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Abstract

Let F be a non-Archimedean local field, and let $G = \mathrm{SL}(n, F)$. Let (π, V) be an irreducible (smooth, complex) self-dual representation of G. The underlying space V admits a unique (up to scaling) non-degenerate G-invariant bilinear form, which is necessarily either symmetric or skew-symmetric. We define $\epsilon(\pi) = \pm 1$ accordingly. We determine $\epsilon(\pi)$ in almost all cases. We show that if n is odd or $n \equiv 0 \mod 4$, then $\epsilon(\pi) = 1$. In addition, if $n \equiv 2 \pmod 4$ and F contains a square root of -1, we prove that then $\epsilon(\pi) = \omega_\pi(-I)$, where ω_π is the central character of π and I is the identity element of $\mathrm{SL}(n, F)$.

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Chapter 1

Introduction

Let G be a topological group and let (π, V) be an irreducible complex representation of G. Suppose that π is *self-dual*, i.e.

$$\pi \simeq \pi^\vee,$$

where $\pi^\vee$ denotes the contragredient representation. In the presence of Schur's lemma, it is easy to see that there exists a non-degenerate G-invariant bilinear form

$$B : V \times V \longrightarrow \mathbb{C},$$

unique up to scalar multiples. Such a form is necessarily either symmetric or skew-symmetric. We say that π is *orthogonal* if B is symmetric and *symplectic* if B is skew-symmetric.

Accordingly, we set

$$\epsilon(\pi) := \begin{cases} 1 & \text{if } B \text{ is symmetric,} \\ -1 & \text{if } B \text{ is skew-symmetric.} \end{cases}$$

The scalar $\epsilon(\pi)$ is called the *sign* of π or, in more classical language, the *Frobenius–Schur indicator* of π. For finite groups, the Frobenius–Schur indicator goes back to Frobenius and Schur and has numerous applications in the structure theory of groups, the classification of real representations, and the theory of characters.

For finite groups of Lie type, the explicit determination of the Frobenius–Schur indicator often proceeds via detailed character-theoretic computations. In [Pra98], Prasad introduced an elegant idea to compute the sign for representations of finite groups of Lie type that admits a special model, called a *Whittaker model*. He used this idea to determine the sign for many classical groups of Lie type, avoiding difficult computations. In [Pra99], Prasad extended the results of [Pra98] to the case of reductive p-adic groups and computed the sign of certain classical groups.

In this thesis we study the sign $\epsilon(\pi)$ for irreducible self-dual (smooth, complex) representations of

$$G \;=\; \mathrm{SL}(n, F),$$

where F is a non-Archimedean local field. Our main results provide a nearly complete description of $\epsilon(\pi)$ in terms of the residue class of n modulo 4.

1.1 Self-dual representations and signs for p-adic groups

In the case $G = \mathrm{GL}(n, F)$, Prasad's approach can be combined with the rich structure of the representation theory of $\mathrm{GL}(n)$ to obtain a particularly clean result: every irreducible self-dual smooth representation of $\mathrm{GL}(n, F)$ is orthogonal, [Vin06, Theorem 8] i.e.

$$\epsilon(\pi) \;=\; 1 \qquad \text{for all irreducible self-dual } \pi \text{ of } \mathrm{GL}(n, F).$$

For $G = \mathrm{SL}(n, F)$, the situation is more subtle. The analogy with the finite-field case already indicates that the sign need not be uniformly equal to 1. Lansky and

Vinroot studied self-dual unitarizable representations of $\mathrm{SL}(n,F)$ using *Klyachko models* [LV11]. Their methods yield strong results in terms of the existence of Klyachko model, and relate the sign to the size and structure of L-packets. However, Klyachko models are quite specific to general and special linear groups.

In this thesis we take a different point of view. Rather than working with a specific model, we treat the sign as a representation-theoretic invariant which behaves well under natural operations such as parabolic induction and passage to Levi subgroups. The underlying method is general and should apply to other classes of groups.

1.2 Strategy and methods

Our strategy for understanding $\epsilon(\pi)$ for irreducible self-dual representations of $\mathrm{SL}(n,F)$ rests on three main pillars:

1. **Reduction to the tempered case.** In [RS12], Roche and Spallone have established a descent result which relate the sign for a representation π of G to the sign of certain representation of a Levi subgroup appearing in the Langlands data of π.
2. **Lifting from the Levi subgroup M to $\widetilde{M}$.** After reducing to a tempered representation σ of a Levi subgroup M of $\mathrm{SL}(n,F)$, we lift the problem to the corresponding Levi subgroup $\widetilde{M}$ inside $\mathrm{GL}(n,F)$, which is (up to isomorphism) a direct product of general linear groups.

   The transfer is implemented by choosing an irreducible representation $\widetilde{\sigma}$ of $\widetilde{M}$ whose restriction to M contains the tempered representation σ we wish to study. This passage introduces a character χ of $\widetilde{M}/M$.

   The twisted sign for $\widetilde{\sigma}$ agrees with the twisted sign for σ on M, and in fact reduces to the contribution of the middle block: if there is no middle block, then the sign is forced to be 1; if a middle block is present, then the entire problem reduces to computing a twisted sign for the representation carried by that single block, now a representation of a general linear group. In this way, the sign problem on $\mathrm{SL}(n, F)$ is transformed into a concrete computation on one general linear factor.

3. **Adapting Prasad's method to incorporate the twist χ.**

   The final step is to compute the twisted sign for the middle block. Here we adapt Prasad's method for generic representations [Pra99], with an additional modification to incorporate the twisting character χ. He shows the sign can be read off from a central-character value at a specific element, and in our setup we get the same phenomenon with an extra χ factor.

1.3 Main results

We now state our main theorem.

