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ROBERT GERALD McINTYRE

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THE DIFFUSION OF RESONANCE

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APPROVED BY

R. G. Fowler

William N. Huff

Otto J. Hennie

R. G. Zimmer

P. T. Little

DISSERTATION COMMITTEE

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THE DIFFUSION OF RESONANCE EXCITATION THROUGH A GAS

CHAPTER I

INTRODUCTION

"Resonance radiation" is the term applied to radiation emitted by an atom in a transition from an excited state to the ground state.

Resonance radiation is much more absorbable than the other components of atomic spectra, and therefore the mechanisms by which it propagates to the walls of a vessel are considerably more complex.

For example, the 2537A line of mercury at 1 mm. Hg. is appreciably absorbed in a distance of .001 cm. It is seen, then, that the transfer is a two-stage phenomenon alternately involving the transmission of a free photon and a displacement of the excitation energy as captured energy in an excited atom.

The phenomenon was first studied by K. T. Compton.[1] Compton gave a theoretical treatment which regarded the transfer of excitation as a type of Brownian motion. His result was a diffusion-type equation for the density of excited atoms, "n", of the form

$$\frac{\partial n}{\partial t} = D \nabla^2 n, \tag{1.1}$$

where D involves λ, the mean free path of the photon, and τ, the life-time of the individual atom, and is expressed by

$$D = \lambda^2/3\tau \, . \tag{1.2}$$

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1. K. T. Compton, Phys. Rev. 20, 283 (1922).

Compton ignored the displacement of the excitation energy in an excited atom. Later authors do not criticize him for this. However, λ is assumed to be a constant and frequency independent, whereas, in reality, if a mean free path does exist it should, by virtue of the Doppler effect, be extremely frequency dependent.

The next attack on the problem by E. A. Milne[1], did not yield results significantly different from Compton's.

The frequency spectrum of the line was first taken into account by C. Kenty[2], whose attempt to calculate an average diffusion coefficient, on assuming a Doppler line emitted from each volume element of the gas, and essentially averaging (1.2) over the spectrum of the emitted radiation, yielded the untenable result that for an enclosure of infinite size this average was infinite. Kenty confined his attention to the calculation of a diffusion coefficient for the finite case and arrived at much better results[3] than previous workers. However, he too ignored the displacement of excited states and, therefore, overlooked what is, in many cases, the most important mechanism in the transport of resonance radiation.

Much greater complexity was added to the theory by T. Holstein[4] who argued that beginning with Compton's uniform absorption coefficient $1/\lambda$, leading to a probability, $T(\rho)$, of the radiation travelling a distance ρ, of the form

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1. E. A. Milne, J. Lond. Math. Soc. 1, 1 (1926).
2. C. Kenty, Phys. Rev. 42, 823 (1932).
3. M. W. Zemansky, Phys. Rev. 42, 843 (1932).
4. T. Holstein, Phys. Rev. 72, 1212 (1947).

1.3 $$T(\rho) \overset{3}{=} e^{-\rho/\lambda} ,$$

one might express the probability, $T(\rho, \nu)$, that a photon with frequency ν will travel a distance ρ as

1.4 $$T(\rho,\nu) = e^{-k(\nu)} \quad ,$$

and integrating over the frequency spectrum, $P(\nu)$, one arrives at

1.5 $$T(\rho) = \int P(\nu) e^{-k(\nu)\rho} \, d\nu \quad .$$

The term $k(\nu)$ appearing in the above expression is treated extensively in the texts.[1]

The two objections to Holstein's work are the same as for the preceeding authors. Only one of the two important mechanisms is considered, namely the displacement of excitation energy during an emission-absorption process. Holstein and other authors refer to this displacement as the "skip" of the photon. Recently, R. G. Fowler[2] has taken note of the above deficiency and has mentioned that the diffusion term in the Boltzmann Equation[3] must be included. Fowler's argument is quite simple and irrefutable. Since there are two alternately occurring processes in the diffusion of resonance excitation, and since, in many cases, the two are of roughly equal importance, no description that ignores either can be

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1. A. C. G. Mitchell and M. W. Zymansky, Resonance Radiation and Excited Atoms, (The Macmillan Company, 1934).

2. R. G. Fowler, Handbuch der Physik, Encyclopedia of Physics 22, 209 (1954).

3. S. Chapman and T. G. Cowling, The Mathematical Theory of Non-Uniform Gases, (Cambridge University Press, 1939).

considered adequate.

The second general objection to the preceding work is that too early in the theory the authors consider only averages over velocity space.

The purpose of this work has been to arrive at a purely kinetic-theoretical description of the phenomena involved in the transport of resonance excitation in gases without regard for the subjects of existing arguments such as the existence of a diffusion coefficient.

The author feels that the result has been that the truly important questions have been previously unasked, and that these questions appear with their answers in the conclusion of this dissertation.

In order to test the theory, new types of experiments are suggested. Also, new boundary conditions are suggested for the differentio-integral equation, since the results indicate that one could never have confidence that he had obtained those boundary conditions assumed in this dissertation.

CHAPTER II

DESCRIPTION OF THE PROBLEM AND DERIVATION OF THE EQUATION OF TRANSPORT

We consider that only two states of a gas exist and that there is a known initial distribution of each state.

It is assumed that after initial excitation the distribution of neutral atoms is uniform in space and Maxwellian in velocity. We, therefore, are neglecting contributions of photons to the dynamical state of the gas.

Let $f(\vec{r}, \vec{c}, t)dxdydzdudvdw$ be the number of atoms in space-velocity range $(x, x+dx; y, y+dy; z, z+dz; u, u+du; v, v+dv; w, w+dw)$ at a time t, and let $p(\vec{r}, \vec{c}, t)dxdydzdudvdw$ bebthe number of excited atoms in the same range where

$$2.1 \qquad \vec{r} = (x, y, z)$$

represents the position coordinates, and where

$$2.2 \qquad \vec{c} = (u, v, w)$$

represents the velocity coordinates.

Now, let a equal the probability per unit time that an excited atom will radiate. It is assumed that this probability is the same for all atoms regardless of velocity, and is just the Einstein Emission Coefficient.

Similarly, let "B" equal an excitation cross section which is defined as that area which divided by unit area gives the probability of absorption per unit length of path of a photon in a region wherein eligible absorbers have a density of one per unit volume.

The term "eligible absorbers" will be discussed later. The probability of absorption of a photon by an ineligible absorber is, of course, zero.

We assume $p \ll f$.

Now, when an atom with a velocity $\vec{c}$ emits, the photon suffers a Doppler shift, dependent upon the component of velocity, c_ρ, of the emitting atom in the direction of emission. For another atom to absorb this photon it too must have a component of velocity along the line-of-sight of the photon nearly equal to c_ρ. We assume that there is some fixed value Δc such that for an emission-absorption process to take place between atoms with velocities $\vec{c}$ and $\vec{c}\,'$ respectively and displaced a vector $\vec{\rho}$ apart the condition

$$\left| c_\rho - c'_\rho \right| \leq \frac{\Delta c}{2}$$

must be fulfilled, and our absorbing atom is called an eligible receiver. Obviously, the time required for a photon to skip from one atom to another is so short that it may, for our model, be considered an instantaneous process. We may, therefore, neglect the distance travelled by the emitting atom during this process. But, more important, we may associate each photon in the gas with some atom in the gas, and look for the distribution $p(\vec{r}, \vec{c}, t)$ of excited atoms.

Assuming that occulting is negligible, the probability that a photon, emitted along $\vec{\rho}$ from an atom with velocity $\vec{c}$, will be absorbed in a length $d\rho$ will be B times the number of atoms eligible to absorb that photon in a volume element $1 \times 1 \times d\rho$.

The number of atoms capable of absorbing the photon is included in the number of unexcited atoms but since $p \ll f$ we may consider that f is the distribution function for just the unexcited atoms, i.e.

2.3 $$f \cong f - p \quad .$$

Hence, the number of atoms eligible to absorb the photon in a volume element $1 \times 1 \times d\rho$ will be

2.4 $$\int_{x}^{x+1}\int_{y}^{y+1}\int_{\rho}^{\rho+d\rho}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{c_\rho-\frac{\Delta c}{2}}^{c_\rho+\frac{\Delta c}{2}} f(\vec{r},\vec{c},t)\,dW\,dV\,dU\right\}dz\,dy\,dx$$

where the x and y directions are chosen as two orthogonal directions, both normal to $\vec{\rho}$, and z is taken along $\vec{\rho}$; and where f(x, y, z, U, V, W, t) takes on all position coordinates within the space element and all values of velocity components $-\infty$ to ∞ in the x and y directions but is limited to $c_\rho \pm \frac{\Delta c}{2}$ in the z direction.

Now, since $f(\vec{r}, \vec{c}, t)$ is uniform in space, the integrations over space variables amount to multiplying by $1 \times 1 \times d\rho$.

Since $\Delta c/2$ is very small it may be assumed that the variations in $f(\vec{r}, \vec{c}, t)$ as W varies from $c_\rho - \Delta c/2$ to $c_\rho + \Delta c/2$ are small and that the function $f(x, y, z, U, V, c_\rho, t)$ may be used instead. Hence, the integration over W amounts to multiplying by Δc.

Therefore, the number of atoms capable of absorbing the photon is

2.5 $$1 \times 1 \times d\rho \times \Delta c \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, z, U, V, c\ , t)\,dV\,dU$$

and the probability per unit length of absorption may be expressed as

2.6 $$B \Delta c \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y, z, U, V, c_\rho)\,dV\,dU.$$

Now,

2.7 $$f = n\left(\frac{m}{2\pi kt}\right)^{3/2} e^{-mc^2/2kT}$$

for a Maxwellian distribution where $c^2 = U^2 + V^2 + W^2$ and the conventional symbols n, m, k, T are used for gas density, atomic mass, Boltzmann constant and temperature respectively.

On substituting in the double integral, where $W = c_\rho$, we find $Q(c_\rho)$, the probability of absorption per unit length of path of a photon emitted from an atom with velocity component c_ρ in the direction of emission, to be

2.8 $$Q(c_\rho) = nB\Delta c\left(\frac{m}{2\pi kt}\right)^{1/2} e^{-mc^2/2kT}.$$

The probability $P(\rho, c_\rho)$ that a photon emitted from an atom with a velocity c_ρ along the line-of-sight of the photon travels at least a distance ρ before recapture is[1]

2.9 $$P(\rho, c_\rho) = e^{-nB\Delta c\,(m/2\pi kT)^{1/2}\rho\, e^{-mc_\rho^2/2kT}}$$

Now, suppose we wish to determine what happens to our function $p(\vec{r}, \vec{c}, t)$ in time.

The explicit time dependence depends upon three processes:

1. Spontaneous emission.

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1. J. H. Jeans, The Dynamical Theory of Gases, (Cambridge University Press, 1921).

2. Loss and gain of members caused by movement into and out of our space coordinate ranges by excited atoms with the required velocity coordinates.

3. Absorption by unexcited atoms of photons emitted in other regions of the gas.

The term introduced by (1) above is simply $-ap(\vec{r}, \vec{c}, t)$.

The term introduced by (2) above is the usual second term in the Boltzmann Equation, i.e. $-\vec{c} \cdot \nabla_r \, p(\vec{r}, \vec{c}, t)$.

This term has been neglected in previous work, though a need for it has been pointed out by R. G. Fowler, who noted that in many cases the mean half life of an atom multiplied by the mean speed of an atom yields a distance greater than the expected value of the skip distance of a photon in flight. Therefore, the displacement of excitation energy while in captivity becomes the most important mechanism in diffusion.

We neglect the effect of collisions since the mean free path of an atom is much greater than the displacement of an atom while in the excited state and, hence, a photon will almost always escape before its carrier suffers collision. Of course, for denser gases ($n > 10^{18}$) this assumption will not hold and the Boltzmann exchange integral would have to be added to our equation.

In determining the form of (3) above, we wish to see what contributions to $p(\vec{r}, \vec{c}, t)$ are made by absorption of photons emitted throughout the region.

Let $\vec{r}\,'$ be any point in our region at which an emission might take place. Let us first look for the number of photons per unit volume emitted in the vicinity of $\vec{r}\,'$ which could be absorbed at $\vec{r}$ by those atoms having a velocity component c_ρ' within $\pm\frac{\Delta c}{2}$ of c_ρ. More precisely, we may consider that the number of atoms about $\vec{r}\,'$ capable of emitting photons which could, if emitted in the proper direction, be absorbed in the neighborhood of $\vec{r}$ by atoms whose velocities have the range $c_\rho \pm \frac{\Delta c}{2}$ is just the number of excited atoms in the neighborhood of $\vec{r}\,'$ whose velocity component in the direction $\vec{r} - \vec{r}\,'$ is also $c_\rho \pm \frac{\Delta c}{2}$.

Hence, the number per unit volume of excited atoms capable of emitting such a photon is

2.10 $$\oint p(\vec{r}\,', \vec{c}\,', t)\, d\tau_{v'} ,$$

where the integration is over velocity space $-\infty$ to $+\infty$ in two orthogonal directions both normal to $\vec{r} - \vec{r}\,'$ and over $c_\rho \pm \frac{\Delta c}{2}$ in the direction $\vec{r} - \vec{r}\,'$. The last integration is equivalent to multiplying by Δc and the integral becomes

2.11 $$\Delta c \oiint_{S_{v'}} p(\vec{r}\,', \vec{c}\,', t)\, dS_{v'} ,$$

where the integration is over a plane in velocity space normal to $\vec{\rho}$ and from the origin of velocity coordinates, displaced a distance c_ρ.

The number of photons emitted per unit time per unit volume about $\vec{r}\,'$ capable of being absorbed in the required velocity-component range about $\vec{r}$ is, then,

2.12 $$\frac{a\, \Delta c \oiint_{S_{v'}} p(\vec{r}\,', \vec{c}\,', t)\, dS_{v'}}{4\pi\rho^2}$$

where

2.13 $$\rho = |\vec{r} - \vec{r}\,'|$$

and the restriction on $\vec{c}'$ that it also has a component along $\vec{\rho}$ between $c_\rho \pm \frac{\Delta c}{2}$.

The number of photons that actually do cross a unit area about $\vec{r}$ normal to $\vec{\rho}$ is the above number multiplied by $P(\rho, c_\rho)$, or

2.14 $$\frac{P(\rho, c_\rho)\, a \Delta c \oint\!\!\oint_{S_{V'}} p(x', y', z', U', V', c_\rho)\, dS_{V'}}{4\pi\rho^2}.$$

Next, the fraction of these absorbed per unit volume about $\vec{r}$, $\vec{c}$ in a velocity range $d\vec{c} = dUdVdW$ such that $\vec{c}$ has component c_ρ along $\vec{\rho}$ is $B\, f(\vec{r}, \vec{c}, t)\, d\vec{r} d\vec{c}$. Therefore, the total contribution of (3) to the explicit time dependence of $p(\vec{r}, \vec{c}, t)$ is

2.15 $$\oint_{R'} \frac{a \Delta c\, B\, f(\vec{r}, \vec{c}, t)\, P(\rho, c_\rho) \oint\!\!\oint_{S_V} p(\vec{r}', \vec{c}', t)\, dS_{V'}}{4\pi\rho^2}$$

since obviously every point in the region is a point at which an emission can occur.

Therefore, adding the three effects and transposing the first two to the left side of the equation, we arrive at our complete equation,

2.16 $$\frac{\partial p(\vec{r}, \vec{c}, t)}{\partial t} + \vec{c}\cdot\nabla_r\, p(\vec{r}, \vec{c}, t) + a p(\vec{r}, \vec{c}, t) =$$

$$\frac{a\Delta c B}{4\pi} \oint_{R'} \frac{f(\vec{r}, \vec{c}, t) P(\rho, c_\rho)}{\rho^2} \oint\!\!\oint_{S_{V'}} p(\vec{r}', \vec{c}', t)\, dS_{V'}\, d\tau_{R'} .$$

Finally, on substituting for $f(\vec{r}, \vec{c}, t)$ and $P(\rho, c_\rho)$, the modified Boltzmann Equation for the problem takes the explicit form:

2.17 $$\partial p(\vec{r}, \vec{c}, t) + c\cdot\nabla_r\, p(\vec{r}, \vec{c}, t) + a p(\vec{r}, \vec{c}, t) = \frac{a\Delta c B}{4\pi} \oint_{R'} n\left(\frac{m}{2\pi kT}\right)^{3/2} \cdot$$

$$e^{-\frac{mc^2}{2kT} - nB\Delta c\left(\frac{m}{2\pi kT}\right)^{1/2} \rho\, e^{-\frac{mc_\rho^2}{2kT}}} \oint\!\!\oint_{S_{V'}} p(\vec{r}', \vec{c}', t)\, dS_{V'}\, d\tau_{R'} .$$

CHAPTER III

APPLICATION OF THE EQUATION TO AN INFINITE SLAB

Now let us apply our equation to the case of an infinite slab such that

3.1
$$\begin{gathered} -L \leq x \leq L \\ -\infty \leq y \leq \infty \\ -\infty \leq z \leq \infty \end{gathered}$$

Since $p(\vec{r}, \vec{c}, t)$ and $f(\vec{r}, \vec{c}, t)$ do not depend upon y and z, we might just as well look for $p(x, o, o, U, V, W, t)$ and denote it $p(x, \vec{c}, t)$.

Also,

3.2
$$\frac{\partial p}{\partial y}(x, \vec{c}, t) = \frac{\partial p}{\partial z}(x, \vec{c}, t) = 0 ,$$

hence, $\vec{c} \cdot \nabla_r \, p(x, \vec{c}, t)$ becomes $U \frac{\partial p}{\partial x}(x, \vec{c}, t)$ and our equation becomes

3.3
$$\frac{\partial p(x,\vec{c},t)}{\partial t} + U\frac{\partial p(x,\vec{c},t)}{\partial x} + a\,p(x,\vec{c},t) = \frac{a \Delta c B n}{4\pi}\left(\frac{m}{2\pi kT}\right)^{3/2} \oint_{R'} \frac{1}{\rho^2} \cdot$$

$$e^{-\frac{mc^2}{2kT} - n\left(\frac{m}{2\pi kT}\right)^{1/2} \rho B \Delta c \, e^{-\frac{mc_R^2}{2kT}}} \iint_{S_V'} p(x', \vec{c}', t)\, dS_V' \, d\tau_R'$$

which we now wish to simplify by integrating over the $S_{V'}$ plane.

We assume that $p(x, \vec{c}, t)$, being Maxwellian in velocity initially, departs from this type distribution so much more rapidly in the x direction than in the y and z directions that it remains Maxwellian in velocity in the y and z directions during the time of interest to us. Hence, $p(x, c, t)$ takes the form

3.4
$$F(x, U, t)e^{-m(V^2+ W^2)/2kT}$$

and since we are interested in seeing how $F(x, \vec{c}, t)$ departs from a Maxwellian form in time we let $F(x, U, t)$ take the form

3.5 $$F(x, U, t) = G(x, U, t)e^{-mU^2/2kT}$$

and since $c^2 = U^2 + V^2 + W^2$ we find

3.6 $$p(x, \vec{c}, t) = G(x, U, t)e^{-mc^2/2kT}.$$

In performing the integration over $S_{V'}$, we must make some assumption as to the form of $G(x, U, t)$. The simplest assumption is a Taylor's expansion in powers of U. Any other expansion, such as Hermite Polynomials, would lead to extreme complications in rotating coordinates although some simplification in evaluating coefficients could be gained by virtue of their orthogonality. Hence, we assume

3.7 $$G(x, U, t) = n'(m/2\pi kT)^{3/2}(\alpha(x, t) + \beta(x, t)U + \gamma(x, t)U^2),$$

where n' is the initial density of excited gas.

In rotating our primed velocity coordinates, we wish one axis to have the direction of $\vec{\rho}$, hence its direction cosines are $(x'-x)/\rho$, y'/ρ, z'/ρ. Since U' will have no component along an axis paralell to the yz plane, we make this requirement of our V" axis, and we find its direction cosines to be

$$0, \quad \frac{-z'}{(y'^2 + z'^2)^{1/2}}, \quad \frac{y'}{(y'^2 + z'^2)^{1/2}}$$

and our third axis then has direction cosines

$$\frac{y'^2 + z'^2}{\rho(y'^2 + z'^2)^{1/2}}, \quad \frac{-y'(x'-x)}{\rho(y'^2 + z'^2)^{1/2}}, \quad \frac{-z'(x'-x)}{\rho(y'^2 + z'^2)^{1/2}}.$$

Obviously, the direction cosines of the x axis in terms of the new coordinates are

$$\frac{x'-x}{\rho}\,,\quad 0\,,\quad \frac{(y'^2+z'^2)^{1/2}}{\rho}.$$

Now we require that the component of $\vec{c}\,'$ along $\vec{\rho}$ be c_ρ and this is just U", i.e.

3.8 $$U'' = c_\rho\,.$$

So we have

3.9 $$U' = \frac{x'-x}{\rho}\,c_\rho + \frac{(y'^2+z'^2)^{1/2}}{\rho}\,W''\,,$$

and

3.10 $$c'^2 = c_\rho^2 + V''^2 + W''^2\,.$$

Substituting for U' in G(x', U', t), we have

3.11 $$G(x', U', t) = n'\left(\frac{m}{2\pi kT}\right)^{3/2}\Big\{\Big(\alpha(x', t) + \beta(x', t)\frac{(x'-x)}{\rho}\,c_\rho + \gamma(x', t)\frac{(x'-x)^2}{\rho^2}\,c_\rho^2\Big) + \Big(\beta(x', t)\frac{(y'^2+z'^2)^{1/2}}{\rho} + 2\gamma(x', t)\frac{(x'-x)(y'^2+z'^2)^{1/2}}{\rho^2}\Big)W'' + \gamma(x', t)\frac{(y'^2+z'^2)}{\rho^2}\,W''^2\Big\}.$$

Substituting for $p(x', \vec{c}\,', t)$, our equation becomes

3.12 $$\frac{\partial p(x,\vec{c},t)}{\partial t} + U\frac{\partial p(x,\vec{c},t)}{\partial x} + a\,p(x,\vec{c},t) = n'\frac{a_{\Delta}cB}{4\pi}\,n\left(\frac{m}{2\pi kT}\right)^{3/2}\oint_{R'}\Big[\frac{1}{\rho^2}e^{-\frac{mc^2}{2kT}} - nB_{\Delta}c\left(\frac{m}{2\pi kT}\right)^{1/2}\rho\, e^{-\frac{mc_\rho^2}{2kT}}$$

$$\cdot \iint \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m(c_p^2+V''^2+W''^2)/2kT} \Big\{ \Big(\alpha(x',t)$$

$$+\beta(x',t)\frac{(x'-x)}{\rho}c_p + \gamma(x',t)\frac{(x'-x)^2}{\rho^2}c_p^2\Big) + \Big(\beta(x',t)\frac{(y'^2+z'^2)^{1/2}}{\rho}$$

$$+2\gamma(x',t)\frac{(x'-x)(y'^2+z'^2)^{1/2}}{\rho^2}\Big)W'' + \gamma(x',t)\frac{(y'^2+z'^2)}{\rho^2}\Big)W''^2\Big\} dW''dV'd\tau_{R'}.$$

The second term above is an odd function in W" and substituting limits $-\infty < W'' < \infty$ reduces it to zero. We may then rewrite the $S_{V'}$ integral as

3.13
$$\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mc_p^2/2kT} \Big\{\Big(\alpha(x',t)+\beta(x',t)\frac{(x'-x)}{\rho}c_p + \gamma(x',t)\frac{(x'-x)^2}{\rho^2}c_p^2\Big)\cdot$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-m(V''^2+W''^2)}{2kT}} dW''dV' + \gamma(x',t)(y'^2+z'^2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W''^2 e^{\frac{-m(V''^2+W''^2)}{2kT}} dW''dV''$$

The two integrals above are evaluated as $2\pi kT/m$ and $(kT/m)/(2\pi kT/m)$ respectively. So our equation reduces to

3.14
$$\frac{\partial p(x,\vec{c},t)}{\partial t} + U\frac{\partial p(x,\vec{c},t)}{\partial x} + ap(x,\vec{c},t) =$$

$$n'\frac{a\Delta cB}{4\pi}n\left(\frac{m}{2\pi kT}\right)^{3/2}\oint_{R'}\frac{1}{\rho^2}\, e^{-\frac{mc_p^2}{2kT}-\frac{mc^2}{2kT}-n\Delta cB\left(\frac{m}{2\pi kT}\right)^{1/2}\rho\, e^{\frac{-mc_p^2}{2kT}}}$$

$$\left(\frac{m}{2\pi kT}\right)^{1/2}\Big\{\alpha(x',t)+\beta(x',t)\frac{(x'-x)}{\rho}c_p + \gamma(x',t)\Big[\frac{(x'-x)^2}{\rho^2}c_p^2 + \frac{kT}{m}\frac{(y'^2+z'^2)}{\rho^2}\Big]\Big\}d\tau_{R'}.$$

Now we may substitute for $p(x, \vec{c}, t)$. To find three equations in the three unknowns $\alpha(x, t)$, $\beta(x, t)$ and $\gamma(x,t)$, we multiply the above equation through by 1, U, and U^2 and each time integrate over velocity space, this time in the unprimed coordinates.

