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THEORY OF PRE-CLOSURES

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THEORY OF PRE-CLOSURES

CHAPTER I

Introduction

Closure operators have appeared throughout the literature of mathematics in many guises. The first formulation of closure operators was given by E.H. Moore in 1910 in his investigations of general analysis [11] when he spoke of a property π of subsets of a set S being extensionally attainable if (i) S had the property π, and (ii) if a family $\{A_j\}_{j\varepsilon J}$ of subsets of S had the property π, then so did the set $\bigcap\{A_j: j\varepsilon J\}$. With this definition Moore called the least subset having the property π and containing a given set A the closure of A (with respect to π). If we construe the notion of satisfying a property π to simply mean membership in a given subfamily of the power set of S, then Moore's definition is in perfect agreement with that used today. As examples of such closure operations we can look at algebraische Hüllensystemen [17], the well-known Kuratowski closure operator, Lorenzen's proof of the General Existence Theorem for primitive recursion, etc.

In 1930 Tarski [21] axiomatized some of the fundamental concepts used in logic by means of closures which he called

Folgerungen--the fact that Folgerungen were closures was first noted by J. Schmidt in [16]. Tarski's work employed the current definition of a closure operator (if we drop his requirement that the domain be countable), and he developed most of the elementary properties of closed sets. His most important theorem--the theoretical possibility of extending a consistent set of statements to a complete system--failed to provide an approach toward finding an effective axiomatization of number theory because of Gödel's incompleteness theorem which appeared the following year, and unfortunately this seems to have ended Tarski's research into the structure of closure operators. G. Birkhoff brought closures back in his 1940 Lattice Theory as a way of looking at complete lattices and investigated (also later with O. Frink, Jr.) the lattice of closed subsets, and in the early forties O. Ore did intensive work in extending the theory of topological spaces to closure spaces as well as studying the lattice of all closures on a given set. In 1952 J. Schmidt gave his famous axiomatic characterization of closed sets of an algebraic closure. More recent papers on closure theory have been rather limited, so in this thesis some interesting new directions will be explored.

The approach used here will revolve about a less popular set mapping which I will call a pre-closure (the same name is used by D. Higgs; P. Hammer calls them expansive in [10]) which one encounters frequently--for example in the constructive definition of a subalgebra generated by a set of elements (see [20]), in the extension of a set of consistent statements, in the theory of semi-spaces, etc. Although pre-closures are usually associated with algebras, J. Schmidt points out their origin in Analysis [16]:

> "Im allgemeineren Fall wird man die Folge (M_n), die einer sukzessiven Hintereinanderausführung des Operators Q ihre Entstehung verdankt, durch eine transfinite in allgemeinen überzahlbaren Ordinaltypus zu ersetzen haben; Musterbeispiel hierfür sind die Baireschen Funktionenklassen, aus denen auch die Idee der gestuften Räume entsprungen ist."

We will be primarily working with pre-closures for general algebraic considerations--as tools to analyze closures as well as objects of interest in their own right. A probable reason for their lack of success to date has been their relegation to a secondary role, as for example the following passage by J. Schmidt states (see [16]):

> "Dieser Fundamentale Charakter kann meines Erachtens den sogenannten 'gestuften' oder 'mehrstufigen' Räumen, bei den der Begriff des Hüllenoperators durch weglassen des Axioms der Idempotenz abgeschwächt wird, nicht mehr zugebilligt werden. Jene allgemeineren Operatoren stellen im Grunde nichts als eine spezielle unter zahlreichen anderen Erzeugungsweisen von Hüllenstrukturen dar;

ein selbstständiges Interesse bieten sie darüber hinaus wohl kaum."

Pre-closures could be dispensed with in parts of the following just as almost any definition in mathematics can be eliminated at the expense of lengthier and more cumbersome proofs. But without strategically selected intermediate definitions it is often quite difficult to formulate an attack on a problem in a comprehensible manner, and many of the theorems in the following will show just how pleasant it is to have a calculus of pre-closures.

After examining the basic structure of the lattice of pre-closures we define several special families of pre-closures and look at their respective structures within the lattice of pre-closures, in particular whether the families are closed under union or intersection. Following this we look at how the various families are related, deducing an important theorem which will be needed in the representation theorems, and then Chapter I concludes with some results concerning the ties between supremum and composition.

Chapter II parallels some of the standard questions in general algebra and shows, for example, that every algebraic closure space can be isomorphically embedded in an N-ary closure space. Later closure congruences are characterized in a simple and elegant manner, and in the concluding section two interesting representation theorems are given. The author has attempted to give appropriate references to all theorems which are not original with this paper.

1. Structure of the Lattice of Pre-closures

Let S be a fixed set. A pre-closure on S is a set-mapping C from $\mathcal{P}(S)$ to $\mathcal{P}(S)$ which satisfies the two conditions:

(i) $P \subseteq C(P)$ for all P contained in S, and

(ii) $P \subseteq Q$ implies $C(P) \subseteq C(Q)$ for all P,Q in S.

A set mapping with property (i) is called extensive, and with (ii) isotone.

It is convenient to formulate the study of the family $\mathcal{C}$ of pre-closures on S in lattice-theoretical terms, so we will begin by defining a partial ordering $\subseteq$ on $\mathcal{C}$ by: $C_1 \subseteq C_2$ if and only if $C_1(P) \subseteq C_2(P)$ for all P contained in S. It is trivial to verify that $\subseteq$ actually is a partial ordering, and furthermore that with respect to this partial ordering the supremum and infimum (denoted by $\cup$ and $\cap$ respectively) of any two pre-closures exist, and are defined pointwise by $(C_1 \cup C_2)(P) = C_1(P) \cup C_2(P)$, $(C_1 \cap C_2)(P) = C_1(P) \cap C_2(P)$. Thus $(\mathcal{C}, \subseteq)$ defines a lattice which shall be denoted by $\mathcal{L}$. The first proposition shows that $\mathcal{L}$ inherits many of the properties of $\mathcal{P}(S)$.

PROPOSITION 1: $\mathcal{L}$ is a complete lattice satisfying the infinite distributive laws.

Proof: The proof is a straightforward consequence of the fact that $\mathcal{P}(S)$ has these properties and noting that if $\{C_a\}_{a\varepsilon A}$ is a subset of $\mathcal{L}$, then $\cup\{C_a: a\varepsilon A\}$ maps a subset of S, say P, into

$\bigcup\{C_a(P)\colon a\varepsilon A\}$, and dually $\bigcap\{C_a\colon a\varepsilon A\}$ maps the set P into $\bigcap\{C_a(P)\colon a\varepsilon A\}$.

Suppose I is an initial segment of the ordinals, and $\{C_\alpha\colon \alpha\varepsilon I\}$ is a family of pre-closures indexed by I. Then we define the <u>ordinal</u> <u>composition</u> $\mu\{C_\alpha\colon \alpha\varepsilon I\}$ recursively by $\mu\{C_\alpha\colon \alpha\leq\beta\} = C_\beta(\mu\{C_\alpha\colon \alpha<\beta\})$ for β in I, and $\mu\{C_\alpha\colon \alpha<\beta\} = \bigcup\{\mu\{C_\gamma\colon \gamma\leq\alpha\}\colon \alpha<\beta\}$ for all limit ordinals β which satisfy the condition that $\alpha<\beta$ implies $\alpha\varepsilon I$. For the special case in which all of the C_α are equal we will adopt a notation suggested by exponentiation, namely $C^\beta = \mu\{C_\alpha\colon \alpha\leq\beta\}$ if β is not a limit ordinal, and $C^\beta = \mu\{C_\alpha\colon \alpha<\beta\}$ if β is a limit ordinal. One of the most important properties of $\mathcal{L}$ is:

PROPOSITION 2: $\mathcal{L}$ is closed under ordinal composition.

Proof: (By transfinite induction on the length of the initial segment I.) If $I = \{1\}$, then $\mu\{C_\alpha\colon \alpha\varepsilon I\} = C_1$ which is a member of $\mathcal{L}$. Now suppose the theorem is true for all families with index set properly contained in I. Then we will consider two cases to show that it is also true for I.

Case (i). Suppose I has a maximal member β. Let $\{C_\alpha\colon \alpha\varepsilon I\}$ be a family of pre-closures indexed by I. Since $\mu\{C_\alpha\colon \alpha\varepsilon I\} = C_\beta(\mu\{C_\alpha\colon \alpha<\beta\})$, and since $\mu\{C_\alpha\colon \alpha<\beta\}$ is a member of $\mathcal{L}$ by the induction hypothesis, it suffices to show that the composition of two pre-closures is again a pre-closure. So suppose C_1 and C_2 are any two members of $\mathcal{I}$, and $P \subseteq S$. From $C_2C_1(P) \supseteq C_1(P) \supseteq P$ we have the extensive property, and if $A \subseteq B \subseteq S$, then $C_1(A) \subseteq C_1(B)$, so $C_2C_1(A) \subseteq C_2C_1(B)$, and thus C_2C_1 is a member of $\mathcal{I}$.

Case (ii). Suppose I does not have a maximal member. Then $\mu\{C_\alpha : \alpha\varepsilon I\} = \bigcup\{\mu\{C_\alpha : \alpha\varepsilon I_1, I_1$ is an initial segment of the ordinals properly contained in $I\}\}$, and combining the induction hypothesis with the completeness of $\mathcal{I}$ gives the conclusion.

Evelyn Nelson [12] has pointed out that, considering composition as a binary operation, $(\mathcal{I}, \cdot)$ is a positively ordered semi-group, and from this she deduces some interesting results related to those which will appear in section 4 of this chapter.

Although $\mathcal{P}(S)$ is a Boolean algebra, such is not the case with $\mathcal{I}$ because of:

PROPOSITION 3: A pre-closure C has a complement in $\mathcal{I}$ if and only if $C(A \cup B) = C(A) \cup B$ for all A and B in $\mathcal{P}(S)$.