Theorem 1.3.1. *Let π be an irreducible self-dual smooth complex representation of $G = \mathrm{SL}(n, F)$, where F is a non-Archimedean local field. Then:*

1. *If n is odd, then $\epsilon(\pi) = 1$.*
2. *If $n \equiv 0 \pmod 4$, then $\epsilon(\pi) = 1$.*
3. *If $n \equiv 2 \pmod 4$ and F contains a square root of -1, then*

$$\epsilon(\pi) \;=\; \omega_\pi(-I),$$

   where ω_π is the central character of π and I is the identity matrix in $\mathrm{SL}(n, F)$.

The theorem leaves one case open: when $n \equiv 2 \pmod 4$ and F does *not* contain a square root of -1. Thus our methods do not yet yield a general formula for $\epsilon(\pi)$.

Finite-field analogues due to Prasad show that naive generalizations of the central-character formula can fail in this situation, and suggest that new ideas are required to handle this case.

The theorem above extends and reinterprets earlier work in several ways. It can be viewed as a conceptual refinement of results of Prasad in the generic case for split groups, recast in the specific setting of $\mathrm{SL}(n, F)$ and extended to all irreducible self-dual representations via the Langlands classification and descent of signs. It is also closely related to the work of Lansky and Vinroot [LV11], who studied self-dual unitarizable representations of $\mathrm{SL}(n, F)$ via Klyachko models; our approach avoids models and instead proceeds through structural properties of induced representations and Jacquet modules. In particular, the use of Roche–Spallone descent [RS12] and Prasad's method in [Pra99] together yields a framework which, at least in principle, extends beyond the case of general and special linear groups.

1.4 Future directions and open problems

The most immediate open problem suggested by Theorem 1.3.1 is to resolve the remaining case $n \equiv 2 \pmod 4$ when -1 is not a square in F. In this situation, neither our methods nor those of Lansky and Vinroot [LV11] currently determine $\epsilon(\pi)$ for all self-dual representations of $\mathrm{SL}(n, F)$. In this case, Prasad [Pra98, Pra99] has constructed explicit counterexamples showing that no simple criterion based solely on the central character can exist.

Theorem 1.4.1 (Prasad)**.** *Let F be a non-Archimedean local field whose residue field has cardinality $q \equiv 3 \pmod 4$. Then for $n = 4m + 2$ with $m \geq 1$, there exist irreducible, self-dual, generic representations π of $\mathrm{SL}(n, F)$ such that:*

1. *π is symplectic ($\epsilon(\pi) = -1$) even though $\omega_\pi(-I) = 1$*

2. *π is orthogonal ($\epsilon(\pi) = 1$) even though $\omega_\pi(-I) = -1$*

The Remaining Arithmetic Case

The most immediate open problem is the complete resolution of the $n \equiv 2 \pmod 4$ case:

Problem 1.4.2. *Determine a computable formula for $\epsilon(\pi)$ when π is an irreducible self-dual representation of $\mathrm{SL}(n, F)$ with $n \equiv 2 \pmod 4$ and F does not contain a square root of -1.*

One natural direction for further research is to seek a description of $\epsilon(\pi)$ in terms of the local Langlands parameter of π, perhaps involving epsilon-factors.

Extension to Other Classical Groups

Problem 1.4.3. *Extend our methods to determine the signs of self-dual representations for other classical groups over non-Archimedean local fields, particularly:*

- *The symplectic groups $Sp(2n, F)$*
- *The special orthogonal groups $SO(n, F)$*
- *The unitary groups $SU(n, F)$*

With our method, one can verify that the sign of any irreducible self-dual representation of $\mathrm{GL}(n, F)$ has sign 1.

Chapter 2

Preliminaries and Notation

In this chapter we collect the notation and standard facts from the representation theory of p-adic groups that will be used throughout the dissertation. We first fix notation for local fields, parabolic subgroups, Levi factors, and modulus characters. We then review smooth representations, contragredients, parabolic induction, Jacquet modules, and Frobenius reciprocity. We also record the necessary facts for $\mathrm{GL}(n, F)$, including generic representations, Whittaker models, and Rodier's compact approximation. We conclude with the Langlands classification for $\mathrm{SL}(n, F)$, which will be used in the reduction to tempered representations.

2.1 Local fields and basic notation

Throughout this dissertation, F denotes a non-Archimedean local field. We use the term p-adic group to mean the group of F-points of an algebraic group defined over F, regardless of the characteristic of F. In the representation-theoretic results below, we will mainly consider reductive p-adic groups, that is, groups of the form

$$G = \mathbf{G}(F),$$

where $\mathbf{G}$ is a connected reductive algebraic group defined over F. The group $G = \mathbf{G}(F)$, equipped with the topology inherited from F, is a totally disconnected locally compact group.

In most of the work we will be concerned with the p-adic special linear group

$$\mathrm{SL}(n, F) := \{g \in \mathrm{GL}(n, F) : \det(g) = 1\},$$

which is the group of F-points of the algebraic group $\mathbf{SL}_n$. However, certain intermediate results are formulated for a general connected reductive group.

We write:

- $\mathcal{O}$ for the ring of integers of F,
- $\mathfrak{p}$ for the maximal ideal of $\mathcal{O}$,
- ϖ for a fixed uniformizer, so that $\mathfrak{p} = \varpi\mathcal{O}$,
- $k = \mathcal{O}/\mathfrak{p}$ for the residue field, with cardinality $q = \#k$,
- $v_F \colon F^\times \to \mathbb{Z}$ for the normalized valuation, so that $v_F(\varpi) = 1$,
- $|\cdot|_F \colon F^\times \to \mathbb{R}_{>0}$ for the normalized absolute value given by

$$|x|_F = q^{-v_F(x)} \qquad \text{for } x \in F^\times.$$

We extend $|\cdot|_F$ to F by setting $|0|_F = 0$. We will also use the notation

$$\mathrm{GL}(n, F) = \{g \in M_n(F) : \det(g) \neq 0\}.$$

If H is a subgroup of G, we denote by $Z(H)$ the center of H, and by

$$N_G(H) = \{g \in G : gHg^{-1} = H\}$$

the normalizer of H in G.