CHAPTER IV

DERIVATION OF EQUATIONS FOR THE FOURIER COEFFICIENTS $\alpha(x, t)$, $\beta(x, t)$ and $\gamma(x, t)$

To arrive at our first of three equations in $\alpha(x, t)$, $\beta(x, t)$ and $\gamma(x, t)$, we integrate over all velocity space term by term in our equation 3.14. Let us first look at the integral over the left-hand side of the equation, i.e.

$$4.1 \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\partial p(x,\vec{c},t)}{\partial t} + U\frac{\partial p(x,\vec{c},t)}{\partial x} + a p(x,\vec{c},t)\right] dW dV dU$$

or, since the integrations are not over x or t,

$$4.2 \qquad \frac{\partial}{\partial x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U p(x,\vec{c},t)\, dW dV dU$$

$$+ \left(\frac{\partial}{\partial t} + a\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,\vec{c},t)\, dW dV dU$$

where $p(x, \vec{c}, t)$ has the form

$$4.3 \qquad p(x,\vec{c},t) = n'\left(\frac{m}{2\pi kT}\right)^{3/2}\left\{\alpha(x,t) + \beta(x,t)\,U + \gamma(x,t)\,U^2\right\} e^{-\frac{mc^2}{2kT}}.$$

In integrating, we will need the integrals of $U^i e^{-mU^2/2kT}$ $i = 0,1\ldots,5.$ The integrals over V and W will in every case yield

$$4\int_0^{\infty} e^{-\frac{mV^2}{2kT}}\, dV \int_0^{\infty} e^{-\frac{mW^2}{2kT}}\, dW$$

or

$$\frac{2\pi kT}{m}.$$

Obviously, integrations where the power of U is odd will vanish. Knowing

4.4 $$\int_{-\infty}^{\infty} e^{-\frac{mU^2}{2kT}} = \left(\frac{2\pi kT}{m}\right)^{1/2},$$

4.5 $$\int_{-\infty}^{\infty} U^2 e^{-\frac{mU^2}{2kT}} = \frac{kT}{m}\left(\frac{2\pi kT}{m}\right)^{1/2}$$

and

4.6 $$\int_{-\infty}^{\infty} U^4 e^{-\frac{mU^2}{2kT}} = 3\left(\frac{kT}{m}\right)^2\left(\frac{2\pi kT}{m}\right)^{1/2}$$

we immediately derive the formulas

4.7 $$\oint_{R_V} p\, d\tau_V = n'\left\{\alpha(x,t) + \frac{kT}{m}\gamma(x,t)\right\},$$

4.8 $$\oint_{R_V} Up\, d\tau_V = n'\left\{\frac{kT}{m}\beta(x,t)\right\},$$

4.9 $$\oint_{R_V} U^2 p\, d\tau_V = n'\left\{\frac{kT}{m}\alpha(x,t) + 3\left(\frac{kT}{m}\right)^2\gamma(x,t)\right\}$$

and

4.10 $$\oint_{R_V} U^3 p\, d\tau_V = n'\left\{3\left(\frac{kT}{m}\right)^2\beta(x,t)\right\}$$

So the left side of our first equation becomes

4.11 $$n'\left[\frac{\partial\alpha}{\partial t}(x,t) + \frac{kT}{m}\frac{\partial\gamma}{\partial t}(x,t) + \frac{kT}{m}\frac{\partial\beta}{\partial x}(x,t) + a\,\alpha(x,t) + a\frac{kT}{m}\gamma(x,t)\right].$$

Since multiplying through in the second case by U gives an equation similar to (4.2), the integration may be written down immediately as

4.12 $$n'\left[\frac{kT}{m}\frac{\partial\beta}{\partial t}(x,t) + \frac{kT}{m}\frac{\partial\alpha}{\partial x}(x,t) + 3\left(\frac{kT}{m}\right)^2\frac{\partial\gamma}{\partial x}(x,t) + a\frac{kT}{m}\beta(x,t)\right]$$

and similarly the left side of our third equation, obtained by multiplying through by U^2 and integrating, becomes

4.13 $$n'\left[\frac{kT}{m}\frac{\partial\alpha}{\partial t}(x,t) + 3\left(\frac{kT}{m}\right)^2\frac{\partial\gamma}{\partial t}(x,t) + 3\left(\frac{kT}{m}\right)^2\frac{\partial\beta}{\partial x}(x,t) + a\frac{kT}{m}\alpha(x,t) + 3a\left(\frac{kT}{m}\right)^2\gamma(x,t)\right].$$

The right-hand side of our first equation is equal to

4.14
$$\frac{n' a \Delta c B n}{4\pi}\left(\frac{m}{2\pi kT}\right)^{3/2}\oint_{R'}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\rho^2}e^{-\frac{mc_\rho^2}{2kT}-\frac{mc^2}{2kT}-n\Delta cB\left(\frac{m}{2\pi kT}\right)^{1/2}\rho e^{-\frac{mc_\rho^2}{2kT}}}\,\cdot$$
$$\left(\frac{m}{2\pi kT}\right)^{1/2}\left\{\alpha(x',t)+\beta(x',t)\frac{(x'-x)}{\rho}c_\rho+\left[\frac{(x'-x)^2}{\rho^2}c_\rho^2+\frac{kT}{m}\frac{(y'^2+z'^2)}{\rho^2}\right]\gamma(x',t)\right\}d\tau_{R'}\,d\tau_v$$

Now, to integrate over velocity space, let us transform to spherical coordinates with the axis along $\vec{\rho}$ where the new variables are θ, the polar angle; φ, the azimuth angle; and c, the speed. Our Jacobian for such a transformation is $c^2\sin\theta$. Under the integration there are only two symbols appearing, namely c and c_ρ. Since $c_\rho = c\cos\theta$ the right-hand side transforms to $nn'\frac{a\Delta cB}{4}\left(\frac{m}{2\pi kT}\right)^2$ times

4.15
$$\oint_{R'}\frac{1}{\rho^2}\left[\alpha(x',t)+\frac{kT}{m}\frac{(y'^2+z'^2)}{\rho^2}\gamma(x',t)\right]\int_0^\infty\int_0^\pi\int_0^{2\pi}c^2\sin\theta\,\cdot$$
$$e^{-\frac{mc^2}{2kT}-\frac{mc^2}{2kT}\cos^2\theta-n\Delta cB\left(\frac{m}{2\pi kT}\right)^{1/2}\rho e^{-\frac{mc^2\cos^2\theta}{2kT}}}\,d\varphi\,d\theta\,dc\,d\tau_{R'}$$

$$+\oint_{R'}\frac{1}{\rho^2}\beta(x',t)\frac{(x'-x)}{\rho}\int_0^\infty\int_0^\pi\int_0^{2\pi}c^3\cos\theta\sin\theta\,\cdot$$
$$e^{-\frac{mc^2}{2kT}-\frac{mc^2}{2kT}\cos^2\theta-n\Delta cB\left(\frac{m}{2\pi kT}\right)^{1/2}\rho e^{-\frac{mc^2\cos^2\theta}{2kT}}}\,d\varphi\,d\theta\,dc\,d\tau_{R'}$$

$$+\oint_{R'}\frac{1}{\rho^2}\frac{(x'-x)^2}{\rho^2}\gamma(x,t)\int_0^\infty\int_0^\pi\int_0^{2\pi}c^4\cos^2\theta\sin\theta\,\cdot$$
$$e^{-\frac{mc^2}{2kT}-\frac{mc^2}{2kT}\cos^2\theta-n\Delta cB\left(\frac{m}{2\pi kT}\right)^{1/2}\rho e^{-\frac{mc^2\cos^2\theta}{2kT}}}\,d\varphi\,d\theta\,dc\,d\tau_{R'}.$$

The second term, being odd in θ about $\theta = \pi/2$, is zero. Integration yields simply a factor 2π over φ in the first and third terms of (4.15).

Letting $M = n\Delta cB\left(\frac{m}{2\pi kT}\right)^{1/2}$, we may write the right-hand side of our first equation as

4.16
$$nn'\frac{a\Delta cB}{2}\left(\frac{m}{2\pi kT}\right)^{2}\left\{\oint_{R'}\frac{1}{\rho^{2}}\left[\alpha(x,t)+\frac{kT}{m}(y'^{2}+z'^{2})\gamma(x,'t)\right]\mathcal{J}_1\,d\tau_{R'}\right.$$
$$\left.+\oint_{R'}\frac{1}{\rho^{2}}\frac{(x'-x)^{2}}{\rho^{2}}\gamma(x,t)\,\mathcal{J}_3\,d\tau_{R'}\right.$$

where

4.17
$$\mathcal{J}_1 = \int_0^{\infty}\int_0^{\pi} c^{2}\sin\theta\, e^{-\frac{mc^{2}}{2kT}-\frac{mc^{2}}{2kT}\cos^{2}\theta-\rho M e^{-\frac{mc^{2}\cos^{2}\theta}{2kT}}}\,d\theta dc$$

and

4.18
$$\mathcal{J}_3 = \int_0^{\infty}\int_0^{\pi} c^{4}\sin\theta\cos^{2}\theta\, e^{-\frac{mc^{2}}{2kT}-\frac{mc^{2}}{2kT}\cos^{2}\theta-\rho M e^{-\frac{mc^{2}\cos^{2}\theta}{2kT}}}\,d\theta dc\,.$$

In evaluating $\mathcal{J}_1$ and $\mathcal{J}_3$, let us expand the term

$$e^{-\rho M e^{-\frac{mc^{2}}{2kT}\cos^{2}\theta}}$$

as e^{-x}. $\mathcal{J}_1$ then becomes

4.19
$$\mathcal{J}_1 = 2\int_0^{\pi/2}\sin\theta\int_0^{\infty} c^{2}\, e^{-\frac{mc^{2}}{2kT}(1+\cos^{2}\theta)}\left\{1-\rho M e^{-\frac{mc^{2}}{2kT}\cos^{2}\theta}\right.$$
$$\left.+\frac{\rho^{2}M^{2}}{2}e^{-2\frac{mc^{2}}{2kT}\cos^{2}\theta}-\frac{\rho^{3}M^{3}}{3!}e^{-3\frac{mc^{2}}{2kT}\cos^{2}\theta}\cdots\right\}dc\,d\theta$$

and similarly

4.20
$$\mathcal{J}_3 = 2\int_0^{\pi/2}\sin\theta\cos^{2}\theta\int_0^{\infty} c^{4} e^{-\frac{mc^{2}}{2kT}(1+\cos^{2}\theta)}\left\{1-\rho M e^{-\frac{mc^{2}}{2kT}\cos^{2}\theta}\right.$$

$$+\frac{\rho^2 M^2}{2} e^{-\frac{2mc^2\cos^2\theta}{2kT}} - \frac{\rho^3 M^3}{3!} e^{-\frac{3mc^2\cos^2\theta}{2kT}} \ldots \Big\} dc\, d\theta.$$

Integrating first with respect to c, after simplifying by collecting exponents of e, $\mathcal{J}_1$ becomes

4.21
$$\mathcal{J}_1 = 2\int_0^{\pi/2} \sin\theta \, \frac{\sqrt{\pi}}{2} \cdot \frac{1}{2} \cdot \left(\frac{2kT}{m}\right)^{3/2} \Big\{ \frac{1}{(1+\cos^2\theta)^{3/2}}$$

$$\frac{-\rho M}{(1+2\cos^2\theta)^{3/2}} + \frac{\rho^2 M^2}{2(1+3\cos^2\theta)^{3/2}} - \frac{\rho^3 M^3}{3!(1+4\cos^2\theta)^{3/2}} \ldots \Big\} d\theta$$

which easily integrates to

4.22
$$\mathcal{J}_1 = \frac{1}{2\pi}\left(\frac{2\pi kT}{m}\right)^{3/2} \Big\{ \frac{1}{2^{1/2}} - \frac{\rho M}{3^{1/2}} + \frac{\rho^2 M^2}{2\cdot 4^{1/2}} - \frac{\rho^3 M^3}{3!\, 5^{1/2}} \cdots \Big\}$$

and similarly $\mathcal{J}_3$ integrates to

4.23
$$\mathcal{J}_3 = \frac{1}{2\pi}\left(\frac{kT}{m}\right)\left(\frac{2\pi kT}{m}\right)^{3/2} \Big\{ \frac{1}{2^{3/2}} - \frac{\rho M}{3^{3/2}} + \frac{\rho^2 M^2}{2\cdot 4^{3/2}} - \frac{\rho^3 M^3}{3!\, 5^{3/2}} \cdots \Big\}$$

and our complete first equation becomes, dividing out n',

4.24
$$\frac{\partial\alpha(x,t)}{\partial t} + \frac{kT}{m}\frac{\partial\gamma(x,t)}{\partial t} + \frac{kT}{m}\frac{\partial\beta(x,t)}{\partial x} + a\,\alpha(x,t) + a\frac{kT}{m}\gamma(x,t) =$$

$$\frac{Ma}{4\pi}\Big\{ \oint_{R'} \frac{1}{\rho^2}\left[\alpha(x',t) + \frac{kT}{m}\frac{(y'^2+z'^2)}{\rho^2}\gamma(x',t)\right]\left[\frac{1}{\sqrt{2}} - \frac{\rho M}{\sqrt{3}} + \frac{\rho^2 M^2}{2\sqrt{4}} - \frac{\rho^3 M^3}{3!\sqrt{5}} \ldots\right] d\tau_{R'}$$

$$+ \frac{kT}{m}\oint_{R'} \frac{1}{\rho^2}\frac{(x'-x)^2}{\rho^2}\gamma(x',t)\left[\frac{1}{2^{3/2}} - \frac{\rho M}{3^{3/2}} + \frac{\rho^2 M^2}{2\cdot 4^{3/2}} - \frac{\rho^3 M^3}{3!\, 5^{3/2}} \ldots\right] d\tau_{R'}\Big\}.$$

In evaluating the right-hand side of the second equation, obtained by multiplying the right-hand side of our original equation by U and integrating over all velocity space, let us perform two transformations of velocity coordinates.

First let us perform a rigid rotation as before when we integrated over velocity space in the primed coordinates. The transformation to these coordinates will transform U to $\frac{(x'-x)}{\rho}c_\rho + \frac{(y'^2+z'^2)^{1/2}}{\rho}W'''$. Since our W''' axis is perpendicular to $\vec{\rho}$, we can see immediately that the second term above will introduce an integral odd in W''' and hence may be dropped, since U appears linearly in the integrand. Since our integrand will now have terms only in c and $c_\rho^2 = c^2\cos^2\theta$ as before, we may transform to spherical coordinates and arrive at the expression for the right-hand side of the second equation,

4.25
$$n'n\frac{a\Delta cB}{4\pi}\left(\frac{m}{2\pi kT}\right)^2\Bigg\{\oint_{R'}\frac{(x'-x)}{\rho^3}\left[\alpha(x',t)+\frac{kT}{m}\frac{(y'^2+z'^2)}{\rho^2}\gamma(x',t)\right]\cdot$$
$$\int_0^\infty\int_0^\pi\int_0^{2\pi} c^3\cos\theta\sin\theta\; e^{-\frac{mc^2}{2kT}-\frac{mc^2\cos^2\theta}{2kT}}\cdot$$
$$e^{-\rho M e^{-\frac{mc^2}{2kT}\cos^2\theta}}\,d\varphi\, d\theta\, dc\, d\tau_{R'}$$

$$+\oint_{R'}\frac{(x'-x)}{\rho^3}\left[\beta(x',t)\frac{(x'-x)}{\rho}\right]\int_0^\infty\int_0^\pi\int_0^{2\pi} c^4\cos^2\theta\sin\theta\;\cdot$$
$$e^{-\frac{mc^2}{2kT}(1+\cos^2\theta)-\rho M e^{-\frac{mc^2}{2kT}\cos^2\theta}}\,d\varphi\, d\theta\, dc\, d\tau_{R'}$$

$$+\oint_{R'}\frac{(x'-x)}{\rho^3}\left[\gamma(x',t)\frac{(x'-x)}{\rho}\right]\int_0^\infty\int_0^\pi\int_0^{2\pi} c^5\cos^3\theta\sin\theta\;\cdot$$
$$e^{-\frac{mc^2}{2kT}(1+\cos^2\theta)-\rho M e^{-\frac{mc^2}{2kT}\cos^2\theta}}\,d\varphi\, d\theta\, dc\, d\tau_{R'}.$$

In this case the first and third integrals vanish, being odd about $\pi/2$. The triple integral over velocity space can be recognized as $2\pi\mathcal{I}_3$, evaluated previously, hence our complete second equation becomes, on dividing through by $n'kT/m$,

4.26
$$\frac{\partial\alpha}{\partial x}(x,t) + \frac{3kT}{m}\frac{\partial\gamma}{\partial x}(x,t) + \frac{\partial\beta}{\partial t}(x,t) + a\beta(x,t) =$$
$$\frac{aM}{4\pi}\oint_{R'} \frac{1}{\rho^2}\frac{(x'-x)^2}{\rho^2}\beta(x',t)\left[\frac{1}{2^{3/2}} - \frac{\rho M}{3^{3/2}} + \frac{\rho^2 M^2}{2\cdot 4^{3/2}} - \frac{\rho^3 M^3}{3!5^{3/2}}\cdots\right]d\tau_{R'}.$$

Since the right-hand side of the third equation is obtained by multiplying our original equation by U^2 and integrating over all velocity space, the integrand will contain, after a transformation by rotation as in our second equation,

4.27
$$U^2 = \frac{(x'-x)^2}{\rho^2}c_\rho^2 + 2\frac{(x'-x)(y'^2+z'^2)^{1/2}}{\rho^2}W'''c_\rho + \frac{(y'^2+z'^2)}{\rho^2}W'''^2 .$$

The second term, linear in W''', yields an integrand odd in W''' and, therefore, drops out. The right-hand side of the third equation then becomes, after rearranging terms, knowing $c^2 = U'''^2 + V'''^2 + W'''^2$ and $U''' = c_\rho$,

4.28
$$n'n\frac{a\Delta cB}{4\pi}\left(\frac{m}{2\pi kT}\right)^2\oint_{R'}\int_{-\infty}^{\infty} e^{-\frac{mc_\rho^2}{kT}-\rho M e^{-\frac{mc_\rho^2}{2kT}\rho}}\Big\{\alpha(x',t)$$
$$+\beta(x',t)\frac{(x'-x)}{\rho}c_\rho + \left[\frac{(x'-x)^2}{\rho^2}c_\rho^2 + \frac{kT}{m}\frac{(y'^2+z'^2)}{\rho^2}\right]\gamma(x',t)\Big\}\cdot$$
$$\left\{\frac{(x'-x)^2}{\rho^2}c_\rho^2\int_{-\infty}^{\infty} e^{-\frac{mW'''^2}{2kT}}dW''' + \frac{(y'^2+z'^2)}{\rho^2}\int_{-\infty}^{\infty} W'''e^{-\frac{mW'''^2}{2kT}}dW'''\right\}\cdot$$
$$\int_{-\infty}^{\infty} e^{-\frac{mV'''^2}{2kT}}dV'''\,dc_\rho\,d\tau_{R'} .$$

Only one term contains an odd power of c_ρ. Since this will yield an odd integrand in c_ρ, it may be dropped. Performing the integrations over V''' and W''', the expression becomes

$$4.29 \qquad nn'a\Delta cB\left(\frac{m}{2\pi kT}\right)\oint_{R'}\frac{1}{\rho^2}\left\{\left[\frac{kT}{m}\frac{(y'^2+z'^2)}{\rho^2}\alpha(x',t)\right.\right.$$

$$+\left(\frac{kT}{m}\right)^2\frac{(y'^2+z'^2)^2}{\rho^4}\gamma(x',t)\Big]J_1 \;+\Big[\frac{(x'-x)^2}{\rho^2}\alpha(x',t)$$

$$\left.+2\frac{kT}{m}\frac{(x'-x)^2(y'^2+z'^2)}{\rho^4}\gamma(x',t)\right]J_2 + \frac{(x'-x)^4}{\rho^4}\gamma(x',t)\,J_3\Big\}\,d\tau_{R'}$$

where

$$4.30 \qquad J_1 = \int_{-\infty}^{\infty} e^{-\frac{mc_\rho^2}{kT}-\rho M e^{-\frac{mc_\rho^2}{2kT}}}\,dc_\rho\,,$$

$$4.31 \qquad J_2 = \int_{-\infty}^{\infty} c_\rho^2\, e^{-\frac{mc_\rho^2}{kT}-\rho M e^{-\frac{mc_\rho^2}{2kT}}}\,dc_\rho$$

and

$$4.32 \qquad J_3 = \int_{-\infty}^{\infty} c_\rho^4\, e^{-\frac{mc_\rho^2}{kT}-\rho M e^{-\frac{mc_\rho^2}{2kT}}}\,dc_\rho\,.$$

The above three integrals are evaluated in the same manner as $\mathcal{I}_1$ as

$$4.33 \qquad J_1 = \left(\frac{2\pi kT}{m}\right)^{1/2}\left\{\frac{1}{\sqrt{2}}-\frac{\rho M}{\sqrt{3}}+\frac{\rho^2M^2}{2\sqrt{4}}-\frac{\rho^3M^3}{3!\sqrt{5}}\cdots\right\},$$

$$4.34 \qquad J_2 = \left(\frac{2\pi kT}{m}\right)^{1/2}\frac{kT}{m}\left\{\frac{1}{2^{3/2}}-\frac{\rho M}{3^{3/2}}+\frac{\rho^2M^2}{2\cdot 4^{3/2}}-\frac{\rho^3M^3}{3!\,5^{3/2}}\cdots\right\}$$

and

$$4.35 \qquad J_3 = 3\left(\frac{2\pi kT}{m}\right)^{1/2}\left(\frac{kT}{m}\right)^2\left\{\frac{1}{2^{5/2}}-\frac{\rho M}{3^{5/2}}+\frac{\rho^2M^2}{2\cdot 4^{5/2}}-\frac{\rho^3M^3}{3!\,5^{5/2}}\cdots\right\}.$$