Proof: Noting that the all element of $\mathcal{I}$ is given by the mapping which takes P into S, and the null element of $\mathcal{I}$ by the mapping which takes P into P, we have the result that $\overline{C}$ is the complement of C in $\mathcal{I}$ implies $C(P) \cup \overline{C}(P) = S$ and $C(P) \cap \overline{C}(P) = P$ (for all P contained in S), and this is true if and only if $C(P) = P \cup (S - \overline{C}(P))$. Thus $C(A \cup B) = A \cup B \cup$

$(S - \overline{C}(A \cup B)) \subseteq A \cup B \cup (S - \overline{C}(A)) = C(A) \cup B.$

Conversely, $C(A \cup B) = C(A) \cup B$ for $A,B \subseteq S$ implies $C(P) = C(\emptyset) \cup P$, and letting $\overline{C}(P)$ equal $P \cup (S - C(\emptyset))$ we have $C(P) \cup \overline{C}(P) = S$ and $C(P) \cap \overline{C}(P) = P$, i.e. $\overline{C}$ is indeed the complement of C.

Since $\mathcal{L}$ is distributive, it follows that an element with a complement necessarily has a unique complement, and further the complemented elements form a sublattice. The next proposition shows the structure of this sublattice.

PROPOSITION 4: The sublattice of complemented elements of $\mathcal{L}$ is isomorphic to the Boolean of S.

Proof: Consider the canonical mapping φ from $\mathcal{P}(S)$ to $\mathcal{L}$ where $(\varphi(P))(Q) = P \cup Q$.

G. Birkhoff (see pg. 38 in [4]) defines a lattice to be $\uparrow$-atomic if, for any two elements a and b with $a<b$, there is at least one maximal well-ordered chain connecting the points a and b. A lattice is $\downarrow$-atomic if its dual is $\uparrow$-atomic.

PROPOSITION 5: The lattice $\mathcal{L}$ is $\uparrow$-atomic and $\downarrow$-atomic.

Proof: Let C and C^o be two members of $\mathcal{L}$ with $C \subseteq C^o$. Let $A_1, \ldots, A_\alpha$ be a well-ordering of the subsets of S in such a manner that $A_\beta \subseteq A_\gamma$ implies $\gamma \leq \beta$. (The existence of such an ordering can be shown by Zorn's Lemma.) Let $s_1, \ldots, s_\beta$ be a well-ordering of the elements of S.

Define a well-ordered sequence $\{C_\gamma\}$ of pre-closures as follows. Let $C_1 = C$, and if C_γ is defined, let $C_{\gamma+1}(P) = C_\gamma(P)$ if $P \neq A_\delta$, where δ is the least ordinal (if it exists) such that $C^o(A_\delta) \neq C_\gamma(A_\delta)$, and let $C_{\gamma+1}(A_\delta) = C_\gamma(A_\delta) \cup \{\inf\{C^o(A_\delta) - C_\gamma(A_\delta)\}\}$. And for limit ordinals σ simply set $C_\sigma = \bigcup\{C_\gamma: \gamma<\sigma\}$.

C_1 is clearly a pre-closure, and if C_γ is a member of $\mathfrak{L}$, then $C_{\gamma+1}$ is easily seen to be extensive. Suppose A and B are subsets of S with A properly contained in B. If neither A nor B is equal to the A_δ described above, then it is also clear that $C_{\gamma+1}(A) \subseteq C_{\gamma+1}(B)$, and if $B = A_\delta$, then it follows that $C_{\gamma+1}(A) = C_\gamma(A) \subseteq C_\gamma(B) \subseteq C_\gamma(B) \cup \{\inf\{C^o(A_\delta) - C_\gamma(A_\delta)\}\} = C_{\gamma+1}(B)$. Suppose now that $A = A_\delta$. Then $B = A_\tau$ for some ordinal τ which is less than δ, and thus $C_{\gamma+1}(B) = C_{\gamma+1}(A_\tau) = C_\gamma(A_\tau) = C^o(A_\tau) = C^o(B)$, so clearly $C_{\gamma+1}(A) \subseteq C_{\gamma+1}(B)$. For limit ordinals σ we simply apply the completeness of $\mathfrak{L}$ to show that C_σ is in $\mathfrak{L}$ if C_γ is in $\mathfrak{L}$ for all $\gamma<\sigma$. This shows that $\{C_\gamma\}$ is a chain of pre-closures between C and C^o, and by construction it is well-ordered.

If C_γ is in the above-mentioned chain and $C_\gamma \neq C^o$, Then for some δ we have $C_\gamma(A_\delta) \neq C^o(A_\delta)$, so $C_\gamma \neq C_{\gamma+1}$. Therefore there is an ε such that ε is the least ordinal for which $C_\varepsilon = C^o$. For $\gamma<\varepsilon$, C_γ and

$C_{\gamma+1}$ differ on exactly one subset of S, and then by only one element of S, so $C_{\gamma+1}$ covers C_γ. With this it is easy to see that the chain $C_1,\ldots,C_\varepsilon$ is maximal, completing the first part of the proof.

A parallel argument shows that $\mathcal{L}$ is $\downarrow$-atomic.

2. Special Pre-closures and Their Lattice Properties

We will work with the following basic types of pre-closures throughout the rest of the paper. If $\aleph$ is a cardinal number, then an _$\aleph$ pre-closure_ satisfies the condition $C(P) = \bigcup\{C(Q): Q \subseteq P, \overline{\overline{Q}} < \aleph\}$. (A double-bar over a set symbol is to be read "the cardinality of...".) If C is an $\aleph_0$ pre-closure, then we say it is _algebraic_ (also _domain finite_ or _compact_)--the choice of the word algebraic is motivated by the fact that the natural closure associated with a general algebra is an $\aleph_0$ pre-closure (see [17]). A pre-closure is _$\aleph$-ary_ if $C = (C_\aleph)^\alpha$ for some ordinal α, where $C_\aleph(P) = \bigcup\{C(Q): Q \subseteq P \text{ and } \overline{\overline{Q}} < \aleph\}$. Thus an $\aleph$ pre-closure is an $\aleph$-ary pre-closure. For the special case that $\aleph$ is finite we will call such a pre-closure by the generic name _enary_. Finally, a _closure_ is an idempotent pre-closure, i.e. $C^2 = C$.

We will need a function from the cardinals to the ordinals defined as follows: if $\aleph$ is a cardinal number, then $\alpha(\aleph)$ is the least ordinal such that the cardinality of all ordinals less than or equal to it is $\aleph$. Also we recall that an ordinal α is <u>regular</u> provided no smaller ordinal is cofinal with it.

PROPOSITION 6: The $\aleph$ pre-closures are closed under arbitrary supremum, and they form a sublattice if $\aleph_0 \leq \aleph$; furthermore, if $\alpha(\aleph)$ is regular then they are closed under ordinal composition.

Proof: Let $\{C_a\}_{a\varepsilon A}$ be a family of $\aleph$ pre-closures. If $x \varepsilon \bigcup C_a(P)$, then for some a, $x \varepsilon C_a(P)$. Thus there is a $Q \subseteq P$ such that $\overline{\overline{Q}} < \aleph$ and $x \varepsilon C_a(Q)$. From this we see that $x \varepsilon \bigcup \{C_a(Q)\colon a\varepsilon A\}$.

Suppose now that $\aleph_0 \leq \aleph$, and C_1 and C_2 are pre-closures. If $x \varepsilon C_1(P) \cap C_2(P)$, then $x \varepsilon C_1(Q_1)$ and $x \varepsilon C_2(Q_2)$, where Q_1 and Q_2 are contained in P and have cardinality less than $\aleph$. But then x is in $(C_1 \cap C_2)(Q_1 \cup Q_2)$ and $Q_1 \cup Q_2$ has cardinality less than $\aleph$.

If $\aleph$ is such that $\alpha(\aleph)$ is regular, then let C_1 and C_2 be two $\aleph$ pre-closures and suppose $x \varepsilon C_1C_2(P)$. Then $x \varepsilon C_1(Q)$ for some Q contained in $C_2(P)$ with $\overline{\overline{Q}} < \aleph$. If $y \varepsilon Q$, then $y \varepsilon C_2(Q_y)$ for some Q_y contained in P with $\overline{\overline{Q}}_y < \aleph$. Therefore $x \varepsilon C_1C_2(\bigcup\{Q_y\colon y \varepsilon Q\})$, and since the cardinality of $\bigcup\{Q_y\colon y\varepsilon Q\}$ is less than $\aleph$, it follows that C_1C_2 is an $\aleph$ pre-closure. Now from the fact that the union of $\aleph$ pre-closures is again an $\aleph$ pre-closure, we can conclude that the ordinal composition of $\aleph$ pre-closures is again an $\aleph$ pre-closure if $\alpha(\aleph)$ is regular.

The family of $\aleph$ pre-closures is not in general closed under arbitrary intersection as the following example indicates (where $\aleph$ is infinite). Let S be the set of ordinals less than or equal to $\alpha(\aleph)$, and for each ordinal β less than $\alpha(\aleph)$ define $C_\beta(P) = P$ if $\{1,2,\ldots,\beta\} \not\subseteq P$, and otherwise $C_\beta(P) = S$. Then with $A = \{s \varepsilon S: s < \alpha(\aleph)\}$ we have $(\bigcap C_\beta)(A) = S$, but $\alpha(\aleph)$ does not belong to $(\bigcap C_\beta)(B)$ for any $B \subseteq A$ such that $\overline{\overline{B}} < \alpha(\aleph)$.

The family of $\aleph$-ary pre-closures will be examined in the next section, but for the present we will see that it is possible that two enary closures may fail to have an enary intersection. Consider the following construction on $\mathcal{P}(I \times I)$, where I is the set of positive integers:

$\sigma_1 \{(1,1),(1,3)\} = \{(1,2)\}$

$\sigma_1 \{(1,2n),(1,2n+3)\} = \{(1,2n+2)\}$, $n=1,2,\ldots,$

$\sigma_1(P) = P$ otherwise;

$\sigma_2 \{(k,1),(k,3)\} = \{(k,2)\}$, $k=2,3,\ldots$

$\sigma_2 \{(k,2n),(k,2n+3)\} = \{(k,2n+2)\}$, $n=1,2,\ldots;$ $k=2,3,\ldots$

$\sigma_2(P) = P$ otherwise.

