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VARIATIONAL AND OPTIMAL CONTROL PROBLEMS

INVOLVING UNBOUNDED OPERATORS

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VARIATIONAL AND OPTIMAL CONTROL PROBLEMS INVOLVING UNBOUNDED OPERATORS

CHAPTER I

UNBOUNDED LINEAR OPERATORS

1.1. Introduction. In recent years considerable attention has been given to a class of linear control problems known as minimum effort problems. In 1962 L. W. Neustadt [23] formulated the minimum effort problem, and in 1964 he extended this problem to a formulation in Lebesgue spaces. Also in 1962, W. T. Reid [30] considered a problem of this type and again the setting was in Lebesgue spaces.

Reid [30] and Neustadt [24] both attacked the problem by transforming the minimum norm problem into an equivalent finite moment problem. It should be noted that Neustadt used a slightly different method of analysis than that of Reid.

In 1966 W. A. Porter and J. P. Williams [25] formulated the minimum effort control problem in an abstract Banach space, and applied classical techniques of functional analysis to analyze the problem. The basic problem considered by Porter and Williams [25] may be stated as follows.

Let X and Y be Banach spaces and $T:X \to Y$ a continuous linear transformation with range $R(T) = Y$. Given $\xi \in Y$, find an element $x \in X$ satisfying $Tx = \xi$ which minimizes $\|x\|$.

In [26], Porter and Williams considered extensions and variations of the basic problem. In [27], Porter noted that the assumption that T be onto Y needed to be removed when considering distributive systems, and in this case the problem of existence becomes more difficult. Porter gave in [27] an existence theorem for the case when R(T) is dense in Y. In all of Porter's work, conditions were placed on the Banach spaces to insure the existence of a solution for each continuous transformation. Also, no problems were considered for which the operator was unbounded.

Similar variational problems were analyzed in [9] by Z. I. Halilov and È. Dž. Aslanov. In 1969, Halilov and B. N. Panaioti considered the following problem in Hilbert spaces.

Let X and Y be Hilbert spaces and $T:D(T) \to Y$ a linear operator with $D(T) \subseteq X$. Given $\eta \in X$ and $\xi \in Y$, find an element $x \in D(T)$ which minimizes

$$\|x - \eta\|^2 + \|Tx - \xi\|^2.$$

The operator T was supposed to be such that T^{-1} was a continuous operator. In this case, the problem becomes equivalent to finding an element $y \in R(T)$ which minimizes

$$\|T^{-1}y - \eta\|^2 + \|y - \xi\|^2.$$

In [21], N. Minamide and K. Nakamura considered a problem which includes as special cases the above described problems of Porter and Williams and Halilov and Aslanov. The problem formulated by Minamide and Nakamura shall be generalized in Chapter III. The principal objective of the present paper is to obtain existence theorems for this problem, especially for the case involving unbounded operators. Once existence has been established, many of the results of the previous articles may be applied to obtain necessary conditions, and to characterize solutions.

Because in both classical variational problems, and modern control problems, differential operators frequently appear, it is of considerable interest to analyze problems involving unbounded operators. Since a bounded linear operator with closed domain is a closed operator, it seems natural to extend the above mentioned problems to the case where the operators are closed. In most of the previous articles, the spaces considered were usually reflexive Banach spaces or Hilbert spaces. Because many of the normed linear spaces that occur in classical variational problems are not reflexive, it is desirable to discuss problems in non-reflexive spaces.

Since differential operators have many special properties, Chapter I is devoted to a study of a certain class of linear operators which includes many differential operators. A very useful generalization of the classical Minkowski functional is presented in Chapter II. Chapter III is devoted to the formulation of a variational problem which is a direct generalization of Problem (P) in [21], and to the proof of existence theorems. Also, in Chapter III much of the theory developed in the previous chapters is used in considering a few specific problems.

1.2. Co-compact operators. The following terms and definitions are motivated by the definitions of Kato [16]. Suppose X and Y are normed linear spaces. We shall assume that the scalars are always the real numbers. If $D(T) \subseteq X$ is the domain of the linear operator T from X to Y, then this will be denoted by $T:D(T) \to Y$, and R(T) will represent the range of T. The symbol $\|\cdot\|$ is used to denote the norm in a normed linear space, the precise space involved being clear from the context.

If $\{x_n\}$ is a sequence in X, then we shall use the symbol $x_n \to x$ to mean that $\{x_n\}$ converges to $x \in X$ in the norm topology. The normed conjugate of X will be denoted by X^*, and x^* will represent an element of X^*. If $x \in X$ and $x^* \in X^*$, then $\langle x,x^*\rangle$ shall be used for $x^*(x)$. Although the symbol $\langle , \rangle$ is also used to denote an inner product, it will be clear from the context which interpretation is to be made. If X is a Hilbert space with inner product $\langle , \rangle$, then we may identify X^* with X and with this identification the double use of the symbol $\langle , \rangle$ will not present any ambiguity.

If X and Y are normed linear spaces, then the product space is the space $X \times Y$ of ordered pairs (x,y) with norm defined by

$$\|(x,y)\| = \sqrt{\|x\|^2 + \|y\|^2}\ , \quad (x,y) \in X \times Y. \tag{1.2.1}$$

It should be noted that other choices of the norm are possible. However, with norm defined by (1.2.1) we have that the spaces $(X \times Y)^*$ and $X^* \times Y^*$ are equivalent. To say that $(X \times Y)^* = X^* \times Y^*$ means the following:

(i) each element $(x^*,y^*) \in X^* \times Y^*$ defines an element $u^* \in (X \times Y)^*$ by

$$\langle (x,y),u^*\rangle = \langle x,x^*\rangle + \langle y,y^*\rangle \tag{1.2.2}$$

and conversely, each $u^* \in (X \times Y)^*$ may be expressed in the form (1.2.2) by a unique $(x^*,y^*) \in X^* \times Y^*$;

(ii) The norm of u^* is equal to the norm of the corresponding (x^*,y^*).

For a more detailed discussion of this matter one may see Kato [16; p. 164].

If $\{x_n\}$ is a sequence in X, then we shall say that $\{x_n\}$ <u>converges weakly</u> to $x \in X$ if $\langle x_n, x^* \rangle \to \langle x, x^* \rangle$ for each $x^* \in X^*$. If $\{x_n\}$ converges weakly to $x \in X$, then we shall denote this by "$x_n \to x$ weakly".

A set will be called <u>open</u> (<u>closed</u>, <u>compact</u>) if it is open (closed, compact) in the norm topology. A set will be called <u>weakly open</u> (<u>weakly closed</u>, <u>weakly compact</u>) if it is open (closed, compact) in the weak topology. The symbol θ will represent the zero element of a normed linear space, the precise space involved being clear from the context.

The interior of a set K will be denoted by int K, the boundary of K by ∂K, and the closure of K by cl K. The letter R will be used for the real numbers, and real n-dimensional space shall be denoted by R^n.

DEFINITION 1.2.1. *Let X and Y be normed linear spaces, and $T:D(T) \to Y$ a linear operator with $D(T) \subseteq X$. A sequence $\{x_n\}$ in D(T) is called semi-T-convergent if there is some element $y \in Y$ with $Tx_n \to y$. A sequence $\{x_n\}$ in D(T) is called T-convergent if there is an $x \in X$ and $y \in Y$ with $x_n \to x$ and $Tx_n \to y$.*

It should be noted that each T-convergent sequence is a semi-T-convergent sequence, but not conversely. In fact, a semi-T-convergent sequence need not be bounded. The following definition is standard and may be found in most functional analysis books.

DEFINITION 1.2.2. *Let $T:D(T) \to Y$ be a linear operator with $D(T) \subseteq X$. The operator T is said to be closed, if for each T-convergent sequence $\{x_n\}$ with $Tx_n \to y$ and $x_n \to x$, we have that $x \in D(T)$ and $Tx = y$.*

If $G(T) = \{(x,Tx) : x \in D(T)\}$ is the graph of T, then it follows that T is closed if and only if G(T) is closed as a subspace of $X \times Y$.

The following lemma may be found in Dunford and Schwartz [5; p. 422].

LEMMA 1.2.1. *If* K *is a convex subset of the normed linear space* X, *then* K *is closed if and only if it is weakly closed*.

If Lemma 1.2.1 is applied to $G(T)$, then we obtain the following result.

LEMMA 1.2.2. *Let* X *and* Y *be normed linear spaces and* $T:D(T) \to Y$ *a linear operator with* $D(T) \subseteq X$. *The operator* T *is closed if and only if for each sequence* $\{x_n\}$ *in* $D(T)$ *with* $x_n \to x$ *weakly and* $Tx_n \to y$ *weakly we have that* $x \in D(T)$ *and* $Tx = y$.

The concept of a co-compact operator will now be introduced.

DEFINITION 1.2.3. *Let* X *and* Y *be normed linear spaces and* $T:D(T) \to Y$ *a linear operator with* $D(T) \subseteq X$. *The operator* T *is said to be co-compact (weakly co-compact) if each bounded semi-T-convergent sequence* $\{x_n\}$, *with* $Tx_n \to y$, *has a subsequence* $\{z_k\}$ *such that* $z_k \to x$ ($z_k \to x$ *weakly*), $x \in D(T)$, *and* $Tx = y$.

In order to indicate the motivation behind the use of the terms co-compact and weakly co-compact, we shall review a few concepts concerning operator theory.

A linear operator $T:D(T) \to Y$ is said to be *weakly compact* if T takes bounded sequences onto sequences which have a weakly convergent subsequence. The operator T is said to be *compact* if T takes bounded sequences onto sequences which have a convergent subsequence. It is clear that every compact operator is weakly compact. However, there are many examples of weakly compact operators which are not compact. Examples of weakly compact operators which are not compact may be found in Goldberg [8; pp. 88-92].

If $T:D(T) \to Y$ is a linear operator with $D(T) \subseteq X$, then a linear operator $T^{\#}$ is said to be a <u>generalized inverse</u> of T if

$$R(T) \subseteq D(T^{\#}) \subseteq Y, \tag{1.2.3}$$

$$T^{\#}y = x \in D(T) \text{ and } Tx = y, \text{ for } y \in R(T). \tag{1.2.4}$$

In [32] by Reid and [2] by Beutler, one may find a discussion of generalized inverses and a set of references concerning generalized inverses.

If X is a reflexive Banach space, then each bounded sequence contains a weakly convergent subsequence. Thus, we have the following result.

LEMMA 1.2.3. *Suppose X is a reflexive Banach space and Y is a normed linear space. If $T:D(T) \to Y$ is closed and $D(T) \subseteq X$, then T is weakly co-compact.*

It should also be noted that a continuous operator with closed domain is closed. In particular, if T is continuous and $D(T) = X$, then T is closed.

THEOREM 1.2.1. *Let X and Y be normed linear spaces and $T:D(T) \to Y$ a closed linear operator with $D(T) \subseteq X$. If T^{-1} exists, then:*

(i) *if T^{-1} is compact, then T is co-compact;*

(ii) *if T^{-1} is continuous and R(T) is closed, then T is co-compact;*

(iii) *if T^{-1} is weakly compact, then T is weakly co-compact.*

Suppose T^{-1} is compact and $\{x_n\}$ is a bounded semi-T-convergent sequence with $Tx_n \to y \in Y$. The sequence $\{Tx_n\}$ is bounded and $x_n = T^{-1}(Tx_n)$; therefore there exists a subsequence $\{z_k\}$ of $\{x_n\}$ which converges to $x \in X$. Since T is closed and $\{z_k\}$ is T-convergent with $z_k \to x$ and $Tz_k \to y$, we have that $x \in D(T)$ and $Tx = y$. This completes the proof of (i), and with the use of Lemma 1.2.2 the proof of (iii) is similar.

If T^{-1} is continuous and $R(T)$ is closed, then $Tx_n \to y$ implies $y \in R(T)$. Therefore, if $\{x_n\}$ is a semi-T-convergent sequence with $Tx_n \to y$, then there exists an $x \in D(T)$ with $Tx = y$. Since $x_n = T^{-1}(Tx_n)$ and T^{-1} is continuous, we have that $x_n \to T^{-1}y = x$, and $Tx = y$.

The following result is a direct consequence of Definition 1.2.3, and its proof will be omitted.

LEMMA 1.2.4. <u>Let</u> X <u>and</u> Y <u>be normed linear spaces and</u> $T:D(T) \to Y$ <u>a linear operator with</u> $D(T) \subseteq X$. <u>If</u> T <u>is weakly co-compact, then</u> T <u>is closed, and if</u> T <u>is co-compact, then</u> T <u>is weakly co-compact.</u>

EXAMPLE 1.2.1. Let $1 \leq p \leq +\infty$ and μ be a measure defined on a σ-ring Σ of subsets of a set S. The set of all μ-measurable functions equivalent to f shall be denoted by $[f]$.

If $1 \leq p < \infty$, then $L_p(S,\Sigma,\mu)$ will denote the normed linear space of equivalent classes $[f]$ for which $|f|^p$ is integrable, with norm given by

$$\|[f]\|_p = \left(\int_S |f|^p d\mu\right)^{1/p}.$$

As is customary, we shall use f instead of $[f]$ as an element in $L_p(S,\Sigma,\mu)$.

If $p = \infty$, then $L_\infty(S,\Sigma,\mu)$ will denote the normed linear space of equivalent classes $[f]$ for which f is essentially bounded on S, with norm given by

$$\|[f]\|_\infty = \text{ess sup } |f(s)|.$$

If $S \subseteq R^n$, μ is Lebesgue measure, and Σ is the class of Lebesgue measurable sets, then we write $L_p(S)$ instead of $L_p(S,\Sigma,\mu)$.

If S is the set of positive integers, Σ the class of all subsets of S, and μ the counting measure, then we shall write ℓ_p instead of $L_p(S,\Sigma,\mu)$. In this case, elements of ℓ_p are sequences $\{\alpha_n\}$ with $\|\{\alpha_n\}\|_p = \left(\sum_{n=1}^{\infty} |\alpha_n|^p\right)^{1/p}$.

A complete discussion of these spaces may be found in Dunford and Schwartz [5; p. 285].

Let R denote the real numbers and suppose $x = \{\lambda_n\}$ is an element of ℓ_2. Define $T:\ell_2 \to R$ by $Tx = \lambda_1$, and note that T is a continuous operator. Since ℓ_2 is norm reflexive, it follows from Lemma 1.2.3 that T is weakly co-compact. Let $e^n \in \ell_2$ denote the sequence given by $e^n = \{\xi_k^n\}$, where

$$\xi_k^n = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases} \tag{1.2.5}$$

Because $Te^n = 0$ for each $n \geq 2$, we have that $\{e^n\}$ is a bounded semi-T-convergent sequence. However, $\{e^n\}$ does not have a convergent subsequence, and hence T is not a co-compact operator.

If $x = \{\lambda_n\}$ is an element of ℓ_1, then define $T:\ell_1 \to R$ by $Tx = \lambda_1$. The operator T is continuous with $D(T) = \ell_1$, and hence T is closed. If $e^n \in \ell_1$ is defined by relation (1.2.5), then $\{e^n\}$ is a bounded semi-T-convergent sequence with $Te^n \to 0$. However, in ℓ_1 there is no weakly convergent subsequence of $\{e^n\}$. Consequently, we have that T is not weakly co-compact.