2.2 Reductive groups, parabolic subgroups, and Levi factors

We fix once and for all a minimal parabolic subgroup

$$P_0 = M_0 N_0$$

of G, where M_0 is a Levi factor and N_0 is its unipotent radical. The choice of P_0 determines a set of standard parabolic subgroups: a parabolic subgroup P of G is called *standard* if it contains P_0. Every parabolic subgroup P has a Levi decomposition

$$P = MN,$$

where M is a Levi factor and N is the unipotent radical; when P is standard we also call M standard.

For $G = \mathrm{SL}(n, F)$ we take B to be the upper triangular matrices in $\mathrm{SL}(n, F)$ and T the diagonal matrices in $\mathrm{SL}(n, F)$. Standard parabolic subgroups of $\mathrm{SL}(n, F)$ are then the intersections with $\mathrm{SL}(n, F)$ of block upper triangular parabolic subgroups of $\mathrm{GL}_n(F)$. In particular, for a composition

$$n = n_1 + \cdots + n_r$$

we obtain a standard Levi subgroup

$$M = \Big\{ \mathrm{diag}(g_1, \ldots, g_r) \in \mathrm{GL}_n(F) : g_i \in \mathrm{GL}_{n_i}(F) \Big\} \cap \mathrm{SL}_n(F).$$

Modulus character

Let $\mathbf{G}$ be a connected reductive algebraic group defined over F, and let $G = \mathbf{G}(F)$. Let $P = MN$ be a parabolic subgroup of G with Levi factor M and unipotent radical N. We denote by

$$\delta_P : P \to \mathbb{R}_{>0}$$

the modulus character of P, characterized by

$$\int_P f(p_1 p)\, dp_1 = \delta_P(p)^{-1} \int_P f(p_1)\, dp_1$$

for all $p \in P$ and all $f \in C_c^\infty(P)$, where dp_1 is a left Haar measure on P (see [Cas95, §1.5]).

Modulus character via the adjoint action. We write $\mathrm{Lie}(G)$ and $\mathrm{Lie}(N)$ for the Lie algebras of the corresponding algebraic groups, viewed as finite-dimensional F-vector spaces.

For each $g \in G$, conjugation by g defines an automorphism

$$\mathrm{Int}(g) : G \to G, \qquad x \mapsto gxg^{-1}.$$

Differentiating at the identity gives an F-linear automorphism

$$\mathrm{Ad}(g) : \mathrm{Lie}(G) \to \mathrm{Lie}(G),$$

called the adjoint action of g. Since N is normalized by P, and in particular by M, the subspace $\mathrm{Lie}(N) \subseteq \mathrm{Lie}(G)$ is stable under $\mathrm{Ad}(m)$ for every $m \in M$.

The modulus character $\delta_P : P \to \mathbb{R}_{>0}$ is trivial on N (see [Cas95, §1.5]), so it factors through the quotient $P/N \simeq M$. Thus we may regard δ_P as a character of M. For $m \in M$, one has

$$\delta_P(m) = \left|\det\!\big(\mathrm{Ad}(m)\,|_{\mathrm{Lie}(N)}\big)\right|_F .$$

In particular, if $z \in Z(G)$, then conjugation by z is trivial on G, so

$$\mathrm{Int}(z) = \mathrm{Id}_G.$$

Therefore its differential is also trivial:

$$\mathrm{Ad}(z) = \mathrm{Id}_{\mathrm{Lie}(G)}.$$

Restricting to the F-subspace $\mathrm{Lie}(N)$, we obtain

$$\mathrm{Ad}(z) \mid_{\mathrm{Lie}(N)} = \mathrm{Id}_{\mathrm{Lie}(N)}.$$

Hence, whenever $z \in Z(G) \cap M$, we have

$$\delta_P(z) = \left|\det\!\big(\mathrm{Ad}(z) \mid_{\mathrm{Lie}(N)}\big)\right|_F = |\det(\mathrm{Id}_{\mathrm{Lie}(N)})|_F = 1.$$

2.3 Smooth representations, contragredients, and central characters

Let G be a totally disconnected locally compact group. A (complex) representation of G is a pair (π, V) where V is a complex vector space and

$$\pi \colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

is a group homomorphism.

Definition 1. *A representation (π, V) of G is called* smooth *if for every $v \in V$ the stabilizer*

$$\mathrm{Stab}_G(v) = \{g \in G : \pi(g)v = v\}$$

is an open subgroup of G.

Definition 2. *A smooth representation (π, V) of G is called* admissible *if for every compact open subgroup K of G, the subspace*

$$V^K := \{v \in V : \pi(k)v = v \text{ for all } k \in K\}$$

is finite-dimensional.

Definition 3. *A smooth representation* (π, V) *of* G *is called* irreducible *if* V *has no nonzero proper* G*-stable subspaces.*

Definition 4. *A character of* G *is a one-dimensional smooth representation of* G*. Equivalently, it is a locally constant group homomorphism* $\chi : G \to \mathbb{C}^\times$*; equivalently,* $\ker(\chi)$ *is an open subgroup of* G*.*

If (π, V) is a smooth representation of G and χ is a smooth character of G, we write $\pi \otimes \chi$ (or simply $\pi\chi$) for the twist defined by

$$(\pi \otimes \chi)(g) \;=\; \chi(g)\,\pi(g), \qquad g \in G.$$

Lemma 2.3.1 (Schur's lemma)**.** *Let* (π, V) *and* (τ, W) *be irreducible smooth representations of* G*. Then*

$$\mathrm{Hom}_G(\pi, \tau) = \begin{cases} 0, & \text{if } \pi \not\simeq \tau, \\ \mathbb{C}, & \text{if } \pi \simeq \tau. \end{cases}$$