Our complete third equation becomes on dividing through by $n'kT/m$

$$4.36 \qquad \frac{\partial\alpha(x,t)}{\partial t}+3\frac{kT}{m}\frac{\partial\gamma(x,t)}{\partial t}+3\frac{kT}{m}\frac{\partial\beta(x,t)}{\partial x}+a\alpha(x,t)+3a\frac{kT}{m}\gamma(x,t)$$

$$=\frac{aM}{4\pi}\oint_{R'}\left\{\left[\frac{(y'^2+z'^2)}{\rho^2}\alpha(x',t)+\frac{kT}{m}\frac{(y'^2+z'^2)}{\rho^4}\gamma(x',t)\right]\left[\frac{1}{\sqrt{2}}-\frac{\rho M}{\sqrt{3}}\right.\right.$$

$$+\frac{\rho^2 M^2}{2!\sqrt{4}} - \frac{\rho^3 M^3}{3!\sqrt{5}} \cdots \Big] + \left[\frac{(x'-x)^2}{\rho^2}\alpha(x',t) + 2\frac{kT}{m}\frac{(x'-x)^2(y^2+z^2)}{\rho^4}\gamma(x',t)\right]\cdot$$

$$\left[\frac{1}{2^{3/2}} - \frac{\rho M}{3^{3/2}} + \frac{\rho^2 M^2}{2!\cdot 4^{3/2}} - \frac{\rho^3 M^3}{3!\,5^{3/2}} \cdots\right] + \left[3\frac{kT}{m}\frac{(x'-x)^4}{\rho^4}\gamma(x',t)\right]\cdot$$

$$\left[\frac{1}{2^{5/2}} - \frac{\rho M}{3^{5/2}} + \frac{\rho^2 M^2}{2!\cdot 4^{5/2}} - \frac{\rho^3 M^3}{3!\,5^{5/2}} \cdots\right]\Big\}\, d\tau_{R'}\,.$$

CHAPTER V

SIMPLIFICATION OF EQUATIONS. MacLAURIN EXPANSION OF FOURIER COEFFICIENTS IN x

Now, let us further simplify calculations by assuming that $\gamma(x, t)$ equals zero. This limits the time over which our approximate solution will be valid. With this assumption our equations become

5.1 $$\frac{\partial\alpha}{\partial t}(x,t) - \frac{kT}{m}\frac{\partial\beta}{\partial x}(x,t) + a\,\alpha(x,t) = \frac{aM}{4\pi}\oint_{R'}\frac{\alpha(x',t)}{\rho^2}\cdot \left\{\frac{1}{\sqrt{2}} - \frac{\rho M}{\sqrt{3}} + \frac{\rho^2M^2}{2\sqrt{4}} - \frac{\rho^3M}{3!\sqrt{5}}\cdots\right\} d\tau_{R'}$$

and

5.2 $$\frac{\partial\beta}{\partial t}(x,t) + \frac{\partial\alpha}{\partial x}(x,t) + a\,\beta(x,t) = \frac{aM}{4\pi}\oint_{R'}\frac{(x'-x)^2}{\rho^2},\frac{\beta(x',t)}{\rho^2}\cdot \left\{\frac{1}{2^{3/2}} - \frac{\rho M}{5^{3/2}} + \frac{\rho^2M^2}{2\cdot 4^{3/2}} - \frac{\rho^3M^3}{3!5^{3/2}}\cdots\right\} d\tau_{R'}.$$

Now, in order to simplify the form of the equation, let us assume $\alpha(x, t)$ and $\beta(x, t)$ may be approximated by the following form:

5.3 $$\alpha(x,t) = \alpha_0(t) + x\,\alpha_1(t) + x^2\alpha_2(t)$$

and

5.4 $$\beta(x,t) = \beta_0(t) + x\,\beta_1(t) + x^2\beta_2(t).$$

We notice that we have a predominance of terms in ρ, the distance $\{(x'-x)^2 + y'^2 + z'^2\}^{1/2}$ from the fixed point (x, 0, 0) to the variable point (x', y', z') with the polar axis in the positive x-direction, θ being the polar angle and φ the azimuth angle. We remember that $(x'-x)/\rho = \cos\theta$. The Jacobian of the transformation is $\rho^2\sin\theta$. We see that, since $x' = x + \rho\cos\theta$, 5.5 $$\alpha(x',t) = (\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)) + (\alpha_1(t) + 2x\alpha_2(t))\rho\cos\theta + (\alpha_2(t))\rho^2\cos^2\theta$$

and, similarly,

5.6 $$\beta(x',t) = (\beta_0(t) + x\beta_1(t) + x^2\beta_2(t)) + (\beta_1(t) + 2\beta_2(t)x)\rho\cos\theta + (\beta_2(t))\rho^2\cos^2\theta\,.$$

We have no dependence on φ, hence the integration over this variable yields a factor 2π. The limits over each integration would be

5.7 $$2\pi\int_0^{\pi/2}\int_0^{(L-x)\sec\theta}\ldots\ldots d\rho\, d\theta + 2\pi\int_{\pi/2}^{\pi}\int^{(-L-x)\sec\theta}\ldots\ldots d\rho\, d\theta.$$

Now, the series expansion in each equation is obviously absolutely convergent for all ρ but as ρ tends to infinity the integration would diverge term by term. However, since a photon "skips" but a short distance, ρ may be considered to have an upper bound K. In the following, let us assume that K exists and is greater than or equal to 2L, the thickness of the slab. Then each of the integrals on the right of our two equations assume the form of the sum of three integrals with limits

5.8 $$\int_0^{\operatorname{arc\,sec}(K/L-x)}\int_0^{(L-x)\sec\theta}\ldots\ldots d\rho\, d\theta,$$

5.9 $$\int_{\operatorname{arc\,sec}(K/L-x)}^{\operatorname{arc\,sec}(K/-L-x)}\int_0^{K}\ldots\ldots d\rho\, d\theta$$

and

5.10 $$\int_{\operatorname{arc\,sec}(K/-L-x)}^{\pi}\int_0^{(-L-x)\sec\theta}\ldots\ldots d\rho\, d\theta.$$

Therefore, our two equations become

5.11 $$\frac{\partial\alpha}{\partial t}(x,t) + \frac{kT}{m}\frac{\partial\beta}{\partial x}(x,t) + a\,\alpha(x,t) = \frac{aM}{2}\left\{J_1 + J_2 + J_3\right\},$$

and

5.12 $$\frac{\partial\beta}{\partial t}(x,t) + \frac{\partial\alpha}{\partial x}(x,t) + a\beta(x,t) = \frac{aM}{2}\left\{F_1 + F_2 + F_3\right\},$$

where

5.13 $$J_1 = \int_0^{\operatorname{arc\,sec}(K/L-x)}\int_0^{(L-x)\sec\theta} \sin\theta\left\{\left[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\right] + \left[\alpha_1(t) + 2x\alpha_2(t)\right]\cos\theta\,\rho + \alpha_2(t)\cos^2\theta\,\rho^2\right\}\cdot \left\{\frac{1}{\sqrt{2}} - \frac{\rho M}{\sqrt{3}} + \frac{(\rho M)^2}{2\sqrt{4}} - \frac{(\rho M)^3}{3!\sqrt{5}}\cdots\right\}d\rho\, d\theta$$

and

5.14 $$F_1 = \int_0^{\operatorname{arc\,sec}(K/L-x)}\int_0^{(L-x)\sec\theta} \sin\theta\left\{\cos^2\theta\left[\beta_0(t) + x\beta_1(t) + x^2\beta_2(t)\right] + \left[\beta_1(t) + 2x\beta_2(t)\right]\cos^3\theta\,\rho + \beta_2(t)\cos^4\theta\,\rho^2\right\}\cdot \left\{\frac{1}{2^{3/2}} - \frac{\rho M}{3^{3/2}} + \frac{(\rho M)^2}{2\cdot 4^{3/2}} - \frac{(\rho M)^3}{3!\,5^{3/2}}\cdots\right\}d\rho\, d\theta,$$

and J_2 and J_3 have the same integrands as J_1, but with limits of integration given in 5.9 and 5.10 respectively. Similarly, F_2 and F_3 have the same integrands as F_1, but with limits of integration given in 5.9 and 5.10 respectively.

Integrating J_1 first with respect to ρ, we have

5.15 $$J_1 = \int_0^{\operatorname{arc\,sec}(K/L-x)} \sin\theta\left\{\left[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\right]\cdot\right.$$

$$\left[\frac{\rho}{\sqrt{2}} - \frac{\rho^2 M}{2\sqrt{3}} + \frac{\rho^3 M^2}{3!\sqrt{4}} - \frac{\rho^4 M^3}{4!\sqrt{5}} \cdots\right]$$

$$+\cos\theta\left[\alpha_1(t) + 2x\alpha_2(t)\right]\left[\frac{\rho^2}{2\sqrt{2}} - \frac{\rho^3 M}{3\sqrt{3}} + \frac{\rho^4 M^2}{2\cdot 4^{3/2}} - \frac{\rho^5 M^3}{3!\,5^{3/2}}\cdots\right]$$

$$+\cos^2\theta\,\alpha_2(t)\left[\frac{\rho^3}{3\sqrt{2}} - \frac{\rho^4 M}{4\sqrt{3}} + \frac{\rho^5 M^2}{5\cdot 2\sqrt{4}} - \frac{\rho^6 M^3}{6\cdot 3!\sqrt{5}}\cdots\right]\Bigg\}\Bigg|_0^{(L-x)\sec\theta} d\theta$$

Evaluation at the lower limit yields zero, so we have on substituting = (L-x)/cosθ

5.16
$$J_1 = \int_0^{\operatorname{arc\,sec}(K/L-x)} \sin\theta\left\{\left[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\right]\cdot\right.$$

$$\left[\frac{(L-x)}{\sqrt{2}\cos\theta} - \frac{(L-x)^2 M}{2\sqrt{3}\cos^2\theta} + \frac{(L-x)^3 M^2}{3!\sqrt{4}\cos^3\theta} - \frac{(L-x)^4 M^3}{4!\sqrt{5}\cos^4\theta}\cdots\right]$$

$$+\left[\alpha_1(t) + 2x\alpha_2(t)\right]\left[\frac{(L-x)^2}{2\sqrt{2}\cos\theta} - \frac{(L-x)^3 M}{3\sqrt{3}\cos^2\theta}\right.$$

$$\left.+\frac{(L-x)^4 M^2}{2\cdot 4^{3/2}\cos^3\theta} - \frac{(L-x)^5 M^3}{3!\,5^{3/2}\cos^4\theta}\cdots\right] + \alpha_2(t)\left[\frac{(L-x)^3}{3\sqrt{2}\cos\theta}\right.$$

$$\left.\left.-\frac{(L-x)^4 M}{4\sqrt{3}\cos^2\theta} + \frac{(L-x)^5 M^2}{5\cdot 2\sqrt{4}\cos^3\theta} - \frac{(L-x)^6 M^3}{6\cdot 3!\sqrt{5}\cos^4\theta}\cdots\right]\right\} d\theta,$$

and integrating over θ,

5.17
$$J_1 = \left\{\left[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\right]\left[-\frac{(L-x)}{\sqrt{2}}\log\cos\theta\right.\right.$$

$$\left.-\frac{(L-x)^2 M}{2\sqrt{3}\cos\theta} + \frac{(L-x)^3 M^2}{2\cdot 3!\sqrt{4}\cos^2\theta} - \frac{(L-x)^4 M^3}{3\cdot 4!\sqrt{5}\cos^3\theta}\cdots\right]$$

$$+\left[\alpha_1(t) + 2x\alpha_2(t)\right]\left[-\frac{(L-x)^2}{2\sqrt{2}}\log\cos\theta - \frac{(L-x)^3 M}{3\sqrt{3}\cos\theta}\right.$$

$$\left.+\frac{(L-x)^4 M^2}{2\cdot 2!\,4^{3/2}\cos^2\theta} - \frac{(L-x)^5 M^3}{3\cdot 3!\,5^{3/2}\cos^3\theta}\cdots\right] + \alpha_2(t)\left[-\frac{(L-x)^3}{3\sqrt{2}}\log\cos\theta\right.$$

$$\left.\left.-\frac{(L-x)^4 M}{4\sqrt{3}\cos\theta} + \frac{(L-x)^5 M^2}{2\cdot 5\cdot 2!\sqrt{4}\cos^2\theta} - \frac{(L-x)^6 M^3}{3\cdot 6\cdot 3!\sqrt{5}\cos^3\theta}\right]\right\}\Bigg|_{\theta=0}^{\cos^{-1}\left(\frac{L-x}{K}\right)}$$

and substituting our limits

5.18

$$J_1 = \Big\{ \Big[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\Big]\Big[-\frac{(L-x)}{\sqrt{2}}\log\Big(\frac{L-x}{K}\Big) - \frac{K(L-x)M}{2\sqrt{3}} + \frac{K^2(L-x)M^2}{2\cdot 3!\sqrt{4}} - \frac{K^3(L-x)M^3}{3\cdot 4!\sqrt{5}} \cdots\Big]$$

$$+\Big[\alpha_1(t) + 2x\alpha_2(t)\Big]\Big[-\frac{(L-x)^2}{2\sqrt{2}}\log\Big(\frac{L-x}{K}\Big) - \frac{K(L-x)^2M}{3\sqrt{3}} + \frac{K^2(L-x)^2M^2}{2\cdot 2!\,4^{3/2}} - \frac{K^3(L-x)^2M^3}{3\cdot 3!\,5^{3/2}} \cdots\Big] + \alpha_2(t)\Big[\frac{(L-x)^3}{3\sqrt{2}}\log\Big(\frac{L-x}{K}\Big)$$

$$-\frac{K(L-x)^3M}{4\sqrt{3}} + \frac{K^2(L-x)^3M^2}{2\cdot 5\cdot 2!\sqrt{4}} - \frac{K^3(L-x)^3M^3}{3\cdot 6\cdot 3!\sqrt{5}} \cdots\Big]$$

$$-\Big[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\Big]\Big[-\frac{(L-x)^2M}{2\sqrt{3}} + \frac{(L-x)^3M^2}{2\cdot 3!\sqrt{4}} - \frac{(L-x)^4M^3}{3\cdot 4!\sqrt{5}} \cdots\Big] - \Big[\alpha_1(t) + 2x\alpha_2(t)\Big]\Big[-\frac{(L-x)^3M}{3\sqrt{3}} + \frac{(L-x)^4M^2}{2\cdot 2!\,4^{3/2}}$$

$$-\frac{(L-x)^5M^3}{3\cdot 3!\,5^{3/2}} \cdots\Big] - \alpha_2(t)\Big[-\frac{(L-x)^4M}{4\sqrt{3}} + \frac{(L-x)^5M^2}{2\cdot 5\cdot 2!\sqrt{4}} - \frac{(L-x)^6M^3}{3\cdot 6\cdot 3!\sqrt{5}} \cdots\Big]\Big\}.$$

Now,

5.19

$$J_2 = \int_{\operatorname{arc\,sec}(K/L-x)}^{\operatorname{arc\,sec}(K/-L-x)} \int_0^K (\text{same integrand as } J_1 \ldots)\, d\rho\, d\theta .$$

Integrating first with respect to ρ, we have

5.20

$$J_2 = \int_{\operatorname{arc\,sec}(K/L-x)}^{\operatorname{arc\,sec}(K/-L-x)} \sin\theta\Big\{\Big[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\Big]\Big[\frac{K}{\sqrt{2}} - \frac{K^2M}{2\sqrt{3}} + \frac{K^3M^2}{3!\sqrt{4}} - \frac{K^4M^3}{4!\sqrt{5}} \cdots\Big] + \cos\theta\Big[\alpha_1(t) + 2x\alpha_2(t)\Big]\cdot$$

$$\Big[\frac{K^2}{2\sqrt{2}} - \frac{K^3M}{3\sqrt{3}} + \frac{K^4M^2}{2\cdot 4^{3/2}} - \frac{K^5M^3}{3!\,5^{3/2}} \cdots\Big] + \cos^2\theta\,\alpha_2(t)\Big[\frac{K^3}{3\sqrt{2}} - \frac{K^4M}{4\sqrt{3}} + \frac{K^5M^2}{5\cdot 2\sqrt{4}} - \frac{K^6M^3}{6\cdot 3!\sqrt{5}} \cdots\Big]\Big\}\, d\theta ,$$

and, since

5.21 $$\int_{\operatorname{arc\,sec}(K/L-x)}^{\operatorname{arc\,sec}(K/-L-x)} \sin\theta\,d\theta = \frac{2L}{K};$$

5.22 $$\int_{\operatorname{arc\,sec}(K/L-x)}^{\operatorname{arc\,sec}(K/-L-x)} \cos\theta\sin\theta\,d\theta = -\frac{2Lx}{K^2}$$

5.23 $$\int_{\operatorname{arc\,sec}(K/L-x)}^{\operatorname{arc\,sec}(K/-L-x)} \cos^2\theta\sin\theta\,d\theta = \frac{2}{3}\left(\frac{L^3+3Lx^2}{K^3}\right),$$

we have, on integrating over θ,

5.24 $$J_2 = 2L\left[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\right]\left[\frac{1}{\sqrt{2}} - \frac{KM}{2\sqrt{3}} + \frac{K^2M^2}{3!\sqrt{4}} - \frac{K^3M^3}{4!\sqrt{5}}\dots\right] - 2Lx\left[\alpha_1(t) + 2x\alpha_2(t)\right]\left[\frac{1}{2\sqrt{2}} - \frac{KM}{3\sqrt{3}} + \frac{K^2M^2}{2\cdot 4^{3/2}} - \frac{K^3M^3}{3!\,5^{3/2}}\dots\right] + \frac{2}{3}(L^3+3Lx^2)\alpha_2(t)\left[\frac{1}{3\sqrt{2}} - \frac{KM}{4\sqrt{3}} + \frac{K^2M^2}{5\cdot 2\sqrt{4}} - \frac{K^3M^3}{6\cdot 3!\sqrt{5}}\dots\right]$$

Now, J_3 is treated very similarly to J_1. Remembering that

5.25 $$J_3 = \int_{\operatorname{arc\,sec}(K/-L-x)}^{\pi} \int_{c}^{(-L-x)\sec\theta} (\text{ same integrand as } J_1)\,d\rho\,d\theta,$$

and integrating first with respect to ρ, we have,

5.26 $$J_3 = \int_{\operatorname{arc\,sec}(K/-L-x)}^{\pi} \sin\theta\left\{\left[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\right]\left[\frac{(-L-x)}{\sqrt{2}\cos\theta} - \frac{(L-x)^2M}{2\sqrt{3}\cos^2\theta} + \frac{(L-x)^3M^2}{3!\sqrt{4}\cos^3\theta} - \frac{(L-x)^4M^3}{4!\sqrt{5}\cos^4\theta}\dots\right] + \left[\alpha_1(t) + 2x\alpha_2(t)\right]\left[\frac{(-L-x)^2}{2\sqrt{2}\cos\theta} - \frac{(-L-x)^3M}{3\sqrt{3}\cos^2\theta} + \frac{(-L-x)^4M^2}{2\cdot 4^{3/2}\cos^3\theta} - \frac{(-L-x)^5M^3}{3!\,5^{3/2}\cos^4\theta}\dots\right] + \alpha_2(t)\left[\frac{(-L-x)^3}{3\sqrt{2}\cos\theta} - \frac{(-L-x)^4M}{4\sqrt{3}\cos^2\theta}\right.\right.$$

$$+\frac{(-L-x)^5M^2}{5\cdot 2\sqrt{4}\cos^3\theta} - \frac{(-L-x)^6M^3}{6\cdot 3!\sqrt{5}\cos^4\theta}\ \ldots\Big]\Big\}\,d\theta$$

where the only difference in the integrand at a similar step in J_1 is re replacing of quantities (L-x) by (-L-x).

Integrating over θ, and noting that since $\pi/2 < 0 < \pi$ we must integrate $\sin\theta/\cos\theta d\theta$ to $-\log(-\cos\theta)$ in order that the logarithm term have meaning, we have, on integrating over θ,

5.27

$$J_3 = \Big\{\Big[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\Big]\Big[-\frac{(-L-x)}{\sqrt{2}}\log(-\cos\theta) - \frac{(-L-x)^2M}{2\sqrt{3}\cos\theta} + \frac{(-L-x)^3M^2}{2\cdot 3!\sqrt{4}\cos^2\theta} - \frac{(-L-x)^4M^3}{3\cdot 4!\sqrt{5}\cos^3\theta}\ldots\Big]$$

$$+\Big[\alpha_1(t) + 2x\alpha_2(t)\Big]\Big[-\frac{(-L-x)^2}{2\sqrt{2}}\log(-\cos\theta) - \frac{(-L-x)^3M}{3\sqrt{3}\cos\theta} + \frac{(-L-x)^4M^2}{2\cdot 2!4^{3/2}\cos^2\theta} - \frac{(-L-x)^5M^3}{3\cdot 3!5^{3/2}\cos^3\theta}\ldots\Big]$$

$$+\alpha_2(t)\Big[-\frac{(-L-x)^3}{3\sqrt{2}}\log(-\cos\theta) - \frac{(-L-x)^4M}{4\sqrt{3}\cos\theta} + \frac{(-L-x)^5M^2}{2\cdot 5\cdot 2!\sqrt{4}\cos^2\theta} - \frac{(-L-x)^6M^3}{3\cdot 6\cdot 3!\sqrt{5}\cos^3\theta}\ldots\Big]\Big\}\Bigg|_{\theta=\cos^{-1}((-L-x)/K)}^{\pi}$$

and substituting our limits, lower limit first, we find

5.28

$$J_3 = \Big[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\Big]\Big[-\frac{(L+x)}{\sqrt{2}}\log\Big(\frac{L+x}{K}\Big) - \frac{(L+x)KM}{2\sqrt{3}} + \frac{(L+x)K^2M^2}{2\cdot 3!\sqrt{4}} - \frac{(L+x)K^3M^3}{3\cdot 4!\sqrt{5}}\ldots\Big]$$

$$+\Big[\alpha_1(t) + 2x\alpha_2(t)\Big]\Big[\frac{(L+x)^2}{2\sqrt{2}}\log\Big(\frac{L+x}{K}\Big) + \frac{(L+x)^2KM}{3\sqrt{3}} - \frac{(L+x)^2K^2M^2}{2\cdot 2!\,4^{3/2}} + \frac{(L+x)^2K^3M^3}{3\cdot 3!\,5^{3/2}}\ldots\Big] + \alpha_2(t)\Big[-\frac{(L+x)^3}{3\sqrt{2}}\log\Big(\frac{L+x}{K}\Big)$$

$$-\frac{(L+x)^3KM}{4\sqrt{3}} + \frac{(L+x)^3K^2M^2}{2\cdot 5\cdot 2!\sqrt{4}} - \frac{(L+x)^3K^3M^3}{3\cdot 6\cdot 3!\sqrt{5}}\ldots\Big]$$

$$-\Big[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\Big]\Big[-\frac{(L+x)^2M}{2\sqrt{3}} + \frac{(L+x)^3M^2}{2\cdot 3!\sqrt{4}}$$

$$-\frac{(L+x)^4M^3}{3\cdot 4!\sqrt{5}}\cdots\Big] - \Big[\alpha_1(t) + 2x\alpha_2(t)\Big]\Big[\frac{(L+x)^3M}{3\sqrt{3}} - \frac{(L+x)^4M^2}{2\cdot 2!\,4^{3/2}}$$

$$+\frac{(L+x)^5M^3}{3\cdot 3!\,5^{3/2}}\cdots\Big] - \alpha_2(t)\Big[-\frac{(L+x)^4M}{4\sqrt{3}} + \frac{(L+x)^5M^2}{2\cdot 5\cdot 2!\sqrt{4}} - \frac{(L+x)^6M^3}{3\cdot 6\cdot 3!\sqrt{5}}\cdots\Big].$$