Let X and Y be normed linear spaces and $T:D(T) \to Y$ a closed linear operator with $D(T) \subseteq X$. The null space of T is given by

$$N(T) = \{x \in D(T) : Tx = \theta\}.$$

If T is a closed linear operator, then N(T) is a closed subspace of X. Let X/N(T) be the corresponding quotient space, and denote the equivalence class that contains x by [x]. With norm defined by

$$\|[x]\| = \inf\{\|z\| : z \in [x]\},$$

the quotient space is a normed linear space. Let

$$D(\hat{T}) = \{[x] \in X/N(T) : x \in D(T)\},$$

and define $\hat{T}:D(\hat{T}) \to Y$ by

$$\hat{T}([x]) = Tx. \tag{1.2.6}$$

The following results are well known and may be found in Goldberg [8] on pages 8, 9, 63, 64 and 99.

LEMMA 1.2.5. *Let* X *and* Y *be normed linear spaces and* $T:D(T) \to Y$ *a linear operator with* $D(T) \subseteq X$. *If* $\hat{T}$ *is defined by* (1.2.6), *then:*

(i) $\hat{T}$ *is closed if* T *is closed;*

(ii) $X/N(T)$ *is a Banach space if* T *is closed and* X *is a Banach space;*

(iii) *if* $\{[x_n]\}$ *is a sequence converging to* $[x]$, *then there exists a sequence* $\{z_n\}$ *in* $N(T)$ *such that* $(x_n + z_n)$ *converges to* x;

(iv) *if* X *and* Y *are Banach spaces,* T *is closed, and* $R(T)$ *is closed, then* $\hat{T}^{-1}$ *is a continuous operator.*

LEMMA 1.2.6. *Suppose* X *and* Y *are Banach spaces and* $T:D(T) \to Y$ *is closed with* $D(T) \subseteq X$. *If* $T(K \cap D(T))$ *is closed for each closed and bounded set* K, *then* $R(T)$ *is closed. Also, if* $N(T)$ *is finite dimensional and* $R(T)$ *is closed, then* $T(K \cap D(T))$ *is closed for each closed and bounded set* K.

Conclusions (i) and (iv) of Lemma 1.2.5 may be found on pages 8 and 9 in [8], while conclusions (ii) and (iii) may be found on pages 63 and 64. The statement and proof of Lemma 1.2.6 may be found on page 99 in [8].

The following result gives a complete characterization of co-compact operators in Banach spaces.

THEOREM 1.2.2. *Let* X *and* Y *be Banach spaces and* $T:D(T) \to Y$ *a linear operator with* $D(T) \subseteq X$. *The operator* T *is co-compact if and only if* T *is*

<u>closed, with closed range and finite dimensional null space</u>.

If T is co-compact, then it follows from Lemma 1.2.4 that T is closed. Suppose $K \subseteq X$ is closed and bounded, and $y \in \text{cl}\,[T(K \cap D(T))]$. Because $y \in \text{cl}\,[T(K \cap D(T))]$, we have that there exists a sequence $\{x_n\}$ in $K \cap D(T)$, with $Tx_n \to y$. The sequence $\{x_n\}$ is bounded and T is co-compact; consequently, we have that there is an element $x \in D(T)$ and a subsequence $\{z_k\}$ of $\{x_n\}$ such that $z_k \to x$ and $Tx = y$. Moreover, since K is closed, it follows that $x \in K$, which implies that $y = Tx \in T(K \cap D(T))$. Consequently, $T(K \cap D(T))$ is closed for each closed and bounded set K, and from Lemma 1.2.6 it follows that $R(T)$ is closed.

Now suppose $\{x_n\}$ is a sequence in $N(T)$ with $\|x_n\| \leq 1$. Because $Tx_n = \theta$, we have that $\{x_n\}$ is a bounded semi-T-convergent sequence with $Tx_n \to \theta$. Therefore, $\{x_n\}$ has a subsequence $\{z_k\}$ with $z_k \to x$ and $Tx = \theta$, or equivalently, $x \in N(T)$. Consequently, each sequence in the unit ball of $N(T)$ has a subsequence converging to an element in $N(T)$, and hence $N(T)$ is finite dimensional.

Conversely, suppose T is a closed operator with closed range and finite dimensional null space. If $\{x_n\}$ is a bounded semi-T-convergent sequence with $Tx_n \to y$, then y must belong to $R(T)$. Let $x \in D(T)$ be such that $Tx = y$. It follows from (iv) of Lemma 1.2.5 that $\hat{T}^{-1}$ is continuous, and since $Tx_n \to y$ we have that

$$[x] = \hat{T}^{-1}(y) = \hat{T}^{-1}(\lim_{n\to\infty} Tx_n) = \lim_{n\to\infty} [x_n].$$

By (iii) of Lemma 1.2.5, there is a sequence $\{z_n\}$ in $N(T)$ such that $(x_n + z_n) \to x$. The sequence $\{x_n\}$ is bounded and $\{x_n + z_n\}$ is bounded; consequently, the sequence $\{z_n\}$ is bounded. Since $N(T)$ is finite dimensional, it follows that there exists a subsequence $\{\eta_k\}$ of $\{z_n\}$ and an element $z \in N(T)$ with $\eta_k \to z$. Let $\{w_k\}$ be the corresponding subsequence of $\{x_n\}$ and note that $w_k \to (x - z)$. Both x and z belong to $D(T)$ and $z \in N(T)$, so we have $T(x-z) = Tx = y$, and the theorem is proved.

If X and Y are normed linear spaces and $T:D(T) \to Y$ is closed, then the *kernel index* of T is the dimension of $N(T)$. The *deficiency index* of T is the dimension of the quotient space $Y/R(T)$. The kernel index of T is denoted by $\alpha(T)$ and the deficiency index of T by $\beta(T)$.

DEFINITION 1.2.4. *Let* X *and* Y *be normed linear spaces and* $T:D(T) \to Y$ *a linear operator with* $D(T) \subseteq X$. *The operator* T *is said to be a Fredholm operator if* T *is closed and both* $\alpha(T)$ *and* $\beta(T)$ *are finite*.

In Goldberg [8; p. 103] it is shown that if X and Y are Banach spaces and T is a Fredholm operator, then $R(T)$ is closed. From this remark and Theorem 1.2.2, it follows that in Banach spaces Fredholm operators are co-compact. There are many differential operators which are Fredholm, and hence the class of co-compact operators contains many important operators.

Theorem 1.2.2 gives a complete characterization of co-compact operators with domain and range in Banach spaces. We have the following corresponding result concerning weakly co-compact operators.

THEOREM 1.2.3. *Let* X *and* Y *be Banach spaces and* $T:D(T) \to Y$ *a linear operator with* $D(T) \subseteq X$. *If* T *is closed with closed range and reflexive null space, then* T *is weakly co-compact*.

Suppose T is a closed operator, $R(T)$ is closed, and $N(T)$ is reflexive. If $\{x_n\}$ is a bounded semi-T-convergent sequence such that $Tx_n \to y$, then $y \in R(T)$. Let $x \in D(T)$ be such that $Tx = y$. It follows from (iv) of Lemma 1.2.5 that $\hat{T}^{-1}$ is continuous, and therefore we have that

$$[x] = \hat{T}^{-1}(y) = \hat{T}^{-1}(\lim_{n\to\infty} Tx_n) = \lim_{n\to\infty} [x_n].$$

From (iii) of Lemma 1.2.5, we obtain a sequence $\{z_n\}$ in $N(T)$ such that $(x_n + z_n) \to x$. Since $\{z_n\}$ is bounded and $N(T)$ is reflexive, there exists a subsequence $\{\eta_k\}$ of $\{z_n\}$ and an element $z \in N(T)$ with $\eta_k \to z$ weakly. Let $\{w_k\}$ denote the corresponding subsequence of $\{x_n\}$, and note that $w_k \to (x - z)$ weakly, while $Tw_k \to y$. Since T is a closed operator, it follows that $(x - z) \in D(T)$ and $T(x - z) = Tx = y$.

It should be noted that Theorem 1.2.3 is only a sufficient condition for an operator to be weakly co-compact. A partial converse to this theorem is given by the following result.

THEOREM 1.2.4. Let X and Y be normed linear spaces and $T:D(T) \to Y$ a linear operator with $D(T) \subseteq X$. If T is weakly co-compact, then N(T) is a reflexive Banach space.

First it shall be shown that N(T) is complete. Let $\{x_n\}$ be a Cauchy sequence in N(T), and suppose M is the completion of N(T). Since $\{x_n\}$ is Cauchy, there is an element $x_o \in M$ such that $x_n \to x_o$ in M. However, $Tx_n = \theta$ and $\{x_n\}$ is bounded, so there exists an element $x \in N(T)$ and a subsequence $\{z_k\}$ of $\{x_n\}$ such that $z_k \to x$ weakly in N(T). If $m^* \in M^*$ is a continuous linear functional on M, then the restriction of m^* to N(T) is a continuous linear functional on N(T). Therefore, it follows that $\langle z_k, m^*\rangle \to \langle x, m^*\rangle$, and hence the sequence $\{z_k\}$ converges weakly to x in M. Since the sequence $\{z_k\}$ converges to x_o in M, we have that $x = x_o$, and hence the sequence $\{x_n\}$ converges to $x \in N(T)$. This proves that N(T) is a complete space.

In view of the weak co-compactness of T, each bounded sequence in

$N(T)$ has a weakly convergent subsequence, and consequently $N(T)$ is a reflexive Banach space.

The conclusion of Theorem 1.2.4 indicates nothing about the range of T. Therefore, even the combination of Theorem 1.2.3 with Theorem 1.2.4 does not provide a specific characterization of weakly co-compact operators. If in Theorem 1.2.3 we could delete the assumption that $R(T)$ is closed, then Theorem 1.2.3 and Theorem 1.2.4 would provide a complete characterization of weakly co-compact operators in Banach spaces. On the other hand, if in Theorem 1.2.4 we could prove that $R(T)$ is closed under the additional assumption that X and Y are Banach spaces, then we would have a complete characterization. The following examples will show that neither of these cases is possible.

EXAMPLE 1.2.2. Let $I = [0,1]$ and $X = Y = L_2(I)$, and define $T:X \to Y$ by

$$[Tx](t) = \int_0^t x(s)ds.$$

Because T is a continuous operator defined on all of X, we have that T is closed. The space $L_2(I)$ is reflexive, and therefore in view of Lemma 1.2.3 we have that T is weakly co-compact. The range of T is the set of all $x \in L_2(I)$ which are absolutely continuous, with derivative belonging to $L_2(I)$, and satisfying $x(0) = 0$. Thus $R(T)$ is not closed and T is weakly co-compact. It is to be noted that both X and Y are Banach spaces.

EXAMPLE 1.2.3. Let c_o be the subspace of ℓ_∞ consisting of the sequences which converge to 0. Let $x = \{\lambda_n\}$ be an element of c_o and define $T:c_o \to \ell_2$ by $Tx = y$, where $y = \{\lambda_n/n\}$. The domain of T is a Banach space and T is continuous, and thus we have that T is closed. Moreover, the null space of T is only $\{\theta\}$ because T is one-to-one, and consequently T is a closed linear operator with reflexive null space. Let $\tau_n \in c_o$ be defined by $\tau_n = \{\lambda_k^n\}$ where

$$\lambda_k^n = \begin{cases} 1, & \text{if } 1 \le k \le n; \\ 0, & \text{if } k > n. \end{cases}$$

Then $\|\tau_n\| = 1$ in c_o, and $T\tau_n \to y_1$ in ℓ_2, where $y_1 = \{1/n\}$. Now y_1 is not in R(T), since $Tx = y_1$ implies that $x = \{1,1,1,\cdots\}$, which is not in $D(T) = c_o$. Hence the operator T can not be weakly co-compact.

The following result about weakly co-compact operators is basic, when questions of existence are considered.

THEOREM 1.2.5. Let X and Y be normed linear spaces and $T:D(T) \to Y$ a weakly co-compact operator with $D(T) \subseteq X$. If $K \subseteq X$ is closed, convex and bounded, then $T(K \cap D(T))$ is closed and convex.

Suppose $K \subseteq X$ is closed, bounded, and convex, and $y \in \text{cl}\,[T(K \cap D(T))]$. There is a sequence $\{x_n\}$ in $K \cap D(T)$ with $Tx_n \to y$, and hence $\{x_n\}$ is a bounded semi-T-convergent sequence. Let $\{z_k\}$ be a subsequence of $\{x_n\}$ with $z_k \to x$ weakly and $Tx = y$. Since K is closed and convex, we have by Lemma 1.2.1 that K is weakly closed and therefore $x \in K$. Thus, we have that $y = Tx \in T(K \cap D(T))$, which proves that $T(K \cap D(T))$ is closed.

A theorem which is fundamental in the study of linear operators is the open-mapping theorem. The open-mapping theorem was used by Porter and Williams [25,26], and Minamide and Nakamura [21,22], in obtaining existence for the minimum effort control problem. The following theorem is an open-mapping theorem for which neither the domain space nor the range space need be complete.

If X is a normed linear space, and $\varepsilon > 0$, then we shall let

$$B(X,\varepsilon) = \{x \in X : \|x\| \leq \varepsilon\},$$

and

$$B^o(X,\varepsilon) = \{x \in X : \|x\| < \varepsilon\}.$$

THEOREM 1.2.6. *Let X and Y be normed linear spaces and $T:D(T) \to Y$ a weakly co-compact operator with $D(T) \subseteq X$. If $R(T)$ is of the second category in Y, then $T(K \cap D(T))$ is open for each open set $K \subseteq X$, and* $R(T) = Y$.

Because of translation, it suffices to show that for each $\varepsilon > 0$ the set $T(B(X,\varepsilon) \cap D(T))$ contains an open set of the form $B^o(Y,r)$, for some $r > 0$. Since $R(T)$ is of the second category in Y, and

$$R(T) = \bigcup_{n\geq 1} n\cdot T(B(X,1) \cap D(T)),$$

there is a positive integer m such that cl $[m\cdot T(B(X,1) \cap D(T))]$ contains a non-empty open set. Therefore the set

$$\text{cl } [T(B(X,\varepsilon/2) \cap D(T))] = (\varepsilon/2m) \text{ cl } [m\cdot T(B(X,1) \cap D(T))]$$

also contains a non-empty open set V. Thus we have that

$$\theta \in V - V \subseteq \text{cl } [T(B(X,\varepsilon) \cap D(T))].$$

Since $V - V$ is an open set containing θ, there is an $r > 0$ such that

$$\theta \in B^o(Y,r) \subseteq V - V \subseteq \text{cl } [T(B(X,\varepsilon) \cap D(T))].$$

Because $B(X,\varepsilon)$ is a closed and bounded convex set, it follows from Theorem 1.2.5 that

$$\text{cl } [T(B(X,\varepsilon) \cap D(T))] = T(B(X,\varepsilon) \cap D(T)).$$

Thus for each $\varepsilon > 0$ the set $T(B(X,\varepsilon) \cap D(T))$ contains a set of the form $B^o(Y,r)$ for some $r > 0$. In particular, $T(B(X,1) \cap D(T))$ is a neighborhood of θ in Y, and hence $R(T) = Y$.

One important thing to note about Theorem 1.2.6 is that there is no assumption concerning the completeness of X or Y. There are examples of normed linear spaces that are of the second category, which are not complete. For such an example see, Bourbaki [4; p. 3]. Also, Theorem 1.2.6 differs from the standard open-mapping theorem in that X need not be complete.

EXAMPLE 1.2.4. Let X denote the space of all continuous real-valued functions defined on [0,1] with norm

$$\|x\| = |x(0)| + \left(\int_0^1 x^2(s)ds\right)^{1/2}.$$

Let $Y = C[0,1]$ be the space of continuous real-valued functions with uniform norm,

$$\|x\| = \sup \{|x(s)| : 0 \leq s \leq 1\}.$$

The derivative of x at the point t shall be denoted by $x'(t)$. Let

$$D(T) = \{x \in X : x'(t) \text{ exists and is continuous}\},$$

and define $T:D(T) \to Y$ by $[Tx](t) = x'(t)$.