Let $\varphi_i(P) = \bigcup\{\sigma_i(Q): Q \subseteq P\} \cup P$, for $i=1,2$. It is easy to see that φ_i is a pre-closure, actually a 3 pre-closure. Then C_i defined by $C_i = \varphi_i^{\omega}$ is a 3-ary closure for $i = 1,2$. Now, for $k = 1,2,\ldots$ identify the point (k,k) of $\mathcal{P}(I \times I)$ with $(1,k)$, and let A be all points of the form $(k,2n-1)$ where k and n range over the positive integers. Then $\{(1,2n): n=1,2,\ldots\}$ is a subset of $C_i(A)$ for $i=1,2$. But if $2N$ is an even integer, then $(1,2N)$ does not belong to $(C_2)_N(A)$, and since $(2N,2n)$ is not in $C_1(A)$ for any $n \neq N$, it follows that for $n \neq N$ the point $(2N,2n)$ does not belong to $C_1 \cap C_2(A)$, and therefore $(1,2N)$ is not in $(C_1 \cap C_2)_N{}^{\alpha}$ for any ordinal α, i.e. $C_1 \cap C_2$ is not enary.

For future reference Proposition 7 will be stated--its content has been known for some time.

PROPOSITION 7: The intersection of an arbitrary family of closures is a closure, but not necessarily a finite union or composition.

Proof: (see, e.g., [7])

Another important classification is that of a <u>topological pre-closure</u>, namely one which satisfies the additive property $C(A \cup B) = C(A) \cup C(B)$.

PROPOSITION 8: The topological pre-closures are closed under arbitrary union and ordinal composition.

Proof: If $\{C_a\}_{a\varepsilon A}$ is a family of topological pre-closures, then $(\bigcup\{C_a : a\varepsilon A\})(P \cup Q) =$

$(\bigcup\{C_a : a\varepsilon A\})(P) \cup (\bigcup\{C_a : a\varepsilon A\})(Q)$, and

$C_1C_2(P \cup Q) = C_1(C_2(P) \cup C_2(Q)) = C_1C_2(P) \cup C_1C_2(Q)$.

The topological pre-closures do not form a sublattice, for consider C_1 and C_2 defined on $S = \{1,2,3\}$ by $C_1(\{1\}) = \{1\}$, $C_1(\{2\}) = \{2,3\}$, and $C_1(P) = S$ otherwise, $C_2(\{1\}) = \{1,3\}$, $C_2(\{2\}) = \{2\}$, $C_2(P) = P$ otherwise. Then

$(C_1 \cap C_2)(\{1,2\}) = \{1,2,3\}$, but $(C_1 \cap C_2)(\{1\}) \cup (C_1 \cap C_2)(2) = \{1,2\}$.

3. Relationships between the Special Types of Pre-closures

PROPOSITION 9: (Schmidt-Hammer). If C is a pre-closure, then for some ordinal α we actually have C^α is a closure.

Proof: (Straightforward).

PROPOSITION 10: If C is an $\aleph$ pre-closure, then $C^{\alpha(\aleph)}$ is an $\aleph$ closure, provided $\alpha(\aleph)$ is regular.

Proof: By Proposition 6 it is immediate that C^α is an $\aleph$ pre-closure, where $\alpha = \alpha(\aleph)$. Therefore we only need to show that C^α is idempotent. Let x be in $C^\alpha C^\alpha(P)$. Since C^α is an $\aleph$ pre-closure, there is a Q in $C^\alpha(P)$ such that $x \in C^\alpha(Q)$ and $\overline{\overline{Q}} < \aleph$. For each y in Q there is an ordinal α_y less than α such that $y \in C^{\alpha_y}(P)$ (because α is necessarily a limit ordinal). Let β be the supremum of the α_y. From the fact that $\overline{\overline{Q}} < \aleph$ we can conclude that β is less than α. Therefore $C^\alpha(Q) \subseteq C^\alpha(C^\beta(P))$ $= \bigcup \{C^{\beta+\gamma}(P): \gamma < \alpha\} \subseteq C^\alpha(P)$. Thus x is in $C^\alpha(P)$, and this gives $C^{2\alpha} \subseteq C^\alpha$.

As an easy consequence of Proposition 10 we have the following three results.

COROLLARY 10.1: Suppose C is an $\aleph^*$ pre-closure and $\alpha(\aleph)$ is a regular ordinal for some $\aleph$ greater than or equal to $\aleph^*$. Then $C^{\alpha(\aleph)}$ is an $\aleph$ closure.

Proof: We need only note that C is an $\aleph$ pre-closure.

REMARK: If C is the natural pre-closure associated with a partial algebra, then the least such regular ordinal in the above corollary is called the <u>ordinal dimension</u> of the partial algebra by Jürgen Schmidt [20].

COROLLARY 10.2: Let C be any $\aleph$-ary closure. If $\alpha(\aleph)$ is a regular ordinal, then $C = (C_{\aleph})^{\alpha(\aleph)}$.

Proof: Let $\alpha(\aleph)$ be a regular ordinal and C an $\aleph$-ary closure. Then for some ordinal σ we have $C = (C_{\aleph})^{\sigma}$. By Proposition 10 we know that $(C_{\aleph})^{\alpha(\aleph)}$ is a closure. Since C is a closure we have the inequality $(C_{\aleph})^{\alpha} \subseteq C$. Now $C_{\aleph} \subseteq (C_{\aleph})^{\alpha(\aleph)}$ leads to $(C_{\aleph})^{\sigma} \subseteq (C_{\aleph})^{\alpha(\aleph)}$, and the conclusion follows.

COROLLARY 10.3: For every cardinal $\aleph$ such that $\alpha(\aleph)$ is regular the family of $\aleph$-ary closures is identical with the family of $\aleph$ closures.

Proof: Clearly every $\aleph$ closure is an $\aleph$-ary closure. Conversely, if C is $\aleph$-ary then from Proposition 10 and Corollary 10.2 it follows that C is an $\aleph$ closure.

PROPOSITION 11: Suppose C is an $\aleph$ closure with $\alpha(\aleph)$ regular. Then for any cardinal $\aleph^*$ it is true that C is $\aleph^*$-ary if and only if $C(Q) = (C_{\aleph^*})^{\alpha(\aleph)}(Q)$ for all sets with cardinality less than $\aleph^*$.

Proof: Suppose C satisfies the hypothesis of the first sentence. Then for any P, $C(P) = \bigcup\{C(Q): Q \subseteq P,$

$\overline{\overline{Q}} < \aleph \}$. If the second part of the bi-implication is true, we can rewrite the last equation as $C(P) = \bigcup \{ (C_{\aleph^*})^{\alpha(\aleph)}(Q) : Q \subseteq P, \overline{\overline{Q}} < \aleph \}$, and this in turn is clearly contained in $(C_{\aleph^*})^{\alpha(\aleph)}(P)$. Thus $C \subseteq (C_{\aleph^*})^{\alpha(\aleph)}$, and since C is a closure we can rewrite the last expression using equality.

For the converse we consider the following two cases. <u>(i)</u> If $\aleph^* \leq \aleph$, then the conclusion follows from Corollary 10.1 if we note that $C_{\aleph^*}$ is an $\aleph^*$ pre-closure. For then from the hypothesis we have $C = (C_{\aleph^*})^{\alpha}$ for some ordinal α, and from the corollary we see that $(C_{\aleph^*})^{\alpha(\aleph)}$ is a closure, so necessarily $C = (C_{\aleph^*})^{\alpha(\aleph)}$, so in particular $C(Q) = (C_{\aleph^*})^{\alpha(\aleph)}(Q)$ for all sets Q with cardinality less than $\aleph^*$. <u>(ii)</u> If $\aleph \leq \aleph^*$, then $C_{\aleph} \subseteq C_{\aleph^*}$ implies $(C_{\aleph})^{\alpha(\aleph)} \subseteq (C_{\aleph^*})^{\alpha(\aleph)}$, and then from the fact that $C_{\aleph}^{\alpha} = C$ we easily conclude that $C = (C_{\aleph^*})^{\alpha(\aleph)}$.

PROPOSITION 12: If C is an $\aleph$-ary closure and C^* is an $\aleph^*$ pre-closure with $C^* \subseteq C$, where $\alpha(\aleph)$ is regular and $\alpha(\aleph)$ is no less than $\alpha(\aleph^*)$, then $C = C^{*\alpha(\aleph)}$ if and only if $C^{*\alpha(\aleph)}(Q) = C(Q)$ for all Q such that $\overline{\overline{Q}} < \aleph$.

Proof: Letting α denote $\alpha(\aleph)$, first we note that $C^* \subseteq C$, hence $C^{*\alpha} \subseteq C$ since C is a closure. The proof in one direction is trivial, so we assume $C(Q) = C^{*\alpha}(Q)$ for all Q such that $\overline{\overline{Q}} < \aleph$. Then, for any P, $C_{\aleph}(P) = \bigcup\{C(Q): Q \subseteq P, \bar{\bar{Q}} < \aleph\} = \bigcup\{C^{*\alpha}(Q): Q \subseteq P, \bar{\bar{Q}} < \aleph\} \subseteq C^{*\alpha}(P)$. Thus we have $C_{\aleph} \subseteq C^{*\alpha}$, and since $C^{*\alpha}$ is a closure by Corollary 10.1 we can deduce $(C_{\aleph})^{\alpha} \subseteq C^{*\alpha}$. Since C is $\aleph$-ary, we can apply Corollary 10.2 to obtain the desired conclusion.

PROPOSITION 13: Let C be a closure. Then C is $\aleph$-ary if and only if there exists an $\aleph$ pre-closure C^* such that $C = C^{*\sigma}$ for some ordinal σ.

Proof: If C is $\aleph$-ary, then we can simply choose C to be $C_{\aleph}$. For the converse, since $C = C_{\aleph}^{\sigma}$, then $C^* \subseteq C$. Thus $C^*_{\aleph} \subseteq C_{\aleph}$, so $C^{*\sigma} \subseteq (C_{\aleph})^{\sigma}$, i.e., $C \subseteq (C_{\aleph})^{\sigma}$. Now, since C is a closure and $C_{\aleph} \subseteq C$ we have $(C_{\aleph})^{\sigma} \subseteq C$, and the proposition is proved.