In particular, if (π, V) *is irreducible, then*

$$\mathrm{End}_G(V) = \mathbb{C} \cdot \mathrm{Id}_V.$$

Proof. We refer to [Ren10, III.1.8]. $\square$

Contragredient and central character

Let (π, V) be a smooth representation of G, and let

$$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$$

be the algebraic dual of V. The group G acts on V^* by

$$(g \cdot \lambda)(v) = \lambda(\pi(g^{-1})v), \qquad g \in G,\ \lambda \in V^*,\ v \in V.$$

A vector $\lambda \in V^*$ is called *smooth* if its stabilizer

$$Stab_G(\lambda) = \{g \in G : g \cdot \lambda = \lambda\}$$

is an open subgroup of G. The *smooth dual* of V is the subspace

$$V^\vee \subseteq V^*$$

of smooth vectors in V^*. Then $(\pi^\vee, V^\vee)$ is called the **contragredient** of (π, V).

If $Z(G)$ denotes the center of G, then for any irreducible smooth representation (π, V) of G there exists a character

$$\omega_\pi \colon Z(G) \to \mathbb{C}^\times$$

such that $\pi(z)$ acts as the scalar $\omega_\pi(z)$ on V for all $z \in Z(G)$. We call ω_π the **central character** of π.

Tempered representations

Let (π, V) be an irreducible admissible representation of G with central character ω_π. For $v \in V$ and $\lambda \in V^\vee$, define the *matrix coefficient*

$$f_{v,\lambda}(g) \;=\; \langle \pi(g)v, \lambda \rangle \qquad \text{for } g \in G.$$

Haar measures on $G/Z(G)$. Fix a left Haar measure dg on G and a left Haar measure dz on $Z(G)$. We equip the quotient $G/Z(G)$ with the corresponding quotient measure $d\dot{g}$. For $1 \leq p < \infty$, we denote by $L^p(G/Z(G))$ the usual L^p-space with respect to $d\dot{g}$.

Note that for every $z \in Z(G)$ we have

$$f_{v,\lambda}(gz) = \omega_\pi(z)\, f_{v,\lambda}(g).$$

Hence, if ω_π is unitary, then

$$|f_{v,\lambda}(gz)| = |f_{v,\lambda}(g)|,$$

so $|f_{v,\lambda}|$ descends to a well-defined function on $G/Z(G)$.

Definition 5 (Tempered)**.** *Let (π, V) be an irreducible admissible representation of G with* unitary *central character. We say that π is* tempered *if for all $v \in V$, $\lambda \in V^\vee$, and every $\varepsilon > 0$, the function*

$$gZ(G) \longmapsto |f_{v,\lambda}(g)|$$

belongs to $L^{2+\varepsilon}(G/Z(G))$.

Representations of direct products

We record the standard description of irreducible representations of direct products.

Lemma 2.3.2. *Let $H_1, \ldots, H_r$ be reductive p-adic groups, and set*

$$H = \prod_{i=1}^{r} H_i.$$

For each i, let π_i be an irreducible admissible representation of H_i. Then the external tensor product

$$\pi = \otimes_{i=1}^{r} \pi_i$$

is an irreducible admissible representation of H.

Conversely, every irreducible smooth representation of H is, up to equivalence, of the form

$$\otimes_{i=1}^{r} \pi_i,$$

where each π_i is an irreducible smooth representation of H_i.

Moreover,

$$\pi = \otimes_{i=1}^r \pi_i \textit{ is tempered} \iff \pi_i \textit{ is tempered for all } i.$$

Proof. If each π_i is an irreducible admissible representation of H_i, then $\otimes_{i=1}^r \pi_i$ is an irreducible admissible representation of H; conversely, every irreducible admissible representation of H is of this form, uniquely up to equivalence. See [Ren10, III.1.14]; see also [Cas95, §2.6].

It remains to prove the temperedness statement. Let $V = \bigotimes_{i=1}^r V_i$ where V_i is the underlying vector space of π_i. Since

$$Z(H) = \prod_{i=1}^r Z(H_i),$$

we have a natural identification

$$H/Z(H) \;\simeq\; \prod_{i=1}^r H_i/Z(H_i).$$

Assume first that each π_i is tempered. Since each π_i is admissible, we may identify

$$\pi^\vee \simeq \otimes_{i=1}^r \pi_i^\vee,$$

by [Ren10, III.1.14]. Fix $\varepsilon > 0$. Let

$$v = \otimes_{i=1}^r v_i \in V, \qquad \lambda = \otimes_{i=1}^r \lambda_i \in V^\vee.$$

The matrix coefficient $f_{v,\lambda}$ of π is given by

$$f_{v,\lambda}(h_1, \ldots, h_r) = \langle \pi(h_1, \ldots, h_r)v, \lambda\rangle = \prod_{i=1}^r \langle \pi_i(h_i)v_i, \lambda_i\rangle \qquad \text{for } h_i \in H_i.$$

Write

$$f^{(i)}_{v_i,\lambda_i}(h_i) = \langle \pi_i(h_i)v_i, \lambda_i\rangle\,, \qquad \text{where } 1 \le i \le r.$$

Since each π_i is tempered, every matrix coefficient of π_i belongs to $L^{2+\epsilon}\big(H_i/Z(H_i)\big)$ for every $\epsilon > 0$. Therefore, by Fubini's theorem,

$$\int_{H/Z(H)} |f_{v,\lambda}(h)|^{2+\epsilon}\, dh = \prod_{i=1}^{r} \int_{H_i/Z(H_i)} \left|f^{(i)}_{v_i,\lambda_i}(h_i)\right|^{2+\epsilon} dh_i < \infty.$$

Hence every matrix coefficient attached to pure tensors lies in $L^{2+\epsilon}\big(H/Z(H)\big)$. Since arbitrary vectors in V and arbitrary smooth linear forms in $V^\vee$ are finite sums of pure tensors, the same holds for all matrix coefficients of π. Thus π is tempered.