If $\{x_n\}$ is a bounded semi-T-convergent sequence with $Tx_n \to y$, then $\{|x_n(0)|\}$ is bounded and $\{x_n'\}$ converges uniformly to y on [0,1]. Let $\{z_k\}$ be a subsequence of $\{x_n\}$ such that $z_k(0) \to \alpha$, and let $x(t) = \alpha + \int_0^t y(s)ds$. Since $\{z_k'\}$ converges uniformly to y and $\{z_k(0)\} \to x(0)$, the sequence $\{z_k\}$ converges uniformly to $x(t) = \alpha + \int_0^t y(s)ds$ on [0,1], and $x'(t) = y(t)$. But uniform convergence of $\{z_k\}$ to x implies that $\{z_k\}$ converges to x in X. This proves that T is a co-compact operator, and hence weakly co-compact. By Theorem 1.2.6 we have that T is an open mapping; however, the space X is not complete.

Let X, Y, and Z be normed linear spaces and $T:D(T) \to Y$, $S:D(S) \to Z$ linear operators with $D(T) \subseteq X$ and $D(S) \subseteq X$. Let $U = X \times Y$ and $W = Y \times Z$ be the product spaces with corresponding norms defined by (1.2.1). Let

$$D(Q) = [D(T) \cap D(S)] \times Y, \tag{1.2.7}$$

and $Q:D(Q) \to W$ be defined by

$$Q(x,y) = (Tx + y, Sx). \tag{1.2.8}$$

If both T and S are closed operators, then it follows readily that Q is closed. However, it is possible for Q to be closed without S being closed.

EXAMPLE 1.2.5. Let $X = Y = L_2(I)$, where $I = [0,1]$ and $Z = R^2 = R \times R$. Let $D(S) = \{x \in X : x \text{ a.c.}\}$, and define $S:D(S) \to Z$ by

$$Sx = \begin{pmatrix} x(0) \\ x(1) \end{pmatrix};$$

also, for $D(T) = \{x \in X : x \text{ a.c.}, x' \in L_2(I)\}$, define $T:D(T) \to Y$ by $[Tx](t) = x'(t)$. In the above, and corresponding future statements the abbreviation "a.c." is used for "absolutely continuous". It is easy to establish that Q is a closed operator, but S is not closed; moreover, the operator Q is also weakly co-compact.

THEOREM 1.2.7. *Let* X, Y, *and* Z *be normed linear spaces and* $T:D(T) \to Y$, $S:D(S) \to Z$ *closed linear operators with* $D(T) \subseteq X$ *and* $D(S) \subseteq X$. *If* $Q:D(Q) \to W$ *is defined by* (1.2.8), *then*:

(i) *if* X *and* Y *are reflexive Banach spaces, then* Q *is weakly co-compact*;

(ii) *if* T *is bounded with* $D(T) = X$, *and* S *is weakly co-compact, then*

Q *is weakly co-compact*;

(iii) *if* T *is bounded with* $D(T) = X$, *and* X *is reflexive, then* Q *is weakly co-compact*.

If $\{(x_n,y_n)\}$ is a sequence in U such that $x_n \to \xi$ weakly in X and $y_n \to \eta$ weakly in Y, then it follows from condition (1.2.2) that the sequence $\{(x_n,y_n)\}$ converges weakly to (ξ,η) in U. Therefore, if X and Y are reflexive Banach spaces, then U is reflexive, and hence conclusion (i) follows from Lemma 1.2.3.

In order to establish conclusion (ii), suppose $\{(x_n,y_n)\}$ is a bounded semi-Q-convergent sequence with $Q(x_n,y_n) \to (y_o,z_o)$. Then we have that $Sx_n \to z_o$, and $\{x_n\}$ is bounded. Hence there is a subsequence $\{\xi_k\}$ of $\{x_n\}$ and an element $x_o \in D(S)$ with $\xi_k \to x_o$ weakly and $Sx_o = z_o$. Since T is continuous, we have that $T\xi_k \to Tx_o$ weakly. Let $\{\eta_k\}$ be the subsequence of $\{y_n\}$ corresponding to $\{\xi_k\}$ and note that $\eta_k \to (y_o - Tx_o)$ weakly. Therefore $\{(\xi_k,\eta_k)\}$ is a subsequence of $\{(x_n,y_n)\}$ such that $(\xi_k,\eta_k) \to (x_o,y_o - Tx_o)$ weakly, and $Q(x_o,y_o - Tx_o) = (y_o,Sx_o) = (y_o,z_o)$.

Conclusion (iii) follows from (ii) and Lemma 1.2.3, since the operator S is closed and X is reflexive.

The following is an example of a weakly co-compact operator L which is not co-compact and such that $D(L)$ is not in a reflexive space.

EXAMPLE 1.2.6. Let $X = L_2(I)$ and $Y = L_1(I)$, where $I = [0,1]$. The product space $U = X \times Y$ is not reflexive. If E is the linear operator from X to Y defined by $Ex = x$, then it follows from Lemma 1 in [12] by Hemming and Vandelinde, that the graph of E is a reflexive Banach space.

Let $L:U \to L_1(I)$ be defined by

$$L(x,y) = x - y;$$

since L is continuous, it follows that L is a closed operator. Moreover, the range of L is all of $L_1(I)$, while the null space of L is the graph of E. By Theorem 1.2.3 we have that L is weakly co-compact. Clearly the operator L is not co-compact.

CHAPTER II

A MINKOWSKI FUNCTIONAL

2.1. Introduction. The Minkowski functional plays an important role in modern functional analysis, including applications to control problems. For example, in minimum norm problems the Minkowski functional produces the value of the norm of the optimal solutions. Moreover, Porter and Williams [25], and Minamide and Nakamura [21], were able to prove existence of solutions for minimum effort control problems provided that a Minkowski functional was positive at particular points. In order to obtain similar results for problems involving convex functionals, a generalization of the classical Minkowski functional is desired.

2.2. Convex functions and convex chains. The minimum effort control problem is concerned with the minimization of a convex function. In order to be precise, we present the following definition.

DEFINITION 2.2.1. *Let* X *be a normed linear space and* $J: X \to (-\infty, +\infty]$ *a function such that* $J(x) < +\infty$ *for at least one* $x \in X$. *Such a function* J *is said to be convex if*

$$J(\lambda x_1 + [1-\lambda]x_2) \le \lambda J(x_1) + [1-\lambda]J(x_2), \tag{2.2.1}$$

for all $x_1, x_2 \in X$, *and* λ *such that* $0 < \lambda < 1$. *The function* J *is said to be subadditive if*

$$J(x_1 + x_2) \leq J(x_1) + J(x_2), \qquad (2.2.2)$$

for all $x_1, x_2 \in X$. *The function* J *is said to be positively homogeneous of degree* 1 *if*

$$J(\lambda x) = \lambda J(x), \qquad (2.2.3)$$

for all $x \in X$, *and* λ *such that* $\lambda > 0$.

A function J which is positively homogeneous of degree 1 will be referred to as simply positively homogeneous. Whenever terms such as $\lambda(+\infty)$ appear in (2.2.1) we shall always set $\lambda(+\infty)$ equal to $+\infty$, if $\lambda > 0$. Also, only those functions J which have at least one finite value shall be considered in this paper.

The following two results may be found in Rockafellar [34] for the special case when X is finite dimensional. However, the proofs found on pages 30 and 83 of [34] may be applied, without change, to prove the following results.

LEMMA 2.2.1. *Let* X *be a normed linear space and* $J:X \to (-\infty,+\infty]$ *a positively homogeneous function. The function* J *is convex if and only if* J *is subadditive.*

LEMMA 2.2.1. *If* X *is a finite dimensional normed linear space and* $J:X \to R$ *is convex, then* J *is continuous.*

In infinite dimensional spaces, Lemma 2.2.1 is not valid. In a large number of classical variational problems the functions to be minimized are convex, but not continuous.

EXAMPLE 2.2.1. Let C[0,1] be the space of continuous functions as defined in Example 1.2.4. Let X be the subspace of C[0,1] consisting of all functions with continuous derivatives; in particular, if $x \in X$ then the derivative function x' belongs to C[0,1]. Let $J_i : X \to R$, $(i = 1,2,3)$, be defined by

$$J_1(x) = \int_0^1 [x(s)]^2 ds,$$

$$J_2(x) = \int_0^1 ([x(s)]^2 + [x'(s)]^2) ds,$$

$$J_3(x) = \|x\| + \|x'\|.$$

The function J_1 is continuous and convex, while J_2 and J_3 are not continuous. Also, the function J_2 is convex and not subadditive, while J_3 is subadditive.

As Example 2.2.1 indicates, convex functions are not necessarily continuous. However, convex functions do have the following property.

LEMMA 2.2.3. *Let X be a normed linear space and $J:X \to R$ a convex function. If $x \in X$, then we have that $J(\lambda x) \to J(x)$ as $\lambda \to 1$.*

If $\phi:R \to R$ is given by $\phi(t) = J(tx)$, then ϕ is a real-valued convex function defined on R. By Lemma 2.2.2 we have that ϕ is continuous, from which Lemma 2.2.3 easily follows.

Given a convex function $J:X \to (-\infty,+\infty]$ and $\alpha \in R$, we shall let $L(\alpha)$ denote the set

$$L(\alpha) = \{x \in X : J(x) \leq \alpha\}. \tag{2.2.4}$$

It should be noted that $L(\alpha)$ is a convex subset of X. Moreover, if $x_1 \in L(\alpha_1)$ and $x_2 \in L(\alpha_2)$, then $J(x_1) \leq \alpha_1$ and $J(x_2) \leq \alpha_2$. Consequently, if $0 < \lambda < 1$, then it follows that

$$J(\lambda x_1 + [1-\lambda]x_2) \leq \lambda\alpha_1 + [1 - \lambda]\alpha_2,$$

and hence $\lambda x_1 + [1-\lambda]x_2$ belongs to $L(\lambda\alpha_1 + [1 - \lambda]\alpha_2)$. Therefore, if J is a convex function, for α_1, α_2 in R and $0 < \lambda < 1$, we have that

$$\lambda L(\alpha_1) + [1-\lambda]L(\alpha_2) \subseteq L(\lambda\alpha_1 + [1-\lambda]\alpha_2).$$

DEFINITION 2.2.2. *Let X be a normed linear space. A collection*

$$C = \{A(\alpha) : 0 < \alpha < +\infty\}$$

of convex subsets of X is said to be a convex chain in X if each set $A(\alpha)$ *is non-empty, and C satisfies the conditions*

(2.2.5) $$\text{if } \alpha_1 \le \alpha_2, \text{ then } A(\alpha_1) \subseteq A(\alpha_2),$$

(2.2.6) $$\lambda A(\alpha_1) + [1-\lambda]A(\alpha_2) \subseteq A(\lambda\alpha_1 + [1-\lambda]\alpha_2),$$

for all $\alpha_1 > 0$, $\alpha_2 > 0$ *and* λ *such that* $0 < \lambda < 1$. *A convex chain C is said to be positively homogeneous if for each* $\lambda > 0$ *and* $\alpha > 0$, *C satisfies*

(2.2.7) $$A(\lambda\alpha) = \lambda A(\alpha).$$

For simplicity in notation, the set $\bigcup_{\alpha > 0} A(\alpha)$ shall be represented by $A(\infty)$ and $A(0)$ shall be used to denote the set $\bigcap_{\alpha > 0} A(\alpha)$.

Given a convex chain C, it follows from condition (2.2.5) that $A(\infty)$ is a convex set in X. If C is a positively homogeneous chain and $\lambda > 0$, then it follows that $\lambda A(\infty) \subseteq A(\infty)$, and consequently, $\lambda A(\infty) = A(\infty)$. Moreover, if C is a positively homogeneous chain, then each set $A(\alpha)$ is given by

(2.2.8) $$A(\alpha) = \alpha A(1).$$

DEFINITION 2.2.3. *Let X be a normed linear space and* $C = \{A(\alpha) : 0 < \alpha < +\infty\}$ *a convex chain in X. The convex chain C is said to be nested at* θ *if* $\theta \in A(0)$.

The following example is given to indicate the motivation behind the definition of a convex chain.

EXAMPLE 2.2.2. Let X be a normed linear space, and Γ a convex subset of X. Suppose $J:X \to (-\infty,+\infty]$ is a convex function and there exists an element $\gamma_o \in \Gamma$ such that $J(\gamma_o) = 0$. For each $\alpha > 0$, let $A(\alpha)$ be the convex set

$$A(\alpha) = \Gamma \cap L(\alpha). \tag{2.2.9}$$

It is easy to verify that $C = \{A(\alpha) : 0 < \alpha < +\infty\}$ is a convex chain in X. Moreover, if J is positively homogeneous and $\lambda\Gamma \subseteq \Gamma$ for all $\lambda > 0$, then it follows that C is a positively homogeneous chain. If $\gamma_o = \theta$, then C is nested at θ.

Suppose C is a convex chain nested at θ and $0 < \lambda < 1$. If $x \in A(\alpha)$, then it follows from condition (2.2.6) that

$$\lambda x = \lambda x + [1-\lambda]\theta \in A(\lambda\alpha + [1-\lambda]\beta)$$

for each $\beta > 0$, and hence $\lambda A(\alpha) \subseteq A(\varepsilon)$ for all $\varepsilon > \lambda\alpha$. Consequently, we have the following useful result.

LEMMA 2.2.4. *Let X be a normed linear space and $C = \{A(\alpha) : 0 < \alpha < +\infty\}$ a convex chain which is nested at θ. If $0 < \lambda \leq 1$ and $\lambda\alpha < \varepsilon$, then $\lambda A(\alpha) \subseteq A(\varepsilon)$. In particular, if $0 < \lambda \leq 1$, then $\lambda A(\alpha) \subseteq A(\alpha)$.*

The following theorem is fundamental to the results in Chapter III.

THEOREM 2.2.1. *Let X be a normed linear space and $C = \{A(\alpha) : \alpha > 0\}$ a convex chain in X. Suppose Y is a normed linear space and $T:D(T) \to Y$ is a linear operator with $D(T) \subseteq X$, and $[A(\alpha) \cap D(T)]$ is non-empty for each $\alpha > 0$. If D denotes the collection of non-empty convex sets*

$$D = \{B(\alpha) = T(A(\alpha) \cap D(T)) : \alpha > 0\},$$

then:

(i) D *is a convex chain in* Y;

(ii) *if* C *is positively homogeneous*, *then* D *is positively homogeneous*;

(iii) *if* C *is nested at* θ, *then* D *is nested at* θ.

Theorem 2.2.1 is an easy consequence of the linearity of the operator T, and the proof will be omitted.

EXAMPLE 2.2.3. Let X = R, and define $C = \{A(\alpha) : \alpha > 0\}$ by

$$A(\alpha) = \{x \in R : |x| \leq 1 + \alpha\}.$$

Since $A(\alpha) = \{x \in R : |x| - 1 \leq \alpha\}$, we have that $A(\alpha)$ is the level set of the convex function $J:R \to R$ defined by $J(x) = |x| - 1$. It is easy to verify that C is a convex chain. However, A(1) is the closed interval [-2,2], A(1/2) is the closed interval [-3/2,3/2], and (1/2)A(1) is the interval [-1,1]. Hence, C is not a positively homogeneous chain.