A fifth category of pre-closure which has not explicitly appeared in the literature but will play an important role in this paper is the <u>operational</u> <u>closure</u>. If C is an enary closure and C(Q) is countable (possible finite) for all finite Q, then C will be called operational, and if in particular C is N-ary, then N will be called an <u>index</u> for C. It is an easy consequence of the definition that if C is N-ary for some finite N and C(Q) is countable for all Q of cardinality less than N, then C is operational with index N. If we restrict our attention to operational closures, then we can state a result parallel to Proposition 13, but considerably stronger. The next theorem will be the key to one of the representation theorems. The value of Proposition 14 is the availability of a slow growth pre-closure to replace an operational closure.

PROPOSITION 14: Let C be a closure. Then C is operational if and only if there exist a pre-closure C^* and positive integers M and L such that:

(i) $C = C^{*^{\omega}}$

(ii) C^* is M-ary

(iii) $\overline{\overline{C(Q)}} < L$ for all Q such that $\overline{\overline{Q}} < M$.

Proof: Suppose we have a C^* such that (i)-(iii) are satisfied. Then from (i) and (ii) and Proposition 13 it follows that C is M-ary. From (ii) and (iii) it follows that $C^*(P)$ is finite when P is finite. Thus C is operational with index M.

The proof of the converse is a little more involved. Let C be operational (with index N). If Q is any subset of S such that $\overline{\overline{Q}} < N$, then we know by the remark above that C(Q) is countable and therefore we can assume that the elements of C(Q) have been indexed as follows: $C(Q) = \{a_1^Q, a_2^Q, \ldots\}$ (where the sequence might be finite). Furthermore it is consistent to require that the above sequences satisfy: 1) all members of a given sequence are distinct, and 2) the members of Q appear first in a_1^Q, $a_2^Q, \ldots$.

Now, for any subset R of S such that $\overline{\overline{R}} < N+1$

define $C^*(R)$ to be the set $R \cup \{a_1^{\emptyset}\} \cup \{a_{n+1}^{Q}:$ $Q \subseteq R$, $\overline{\overline{Q}} < N$, and $a_n^Q \in R\}$ with the understanding that the expression $a_1^{\emptyset}$ is to be deleted from the above if $C(\emptyset)$ is empty.

The following three properties of C^* are easy consequences of the definition of C^*, where, of course, R has less than $N+1$ elements.

$C^*.1.$ $R \subseteq C^*(R)$

$C^*.2.$ $C^*(R) \subseteq C(R)$

$C^*.3.$ $C^*(R) = \bigcup\{C^*(Q): Q \subseteq R\}$.

Because of $C^*.3$ it is possible to extend the domain of definition of C^* to all of 2^S by

$C^*.4.$ $C^*(P) = \bigcup\{C^*(R): R \subseteq P, \overline{\overline{R}} < N+1\}$.

We will assume C^* to be so extended, and then note that from the four properties above it easily follows that C is an algebraic pre-closure. Then from $C^*.2.$ and $C^*.4.$ and the monotonicity condition for C we conclude $C^*(P) \subseteq C(P)$ for any P, i.e., $C^* \subseteq C$.

Let Q be such that $\overline{\overline{Q}} < N$. If $C(Q) = \emptyset$, then necessarily $C^*(Q) = C(Q)$. For this special case it is immediate that $C(Q) = (C^*)_N^{\omega}(Q)$. So, now we will assume that $C(Q) \neq \emptyset$. Then from the restriction (2) on the sequence a_i^Q and from the definition of $C^*(Q)$ we have $a_1^Q \in C^*(Q)$. Suppose n is a positive integer and $a_n^Q \in C^{*n}(Q)$. Then $a_{n+1}^Q \in C^{*(n+1)}(Q)$, so, by a simple induction argument we have $\{a_1^Q, a_2^Q, \ldots\}$ is a subset of $C^{*\omega}(Q)$, i.e. $C(Q) \subseteq C^{*\omega}(Q)$. But since $C^* \subseteq C$, it follows that $C^{*\omega}(Q) = C(Q)$ for all Q such that $\overline{\overline{Q}} < N$, and then from Proposition 12 we have $C = C^{*\omega}$. Now if we let $M = N+1$ and $L = N+2+2^{N+1}(N+1)$, then it is straightforward to show that the three conditions of Proposition 14 are satisfied by C.

PROPOSITION 15: In general we have C is operational implies C is enary, and C is enary implies C is algebraic, but not conversely.

Proof: The implications are clear. To show that enary does not imply operational we need only to find an enary closure C such that $C(Q)$ is uncountable for some finite Q. Let S be the real line. Define $C(A) = S$ for all A. To show algebraic does not imply enary, let S be the positive integers. Define: $C(A) = A \cup \{2n+2: 2,4,\ldots,2n \in A\}$. Then C is algebraic, but not enary.

4. Composition and Supremum

The relationship between composition and supremum is a rather difficult one to characterize. From the Schmidt-Hammer theorem we know that given a pre-closure C there is some ordinal α such that C^α is a closure, and indeed C^α is the least closure greater than or equal to C. We may also say that C^{α} is the least closure with the same fix-points as C. Let us designate the minimal such ordinal by O(C).

PROPOSITION 16: Let C_1 and C_2 be two pre-closures. Then $(C_1 \cup C_2)^{\alpha} \subseteq (C_1C_2)^{\alpha}$ for all ordinals α; and if α is a limit ordinal, then $(C_1 \cup C_2)^{\alpha} = (C_1C_2)^{\alpha}$.

Proof: Since $(C_1 \cup C_2) \subseteq C_1C_2$, it easily follows that $(C_1 \cup C_2)^{\alpha} \subseteq (C_1C_2)^{\alpha}$ for all ordinals α. To show the second part we note that the limit ordinals are well-ordered and use transfinite induction as follows. For N an integer, $(C_1C_2)^N \subseteq (C_1 \cup C_2)^{2N}$, and thus $(C_1C_2)^{\omega} \subseteq (C_1 \cup C_2)^{\omega}$. Combining this with the above gives $(C_1 \cup C_2)^{\omega} = (C_1C_2)^{\omega}$. Now suppose $(C_1 \cup C_2)^{\alpha} = (C_1C_2)^{\alpha}$ for a limit ordinal α. Then for N a positive integer, $(C_1C_2)^N(C_1C_2)^{\alpha} \subseteq (C_1 \cup C_2)^{2N}(C_1 \cup C_2)^{\alpha}$, so $(C_1C_2)^{\alpha+\omega} \subseteq (C_1 \cup C_2)^{\alpha+\omega}$, and since $\alpha+\omega$ is a limit ordinal, then $(C_1C_2)^{\alpha+\omega} = (C_1 \cup C_2)^{\alpha+\omega}$. Now suppose α is a limit of limit ordinals and for β a limit ordinal less than α assume $(C_1 \cup C_2)^{\beta} = (C_1C_2)^{\beta}$. Then $(C_1 \cup C_2)^{\alpha} = \bigcup \{(C_1 \cup C_2)^{\beta}: \beta \text{ is a limit ordinal less than } \alpha\}$ $= \bigcup \{(C_1C_2)^{\beta}: \beta \text{ is a limit ordinal less than } \alpha\}$ $= (C_1C_2)^{\alpha}$.

From Proposition 6 and Corollary 10.1 we can make upper estimates of $O(C_1 \cup C_2)$, and from Proposition 16 we have a lower bound as stated in the following.

COROLLARY 16.1: $O(C_1 \cup C_2) \geq O(C_1C_2)$, with equality if $O(C_1C_2)$ is a limit ordinal.

Proof: Suppose $\alpha < O(C_1C_2)$. Then $(C_1C_2)^{2\alpha} \neq (C_1C_2)^{\alpha}$. Therefore $(C_1C_2)^{\alpha} \neq (C_1C_2)^{\beta}$, where $\beta = O(C_1C_2)$. Now $(C_1 \cup C_2)^{\alpha} \subseteq (C_1C_2)^{\alpha}$, so $(C_1 \cup C_2)^{\alpha} \neq (C_1C_2)^{\beta}$. But $(C_1 \cup C_2)^{\gamma} = (C_1C_2)^{\beta}$, where $\gamma = O(C_1 \cup C_2)$, so $\gamma \geq \alpha$. and this gives $O(C_1 \cup C_2) \geq O(C_1C_2)$.

We can deduce considerably stronger conditions if the pre-closures satisfy some sort of commutative condition, but first let us look at the awkward behavior of simple commutativity. Suppose $C_1C_2 = C_2C_1$. Then we can easily conclude $C_1{}^nC_2{}^m = C_2{}^mC_1{}^n$ if m and n are finite, but for arbitrary ordinals this relationship fails as the following counter-example shows. Consider φ_1 and φ_2 defined on $\mathcal{P}(S)$, where $S = \{x \leq \omega\}$, by $\varphi_1(\{x:x\leq n\}) = \{x: x\leq n+1\}$, for $n < \omega$, and $\varphi_1(P) = P$ otherwise; $\varphi_2(P) = S$ if $\mathrm{Sup}(P) = \omega$, otherwise $\varphi_2(P) = P$. Then define the pre-closures C_i by $C_i(P) = \bigcup\{\varphi_i(Q): Q \subseteq P\}$, for $i = 1,2$. (It is readily seen that these are indeed pre-closures). It is easy to verify that C_1 and C_2 commute, but $C_1^{\omega}C_2 \neq C_2C_1^{\omega}$ since $C_2C_1^{\omega}(\{1\}) = S$ and $C_1^{\omega}C_2(\{1\}) = S - \{\omega\}$. Thus the hypothesis of the following proposition should not appear unduly restrictive.

PROPOSITION 17: Let C_1 and C_2 be pre-closures. If C_1 and C_2 are totally commutative (i.e., $C_1{}^{\alpha}C_2{}^{\beta} = C_2{}^{\beta}C_1{}^{\alpha}$ for all ordinals α,β), then $(C_1 \cup C_2)^{\alpha} \subseteq C_1{}^{\alpha}C_2{}^{\alpha}$ for any ordinal α, with equality if α is a limit ordinal.