Conversely, assume that $\pi = \otimes_{i=1}^{r}\pi_i$ is tempered. We show that each π_i is tempered. Since π is tempered, its central character is unitary. Hence each π_i also has unitary central character.

Fix $i_0 \in \{1, \dots, r\}$. Let $\varepsilon > 0$, and let

$$f_{i_0}(h) = \langle \pi_{i_0}(h) v_{i_0}, \lambda_{i_0} \rangle$$

be an arbitrary matrix coefficient of π_{i_0}. For each $j \neq i_0$, choose a nonzero vector $v_j \in V_j$. Then there exists $\lambda_j \in V_j^\vee$ such that

$$\lambda_j(v_j) \neq 0.$$

Define

$$f_j(h_j) = \langle \pi_j(h_j) v_j, \lambda_j \rangle.$$

Then

$$f_j(1) = \lambda_j(v_j) \neq 0.$$

Since π_j is smooth, f_j is locally constant on H_j. Since ω_{π_j} is unitary,

$$|f_j(h_j z)|^{2+\varepsilon} = |f_j(h_j)|^{2+\varepsilon} \qquad \text{for } z \in Z(H_j),$$

so $|f_j|^{2+\varepsilon}$ descends to a locally constant function on $H_j/Z(H_j)$. Hence there exists a compact open neighborhood $U_j \subset H_j/Z(H_j)$ of the identity on which $|f_j|^{2+\varepsilon}$ is strictly positive. Therefore

$$\int_{H_j/Z(H_j)} |f_j(h_j)|^{2+\varepsilon}\, dh_j > 0,$$

possibly with value $+\infty$.

Now let

$$v = \otimes_{i=1}^r v_i \in V, \qquad \lambda = \otimes_{i=1}^r \lambda_i \in V^\vee.$$

Then

$$\langle \pi(h_1, \ldots, h_r)v, \lambda\rangle = \prod_{i=1}^r \langle \pi_i(h_i)v_i, \lambda_i\rangle.$$

Therefore, by Tonelli's theorem,

$$\int_{H/Z(H)} |\langle \pi(h)v, \lambda\rangle|^{2+\varepsilon}\, dh = \prod_{i=1}^r \int_{H_i/Z(H_i)} |\langle \pi_i(h_i)v_i, \lambda_i\rangle|^{2+\varepsilon}\, dh_i.$$

Every factor with $i \neq i_0$ is strictly positive, and the left-hand side is finite because π is tempered. Hence

$$\int_{H_{i_0}/Z(H_{i_0})} |\langle \pi_{i_0}(h)v_{i_0}, \lambda_{i_0}\rangle|^{2+\varepsilon}\, dh < \infty.$$

Since the matrix coefficient of π_{i_0} was arbitrary, π_{i_0} is tempered. This proves the converse.

□

2.4 Induced representations and Jacquet modules

In this section we collect the induction constructions that will be used throughout the dissertation. We begin with smooth induction and compact induction for general closed subgroups of a totally disconnected locally compact group, and then recall unnormalized and normalized parabolic induction for reductive p-adic groups. We also define Jacquet modules and record Frobenius reciprocity in its normalized form.

Smooth induction

Let H be a closed subgroup of G, and let (ρ, W) be a smooth representation of H.

Definition 6. *The* smoothly induced representation *of ρ from H to G, denoted by*

$$\mathrm{Ind}_H^G(\rho),$$

is the space of all locally constant functions

$$f : G \to W$$

such that

$$f(hg) = \rho(h)f(g) \qquad \textit{for all } h \in H,\ g \in G.$$

The group G acts on $\mathrm{Ind}_H^G(\rho)$ by right translation:

$$(\pi(g)f)(x) = f(xg), \qquad g, x \in G.$$

See [Cas95, §2.4].

Compact induction

Definition 7. *The* compactly induced representation *of ρ from H to G, denoted by*

$$\text{c-Ind}_H^G(\rho),$$

is the space of all locally constant functions

$$f : G \to W$$

such that

$$f(hg) = \rho(h)f(g) \qquad \textit{for all } h \in H,\ g \in G,$$

and such that the support of f is compact modulo H. See [Cas95, §2.4].

Remark 2.4.1. *Compact induction will reappear in Section 2.7 in connection with Rodier's compact approximation theorem.*

2.5 Parabolic induction

Let G be the group of F-points of a connected reductive algebraic group defined over F, and let

$$P = MN$$

be a parabolic subgroup of G, where M is a Levi factor and N is the unipotent radical. Let (σ, V_σ) be a smooth representation of M. We write

$$i_P^G(\sigma)$$

for the *normalized parabolically induced* representation of G with respect to σ, defined as follows. As a vector space, $i_P^G(\sigma)$ consists of the locally constant functions

$$f\colon G \to V_\sigma$$

such that

$$f(mng) = \delta_P(m)^{1/2}\,\sigma(m)\,f(g)$$

for all $m \in M$, $n \in N$ and $g \in G$. The action of G is given by right translation:

$$\big(\pi(g)f\big)(x) = f(xg), \qquad g, x \in G.$$

The normalized induced representation $i_P^G(\sigma)$ is the version compatible with contragredients, Jacquet modules, and the Langlands classification.