Adding $J_1 + J_3$

5.29
$$J_1 + J_3 = \Big[\alpha_0(t) + x\alpha_1(t) + x^2\alpha_2(t)\Big]\Big[-\frac{(L+x)}{\sqrt{2}}\log\Big(\frac{L+x}{K}\Big)$$

$$-\frac{(L-x)}{\sqrt{2}}\log\Big(\frac{L-x}{K}\Big) - \frac{KLM}{\sqrt{3}} + \frac{LK^2M^2}{3!\sqrt{4}} - \frac{2LK^3M^3}{3\cdot 4!\sqrt{5}}$$

$$+\frac{L^2M}{\sqrt{3}} + \frac{x^2M}{\sqrt{3}} - \frac{L^3M^2}{3!\sqrt{4}} - \frac{3Lx^2M^2}{3!\sqrt{4}} + \frac{2L^4M^3}{3\cdot 4!\sqrt{5}} + \frac{12L^2x^2M^3}{3\cdot 4!\sqrt{5}}$$

$$+\frac{2x^4M^3}{3\cdot 4!\sqrt{5}}\Big] + \Big[\alpha_1(t) + 2x\alpha_2(t)\Big]\Big[\frac{(L+x)^2}{2\sqrt{2}}\log\Big(\frac{L+x}{K}\Big)$$

$$-\frac{(L-x)^2}{2\sqrt{2}}\log\Big(\frac{L-x}{K}\Big) + \frac{4LxKM}{3\sqrt{3}} - \frac{LxK^2M^2}{4^{3/2}} + \frac{4LxK^3M^3}{3\cdot 3!\,5^{3/2}}$$

$$-\frac{2L^2xM}{\sqrt{3}} - \frac{2x^3M}{3\sqrt{3}} + \frac{L^3xM^2}{4} + \frac{Lx^3M^2}{4} - \frac{2L^4xM^3}{3\cdot 3!\sqrt{5}}$$

$$-\frac{4L^2x^3M^3}{3\cdot 3!\sqrt{5}} - \frac{2x^5M^3}{3\cdot 3!\,5^{3/2}}\Big] + \alpha_2(t)\Big[-\frac{(L+x)^3}{3\sqrt{2}}\log\Big(\frac{L+x}{K}\Big)$$

$$-\frac{(L-x)^3}{3\sqrt{2}}\log\Big(\frac{L-x}{K}\Big) - \frac{L^3KM}{2\sqrt{3}} - \frac{3Lx^2KM}{2\sqrt{3}} + \frac{L^3K^2M^2}{20}$$

$$+\frac{3Lx^2K^2M^2}{20} - \frac{L^3K^3M^3}{54\sqrt{5}} - \frac{Lx^2K^3M^3}{18\sqrt{5}} + \frac{L^4M}{2\sqrt{3}}$$

$$+\frac{3L^2x^2M}{\sqrt{3}} + \frac{x^4M}{2\sqrt{3}} - \frac{L^5M^2}{20} - \frac{L^3x^2M^2}{2} - \frac{Lx^4M^2}{4}$$

$$+\frac{L^6M^3}{54\sqrt{5}} + \frac{5L^4x^2M^3}{18\sqrt{5}} + \frac{5L^2x^4M^3}{18\sqrt{5}} + \frac{x^6M^3}{54\sqrt{5}}\Big].$$

Adding J_2 and grouping coefficients of the $\alpha_i(t)$'s, we have

5.30
$$J_1 + J_2 + J_3 = \alpha_0(t)\Big[-\frac{(L+x)}{\sqrt{2}}\log(L+x) - \frac{(L-x)}{\sqrt{2}}\log(L-x)$$

$$+\frac{(L+x)}{\sqrt{2}}\log K + \frac{(L-x)}{\sqrt{2}}\log K + \frac{2L}{\sqrt{2}} - \frac{2LKM}{\sqrt{3}} + \frac{LK^2M^2}{4}$$

$$-\frac{LK^3M^3}{3\sqrt{5}} + \frac{L^2M}{\sqrt{3}} + \frac{x^2M}{\sqrt{3}} - \frac{L^3M^2}{3!\sqrt{4}} - \frac{3Lx^2M^2}{3!\sqrt{4}}$$

$$+\frac{2L^4M^3}{3\cdot 4!\sqrt{5}} + \frac{12L^2x^2M^3}{3\cdot 4!\sqrt{5}} + \frac{2x^4M^3}{3\cdot 4!\sqrt{5}}\Big] + \alpha_1(t)\Big[$$

$$-\frac{x(L+x)}{\sqrt{2}}\log(L+x) - \frac{x(L-x)}{\sqrt{2}}\log(L-x) + \frac{x(L+x)}{\sqrt{2}}\log K$$

$$+\frac{x(L-x)}{\sqrt{2}}\log K + \frac{2Lx}{\sqrt{2}} - \frac{2LxKM}{\sqrt{3}} + \frac{LxK^2M^2}{4} - \frac{LxK^3M^3}{9\sqrt{5}}$$

$$+\frac{L^2xM}{\sqrt{3}} + \frac{x^3M}{\sqrt{3}} - \frac{L^3xM^2}{3!\sqrt{4}} - \frac{3Lx^3M^2}{3!\sqrt{4}} + \frac{2L^4xM^3}{3\cdot 4!\sqrt{5}}$$

$$+\frac{12L^2x^3M^3}{3\cdot 4!\sqrt{5}} + \frac{2x^5M^3}{3\cdot 4!\sqrt{5}} + \frac{(L+x)^2}{2\sqrt{2}}\log(L+x) - \frac{(L-x)^2}{2\sqrt{2}}\log(L-x)$$

$$-\frac{(L+x)^2}{2\sqrt{2}}\log K + \frac{(L-x)^2}{2\sqrt{2}}\log K - \frac{Lx}{\sqrt{2}} + \frac{2LxKM}{\sqrt{3}} - \frac{LxK^2M^2}{4}$$

$$+\frac{LxK^3M^3}{9\sqrt{5}} - \frac{2L^2xM}{\sqrt{3}} - \frac{2x^3M}{3\sqrt{3}} + \frac{L^3xM^2}{4} + \frac{Lx^3M^2}{4}$$

$$-\frac{2L^4xM^3}{3\cdot 3!\sqrt{5}} - \frac{4L^2x^3M^3}{3\cdot 3!\sqrt{5}} - \frac{2x^5M^3}{3\cdot 3!5^{3/2}}\Big] + \alpha_2(t)\Big[$$

$$-\frac{x^2(L+x)}{\sqrt{2}}\log(L+x) - \frac{x^2(L-x)}{\sqrt{2}}\log(L-x) + \frac{x^2(L+x)}{\sqrt{2}}\log K$$

$$+\frac{x^2(L-x)}{\sqrt{2}}\log(L-x) + \frac{2Lx^2}{\sqrt{2}} - \frac{2Lx^2KM}{\sqrt{3}} + \frac{Lx^2K^2M^2}{4}$$

$$-\frac{Lx^2K^3M^3}{9\sqrt{5}} + \frac{L^2x^2M}{\sqrt{3}} + \frac{x^4M}{\sqrt{3}} - \frac{L^3x^2M^2}{3!\sqrt{4}} - \frac{3Lx^4M^2}{3!\sqrt{4}}$$

$$+\frac{2L^4x^2M^3}{3\cdot 4!\sqrt{5}} + \frac{12L^2x^4M^3}{3\cdot 4!\sqrt{5}} + \frac{2x^6M^3}{3\cdot 4!\sqrt{5}} + \frac{x(L+x)^2}{\sqrt{2}}\log(L+x)$$

$$-\frac{x(L-x)^2}{\sqrt{2}}\log(L-x) - \frac{x(L+x)^2}{\sqrt{2}}\log K + \frac{x(L-x)^2}{\sqrt{2}}\log K - \frac{2Lx^2}{\sqrt{2}}$$

$$+\frac{4Lx^2KM}{\sqrt{3}} - \frac{2Lx^2K^2M^2}{4} + \frac{2Lx^2K^3M^3}{9\sqrt{5}} - \frac{4L^2x^2M}{\sqrt{3}}$$

$$-\frac{4x^4M}{3\sqrt{3}} + \frac{L^3x^2M^2}{2} + \frac{Lx^4M^2}{2} - \frac{4L^4x^2M^3}{3\cdot 3!\sqrt{5}} - \frac{8L^2x^4M^3}{3\cdot 3!\sqrt{5}}$$

$$-\frac{4x^6M^3}{3\cdot 3!5^{3/2}} - \frac{(L+x)^3}{3\sqrt{2}}\log(L+x) - \frac{(L-x)^3}{3\sqrt{2}}\log(L-x)$$

$$+\frac{(L+x)^3}{3\sqrt{2}}\log K + \frac{(L-x)^3}{3\sqrt{2}}\log K + \frac{2L^3}{9\sqrt{2}} + \frac{2Lx^2}{3\sqrt{2}} - \frac{2L^3KM}{3\sqrt{3}}$$

$$-\frac{2Lx^2KM}{\sqrt{3}} + \frac{L^3K^2M^2}{12} + \frac{Lx^2K^2M^2}{4} - \frac{L^3K^3M^3}{27\sqrt{5}}$$

$$-\frac{Lx^2K^3M^3}{9\sqrt{5}} + \frac{L^4M}{2\sqrt{3}} + \frac{3L^2x^2M}{\sqrt{3}} + \frac{x^4M}{2\sqrt{3}} - \frac{L^5M^2}{20} - \frac{L^3x^2M^2}{2}$$

$$-\frac{Lx^4M^2}{4} + \frac{L^6M^3}{54\sqrt{5}} + \frac{5L^4x^2M^3}{18\sqrt{5}} + \frac{5L^2x^4M^3}{18\sqrt{5}} + \frac{x^6M^3}{54\sqrt{5}}\Bigg].$$

Collecting like terms, the above equation reduces to

5.31 $$J_1 + J_2 + J_3 = \alpha_0(t)\Bigg[-\frac{(L+x)}{\sqrt{2}}\log(L+x) - \frac{(L-x)}{\sqrt{2}}\log(L-x)$$

$$+\frac{2L}{\sqrt{2}}\log K + \frac{2L}{\sqrt{2}} - \frac{2LKM}{\sqrt{3}} + \frac{LK^2M^2}{4} - \frac{LK^3M^3}{9\sqrt{5}} + \frac{L^2M}{\sqrt{3}}$$

$$+\frac{x^2M}{\sqrt{3}} - \frac{L^3M^2}{3!\sqrt{4}} - \frac{Lx^2M^2}{4} + \frac{2L^4M^3}{3\cdot 4!\sqrt{5}} + \frac{12L^2x^2M^3}{3\cdot 4!\sqrt{5}} + \frac{2x^4M^3}{3\cdot 4!\sqrt{5}}\Bigg]$$

$$+\alpha_1(t)\Bigg[-\frac{x(L+x)}{\sqrt{2}}\log(L+x) - \frac{x(L-x)}{\sqrt{2}}\log(L-x)$$

$$+\frac{(L+x)^2}{2\sqrt{2}}\log(L+x) - \frac{(L-x)^2}{2\sqrt{2}}\log(L-x) + \frac{Lx}{\sqrt{2}} - \frac{L^2xM}{\sqrt{3}} + \frac{x^3M}{3\sqrt{3}}$$

$$+\frac{L^3xM^2}{6} - \frac{L^4xM^3}{12\sqrt{5}} - \frac{L^2x^3M^3}{18\sqrt{5}} + \frac{x^5M^3}{180\sqrt{5}}\Bigg] + \alpha_2(t)\Bigg[$$

$$-\frac{x^2(L+x)}{\sqrt{2}}\log(L+x) - \frac{x^2(L-x)}{\sqrt{2}}\log(L-x) + \frac{x(L+x)^2}{\sqrt{2}}\log(L+x)$$

$$-\frac{x(L-x)^2}{\sqrt{2}}\log(L-x) - \frac{(L+x)^3}{3\sqrt{2}}\log(L+x) - \frac{(L-x)^3}{3\sqrt{2}}\log(L-x)$$

$$+\frac{2L^3}{3\sqrt{2}}\log K + \frac{2L^3}{9\sqrt{2}} + \frac{2Lx^2}{3\sqrt{2}} - \frac{2L^3KM}{3\sqrt{3}} + \frac{L^3K^2M^2}{12}$$

$$-\frac{L^3K^3M^3}{27\sqrt{5}} + \frac{L^4M}{2\sqrt{3}} + \frac{x^4M}{6\sqrt{3}} - \frac{L^5M^2}{20} - \frac{L^3x^2M^2}{12}$$

$$+\frac{L^6M^3}{54\sqrt{5}} + \frac{L^4x^2M^3}{12\sqrt{5}} + \frac{x^6M^3}{540\sqrt{5}}\Bigg].$$

Let us now evaluate

5.32 $$F_1 = \int_0^{\operatorname{arc\,sec}(K/L-x)} \int_0^{(L-x)\sec\theta} \sin\theta\Big\{\Big[\beta_0(t) + x\beta_1(t)$$

$$+ x^2\beta_2(t)\Big]\cos^2\theta + \Big[\beta_1(t) + 2x\beta_2(t)\Big]\rho\cos^3\theta$$

$$+\beta_2(t)\cos^4\theta\,\rho^2\Big\}\Big\{\frac{1}{2^{3/2}} - \frac{\rho M}{3^{3/2}} + \frac{(\rho M)^2}{2\cdot 4^{3/2}} - \frac{(\rho M)^3}{3!\,5^{3/2}}\cdots\Big\}d\rho d\theta$$

Integrating first with respect to ρ,

5.33
$$F_1 = \int_0^{\operatorname{arc\,sec}(K/L-x)} \sin\theta\Big\{\Big[\beta_0(t) + x\beta_1(t) + x^2\beta_2(t)\Big]\bullet$$

$$\cos^2\theta\Big[\frac{\rho}{2^{3/2}} - \frac{\rho^2 M}{2\cdot 3^{3/2}} + \frac{\rho^3 M^2}{3!\,4^{3/2}} - \frac{\rho^4 M^3}{4!\,5^{3/2}}\Big] + \Big[\beta_1(t)$$

$$+ 2x\beta_2(t)\Big]\cos^3\theta\Big[\frac{\rho^2}{2\cdot 2^{3/2}} - \frac{\rho^3 M}{3\cdot 3^{3/2}} + \frac{\rho^4 M^2}{2\cdot 4^{5/2}} - \frac{\rho^5 M^3}{3!\,5^{5/2}}\Big]$$

$$+\beta_2(t)\cos^4\theta\Big[\frac{\rho^3}{3\cdot 2^{3/2}} - \frac{\rho^4 M}{4\cdot 3^{3/2}} + \frac{\rho^5 M^2}{5\cdot 2\cdot 4^{3/2}}$$

$$- \frac{\rho^6 M^3}{6\cdot 3!\,5^{3/2}}\Big]\Big\}\Big|_0^{(L-x)\sec\theta} d\theta$$

and substituting limits, we have

5.34
$$F_1 = \int_0^{\operatorname{arc\,sec}(K/L-x)} \sin\theta\Big\{\Big[\beta_0(t) + x\beta_1(t) + x^2\beta_2(t)\Big]\Big[\frac{(L-x)\cos\theta}{2^{3/2}}$$

$$- \frac{(L-x)^2 M}{2\cdot 3^{3/2}} + \frac{(L-x)^3 M^2}{3!\,4^{3/2}\cos\theta} + \frac{(L-x)^4 M^3}{4!\,5^{3/2}\cos^2\theta}\Big] + \Big[\beta_1(t)$$

$$+ 2x\beta_2(t)\Big]\Big[\frac{(L-x)^2\cos\theta}{2^{5/2}} - \frac{(L-x)^3 M}{3^{5/2}} + \frac{(L-x)^4 M^2}{2\cdot 4^{5/2}\cos\theta}$$

$$-\frac{(L-x)^5 M^3}{3!\,5^{5/2}\cos^2\theta}\Big] + \beta_2(t)\Big[\frac{(L-x)^3\cos\theta}{3\cdot 2^{3/2}} - \frac{(L-x)^4 M}{4\cdot 3^{3/2}}$$

$$+\frac{(L-x)^5 M^2}{5\cdot 2\cdot 4^{3/2}\cos\theta} - \frac{(L-x)^6 M^3}{6\cdot 3!\,5^{3/2}\cos^2\theta}\Big]\Big\} d\theta,$$

and integrating over θ,

5.35
$$F_1 = \Big\{\Big[\beta_0(t) + x\beta_1(t) + x^2\beta_2(t)\Big]\Big[-\frac{(L-x)\cos^2\theta}{2\cdot 2^{3/2}} + \frac{(L-x)^2 M\cos\theta}{2\cdot 3^{3/2}}$$

$$-\frac{(L-x)^3 M^2\log\cos\theta}{3!\,4^{3/2}} - \frac{(L-x)^4 M^3}{4!\,5^{3/2}\cos\theta}\Big] + \Big[\beta_1(t)$$

$$+ 2x\beta_2(t)\Big]\Big[-\frac{(L-x)^2\cos^2\theta}{2^{7/2}} + \frac{(L-x)^3 M\cos\theta}{3^{5/2}} - \frac{(L-x)^4 M^2 \log\cos\theta}{2\cdot 4^{5/2}}$$

$$-\frac{(L-x)^5 M^3}{3!\,5^{5/2}\cos\theta}\Big] + \beta_2(t)\Big[-\frac{(L-x)^3\cos^2\theta}{3\cdot 2^{5/2}} + \frac{(L-x)^4 M\cos\theta}{4\cdot 3^{3/2}}$$

$$-\frac{(L-x)^5 M^2 \log\cos\theta}{5\cdot 2\cdot 4^{3/2}} - \frac{(L-x)^6 M^3}{6\cdot 3!\,5^{3/2}\cos\theta}\Big]\Big\}\Bigg|_{\cos\theta = 1}^{(L-x)/K}$$

Substituting limits, we have

5.36
$$F_1 = \Big[\beta_0(t) + x\beta_1(t) + x^2\beta_2(t)\Big]\Big[-\frac{(L-x)^3}{2\cdot 2^{3/2}K^2} + \frac{(L-x)^3 M}{2\cdot 3^{3/2}K}$$

$$-\frac{(L-x)^3 M^2 \log(L-x)}{3!\,4^{3/2}} + \frac{(L-x)^3 M^2 \log K}{3!\,4^{3/2}} - \frac{(L-x)^3 M^3 K}{4!\,5^{3/2}}$$

$$+\frac{(L-x)}{2\cdot 2^{3/2}} - \frac{(L-x)^2 M}{2\cdot 3^{3/2}} + \frac{(L-x)^4 M^3}{4!\,5^{3/2}}\Big] + \Big[\beta_1(t) + 2x\beta_2(t)\Big]\cdot$$

$$\Big[-\frac{(L-x)^4}{2^{7/2}K^2} + \frac{(L-x)^4 M}{3^{5/2}K} - \frac{(L-x)^4 M^2 \log(L-x)}{2\cdot 4^{5/2}}$$

$$+\frac{(L-x)^4 M^2 \log K}{2\cdot 4^{5/2}} - \frac{(L-x)^4 M^3 K}{3!\,5^{5/2}} + \frac{(L-x)^2}{2^{7/2}}$$

$$-\frac{(L-x)^3 M}{3^{5/2}} + \frac{(L-x)^5 M^3}{3!\,5^{5/2}}\Big] + \beta_2(t)\Big[-\frac{(L-x)^5}{3\cdot 2^{5/2}K^2} + \frac{(L-x)^5 M}{4\cdot 3^{3/2}K}$$

$$-\frac{(L-x)^5 M^2 \log(L-x)}{5\cdot 2\cdot 4^{3/2}} + \frac{(L-x)^5 M^2 \log K}{5\cdot 2\cdot 4^{3/2}} - \frac{(L-x)^5 M^3 K}{6\cdot 3!\,5^{3/2}}$$

$$+\frac{(L-x)^3}{3\cdot 2^{5/2}} - \frac{(L-x)^4 M}{4\cdot 3^{3/2}} + \frac{(L-x)^6 M^3}{6\cdot 3!\,5^{3/2}}\Big]$$

Let us now evaluate

5.37
$$F_2 = \int_{\mathrm{arc\,sec}(K/L-x)}^{\mathrm{arc\,sec}(K/-L-x)} \int_0^K (\text{same integrand as } F_1)\, d\rho\, d\theta.$$

Integrating first with respect to ρ, we have

5.38
$$F_2 = \int_{\mathrm{arc\,sec}(K/L-x)}^{\mathrm{arc\,sec}(K/-L-x)} \sin\theta\Big\{\cos^2\theta\Big[\beta_0(t) + x\beta_1(t) + x^2\beta_2(t)\Big]\Big[\frac{K}{2^{3/2}}$$

$$-\frac{K^2 M}{2\cdot 3^{3/2}} + \frac{K^3 M^2}{3!\,4^{3/2}} - \frac{K^4 M^3}{4!\,5^{3/2}}\Big] + \cos^3\theta\Big[\beta_1(t) + 2x\beta_2(t)\Big]\Big[\frac{K^2}{2^{5/2}}$$

$$-\frac{K^3M}{3^{5/2}}+\frac{K^4M^2}{2\cdot 4^{5/2}}-\frac{K^5M^3}{3!5^{5/2}}\Bigg]+\cos^4\theta\beta_2(t)\Bigg[\frac{K^3}{3\cdot 2^{3/2}}$$

$$-\frac{K^4M}{4\cdot 3^{3/2}}+\frac{K^5M^2}{5\cdot 2\cdot 4^{3/2}}-\frac{K^6M^3}{6\cdot 3!5^{3/2}}\Bigg]\Bigg\}d\theta\,.$$

Remembering 5.23, and since

5.39 $$\int_{\operatorname{arc\,sec}(K/L-x)}^{\operatorname{arc\,sec}(K/-L-x)}\cos^3\theta\sin\theta\,d\theta=-\frac{2}{K}(L^3x+Lx^3),$$

and

5.40 $$\int_{\operatorname{arc\,sec}(K/L-x)}^{\operatorname{arc\,sec}(K/-L-x)}\cos^4\theta\sin\theta=\frac{1}{5K^5}(2L^5+20L^3x+10Lx^4),$$

our integral for F_2 becomes

5.41 $$F_2=\frac{2}{3}\Big[L^3+3Lx^2\Big]\Big[\beta_0(t)+x\beta_1(t)+x^2\beta_2(t)\Big]\Big[\frac{1}{2^{3/2}K^2}-\frac{M}{2\cdot 3^{3/2}K}$$
$$+\frac{M^2}{3!4^{3/2}}-\frac{KM^3}{4!5^{3/2}}\Big]-2\Big[L^3x+Lx^3\Big]\Big[\beta_1(t)+2x\beta_2(t)\Big]\Big[\frac{1}{2^{5/2}K^2}$$
$$-\frac{M}{3^{5/2}K}+\frac{M^2}{2\cdot 4^{5/2}}-\frac{KM^3}{3!5^{5/2}}\Big]+\frac{2}{5}\Big[L^5+10L^3x^2+5Lx^4\Big]\beta_2(t)\cdot$$
$$\Big[\frac{1}{3\cdot 2^{3/2}K^2}-\frac{M}{4\cdot 3^{3/2}K}+\frac{M^2}{5\cdot 2\cdot 4^{3/2}}-\frac{KM^3}{6\cdot 3!5^{3/2}}\Big].$$