Proof: By transfinite induction we demonstrate that

$(C_1C_2)^\alpha = C_1{}^\alpha C_2{}^\alpha$, and then the conclusion is immediate from Proposition 16. Suppose equality holds for some particular ordinal α. Then $(C_1C_2)^{\alpha+1} = (C_1C_2)(C_1C_2)^\alpha = (C_1C_2)(C_1{}^\alpha C_2{}^\alpha) = C_1{}^{\alpha+1}C_2{}^{\alpha+1}$ (by total commutativity). Now suppose $(C_1C_2)^\beta = C_1{}^\beta C_2{}^\beta$ for $\beta < \alpha$, where α is a limit ordinal. Then from $C_1{}^\beta C_2{}^\beta \subseteq C_1{}^\alpha C_2{}^\alpha$, for $\beta<\alpha$, it follows that $(C_1C_2)^\beta \subseteq C_1{}^\alpha C_2{}^\alpha$ for all $\beta<\alpha$, so $(C_1C_2)^\alpha \subseteq (C_1C_2)^\alpha$. For inclusion in the other direction, we have first $C_1{}^\beta C_2{}^\beta \subseteq (C_1C_2)^\alpha$ for $\beta<\alpha$, and thus $C_1{}^\alpha C_2{}^\beta \subseteq (C_1C_2)^\alpha$ and by total commutativity $C_2{}^\beta C_1{}^\alpha \subseteq (C_1C_2)^\alpha$, so $C_2{}^\alpha C_1{}^\alpha \subseteq (C_1C_2)^\alpha$.

COROLLARY 17.1: Suppose C_1 and C_2 are totally commutative. Then $(C_1 \cup C_2)^{O(C_1 \cup C_2)} = (C_1)^{O(C_1)}(C_2)^{O(C_2)}$.

COROLLARY 17.2: If $C_1,\ldots,C_n$ are pairwise totally commutative, then for α sufficiently large we have $(C_1 \cup \ldots \cup C_n)^\alpha = C_1{}^\alpha\ldots C_n{}^\alpha$.

PROPOSITION 18: If C_1 and C_2 are algebraic pre-closures with $C_1C_2 \subseteq C_2C_1$, then $(C_1 \cup C_2)^\omega = C_1^\omega C_2^\omega$.

Proof: $C_1C_2 \subseteq C_2C_1$ implies $C_1{}^n \subseteq (C_1C_2)^n \subseteq C_2{}^nC_1{}^n$ for n finite. Therefore $C_1^\omega \subseteq (C_1C_2)^\omega \subseteq \bigcup\{C_2{}^nC_1{}^n: n<\omega\} \subseteq C_2^\omega C_1^\omega$. Since $(C_1C_2)^\omega$ is a closure by Proposition 10, it follows that $C_2^\omega C_1^\omega \subseteq (C_1C_2)^\omega$. Therefore $(C_1C_2)^\omega = C_2^\omega C_1^\omega$, so by Propositions 10 and 16 we have the conclusion.

COROLLARY 18.1: If $C_1, \ldots, C_n$ is a family of algebraic pre-closures with $C_iC_j \subseteq C_jC_i$ for $i \leq j$, then $(C_1 \cup \ldots \cup C_n)^{\omega} = C_n^{\omega} \ldots C_1^{\omega}$.

PROPOSITION 19: If C_1 and C_2 are closures and $C_1C_2 \subseteq C_2C_1$, then $(C_1 \cup C_2)^{\alpha} \subseteq C_2C_1$ for all α, with equality if $\alpha = O(C_1 \cup C_2)$.

Proof: Straightforward.

In the more commonly studied lattice $(L, \vee, \wedge)$ of closures there have been questions of considerable importance involving the relationship of supremum and composition which can be phrased as follows: if we close a collection of elements under two properties, when is this equivalent to closing them under the first property followed by closing them under the second property? In L this would translate into: when does $C_1 \vee C_2 = C_2C_1$? We have just seen that $C_1C_2 \subseteq C_2C_1$ is a sufficient condition (and clearly necessary)(also see e.g. [20]). Another formulation for dealing with this question is given by the following simple generalization of a theorem due to Baumann and Pfanzagl [1]:

PROPOSITION 20: If $C, C_1, \ldots, C_k$ are algebraic pre-closures satisfying the inequalities $C_iC^{\omega} \subseteq (C \cup C_1 \cup \ldots \cup C_{i-1})^{\omega} \cdot (C_1 \cup \ldots \cup C_k)^{\omega}$, then $(C \cup C_1 \cup \ldots \cup C_k)^{\omega} = C^{\omega}(C_1 \cup \ldots \cup C_k)^{\omega}$.

Proof: First we show that for any integer m, $C_i{}^mC$ is contained in $C^{\omega}(C_1 \cup \ldots \cup C_k)^{\omega}$, for $1 \leq i \leq k$. But this will easily follow from noting that

$C^{\omega}(C_1 \cup \ldots \cup C_k)^{\omega} = (C \cup C_1 \cup \ldots \cup C_k)^{\omega}$ $\cdot (C_1 \cup \ldots \cup C_k)^{\omega}$. As an easy consequence $(C_1 \cup \ldots \cup C_k)^{\omega} C^{\omega} \subseteq C^{\omega}(C_1 \cup \ldots \cup C_k)^{\omega}$, and then from Proposition 18 we have the conclusion.

The chapter will conclude with a couple of remarks on the lattice L of closure operators.

PROPOSITION 21: The family of enary closures is closed in L with respect to finite supremum.

Proof: Suppose C and C^* are N_1-ary and N_2-ary respectively. Let $N = \max(N_1, N_2)$. Then $C_N \subseteq (C \cup C^*)_N$ implies $C_N^{\omega} \subseteq (C \cup C^*)_N^{\omega}$, i.e. $C \subseteq (C \cup C^*)_N^{\omega}$. Likewise $C^{*\omega} \subseteq (C \cup C^*)_N^{\omega}$. Thus $(C \cup C^*)^{\omega} \subseteq (C \cup C^*)_N^{\omega}$, i.e., $(C \cup C^*)^{\omega}$ is N-ary.

PROPOSITION 22: If $\{C_a\}_{a \varepsilon A}$ is a family of N-ary closures for a given integer N, then $\bigvee \{C_a : a \varepsilon A\}$ is also N-ary.

Proof: For a given member C_a of the above family we have $(C_a)_N \subseteq (\bigvee \{C_a : a \varepsilon A\})_N$, and thus C_a is contained in $(\bigvee \{C_a : a \varepsilon A\})_N^{\omega}$ which easily leads to $\bigcup \{C_a : a \varepsilon A\} \subseteq (\bigvee \{C_a : a \varepsilon A\})_N^{\omega}$, and since the right hand side of this inequality is a closure we have $(\bigvee \{C_a : a \varepsilon A\}) \subseteq (\bigvee \{C_a : a \varepsilon A\})_N^{\omega}$, i.e., $\bigvee \{C_a : a \varepsilon A\}$ is an N-ary closure.

CHAPTER II

1. Extensions and Restrictions of Pre-closure Spaces

In the formal study of any particular species of mathematical systems we are interested in knowing what properties of the systems will be inherited by subsystems, and what desirable properties we might gain by embedding in a larger system. We will consider the same questions with respect to pre-closures.

Since the fundamental set S will no longer be fixed, define a <u>pre-closure space</u> as an ordered pair (C,S), where C is a pre-closure on S. Suppose (C,S) and (C_1,S_1) are two pre-closure spaces. If there is a one-one mapping from S into S_1, say φ, such that for any A contained in S we have $\varphi C(A) = C_1(\varphi(A)) \cap \varphi(S)$, then (C_1,S_1) will be called an <u>extension</u> of (C,S), and (C,S) will be called a <u>restriction</u> of (C_1,S_1). In what follows we can assume without loss of generality that S is a subset of S_1, and the condition defining an extension is simply $C_1(A) \cap S = C(A)$.

PROPOSITION 1: If (C_1,S_1) is a topological pre-closure space, then any restriction (C,S) is topological.

Proof: $C(A \cup B) = C_1(A \cup B) \cap S = (C_1(A) \cup C_1(B)) \cap S$
$$= (C_1(A) \cap S) \cup (C_1(B) \cap S) = C(A) \cup C(B).$$

PROPOSITION 2: If (C_1,S_1) is an $\mathcal{X}$ pre-closure space, then any restriction (C,S) is an $\aleph$ pre-closure space.

Proof: If $x \in C(P)$, then $x \in C_1(P) \cap S$, so for some Q contained in P we have $x \in C_1(Q) \cap S$, where the cardinality of Q is less than $\aleph$, i.e., $x \in C(Q)$.

We can rephrase the above two propositions to say that any pre-closure space which is not topological (or $\aleph$) cannot be extended to a pre-closure space which is topological (or $\aleph$). The following proposition is the most important of this section in view of the representation theorem for operational closures which will be proved later.

PROPOSITION 3: Every algebraic closure space (C,S) can be embedded in an N-ary closure space (C_1,S_1) (where N is a positive integer greater than two). Further, if S is at least denumerable, then S_1 can be chosen so that it has the same cardinality as S.

Proof: Let (C,S) be an algebraic closure space and N a positive integer, $N \geq 2$. If P is a finite subset of S, let $n(P) = \overline{\overline{P}}$, $\sigma(P) = \left[\frac{n(P)}{N-1}\right]$ (i.e., $\sigma(P)$ is the greatest integer less than or equal to $\frac{n(P)}{N-1}$). Let $x^P_{0,1},\ldots,x^P_{0,n(P)}$ be an ordering of the elements of P. Add new elements $x^P_{i,j}$ to S, where $1 \leq i \leq \sigma(P)$, $1 \leq j \leq n(P) - i(N-1)$, and designate the set $S \cup \{x^P_{i,j}: P \text{ is a finite subset of } S,\ 1 \leq i \leq \sigma(P),\ 1 \leq j \leq n(P) - i(N-1)\}$ by S_1. Define a set function φ_P from S_1 to S_1 as follows:

$\varphi_P(\{x^P_{i,k+1},\ldots,x_{i,k+N}) = \{x^P_{i+1,k+1}\}$ for $0 \leq i \leq \sigma(P)$

and $0 \leq k \leq n(P) - (i+1)(N-1)$,

$\varphi_P(\{x^P_{\sigma(P),j}: 0 \leq j \leq n(P) - \sigma(P)(N-1)\}) = C(P)$

$\varphi_P(R) = R$ for all other R contained in S_1.