When working with induced representations, it is often convenient to fix a maximal compact subgroup K and use the Iwasawa decomposition

$$G = PK.$$

Then every function $f \in i_P^G(\sigma)$ is determined by its restriction to K, and the defining relation may be rewritten as

$$f(pk) = \delta_P(p)^{1/2}\sigma(p)f(k), \qquad p \in P,\ k \in K.$$

Jacquet modules

Let (π, V) be a smooth representation of G. Define $V(N)$ to be the subspace of V generated by all vectors of the form

$$\pi(n)v - v, \qquad n \in N, \ v \in V.$$

Definition 8. *The* Jacquet module *of π with respect to $P = MN$ is the quotient*

$$V_N := V/V(N).$$

The normalized action of M on V_N is defined by

$$\pi_N(m)(\overline{v}) = \delta_P(m)^{-1/2}\,\overline{\pi(m)v}, \qquad m \in M,$$

where $\overline{v}$ denotes the image of v in V_N. This makes (π_N, V_N) into a smooth representation of M.

We refer to [Cas95, §3.2] for the definition of the Jacquet module and Frobenius reciprocity. For the normalized parabolic induction and Jacquet functors for reductive p-adic groups, see also [Ren10, VI.1].

Theorem 2.5.1 (Frobenius reciprocity)**.** *Let (π, V) be a smooth representation of G and let (σ, W) be a smooth representation of M. Then there is a natural isomorphism*

$$\mathrm{Hom}_G\big(\pi, i_P^G(\sigma)\big) \simeq \mathrm{Hom}_M(\pi_N, \sigma),$$

where π_N denotes the normalized Jacquet module of π with respect to P.

Proof. This is the normalized form of the second Frobenius reciprocity. We refer to [Cas95, §3.2, especially (3.2.4)]. $\square$

Contragredients of normalized induced representations

We also record the compatibility of normalized parabolic induction with contragredients.

Lemma 2.5.2. *Let G be a p-adic reductive group and let $P = MN$ be a standard parabolic subgroup of G with Levi component M. Let σ be a smooth representation of M. Then*

$$\left(i_P^G(\sigma)\right)^\vee \simeq i_P^G(\sigma^\vee).$$

Proof. We refer to [Ren10, (IV.2.1.2)]. □

2.6 Root systems for $\mathrm{GL}(n, F)$

Let $G = \mathrm{GL}(n, F)$ and let $B = TU$ be the standard Borel subgroup of upper triangular matrices. Then

$$T = \left\{ \mathrm{diag}(t_1, \ldots, t_n) : \; t_i \in F^\times \right\}$$

For $1 \leq i \leq n$, define the characters $e_i : T \to F^\times$ by

$$e_i(\mathrm{diag}(t_1, \ldots . t_n)) = t_i.$$

Then the **root system** of G with respect to T is

$$\Phi = \{e_i - e_j : 1 \leq i \neq j \leq n\}.$$

The **simple roots** are given by

$$\alpha_i := e_i - e_{i+1} \qquad \text{for } i = 1, \ldots, n-1$$

and as characters of T they have the form

$$\alpha_i\big(\mathrm{diag}(t_1, \ldots, t_n)\big) \;=\; \frac{t_i}{t_{i+1}}.$$

For each $1 \leq i < j \leq n$, let E_{ij} be the matrix unit and define the **root subgroup** of U by

$$U_{i,j} \ := \ \{\, I + uE_{ij} : \ u \in F \,\}.$$

Then $U_{\alpha_i} = U_{i,i+1}$ is called the **simple root subgroup** corresponding to α_i.

Note that if $s = \mathrm{diag}(s_1, \ldots, s_n) \in T$, then

$$s\,(I + uE_{i,i+1})\,s^{-1} \ = \ I + \Big(\frac{s_i}{s_{i+1}}\Big) u\,E_{i,i+1} \ = \ I + \alpha_i(s)\,u\,E_{i,i+1}. \tag{2.1}$$

Thus conjugation by s acts on the element $u \in F \simeq U_{i,i+1}$ via multiplication by $\alpha_i(s)$.

Remark 2.6.1. *With notation as above, the following are equivalent:*

1. *T has an element s which acts by -1 on each simple root subgroup of U.*
2. *$\alpha_i(s) = -1 \qquad$ for every $i = 1, \ldots, n-1$.*
3. *$s \in \{a \cdot \mathrm{diag}(1, -1, 1, -1, \ldots, (-1)^{n-1}) : a \in F^\times\}$*
4. *$s\,(I + uE_{i,i+1})\,s^{-1} \ = \ I - uE_{i,i+1} \qquad$ for all $u \in F, \ \ i = 1, \ldots, n-1$.*

2.7 Generic representations and Whittaker models

Let G be the group of F-points of a connected reductive algebraic group defined over a non-Archimedean local field F. Assume that G is quasi-split over F. Fix a Borel subgroup $B = TU$ of G defined over F with unipotent radical U.

Definition 9. *A smooth character $\psi : U \to \mathbb{C}^\times$ is called* nondegenerate *if its restriction to each simple root subgroup $U_\alpha \subset U$ is nontrivial (with respect to (G, T, B)).*

Let (π, V) be a smooth representation of G. For a character ψ of U, define the space of *Whittaker functionals* by

$$\mathrm{Hom}_U(\pi, \psi) \ := \ \{\, \ell : V \to \mathbb{C} \ \mid \ \ell(\pi(u)v) = \psi(u)\,\ell(v) \text{ for all } u \in U, \ v \in V \,\}.$$

Definition 10. *An irreducible smooth representation* (π, V) *of* G *is called* ψ-generic *if* $\mathrm{Hom}_U(\pi, \psi) \neq 0$ *for some nondegenerate character* ψ *of* U*. In this case, we also say that* π *admits a* Whittaker model.

Theorem 2.7.1 (Multiplicity one for Whittaker models)**.** *Let* G *be the group of* F*-points of a connected reductive group defined and split over* F*. Let* π *be an irreducible admissible representation of* G*, and let* $\psi : U \to \mathbb{C}^\times$ *be a nondegenerate character. Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(\pi, \psi) \leq 1.$$

In particular, π *is generic if and only if*

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(\pi, \psi) = 1$$

for some nondegenerate character ψ *of* U*.*

Proof. See [Rod73, Theorem 3]. □

2.8 Rodier's compact approximation

In this section we recall a theorem of Rodier which allows one to approximate Whittaker models by characters of compact open subgroups. This will be a key input later in our adaptation of Prasad's argument.