EXAMPLE 2.2.4. Let X be a normed linear space and $K \subseteq X$ a convex, absorbing set containing θ. Define $C = \{A(\alpha) : \alpha > 0\}$ by $A(\alpha) = \alpha K$. In this example C is a positively homogeneous chain, $A(\infty) = X$, and since $\theta \in A(0)$, we have that C is nested at θ.

2.3. *The extended Minkowski functional*. The objective of this section is to define a function which generalizes the classical Minkowski functional and to study a few of its properties.

DEFINITION 2.3.1. *Let* X *be a normed linear space and* $C = \{A(\alpha) : \alpha > 0\}$ *a convex chain in* X. *The extended Minkowski functional of* C *is the function*

$$\rho(\ ;C) : X \to [0,+\infty]$$

defined by

$$\text{(2.3.1)} \qquad \rho(x;C) = \begin{cases} +\infty, & \text{if } x \notin A(\infty); \\ \inf\{\alpha > 0 : x \in A(\alpha)\}, & \text{if } x \in A(\infty). \end{cases}$$

As is customary in such cases, we shall use ρ instead of $\rho(\ ;C)$, once the convex chain C has been given. The function ρ is a generalization of the classical Minkowski functional for an absorbing convex set K containing θ, in which case the convex chain is defined as in Example 2.2.4.

THEOREM 2.3.1. _Let_ X _be a normed linear space and_ $C = \{A(\alpha) : \alpha > 0\}$ _a convex chain in_ X. _If_ $\rho : X \to [0,+\infty]$ _denotes the corresponding extended Minkowski functional of_ C, _then_:

(i) ρ _is a convex function_;

(ii) _if_ C _is positively homogeneous, then_ ρ _is a subadditive function which is positively homogeneous_;

(iii) _if_ C _is nested at_ θ, _then_ $\rho(\theta) = 0$.

Suppose C is a convex chain, $x_1, x_2 \in X$, and $0 < \lambda < 1$. If either x_1 or x_2 does not belong to $A(\infty)$, then we have that

$$\rho(\lambda x_1 + [1-\lambda]x_2) \leq +\infty = \lambda\rho(x_1) + [1-\lambda]\rho(x_2).$$

If both x_1 and x_2 belong to $A(\infty)$, then it follows from the convexity of $A(\infty)$ that $\lambda x_1 + [1-\lambda]x_2$ also belongs to $A(\infty)$. If $x_1 \in A(\alpha_1)$ and $x_2 \in A(\alpha_2)$, then it follows that

$$\lambda x_1 + [1-\lambda]x_2 \in \lambda A(\alpha_1) + [1-\lambda]A(\alpha_2) \subseteq A(\lambda\alpha_1 + [1-\lambda]\alpha_2).$$

Therefore, we have that

$$\rho(\lambda x_1 + [1-\lambda]x_2) \leq \lambda\rho(x_1) + [1-\lambda]\rho(x_2),$$

which completes the proof of (i).

If C is a positively homogeneous chain, then recall that for $\lambda > 0$ we have that $\lambda A(\infty) = A(\infty)$. In particular, if $x \in X$ and $\lambda > 0$, then $x \in A(\infty)$ if and only if $\lambda x \in A(\infty)$. Consequently, if $x \notin A(\infty)$, then $\rho(\lambda x) = +\infty = \lambda(+\infty) = \lambda\rho(x)$. On the other hand, if $x \in A(\infty)$ then both $\rho(x)$ and $\rho(\lambda x)$ are finite for each $\lambda > 0$. Suppose $\rho(x) = \gamma$. If $\alpha > \gamma$, then $x \in A(\alpha)$, and hence $\lambda x \in \lambda A(\alpha) = A(\lambda\alpha)$. Therefore, it follows that $\rho(\lambda x) \leq \lambda\alpha$ and since $\alpha > \gamma$ was arbitrarily chosen, we have that $\rho(\lambda x) \leq \lambda\gamma = \lambda\rho(x)$. If we replace x by λx and λ by $1/\lambda$, then we obtain $\rho(x) \leq (1/\lambda)\rho(\lambda x)$, which is equivalent to $\lambda\rho(x) \leq \rho(\lambda x)$. Thus we have established that the function ρ is positively homogeneous, and from Lemma 2.2.1 it follows that ρ is subadditive.

Conclusion (iii) is obvious from the definition (2.3.1) of the function ρ.

It should be noted that even if $C = \{A(\alpha) : \alpha > 0\}$ is positively homogeneous, then ρ does not have to be the classical Minkowski functional of the set $A(1)$. In fact, the classical Minkowski functional is not defined for convex sets not containing θ.

EXAMPLE 2.3.1. Let $X = R^2$, and define K to be the convex set

$$K = \{(x,y) \in R^2 : x > 0,\ y > 0,\ x^2 + y^2 \leq 1\}.$$

Moreover, let $C = \{A(\alpha) : \alpha > 0\}$ be defined by $A(\alpha) = \alpha K$, and note that C is a positively homogeneous chain. The corresponding functional ρ is given by

$$\rho(x,y) = \begin{cases} +\infty, & \text{if } x \leq 0 \text{ or } y \leq 0; \\ \sqrt{x^2 + y^2}, & \text{if } x > 0 \text{ and } y > 0. \end{cases}$$

Also, it is to be noted that $\rho(0,0) = +\infty$, and clearly ρ is not continuous.

LEMMA 2.3.1. *Let* X *be a normed linear space,* $C = \{A(\alpha) : \alpha > 0\}$ *a convex chain, and* ρ *the corresponding extended Minkowski functional. If* C *is nested at* θ, *then* $\rho(\lambda x) \leq \lambda\rho(x)$ *for all* $x \in X$ *and* λ *such that* $0 < \lambda \leq 1$.

If $x \notin A(\infty)$, then clearly we have that $\rho(\lambda x) \leq +\infty = \lambda\rho(x)$. Suppose $x \in A(\infty)$ and let $\rho(x) = \gamma < +\infty$. To prove that $\rho(\lambda x) \leq \lambda\gamma$, it suffices to show that $\lambda x \in A(\lambda\gamma + \varepsilon)$ for all $\varepsilon > 0$.

Since $\rho(x) = \gamma$, for $\varepsilon > 0$ and $0 < \lambda \leq 1$ we have that $x \in A(\gamma + (\varepsilon/2\lambda))$ and hence $\lambda x \in \lambda A(\gamma + (\varepsilon/2\lambda))$. As $\lambda[\gamma + (\varepsilon/2\lambda)] < \lambda\gamma + \varepsilon$, it then follows from Lemma 2.2.4 that

$$\lambda x \in \lambda A(\gamma + (\varepsilon/2\lambda)) \subseteq A(\lambda\gamma + \varepsilon).$$

Example 2.2.2 illustrates how the level sets $L(\alpha)$ of certain convex functions may generate a convex chain. Conversely, given a convex chain C, we have the corresponding functional $\rho(\ ;C)$, which is convex. Of primary importance is the relationship between the sets $A(\alpha)$ in C and the level sets $\{x \in X : \rho(x;C) \leq \alpha\}$. In order to investigate this relationship, we shall introduce the following definition which may be found in Valentine [37; p. 11].

DEFINITION 2.3.2. *Let* X *be a normed linear space and* $K \subseteq X$ *a convex set. A point* $\xi \in K$ *is said to be a core point of* K *if for each* $x \in X$ *with* $x \neq \xi$, *there is a* γ *satisfying* $0 < \gamma \leq 1$ *such that if* $0 \leq \lambda < \gamma$, *then* $\lambda x + [1-\lambda]\xi$ *belongs to* K. *The set of all core points of* K *is called the core of* K, *and will be denoted by* core K.

As a special case of Theorem 1.14 in Valentine [37; p. 12], we have the following useful result.

LEMMA 2.3.2. *Let* X *be a normed linear space and* $K \subseteq X$ *a convex set.*

<u>If</u> $\xi \in$ <u>core</u> K <u>and</u> $x \in K$, <u>then for all</u> λ <u>satisfying</u> $0 \leq \lambda < 1$ <u>the element</u> $\lambda x + [1-\lambda]\xi$ <u>also belongs to</u> core K.

Valentine [37; p. 16] notes that for a convex set K the interior of K is always a subset of the core of K. Moreover, if K is convex and int K is non-empty, then we have that int K = core K. However, there are many cases when a convex set will have an empty interior but non-empty core.

EXAMPLE 2.3.2. Let $Y = C[0,1]$ be the space of real-valued continuous functions on $[0,1]$ with uniform norm, and let X be the subspace of real-valued continuously differentiable functions on $[0,1]$. Let $D(T) = X$, and define $T:X \to Y$ by $[Tx](t) = x'(t)$. Let $B(Y,1)$ denote the unit ball in Y and suppose K is the convex set given by

$$K = \{x \in X : Tx \in B(Y,1)\}.$$

If $x \in X$ and $Tx \neq \theta$, then let $\gamma = \min 1,\{1/\|Tx\|\}$, while if $Tx = \theta$, then let $\gamma = 1$. In any case, if $0 \leq \lambda < \gamma$ it follows that $\lambda x + [1-\lambda]\theta$ belongs to K, and hence θ is a core point of K. To show that θ is not an interior point of K, it suffices to show that for each $\varepsilon > 0$, there exists a function $g \in X$, such that $\|g\| < \varepsilon$ and g does not belong to K. For $\varepsilon > 0$, let $g(t) = (\varepsilon/2)\sin[4t/\varepsilon]$. Clearly $\|g\| = \varepsilon/2 < \varepsilon$, with $g'(t) = 2\cos[(4/\varepsilon)t]$, and for $t = 0$ we have $g'(0) = 2 > 1$, so that $g \notin K$.

In general, if X and Y are normed linear space, while $T:X \to Y$ is a linear operator onto Y, and $V \subsetneq Y$ is convex with $\theta \in$ core V, then θ will be a core point of $\{x \in X : Tx \in V\}$.

It should be noted that the set K in Example 2.3.2 does not have θ as a core point, if K is considered as a subset of $C[0,1]$. For this reason we give the following definition.

DEFINITION 2.3.3. <u>Let</u> X <u>be a normed linear space and</u> $D \subsetneq X$ <u>a subspace</u>

of X. *If* $K \subseteq X$ *is a convex set, then the set* $K \cap D$ *is a convex set in* D. *A point* ξ *in* $K \cap D$ *is said to be a core point of* K *relative to* D *if for each* $x \in D$ *with* $x \neq \xi$ *there is a* γ *satisfying* $0 < \gamma \leq 1$ *such that* $\lambda x + [1-\lambda]\xi$ *belongs to* K *for each* $0 \leq \lambda < \gamma$.

If X and Y are normed linear spaces, while $T:D(T) \to Y$ is a linear operator with $D(T) \subseteq X$, and $K \subseteq X$, then the set of all core points of K relative to D(T) will be denoted by $\text{core}_T K$.

LEMMA 2.3.3. *Suppose* X *is a normed linear space and* $K \subseteq X$ *is convex. If* $\xi \in \text{core } K$, *then there exists a* $\gamma > 1$ *such that if* $1 \leq \lambda < \gamma$ *then* $\lambda\xi \in K$.

The conclusion follows readily if $\xi = \theta$; therefore, suppose $\xi \neq \theta$. Let $x = 2\xi$, and note that since $\xi \in \text{core } K$ there exists a γ_1 with $0 < \gamma_1 \leq 1$, and such that $\lambda x + [1-\lambda]\xi \in K$ for all λ satisfying $0 \leq \lambda < \gamma_1$. Let $\gamma = 1 + \gamma_1$, and suppose that $1 \leq \lambda < \gamma$. It follows that $0 \leq \lambda - 1 < \gamma - 1 = \gamma_1 \leq 1$, and hence $\lambda\xi = (\lambda-1)2\xi + (1 - [\lambda-1])\xi = (\lambda-1)x + (1 - [\lambda-1])\xi \in K$.

If X and Y are normed linear spaces, while $T:D(T) \to Y$ is a linear operator with $D(T) \subseteq X$, and $K \subseteq X$ is convex, then $\text{core}_T K$ is exactly the set of all core points of $[K \cap D(T)]$ when $[K \cap D(T)]$ is considered as a subset of the normed linear space D(T). Consequently, by an application of Lemma 2.3.2 and Lemma 2.3.3 we obtain the following result.

LEMMA 2.3.4. *Let* X *and* Y *be normed linear spaces,* $T:D(T) \to Y$ *a linear operator with* $D(T) \subseteq X$, *and* $K \subseteq X$ *a convex set. If* $\xi \in \text{core}_T K$, *then:*

(i) *if* $x \in K \cap D(T)$, *then* $\lambda x + [1-\lambda]\xi$ *belongs to* $\text{core}_T K$ *for all* λ *satisfying* $0 \leq \lambda < 1$;

(ii) *there exists a* $\gamma > 1$ *such that if* $1 \leq \lambda < \gamma$, *then* $\lambda\xi \in K \cap D(T)$.

THEOREM 2.3.2. *Let* X *be a normed linear space, and* $C = \{A(\alpha) : 0 < \alpha < +\infty\}$ *a convex chain nested at* θ. *If* $\rho : X \to [0,+\infty]$ *is the corresponding extended Minkowski functional for* C, *then:*

(i) $\text{core } A(\alpha) \subseteq \{x \in X : \rho(x) < \alpha\} \subseteq A(\alpha)$;

(ii) $A(\alpha) \subseteq \{x \in X : \rho(x) \leq \alpha\} \subseteq \text{cl } [A(\alpha)]$.

In particular, if $A(\alpha)$ *is closed, then* $A(\alpha) = \{x \in X : \rho(x) \leq \alpha\}$.

The relations $\{x \in X : \rho(x) < \alpha\} \subseteq A(\alpha)$ and $A(\alpha) \subseteq \{x \in X : \rho(x) \leq \alpha\}$ are obvious from the definition of ρ. If $x \in \text{core } A(\alpha)$, then by Lemma 2.3.3 there is a $\lambda > 1$ such that $\lambda x \in A(\alpha)$. Consequently, we have that $(1/\lambda) < 1$ and $\rho(\lambda x) \leq \alpha$. Thus it follows from Lemma 2.3.1 that

$$\rho(x) = \rho((\lambda/\lambda)x) \leq (1/\lambda)\rho(\lambda x) \leq (1/\lambda)\alpha < \alpha,$$

and conclusion (i) is proved.

If $x \in \{x \in X : \rho(x) \leq \alpha\}$, then $\rho(x) < \alpha$ or $\rho(x) = \alpha$. If $\rho(x) < \alpha$, then clearly $x \in A(\alpha)$. Suppose $\rho(x) = \alpha$, and let $\{\lambda_n\}$ be a sequence such that $0 < \lambda_n < \lambda_{n+1}$, $n = 1,2,\cdots$, and $\lambda_n \to 1$. By Lemma 2.3.1 we have that

$$\rho(\lambda_n x) \leq \lambda_n \rho(x) = \lambda_n \alpha < \alpha,$$

and therefore $\lambda_n x$ belongs to $A(\alpha)$ for each $n = 1,2,3,\cdots$. Since $\lambda_n x \to x$, we have that x must belong to $\text{cl } [A(\alpha)]$, and conclusion (ii) is proved.

It should be noted that if $\text{int } A(\alpha)$ is non-empty, then conclusion (i) implies that $\text{int } \{x \in X : \rho(x) \leq \alpha\}$ is also non-empty. Also, if $\rho(x) = \alpha$ then x cannot belong to $\text{core } A(\alpha)$.