Treating F_3 similarly to F_1,

5.42 $$F_3=\int_{\operatorname{arc\,sec}(K/-L-x)}^{\pi}\int_0^{(-L-x)\sec\theta}(\text{same integrand as } F_1)\;d\rho\,d\theta$$

integrating with respect to ρ,

5.43 $$F_3=\int_{\operatorname{arc\,sec}(K/-L-x)}^{\pi}\sin\theta\Big\{\Big[\beta_0(t)+x\beta_1(t)+x^2\beta_2(t)\Big]\Big[\frac{(-L-x)\cos\theta}{2^{3/2}}$$
$$-\frac{(-L-x)^2M}{2\cdot 3^{3/2}}+\frac{(-L-x)^3M^2}{3!4^{3/2}\cos\theta}-\frac{(-L-x)^4M^3}{4!5^{3/2}\cos^2\theta}\Big]+\Big[\beta_1(t)+2x\beta_2(t)\Big]\cdot$$

$$\left[\frac{(-L-x)^2\cos\theta}{2^{5/2}} - \frac{(-L-x)^3 M}{3^{5/2}} + \frac{(-L-x)^4 M^2}{2\cdot 4^{5/2}\cos\theta} - \frac{(-L-x)^5 M^3}{3!\,5^{5/2}\cos^2\theta}\right]$$

$$+\beta_2(t)\left[\frac{(-L-x)^3\cos\theta}{3\cdot 2^{3/2}} - \frac{(L-x)^4 M}{4\cdot 3^{3/2}} + \frac{(L-x)^5 M^2}{5\cdot 2\cdot 4^{3/2}\cos\theta}\right.$$

$$\left.\left. - \frac{(-L-x)^6 M^3}{5\cdot 3!\,5^{3/2}\cos^2\theta}\right]\right\} d\theta ,$$

and integrating over θ, we have

5.44
$$F_3 = \left[\beta_0(t) + x\beta_1(t) + x^2\beta_2(t)\right]\left[\frac{-(-L-x)\cos^2\theta}{2\cdot 2^{3/2}} + \frac{(-L-x)^2 M\cos\theta}{2\cdot 3^{3/2}}\right.$$

$$\left. - \frac{(-L-x)^3 M^2}{3!\,4^{3/2}}\log(-\cos\theta) - \frac{(-L-x)^4 M^3}{4!\,5^{3/2}\cos\theta}\right] + \left[\beta_1(t)\right.$$

$$\left. + 2x\beta_2(t)\right]\left[-\frac{(-L-x)^2\cos^2\theta}{2^{7/2}} + \frac{(-L-x)^3 M\cos\theta}{3^{5/2}}\right.$$

$$\left. - \frac{(-L-x)^4 M^2}{2\cdot 4^{5/2}}\log(-\cos\theta) - \frac{(-L-x)^5 M^3}{3!\,5^{5/2}\cos\theta}\right] + \beta_2(t)\left[\right.$$

$$- \frac{(-L-x)^3\cos^2\theta}{3\cdot 2^{5/2}} + \frac{(-L-x)^4 M\cos\theta}{4\cdot 3^{3/2}} - \frac{(-L-x)^5 M^2}{5\cdot 2\cdot 4^{3/2}}\log(-\cos\theta)$$

$$\left.\left. - \frac{(-L-x)^6 M^3}{6\cdot 3!\,5^{3/2}}\right]\right\}\Bigg|_{\theta=\cos^{-1}((-L-x)/K)}^{\pi} .$$

Substituting limits, we have

5.45
$$F_3 = \left[\beta_0(t) + x\beta_1(t) + x^2\beta_2(t)\right]\left[-\frac{(L+x)^3}{2\cdot 2^{3/2}K^2} + \frac{(L+x)^3 M}{2\cdot 3^{3/2}K}\right.$$

$$- \frac{(L+x)^3 M^2}{3!\,4^{3/2}}\log(L+x) + \frac{(L+x)^3 M^2}{3!\,4^{3/2}}\log K - \frac{(L+x)^3 M^3 K}{4!\,5^{3/2}}$$

$$\left. + \frac{(L+x)}{2\cdot 2^{3/2}} - \frac{(L+x)^2 M}{2\cdot 3^{3/2}} + \frac{(L+x)^4 M^3}{4!\,5^{3/2}}\right] + \left[\beta_1(t) + 2\beta_2(t)\right]\cdot$$

$$\left[+\frac{(L+x)^4}{2^{9/2}K^2} - \frac{(L+x)^4 M}{3^{5/2}K} + \frac{(L+x)^4 M^2}{2\cdot 4^{5/2}}\log(L+x)\right.$$

$$- \frac{(L+x)^4 M^2}{2\cdot 4^{5/2}}\log K + \frac{(L+x)^4 M^3 K}{3!\,5^{5/2}} - \frac{(L+x)^2}{2^{7/2}} + \frac{(L+x)^3 M}{3^{5/2}}$$

$$\left. - \frac{(L+x)^5 M^3}{3!\,5^{5/2}}\right] + \beta_2(t)\left[-\frac{(L+x)^5}{3\cdot 2^{5/2}K^2} + \frac{(L+x)^5 M}{4\cdot 3^{3/2}K}\right.$$

$$-\frac{(L+x)^5M^2}{5\cdot2\cdot4^{3/2}}\log(L+x) + \frac{(L+x)^5M^2}{5\cdot2\cdot4^{3/2}}\log K - \frac{(L+x)^5M^3K}{6\cdot3!5^{3/2}}$$

$$+\frac{(L+x)^3}{3\cdot2^{3/2}} - \frac{(L+x)^4M}{4\cdot3^{3/2}} + \frac{(L+x)^6M^3}{6\cdot3!5^{3/2}}\Bigg].$$

Adding $F_1 + F_3$, we have

5.46
$$F_1+F_3 = \left[\beta_0(t) + x\beta_1(t) + x^2\beta_2(t)\right]\Bigg[-\frac{L^3}{2^{3/2}K^2} - \frac{3Lx^2}{2^{3/2}K^2} + \frac{L^3M}{3^{3/2}K}$$

$$+\frac{Lx^2M}{\sqrt{3}K} - \frac{(L+x)^3M^2\log(L+x)}{3!4^{3/2}} - \frac{(L-x)^3M^2\log(L-x)}{3!4^{3/2}}$$

$$+\frac{L^3M^2\log K}{24} + \frac{Lx^2M^2\log K}{8} - \frac{2L^3M^3K}{4!5^{3/2}} - \frac{6Lx^2M^3K}{4!5^{3/2}}$$

$$+\frac{L}{2^{3/2}} - \frac{L^2M}{3^{3/2}} - \frac{x^2M}{3^{3/2}} + \frac{2L^4M^3}{4!5^{3/2}} + \frac{12L^2x^2M^3}{4!5^{3/2}} + \frac{2x^4M^3}{4!5^{3/2}}\Bigg]$$

$$+\left[\beta_1(t)+2x\beta_2(t)\right]\Bigg[\frac{L^3x}{\sqrt{2}K^2} + \frac{Lx^3}{\sqrt{2}K^2} - \frac{8L^3xM}{3^{5/2}K} - \frac{8Lx^3M}{3^{5/2}K}$$

$$+\frac{(L+x)^4M^2\log(L+x)}{2\cdot4^{5/2}} - \frac{(L-x)^4M^2\log(L-x)}{2\cdot4^{5/2}} - \frac{L^3xM^2\log K}{8}$$

$$-\frac{Lx^3M^2\log K}{8} + \frac{8L^3xM^3K}{3!5^{5/2}} + \frac{8Lx^3M^3K}{3!5^{5/2}} - \frac{Lx}{2^{3/2}}$$

$$+\frac{6L^2xM}{3^{5/2}} + \frac{2x^3M}{3^{5/2}} - \frac{10L^4xM^3}{3!5^{5/2}} - \frac{20L^2x^3M^3}{3!5^{5/2}}$$

$$-\frac{2x^5M^3}{3!5^{5/2}}\Bigg] + \beta_2(t)\Bigg[-\frac{L^5}{3\cdot2^{3/2}K^2} - \frac{10L^3x^2}{3\cdot2^{3/2}K^2} - \frac{5Lx^4}{3\cdot2^{3/2}K^2}$$

$$-\frac{(L+x)^5M^2\log(L+x)}{5\cdot2\cdot4^{3/2}} - \frac{(L-x)^5M^2\log(L-x)}{5\cdot2\cdot4^{3/2}}$$

$$+\frac{(L^5+10L^3x^2+5Lx^4)M^2\log K}{5\cdot4^{3/2}} - \frac{L^5M^3K}{2\cdot45\sqrt{5}} - \frac{L^3x^2M^3K}{9\sqrt{5}}$$

$$-\frac{Lx^4M^3K}{18\sqrt{5}} + \frac{L^3}{3\cdot2^{3/2}} + \frac{Lx^2}{2^{3/2}} - \frac{L^4M}{2\cdot3^{3/2}} - \frac{L^2x^2M}{\sqrt{3}}$$

$$-\frac{x^4M}{2\cdot3^{3/2}} + \frac{L^6M^3}{90\sqrt{5}} + \frac{L^4x^2M^3}{6\sqrt{5}} + \frac{L^2x^4M^3}{6\sqrt{5}} + \frac{x^6M^3}{90\sqrt{5}}$$

$$+\frac{L^5M}{2\cdot3^{3/2}K} + \frac{5L^3x^2M}{3^{3/2}K} + \frac{5Lx^4M}{2\cdot3^{3/2}K}\Bigg]$$

and, on adding F_2 to the above, we have

5.47
$$F_1+F_2+F_3 = \left[\beta_0(t)+x\beta_1(t)+x^2\beta_2(t)\right]\left[-\frac{L^3}{6\sqrt{2}K^2} - \frac{Lx^2}{2\sqrt{2}K^2} + \frac{2L^3M}{3^{5/2}K}\right.$$
$$+\frac{2Lx^2M}{3^{3/2}K} - \frac{(L+x)^3M^2\log(L+x)}{48} - \frac{(L-x)^3M^2\log(L-x)}{48}$$
$$+\frac{L^3M^2\log K}{24} + \frac{Lx^2M^2\log K}{8} + \frac{L^3M^2}{72} + \frac{Lx^2M^2}{24}$$
$$-\frac{L^3KM^3}{45\sqrt{5}} - \frac{Lx^2KM^3}{15\sqrt{5}} + \frac{L}{2\sqrt{2}} - \frac{L^2M}{3\sqrt{3}} - \frac{x^2M}{3\sqrt{3}} + \frac{L^4M^3}{60\sqrt{5}}$$
$$\left.+\frac{L^2x^2M^3}{10\sqrt{5}} + \frac{x^4M^3}{60\sqrt{5}}\right] + \left[\beta_1(t)+2x\beta_2(t)\right]\left[\frac{L^3x}{2\sqrt{2}K^2}\right.$$
$$+\frac{Lx^3}{2\sqrt{2}K^2} - \frac{2L^3xM}{3\sqrt{3}K} - \frac{2Lx^3M}{3\sqrt{3}K} + \frac{(L+x)^4M^2\log(L+x)}{64}$$
$$-\frac{(L-x)^4M^2\log(L-x)}{64} - \frac{L^3xM^2\log K}{8} - \frac{Lx^3M^2\log K}{8}$$
$$-\frac{L^3xM^2}{32} - \frac{Lx^3M^2}{32} + \frac{L^3xKM^3}{15\sqrt{5}} + \frac{Lx^3KM^3}{15\sqrt{5}} - \frac{Lx}{2\sqrt{2}}$$
$$\left.+\frac{6L^2xM}{9\sqrt{3}} + \frac{2x^3M}{9\sqrt{3}} - \frac{10L^4xM^3}{150\sqrt{5}} - \frac{2L^2x^3M^3}{15\sqrt{5}} - \frac{x^5M^3}{75\sqrt{5}}\right]$$
$$+\beta_2(t)\left[-\frac{L^5}{10\sqrt{2}K^2} - \frac{L^3x^2}{\sqrt{2}K^2} - \frac{Lx^4}{2\sqrt{2}K^2} + \frac{2L^5M}{15\sqrt{3}K} + \frac{4L^3x^2M}{3\sqrt{3}K}\right.$$
$$+\frac{2Lx^4M}{3\sqrt{3}K} - \frac{(L+x)^5M^2\log(L+x)}{80} - \frac{(L-x)^5M^2\log(L-x)}{80}$$
$$+\frac{(L^5+10L^3x^2+5Lx^4)M^2\log K}{40} + \frac{(L^5+10Lx^3+5Lx^4)M^2}{200}$$
$$-\frac{L^5KM^3}{75\sqrt{5}} - \frac{2L^3x^2KM^3}{15\sqrt{5}} - \frac{Lx^4KM^3}{15\sqrt{5}} + \frac{L^3}{3\cdot 2^{3/2}} + \frac{Lx^2}{2^{3/2}}$$
$$-\frac{L^4M}{2\cdot 3^{3/2}} - \frac{L^2x^2M}{\sqrt{3}} - \frac{x^4M}{6\sqrt{3}} + \frac{L^6M^3}{90\sqrt{5}} + \frac{L^4x^2M^3}{6\sqrt{5}}$$
$$\left.+\frac{L^2x^4M^3}{6\sqrt{5}} + \frac{x^6M^3}{90\sqrt{5}}\right].$$

Grouping coefficients of the $\beta_i(t)$'s, we find

5.48
$$F_1+F_2+F_3 = \beta_0(t)\left[-\frac{L^3}{6\sqrt{2}K^2} - \frac{Lx^2}{2\sqrt{2}K^2} + \frac{2L^3M}{3^{5/2}K} + \frac{2Lx^2M}{3^{3/2}K}\right.$$
$$-\frac{(L+x)^3M^2\log(L+x)}{48} - \frac{(L-x)^3M^2\log(L-x)}{48} + \frac{L^3M^2\log K}{24}$$

$$+\frac{Lx^2M^2}{8}\log K+\frac{L^3M^2}{72}+\frac{Lx^2M^2}{24}-\frac{L^3KM^3}{45\sqrt{5}}$$

$$-\frac{Lx^2KM^3}{15\sqrt{5}}+\frac{L}{2\sqrt{2}}-\frac{L^2M}{3\sqrt{3}}-\frac{x^2M}{3\sqrt{3}}+\frac{L^4M^3}{60\sqrt{5}}$$

$$+\frac{L^2x^2M^3}{10\sqrt{5}}+\frac{x^4M^3}{60\sqrt{5}}\Bigg]+\beta_1(t)\Bigg[-\frac{L^3x}{6\sqrt{2}K^2}-\frac{Lx^3}{2\sqrt{2}K^2}$$

$$+\frac{2L^3xM}{3^{5/2}K}+\frac{2Lx^3M}{3^{3/2}K}-\frac{x(L+x)^3M^2}{48}\log(L+x)$$

$$-\frac{x(L-x)^3M^2}{48}\log(L-x)+\frac{L^3xM^2}{24}\log K+\frac{Lx^3M^2}{8}\log K$$

$$+\frac{L^3xM^2}{72}+\frac{Lx^3M^2}{24}-\frac{L^3xKM^3}{45\sqrt{5}}-\frac{Lx^3KM^3}{15\sqrt{5}}+\frac{Lx}{2\sqrt{2}}$$

$$-\frac{L^2xM}{3\sqrt{3}}-\frac{x^3M}{3\sqrt{3}}+\frac{L^4xM^3}{60\sqrt{5}}+\frac{L^2x^3M^3}{10\sqrt{5}}+\frac{x^5M^3}{60\sqrt{5}}+\frac{L^3x}{2\sqrt{2}K^2}$$

$$+\frac{Lx^3}{2\sqrt{2}K^2}-\frac{2L^3xM}{3\sqrt{3}K}-\frac{2Lx^3M}{3\sqrt{3}K}+\frac{(L+x)^4M^2}{64}\log(L+x)$$

$$-\frac{(L-x)^4M^2}{64}\log(L-x)-\frac{L^3xM^2}{8}\log K-\frac{Lx^3M^2}{8}\log K$$

$$-\frac{L^3xM^2}{32}-\frac{Lx^3M^2}{32}+\frac{L^3xKM^3}{15\sqrt{5}}+\frac{Lx^3KM^3}{15\sqrt{5}}-\frac{Lx}{2\sqrt{2}}$$

$$+\frac{6L^2xM}{9\sqrt{3}}+\frac{2x^3M}{9\sqrt{3}}-\frac{10L^4xM^3}{150\sqrt{5}}-\frac{2L^2x^3M^3}{15\sqrt{5}}-\frac{x^5M^3}{75\sqrt{5}}\Bigg]$$

$$+\beta_2(t)\Bigg[-\frac{L^3x^2}{6\sqrt{2}K^2}-\frac{Lx^4}{2\sqrt{2}K^2}+\frac{2L^3x^2M}{3^{5/2}K}+\frac{2Lx^4M}{3^{3/2}K}$$

$$-\frac{x^2(L+x)^3M^2}{48}\log(L+x)-\frac{x^2(L-x)^3M^2}{48}\log(L-x)+\frac{L^3x^2M^2}{24}\log K$$

$$+\frac{Lx^4M^2}{8}\log K+\frac{L^3x^2M^2}{72}+\frac{Lx^4M^2}{24}-\frac{L^3x^2KM^3}{45\sqrt{5}}$$

$$-\frac{Lx^4KM^3}{15\sqrt{5}}+\frac{Lx^2}{2\sqrt{2}}-\frac{L^2x^2M}{3\sqrt{3}}-\frac{x^4M}{3\sqrt{3}}+\frac{L^4x^2M^3}{60\sqrt{5}}+\frac{L^2x^4M^3}{10\sqrt{5}}$$

$$+\frac{x^6M^3}{60\sqrt{5}}+\frac{L^3x^2}{\sqrt{2}K^2}+\frac{Lx^4}{\sqrt{2}K^2}-\frac{4L^3x^2M}{3\sqrt{3}K}-\frac{4Lx^4M}{3\sqrt{3}K}$$

$$+\frac{x(L+x)^4M^2}{32}\log(L+x) - \frac{x(L-x)^4M^2}{32}\log(L-x)$$

$$-\frac{L^3x^2M^2}{4}\log K - \frac{Lx^4M^2}{4}\log K - \frac{L^3x^2M^2}{16} - \frac{Lx^4M^2}{16}$$

$$+\frac{2L^3x^2KM^3}{15\sqrt{5}} + \frac{2Lx^4KM^3}{15\sqrt{5}} - \frac{Lx^2}{\sqrt{2}} + \frac{12L^2x^2M}{9\sqrt{3}}$$

$$+\frac{4x^4M}{9\sqrt{3}} - \frac{2L^4x^2M^3}{15\sqrt{5}} - \frac{4L^2x^4M^3}{15\sqrt{5}} - \frac{2x^6M^3}{75\sqrt{5}} - \frac{L^5}{10\sqrt{2}K^2}$$

$$-\frac{L^3x^2}{\sqrt{2}K^2} - \frac{Lx^4}{2\sqrt{2}K^2} + \frac{2L^5M}{15\sqrt{3}K} + \frac{4L^3x^2M}{3\sqrt{3}K} + \frac{2Lx^4M}{3\sqrt{3}K}$$

$$-\frac{(L+x)^5M^2}{80}\log(L+x) - \frac{(L-x)^5M^2}{80}\log(L-x)$$

$$+\frac{(L^5+10L^3x^2+5Lx^4)M^2}{40}\log K + \frac{(L^5+10L^3x^2+5Lx^4)M^2}{200}$$

$$-\frac{L^5KM^3}{75\sqrt{5}} - \frac{2L^3x^2KM^3}{15\sqrt{5}} - \frac{Lx^4KM^3}{15\sqrt{5}} + \frac{L^3}{3\cdot 2^{3/2}}$$

$$+\frac{Lx^2}{2^{3/2}} - \frac{L^4M}{6\sqrt{3}} - \frac{L^2x^2M}{\sqrt{3}} - \frac{x^4M}{6\sqrt{3}} + \frac{L^6M^3}{90\sqrt{5}}$$

$$+\frac{L^4x^2M^3}{6\sqrt{5}} + \frac{L^2x^4M^3}{6\sqrt{5}} + \frac{x^6M^3}{90\sqrt{5}}\Big].$$

Collecting like terms, the above reduces to

5.49 $$F_1+F_2+F_3 = \beta_0(t)\Big[-\frac{L^3}{6\sqrt{2}K^2} - \frac{Lx^2}{2\sqrt{2}} + \frac{2L^3M}{9\sqrt{3}K} + \frac{2Lx^2M}{3\sqrt{3}K}$$

$$-\frac{(L+x)^3M^2}{48}\log(L+x) - \frac{(L-x)^3M^2}{48}\log(L-x)$$

$$+\frac{L^3M^2}{24}\log K + \frac{Lx^2M^2}{8}\log K + \frac{L^3M^2}{72} + \frac{Lx^2M^2}{24}$$

$$-\frac{L^3KM^3}{45\sqrt{5}} - \frac{Lx^2KM^3}{15\sqrt{5}} + \frac{L}{2\sqrt{2}} - \frac{L^2M}{3\sqrt{3}} - \frac{x^2M}{3\sqrt{3}}$$

$$+\frac{L^4M^3}{60\sqrt{5}} + \frac{L^2x^2M^3}{10\sqrt{5}} + \frac{x^4M^3}{60\sqrt{5}}\Big] + \beta_1(t)\Big[$$

$$-\frac{x(L+x)^3M^2}{48}\log(L+x) - \frac{x(L-x)^3M^2}{48}\log(L-x)$$

$$+\frac{(L+x)^4M^2}{64}\log(L+x) - \frac{(L-x)^4M^2}{64}\log(L-x)$$

$$-\frac{L^3xM^2}{12}\log K+\frac{L^3x}{3\sqrt{2}K^2}-\frac{4L^3xM}{9\sqrt{3}K}-\frac{5L^3xM^2}{288}$$

$$+\frac{Lx^3M^2}{96}+\frac{2L^3xKM^3}{45\sqrt{5}}+\frac{L^2xM}{3\sqrt{3}}-\frac{x^3M}{9\sqrt{3}}-\frac{L^4xM^3}{20}$$

$$-\frac{L^2x^3M^3}{30\sqrt{5}}+\frac{x^5M^3}{300\sqrt{5}}\Big]+\beta_2(t)\Big[-\frac{x^2(L+x)^3M^2}{48}\log(L+x)$$

$$-\frac{x^2(L-x)^3M^2}{48}\log(L-x)+\frac{x(L+x)^4M^2}{32}\log(L+x)$$

$$-\frac{x(L-x)^4M^2}{32}\log(L-x)-\frac{(L+x)^5M^2}{80}\log(L+x)$$

$$-\frac{(L-x)^5M^2}{80}\log(L-x)+\frac{L^5M^2}{40}\log K+\frac{L^3x^2M^2}{24}\log K$$

$$-\frac{L^5}{10\sqrt{2}K^2}-\frac{L^3x^2}{6\sqrt{2}K^2}+\frac{2L^5M}{15\sqrt{3}K}+\frac{2L^3x^2M}{9\sqrt{3}K}+\frac{L^5M^2}{200}$$

$$+\frac{L^3x^2M^2}{720}+\frac{Lx^4M^2}{240}-\frac{L^5KM^3}{75\sqrt{5}}-\frac{L^3x^2KM^3}{45\sqrt{5}}$$

$$+\frac{L^3}{6\sqrt{2}}-\frac{L^4M}{6\sqrt{3}}-\frac{x^4M}{18\sqrt{3}}+\frac{L^6M^3}{90\sqrt{5}}+\frac{L^4x^2M^3}{20\sqrt{5}}+\frac{x^6M^3}{900\sqrt{5}}\Big].$$

CHAPTER VI

EVALUATION OF MacLAURIN SERIES COEFFICIENTS $\alpha_i(t)$'s and $\beta_i(t)$'s; $i = 1,2,3$. FINAL SOLUTION

Now our two equations, 5.11 and 5.12 involve six coefficients, the $\alpha_i(t)$'s and the $\beta_i(t)$'s, $i = 1,2,3$, which we evaluate as follows.

We multiply through each equation by 1, x and x and each time integrate term by term, with respect to x between the limits (-L, L). We obtain six linear first order differential equations in the time-dependent coefficients. The six boundary conditions required for a solution are, of course, the values of each at time zero.