Next we define a function φ from S_1 to S_1 by

$$\varphi(R) = \bigcup\{\varphi_P(R_1): R_1 \subseteq R, P \text{ a finite subset of } S\} \cup R.$$

Clearly φ is extensive and isotone, i.e., φ is a pre-closure, and indeed an N+1 pre-closure.

Let Q be a subset of S and suppose $x \in \varphi(Q) \cap S$. Then either $x \in Q$ or for some finite subset P of S we have x is in $\varphi_P(Q_1)$ where $Q_1 \subseteq Q$, and if x is not in Q_1, then $P = Q_1$ and $x \in C(Q)$. Now by induction we will show that $x \in \varphi^n(Q) \cap S$ implies x is in C(Q) for any positive integer n. Suppose the assertion is true for $\varphi^n(Q) \cap S$. Then if $x \in \varphi^{n+1}(Q) \cap S$, either $x \in Q$ or for some finite P contained in S, x is in $\varphi_P(R_1) \cap S$, where $R_1 \subseteq \varphi^n(Q)$. We will consider three cases. (i) Suppose $R_1 \subseteq S$ and $R_1 \neq P$. Then $x \in R_1$, so $x \in \varphi^n(Q)$ and by the induction hypothesis $x \in C(Q)$. (ii) $R_1 \subseteq S$ and $R_1 = P$. Then $x \in \varphi_P(P)$ which implies $x \in C(P)$. Now $P \subseteq \varphi^n(Q)$ implies $x^P_{0,j} \in \varphi^n(Q)$, $1 \leq j \leq n(P)$, and by the induction hypothesis $x^P_{0,j} \in C(Q)$ for $1 \leq j \leq n(P)$, i.e., $P \subseteq C(Q)$. But then $C(P) \subseteq C(Q)$, so $x \in C(Q)$. (iii) $R_1 \not\subseteq S$. Then $x \in \varphi_P(R_1) \cap S$ implies $R_1 = \{x^P_{\sigma(P),j}$:

$1 \leq j \leq n(P) - \sigma(P)(N-1)\}$ and $\sigma(P) \geq 1$, i.e., $x^P_{\sigma(P),j} \varepsilon \varphi^n(Q)$, $1 \leq j \leq n(P) - \sigma(P)(N-1)$. This necessarily implies $x^P_{\sigma(P)-1,j} \varepsilon \varphi^{n-1}(Q)$, $1 \leq j \leq n(P) - (\sigma(P)-1)(N-1)$, and in a finite number of steps we have $x^P_{0,j} \varepsilon \varphi^{n-\sigma(P)}(Q)$, $1 \leq j \leq n(P)$, i.e., $P \subseteq \varphi^{n-\sigma(P)}(Q)$. But now the induction hypothesis applies (since $P \subseteq S$) and we have $x^P_{0,j} \varepsilon C(Q)$ for $1 \leq j \leq n(P)$, i.e. $P \subseteq C(Q)$, so $C(P) \subseteq C(Q)$. Now x is in $\varphi_P(R_1) \cap S$ implies x is in $C(P)$, and thus $x \varepsilon C(Q)$.

Thus for all $Q \subseteq S$, $\varphi^n(Q) \cap S \subseteq C(Q)$ for $n = 1,2,\ldots.$ Now define C_1 to be φ^ω. By Proposition 13, Chapter I we see that C_1 is an N+1-ary closure, and from the above results $C_1(Q) \cap S \subseteq C(Q)$ for all Q contained in S.

Suppose now that $Q \subseteq S$ and $x \varepsilon C(Q)$. Then for some finite P contained in Q, $x \varepsilon C(P)$. Now, for $1 \leq j \leq n(P) - N + 1$ we have

$x^P_{1,j} \varepsilon \varphi_P(\{x^P_{0,j},\ldots,x^P_{0,j+N-1}\}) \subseteq \varphi(P) \subseteq \varphi(Q) \subseteq C_1(Q)$. Thus $\varphi(\{x^P_{1,j}: 1 \leq j \leq n(P)-N+1\}) \subseteq C_1(Q)$, which implies $x^P_{2,j}$ is a member of $C_1(Q)$ for $1 \leq j \leq n(P)-2(N-1)$, and in a finite number of steps we have $x^P_{\sigma(P),j} \varepsilon C_1(Q)$ for $1 \leq j \leq n(P)-\sigma(P)(N-1)$, thus $C(P) = \varphi_P(\{x^P_{\sigma(P),j}: 1\leq j\leq n(P)-\sigma(P)(N-1)\}) \subseteq \varphi(\{x^P_{\sigma(P),j}: 1 \leq j \leq n(P) - \sigma(P)(N-1)\}) \subseteq C_1(Q)$. Therefore $x \varepsilon C(Q)$ implies $x \varepsilon C_1(Q)$, so $C(Q) \subseteq$

$C_1(Q) \cap S$. Combining this with the above gives $C(Q) = C_1(Q) \cap S$, which means (C_1, S_1) is the desired extension of (C,S).

Finally, in the construction of S_1 we adjoined to each finite subset of S a finite number of new elements, so if $\overline{\overline{S}} \geq \aleph_0$, then $\overline{\overline{S}} = \overline{\overline{S}}_1$.

COROLLARY: Let (C,S) be an algebraic closure space. Then (C,S) has an operational extension (C_1,S_1) if and only if $C(P)$ is countable for all finite subsets P.

Proof: Suppose there is a finite P such that $C(P)$ is not countable. Let (C_1,S_1) be an extension of (C,S). Then $C_1(P) \cap S = C(P)$ implies $C_1(P)$ is not countable, so (C_1,S_1) cannot be operational.

Conversely, suppose $C(P)$ is countable for each finite P. Let (C_1,S_1) be the extension constructed in Proposition 3, and suppose P_1 is a finite subset of S_1. Clearly $\varphi_P(P_1)$ is countable for each finite P contained in S, and from this we have $\varphi(P_1)$ is countable. Now suppose R is any (countable) subset of $\varphi(P_1)$. Then $\varphi(R) = \bigcup\{\varphi_P(R_1): R_1 \subseteq R$, P is a finite subset of $S\}$, and since only a countable number of the R_1 have cardinality N, it follows that $\varphi(R)$ is countable. Then by a simple induction we have $\varphi^n(P_1)$ is countable, so $C_1(P_1)$ is countable, i.e., C_1 is operational.

PROPOSITION 4: An N-ary closure space (C,S) can be extended to an operational closure space if and only if (C,S) is an operational closure space.

Proof: (Apply the preceeding corollary.)

PROPOSITION 5: The restriction of an enary closure is not necessarily enary.

Proof: Combine the results of Proposition 3 with the second example in the proof of Proposition 15, Chapter I.

PROPOSITION 6: The restriction of an operational closure space is not necessarily operational.

Proof: (Same as above.)

PROPOSITION 7: The restriction of any of the special closure operators to a closed subset (i.e., a subset P such that C(P) = P) yields the same kind of closure.

Proof: We have already proved it for topological and $\aleph$ closures. So suppose (C_1, S_1) is an $\aleph$-ary closure and P_1 is a closed subset. Let (C, P_1) be the restriction to P_1, and suppose Q is a subset of P_1. Then $C(Q) = C_1(Q) \cap P_1$. But $C_1(Q) \subseteq C_1(P_1) = P_1$, and thus $C_{\aleph}{}^{\alpha}(Q) = (C_1)_{\aleph}{}^{\alpha}(Q)$ for any ordinal α, and thus it follows that C is $\aleph$-ary. With this we can easily see that the operational property is preserved.

2. Algebraic Homomorphisms, Congruences and Factor Spaces

The theory of algebraic closure homomorphisms will be patterned after the theory of homomorphisms of algebraic systems. In particular, from the property that algebraic homomorphisms map subalgebras onto subalgebras we abstract and give the following definition. Suppose (C,S) and (C_1,S_1) are closure spaces and φ is a map from S into S_1 such that $\varphi(C(A)) = C_1(\varphi(A))$ for all A contained in S. Then we will call φ an algebraic closure homomorphism and we use the obvious generalizations of isomorphism, homomorphic image, etc. For the remainder of this chapter we will drop the word algebraic and simply speak of closure homomorphism.

PROPOSITION 8: (i) Compositions of homomorphisms are homomorphisms, and (ii) inverse and direct images of closed subsets are closed under homomorphisms.

Proof: Part (i) is straightforward. So suppose φ is a closure homomorphism from (C,S) to (C_1,S_1). The direct image of a closed subset of S is a closed subset of S_1 as is clear from the definition of a closure homomorphism. Let us now consider the inverse image of a closed subset of S_1, say $\varphi^{-1}(C_1(A_1))$, where A_1 is any given subset of S_1. Then we have $C(\varphi^{-1}(C_1(A_1)) \subseteq \varphi^{-1}\varphi C(\varphi^{-1}C_1(A_1))$ $= \varphi^{-1}C_1\varphi(\varphi^{-1}C_1(A_1)) \subseteq \varphi^{-1}C_1C_1(A_1) = \varphi^{-1}C_1(A_1)$, and thus $\varphi^{-1}(C_1(A_1)$ is closed.

Now, for any mapping φ from S to S_1 we have $\varphi^{-1}\varphi$ is a topological closure operator on S since $\varphi^{-1}\varphi(A \cup B) = \varphi^{-1}(\varphi(A) \cup \varphi(B)) = \varphi^{-1}\varphi(A) \cup \varphi^{-1}\varphi(B)$. Another important property of $\varphi^{-1}\varphi$ is that if we view it as a subset of $S \times S$, then it is an equivalence relation. (As a side remark we can easily see that any relation on S, considered as a set mapping, is a topological pre-closure, and this immediately leads to the possibility of studying the uniform structures of topology in terms of filters in the lattice $\mathfrak{L}$.) Let us denote the family of equivalence relations on S by $\mathcal{E}(S)$. The equivalence relations on a closure space (C,S) which are formed as above by closure homomorphisms will be called <u>closure congruences</u>, and the family of congruences for (C,S) will be denoted by $\mathcal{K}(C,S)$. The next three propositions will give an axiomatic characterization of $\mathcal{K}(C,S)$.