THEOREM 2.3.3. *Let* X *be a normed linear space,* $C = \{A(\alpha) : \alpha > 0\}$ *a convex chain in* X, *and* ρ *the corresponding extended Minkowski functional. If* $A(\alpha)$ *is closed for each* $\alpha > 0$, *then* ρ *is a lower semi-continuous function.*

It suffices to show that the set $\{x \in X : \rho(x) \leq \alpha\}$ is closed for each $\alpha \in [0,+\infty]$. If $\alpha = +\infty$, then $\{x \in X : \rho(x) \leq \alpha\} = X$ and is closed. Suppose $0 \leq \alpha < +\infty$, and let $\{x_n\}$ be a sequence in $\{x \in X : \rho(x) \leq \alpha\}$ such that $x_n \to x_o$. If $\rho(x_o) = \gamma > \alpha$, then there exists a λ_o such that $0 \leq \alpha < \lambda_o < \gamma$. Since $\rho(x_o) > \lambda_o$, it follows that $x_o \notin A(\lambda_o)$ and because $\rho(x_n) \leq \alpha < \lambda_o$ we have that $x_n \in A(\lambda_o)$ for all $n \geq 1$. Since the set $A(\lambda_o)$ is closed and $x_n \to x$, it follows that x must belong to $A(\lambda_o)$. This is a contradiction, and hence we must have that $\rho(x_o) \leq \alpha$.

2.4. Support properties of convex sets. Conditions which determine when two convex sets can be separated by a hyperplane are very important when one is considering existence problems in modern control theory. As a special case we shall be interested in conditions which determine when a point is a support point of a given convex set.

DEFINITION 2.4.1. *Let X be a normed linear space and $K \subseteq X$ a convex set. A point $\xi \in \text{cl } K$ is said to be a support point of K if there exists a non-zero linear functional ϕ, not necessarily continuous, such that ϕ satisfies*

$$<\xi,\phi> = \sup \{<x,\phi> : x \in K\}, \tag{2.4.1}$$

The set of all support points of K shall be denoted by supp K. A non-zero linear functional ϕ satisfying (2.4.1) is called a support functional of K at the support point $\xi \in \text{cl } K$. If ξ is a support point of K and there exists a non-zero continuous linear functional x^* satisfying condition (2.4.1), then ξ will be called a strong support point. The set of all strong support points will be denoted by $\text{supp}^* K$.

If K is a convex set with $\theta \in \text{int } K$, then it follows from Theorem 2.8 in [36] that $\text{supp}^* K = \partial K$. In general we have that $\text{supp}^* K \subseteq \text{supp } K \subseteq \partial K$

for each convex set K. Even if $\xi \in$ supp K and ϕ is a support functional at the point ξ, it is possible that $\langle\xi,\phi\rangle = 0$.

LEMMA 2.4.1. <u>Let</u> X <u>be a normed linear space and</u> $K \subseteq X$ <u>a convex set with</u> $\theta \in$ core K. <u>If</u> ξ <u>is a support point and</u> ϕ <u>a corresponding support functional, then</u> $\langle\xi,\phi\rangle > 0$.

Since $\theta \in K$, we have that $\langle\xi,\phi\rangle$ must be at least as large as $0 = \langle\theta,\phi\rangle$. Since ϕ is a non-zero functional, there is an element $x \in X$ with $\langle x,\phi\rangle > 0$. Because θ is a core point of K, there is a $\lambda > 0$ such that $\lambda x \in K$, and hence we have that

$$\langle\xi,\phi\rangle \geq \langle\lambda x,\phi\rangle = \lambda\langle x,\phi\rangle > 0.$$

The following result is a direct consequence of Theorem 2.7 in Valentine [37; p. 24].

LEMMA 2.4.2. <u>Suppose</u> X <u>is a normed linear space and</u> $K \subseteq X$ <u>is convex with</u> $\theta \in$ core K. <u>If</u> $\xi \in \partial K$ <u>and</u> ξ <u>is not a core point of</u> K, <u>then</u> ξ <u>is a support point</u>.

If $\theta \in$ int K, then each point in ∂K is a strong support point. On the other hand, if int K is empty, then supp^*K may be empty. The following example shows that even if int K is empty, then there may be points which belong to supp^*K.

EXAMPLE 2.4.1. Let X be the subspace of C[0,1] consisting of functions with continuous derivatives. Let C[0,1] = Y and define $T:X \to Y$ by $[Tx](s) = x'(s)$. Let K be defined by

$$K = \{x \in X : \|x\| + \|Tx\| \leq 1\},$$

and note that K does not contain an interior point. If $\xi(s)$ is defined by $\xi(s) = 1$ for each $s \in [0,1]$, then ξ is not a core point of K. Let x^*

be defined by

$$\langle x,x^*\rangle = \int_0^1 x(s)ds,$$

and note that $x^* \in X^*$. Moreover, if $x \in K$ then

$$|\langle x,x^*\rangle| = \left|\int_0^1 x(s)ds\right| \leq \int_0^1 |x(s)|ds \leq 1.$$

Since we have that $\langle \xi,x^*\rangle = 1$, it follows that ξ belongs to supp^*K.

In [17], Klee has given an example of a closed and bounded convex set K for which supp^*K is empty.

Suppose X is a normed linear space and $C = \{A(\alpha) : 0 < \alpha < +\infty\}$ is a convex chain nested at θ. If ρ is the corresponding extended Minkowski functional and $\rho(\xi) = \alpha_o > 0$, then it follows from Theorem 2.3.2 that ξ is not a core point of $A(\alpha_o)$ and $\xi \in \text{cl } A(\alpha_o)$. Since ξ is not a core point of $A(\alpha_o)$, we have that ξ is not an interior point of $A(\alpha_o)$, and hence ξ belongs to $\partial A(\alpha_o)$. Therefore, if θ belongs to core $A(\alpha_o)$ it follows from Lemma 2.4.2 that ξ is a support point of $A(\alpha_o)$. Moreover, if ϕ is a support functional at ξ, then it follows from Lemma 2.4.1 that $\langle \xi,\phi\rangle > 0$.

For clarity of future reference, the above results are stated as the following theorem.

THEOREM 2.4.1. <u>Let</u> X <u>be a normed linear space</u>, $C = \{A(\alpha) : 0 < \alpha < +\infty\}$ <u>a convex chain nested at</u> θ, <u>and</u> ρ <u>the corresponding extended Minkowski functional</u>. <u>If</u> θ <u>belongs to</u> core $A(\alpha_o)$ <u>and</u> $\rho(\xi) = \alpha_o > 0$, <u>then</u> ξ <u>is a support point of</u> $A(\alpha_o)$. <u>Moreover, if</u> ϕ <u>is any support functional at</u> ξ, <u>then</u> $\langle \xi,\phi\rangle > 0$.

The following result is fundamental to the following discussion.

THEOREM 2.4.2. <u>Let</u> X <u>and</u> Y <u>be normed linear spaces</u>, $T:D(T) \to Y$ <u>a</u>

linear operator with $D(T) \subsetneq X$, *and* $C = \{A(\alpha) : 0 < \alpha < +\infty\}$ *a convex chain nested at* θ. *If* $R(T) = Y$ *and* θ *belongs to* $\text{core}_T A(\alpha)$ *for each* $\alpha > 0$, *then the convex chain*

$$D = \{B(\alpha) = T(A(\alpha) \cap D(T)) : 0 < \alpha < +\infty\}$$

is such that $\theta \in \text{core } B(\alpha)$ *for each* $\alpha > 0$.

If $y \in Y = R(T)$, then there exists an element $x \in D(T)$ such that $Tx = y$. Since θ belongs to $\text{core}_T A(\alpha)$, there is a γ satisfying $0 < \gamma \leq 1$ such that if $0 \leq \lambda < \gamma$, then $\lambda x = \lambda x + [1 - \lambda]\theta \in A(\alpha) \cap D(T)$. Consequently, the conclusion of the theorem follows from the fact that $\lambda y = \lambda Tx = T(\lambda x) \in B(\alpha)$.

CHAPTER III

A MINIMUM EFFORT PROBLEM

3.1. Introduction. The principal objective of this chapter is to obtain existence theorems for a generalized minimum effort control problem. A secondary problem is also briefly considered. These problems are formulated as generalizations of the Problem (P) in [21] by Minamide and Nakamura. Therefore, some terminology shall be introduced that agrees with the terminology employed by Minamide and Nakamura [21].

After existence has been established, a few examples are considered, and remarks are made concerning necessary conditions.

3.2. Formulation of the problems. Throughout this chapter X, Y, and Z will denote normed linear spaces, $T:D(T) \to Y$ and $S:D(S) \to Z$ will be linear operators with $D(T) \subseteq X$ and $D(S) \subseteq X$, and $\Omega \subseteq X$ will be a convex set containing θ. We shall let $U = X \times Y$ and $W = Y \times Z$ be the product spaces with norm defined by (1.2.1). Moreover, J will denote a real-valued convex function defined on U such that

(3.2.1) $$J(u) \geq 0 \quad \text{for all } u \in U,$$

(3.2.2) $$J(\theta) = 0,$$

(3.2.3) $$J(u) \to +\infty, \text{ as } \|u\| \to +\infty .$$

It is to be noted that the set of all convex functions satisfying conditions (3.2.1), (3.2.2), and (3.2.3) forms a positive cone. That is, if J_1 and J_2 satisfy all three conditions, then $J = \alpha J_1 + \beta J_2$ also satisfies all three conditions for each $\alpha > 0$ and $\beta > 0$.

As in Chapter I, we shall let

$$D(Q) = [D(T) \cap D(S)] \times Y, \tag{1.2.7}$$

and define $Q:D(Q) \to W$ by

$$Q(x,y) = (Tx + y, Sx). \tag{1.2.8}$$

Also, to simplify notation we shall define $\Gamma \subseteq X$ by

$$\Gamma = \Omega \cap [D(T) \cap D(S)]. \tag{3.2.4}$$

For clarity of future reference, we shall state the following hypotheses.

(H_o) $\theta \in \text{core}_T\Omega$, $D(S) \subseteq D(T)$, and the range of S is equal to all of Z.

(H_1) $D(Q)$ is dense in U, and $\lambda\Omega \subseteq \Omega$ for each $\lambda \geq 0$.

If $\alpha \in R$, then let

$$L(\alpha) = \{u \in U : J(u) \leq \alpha\}; \tag{3.2.5}$$

also, for $\alpha > 0$ the symbol $A(\alpha)$ will denote the set defined by

$$A(\alpha) = [\Omega \times Y] \cap L(\alpha). \tag{3.2.6}$$

If C denotes the collection

$$C = \{A(\alpha) : 0 < \alpha < +\infty\}, \tag{3.2.7}$$

then it follows from Example 2.2.2 that C is a convex chain in U. Moreover, since $\theta \in [\Omega \times Y] \cap L(\alpha) = A(\alpha)$, we have that C is nested at θ. We shall let D be the convex chain in W defined by

$$D = \{B(\alpha) = Q(A(\alpha) \cap D(Q)) : 0 < \alpha < +\infty\}. \tag{3.2.8}$$

It should be noted that condition (3.2.3) implies that each level set

$L(\alpha)$ is bounded, and hence the set $A(\alpha)$ must also be bounded.

The following result will be very useful.

LEMMA 3.2.1. <u>If hypothesis</u> (H_o) <u>holds and</u> $\alpha > 0$, <u>then</u> $\theta \in \text{core}_Q A(\alpha)$, $R(Q) = W$, <u>and</u> $\theta \in \text{core } B(\alpha)$.

Suppose $u = (x,y)$ is an element in $D(Q)$. To establish the first part of the conclusion, it suffices to show the existence of a γ satisfying $0 < \gamma \leq 1$, and such that if $0 \leq \lambda < \gamma$ then $\lambda u \in A(\alpha)$. Since $\theta \in \text{core}_T \Omega$ and $x \in D(T)$, there is a γ_1 with $0 < \gamma_1 \leq 1$ such that $\lambda x \in \Omega$ for all λ satisfying $0 \leq \lambda < \gamma_1$. Now we have two cases to consider; either $J(u) = 0$ or else $J(u) = M > 0$. If $J(u) = 0$, then it follows that $0 \leq J(\lambda u) \leq \lambda J(u) = 0$ for all λ satisfying $0 \leq \lambda < \gamma_1$. Therefore, we have that $\lambda u \in L(\alpha)$ and $\lambda u = (\lambda x, \lambda y) \in (\Omega \times Y)$ for all $0 \leq \lambda < \gamma_1$, and hence $\lambda u \in A(\alpha)$. On the other hand, if $J(u) = M > 0$ let $\gamma = \min\{\gamma_1, \alpha/M\}$. If $0 \leq \lambda < \gamma$, then we have that $\lambda u \in [\Omega \times Y]$ and

$$J(\lambda u) \leq \lambda J(u) = \lambda M < \alpha.$$

Thus it follows that $\lambda u \in A(\alpha)$, and the proof of the first part of the conclusion is complete.

To complete the proof, it suffices to show that Q has range equal to all of W, because, if this is the case, then θ will be a core point of $B(\alpha)$ by Theorem 2.4.2. If $w = (y,z) \in W$, then there exists an element $x \in D(S)$ such that $Sx = z$. Since $D(S) \subseteq D(T)$, it follows that $x \in D(T)$ and that $Tx \in Y$. Now $u = (x, -Tx + y)$ is such that $u \in D(Q)$. Moreover, by direct calculation we have that

$$Qu = Q(x, -Tx + y) = (Tx - Tx + y, Sx) = (y,z) = w.$$

This proves that $R(Q) = W$, which completes the proof of the lemma.

The following problem is a direct generalization of Problem (P) in [21] by Minamide and Nakamura.

PROBLEM (I). Suppose hypothesis (H_0) is satisfied and $w_0 = (\xi,\eta)$ is a given element in W, with $\eta \in S([\text{core}_T\Omega] \cap D(S))$. Find an element $x_0 \in \Omega \cap D(S)$ satisfying $Sx_0 = \eta$ and such that

$$(3.2.9) \qquad J(x_0,\xi - Tx_0) \le J(x,\xi - Tx)$$

for all $x \in \Omega \cap D(S)$ satisfying $Sx = \eta$.

We shall also consider the following problem.

PROBLEM (II). Suppose hypothesis (H_1) is satisfied, and $w_0 = (\xi,\eta)$ is a given element in W with $\eta \in S[\Omega \cap D(S)]$. Find an element $x_0 \in [\Omega \cap D(T) \cap D(S)] = \Gamma$ such that $Sx_0 = \eta$ and condition (3.2.9) is satisfied for all $x \in \Gamma$ such that $Sx = \eta$.

If there exists an element $x_0 \in \Gamma$ satisfying condition (3.2.9) and $Sx_0 = \eta$, then x_0 is called an optimal solution of the respective problem (I) or (II).

It is to be noted that in the statement of Problem (I) there is no requirement that Y or Z be distinct from the trivial space $\{\theta\}$. For clarity of future reference, Problem (I) will be restated in each of these special cases.

PROBLEM (I-a). Suppose hypothesis (H_0) is satisfied and $Y = \{\theta\}$ with T the zero transformation. Define $J_1: X \to R$ by

$$(3.2.10) \qquad J_1(x) = J(x,\theta),$$

and suppose $\eta \in S[(\text{core }\Omega) \cap D(S)]$ is given. Find an element $x_0 \in [\Omega \cap D(S)]$ such that $Sx_0 = \eta$ and $J_1(x_0) \le J_1(x)$ for all x satisfying $Sx = \eta$.

Problem (I-a) is a direct generalization of the minimum effort control problem as considered in [25] by Porter and Williams. In the case that $Z = \{\theta\}$, and S is the zero transformation, we obtain the following approximation problem.