To group similar proceedures in each equation, let us carry through with the left-hand sides first. The left-hand side of equation 5.11 becomes, on substituting for $\alpha(x, t)$ and $\beta(x, t)$, as given in 5.3 and 5.4,

6.1
$$\frac{d\alpha_0(t)}{dt} + x\frac{d\alpha_1(t)}{dt} + x^2\frac{d\alpha_2(t)}{dt} + \frac{kT}{m}\beta_1(t) + \frac{2x\,kT}{m}\beta_2(t) + a\alpha_0(t) + ax\,\alpha_1(t) + ax^2\alpha_2(t).$$

Equation 6.1 integrated over the limits $-L < x < L$ gives

6.2
$$2L\frac{d\alpha_0(t)}{dt} + \frac{2L^3}{3}\frac{d\alpha_2(t)}{dt} + \frac{2L\,kT}{m}\beta_1(t) + 2aL\alpha_0(t) + \frac{2aL^3}{3}\alpha_2(t).$$

Multiplying 6.1 by x and integrating, we get

6.3
$$\frac{2L^3}{3}\frac{d\alpha_1(t)}{dt} + \frac{4L^3}{3}\frac{kT}{m}\beta_2(t) + \frac{2aL^3}{3}\alpha_1(t).$$

Similarly, multiplying 6.1 by x^2 and integrating yields

6.4
$$\frac{2L^3}{3}\frac{d\alpha_0(t)}{dt} + \frac{2L^5}{5}\frac{d\alpha_2(t)}{dt} + \frac{2L^3}{3}\frac{kT}{m}\beta_1(t)$$

$$+\frac{2aL^3}{3}\alpha_0(t) + \frac{2aL^5}{5}\alpha_2(t).$$

Similarly, the left-hand side of 5.12 becomes, on substituting for $\alpha(x,t)$ and $\beta(x,t)$,

6.5 $$\frac{d\beta_0(t)}{dt} + x\frac{d\beta_1(t)}{dt} + x^2\frac{d\beta_2(t)}{dt} + \alpha_1(t) + 2x\alpha_2(t)$$
$$+a\beta_0(t) + ax\beta_1(t) + ax^2\beta_2(t).$$

Integrating 6.5 over (-L, L) yields

6.6 $$2L\frac{d\beta_0}{dt}(t) + \frac{2L^3}{3}\frac{d\beta_2(t)}{dt} + 2L\alpha_1(t) + 2aL\beta_0(t)$$
$$+\frac{2aL^3}{3}\beta_2(t).$$

Multiplying 6.5 by x and integrating, we have

6.7 $$\frac{2L^3}{3}\frac{d\beta_1(t)}{dt} + \frac{4L^3}{3}\alpha_2(t) + \frac{2aL^3}{3}\beta_1(t).$$

Multiplying 6.5 by x^2 and integrating, we have

6.8 $$\frac{2L^3}{3}\frac{d\beta_0(t)}{dt} + \frac{2L^5}{5}\frac{d\beta_2(t)}{dt} + \frac{2L^3}{3}\alpha_1(t) + \frac{2aL^3}{3}\beta_0(t)$$
$$+\frac{2aL^5}{5}\beta_2(t).$$

Now, 6.2, 6.3, 5.4, 6.6, 6.7 and 6.8 are the left-hand sides, respectively, of the required differential equations.

Before evaluating the right-hand sides, let us evaluate some integrals that will appear. They are:

6.9 $$\int_{-L}^{L}(L+x)\log(L+x)dx = 2L^2\log 2L - L^2$$

6.10 $$\int_{-L}^{L}x(L+x)\log(L+x)\,dx = \frac{2L^3}{3}\log 2L + \frac{L^3}{9}$$

6.11 $$\int_{-L}^{L}x^2(L+x)\log(L+x)\,dx = \frac{2L^4}{3}\log 2L - \frac{2L^4}{9}$$

6.12 $\int_{-L}^{L} x^3 (L+x) \log (L+x) dx = \frac{2L^5}{5} \log 2L + \frac{4L^5}{75}$

6.13 $\int_{-L}^{L} x^4 (L+x) \log (L+x) dx = \frac{2L^6}{5} \log 2L - \frac{23L^6}{225}$

6.14 $\int_{-L}^{L} (L-x) \log (L-x) dx = 2L^2 \log 2L - L^2$

6.15 $\int_{-L}^{L} x (L-x) \log (L-x) dx = -\frac{2L^3}{3} \log 2L - \frac{L^3}{9}$

6.16 $\int_{-L}^{L} x^2 (L-x) \log (L-x) dx = \frac{2L^4}{3} \log 2L - \frac{2L^4}{9}$

6.17 $\int_{-L}^{L} x^3 (L-x) \log (L-x) dx = -\frac{2L^5}{5} \log 2L - \frac{4L^5}{75}$

6.18 $\int_{-L}^{L} x^4 (L-x) \log (L-x) dx = \frac{2L^6}{5} \log 2L - \frac{23L^6}{225}$

6.19 $\int_{-L}^{L} (L+x)^2 \log (L+x) dx = \frac{8L^3}{3} \log 2L - \frac{8L^3}{9}$

6.20 $\int_{-L}^{L} x (L+x)^2 \log (L+x) dx = \frac{4L^4}{3} \log 2L - \frac{L^4}{9}$

6.21 $\int_{-L}^{L} x^2 (L+x)^2 \log (L+x) dx = \frac{16L^5}{15} \log 2L - \frac{38L^5}{225}$

6.22 $\int_{-L}^{L} x^3 (L+x)^2 \log (L+x) dx = \frac{12L^6}{15} \log 2L - \frac{11L^6}{225}$

6.23 $\int_{-L}^{L} (L-x)^2 \log (L-x) dx = \frac{8L^3}{3} \log 2L - \frac{8L^3}{9}$

6.24 $\int_{-L}^{L} x (L-x)^2 \log (L-x) dx = -\frac{4L^4}{3} \log 2L + \frac{L^4}{9}$

6.25 $$\int_{-L}^{L} x^2(L-x)^2\log(L-x)dx = \frac{16L^5}{15}\log 2L - \frac{38L^5}{225}$$

6.26 $$\int_{-L}^{L} x^3(L-x)^2\log(L-x)dx = -\frac{12L^6}{15}\log 2L + \frac{11L^6}{225}$$

6.27 $$\int_{-L}^{L} (L+x)^3\log(L+x)dx = 4L^4\log 2L - L^4$$

6.28 $$\int_{-L}^{L} x(L+x)^3\log(L+x)dx = \frac{12L^5}{5}\log 2L - \frac{7L^5}{25}$$

6.29 $$\int_{-L}^{L} x^2(L+x)^3\log(L+x)dx = \frac{28L^6}{15}\log 2L - \frac{49L^6}{225}$$

6.30 $$\int_{-L}^{L} x^3(L+x)^3\log(L+x)dx = \frac{52L^7}{35}\log 2L - \frac{5L^7}{42}$$

6.31 $$\int_{-L}^{L} x^4(L+x)^3\log(L+x)dx = \frac{44L^8}{35}\log 2L - \frac{17L^8}{175}$$

6.32 $$\int_{-L}^{L} (L-x)^3\log(L-x)dx = 4L^4\log 2L - L^4$$

6.33 $$\int_{-L}^{L} x(L-x)^3\log(L-x)dx = -\frac{12L^5}{5}\log 2L + \frac{7L^7}{25}$$

6.34 $$\int_{-L}^{L} x^2(L-x)^3\log(L-x)dx = \frac{28L^6}{15}\log 2L - \frac{49L^6}{225}$$

6.35 $$\int_{-L}^{L} x^3(L-x)^3\log(L-x)dx = -\frac{52L^7}{35}\log 2L + \frac{5L^7}{42}$$

6.36 $$\int_{-L}^{L} x^4(L-x)^3\log(L-x)dx = \frac{44L^8}{35}\log 2L - \frac{17L^8}{175}$$

6.37 $$\int_{-L}^{L} (L+x)^4\log(L+x)dx = \frac{32L^5}{5}\log 2L - \frac{32L^5}{25}$$

6.38 $$\int_{-L}^{L} x(L+x)^4 \log(L+x)\,dx = \frac{64L^6}{15}\log 2L - \frac{112L^6}{225}$$

6.39 $$\int_{-L}^{L} x^2(L+x)^4 \log(L+x)\,dx = \frac{352L^7}{105}\log 2L - \frac{71L^7}{210}$$

6.40 $$\int_{-L}^{L} x^3(L+x)^4 \log(L+x)\,dx = \frac{96L^8}{35}\log 2L - \frac{3L^8}{14}$$

6.41 $$\int_{-L}^{L} (L-x)^4 \log(L-x)\,dx = \frac{32L^5}{5}\log 2L - \frac{32L^5}{25}$$

6.42 $$\int_{-L}^{L} x(L-x)^4 \log(L-x)\,dx = -\frac{64L^6}{15}\log 2L + \frac{112L^6}{225}$$

6.43 $$\int_{-L}^{L} x^2(L-x)^4 \log(L-x)\,dx = \frac{352L^7}{105}\log 2L - \frac{71L^7}{210}$$

6.44 $$\int_{-L}^{L} x^3(L-x)^4 \log(L-x)\,dx = -\frac{96L^8}{35}\log 2L + \frac{3L^8}{14}$$

6.45 $$\int_{-L}^{L} (L+x)^5 \log(L+x)\,dx = \frac{32L^6}{3}\log 2L - \frac{16L^6}{9}$$

6.46 $$\int_{-L}^{L} x(L+x)^5 \log(L+x)\,dx = \frac{160L^7}{21}\log 2L - \frac{368L^7}{441}$$

6.47 $$\int_{-L}^{L} x^2(L+x)^5 \log(L+x)\,dx = \frac{128L^8}{21}\log 2L - \frac{244L^8}{441}$$

6.48 $$\int_{-L}^{L} (L-x)^5 \log(L-x)\,dx = \frac{32L^6}{3}\log 2L - \frac{16L^6}{9}$$

6.49 $$\int_{-L}^{L} x(L-x)^5 \log(L-x)\,dx = -\frac{160L^7}{21}\log 2L + \frac{368L^7}{441}$$

6.50 $$\int_{-L}^{L} x^2(L-x)^5 \log(L-x)\,dx = \frac{128L^8}{21}\log 2L - \frac{244L^8}{441}$$

Now we are ready to determine the right-hand side of our six equations in the time-dependent coefficients. These will be aM/2 times the respective one of the six integrals we now evaluate.

From 5.31 and the necessary integrals in our preceeding list, we s see that

6.51
$$\int_{-L}^{L}(v_1+v_2+v_3)\,dx = \alpha_0(t)\Big[-\frac{2L^2}{\sqrt{2}}\log 2L + \frac{L^2}{\sqrt{2}} - \frac{2L^2}{\sqrt{2}}\log 2L + \frac{L^2}{\sqrt{2}}$$
$$+\frac{4L^2}{\sqrt{2}}\log K + \frac{4L^2}{\sqrt{2}} - \frac{4L^2KM}{\sqrt{3}} + \frac{L^2K^2M^2}{2} - \frac{2L^2K^3M^3}{9\sqrt{5}}$$
$$+\frac{2L^3M}{\sqrt{3}} + \frac{2L^3M}{3\sqrt{3}} - \frac{L^4M^2}{6} - \frac{L^4M^2}{6} + \frac{L^5M^3}{18\sqrt{5}} + \frac{L^5M^3}{9\sqrt{5}}$$
$$+\frac{L^5M^3}{90\sqrt{5}}\Big] + \alpha_1(t)\Big[-\frac{2L^3}{3\sqrt{2}}\log 2L - \frac{L^3}{9\sqrt{2}} + \frac{2L^3}{3\sqrt{2}} + \frac{L^3}{9\sqrt{2}} + \frac{8L^3}{6\sqrt{2}}\log 2L$$
$$-\frac{8L^3}{18\sqrt{2}} - \frac{8L^3}{6\sqrt{2}} + \frac{8L^3}{18\sqrt{2}}\Big] + \alpha_2(t)\Big[-\frac{2L^4}{3\sqrt{2}}\log 2L + \frac{2L^4}{9\sqrt{2}}$$
$$-\frac{2L^4}{3\sqrt{2}}\log 2L + \frac{2L^4}{9\sqrt{2}} + \frac{4L^4}{3\sqrt{2}}\log 2L - \frac{L^4}{9\sqrt{2}} + \frac{4L^4}{3\sqrt{2}}\log 2L$$
$$-\frac{L^4}{9\sqrt{2}} - \frac{4L^4}{3\sqrt{2}}\log 2L + \frac{L^4}{3\sqrt{2}} - \frac{4L^4}{3\sqrt{2}}\log 2L + \frac{L^4}{3\sqrt{2}} + \frac{4L^4}{3\sqrt{2}}\log K$$
$$+\frac{4L^4}{9\sqrt{2}} + \frac{4L^4}{9\sqrt{2}} - \frac{4L^4KM}{3\sqrt{3}} + \frac{L^4K^2M^2}{6} - \frac{2L^4K^3M^3}{27\sqrt{5}}$$
$$+\frac{L^5M}{\sqrt{3}} + \frac{2L^5M}{30\sqrt{3}} - \frac{L^6M^2}{10} - \frac{L^6M^2}{18} + \frac{L^7M^3}{27\sqrt{5}} + \frac{L^7M^3}{18\sqrt{5}} + \frac{L^7M^3}{1890\sqrt{5}}\Big],$$

which, on collecting terms, reduces to

6.52
$$\int_{-L}^{L}(v_1+v_2+v_3)\,dx = \alpha_0(t)\Big[-\frac{4L^2}{\sqrt{2}}\log 2L + \frac{4L^2}{\sqrt{2}}\log K + \frac{6L^2}{\sqrt{2}}$$
$$-\frac{4L^2KM}{\sqrt{3}} + \frac{L^2K^2M^2}{2} - \frac{2L^2K^3M^3}{9\sqrt{5}} + \frac{8L^3M}{3\sqrt{3}} - \frac{L^4M^2}{3}$$
$$+\frac{8L^5M^3}{45\sqrt{5}}\Big] + \alpha_2(t)\Big[-\frac{4L^4}{3\sqrt{2}}\log 2L + \frac{4L^4}{3\sqrt{2}}\log K + \frac{16L^4}{9\sqrt{2}}$$
$$-\frac{4L^4KM}{3\sqrt{3}} + \frac{L^4K^2M^2}{6} - \frac{2L^4K^3M^3}{27\sqrt{5}} + \frac{16L^5M}{15\sqrt{3}} - \frac{7L^6M^2}{45} + \frac{88L^7M^3}{945\sqrt{5}}\Big].$$

Similarly,

6.53

$$\int_{-L}^{L} x(\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3)\,dx = \alpha_0(t)\Big[-\frac{2L^3}{3\sqrt{2}}\log 2L + \frac{L^3}{9\sqrt{2}} + \frac{2L^3}{3\sqrt{2}}\log 2L - \frac{L^3}{9\sqrt{2}}\Big] + \alpha_1(t)\Big[-\frac{2L^4}{3\sqrt{2}}\log 2L + \frac{2L^4}{9\sqrt{2}} - \frac{2L^4}{3\sqrt{2}}\log 2L + \frac{2L^4}{9\sqrt{2}} + \frac{2L^4}{3\sqrt{2}}\log 2L - \frac{L^4}{18\sqrt{2}} + \frac{2L^4}{3\sqrt{2}}\log 2L - \frac{L^4}{18\sqrt{2}} + \frac{2L^4}{3\sqrt{2}} - \frac{2L^5M}{3\sqrt{3}} + \frac{2L^5M}{15\sqrt{3}} + \frac{L^6M^2}{9} - \frac{L^7M^3}{18\sqrt{5}} - \frac{L^7M^3}{45\sqrt{5}} + \frac{x^7M^3}{1260\sqrt{5}}\Big] + \alpha_2\big[0\big],$$

which reduces to

6.54

$$\int_{-L}^{L} x(\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3)\,dx = \alpha_1(t)\Big[\frac{L^4}{\sqrt{2}} - \frac{8L^5M}{15\sqrt{3}} + \frac{L^6M^2}{9} - \frac{97L^7M^3}{1260\sqrt{5}}\Big].$$

Similarly,

6.55

$$\int_{-L}^{L} x^2(\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3)\,dx = \alpha_0(t)\Big[-\frac{2L^4}{3\sqrt{2}}\log 2L + \frac{2L^4}{9\sqrt{2}} - \frac{2L^4}{3\sqrt{2}}\log 2L + \frac{2L^4}{9\sqrt{2}} + \frac{4L^4}{3\sqrt{2}}\log K + \frac{4L^4}{3\sqrt{2}} - \frac{4L^4KM}{3\sqrt{3}} + \frac{L^4K^2M^2}{6} - \frac{2L^4K^3M^3}{27\sqrt{5}} + \frac{2L^5M}{3\sqrt{3}} + \frac{2L^5M}{5\sqrt{3}} - \frac{L^6M^2}{18} - \frac{L^6M^2}{10} + \frac{L^7M^3}{54\sqrt{5}} + \frac{L^3M^3}{15\sqrt{5}} + \frac{L^7M^3}{126\sqrt{5}}\Big] + \alpha_1(t)\big[0\big] + \alpha_2(t)\Big[-\frac{2L^6}{5\sqrt{2}}\log 2L + \frac{23L^6}{225\sqrt{2}} - \frac{2L^6}{5\sqrt{2}}\log 2L + \frac{23L^6}{225\sqrt{2}} + \frac{12L^6}{15\sqrt{2}}\log 2L - \frac{11L^6}{225\sqrt{2}} + \frac{12L^6}{15\sqrt{2}}\log 2L - \frac{11L^6}{225\sqrt{2}} - \frac{28L^6}{45\sqrt{2}}\log 2L + \frac{49L^6}{675\sqrt{2}} - \frac{28L^6}{45\sqrt{2}}\log 2L$$

$$+\frac{49L^6}{675\sqrt{2}}+\frac{4L^6}{9\sqrt{2}}\log K+\frac{4L^6}{27\sqrt{2}}+\frac{4L^6}{15\sqrt{2}}-\frac{4L^6KM}{9\sqrt{3}}$$

$$+\frac{L^6K^2M^2}{18}-\frac{2L^6K^3M^3}{81\sqrt{5}}+\frac{L^7M}{3\sqrt{3}}+\frac{L^7M}{21\sqrt{3}}-\frac{L^8M^2}{30}$$

$$-\frac{L^8M^2}{30}+\frac{L^9M^3}{81\sqrt{5}}+\frac{L^9M^3}{30\sqrt{5}}+\frac{L^9M^3}{2430\sqrt{5}}\Big]$$

which reduces to

6.56
$$\int_{-L}^{L}x^2(J_1+J_2+J_3)dx=\alpha_0(t)\Big[-\frac{4L^4}{3\sqrt{2}}\log 2L-\frac{4L^4}{3\sqrt{2}}\log K$$

$$+\frac{16L^4}{9\sqrt{2}}-\frac{4L^4KM}{3\sqrt{3}}+\frac{L^4K^2M^2}{6}-\frac{2L^4K^3M^3}{27\sqrt{5}}$$

$$+\frac{16L^5M}{15\sqrt{3}}-\frac{7L^6M^2}{45}+\frac{176L^7M^3}{1890\sqrt{5}}\Big]+\alpha_2(t)\Big[-\frac{4L^6}{9\sqrt{2}}\log 2L$$

$$+\frac{4L^6}{9\sqrt{2}}\log K+\frac{2L^6}{3\sqrt{2}}-\frac{4L^6KM}{9\sqrt{3}}+\frac{L^6K^2M^2}{18}-\frac{2L^6K^3M^3}{81\sqrt{5}}$$

$$+\frac{8L^7M}{21\sqrt{3}}-\frac{L^8M^2}{15}+\frac{112L^9M^3}{2430\sqrt{5}}\Big].$$

From 5.49 and the necessary integrals in our list, we find

6.57
$$\int_{-L}^{L}(F_1+F_2+F_3)dx=\beta_0(t)\Big[-\frac{L^4M^2}{12}\log 2L+\frac{L^4M^2}{48}$$

$$-\frac{L^4M^2}{12}\log 2L+\frac{L^4M^2}{48}-\frac{L^4}{3\sqrt{2}K^2}-\frac{L^4}{3\sqrt{2}K^2}+\frac{4L^4M}{9\sqrt{3}K}$$

$$+\frac{4L^4M}{9\sqrt{3}K}+\frac{L^4M^2}{12}\log K+\frac{L^4M^2}{12}\log K+\frac{L^4M^2}{36}$$

$$+\frac{L^4M^2}{36}-\frac{2L^4KM^3}{45\sqrt{5}}-\frac{2L^4KM^3}{45\sqrt{5}}+\frac{L^2}{\sqrt{2}}-\frac{2L^3M}{3\sqrt{3}}$$

$$-\frac{2L^3M}{9\sqrt{3}}+\frac{L^5M^3}{30\sqrt{5}}+\frac{L^5M^3}{15\sqrt{5}}+\frac{L^5M^3}{150\sqrt{5}}\Big]+\beta_1(t)\Big[0\Big]$$

$$+\beta_2(t)\Big[-\frac{7L^6M^2}{180}\log 2L+\frac{49L^6M^2}{10800}-\frac{7L^6M^2}{180}\log 2L$$

$$+\frac{49L^6M^2}{10800}+\frac{2L^6M^2}{15}\log 2L-\frac{7L^6M^2}{450}+\frac{2L^6M^2}{15}\log 2L$$

$$-\frac{7L^6M^2}{450} - \frac{2L^6M^2\log 2L}{15} + \frac{L^6M^2}{45} - \frac{2L^6M^2\log 2L}{15}$$

$$+\frac{L^6M^2}{45} + \frac{L^6M^2\log K}{20} + \frac{L^6M^2\log K}{36} - \frac{L^6}{5\sqrt{2}K^2}$$

$$-\frac{L^6}{9\sqrt{2}K^2} + \frac{4L^6M}{15\sqrt{3}K} + \frac{4L^6M}{27\sqrt{3}K} + \frac{L^6M^2}{100} + \frac{L^6M^2}{1080}$$

$$+\frac{L^6M^2}{600} - \frac{2L^6KM^3}{75\sqrt{5}} - \frac{2L^6KM^3}{135\sqrt{5}} + \frac{L^4}{3\sqrt{2}} - \frac{L^5M}{3\sqrt{3}} - \frac{L^5M}{45\sqrt{3}}$$

$$+\frac{L^7M^3}{45\sqrt{5}} + \frac{L^7M^3}{30\sqrt{5}} + \frac{L^7M^3}{3150\sqrt{5}}\Bigg],$$

which reduces to

6.58
$$\int_{-L}^{L}(F_1+F_2+F_3)\,dx = \beta_0(t)\Bigg[-\frac{L^4M^2\log 2L}{6} + \frac{L^4M^2\log K}{6}$$

$$-\frac{2L^4}{3\sqrt{2}K^2} + \frac{8L^4M}{9\sqrt{3}K} - \frac{4L^4KM^3}{45\sqrt{5}} + \frac{L^2}{\sqrt{2}} - \frac{8L^3M}{9\sqrt{3}}$$

$$+\frac{7L^4M^2}{72} + \frac{8L^5M^3}{75\sqrt{5}}\Bigg] + \beta_2(t)\Bigg[-\frac{7L^6M^2\log 2L}{90}$$

$$+\frac{7L^6M^2\log K}{90} - \frac{14L^6}{45\sqrt{2}K^2} + \frac{56L^6M}{135\sqrt{3}K} - \frac{28L^6KM^3}{675\sqrt{5}}$$

$$+\frac{L^4}{3\sqrt{2}} - \frac{16L^5M}{45\sqrt{3}} + \frac{7L^6M^2}{200} + \frac{88L^7M^3}{1575\sqrt{5}}\Bigg].$$

