \to u$ in the f-topology, then the response corresponding to u_k converges to the response of f corresponding to u uniformly on compact intervals.

Let Δ be a countable dense subset of Ω and consider the disjoint union

$$\Lambda = \bigcup_{\omega \in \Delta} (0, \infty) \times \{\omega\}$$

Then Λ is a second countable, one-dimensional manifold, and $\mathcal{U}_{\text{pc}}^{\Delta}$ is dense in $\mathcal{U}_{\text{meas}}(f)$. Define the action Γ as follows

$$\Gamma(x, (t, \omega)) = \mu_f(t, 0, x, \omega).$$

We next consider a pseudo semi-group generated by the global flow μ_f. More precisely, we let

$$\mathcal{S} = \{\mu_f(t_2, t_1, \cdot, u) |\ t_2 \geq t_1,\ u \in \mathcal{U}_{\text{meas}}(f)\}$$

and let

$$\mathcal{S}_\Gamma = \{\mu_f(t_2, t_1, \cdot, u) |\ t_2 \geq t_1 \geq 0,\ u \in \mathcal{U}_{\text{pc}}^{\Delta}\}.$$

Clearly $\mathcal{S}_\Gamma \subseteq \mathcal{S}$. We claim that $\mathcal{S}_\Gamma$ and $\mathcal{S}$ are PSGs. Note that if $\phi = \mu_f(t_2, t_1, \cdot, u)$ and $\psi = \mu_f(s_2, s_1, \cdot, v)$, then because the system is time-invariant one can use properties of the global flow to obtain

$$\phi \circ \psi = \mu_f(t_2 - t_1 + s_2 - s_1, 0, \cdot, w)$$

where

$$w = \begin{cases} v, & 0 \leq t \leq s_2 - s_1 \\ u, & s_2 - s_1 \leq t \leq t_2 - t_1 + s_2 - s_1 \end{cases}$$

This establishes that both $\mathcal{S}$ and $\mathcal{S}_\Gamma$ are pseudo semi-groups. We assume $\mathcal{S}$ has the COT. Consider the map:

$$h : D \longrightarrow \mathcal{S}$$

where $D = \{(s, \tau, u) | s \geq \tau, \ u \in \mathcal{U}_{\text{meas}}(f)\}$ and

$$h(s, \tau, u) = \mu_f(s, \tau, \cdot, u) \tag{14}$$

We give D the topology it inherits as a subspace of $\mathbb{R} \times \mathbb{R} \times \mathcal{U}_{\text{meas}}(f)$. Let $\mathcal{P}$ be the parametrization of $\mathcal{S}$ induced by the map h. Then we have the following two elementary lemmas:

Lemma 3.3.3. The map h given by (14) is continuous.

Proof: Indeed, let $S(K, W)$ a subbasic open neighborhood of $\mu_f(t_2, t_1, \cdot, u_0) \in \mathcal{S}$, where $K \subseteq M$ is compact and W is an open subset of M, so

$$\mu_f(t_2, t_1, K, u_0) \subseteq W.$$

Since the global flow as a function from $\mathbb{R} \times \mathbb{R} \times M \times \mathcal{U}_{\text{meas}}(f)$ into M is continuous, for every $x \in K$ there exist $\epsilon_x > 0$, an open neighborhood V_x of x in M, and open neighborhood $\mathcal{H}_x$ of u_0 in $\mathcal{U}_{\text{meas}}(f)$ such that

$$|s - t_2| < \epsilon_x, \ |\tau - t_1| < \epsilon_x, \ u \in \mathcal{H}_x, s > \tau \Rightarrow \mu_f(s, \tau, V_x, u) \subseteq W$$

Extract a finite subcover $\{V_{x_1}, \ldots, V_{x_r}\} \subseteq \{V_x\}_{x \in K}$ and set

$$\epsilon = \min\{\epsilon_{x_1}, \ldots, \epsilon_{x_r}\}, \ \mathcal{H} = \bigcap_{i=1}^{r} \mathcal{H}_{x_i}.$$

Then we obtain

$$|s - t_2| < \epsilon, \ |\tau - t_1| < \epsilon, \ u \in \mathcal{H}, s > \tau \Rightarrow \mu_f(s, \tau, \cdot, u) \in S(K, W)$$

proving the continuity of h.■

Lemma 3.3.4. Γ is dense in $\mathcal{S}$ relative to the parametrization $\mathcal{P}$.