PROPOSITION 9: If we let E be the equivalence relation associated with the closure homomorphism φ, i.e., $E = \varphi^{-1}\varphi$, then, we necessarily have $ECE = EC$.

Proof: Since φ is a closure homomorphism, then $\varphi C = C_1\varphi$. Thus $\varphi C(\varphi^{-1}\varphi) = C_1\varphi(\varphi^{-1}\varphi) = C_1\varphi = \varphi C$. So we have $\varphi^{-1}\varphi C\varphi^{-1}\varphi = \varphi^{-1}\varphi C$, i.e., $ECE = EC$.

From this proof we will see that not every equivalence E on a closure space (C,S) can be written in the form $\varphi^{-1}\varphi$ for some closure homomorphism φ (i.e., $\mathcal{K}(C,S)$ is properly contained in $\mathcal{E}(S)$. Let $S = \{1,2,3\}$, $E = \{(1,2),(2,1)\} \cup$

Δ (where Δ is the diagonal relation), and define $C(\{1\}) = \{1\}$, otherwise $C(P) = S$. Then $EC(\{1\}) = \{1,2\}$, whereas $ECE(\{1\}) = S$.

Next we will see that the identity $ECE = EC$ actually characterizes the class of congruences among the class of equivalence relations by proving a direct parallel of the fundamental homomorphism theorem of general algebra.

PROPOSITION 10: Let (C,S) be a closure space and E an equivalence relation on S such that $ECE = EC$. Let $S_1 = S/E$, and define C_1 to be the set mapping $\mu C\mu^{-1}$, where μ is the canonical map from S to S_1. Then C_1 is the unique closure on S_1 such that μ is a closure homomorphism with congruence E.

Proof: Let A,B be subsets of S_1. Then $C_1(A) = \mu C\mu^{-1}(A) \supseteq \mu\mu^{-1}A = A$, and if $A \subseteq B$, then $\mu^{-1}A \subseteq \mu^{-1}B$, so $C(\mu^{-1}A) \subseteq C(\mu^{-1}B)$, thus $\mu C\mu^{-1}(A) \subseteq \mu C\mu^{-1}(B)$, and thus $C_1(A) \subseteq C_1(B)$. Finally, $C_1C_1(A) = \mu C\mu^{-1}\mu C\mu^{-1}(A) = \mu\mu^{-1}\mu C\mu^{-1}\mu C\mu^{-1}(A) = \mu ECEC\mu^{-1}(A) = \mu ECC\mu^{-1}(A) = \mu EC\mu^{-1}(A) = \mu\mu^{-1}\mu C\mu^{-1}(A) = \mu C\mu^{-1}(A) = C_1(A)$. Therefore C_1 is a closure operator. Now, to verify that μ is a closure homomorphism we have $C_1\mu = \mu C\mu^{-1}\mu = \mu CE = \mu\mu^{-1}\mu CE = \mu ECE = \mu EC = \mu\mu^{-1}\mu C = \mu C$. The uniqueness of C_1 follows from the fact that $\mu C = C_1\mu$ implies $\mu C\mu^{-1} = C_1\mu\mu^{-1} = C_1$.

PROPOSITION 11: If (C,S) is a closure space and $E \in \mathcal{E}(S)$, then the following are equivalent:

(i) E belongs to $\mathcal{K}(C,S)$

(ii) $ECE = EC$

(iii) $CEC = EC$

(iv) $CE \subseteq EC$

Proof: (i) implies (ii) by Proposition 9. Suppose (ii) is true. Then $ECE = EC$ implies $ECEC = ECC = EC$. But from $EC \subseteq CEC \subseteq ECEC$ we can conclude $CEC = EC$, i.e., (iii) is true. (iii) clearly implies (iv) and from (iv) we in turn obtain (ii) immediately. Finally, by Proposition 10 we have (ii) implies that (i) is true.

We see that $CE = EC$ is too strong by the example: $S = \{1,2,3\}$, $C(\{1\}) = C(\{2\}) = C(\{1,2\}) = \{1,2\}$, otherwise $C(P) = S$; $S_1 = \{1,2\}$, $C_1(P) = P$ for all P contained in S_1; $\varphi(1) = 1$, $\varphi(2) = \varphi(3) = 2$. Also $CEC = ECE$ is too weak by the example: $S = \{1,2,3\}$, $E = \Delta \cup \{(1,2),(2,1)\}$, and define $C(\{1\}) = \{1\}$, $C(P) = S$ otherwise.

From statements (ii) and (iii) of Proposition 11 we easily see that if $E \in \mathcal{K}(C,S)$, then EC is a closure operator, whereas CE may not be (otherwise from (ii) we have $CECE = CEC$, and then $CE = CEC$ which with (iii) would give $CE = EC$).

Using the laws of Proposition 11 we see that if E is in $\mathcal{K}(C,S)$, then using the two operations C and E on S we can construct at most five distinct sets, namely P, $C(P)$, $EC(P)$, $E(P)$ and $CE(P)$. A natural question to ask is: Is it possible to find a closure space (C,S) and a congruence $E \in \mathcal{K}(C,S)$ such that for some P the above five sets are distinct? (The question is an obvious rewording of a well-known problem in topology using the Kuratowski closure operator and complementation.) The answer is shown to be in the affirmitive by the following. Let $S = \{1,2,3,4\}$, $C(\{1\}) = C(\{3\}) = C(\{1,3\}) = \{1,3\}$, $C(P) = \{1,2,3\}$ if 2, but not 4, is in P, and if 4 is in P define $C(P)$ to be S. Then with $E = \Delta \cup \{(1,2),(2,1),(3,4),(4,3)\}$ we have $ECE(P) = EC(P) = S$ for all P, so $E \in \mathcal{K}(C,S)$. Now, choosing P to be the singleton $\{1\}$, then $C(P) = \{1,3\}$, $EC(P) = S$, $E(P) = \{1,2\}$, and $CE(P) = \{1,2,3\}$, giving the desired five sets.

PROPOSITION 12: Suppose E is a member of $\mathcal{E}(S)$. Then the family of C in L such that $E \in \mathcal{K}(C,S)$ is a complete sub-lattice of L.

Proof: Let $\{C_a\}_{a\in A}$ be a subfamily of L such that E is in each $\mathcal{K}(C,S)$. Then $(\bigwedge \{C_a: a\in A\}) = (\bigcap \{C_a: a\in A\}) \subseteq \bigcap \{EC_a: a\in A\} = E(\bigcap \{C_a: a\in A\}) = E(\bigwedge \{C_a: a\in A\})$, so E is in $\mathcal{K}(\bigwedge \{C_a: a\in A\})$.

To show that we have closure under arbitrary supremum, first note that (i) if C is in L and $CE \subseteq EC$, then $C^{\sigma}E \subseteq EC^{\sigma}$ for any ordinal σ, and (ii) if $C_b \in L$ for $b \in B$, then $E(\bigcup \{C_b: b\in B\}) = \bigcup \{EC_b: b\in B\}$. From these two facts we easily deduce that if $\{C_a\}_{a\in A}$ is a subfamily of L such that E is in each $\mathcal{K}(C_a,S)$, then $(\bigcup\{C_a: a\varepsilon A\})E = (\bigcup\{C_aE: a\varepsilon A\}) \subseteq (\bigcup\{EC_a: a\varepsilon A\}) = E(\bigcup\{C_a: a\varepsilon A\})$, and thus $(\bigvee\{C_a: a\varepsilon A\})E = (\bigcup\{C_a: a\varepsilon A\})^{\sigma}E \subseteq E(\bigcup\{C_a: a\varepsilon A\})^{\sigma} = E(\bigvee\{C_a: a\varepsilon A\})$, where $\sigma = O(\bigcup\{C_a: a\varepsilon A\})$.

Although $\xi(S)$ can be canonically embedded in $\mathfrak{L}$ or L, it does not form a sublattice of either. (One should be careful not to confuse the intersections of members of $\xi(S)$ considered as subsets of $S \times S$ with the intersections of the same considered as members of $\mathfrak{L}$.) Still, we have:

PROPOSITION 13: $\xi(S)$ is closed in L under arbitrary supremum.

Proof: Let $E_a \varepsilon \xi(S)$ for a in A. By Proposition 6, Chapter I, it follows that $\bigcup\{E_a: a\varepsilon A\}$ is a 2 pre-closure, and since ω is regular, then $\bigvee\{E_a: a\varepsilon A\}$ is a 2 closure. Now we only need to show that if $(\bigvee\{E_a: a\varepsilon A\})(\{x\}) \cap (\bigvee\{E_a: a\varepsilon A\})(\{y\}) \neq \emptyset$, then $(\bigvee\{E_a: a\varepsilon A\})(\{x\}) = (\bigvee\{E_a: a\varepsilon A\})(\{y\})$. So suppose we have such a non-empty intersection. Then for some positive integer n, $(\bigcup\{E_a: a\varepsilon A\})^n(x) \cap (\bigcup\{E_a: a\varepsilon A\})^n(\{y\}) \neq \emptyset$ which, since the E_a are topological, implies $E_{a_1}\cdots E_{a_n}(\{x\}) \cap E_{b_1}\cdots E_{b_n}(\{y\}) \neq \emptyset$ for some choice of subscripts $a_1,\ldots,a_n,b_1,\ldots,b_n$. From this we can easily see that x is a member of $(E_{a_n}\cdots E_{a_1}E_{b_1}\cdots E_{b_n})(\{y\})$ which is contained in $(\bigcup E_a)^{2n}(\{y\})$, and this in turn is contained in $(\bigvee\{E_a: a\varepsilon A\})(\{y\})$, and by symmetry we can conclude the desired equality, and thus it follows that $\bigvee\{E_a: a\varepsilon A\} \ \varepsilon \ \mathcal{E}(S)$.

In general $\mathcal{K}(C,S)$ is not closed under finite supremum or finite infimum in the lattice $\mathcal{E}(S)$, and even the condition that C be an $\aleph_0$ closure is not sufficient.

PROPOSITION 14: If (C,S) is an algebraic closure space, $\mathcal{K}(C,S)$ is closed under arbitrary supremum in $\mathcal{E}(S)$.