Similarly,

6.59
$$\int_{-L}^{L}x(F_1+F_2+F_3)\,dx = \beta_0(t)\Big[0\Big] + \beta_1(t)\Bigg[-\frac{7L^6M^2\log 2L}{180}$$

$$+\frac{49L^6M^2}{10800} - \frac{7L^6M^2\log 2L}{180} + \frac{49L^6M^2}{10800} + \frac{L^6M^2\log 2L}{15}$$

$$-\frac{7L^6M^2}{900} + \frac{L^6M^2\log 2L}{15} - \frac{7L^6M^2}{900} - \frac{L^6M^2\log K}{18} + \frac{2L^6}{9\sqrt{2}K^2}$$

$$-\frac{8L^6M}{27\sqrt{3}K} - \frac{10L^6M^2}{864} + \frac{L^6M^2}{240} + \frac{4L^6KM^3}{135\sqrt{5}} + \frac{2L^5M}{9\sqrt{3}}$$

$$-\frac{2L^5M}{45\sqrt{3}} - \frac{L^7M^3}{30\sqrt{5}} - \frac{L^7M^3}{75\sqrt{5}} + \frac{L^7M^3}{1050\sqrt{5}}\Bigg] + \beta_2(t)\Big[0\Big],$$

which reduces to

6.60
$$\int_{-L}^{L} x(F_1+F_2+F_3)dx = \beta_1(t)\Big[\frac{L^6M^2}{18}\log 2L - \frac{L^6M^2}{18}\log K + \frac{2L^6}{9\sqrt{2}K^2} - \frac{8L^6M}{27\sqrt{3}K} + \frac{4L^6KM^3}{135\sqrt{5}} + \frac{8L^5M}{45\sqrt{3}} - \frac{L^6M^2}{72} - \frac{8L^7M^3}{175\sqrt{5}}\Big].$$

Similarly,

6.61
$$\int_{-L}^{L} x^2(F_1+F_2+F_3)dx = \beta_0(t)\Big[-\frac{L^6}{9\sqrt{2}K^2} - \frac{L^6}{5\sqrt{2}K^2} + \frac{4L^6M}{27\sqrt{3}K} + \frac{4L^6M}{15\sqrt{3}K} - \frac{7L^6M^2}{180}\log 2L + \frac{49L^6M^2}{10800} - \frac{7L^6M^2}{180}\log 2L + \frac{49L^6M^2}{10800} + \frac{L^6M^2}{36}\log K + \frac{L^6M^2}{20}\log K + \frac{L^6M^2}{108} + \frac{L^6M^2}{60} - \frac{2L^6KM^3}{135\sqrt{5}} - \frac{2L^6KM^3}{75\sqrt{5}} + \frac{L^4}{3\sqrt{2}} - \frac{2L^5M}{9\sqrt{3}} - \frac{2L^5M}{15\sqrt{3}} + \frac{L^7M^3}{90\sqrt{5}} + \frac{L^7M^3}{25\sqrt{5}} + \frac{L^7M^3}{210\sqrt{5}}\Big] + \beta_1(t)\Big[0\Big]$$
$$+\beta_2(t)\Big[-\frac{11L^8M^2}{420}\log 2L + \frac{17L^8M^2}{8400} - \frac{11L^8M^2}{420}\log 2L + \frac{17L^8M^2}{8400} + \frac{3L^8M^2}{35}\log 2L - \frac{3L^8M^2}{448} + \frac{3L^8M^2}{35}\log 2L - \frac{3L^8M^2}{448} - \frac{8L^8M^2}{105}\log 2L + \frac{61L^8M^2}{8820} - \frac{8L^8M^2}{105}\log 2L + \frac{61L^8M^2}{8820} + \frac{L^8M^2}{60}\log K + \frac{L^8M^2}{60}\log K - \frac{L^8}{15\sqrt{2}K^2} - \frac{L^8}{15\sqrt{2}K^2} + \frac{4L^8M}{45\sqrt{3}K} + \frac{4L^8M}{45\sqrt{3}K} + \frac{L^8M^2}{300} + \frac{L^8M^2}{1800} + \frac{L^8M^2}{840} - \frac{2L^8KM^3}{225\sqrt{5}} - \frac{2L^8KM^3}{225\sqrt{5}} + \frac{L^6}{9\sqrt{2}} - \frac{L^7M}{9\sqrt{3}} - \frac{L^7M}{63\sqrt{3}} + \frac{L^9M^3}{135\sqrt{5}} + \frac{L^9M^3}{50\sqrt{5}} + \frac{L^9M^3}{4050\sqrt{5}}\Big],$$

which reduces to

6.62
$$\int_{-L}^{L} x(F_1+F_2+F_3)\,dx = \beta_0(t)\Big[-\frac{7L^6M^2}{90}\log 2L + \frac{7L^6M^2}{90}\log K - \frac{14L^6}{45K^2\sqrt{2}} + \frac{56L^6M}{135\sqrt{3}K} - \frac{28L^6KM^3}{675\sqrt{5}} + \frac{L^4}{3\sqrt{2}} - \frac{16L^5M}{45\sqrt{3}} + \frac{7L^6M^2}{200} + \frac{88L^7M^3}{1575\sqrt{5}}\Big] + \beta_2(t)\Big[-\frac{L^8M^2}{30}\log 2L + \frac{L^8M^2}{30}\log K - \frac{2L^8}{15\sqrt{2}K^2} + \frac{8L^8M}{45\sqrt{3}K} - \frac{4L^8KM^3}{225\sqrt{5}} + \frac{L^6}{9\sqrt{2}} - \frac{8L^7M}{63\sqrt{3}} + \frac{2L^8M^2}{209} + \frac{56L^9M^3}{2025\sqrt{5}}\Big]$$

Hence, 6.2, 6.3, 6.4, 6.6, 6.7 and 6.8 are the left-hand sides, respectively, of six differential equations having as right-hand sides, respectively, $aM/2$ times 6.52, 6.54, 6.56, 6.58, 6.60 and 6.62. Denoting these six equations as I, II, ...,VI, respectively, let us derive six new equations by suitable linear combinations of these, such that we may solve for the derivatives of the $\alpha_i(t)$'s and $\beta_i(t)$'s.

The proper combinations will be:

6.63
$$I' = \frac{\frac{L^2}{5}I - \frac{1}{3}III}{\frac{8L^3}{45}}$$

6.64
$$II' = \frac{II}{\frac{2L^3}{3}}$$

6.65
$$III' = \frac{\frac{-L^2}{3}I + III}{\frac{8L^5}{45}}$$

6.66
$$IV' = \frac{\frac{L^2}{5}IV - \frac{1}{3}VI}{\frac{8L^3}{45}}$$

6.67 $$\mathrm{V}' = \frac{\mathrm{V}}{\frac{2L^3}{3}}$$

6.68 $$\mathrm{VI}' = \frac{-\frac{L^2 \mathrm{IV}}{3} + \mathrm{VI}}{\frac{8L^5}{45}}$$

Carrying out the above operations, we get

6.69 $$\frac{d\alpha_0(t)}{dt} + \frac{kT}{m}\beta_1(t) + a\alpha_0(t) = \frac{45aM}{16L^3}\Big\{\alpha_0(t)\Big[-\frac{4L^4}{5\sqrt{2}}\log 2L$$
$$+\frac{4L^4}{5\sqrt{2}}\log K + \frac{6L^4}{5\sqrt{2}} - \frac{4L^4KM}{5\sqrt{3}} + \frac{L^4K^2M^2}{10}$$
$$-\frac{2L^4K^3M^3}{45\sqrt{5}} + \frac{8L^5M}{15\sqrt{3}} - \frac{L^6M^2}{15} + \frac{8L^7M^3}{225\sqrt{5}}$$
$$+\frac{4L^4}{9\sqrt{2}}\log 2L - \frac{16L^4}{27\sqrt{2}} - \frac{4L^4}{9\sqrt{2}}\log K + \frac{4L^4KM}{9\sqrt{3}}$$
$$-\frac{L^4K^2M^2}{18} + \frac{2L^4K^3M^3}{81\sqrt{5}} - \frac{16L^5M}{45\sqrt{3}} + \frac{7L^6M^2}{135}$$
$$-\frac{176L^7M^3}{5670\sqrt{5}}\Big] + \alpha_2(t)\Big[-\frac{4L^6}{15\sqrt{2}}\log 2L + \frac{4L^6}{15\sqrt{2}}\log K + \frac{16L^6}{45\sqrt{2}}$$
$$-\frac{4L^6KM}{15\sqrt{3}} + \frac{L^6K^2M^2}{30} - \frac{2L^6K^3M^3}{135\sqrt{5}} + \frac{16L^7M}{75\sqrt{3}} - \frac{7L^8M^2}{225}$$
$$+\frac{88L^9M^3}{4725\sqrt{5}} + \frac{4L^6}{27\sqrt{2}}\log 2L - \frac{4L^6}{27\sqrt{2}}\log K - \frac{2L^6}{9\sqrt{2}}$$
$$+\frac{4L^6KM}{27\sqrt{3}} - \frac{L^6K^2M^2}{54} + \frac{2L^6K^3M^3}{243\sqrt{5}} - \frac{8L^7M}{63\sqrt{3}} + \frac{L^8M^2}{45}$$
$$-\frac{112L^9M^3}{7290\sqrt{5}}\Big]\Big\},$$

6.70 $$\frac{d\alpha_1(t)}{dt} + 2\frac{kT}{m}\beta_2(t) + a\alpha_1(t) = \frac{3aM}{4L^3}\alpha_1(t)\Big[\frac{L^4}{\sqrt{2}}$$
$$-\frac{8L^5M}{15\sqrt{3}} + \frac{L^6M^2}{9} - \frac{97L^7M^3}{1260\sqrt{5}}\Big],$$

6.71 $$\frac{d\alpha_2(t)}{dt} + a\alpha_2(t) = \frac{45aM}{16L^5}\Big\{\alpha_0(t)\Big[\frac{4L^4}{3\sqrt{2}}\log 2L$$
$$-\frac{4L^4}{3\sqrt{2}}\log K - \frac{2L^4}{\sqrt{2}} + \frac{4L^4KM}{3\sqrt{3}} - \frac{L^4K^2M^2}{6} + \frac{2L^4K^3M^3}{27\sqrt{5}}$$

$$-\frac{8L^5M}{9\sqrt{3}}+\frac{L^6M^2}{9}-\frac{8L^7M^3}{135\sqrt{5}}-\frac{4L^4}{3\sqrt{2}}\log 2L+\frac{4L^4}{3\sqrt{2}}\log K$$

$$+\frac{16L^4}{9\sqrt{2}}-\frac{4L^4KM}{3\sqrt{3}}+\frac{L^4K^2M^2}{6}-\frac{2L^4K^3M^3}{27\sqrt{5}}$$

$$+\frac{16L^5M}{15\sqrt{3}}-\frac{7L^6M^2}{45}+\frac{176L^7M^3}{1890\sqrt{5}}\Bigg]+\alpha_2(t)\Bigg[$$

$$\frac{4L^6}{9\sqrt{2}}\log 2L-\frac{4L^6}{9\sqrt{2}}\log K-\frac{16L^6}{27\sqrt{2}}+\frac{4L^6KM}{9\sqrt{3}}$$

$$-\frac{L^6K^2M^2}{18}+\frac{2L^6K^3M^3}{81\sqrt{5}}-\frac{16L^7M}{45\sqrt{3}}+\frac{7L^8M^2}{135}-\frac{88L^9M^3}{2835\sqrt{5}}$$

$$-\frac{4L^6}{9\sqrt{2}}\log 2L+\frac{4L^6}{9\sqrt{2}}\log K+\frac{2L^6}{3\sqrt{2}}-\frac{4L^6KM}{9\sqrt{3}}+\frac{L^6K^2M^2}{18}$$

$$-\frac{2L^6K^3M^3}{81\sqrt{5}}+\frac{8L^7M}{21\sqrt{3}}-\frac{L^8M^2}{15}+\frac{112L^9M^3}{2430\sqrt{5}}\Bigg]\Bigg\},$$

6.72

$$\frac{d\beta_0(t)}{dt}+\alpha_1(t)+a\beta_0(t)=\frac{45aM}{16L^3}\Bigg\{\beta_0(t)\Bigg[-\frac{L^6M^2}{30}\log 2L$$

$$+\frac{L^6M^2}{30}\log K-\frac{2L^6}{15\sqrt{2}K^2}+\frac{8L^6M}{45\sqrt{3}K}-\frac{4L^6KM}{225\sqrt{5}}+\frac{L^4}{5\sqrt{2}}$$

$$-\frac{8L^5M}{45\sqrt{3}}+\frac{7L^6M^2}{360}+\frac{8L^7M^3}{375\sqrt{5}}+\frac{7L^6M^2}{270}\log 2L$$

$$-\frac{7L^6M^2}{270}\log K+\frac{14L^6}{135\sqrt{2}K^2}-\frac{56L^6M}{405\sqrt{3}K}+\frac{28L^6KM^3}{2025\sqrt{5}}$$

$$-\frac{L^4}{9\sqrt{2}}+\frac{16L^5M}{135\sqrt{3}}-\frac{7L^6M^2}{600}-\frac{88L^7M^3}{4725\sqrt{5}}\Bigg]+\beta_2(t)\Bigg[$$

$$-\frac{7L^8M^2}{450}\log 2L+\frac{7L^8M^2}{450}\log K-\frac{14L^8}{225\sqrt{2}K^2}+\frac{56L^8M}{675\sqrt{3}K}$$

$$-\frac{28L^8KM^3}{3375\sqrt{5}}+\frac{L^6}{15\sqrt{2}}-\frac{16L^7M}{225\sqrt{3}}+\frac{7L^8M^2}{1000}+\frac{88L^9M^3}{7875\sqrt{5}}$$

$$+\frac{L^8M^2}{90}\log 2L-\frac{L^8M^2}{90}\log K+\frac{2L^8}{45\sqrt{2}K^2}-\frac{8L^8M}{135\sqrt{3}K}$$

$$+\frac{4L^8KM^3}{675\sqrt{5}}-\frac{L^6}{27\sqrt{2}}+\frac{8L^7M}{189\sqrt{3}}-\frac{2L^8M^2}{627}$$

$$-\frac{56L^9M^3}{6075\sqrt{5}}\Bigg]\Bigg\},$$

6.73
$$\frac{d\beta_1(t)}{dt} + 2\alpha_2(t) + a\beta_1(t) = \frac{3aM}{4L^3}\beta_1(t)\Big[\frac{L^6M^2}{18}\log 2L - \frac{L^6M^2}{18}\log K + \frac{2L^6}{9\sqrt{2}K} - \frac{8L^6M}{27\sqrt{3}K} + \frac{4L^6KM^3}{135\sqrt{5}} + \frac{8L^5M}{45\sqrt{3}} - \frac{L^6M^2}{72} - \frac{8L^7M^3}{175\sqrt{5}}\Big],$$

and

6.74
$$\frac{d\beta_2(t)}{dt} + a\beta_2(t) = \frac{45aM}{16L^5}\Big\{\beta_0(t)\Big[\frac{L^6M^2}{18}\log 2L - \frac{L^6M^2}{18}\log K + \frac{2L^6}{9\sqrt{2}K^2} - \frac{8L^6M}{27\sqrt{3}K} + \frac{4L^6KM^3}{135\sqrt{5}} - \frac{L^4}{3\sqrt{2}}$$
$$+ \frac{8L^5M}{27\sqrt{3}} - \frac{7L^6M^2}{216} - \frac{8L^7M^3}{225\sqrt{5}} - \frac{7L^6M^2}{90}\log 2L + \frac{7L^6M^2}{90}\log K$$
$$- \frac{14L^6}{45\sqrt{2}K^2} + \frac{56L^6M}{135\sqrt{3}K} - \frac{28L^6KM^3}{675\sqrt{5}} + \frac{L^4}{3\sqrt{2}} - \frac{16L^5M}{45\sqrt{3}}$$
$$+ \frac{7L^6M^2}{200} + \frac{88L^7M^3}{1575\sqrt{5}}\Big] + \beta_2(t)\Big[\frac{7L^8M^2}{270}\log 2L$$
$$- \frac{7L^8M^2}{270}\log K + \frac{14L^8}{135\sqrt{2}K^2} - \frac{56L^8M}{405\sqrt{3}K} + \frac{28L^8KM^3}{2025\sqrt{5}}$$
$$- \frac{L^6}{9\sqrt{2}} + \frac{16L^7M}{135\sqrt{3}} - \frac{7L^8M^2}{600} - \frac{88L^9M^3}{4725\sqrt{5}} - \frac{L^8M^2}{30}\log 2L$$
$$+ \frac{L^8M^2}{30}\log K - \frac{2L^8}{15\sqrt{2}K^2} + \frac{8L^8M}{45\sqrt{3}K} - \frac{4L^8KM^3}{225\sqrt{5}}$$
$$+ \frac{L^6}{9\sqrt{2}} - \frac{8L^7M}{63\sqrt{3}} + \frac{2L^8M^2}{209} + \frac{56L^9M^3}{2025\sqrt{5}}\Big]\Big\}.$$

The above six equations reduce to

6.75
$$\frac{d\alpha_0(t)}{dt} + \frac{kT}{m}\beta_1(t) + a\alpha_0(t) = \frac{45aM}{16}\Big\{\alpha_0(t)\Big[-\frac{16}{45\sqrt{2}}\log\frac{2L}{K}$$
$$+ \frac{82}{135\sqrt{2}} - \frac{16KM}{45\sqrt{3}} + \frac{2K^2M^2}{45} + \frac{8K^3M^3}{405\sqrt{5}} + \frac{8LM}{45\sqrt{3}}$$
$$- \frac{2L^2M^2}{135} + \frac{64L^3M^3}{14175\sqrt{5}}\Big] + \alpha_2(t)\Big[-\frac{16L^2}{135\sqrt{2}}\log\frac{2L}{K} + \frac{2L^2}{15\sqrt{2}}$$
$$- \frac{16L^2KM}{135\sqrt{3}} + \frac{2L^2K^2M^2}{135} - \frac{8L^2K^3M^3}{1215\sqrt{5}} + \frac{136L^3M}{1575\sqrt{3}} - \frac{3L^4M^2}{225}$$

$$+ \frac{3L^5M^3}{920\sqrt{5}}\Big]\Big\},$$

6.76
$$\frac{d\alpha_1(t)}{dt} + \frac{2kT}{m}\beta_2(t) + a\alpha_1(t) = \frac{3aLM}{4}\alpha_1(t)\Big[\frac{1}{\sqrt{2}} - \frac{8LM}{15\sqrt{3}} + \frac{L^2M^2}{9} - \frac{97L^3M^3}{1260\sqrt{5}}\Big],$$

6.77
$$\frac{d\alpha_2(t)}{dt} + a\alpha_2(t) = \frac{45aM}{16L}\Big\{\alpha_0(t)\Big[\frac{-2}{9\sqrt{2}} + \frac{8LM}{45\sqrt{3}} - \frac{2L^2M^2}{45} + \frac{32L^3M^3}{945\sqrt{5}}\Big] + \alpha_2(t)\Big[\frac{2L^2}{27\sqrt{2}} + \frac{8L^3M}{315\sqrt{3}} - \frac{2L^4M^2}{135} + \frac{128L^5M^3}{9505\sqrt{5}}\Big]\Big\},$$

6.78
$$\frac{d\beta_0(t)}{dt} + \alpha_1(t) + a\beta_0(t) = \frac{45aLM}{16}\Big\{\beta_0(t)\Big[-\frac{L^2M^2}{135}\log\frac{2L}{K} - \frac{4L^2}{135\sqrt{2}K^2} + \frac{16L^2M}{405\sqrt{3}K} - \frac{8L^2KM^3}{2025\sqrt{5}} + \frac{4}{45\sqrt{2}} - \frac{16LM}{135\sqrt{3}} + \frac{7L^2M^2}{900} + \frac{64L^3M^3}{23625\sqrt{5}}\Big] + \beta_2(t)\Big[-\frac{L^4M^2}{225}\log\frac{2L}{K} - \frac{4L^4}{225\sqrt{2}K^2} + \frac{64L^4M}{3375\sqrt{3}K} - \frac{8L^4KM}{3375\sqrt{5}} + \frac{4L^2}{135\sqrt{2}} - \frac{136L^3M}{4725\sqrt{3}} + \frac{3L^4M^2}{787} + \frac{9L^5M^3}{4600\sqrt{5}}\Big]\Big\},$$

6.79
$$\frac{d\beta_1(t)}{dt} + 2\alpha_2(t) + a\beta_1(t) = \frac{3aLM}{4}\beta_1(t)\Big[\frac{L^2M^2}{18}\log\frac{2L}{K} + \frac{2L^2}{9\sqrt{2}K^2} - \frac{8L^2M}{27\sqrt{3}K} + \frac{4L^2KM^3}{135\sqrt{5}} + \frac{8LM}{45\sqrt{3}} - \frac{L^2M^2}{72} - \frac{8L^3M^3}{175\sqrt{5}}\Big],$$

and

6.80
$$\frac{d\beta_2(t)}{dt} + a\beta_2(t) = \frac{45aM}{16L}\Big\{\beta_0(t)\Big[-\frac{L^2M^2}{45}\log\frac{2L}{K} - \frac{4L^2}{45\sqrt{2}K^2} + \frac{16L^2M}{135\sqrt{3}K} - \frac{8L^2KM^3}{675\sqrt{5}} - \frac{8LM}{135\sqrt{3}} + \frac{7L^2M^2}{2700} + \frac{32L^3M^3}{1575\sqrt{5}}\Big] + \beta_2(t)\Big[-\frac{L^4M^2}{135}\log\frac{2L}{K} - \frac{4L^4}{135\sqrt{2}K^2} + \frac{16L^4M}{405\sqrt{3}} - \frac{8L^4KM^3}{2025\sqrt{5}} - \frac{8L^3M}{945\sqrt{3}} - \frac{5L^4M^2}{2384} + \frac{32L^5M^3}{14175\sqrt{5}}\Big]\Big\}.$$

Now let us approximate the $\alpha_i(t)$'s and $\beta_i(t)$'s by the leading terms in a MacLaurin Series Expansion.