Proof: Let $\phi_0 = \mu_f(\bar{t}_2, \bar{t}_1, \cdot, u_0) \in \mathcal{S}$ and W be an open neighborhood of $(\bar{t}_2, \bar{t}_1, u_0)$ in the product topology on $D = \{(s, \tau, u) | s \geq \tau, \ u \in \mathcal{U}_{\text{meas}}(f)\}$, where the $\mathbb{R}$ factors have the usual topology and $\mathcal{U}_{\text{meas}}(f)$ has the f-topology. Then there exists an open neighborhood H of u_0 in $\mathcal{U}_{\text{meas}}(f)$ such that

$$u \in H \Rightarrow (\bar{t}_2, \bar{t}_1, u) \in W.$$

Because $\mathcal{U}_{\text{pc}}^{\Delta}$ is dense in $\mathcal{U}_{\text{meas}}(f)$ in the f-topology $\exists v \in H \cap \mathcal{U}_{\text{pc}}^{\Delta}$. Then clearly

$$h(\bar{t}_2, \bar{t}_1, v) = \mu_f(\bar{t}_2, \bar{t}_1, \cdot, v) \in \mathcal{S}_{\Gamma}.$$

This shows that Γ is dense in $(\mathcal{S}, \mathcal{P})$.■

Among other things K. A. Grasse and H. J. Sussmann proved in Theorem 3.17 of [GS] that if Ω is a separable metric space, $\Delta \subseteq \Omega$ is a countable dense subset and $f: \ M \times \Omega \to TM$ is a C^1 control system on M then

$$\mathcal{R}_f^+(x | \mathcal{U}_{\text{meas}}(f)) = M \ \forall x \in M \Rightarrow \mathcal{R}_f^+(x | \mathcal{U}_{\text{pc}}^{\Delta}) = M \ \forall x \in M.$$

In other words, if f is globally controllable by controls in $\mathcal{U}_{\text{meas}}(f)$ then f is globally controllable by controls in $\mathcal{U}_{\text{pc}}^{\Delta}$. Theorem 1.1.6 is a modification of the above result and our objective now is to provide a new proof of the latter using the formalism of a pseudo semi-group containing an action. This new proof is presented in the following theorem.

Theorem 3.3.5. If $\mathcal{S}$ has the accessibility property, then Γ has the normal accessibility property.

Proof: By Theorem 3.2.2 and Lemma 3.3.4 it suffices to verify the $(\mathcal{S},\mathcal{P})$- stability condition of the imbedded $k(x_0)$-dimensional submanifolds $N_j(x_0)$ defined in Claim 1 of Theorem 2.1.3. The maximality property of k implies that f is tangent to $N_j(x_0)$ [GS]. Given such an imbedded $k(x_0)$-dimensional submanifold N_1, $u \in \mathcal{U}_{\text{meas}}(f)$ and $x_1 \in N_1$ let G denote the set of real numbers defined as follows:

$$\begin{aligned} G = &\{t \in [\bar{t}_0, \bar{t}_1] \mid \forall s \in [\bar{t}_0, t]\ \exists\delta(s) > 0, \text{ an open neighborhood } W_s \text{ of } x_1 \\ &\text{relative to } N_1, \text{and an open neighborhood } A_s \text{ of } u \text{ in } \mathcal{U}_{\text{meas}}(f) \text{ such that} \\ &|\sigma - s| < \delta(s),\ |\tau - \bar{t}_0| < \delta(s),\ x \in W_s,\ v \in A_s \Rightarrow \mu_f(\sigma,\tau,x,v) \in N^s\}, \end{aligned}$$

where $N^s = \mu_f(s,\bar{t}_0,N_1,u)$ (and hence $N^{\bar{t}_0} = N_1$), and we assume N_1 is small enough so that $\mu_f(s,\bar{t}_0,x,u)$ is defined $\forall s \in [\bar{t}_0,\bar{t}_1]$ and $\forall x \in N_1$.