Theorem 2.8.1 (Rodier)**.** *Assume that* F *is a non-archimedean local field of characteristic* 0 *whose residual characteristic* p *is sufficiently large in the sense of Rodier (in particular* $p \geq 2h + 1$*, where* h *is the height of the highest root). Let* G *be the group of* F*–points of a connected reductive group defined and split over* F*, and let* $B = TU$ *be a Borel subgroup with unipotent radical* U*. Fix a non-degenerate character*

$$\psi : U \longrightarrow \mathbb{C}^\times.$$

Let (σ_0, V) be an admissible representation of G. Then there exists a compact open subgroup $K \subset G$ and a character $\psi_K : K \to \mathbb{C}^\times$ such that

$$\dim_{\mathbb{C}} \operatorname{Hom}_K(\sigma_0, \psi_K) \;=\; \dim_{\mathbb{C}} \operatorname{Hom}_U(\sigma_0, \psi).$$

In particular, if σ_0 is irreducible and ψ-generic, then

$$\dim_{\mathbb{C}} \operatorname{Hom}_K(\sigma_0, \psi_K) = 1.$$

Proof. We refer to [Rod75]. □

Remark 2.8.2. *For $G = \mathrm{GL}_n(F)$, Rodier observes that one can avoid the assumption $\mathrm{char}(F) = 0$ and the bound $p \geq 2h + 1$ (where h is the height of the highest root); see [Rod75, §VI].*

Illustration for $G = \mathrm{GL}(n, F)$

Let F be a non-Archimedean local field with ring of integers $\mathcal{O}$, uniformizer ϖ, and maximal ideal $\mathfrak{p} = (\varpi)$. Set $G = \mathrm{GL}_n(F)$ and let $B = TU$ be the standard Borel subgroup of upper triangular matrices, where T is the diagonal torus and U is the upper unitriangular subgroup.

The Whittaker character

Fix a nontrivial additive character

$$\psi_F : F \longrightarrow \mathbb{C}^\times$$

which is trivial on $\mathcal{O}$ but not on $\varpi^{-1}\mathcal{O}$. Define the standard nondegenerate character $\psi_{st} : U \to \mathbb{C}^\times$ by

$$\psi_{st}(u) \;:=\; \psi_F\big(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n}\big), \qquad u \in U.$$

If σ_0 is an irreducible ψ_{st}-generic representation of G, then by Shalika's multiplicity-one theorem [Sha74] we have

$$\dim_{\mathbb{C}} \operatorname{Hom}_U(\sigma_0, \psi_{st}) = 1.$$

A concrete family of compact open pairs

For each integer $m \geq 1$ define the principal congruence subgroup

$$K_m := 1 + \varpi^m M_n(\mathcal{O}) \subset \mathrm{GL}_n(\mathcal{O}).$$

Define a character $\psi_m : K_m \to \mathbb{C}^\times$ by

$$\psi_m(1+X) \;:=\; \psi_F\Big(\varpi^{-2m} \sum_{i=1}^{n-1} X_{i,i+1}\Big), \qquad X \in \varpi^m M_n(\mathcal{O}). \tag{2.2}$$

This is a group homomorphism: if $X, Y \in \varpi^m M_n(\mathcal{O})$ then $(1+X)(1+Y) = 1 + (X + Y + XY)$ with $XY \in \varpi^{2m} M_n(\mathcal{O})$, hence

$$\varpi^{-2m} \sum_{i=1}^{n-1} (XY)_{i,i+1} \in \mathcal{O},$$

so $\psi_F\Big(\varpi^{-2m} \sum_{i=1}^{n-1} (XY)_{i,i+1}\Big) = 1$ since ψ_F is trivial on $\mathcal{O}$. So,

$$\begin{aligned}
\psi_m\big((1+X)(1+Y)\big) &= \psi_F\left(\varpi^{-2m} \sum_{i=1}^{n-1} (X+Y+XY)_{i,i+1}\right) \\
&= \psi_F\left(\varpi^{-2m} \sum_{i=1}^{n-1} X_{i,i+1}\right) \psi_F\left(\varpi^{-2m} \sum_{i=1}^{n-1} Y_{i,i+1}\right) \psi_F\left(\varpi^{-2m} \sum_{i=1}^{n-1} (XY)_{i,i+1}\right) \\
&= \psi_m(1+X)\psi_m(1+Y).
\end{aligned}$$

Next set

$$d := \mathrm{diag}(1, \varpi^2, \varpi^4, \ldots, \varpi^{2(n-1)}) \in T, \qquad K'_m := d^m K_m d^{-m},$$

and define another character $\psi'_m : K'_m \to \mathbb{C}^\times$ by

$$\psi'_m(k) := \psi_m(d^{-m} k d^m), \qquad k \in K'_m.$$

The point of conjugating by d^m is that $K'_m \cap U$ grows with m and eventually exhausts U, and that the character ψ'_m agrees with ψ_{st} on $K'_m \cap U$.