PROBLEM (I-b). _Suppose hypothesis_ (H_o) _is satisfied, and_ $Z = \{\theta\}$ _with_ S _the zero transformation. If_ $\xi \in Y$ _is given, find an element_ $x_o \in [\Omega \cap D(T)]$ _such that_ x_o _minimizes_ $J(x, \xi - Tx)$ _over all_ $x \in [\Omega \cap D(T)]$.

Problem (I-b) is a direct generalization of the problem considered in articles [9] and [10] by Halilov and Aslanov. Specific examples of each problem will be given in Section 3.4. Attention will now be directed toward the existence of optimal solutions.

3.3. _Existence of optimal solutions._ If hypothesis (H_o) holds, then we shall let Γ^o be the set defined by

$$\Gamma^o = [\mathrm{core}_T \Omega] \cap D(S). \tag{3.3.1}$$

The following result will play an important role in the fundamental existence theorem for Problem (I).

LEMMA 3.3.1. _Suppose hypothesis_ (H_o) _is satisfied and_ $\eta \in S[\mathrm{core}_T \Omega \cap D(S)]$. _If_ $J(x, \xi - Tx) \geq \alpha$ _for each_ $x \in \Gamma^o$ _satisfying_ $Sx = \eta$, _then_ $J(x, \xi - Tx) \geq \alpha$ _for each_ $x \in \Gamma$ _satisfying_ $Sx = \eta$.

Suppose that $x \in \Gamma$ and $Sx = \eta$. Since $\eta \in S[(\mathrm{core}_T \Omega) \cap D(S)]$, there is an element $x_o \in (\mathrm{core}_T \Omega) \cap D(S)$ such that $Sx_o = \eta$. From conclusion (i) of Lemma 2.3.4, it follows that $x_\lambda = \lambda x + [1-\lambda]x_o$ belongs to $\mathrm{core}_T \Omega$ for each λ satisfying $0 \leq \lambda < 1$. Also, by direct computation we have that

$$Sx_\lambda = S(\lambda x + [1-\lambda]x_o) = \lambda Sx + [1-\lambda]Sx_o = \eta.$$

Therefore, if $0 \leq \lambda < 1$ we have that

$$\begin{aligned}\alpha &\leq J(x_\lambda, \xi - Tx_\lambda) \\ &= J[(\lambda x + [1-\lambda]x_o, \lambda\xi + [1-\lambda]\xi - \lambda Tx - [1-\lambda]Tx_o)] \\ &= J[\lambda(x, \xi - Tx) + [1-\lambda](x_o, \xi - Tx_o)] \\ &\leq \lambda J(x, \xi - Tx) + [1-\lambda]J(x_o, \xi - Tx_o),\end{aligned}$$

and hence it follows that

$$\alpha \leq \lim_{\lambda \to 1} \{\lambda J(x, \xi - Tx) + [1-\lambda]J(x_o, \xi - Tx_o)\} = J(x, \xi - Tx).$$

The following theorem is the fundamental existence theorem for Problem (I). Subsequent existence theorems will be obtained by placing additional restrictions on the operators T and S, the set Ω, and the function J.

THEOREM 3.3.1. Suppose hypothesis (H_o) is satisfied and $\alpha_o > 0$. If the element $w_o = (\xi, \eta)$ belongs to $[\text{supp } B(\alpha_o)] \cap B(\alpha_o)$, then there exists an optimal solution x_o to Problem (I) with $J(x_o, \xi - Tx_o) = \alpha_o$.

Since $w_o \in B(\alpha_o)$, there is an element $u_o = (x_o, y_o) \in A(\alpha_o) \cap D(Q)$ such that $Qu_o = w_o$. However, since $(\xi, \eta) = w_o = Qu_o = (Tx_o + y_o, Sx_o)$, it follows that $y_o = (\xi - Tx_o)$. Therefore, we have that $J(x_o, \xi - Tx_o) = J(u_o) \leq \alpha_o$, and to complete the proof of the theorem it suffices to show that $J(x, \xi - Tx) \geq \alpha_o$ for each $x \in \Gamma$ satisfying $Sx = \eta$. In light of Lemma 3.3.1 we need only show that $J(x, \xi - Tx) \geq \alpha_o$ for each $x \in \Gamma^o$ satisfying $Sx = \eta$.

If $x \in \Gamma^o$ and $Sx = \eta$, then it follows that $Q(x, \xi - Tx) = (\xi, \eta) = w_o$. Since $w_o \in \text{supp } B(\alpha_o)$, there exists a non-zero linear functional ϕ such that

$$\langle w_o, \phi \rangle = \sup \{\langle w, \phi \rangle : w \in B(\alpha_o)\}.$$

By Lemma 3.2.1 we have that $\theta \in$ core $B(\alpha_o)$, and hence it follows from Lemma 2.4.1 that $\langle w_o,\phi\rangle > 0$. The element x belongs to $\mathrm{core}_T\Omega$; thus by conclusion (ii) of Lemma 2.3.4 there exists a $\gamma > 1$ such that $\lambda x \in \Omega$ for all λ satisfying $1 \leq \lambda < \gamma$. In particular, if $1 \leq \lambda < \gamma$ then we have that $\lambda(x,\xi - Tx)$ belongs to $\Omega \times Y$.

Let $\{\lambda_n\}$ be a sequence which converges to 1 and such that for each $n \geq 1$ we have that $1 < \lambda_{n+1} < \lambda_n < \gamma$. For such a sequence it follows that $\lambda_n(x,\xi - Tx)$ belongs to $\Omega \times Y$, and $Q[\lambda_n(x,\xi - Tx)] = \lambda_n w_o$. On the other hand, we have that $\langle Q[\lambda_n(x,\xi - Tx)],\phi\rangle = \langle\lambda_n w_o,\phi\rangle = \lambda_n\langle w_o,\phi\rangle > \langle w_o,\phi\rangle$, and hence $\lambda_n w_o$ does not belong to $B(\alpha_o)$. This means that for each $n \geq 1$ the element $\lambda_n(x,\xi - Tx)$ does not belong to $A(\alpha_o) \cap D(Q)$. But $\lambda_n(x,\xi - Tx)$ belongs to $\Omega \times Y$, and hence it follows that $\lambda_n(x,\xi - Tx)$ does not belong to $L(\alpha_o)$, or equivalently we have that $J[\lambda_n(x,\xi - Tx)] > \alpha_o$ for all $n \geq 1$. From Lemma 2.2.3 it then follows that $J(x,\xi - Tx) = \lim_{n\to\infty} J[\lambda_n(x,\xi - Tx)] \geq \alpha_o$, and this completes the proof.

The following result is the fundamental existence theorem for Problem (II).

THEOREM 3.3.2. <u>Suppose hypothesis</u> (H_1) <u>is satisfied</u>, $\alpha_o > 0$, <u>and</u> $w_o = (\xi,\eta)$ <u>belongs to</u> [supp $B(\alpha_o)$] $\cap$ $B(\alpha_o)$. <u>If there is a support functional</u> ϕ <u>of</u> $B(\alpha_o)$ <u>at</u> w_o <u>satisfying</u> $0 < \langle w_o,\phi\rangle$, <u>then there exists an optimal solution</u> x_o <u>to Problem</u> (II) <u>with</u> $J(x_o,\xi - Tx_o) = \alpha_o$.

Since $w_o = (\xi,\eta)$ belongs to $B(\alpha_o)$, there is an element $u_o = (x_o,y_o) \in A(\alpha_o) \cap D(Q)$ such that $Qu_o = (Tx_o + y_o,Sx_o) = (\xi,\eta) = w_o$. It then follows that $y_o = \xi - Tx_o$ and $J(x_o,\xi - Tx_o) = J(u_o) \leq \alpha_o$.

To complete the proof it suffices to show that $J(x,\xi - Tx) \geq \alpha_o$ for all $x \in \Gamma$ satisfying $Sx = \eta$. Let $\{\lambda_n\}$ be a sequence converging to 1 such that $1 < \lambda_{n+1} < \lambda_n$ for all $n \geq 1$. Since $Q[\lambda_n(x,\xi - Tx)] = \lambda_n w_o$, we have that

$$<Q[\lambda_n(x,\xi - Tx)],\phi> = \lambda_n <w_o,\phi> > <w_o,\phi>,$$

and hence $\lambda_n(x,\xi - Tx)$ does not belong to $A(\alpha_o)$. On the other hand, since $\lambda_n \Omega = \Omega$ it follows that $\lambda_n(x,\xi - Tx)$ belongs to $\Omega \times Y$ for all $n \geq 1$. Thus we have that $J[\lambda_n(x,\xi - Tx)] > \alpha_o$ for $n \geq 1$, and hence by Lemma 2.2.3 it follows that $J(x,\xi - Tx) \geq \alpha_o$.

It should be noted that although Problem (I) and Problem (II) are different in nature, the proofs of Theorem 3.3.1 and Theorem 3.3.2 are similar. For this reason, we shall concentrate primarily on Problem (I), and simply note that analogous results may be obtained for Problem (II) by using similar methods.

If the collection $D = \{B(\alpha) : 0 < \alpha < +\infty\}$ is the convex chain in W defined by condition (3.2.8), then D is nested at θ. Moreover, if hypothesis (H_o) is satisfied then we have from Lemma 3.2.1 that $\theta \in$ core $B(\alpha)$ for each $\alpha > 0$. The convex chain D generates the corresponding extended Minkowski functional $\rho(\ ;D)$ defined on W. One significance of $\rho(\ ;D)$ may be seen from the following result. As is customary, we shall use ρ instead of $\rho(\ ;D)$ to represent the extended Minkowski functional of the convex chain D.

THEOREM 3.3.3. Suppose hypothesis (H_o) is satisfied, ρ is the extended Minkowski functional of D, and $w_o = (\xi,\eta)$ is given. If $\rho(w_o) = \alpha_o > 0$ and $x \in \Gamma$ satisfies $Sx = \eta$, then $J(x,\xi - Tx) \geq \alpha_o$. In particular, if $\rho(w_o) = \alpha_o > 0$ and $w_o \in B(\alpha_o)$, then there exists an optimal solution x_o to Problem (I) such that $J(x_o,\xi - Tx_o) = \alpha_o$.

By Lemma 3.2.1 we have that $\theta \in$ core $B(\alpha_o)$, and hence an application of Theorem 2.4.1 yields that $w_o \in \text{supp } B(\alpha_o)$. As in the proof of Theorem 3.3.1, it follows that $J(x,\xi - Tx) \geq \alpha_o$ for all $x \in \Gamma$ satisfying $Sx = \eta$. Moreover, if $w_o \in B(\alpha_o)$, then the existence of an optimal solution x_o follows from Theorem 3.3.1.

THEOREM 3.3.4. _Suppose hypothesis_ (H_o) _is satisfied_, Ω _is closed_, J _is lower semi-continuous, and_ Q _is a weakly co-compact operator. If_ ρ _is the extended Minkowski functional for_ D, $w_o = (\xi,\eta)$ _and_ $\rho(w_o) = \alpha_o > 0$, _then there exists an optimal solution_ x_o _to Problem (I) such that_ $J(x_o,\xi - Tx_o) = \alpha_o$.

If Ω is closed, then $\Omega \times Y$ is also closed in U. Since J is lower semi-continuous, we have that the level sets $L(\alpha)$ are closed for each $\alpha > 0$. Thus the sets $A(\alpha) = [\Omega \times Y] \cap L(\alpha)$ are closed and bounded convex subsets of U. From Theorem 1.2.5 it follows that the set $B(\alpha_o) = Q[A(\alpha_o) \cap D(Q)]$ is a closed convex set in W. Moreover, since $\rho(w_o) = \alpha_o$ it follows from (ii) of Theorem 2.3.2 that $w_o \in \text{cl } B(\alpha_o) = B(\alpha_o)$. Thus an application of Theorem 3.3.3 yields the conclusion of the theorem.

As a consequence of Theorem 3.3.4 we have the following result, which is the most direct generalization of the existence theorem found in [21]. However, it is still a slight improvement over the theorem in [21] in several ways. First of all, the operators T and S need not be bounded, as required in [21]. Also, the spaces X, Y, and Z need not be complete, while in [21] the space X was assumed to be reflexive and the spaces Y and Z complete.

THEOREM 3.3.5. _Suppose hypothesis_ (H_o) _is satisfied_, ρ _is the extended Minkowski functional of_ D, _and_ $w_o = (\xi,\eta)$. _Moreover, assume that_ Ω _is a closed convex set with_ $\theta \in \text{int } \Omega$, J _is continuous_, W _is of the second_ _category as a subset of itself, and_ Q _is weakly co-compact. If_ $\rho(w_o) = \alpha_o > 0$, _then there exists an optimal solution_ x_o _to Problem_ (I) _with_ $J(x_o, \xi - Tx_o) = \alpha_o$, _and_ $w_o \in \mathrm{supp}^*B(\alpha_o)$.

The first part of the conclusion is simply a special case of Theorem 3.3.4. Since J is continuous it follows that $\theta \in \mathrm{int}\, A(\alpha_o)$, and hence by Theorem 1.2.6 we have that $\theta \in \mathrm{int}\, B(\alpha_o)$. Since $\rho(w_o) = \alpha_o > 0$, it follows that $w_o \in \partial B(\alpha_o) = \mathrm{supp}^*B(\alpha_o)$.

Even though Theorem 3.3.5 is a special case of Theorem 3.3.4, it is presented as a separate theorem in order to emphasize the fact that there exist continuous support functionals at w_o.

It should be noted that if the conditions in Theorem 3.3.4 hold, then for each $w_o = (\xi, \eta)$ satisfying $\rho(w_o;D) > 0$ there exists an optimal solution to Problem (I).

Attention shall now be directed toward consideration of Problem (I-a) in the special case that $J_1(x) = \|x\|$ and $\Omega = X$. In [25], Porter and Williams considered a problem of this type by defining a pseudo-inverse of the linear operator S. In the special case of X a Hilbert space the pseudo-inverse of S is exactly the generalized inverse of S. To obtain the existence of the pseudo-inverse, Porter and Williams made heavy use of the assumption that X was a reflexive Banach space. The following definition is a generalization of one found in [25].

DEFINITION 3.3.1. _Let_ X _and_ Y _be normed linear spaces and_ $T:D(T) \to Y$ _a linear operator with_ $D(T) \subseteq X$. _A pseudo-inverse of_ T _is an operator_ T^Δ _defined on_ R(T) _into_ D(T) _which satisfies the conditions_:

(i) $T[T^\Delta y] = y$, _if_ $y \in R(T)$;

(ii) $\|T^\Delta y\| \le \|x\|$ _for all_ $x \in D(T)$ _satisfying_ $Tx = y$.

It is important to note that the operator T^{Δ} need not be linear. In the special case of Problem (I-a) when $J_1(x) = \|x\|$ and $\Omega = X$, we have that an optimal solution exists if η belongs to $R(S)$ and S^{Δ} exists, and the optimal solution is given by $S^{\Delta}\eta$.

Once existence of T^{Δ} has been established, then characterizations of T^{Δ} similar to those in [25] can be made, provided care is taken to consider when elements belong to $D(T)$. However, this characterization shall not be pursued in this paper.

DEFINITION 3.3.2. A normed linear space X is said to have Property (P) if each convex subset of X has at most one element of minimum norm.

It is to be remarked that if X is a Banach space, then Property (P) is equivalent to the condition that X be rotund. However, Definition 3.3.2 does not imply the existence of an element of minimum norm.