Comment 3.3.6. If $t \in G$ and V is an open neighborhood of $\mu_f(t,\bar{t}_0,x_1,u)$ relative to N^t, then we can shrink $\delta(t)$, W_t and A_t to obtain

$$|\sigma - t| < \delta(t),\ |\tau - \bar{t}_0| < \delta(t),\ x \in W_t,\ v \in A_t \Rightarrow \mu_f(\sigma,\tau,x,v) \in \mathrm{V}.$$

Proof: Let $\delta(t)$, W_t, A_t be as in the definition of G and let V be an open neighborhood of $\mu_f(t,\bar{t}_0,x_1,u)$ relative to N^t. Since N^t is an imbedded submanifold of M, its topology is the relative topology induced by that of M, so there exists an open neighborhood $\widetilde{V}$ of $\mu_f(t,\bar{t}_0,x_1,u)$ in M such that $V = N^t \cap \widetilde{V}$. Since $\mu_f(t,\bar{t}_0,x_1,u) \in \widetilde{V}$ and the global flow is continuous, there exist $\delta' > 0$, an open neighborhood $\widetilde{W_{t'}}$ of x_1 relative to M, and an open neighborhood A_t' of u in $\mathcal{U}_{\text{meas}}(f)$ such that

$$|\sigma - t| < \delta',\ |\tau - \bar{t}_0| < \delta',\ x \in W_{t'},\ v \in A_{t'} \Rightarrow \mu_f(\sigma,\tau,x,v) \in \widetilde{V}.$$

Then

$$|\sigma - t|,\ |\tau - t_0| < \min\{\delta(t), \delta'\},\ x \in W_t \cap \widetilde{W}_{t'} \text{ and } v \in A_t \cap A_{t'}$$

$$\Rightarrow \mu_f(\sigma, \tau, x, v) \in \widetilde{V} \text{ and } \mu_f(\sigma, \tau, x, v) \in N^t$$

$$\Rightarrow \mu_f(\sigma, \tau, x, v) \in \widetilde{V} \cap N^t = V.$$

This proves 3.3.6.

It is easy to see that $G \neq \emptyset$, and in particular $\bar{t}_0 \in G$. Indeed, since the control vector field f is tangent to N_1 it induces a continuous flow on N_1. From $\mu_f(\bar{t}_0, \bar{t}_0, x_1, u) = x_1$ it follows that there exist $\delta > 0$, an open neighborhood W_{t_0} of x_1 relative to N_1 and an open neighborhood A_{t_0} of u in $\mathcal{U}_{\text{meas}}(f)$ such that

$$|\sigma - \bar{t}_0| < \delta,\ |\tau - \bar{t}_0| < \delta,\ x \in W_{t_0},\ v \in A_{t_0}$$

$$\Rightarrow \mu_f(\sigma, \tau, x, v) \in N_1 = N^{\bar{t}_0}.$$

Thus $\bar{t}_0 \in G$.

Let $t_* = \sup G$. Clearly $[\bar{t}_0, t_*) \subseteq G$. and $\bar{t}_0 \leq t_* \leq \bar{t}_1$.

Claim : $t_* \in G$ and $t_* = \bar{t}_1$ (implying $\bar{t}_1 \in G$).