Our boundary conditions are

6.81
$$\alpha_0(0) = 1 \qquad \beta_0(0) = 0$$
$$\alpha_1(0) = 0 \qquad \beta_1(0) = 0$$
$$\alpha_2(0) = 0 \qquad \beta_2(0) = 0 .$$

Hence, evaluating 6.75-6.80 at t=0 gives

6.82
$$\left(\frac{d\alpha_0(t)}{dt}\right)_{t=0} = -a + \frac{45aLM}{16}\left[-\frac{16}{45\sqrt{2}}\log\frac{2L}{K} + \frac{82}{135\sqrt{2}} - \frac{16KM}{45\sqrt{3}} + \frac{2K^2M^2}{45} - \frac{8K^3M^3}{405\sqrt{5}} + \frac{8LM}{45\sqrt{3}} - \frac{2L^2M^2}{135} + \frac{64L^3M^3}{14175\sqrt{5}}\right],$$

6.83
$$\left(\frac{d\alpha_1(t)}{dt}\right)_{t=0} = 0 ,$$

6.84
$$\left(\frac{d\alpha_2(t)}{dt}\right)_{t=0} = \frac{45aM}{16L}\left[-\frac{2}{9\sqrt{2}} + \frac{8LM}{45\sqrt{3}} - \frac{2L^2M^2}{45} + \frac{32L^3M^3}{945\sqrt{5}}\right],$$

6.85
$$\left(\frac{d\beta_0(t)}{dt}\right)_{t=0} = 0 ,$$

6.86
$$\left(\frac{d\beta_1(t)}{dt}\right)_{t=0} = 0$$

and

6.87
$$\left(\frac{d\beta_2(t)}{dt}\right)_{t=0} = 0 .$$

At this point it is obvious that the equations 6.75-6.80 and boundary conditions 6.81 force the identities

6.88
$$\alpha_1(t) \equiv 0$$
$$\beta_0(t) \equiv 0$$
$$\beta_2(t) \equiv 0 .$$

For an approximation, let us carry only terms to the second power in in M. We then have

6.89 $$\left(\frac{d}{dt}\alpha_0(t)\right)_{t=0} = -a\left[1 + \frac{LM}{\sqrt{2}}\log\frac{2L}{K} - \frac{41LM}{24\sqrt{2}} + \frac{KLM^2}{\sqrt{3}} - \frac{L^2M^2}{2\sqrt{3}}\right],$$

6.90 $$\left(\frac{d\alpha_2(t)}{dt}\right)_{t=0} = -a\left[\frac{5M}{8\sqrt{2}L} - \frac{M^2}{2\sqrt{3}}\right]$$

and

6.91 $$\left(\frac{d\beta_1(t)}{dt}\right)_{t=0} = 0 .$$

Differentiating 6.75, 6.77 and 6.79 and substituting from 6.89-6.91, we see, on evaluating at time t = 0,

6.92 $$\left(\frac{d^2}{dt^2}\alpha_0(t)\right)_{t=0} - a^2\left[1 + \frac{LM}{\sqrt{2}}\log\frac{2L}{K} - \frac{41LM}{24\sqrt{2}} + \frac{KLM^2}{\sqrt{3}} - \frac{L^2M^2}{2\sqrt{3}}\right] = -a^2\left[1 + \frac{LM}{\sqrt{2}}\log\frac{2L}{K} - \frac{41LM}{24\sqrt{2}}\right]\left[-\frac{LM}{\sqrt{2}}\log\frac{2L}{K} + \frac{41LM}{24\sqrt{2}} - \frac{LKM^2}{\sqrt{3}} + \frac{L^2M^2}{2\sqrt{3}}\right] - a^2\left[\frac{5M}{8\sqrt{2}L}\right]\left[-\frac{L^3M}{3\sqrt{2}}\log\frac{2L}{K} + \frac{3L^3M}{8\sqrt{2}}\right],$$

6.93 $$\left(\frac{d^2}{dt^2}\alpha_2(t)\right)_{t=0} - a^2\left[\frac{5M}{8\sqrt{2}L} - \frac{M^2}{2\sqrt{3}}\right] = -a^2\left[1 + \frac{LM}{\sqrt{2}}\log\frac{2L}{K} - \frac{41LM}{24\sqrt{2}}\right]\left[-\frac{5M}{8\sqrt{2}L} + \frac{M^2}{2\sqrt{3}}\right] - a^2\left[\frac{5M}{8\sqrt{2}L}\right]\left[\frac{5LM}{24\sqrt{2}}\right]$$

and

6.94 $$\left(\frac{d^2}{dt^2}\beta_1(t)\right)_{t=0} = -2\left(\frac{d}{dt}\alpha_2(t)\right)_{t=0},$$

which, on retaining only terms to the second order in M and collecting terms, gives

6.95 $$\left(\frac{d^2}{dt^2}\alpha_0(t)\right)_{t=0} = a^2\Big[1 + 1.414LM\log\frac{2L}{K} - 2.416LM + 1.155LKM^2 + 0.765L^2M^2 + 0.500L^2M^2\left(\log\frac{2L}{K}\right)^2 - 1.604L^2M^2\log\frac{2L}{K}\Big],$$

6.96 $$\left(\frac{d^2}{dt^2}\alpha_2(t)\right)_{t=0} = a^2\Big[0.500\frac{M}{L} - 1.163M^2 + 0.313M^2\log\frac{2L}{K}\Big]$$

and
and

6.97 $$\left(\frac{d^2}{dt^2}\beta_1(t)\right)_{t=0} = a\Big[0.884\frac{M}{L} - 0.577M^2\Big].$$

Substituting the above in MacLaurin Series, we have from 6.81 and 6.89-6.97

6.98 $$\alpha_0(t) = 1 - at(1 + 0.707LM\log(2L/K) - 1.208LM + 0.577LKM^2 - 0.289L^2M^2) + (a^2t^2/2)(1 + 1.414LM\log(2L/K) - 2.416LM + 1.155LKM^2 + 0.765L^2M^2 + 0.500L^2M^2(\log(2L/K))^2 - 1.604L^2M^2\log(2L/K)),$$

6.99 $$\alpha_2(t) = -at(0.250M/L - 0.289M^2) + (a^2t^2/2)(M/2L - 1.163M^2 + 0.313M^2\log(2L/K)),$$

6.100 $$\beta_1(t) = (at^2/2)(0.884M/L - 0.577M^2),$$

where M is given as

6.101 $$M = n_{AC}B\left(\frac{m}{2\pi kT}\right)^{1/2}.$$

Hence, our final solution is

6.102 $$p(x,\vec{c},t) = n'\left(\frac{m}{2\pi kT}\right)^{3/2}\Big\{\Big[\alpha_0(t) + x^2\alpha_2(t)\Big] + \Big[x\beta_1(t)\Big]U\Big\}e^{-\frac{mc^2}{2kT}},$$

$\alpha_0(t)$, $\alpha_2(t)$ and $\beta_1(t)$ being given by 6.98, 6.99 and 6.100, respectively.

CHAPTER VII

A SUGGESTED EXPERIMENT: CONCLUSIONS

It is of interest to note that, in the non-exponential factor of the solution given in 6.102, the velocity-free part is an even function of x and the part containing a term U is odd in x. Thus, along with a symmetric distribution function for excited atoms which one might expect by ignoring the diffusion term in the Boltzmann Equation, one arrives at a gradient in populations of cells in phase space, for a given U, increasing in the direction of the x-component of velocity. Thus, the diffusion term is quite important and suggests that experiments to study this gradient would be of considerable interest.

Let us consider and predict results for one possibly interesting experiment. Choosing a point R(X, 0, 0) on the x-axis at a distance X from the center of the slab, let us determine the intensity distribution function $I(\nu, t; X, A)$ when all but a small area A centered on the x-axis is shielded from R. Assume X is large both with respect to the dimensions of A and with respect to the thickness of the slab. Thus, it may be assumed that every incremental volume is at the same distance X from R, and that the angle it makes with the x-axis is zero

Now the number of excited atoms in a small volume dxdydz whose x-component of velocity is in the range (U, U+dU) is

7.1 $$dxdydzdU\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, U, V, W, t)\,dVdW ,$$

which is equal to

7.2 $$dxdydzdUn'\left(\frac{m}{2\pi kT}\right)^{3/2}\left\{\alpha_0(t) + x^2\alpha_2(t) + xU\beta_1(t)\right\}\cdot e^{-mU^2/2kT}\left(\int_{-\infty}^{\infty} e^{-mV^2/2kT}dV\right)\left(\int_{-\infty}^{\infty} e^{-mW^2/2kT}dW\right),$$

which reduces to

7.3 $$dxdydzdUn'\left(\frac{m}{2\ kT}\right)^{1/2}\left\{\alpha_0(t) + x^2\alpha_2(t) + xU\beta_1(t)\right\}e^{-mU^2/2kT}.$$

The number of these excited atoms that will emit photons per unit time is simply the expression 7.3 multiplied by a, the Einstein Emission Coefficient. The fraction of those photons emitted that are in the proper direction to pass through a unit area about R normal to the x-axis is $1/4\pi X^2$. Now, the fraction of those photons emitted in the proper direction that will reach the edge of the slab, and hence the unit area about R is just the probability P(L-x, U) that a photon emitted in the x-direction from an atom with a velocity-component in the direction of emission equal to U will travel at least a distance L-x, the distance to the wall. From 2.9 we see that

7.4 $$P(L-x,\ U) = e^{-M(L-x)e^{-mU^2/2kT}}$$

where M is given in 6.101.

Hence the number of photons emitted per unit time from a volume element dxdydz from atoms with x-component of velocity in the range (U, U + dU) that will pass through a unit area about R normal to the x-axis is

7.5 $$dxdydzdU\,\frac{an'}{4\pi X^2}\left(\frac{m}{2\pi kT}\right)^{1/2} e^{-mU^2/2kT\,-\,M(L-x)e^{-mU^2/2kT}},$$

$$\left\{\,\alpha_0(t) + x^2\alpha_2(t) + xU\beta_1(t)\right\}.$$

The unshielded part of the slab may be approximated by a right circular cylinder with its axis along the x-axis and base of area A. Hence, the integration over the y and z-directions yields simply A. Integrating over the remaining variable x from -L to L, we find the total number of photons emitted per unit time from atoms with an x-axis component of velocity in the range (U, U + dU) that pass through a unit area about R normal to the x-axis to be

7.6 $$dU\,\frac{an'}{4\ X}\left(\frac{m}{2\ kT}\right)^{1/2} e^{-mU^2/2kT\,-\,LMe^{-mU^2/2kT}},$$

$$\int_{-L}^{L}\left\{\alpha_0(t) + x^2\alpha_2(t) + xU\beta_1(t)\right\} e^{Mxe^{-mU^2/2kT}}\,dx.$$

Before proceeding with the integration let us examine the above expressions. The terms will contain a factor $\exp(-mU^2/2kT)$ or $U\exp(-mU^2/2kT)$ which will dominate any integration of 7.6, over U, involving infinite limits. L'Hospital's Rule and Jordan's Lemma assure that 7.6 has values and the integration of 7.6 over infinite limits of U has values. However, we can see immediately that the integrations by parts over x will yield terms which will not have values for all U. This is precisely the problem we encountered earlier, and we shall handle it in the same manner, i.e. expanding $\exp(Mx\exp(-mU^2/2kT))$ in a Taylor's Series about x=0.

Since expanding the exponential term under the integral in 7.6 gives

$$e^{Mxe^{-mU^2/2kT}} \cong 1 + Mxe^{-mU^2/2kT} + (M^2x^2/2)e^{-2mU^2/2kT}, \tag{7.7}$$

we may approximate 7.6 by a similar expression where the integration is over a polynomial in x. Obviously the odd powers of x will integrate to zero, hence the even powers of the polynomial $P_0 + P_2x^2 + P_4x^4$ where

$$P_0 = \alpha_0(t) \tag{7.8}$$

$$P_2x^2 = \alpha_0(t)\frac{M^2x^2}{2}e^{-2mU^2/2kT} + x^2\alpha_2(t) + Mx^2U\beta_1(t)e^{-mU^2/2kT} \tag{7.9}$$

and

$$P_4x^4 = \alpha_2(t)\frac{M^2x^4}{2}e^{-2mU^2/2kT} \tag{7.10}$$

integrate to

$$2L\alpha_0(t) \tag{7.11}$$

$$\frac{L^3\alpha_0(t)M^2e^{-2mU^2/2kT}}{3} + \frac{2L^3\alpha_2(t)}{3} + \frac{2ML^3U\beta_1(t)e^{-mU^2/2kT}}{3} \tag{7.12}$$

and

$$\frac{L^5\alpha_2(t)M^2e^{-2mU^2/2kT}}{5} \tag{7.13}$$

respectively. Hence the integral term in 7.6 may be approximated as

$$\left[2L\alpha_0(t) + \frac{2L^3}{3}\alpha_2(t)\right] + \left[\frac{2L^3}{3}U\beta_1(t)e^{-mU^2/2kT}\right]M + \left[\left(\frac{L^3}{3}\alpha_0(t) + \frac{L^5}{5}\alpha_2(t)\right)e^{-2mU^2/2kT}\right]M^2. \tag{7.14}$$

Now, continuing approximations to terms in M^2,

$$e^{-LM\exp(-mU^2/2kT)} = 1 - LMe^{-mU^2/2kT} + (L^2M^2/2)e^{-2mU^2/2kT}, \tag{7.15}$$

and substituting 7.14 and 7.15 into 7.6, multiplying out and retaining terms only to powers of two in M, we find the total number of photons emitted per unit time from atoms with an x-component of velocity in the range (U, U + dU) that pass through a unit area about R normal to the x-axis to be

7.16
$$\frac{dUan'A}{4\pi X^2}\left(\frac{m}{2\pi kT}\right)^{1/2}\Bigg\{\left[2L\alpha_0(t) + \frac{2L^3}{3}\alpha_2(t)\right] + \left[\frac{2L^3}{3}U\beta_1(t) - L\left(2L\alpha_0(t) + \frac{2L^3}{3}\alpha_2(t)\right)\right]e^{-mU^2/2kT}M + \left[\left(\frac{L^3}{3}\alpha_0(t) + \frac{L^5}{5}\alpha_2(t)\right) - L\left(\frac{2L^3}{3}U\beta_1(t)\right) + \frac{L^2}{2}\left(2L\alpha_0(t) + \frac{2L^3}{3}\alpha_2(t)\right)\right]\cdot e^{-2mU^2/2kT}M^2\Bigg\}e^{-mU^2/2kT}$$

which reduces to

7.17
$$\frac{dUan'A}{4\pi X^2}\left(\frac{m}{2\pi kT}\right)^{1/2}\Bigg\{\left[2L\alpha_0(t) + \frac{2L^3}{3}\alpha_2(t)\right]e^{-mU^2/2kT} + \left[-2L^2\alpha_0(t) - \frac{2L^4}{3}\alpha_2(t) + \frac{2L^3}{3}U\beta_1(t)\right]e^{-2mU^2/2kt}M + \left[\frac{4L^3}{3}\alpha_0(t) + \frac{8L^5}{15}\alpha_2(t) - \frac{2L^4}{3}U\beta_1(t)\right]e^{-3mU^2/2kT}M^2.$$

At this point it is important to note that the distribution over U, equivalent to spectral distribution, is skewed to the right from a Maxwellian distribution. This, of course, suggests that the gas does work on the radiation and is itself cooled in the loss of resonance radiation. Also, since the terms in U are all odd functions of U, it appears that the total number of photons crossing a unit area about R normal to the x-axis is independent of the diffusion term in our Boltzmann-like equation. Thus, it is argued that the important experimental work in this area should be more toward spectroscopic studies rather than integrated intensity measurements.

However, let us look at the form of the integrated intensity expression. Expression 7.17 integrated over U between the limits $(-\infty,\infty)$ becomes

7.18 $$\frac{an'A}{4\pi X^2}\left\{\left[2L\alpha_0(t) + \frac{2L^3}{3}\alpha_2(t)\right] + \frac{1}{\sqrt{2}}\left[-2L^2\alpha_0(t) - \frac{2L^4}{3}\alpha_2(t)\right]M + \frac{1}{\sqrt{3}}\left[\frac{4L^3}{3}\alpha_0(t) + \frac{8L^5}{15}\alpha_2(t)\right]M^2\right\}.$$

Substituting in 7.18 for $\alpha_0(t)$ and $\alpha_2(t)$ from 6.98 and 6.99, we have, on retaining only terms to second power in M

7.19 $$\frac{an'AL}{2\pi X^2}((1 - at(L + 0.707LM\log(2L/K) - 1.125LM + 0.577LKM^2 -0.385L^2M^2)+(a^2t^2/2)(1 + 1.414LM\log(2L/K) - 2.249LM + 1.155LKM^2 + 0.377L^2M^2 + 0.500L^2M^2(\log(2L/K))^2 -1.500L^2M^2\log(2L/K))) + (-0.707LM - at(-0.707LM -0.500L^2M^2\log(2L/K) + 0.796L^2M^2) + (a^2t^2/2)(-0.707LM -L^2M^2\log(2L/K) + 1.590L^2M^2)) + (0.770L^2M^2 -at(0.770L^2M^2) + (a^2t^2/2)(0.770L^2M^2))),$$

which reduces to

7.20 $$\frac{an'AL}{2\pi X^2}((1 - 0.707LM + 0.770L^2M^2) - at(1 - 1.832LM + 1.181L^2M^2 + 0.707LM\log(2L/K) + 0.577LKM^2 - 0.500L^2M^2\log(2L/K)) + (a^2t^2/2)(1 - 2.956LM +1.967L^2M^2 + 1.414LM\log(2L/K) + 0.500L^2M^2(\log(2L/K))^2 - 2.500L^2M^2\log(2L/K))).$$

Note that for small L or K this approaches

7.21 $$\frac{an'AL}{2\pi X^2}\left(1 - at + \frac{a^2t^2}{2}\right)$$

which is an approximation for

7.22 $$\frac{an'(2LA)e^{-at}}{4\pi X^2}.$$

Now, in our expansions, undoubtedly for certain values of K, we have trouble. We have no difficulty in showing that we have absolute convergence, in fact uniform absolute convergence. But, in spite of this fact, the slow convergence gives us difficulty. We can to the degree of a approximation we seek here, look at the special case where K is taken to equal 2L. For this case 7.20 reduces to

7.23 $$\frac{an'AL}{2\pi X^2}\left((1 - 0.707LM + 0.770L^2M^2) - at(1 - 0.678LM + 1.181L^2M^2) + (a^2t^2/2)(1 - 2.956LM + 1.967L^2M^2)\right).$$

Now, the above results are for the initial distribution uniform in space and Maxwellian in velocity. As has been pointed out earlier, the total intensity of emitted radiation does not appear frequency dependent, but the spectral distribution changes with time. This fact makes study and tests of theory extremely difficult.

Therefore, it is suggested that a new kind of experimental set-up be employed in order to allow steady state measurements to be made instead of instaneous measurements. Dr. R. G. Fowler has suggested that a continuous source be used with its spectral distribution unimportant. As the radiation penetrates the gas, all but the resonance radiation would be immediately transmitted. Dr. Fowler's suggestion, then, was to chop the source at a given time and study the relaxation of the gas. This seems far superior to electron-beam excitation or secondary excitation using another vessel of an identical gas as the source. However, it seems that since resonance scattering is much more pronounced than elastic scattering, the possibility of steady state measurements of scattering in the resonance frequencies would yiely much more reliable results.

BIBLIOGRAPHY

Books

Bateman, H. Partial Differential Equations of Mathematical Physics. New York: Dover Publications, 1944.

Carslaw, H. S. Introduction to the Theory of Fourier's Series and Integrals. New York: Dover Publications, 1930.

Chandresekhar, S. Radiative Transfer. London: Oxford University Press, 1950.

Chapman, S. and Cowling, T. G. The Mathematical Theory of Non-uniform Gases. London: Cambridge University Press, 1953.

Courant, R. and Hilbert, D. Methods of Mathematical Physics. New York: Interscience Publishers, Inc., 1953.

Jeans, J. H. The Dynamical Theory of Gases. London: Cambridge University Press, 1921.

Kourganoff, V. Basic Methods in Transfer Problems. London: Oxford University Press, 1952.

Mitchell, A. C. G. and Zymansky, M. W. Resonance Radiation and Excited Atoms. New York: The Macmillian Company, 1934.

Murnaghan, F. D. Introduction to Applied Mathematics. New York: John Wiley and Sons, Inc., 1948.

Articles

Fowler, R. G. " The Blockading of Resonance Radiation", Handbuch der Physik, Encyclopedia of Physics, XXII, 1954, 209.

Holstein, T. " The Imprisonment of Resonance Radiation", Physical Review, LXXII, June, 1947, 1212.

APPENDIX

In deriving our diffusion equation, we assumed that for an emission-absorption process to take place the component of the two atoms' relative velocity along their join must have a value in the range $(-\Delta c/2, \Delta c/2)$. Such an assumption is equivalent to the approximation of a function by a step function. Let us analyze this more closely.

It is obvious that we may examine the case of an absorbing atom at rest, since we are interested only in relative velocities in an emission-absorption process. Let us assume that we have an unexcited atom at rest and an excited atom moving toward it with a speed v.

Now, the probability distribution function of radiation emitted from an atom at rest is given by

A.1 $$P(\nu; \nu_0) = \gamma/2\pi((\nu-\nu_0)^2 + \gamma^2/4)$$

where

A.2 $$\gamma = 2e^2\nu_0^2/3mc^3$$

and ν being the frequency of emitted radiation, ν_0 the maximum of the probability curve and c the speed of light. It will be noted that c is used differently in this one instance, and in all other places in the text c stands for the speed of an atom. The terms e and m are, of course, the charge and mass of an electron. For an atom moving toward an observer at rest with speed v this distribution is, by virtue of the Doppler shift, transformed to

A.3 $$P(\nu;\nu_0+\Delta\nu_0) = \frac{\gamma}{2\pi}\frac{1}{(\nu-\nu_0-\Delta\nu_0)^2+\gamma^2/4}$$

where $\Delta\nu_0$ is the Doppler shift of the maximum ν_0 by virtue of the velocity v. It will be noted that this is not an exact transformation, since each frequency is shifted proportional to its rest value. However the effect is negligible, since the values of $P(\nu;\nu_0)$ are appr appreciable over such a small range of ν. Hence, one may consider that the function is simply shifted to the right by an amount $\Delta\nu_0$ determined by the speed v of the approaching emitter.

Now, our objective is to determene relative probabilities of an emission-absorption event to occur with varying values of speeds of an approaching, or retreating, emitter. To do this, we note that the probability of emission of a photon in a neighborhood of frequency is proportional to the value of $P(\nu;\nu_0+\Delta\nu_0)$ and the probability of this photon being absorbed is proportional to the value of $P(\nu;\nu_0)$, hence, the probability of an emission-absorption process to occur is proportional to

A.4 $$g(\Delta\nu_0) = \int_{-\infty}^{\infty} P(\nu;\nu_0)\, P(\nu;\nu_0+\Delta\nu_0)\, d\nu.$$

The above integral will be a function of $\Delta\nu_0$ and, normalized over the range $(-\infty,\infty)$ will yield the relative probabilities of emission-absorption processes for selected ranges of Doppler shifts of ν_0 or, equivalently, selected ranges of v. In our text we wish to substitute for this function a step function zero over most of the line and uniform in height for a finite interval with ν_0 at its center.

Evaluating term A.4, we find

A.5 $$g(\Delta\nu_0) = \int_{-\infty}^{\infty}\left\{\frac{\gamma}{2\pi}\frac{1}{(\nu-\nu_0)^2+\gamma^2/4}\right\}\cdot\left\{\frac{\gamma}{2\pi}\frac{1}{(\nu-\nu_0-\Delta\nu_0)^2+\gamma^2/4}\right\}d\nu.$$

The above integral is evaluated, using complex variables, integrating around a circular path in the upper half-plane and along the real line, to be

A.6 $$g(\Delta\nu_0) \doteq \frac{\gamma}{\pi}\frac{1}{(\Delta\nu_0)^2+\gamma^2}.$$

It will be seen that the above function integrated from minus infinity to plus infinity yields the value one. Hence, has all the properties of a probability distribution function and is the function we wish to approximate by the step function

A.7 $$\begin{aligned} h(\Delta\nu_0) &= 0 & -\infty < \Delta\nu_0 < -\delta\nu/2 \\ &= 1/\delta\nu & -\delta\nu/2 < \Delta\nu_0 < \delta\nu/2 \\ &= 0 & \delta\nu/2 < \Delta\nu_0 < \infty \end{aligned}$$

To do this we minimize the integral

A.8 $$\int_{-\infty}^{\infty}(g-h)^2 g\, d(\Delta\nu_0)$$

by setting the derivative of the above with respect to $\delta\nu$ equal to zero. Carrying out the steps necessary, noting that A.8 is actually the sum of three integrals with their limits functions of the variable of differentiation, we arrive at the equation

A.9 $$\tan^{-1} q = \frac{q^2}{(q^2 + 1)(\pi - q)}$$

where

A.10 $$q = \delta\nu/2\gamma.$$

We are interested in the smallest non-zero positive root of equation A.10.

After determining our q we find $\delta\nu/2 = q\gamma$. This is the maximum frequency in our step function. The next step is to find the speed $\Delta c/2$ which would give this shift. Multiplying this speed by two gives us our Δc.