EXAMPLE 3.3.1. Let $I = (a,b)$ be an open interval in R, and denote by $C^k(I)$ the set of all real-valued functions defined on I and having continuous derivatives up to order k. If $f \in C^k(I)$, then the support of f is the closure of the set $\{x \in I : f(x) \neq 0\}$. Let $C_o^k(I)$ denote the subset of $C^k(I)$ consisting of all functions with compact support. The set of functions having compact supports and derivatives of all orders will be denoted by $C_o^{\infty}(I)$. For $p \geq 1$, and $f \in C_o^{\infty}(I)$, we shall define $\|f\|$ by

$$\|f\|_p = \left(\int_I |f(s)|^p ds\right)^{1/p}, \tag{3.3.2}$$

and the corresponding normed linear space shall be denoted by $C_{op}^{\infty}(I)$. It should be noted that $C_{op}^{\infty}(I)$ is not a complete space, and it is well known that $L_p(I)$ is the completion of $C_{op}^{\infty}(I)$, where $L_p(I)$ is the Lebesgue space defined in Example 1.2.1. Each of the spaces $C_{op}^{\infty}(I)$ satisfies Property

(P) for $p > 1$; also, the space $C_{o2}^{\infty}(I)$ is a real inner product space which is not complete.

THEOREM 3.3.6. Suppose X and Z are normed linear spaces and $S:D(S) \to Z$ is a linear operator with $D(S) \subseteq X$. If X has Property (P) and S is weakly co-compact, then the pseudo-inverse S^{Δ} exists and is unique.

If $z \in R(S)$, then the set $\{x \in D(S) : Sx = z\}$ is closed and convex. If

$$d = \min \{\|x\| : Sx = z\},$$

consider a sequence $\{x_n\}$ such that for $n \geq 1$ we have $Sx_n = z$ and $d \leq \|x_n\| \leq d + (1/n)$. Since S is weakly co-compact, and $\{x_n\}$ is a bounded semi-S-convergent sequence, there is a subsequence $\{\xi_k\}$ such that $\xi_k \to x_o$ weakly, $x_o \in D(S)$, and $Sx_o = z$, consequently we have that $\|x_o\| \geq d$. Also, since $\{x \in X : \|x\| \leq d + (1/k)\}$ is weakly closed, it follows that $\|x_o\| \leq d + (1/k)$ for all $k \geq 1$, and hence we have that $\|x_o\| = d$. The element x_o is an element of $\{x \in D(S) : Sx = z\}$ of minimum norm, and since X has Property (P) it follows that x_o is unique. If $S^{\Delta}:R(S) \to X$ is defined by $S^{\Delta}z = x_o$, it follows that S^{Δ} is the unique pseudo-inverse of S.

The following result will help in the characterization of the pseudo-inverse of an operator defined on an inner product space.

LEMMA 3.3.2. Suppose X is a real inner product space, Z is a normed linear space, and $S:X \to Z$ is a weakly co-compact operator. If M denotes the orthogonal complement of $N(S)$, then X is the direct sum of $N(S)$ and M.

If $M \oplus N(S)$ denotes the direct sum of M and $N(S)$, then we clearly have that $M \oplus N(S) \subseteq X$. On the other hand, if $\xi \in X$ then $\langle x,\xi \rangle$ defines a continuous linear functional on $N(S)$. In view of Theorem 1.2.4 we have that $N(S)$ is a Hilbert space, and hence there exists an element $x_o \in N(S)$ such that $\langle x,x_o \rangle = \langle x,\xi \rangle$ for each $x \in N(S)$. Therefore, $\langle x,\xi - x_o \rangle = 0$ for each $x \in N(S)$ and we have that $(\xi - x_o) \in M$. Since $\xi = x_o + (\xi - x_o)$ belongs to $M \oplus N(S)$, the proof of the lemma is complete.

It is to be remarked that if $z \in R(S)$ then there exists an element $x_o \in M$ such that $Sx_o = z$. Moreover, if $x \in X$ and $Sx = z$, then it follows that $x = (x - x_o) + x_o \in N(S) \oplus M$, and hence we have

$$\|x\|^2 = \|x - x_o\|^2 + \|x_o\|^2 \geq \|x_o\|^2.$$

Therefore, the pseudo-inverse S^Δ is such that $S^\Delta z = x_o$, and consequently an element $x \in X$ may be decomposed into

$$x = (x - S^\Delta(Sx)) + S^\Delta(Sx).$$

Since the operator S restricted to M is one-to-one, we have that its inverse exists. If P denotes the operator S restricted to M, then it follows that $S^\Delta = P^{-1}$, and hence we have the following result.

THEOREM 3.3.7. _Suppose_ X _is a real inner product space_, Z _is a normed linear space, and_ $S: X \to Z$ _is a weakly co-compact operator. If_ M _is the orthogonal complement of_ N(S) _and_ P _is the restriction of_ S _to_ M, _then_ S^Δ, $S^\#$, _and_ P^{-1} _all exist and are equal_.

It should be noted that there was no assumption about the space Z in Theorem 3.3.7. Moreover, as was illustrated in Example 1.2.2, the range of S need not be closed. Also, it is to be remarked that Theorem 3.3.6 may be established by a method that makes use of the pseudo-inverse S^Δ, and in detail similar to the proof of Problem 4 in Taylor [36; p. 245].

3.4. _A necessary condition_. The objective of this section is to obtain for Problem (I) a necessary condition which is in the form of a duality condition. The following definition may be found in Kato [16; p. 167].

DEFINITION 3.4.1. *Let* F *and* H *be normed linear spaces and* $L:D(L) \to H$ *a linear operator with* $D(L)$ *dense in* F. *Let* $D(L^*)$ *be the set of all* $h^* \in H^*$ *such that there exists an element* $f^* \in F^*$ *satisfying*

$$\langle Lf,h^*\rangle = \langle f,f^*\rangle \text{ for all } f \in D(L). \tag{3.4.1}$$

As $D(L)$ *is dense in* F, *the element* f^* *is determined uniquely by* h^*, *and we define* $L^*:D(L^*) \to F^*$ *by* $L^*h^* = f^*$. *The operator* L^* *is called the adjoint of* L.

If L^* is the adjoint operator of L, then we have that

$$\langle Lf,h^*\rangle = \langle f,L^*h^*\rangle, \tag{3.4.2}$$

for all $f \in D(L)$ and $h^* \in D(L^*)$.

For $L:D(L) \to H$, the *graph* of L is the subspace of $F \times H$ given by $G(L) = \{(f,Lf) : f \in D(L)\}$. For L^* the adjoint of L, we shall let $G^{-1}(-L^*)$ denote the *inverse graph* of $-L^*$, that is, the subspace of $F^* \times H^*$ given by

$$G^{-1}(-L^*) = \{(-L^*h^*,h^*) : h^* \in D(L^*)\}. \tag{3.4.3}$$

If N is a subspace of the normed linear space F, then the *annihilator* of N is the subspace of F^* denoted by ann N, and specified by

$$\text{ann } N = \{f^* \in F^* : \langle f,f^*\rangle = 0 \text{ for all } f \in N\}.$$

If F is an inner product space, then we identify ann N with the orthogonal complement of N, and identify F with F^*.

It is important to note that condition (3.4.2) is equivalent to the statement that ann $G(L) = G^{-1}(-L^*)$.

If $K \subseteq F$ is a convex set and $f_o \in \text{supp}^*K$, then the set of all continuous support functionals corresponding to f_o will be denoted by $E(K;f_o)$. If f_o^* is a non-zero continuous support functional to K, then $E(K;f_o^*)$ will denote the set of all $f_o \in \text{cl } K$ such that f_o satisfies the condition

$$<f_o, f_o^*> = \sup \{<f, f_o^*> : f \in K\}.$$

The following duality result is basic in any consideration of necessary conditions.

THEOREM 3.4.1. *Let F and H be normed linear spaces and $L:D(L) \to H$ a linear operator with $D(L)$ dense in F. Let $K \subseteq F$ be a convex set and $L^*:D(L^*) \to F^*$ the adjoint of L. If $Lf_o = h_o$, then the following conditions are equivalent*:

(i) $h_o^* \in E(L[K \cap D(L)];h_o) \cap D(L^*)$,

(ii) $f_o \in E(K \cap D(L);L^*h_o^*) \cap D(L)$.

In view of condition (3.4.2), the equivalence of (i) and (ii) follows readily from the relations

$$\begin{aligned} &\sup \{<f, L^*h_o^*> : f \in K \cap D(L)\} \\ &= \sup \{<Lf, h_o^*> : f \in K \cap D(L)\} \\ &= \sup \{<h, h_o^*> : h \in L[K \cap D(L)]\}. \end{aligned}$$

The following result is a generalization of Proposition 3.1 in [21].

THEOREM 3.4.2. *Suppose that hypothesis (H_o) is satisfied, ρ is the extended Minkowski functional of D, $w_o = (\xi,\eta)$, and $\rho(w_o) = \alpha_o > 0$. If x_o is an optimal solution to Problem (I), then x_o must satisfy the relation*

$$(3.4.3) \qquad <x, S^*\eta^*> + <\xi, \xi^*> =$$

$$\sup \{<x, T^*\xi^*> + <x, S^*\eta^*> + <y, \xi^*> : (x,y) \in A(\alpha_o) \cap D(Q)\},$$

for all $(\xi^*,\eta^*) \in E[B(\alpha_o);w_o] \cap [D(T^*) \times D(S^*)]$.

If x_o is an optimal solution to Problem (I) and $\rho(w_o) = \alpha_o$, then it follows that $J(x_o, \xi - Tx_o) = \alpha_o$, and hence we have that

$w_o = Q(x_o, \xi - Tx_o) \in B(\alpha_o)$. Moreover, from Theorem 2.4.1 we have that $w_o \in \operatorname{supp} B(\alpha_o)$.

If (ξ^*, η^*) is a non-zero continuous support functional at w_o with $\xi^* \in D(T^*)$ and $\eta^* \in D(S^*)$, then for each $(x,y) \in D(Q)$ we have

$$
\begin{aligned}
\langle Q(x,y), (\xi^*,\eta^*)\rangle &= \langle (Tx+y, Sx), (\xi^*,\eta^*)\rangle \\
&= \langle (Tx+y), \xi^*\rangle + \langle Sx, \eta^*\rangle \\
&= \langle x, T^*\xi^*\rangle + \langle y, \xi^*\rangle + \langle x, S^*\eta^*\rangle \\
&= \langle (x,y), (T^*\xi^* + S^*\eta^*, \xi^*)\rangle .
\end{aligned}
$$

Therefore it follows that (ξ^*,η^*) belongs to $D(Q^*)$, and

$$Q^*(\xi^*,\eta^*) = (T^*\xi^* + S^*\eta^*, \xi^*).$$

From Theorem 3.4.1 we have that $(x_o, \xi - Tx_o)$ belongs to $E[(A(\alpha_o) \cap D(Q)); Q^*(\xi^*,\eta^*)] \cap D(Q)$. By direct calculation we find that

$$
\begin{aligned}
&\langle x_o, S^*\eta^*\rangle + \langle \xi, \xi^*\rangle \\
&= \langle x_o, T^*\xi^*\rangle + \langle x_o, S^*\eta^*\rangle + \langle \xi, \xi^*\rangle - \langle Tx_o, \xi^*\rangle \\
&= \langle (x_o, \xi - Tx_o), Q^*(\xi^*,\eta^*)\rangle \\
&= \sup \{\langle (x,y), Q^*(\xi^*,\eta^*)\rangle : (x,y) \in A(\alpha_o) \cap D(Q)\} \\
&= \sup \{\langle (x,y), (T^*\xi^* + S^*\eta^*, \xi^*)\rangle : (x,y) \in A(\alpha_o) \cap D(Q)\} \\
&= \sup \{\langle x, T^*\xi^*\rangle + \langle x, S^*\eta^*\rangle + \langle y, \xi^*\rangle : (x,y) \in A(\alpha_o) \cap D(Q)\},
\end{aligned}
$$

and this completes the proof of the theorem.

It is to be noted that care should be taken when one considers the domains of adjoint operators. It may happen that the domain of the adjoint operator is the trivial space $\{\theta\}$. Another thing to notice is that condition (3.4.3) gives no information about x_o in the case where $Z = \{\theta\}$

and $S = \theta$. However, for such instances necessary conditions for Problem (I-b) may be obtained by extensions of the method used in [10] and [11]. For this reason attention will now be directed toward Problem (I-b), in the special case that X and Y are Hilbert spaces and J is defined by

$$J(x,y) = \sqrt{\|x\|^2 + \|y\|^2}.$$

In particular, the projection method will be used to obtain a necessary condition similar to the Euler condition.

The following result is due to J. von Neumann, and the essential parts of its proof may be found in Theorem 3.24 in Kato [16; pp. 275-276].

LEMMA 3.4.1. *Let H_1 and H_2 be Hilbert spaces and $L:D(L) \to H_2$ a closed linear operator such that $D(L)$ is dense in H_1. If I_i denotes the identity operator on H_i, $(i = 1,2)$, then $(I_2 + LL^*)$ and $(I_1 + L^*L)$ both have bounded inverses defined everywhere on H_2 and H_1, respectively.*

The following theorem is a slight extension of the result in [11] by Halilov and Panaioti in the sense that ξ does not have to belong to $D(T^*)$.

THEOREM 3.4.3. *Suppose X and Y are real Hilbert spaces, $T:D(T) \to Y$ is a closed linear operator with $D(T)$ dense in X, E is the identity operator in X, and J is defined by $J(x,y) = (\|x\|^2 + \|y\|^2)^{1/2}$. If $\Omega = X$, then Problem (I-b) has a unique optimal solution x_o, where x_o is the solution to the equation*

$$x + T^*(Tx - \xi) = \theta. \tag{3.4.4}$$

Moreover, if $\xi \in D(T^*)$ then x_o is given by

$$x_o = (E + T^*T)^{-1}(T^*\xi). \tag{3.4.5}$$

Since $G(T)$ and $G^{-1}(-T^*)$ are orthogonal complements, the space $X \times Y$ may be decomposed into $G(T) \oplus G^{-1}(-T^*)$. Because $(\theta,\xi) \in X \times Y$, there exist elements $x_o \in D(T)$ and $y_o \in D(T^*)$ such that

$$(\theta,\xi) = (x_o,Tx_o) + (-T^*y_o,y_o).$$

It is clear that Problem (I-b) is equivalent to finding the projection of (θ,ξ) onto $G(T)$, and this projection is (x_o,Tx_o). In view of the above condition, we have that $x_o = T^*y_o$ and $\xi = Tx_o + y_o$ which implies condition (3.4.4).

If $\xi \in D(T^*)$, then we have that $(E + T^*T)x_o = T^*\xi$ and hence condition (3.4.5) follows from Lemma 3.4.1.

Now consider Problem (I-b) when the set $\Omega \subsetneq X$ is not all of X. If Ω is closed, then the set

$$\Pi = \{(x,Tx) : x \in \Omega \cap D(T)\}$$

is a closed convex subset of $G(T)$ in $X \times Y$. If X and Y are Hilbert spaces and $(x_o,Tx_o) \in G(T)$, then it follows that there exists a unique element $x_o^m \in \Omega$ such that

$$\|(x_o,Tx_o) - (x_o^m,Tx_o^m)\|^2 = \inf\{\|(x_o,Tx_o) - (x,Tx)\|^2 : x \in \Omega \cap D(T)\}.$$

That is, (x_o^m,Tx_o^m) is the point of Π nearest to the point $(x_o,Tx_o) \in G(T)$.