Proof of the claim : Let $x_* = \mu_f(t_*, \bar{t}_0, x_1, u) \in N^{t_*} = \mu_f(t_*, \bar{t}_0, N_1, u)$. From [GS] we know that the control vector field f is tangent to N^{t_*} and thus induces a continuous flow on N^{t_*}. Hence there exist $\delta_* > 0$, an open neighborhood V_* of x_* relative to N^{t_*} and an open neighborhood A_* of u in $\mathcal{U}_{\text{meas}}(f)$ such that

$$\begin{aligned} &|t - t_*| < \delta_*,\ |s - t_*| < \delta_*,\ x \in V_*,\ v \in A_* \Rightarrow \\ &\mu_f(t, s, x, v) \text{ is defined and in } N^{t_*}. \end{aligned} \tag{15}$$

Let W_* be the open neighborhood of x_1 relative to N_1 for which

$$\mu_f(t_*, \bar{t}_0, W_*, u) = V_*.$$

Since the map $\mu_f(t, \bar{t}_0, \cdot, u)$ is a diffeomorphism between $N_1 = N^{\bar{t}_0}$ and N^t, and W_* is an open neighborhood of x_1 relative to N_1, for $|t - t_*| < \delta_*$ $\mu_f(t, \bar{t}_0, W_*, u)$ is an open subset of N^t. By transitivity of the flow

$$|t - t_*| < \delta_* \Rightarrow \mu_f(t, \bar{t}_0, W_*, u) = \mu_f(t, t_*, \mu_f(t_*, \bar{t}_0, W_*, u), u) = \mu_f(t, t_*, V_*, u).$$

Moreover, $\mu_f(t, t_*, V_*, u)$ is open in N^{t_*} because V_* is open in N^{t_*} and the flow restricted to N^{t_*} is a local diffeomorphism. Thus for $|t - t_*| < \delta_*$

$$\mu_f(t, \bar{t}_0, W_*, u) \subseteq N^{t_*} \cap N^t, \tag{16}$$

and $\mu_f(t, \bar{t}_0, W_*, u)$ is open relative to both N^{t_*} and N^t. Next, $\mu_f(t, \bar{t}_0, x_1, u) = \mu_f(t, t_*, x_*, u)$ and $\mu_f(t_*, t_*, x_*, u) = x_* \in V_*$. Since V_* is an open subset relative to N^{t_*}, by continuity of the flow we can find a positive δ_0 such that

$$|t - t_*| < \delta_0 \Rightarrow \mu_f(t, \bar{t}_0, x_1, u) \in V_*.$$

Choose $t_1 \in G$ with

$$t_* - \min(\delta_0, \delta_*) < t_1 \leq t_*.$$

We then have

$$|t_1 - t_*| < \delta_* \Rightarrow \mu_f(t_1, \bar{t}_0, W_*, u) \subseteq N^{t_*} \cap N^{t_1},$$

and $\mu_f(t_1, \bar{t}_0, W_*, u)$ is open relative to both N^{t_*} and N^{t_1}. In addition

$$|t_1 - t_*| < \delta_0 \Rightarrow \mu_f(t_1, \bar{t}_0, x_1, u) \in V_*.$$

Since V_* is open relative to N^{t_*}, we can choose an open neighborhood $\widetilde{W}_* \subseteq W_*$ of x_1 relative to N_1 so that

$$\mu_f(t_1, \bar{t}_0, \widetilde{W}_*, u) \subseteq V_*.$$

Clearly $\mu_f(t_1, \bar{t}_0, \widetilde{W}_*, u)$ is an open neighborhood of $\mu_f(t_1, \bar{t}_0, x_1, u)$ relative to N^{t_1}. Because $t_1 \in G$, Comment 3.3.6. implies that $\exists \delta = \delta(t_1) > 0$, an open neighborhood W_{t_1} of x_1 relative to N_1, and an open neighborhood A_{t_1} of u in $\mathcal{U}_{\text{meas}}(f)$ such that

$$|\sigma - t_1| < \delta(t_1),\ |\tau - \bar{t}_0| < \delta(t_1),\ x \in W_{t_1},\ v \in A_{t_1} \Rightarrow$$

$$\mu_f(\sigma, \tau, x, v) \in \mu_f(t_1, \bar{t}_0, \widetilde{W}_*, u) \subseteq V_*.$$