Proof: Let E_α be a subset of $\mathcal{K}(C,S)$ for α in I, and we can assume without loss of generality that I is an initial segment of the ordinals. Then an easy induction establishes the inequalities

$$\bigcup\{E_\alpha:\alpha\varepsilon I\} \subseteq \mu\{E_\alpha: \alpha\varepsilon I\} \subseteq (\bigcup\{E_\alpha : \alpha\varepsilon I\})^\omega$$

and this gives for each finite n the inequality

$$(\bigcup\{E_\alpha: \alpha\varepsilon I\})^n \subseteq (\mu\{E_\alpha:\alpha\varepsilon I\})^n \subseteq (\bigvee\{E_\alpha:\alpha\varepsilon I\}).$$

From the fact that E_α is in $\mathcal{K}(C,S)$ another easy induction leads to $C(\mu\{E_\alpha: \alpha\varepsilon I\})^n \subseteq (\mu\{E_\alpha:\alpha\varepsilon I\})^n C$, so $C(\bigcup\{E_\alpha: \alpha\varepsilon I\})^n \subseteq C(\mu\{E_\alpha:\alpha\varepsilon I\})^n \subseteq (\mu\{E_\alpha: \alpha\varepsilon I\})^n C \subseteq (\bigcup\{E_\alpha: \alpha\varepsilon I\})^\omega C$. Now from the fact that C is algebraic we can conclude $C(\bigcup\{E_\alpha: \alpha\varepsilon I\})^\omega \subseteq (\bigcup\{E_\alpha: \alpha\varepsilon I\})^\omega C$, i.e., $C(\bigvee\{E_\alpha:\alpha\varepsilon I\}) \subseteq (\bigvee\{E_\alpha: \alpha\varepsilon I\})C$, so $\bigvee\{E_\alpha: \alpha\varepsilon I\}$ is a member of $\mathcal{K}(C,S)$.

If (C,S) is an algebraic closure space, then if we use the partial ordering of $\mathcal{L}$ on $\mathcal{K}(C,S)$ we see that $\mathcal{K}(C,S)$ is a complete lattice.

The following proposition gives a nice translation property which will be important in the representation theorem involving congruences.

PROPOSITION 15: Suppose E is a congruence for (C,S). If P and Q are subsets of S satisfying $E(P) = E(Q)$, then for any x in $C(P)$ there is a y in $C(Q)$ such that xEy.

Proof: Note that $C(P) \subseteq CE(P) \subseteq CE(Q) \subseteq EC(Q)$.

The following formulations of the isomorphism theorems of E. Noether (see [6]) will show how closely the theory of closure spaces reflects that of universal algebras.

PROPOSITION 16: Let φ be a closure homomorphism from (C,S) to (C_1,S_1). Then if E is a congruence on (C,S) which is contained in the kernel of φ (i.e., in $\varphi^{-1}\varphi$), then there is a unique homomorphism σ from S/E to S_1 such that $\varphi = \sigma\mu$, where μ is the canonical homomorphism from S onto S/E.

Proof: (An easy rewording of the proof on page 60 of [6]).

PROPOSITION 17: Let (C,S) be a closure space, and A a closed subset of S. If E is a congruence on (C,S), then

$$(A,C_1)/E_1 \approx (EA,C_2)/E_2$$

where C_1 and E_1 are respectively C and E restricted to A, C_2 and E_2 are C and E restricted to EA.

Proof: Let μ be the canonical map from S onto S/E. Then, with μ_1 being the restriction of μ to A we have $\mathrm{Im}(\mu_1)$ is EA/E_2, and since the kernel of μ_1 is A, Proposition 16 gives the result.

PROPOSITION 18: Let (C,S) be a closure space and E_1, E_2 two congruences such that $E_1 \subseteq E_2$. Then there is a unique homomorphism φ from S/E_1 to S/E_2 such that $(\text{nat } E_1)\varphi =$ nat E_2 (where nat means the canonical map to the factor space). If $\ker(\varphi) = E_2/E_1$, then E_2/E_1 is a congruence on S/E_1 and φ induces an isomorphism

$$\sigma\colon (S/E_1)(E_2/E_1) \to S/E_2$$

such that $\varphi = (\text{nat } E_2/E_1)\sigma$.

Proof: (Same as the proof of Theorem 3.11 of [6] on page 61).

3. Representation Theorems

In this section we will be concerned with filling in the various kinds of closures with general algebras, so we will begin with the new notation and definitions needed. A mapping f from a finite Cartesian product of S into S will be called finitary. If F is a family of finitary maps, then (S,F) will be called an algebra (or abstract algebra). An algebra (S,F) is N-ary if every operation belonging to F is at most N-ary. The mapping C defined by C(P) = "the smallest subset of S containing P and closed under the elements of F" is an algebraic closure. The closure so defined is called the closure induced by (S,F). Conversely, given a closure C, then any algebra whose induced closure is C will be called an algebraic representation of C (or of the closure space (C,S)). For the sake of completeness the first two propositions will state known results.

PROPOSITION 19: (Jürgen Schmidt). A closure C is induced by an algebra if and only if C is algebraic.

PROPOSITION 20. (Edward Marczewski). A closure is induced by an N-ary algebra if and only if it is N-ary.

PROPOSITION 21: Let C be an algebraic closure. Then C has an algebraic representation by some (S,F) with F finite if and only if C is operational.

Proof: Assume C has an algebraic representation by (S,F) with F finite. Let M be the maximum n such there is an f in F and f is n-ary. Then for all P contained in S define $C_1(P) = \bigcup\{f(P^n): f\varepsilon F,$ domain $(f) = S^n\} \cup P$. (Jürgen Schmidt uses this particular pre-closure in [17]). If $\overline{\overline{Q}} < M$, then $\overline{\overline{C_1(Q)}} < (\overline{\overline{F}})(M^M)+M$. Let the latter expression be L. Without much difficulty we can show Proposition 14 applies, so C is operational.

For the converse assume that C is operational with index N. Then let C_1, M and L satisfy the conditions of Proposition 14, Chapter I. For all Q such that $\overline{\overline{Q}} < M$ let $a_1^Q, \ldots, a_L^Q$ be some fixed ordering of the elements of $C_1(Q)$, where the elements of $C_1(Q)$ may appear more than once so that the sequence has L members, and (2) all the elements of $C_1(Q)$ must appear in the sequence. Then define the finitary functions f_i from S^M to S for $1 \leq i \leq L$ by $f_i(x_1,\ldots,x_M) = a_i^Q$ where Q is the set of elements $\{x_1,\ldots,x_M\}$. (Note that the operations have been chosen symmetric). Also add the nullary functions $a_1^{\emptyset},\ldots,a_M^{\emptyset}$ (if they exist).

Now using property (ii) of Proposition 14, Chapter I it follows readily that $C_1(P) = \bigcup \{f_i(P^M): 1 \leq i \leq L\} \cup \{a_i^{\emptyset}: 1 \leq i \leq M\}$, and then from property (i) we can verify that C is induced by (S,F), where of course F is the set $\{f_i: 1 \leq i \leq L\}$, and therefore C has an algebraic representation with F finite.

From the proof of Proposition 21 we see that an operational closure with index N has a representation by an algebra (S,F) where F has at most $N+2+2^{(N+1)}(N+1)$ maps, and each of the maps are $(N+1)$-ary. The following example will show that Proposition 21 provides a simple necessary and sufficient condition for an important class of closures to have an algebraic representation by an algebra (S,F) with F finite. Let S be a lattice, and for P a subset of S, let $C(P)$ be the filter generated by P (i.e., the least filter containing P). Then C has an algebraic representation by an algebra with a finite number of maps if and only if for each x in S the set $C(\{x\})$ is countable. To show this we need only note that $C = C_3^{\omega}$, and for x,y in S it is true that $C_3(\{x,y\})$ is equal to $C_3(\inf(x,y))$. From the remark following Proposition 21 we see that if we have such a representation, then F need consist of at most 27 ternary maps.

PROPOSITION 22: Suppose E is a congruence for the closure space (C,S), where C is algebraic. Then there is an abstract algebra (S,F) such that the induced closure is C and such that E is a congruence for (S,F) in the sense of general algebra.

Proof: For each finite set Q and each x in $C(Q)$ define the Q-ary map $f_{Q,x}$ by:

$f_{Q,x}(q_1,\ldots,q_n) = x$ if $\{q_1,\ldots,q_n\} = Q$,

$f_{Q,x}(q_1,\ldots,q_n) = y$

where $q_i \in E(Q)$, q_k is not in Q for some k, and y is some element in $C(\{q_1,\ldots,q_n\})$ which is equivalent to x as guaranteed by Proposition 15,

$f_{Q,x}(q_1,\ldots,q_n) = q_1$ otherwise.

Then letting $F = \{f_{Q,x}\colon Q \subseteq S,\ x \in Q\}$ we have the desired algebra.

COROLLARY: Let φ be a homomorphism from the algebraic closure space (C,S) onto the closure space (C_1,S_1). Then we can fit the spaces (C,S) and (C_1,S_1) with algebraic structures such that φ is an algebraic homomorphism and the algebraic structures induce the respective closure spaces.

Proof: We consider the congruence $\varphi^{-1}\varphi$ on (C,S) and apply Proposition 22 to obtain an algebra (S,F) which induces (C,S) and for which $\varphi^{-1}\varphi$ is a congruence in the sense of general algebra for (S,F). Then we define an algebra (S_1,F_1) on S_1 by requiring that $g \varepsilon F_1$ if and only if there is an $f \varepsilon F$ such that $\varphi f(x_1,\ldots,x_n) = g(\varphi(x_1),\ldots,\varphi(x_n))$ for all $x_1,\ldots,x_n$ in S. With this it is easy to show that (S_1,F_1) induces (C_1,S_1) and that φ is an onto homomorphism from (S,F) to (S_1,F_1).

In conclusion it might be appropriate to remark that much of this paper was motivated by studying the possibilities of a finite functional axiomatization of a logic in the spirit of Tarski's study of the possibility of completing a consistent theory [21], and later by an interest in using closures to circumscribe some important aspects of general algebras.

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