THEOREM 3.4.4. *Suppose* E, X, Y, T, *and* J *are as in Theorem* 3.4.3, *and* $\Omega \subseteq X$ *is closed. Then the optimal solution to Problem* (I-b) *exists and is given by* x_o^m, *where* x_o *is the unique solution to equation* (3.4.4). *In particular, if* $\xi \in D(T^*)$, *then the optimal solution is given by*

$$[(E + T^*T)^{-1}(T\xi)]^m.$$

If x_o is the solution to (3.4.4), then it follows that $(x_o,Tx_o) - (\theta,\xi)$ is orthogonal to $G(T)$. Therefore, we have that

$$\|(x_o^m,Tx_o^m) - (\theta,\xi)\|^2 = \|(x_o^m,Tx_o^m) - (x_o,Tx_o)\|^2 + \|(x_o,Tx_o) - (\theta,\xi)\|^2.$$

On the other hand, if $x \in \Omega \cap D(T)$ then we have that

$$\begin{aligned}
&\|(x,Tx) - (\theta,\xi)\|^2 \\
&\quad = \|(x,Tx) - (x_o,Tx_o)\|^2 + \|(x_o,Tx_o) - (\theta,\xi)\|^2 \\
&\quad \geq \|(x_o^m,Tx_o^m) - (x_o,Tx_o)\|^2 + \|(x_o,Tx_o) - (\theta,\xi)\|^2 \\
&\quad = \|(x_o^m,Tx_o^m) - (\theta,\xi)\|^2.
\end{aligned}$$

Since $J(x,\xi - Tx) = \|(x,Tx) - (\theta,\xi)\|$, it is clear that x_o^m is the optimal solution. The second conclusion follows directly from Theorem 3.4.3.

3.5. Examples. This section is devoted to examples of problems which fall in the realm of the previous theory. The first example will show that an optimal solution to Problem (I) need not exist, even with relatively strong conditions placed on the operators T and S.

EXAMPLE 3.5.1. Find a continuous function $x:[0,1] \to R$ such that x minimizes $J_1(x) = \sup \{|x(t)| : 0 \leq t \leq 1\}$ subject to the constraints $x(0) = 0$, $\int_o^1 x(s)ds = 1$.

It is clear that if x satisfies the constraints then $J_1(x) \geq 1$. Moreover, we have that

$$\inf \{J_1(x) : x \text{ continuous, } x(0) = 0, \int_o^1 x(s)ds = 1\} = 1,$$

but there is no continuous function x satisfying $x(0) = 0$, $\int_o^1 x(s)ds = 1$ and $J_1(x) = 1$. Thus the problem does not have an optimal solution.

We now place the problem in the setting of Problem (I). Let X denote the subspace of C[0,1] of all functions which vanish at 0. Let Y = C[0,1] and Z = R, while $T:X \to Y$ is the identity operator and $S:X \to Z$ is defined by $Sx = \int_o^1 x(s)ds$. Let $\Omega = X$, $\xi = \theta$, $\eta = 1$, and J be defined by

$$J(x,y) = (1/2)(\|x\| + \|y\|).$$

In this case, it follows that

$$J(x,\theta - Tx) = J(x,-x) = \|x\| = J_1(x),$$

and hence the problem is an example of Problem (I). It is to be noted that the operator S is compact, T is continuous, J is continuous, Ω is closed and, all spaces are complete, and yet Problem (I) does not have a solution.

The following example is a problem which first motivated the study of Problem (I).

EXAMPLE 3.5.2. Consider the linear vector ordinary differential operator

(3.5.1) $$[Lx](t) = A_1(t)x'(t) + A_o(t)x(t), \quad t \in I = [a,b],$$

where $x(t) = (x_i(t))$, $(i = 1,2,\cdots,n)$, is an absolutely continuous n-dimensional vector function on I, while the $n \times n$ matrices $A_1(t)$ and $A_o(t)$ are such that $A_1(t)$ is non-singular, with $A_1(t)$, $A_1^{-1}(t)A_o(t)$ and $A_1^{-1}(t)$ measurable and essentially bounded. As is customary, we shall use the abbreviation a.c. for absolutely continuous.

If $x:I \to R^n$ is a.c. with $x(t) = (x_i(t))$, $(i = 1,2,\cdots,n)$, then the 2n-dimensional $\omega = (\omega_\sigma)$, $(\sigma = 1,2,\cdots,2n)$, with $\omega_\alpha = x_\alpha(a)$, $\omega_{\alpha+n} = x_\alpha(b)$, $(\alpha = 1,2,\cdots,n)$, will be denoted by $\hat{x}$. For $p \geq 1$, we shall let $L_p^n(I)$ denote the Lebesgue normed linear space of n-dimensional vector functions defined on I.

If M is a $m \times 2n$ constant matrix and $\eta \in R^m$, then one may consider the problem of determining an absolutely continuous n-dimensional vector function x satisfying $M\hat{x} = \eta$ and such that Lx is an element of $L_p^n(I)$ of minimum norm.

In case $M = E_{2n}$, the identity matrix on R^{2n}, the problem is equivalent to a problem considered in [30] by Reid. If E_n is the identity matrix on R^n, N is an $r \times n$ matrix with $r+n = m$, and

$$M = \begin{bmatrix} E_n & 0 \\ 0 & N \end{bmatrix},$$

then the problem is equivalent to the problem considered in [24] by Neustadt. In [30] and [24], the problem was reduced to a finite moment problem. However, this reduction was possible because the operator L could be inverted, and in general this is not the case. For example, if $u \in L_p^n(I)$, then there may be several solutions to the system

$$\text{(3.5.2)} \qquad \begin{cases} [Lx](t) = u(t), \\ M\hat{x} = \eta. \end{cases}$$

Also, it should be noted that there may be no solutions to the system.

Frequently the formulation of such a problem calls for certain restrictions on Lx. For example, we may require that $\|Lx\|_p \leq 1$.

We shall now formulate the problem in a function space setting as an example of Problem (I). Let $X = L_q^n(I)$, $Y = L_p^n(I)$, $Z = R^m$, and T and S be linear operators defined by

$$D(T) = \{x \in X : x \text{ a.c., } Lx \in Y\},$$

and $Tx = Lx$, while $D(S) = D(T)$ and $Sx = M\hat{x}$. Let ξ be a function in $L_p^n(I)$ and $\eta \in R^m$, and define $\Omega \subseteq X$ by

$$\text{(3.5.3)} \qquad \Omega = \{x \in D(T) : \|Tx\| \leq 1\}.$$

We shall consider a general continuous convex function J defined on $X \times Y$. Thus we have an example of Problem (I) for which hypothesis (H_0)

is satisfied. If $p \geq 1$, then Ω is closed, and if $q > 1$, then the operator Q is a weakly co-compact operator.

For an example of the above form, it should be noted that if the system (3.5.2) is such that L defines a partial differential operator and M defines boundary conditions, then the problem becomes much more involved. For example, the null space of L would possibly be infinite dimensional. However, the results in this paper may still be applied to certain problems involving partial differential operators. In particular, the book by Lions [18] provides discussion of various optimal control problems involving partial differential operators.

EXAMPLE 3.5.3. Let $I = [0,1]$, $X = Y = Z = L_1(I)$, and J be defined by

$$J(x,y) = \alpha \int_0^1 |x(s)|ds + \beta \int_0^1 |y(s)|ds$$

where α and β are non-negative constants not both zero. For $D(T)$ given by

$$D(T) = \{x \in L_1(I) : x \text{ a.c.}\},$$

it follows that $x' \in L_1(I)$ for $x \in D(T)$, and we define $T:D(T) \to L_1(I)$ by $[Tx](t) = x'(t)$. Suppose $k(t,s)$ is a bounded and measurable function defined on $[0,1] \times [0,1]$, and $\phi \in L_1([-1,1])$. Let $D(S) = D(T)$ and define $S:D(S) \to L_1(I)$ by

$$[Sx](t) = x'(t) + \int_0^1 k(t,s)\phi(t-s)x(s)ds,$$

$$= [Tx](t) + [Kx](t).$$

It is easy to show that the operator K is a weakly compact operator, and that T is a Fredholm operator. Since S is the operator T perturbed by the operator K, it follows from a stability theorem due to Kato, (see

Goldberg [8; p. 117]), that $R(S)$ is closed and $N(S)$ is finite dimensional. It then follows from Theorem 1.2.2 that S is a co-compact operator.

Suppose $\{(x_n,y_n)\}$ is a bounded semi-Q-convergent sequence with $Q(x_n,y_n) \to (f,g)$. Since $Sx_n \to g$ and S is co-compact, there is a subsequence $\{\xi_k\}$ and an element $x \in D(S)$ such that $\xi_k \to x$ and $Sx = g$. The operator $K:L_1(I) \to L_1(I)$ is bounded, and hence we have that $K\xi_k \to Kx$. Therefore it follows that $T\xi_k \to (g - Kx)$, and if $\{\eta_k\}$ is the subsequence of $\{y_n\}$ corresponding to $\{\xi_k\}$, then we have $\eta_k \to (f - g + Kx)$. Thus the subsequence $\{(\xi_k,\eta_k)\}$ converges to the element $(x,f - g + K) \in D(Q)$ and $Q(x,f - g + Kx) = (f,g)$, which proves that Q is a co-compact operator.

Let $\Omega \subseteq X = L_1(I)$ be given by

$$\Omega = \{x \in X : |x(t)| \leq 1 \text{ a.e.}\};$$

it is easy to verify that Ω is closed and convex. If $x \in D(T)$ then $x' \in L_1(I)$, and it follows that for $t \in I$

$$|x(t)| = |x(0) + \int_0^t x'(s)ds| \leq |x(0)| + \|x'\|.$$

If $\gamma = \min \{1, 1/(|x(0)| + \|x'\|)\}$ and $0 \leq \lambda < \gamma$, it then follows that $\lambda x \in \Omega$, and hence Ω is a closed convex set with $\theta \in \text{core}_T\Omega$.

Suppose $\xi \in L_1(I)$ and $\eta \in L_1(I)$ are such that there exists a function $x \in [\text{core}_T\Omega] \cap D(S)$ with $Sx = \eta$. The problem to be considered is that of the existence of an $x_o \in [D(T) \cap \Omega]$ satisfying

$$\eta(t) = x'(t) + \int_0^1 k(t,s)\phi(t-s)x(s)ds,$$

and which minimizes

$$\alpha \int_0^1 |x(s)|ds + \beta \int_0^1 |\xi(s) - x'(s)|ds.$$

It follows from Theorem 3.3.4 that an optimal solution x_o exists if ξ and η are such that $\rho((\xi,\eta);D) > 0$.

REFERENCES

1. R. G. Bartle, The Elements of Real Analysis, John Wiley and Sons, New York, 1964.

2. F. J. Beutler, The operator theory of the pseudo-inverse, I: bounded operators, II: unbounded operators with arbitrary range, J. Math. Anal. Appl., Vol. 10 (1965), pp. 451-493.

3. E. Bishop and R. R. Phelps, The support functionals of a convex set, Proceedings of Symposia in Pure Mathematics, Vol. VII, American Mathematical Society, Providence, 1963.

4. N. Bourbaki, Espaces Vectoriels Topologiques, Elements de Mathématique, Livre V, Actualités Sci. Ind., 1229, Hermann et Cie, Paris, 1955.

5. N. Dunford and J. Schwartz, Linear Operators, Vol. I, Interscience, 1958.

6. H. G. Eggleston, Convexity, Cambridge Press, 1958.

7. G. M. Ewing, Calculus of Variations, W. W. Norton and Company, New York, 1969.

8. S. Goldberg, Unbounded Linear Operators, McGraw-Hill, New York, 1966.

9. Z. I. Halilov and È. Dž. Aslanov, A variational problem in Hilbert space with an application to partial differential equations, Dokl. Akad. Nauk. SSSR, Vol. 169 (1966), pp. 1056-1059.

10. Z. I. Halilov and È. Dž. Aslanov, Some variational problems with control values in Hilbert space, Dokl. Akad. Nauk. SSSR, Vol. 182 (1968), pp. 1210-1213.

11. Z. I. Halilov and B. N. Panaioti, Minimization of a quadratic functional induced by linear operators in normed spaces, Dokl. Akad. Nauk. SSSR, Vol. 184 (1969), pp. 212-214.

12. F. Hemming and V. D. Vandelinde, An optimal control problem in Banach space, J. Math. Anal. Appl., Vol. 39 (1972), pp. 647-654.

13. M. R. Hestenes, Calculus of Variations and Optimal Control Theory, John Wiley and Sons, New York, 1966.

14. E. Hewitt and K. Stromberg, Real and Abstract Analysis, Springer-Verlag, New York, 1969.

15. V. K. Ivanov, Incorrect problems in topological spaces, Siberian Math. J., Vol. 10 (1969), pp. 1065-1074.

16. T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1966.

17. V. L. Klee, The support property of a convex set in a normed linear space, Duke Math. J., Vol. 15 (1948), pp. 767-772.

18. J. L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, New York, 1971.

19. A. T. Lonseth, Approximate solutions of Fredholm-type integral equations, Bull. Amer. Math. Soc., Vol. 21 (1967), pp. 113-131.

20. D. G. Luenberger, Optimization by Vector Space Methods, John Wiley and Sons, New York, 1969.

21. N. Minamide and K. Nakamura, A minimum cost problem in Banach space, J. Math. Anal. Appl., Vol. 36 (1971), pp. 73-85.

22. N. Minamide and K. Nakamura, Linear bounded phase coordinate control problems under certain regularity and normality conditions, J. SIAM Control, Vol. 10 (1972), pp. 82-93.

23. L. W. Neustadt, Minimum effort control system, J. SIAM Control, Ser. A, Vol. 1 (1962), pp. 16-31.

24. L. W. Neustadt, Optimization, a moment problem, and non-linear programming, J. SIAM Control, Ser. A, Vol. 2 (1964), pp. 33-53.

25. W. A. Porter and J. P. Williams, A note on the minimum effort control problem, J. Math. Anal. Appl., Vol. 13 (1966), pp. 251-264.

26. W. A. Porter and J. P. Williams, Extensions of the minimum effort control problem, J. Math. Anal. Appl., Vol. 13 (1966), pp. 536-549.

27. W. A. Porter, On the optimal control of distributive systems, J. SIAM. Control, Vol. 4 (1966), pp. 466-471.

28. W. A. Porter, Modern Foundations of System Engineering, MacMillan, New York, 1966.

29. B. N. Pshenichnyi, Necessary Conditions for Extremum, Marcel Dekker, Inc., New York, 1971.

30. W. T. Reid, Ordinary linear differential operators of minimum norm, Duke Math. J., Vol. 29 (1962), pp. 591-606.

31. W. T. Reid, A simple optimal control problem involving approximation by monotone functions, J. Optimization Theory and Appl., Vol. 2 (1968), pp. 365-377.

32. W. T. Reid, Generalized inverses of differential and integral operators, Proceedings of the Symposium on Theory and Application of Generalized Inverses of Matrices, Texas Technological College Mathematics Series No. 4, 1968, pp. 1-25.

33. W. T. Reid, Ordinary Differential Equations, John Wiley and Sons, Inc., New York, 1971.

34. A. T. Rockafellar, Convex Analysis, Princeton Press, 1969.

35. I. Springer, Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces, Springer-Verlag, New York, 1970.

36. A. E. Taylor, Introduction to Functional Analysis, John Wiley and Sons, New York, 1966.

37. F. A. Valentine, Convex Sets, McGraw-Hill, New York, 1964.

38. K. Yosida, Functional Analysis, Springer-Verlag, New York, 1971.