Then

$$|t - t_*| < \delta_*,\ |\tau - \bar{t}_0| < \delta(t_1),\ x \in W_{t_1},\ v \in A_{t_1} \cap A_* \Rightarrow$$

$$\mu_f(t_1, \tau, x, v) \in V_*.$$

We have that $\mu_f(t_1, \tau, x, v) \in V_*$, $|t - t_*| < \delta_*$, $|t_1 - t_*| < \delta_*$ and $v \in A_*$. Recalling the definition of δ_*, V_*, and A_* we infer that

$$\mu_f(t, \tau, x, v) = \mu_f(t, t_1, \mu_f(t_1, \tau, x, v), v) \in N^{t_*}.$$

Thus $t_* \in G$.

If $t_* = \bar{t}_1$, then we are done, so assume that $t_* < \bar{t}_1$. Let $\delta_* > 0$, V_*, $x_* \in V_* \subseteq N^{t_*}$ and $A_* \subseteq \mathcal{U}_{\text{meas}}(f)$ be chosen to satisfy condition (15). Fix $t \in (t_*, t_* + \delta_*)$. Then by the transitivity of the global flow and condition (16) we have

$$\mu_f(t, t_*, V_*, u) = \mu_f(t, t_*, \mu_f(t_*, \bar{t}_0, W_*, u), u) = \mu_f(t, \bar{t}_0, W_*, u) \subseteq N^{t_*} \cap N^t.$$

In addition, the map $\mu_f(t,t_*,\cdot,u)$ is a local diffeomorphism of $V_* \subseteq N^{t_*}$ into N^{t_*} by the choice of δ_* and V_*, and by transitivity $\mu_f(t,t_*,\cdot,u)$ is a local diffeomorphism of N^{t_*} and N^t. Therefore $\mu_f(t,t_*,V_*,u)$ is open relative to both N^{t_*} and N^t in their submanifold topologies. Clearly if $\widetilde{V_*} \subseteq V_*$ is another open neighborhood of x_* relative to N^{t_*}, then

$$\mu_f(t,t_*,\widetilde{V_*},u) \subseteq N^{t_*} \cap N^t$$

and $\mu_f(t,t_*,\widetilde{V_*},u)$ is open relative to each submanifold. Next, the control vector field f is tangent to N^t and induces a continuous flow on N^t. Hence there exist $\delta_t > 0$, an open neighborhood U_t of $x_t = \mu_f(t,\bar{t}_0,x_1,u)$ relative to N^t and an open neighborhood A_t of u in $\mathcal{U}_{\text{meas}}(f)$ such that

$$\begin{aligned} &|\sigma - t| < \delta_t,\ |s-t| < \delta_t,\ x \in U_t,\ v \in A_t \Rightarrow \\ &\mu_f(\sigma,s,x,v) \text{ is defined and in } N^t. \end{aligned} \tag{17}$$

Select an open neighborhood $\widetilde{V_*}$ of x_* relative to N^{t_*} such that $\widetilde{V_*} \subseteq V_*$ and

$$\mu_f(t,t_*,\widetilde{V_*},u) \subseteq U_t \subseteq N^t. \tag{18}$$

This is possible because $\mu_f(t,t_*,x_*,u) = x_t \in U_t$, the map $\mu_f(t,t_*,\cdot,u):\ N^{t_*} \to N^t$ is continuous and U_t is open relative to N^t. In the relative topology inherited from M on N^{t_*} the set $\mu_f(t,t_*,\widetilde{V}_*,u)$ is a relatively open subset of N^{t_*}. Therefore there exists an open subset U of M such that

$$\mu_f(t,t_*,\widetilde{V}_*,u) = N^{t_*} \cap U. \tag{19}$$

Because the control flow μ is continuous in its four variables as a map into M and $\mu_f(t,t_*,x_*,u) \in U$ there exist an open neighborhood V' of x_* in M and an open neighborhood A' of u in $\mathcal{U}_{\text{meas}}(f)$ such that

$$\mu_f(t,t_*,V',A') \subseteq U. \tag{20}$$

Then $V' \cap \widetilde{V_*}$ is an open neighborhood of x_* relative to N^{t_*}.

We already showed that $t_* \in G$. By definition of G there exist $\delta(t_*) > 0$, an open neighborhood W_t of x_1 relative to N_1 and an open neighborhood A_{t_*} of u in $\mathcal{U}\text{meas}(f)$ such that

$$|\tau - \bar{t_0}| < \delta(t_*),\ x \in W_t,\ v \in A_{t_*}$$
$$\Rightarrow \mu_f(t_*, \tau, x, v) \in N^{t_*}.$$

By Comment 3.3.6 $\delta(t_*) > 0$, W_t, and A_{t_*} can be chosen so that

$$\begin{aligned} &|\tau - \bar{t_0}| < \delta(t_*),\ x \in W_t,\ v \in A_{t_*} \\ &\Rightarrow \mu_f(t_*, \tau, x, v) \in V' \cap \widetilde{V}_*. \end{aligned} \tag{21}$$

Then (21) implies that

$$|\sigma - t| < \delta(t),\ |\tau - \bar{t}_0| < \delta(t_*),\ x \in W_t,\ v \in A_{t_*} \cap A_* \cap A_t \cap A' \Rightarrow$$

$$\Rightarrow \mu_f(t_*, \tau, x, v) \in V' \cap \widetilde{V}_*,$$

and consequently, from (15),(20) and the inclusion $\widetilde{V}_* \subseteq V_*$ we obtain

$$\mu_f(t, t_*, \mu_f(t_*, \tau, x, , v), v) \in \mu_f(t, t_*, V', v) \cap \mu_f(t, t_*, \widetilde{V}_*, v) \subseteq U \cap N^{t_*}. \tag{22}$$

¿From (22), recalling (19) and (18), we have

$$\mu_f(t, t_*, \mu_f(t_*, \tau, x, v), v) \in \mu_f(t, t_*, \widetilde{V}_*, u) \subseteq U_t \subseteq N^t. \tag{23}$$

¿From (23) and by the transitivity of the flow

$$|\sigma - t| < \delta(t),\ |\tau - \bar{t}_0| < \delta(t_*),\ x \in W_t,\ v \in A_{t_*} \cap A_* \cap A_t \cap A' \Rightarrow$$

$$\mu_f(\sigma, \tau, x, v) = \mu_f(\sigma, t, \mu_f(t, t_*, \mu_f(t_*, \tau, x, v), v), v) \in \mu_f(\sigma, t, U_t, v) \subseteq N^t.$$

This shows that $t \in G$, which contradicts $t > t_* = \sup G$. Therefore we must have $t_* = \bar{t}_1$ and thus $G = [\bar{t}_0, \bar{t}_1]$ and the claim is proved.

The conclusion of the theorem is now apparent: let $x_0 \in M$ and $\phi_0 = \mu_f(\bar{t}_1, \bar{t}_0, \cdot, u) \in \mathcal{S}$ be given. Pick $x_1 \in N_j(x_0)$. Then by the above claim there exist an open neighborhood V of x_1 relative to $N_j(x_0)$, a positive number δ and an open neighborhood A of u in $\mathcal{U}_{\mathrm{meas}}(f)$ such that

$$|\sigma - \bar{t}_1| < \delta, \ |\tau - \bar{t}_0| < \delta, \ x \in V, \ v \in A \Rightarrow$$

$$\mu_f(\sigma, \tau, x, v) \in \mu_f(\bar{t}_1, \bar{t}_0, N_j(x_0), u) = \phi_0(N_j(x_0))$$

The last statement shows that $N_j(x_0)$ is $(\mathcal{S}, \mathcal{P})$- stable.■

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