Lemma 2.8.3. *With notation as above, the sequence $(K'_m \cap U)_{m\geq 1}$ is increasing and*

$$\bigcup_{m\geq 1} (K'_m \cap U) = U.$$

Proof. Write $d = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_i = \varpi^{2(i-1)}$. For any $X = (x_{ij}) \in M_n(F)$ and any $m \geq 1$,

$$(d^m X d^{-m})_{ij} \;=\; d_i^m \, x_{ij} \, d_j^{-m} \;=\; \varpi^{-2m(j-i)} x_{ij}.$$

If $k \in K_m$, then $k = 1 + X$ with $X \in \varpi^m M_n(\mathcal{O})$, so for $i < j$,

$$(d^m X d^{-m})_{ij} \in \varpi^{-m(2(j-i)-1)}\mathcal{O}.$$

Thus

$$K'_m \cap U = \Big\{ u \in U : \; u_{ij} \in \varpi^{-m(2(j-i)-1)}\mathcal{O} \text{ for all } i < j \Big\}.$$

The ideals $\varpi^{-m(2(j-i)-1)}\mathcal{O}$ increase with m, giving $K'_m \cap U \subset K'_{m+1} \cap U$. Since for each fixed $i < j$ the bound $-m(2(j-i)-1)$ tends to $-\infty$ as $m \to \infty$, every $u \in U$ satisfies $u \in K'_m \cap U$ for $m \gg 0$, hence $\bigcup_m (K'_m \cap U) = U$. $\square$

Lemma 2.8.4. *With notation as above,*

$$\psi'_m|_{K'_m \cap U} = \psi_{st}|_{K'_m \cap U}.$$

Proof. Let $u \in K'_m \cap U$. Then $d^{-m} u d^m \in K_m \cap U$, so write

$$d^{-m} u d^m = 1 + X$$

with $X \in \varpi^m M_n(\mathcal{O})$ strictly upper triangular. Conjugating back gives $u = 1 + d^m X d^{-m}$. For $1 \leq i \leq n-1$,

$$u_{i,i+1} = (d^m X d^{-m})_{i,i+1} = (d_{ii})^m X_{i,i+1} (d_{i+1,i+1})^{-m} = \varpi^{-2m} X_{i,i+1}.$$

Therefore

$$\psi_{st}(u) = \psi_F\Big(\sum_{i=1}^{n-1} u_{i,i+1}\Big) = \psi_F\Big(\varpi^{-2m}\sum_{i=1}^{n-1} X_{i,i+1}\Big) = \psi_m(1+X) = \psi'_m(u).$$

□

Theorem 2.8.5 (Rodier's compact approximation theorem for $\mathrm{GL}(n,F)$)**.** *Let (σ_0, V) be an admissible smooth representation of $G = \mathrm{GL}(n,F)$. For the explicit family (K'_m, ψ'_m) defined above, the compactly induced representations*

$$\text{c-Ind}_{K'_m}^{G}(\psi'_m)$$

form an inductive system, and there is a natural isomorphism

$$\varinjlim_m \text{c-Ind}_{K'_m}^{G}(\psi'_m) \;\cong\; \text{c-Ind}_{U}^{G}(\psi_{st}).$$

Consequently,

$$\varinjlim_m \mathrm{Hom}_{K'_m}(\sigma_0, \psi'_m) \;\cong\; \mathrm{Hom}_U(\sigma_0, \psi_{st}).$$

In particular, for all sufficiently large m,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K'_m}(\sigma_0, \psi'_m) = \dim_{\mathbb{C}} \mathrm{Hom}_U(\sigma_0, \psi_{st}).$$

Hence σ_0 is ψ_{st}-generic if and only if

$$\mathrm{Hom}_{K'_m}(\sigma_0, \psi'_m) \neq 0$$

for all sufficiently large m.

Proof. We refer to [Rod75]. □

Corollary 2.8.6. *Let (σ_0, V) be an irreducible generic representation of $G = \mathrm{GL}(n,F)$. Then, with the pairs (K'_m, ψ'_m) defined above, one has*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K'_m}(\sigma_0, \psi'_m) = 1$$

for all sufficiently large m.

Proof. Since σ_0 is generic, there exists a nondegenerate character $\psi : U \to \mathbb{C}^\times$ such that

$$\mathrm{Hom}_U(\sigma_0, \psi) \neq 0.$$

For $G = \mathrm{GL}(n, F)$, every nondegenerate character of U is T-conjugate to ψ_{st}. Thus there exists $t \in T$ such that

$$\psi(u) = \psi_{\mathrm{st}}(t^{-1}ut) \qquad \text{for all } u \in U.$$

Note that for each $\ell \in \mathrm{Hom}_U(\sigma_0, \psi)$, we have $\ell \circ \sigma_0(t) \in \mathrm{Hom}_U(\sigma_0, \psi_{st})$ since

$$\begin{aligned}(\ell \circ \sigma_0(t)) \circ \sigma_0(u) = \ell \circ \sigma_0(tu) &= (\ell \circ \sigma_0(tut^{-1})) \circ \sigma_0(t) \\ &= (\psi(tut^{-1}) \circ \ell) \circ \sigma_0(t) = \psi_{st}(u) \circ \sigma_0(t)\end{aligned}$$

for all $u \in U$. So we may define a linear map

$$A_t : \mathrm{Hom}_U(\sigma_0, \psi) \longrightarrow \mathrm{Hom}_U(\sigma_0, \psi_{\mathrm{st}})$$

by

$$A_t(\ell) := \ell \circ \sigma_0(t) \qquad \text{for all } \ell \in \mathrm{Hom}_U(\sigma_0, \psi).$$

Also, its inverse is given by

$$A_t^{-1}(\ell) := m \circ \sigma_0(t^{-1}) \qquad \text{for all } \ell \in \mathrm{Hom}_U(\sigma_0, \psi_{\mathrm{st}}),$$

so A_t is an isomorphism. Hence

$$\mathrm{Hom}_U(\sigma_0, \psi_{\mathrm{st}}) \neq 0.$$

By Theorem 2.7.1,

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(\sigma_0, \psi_{\mathrm{st}}) = 1.$$

Now apply Theorem 2.8.5 to σ_0 and ψ_{st}. It follows that for all sufficiently large m,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{K'_m}(\sigma_0, \psi'_m) = \dim_{\mathbb{C}} \mathrm{Hom}_U(\sigma_0, \psi_{\mathrm{st}}) = 1.$$

This proves the corollary. $\square$

2.9 Langlands Classification for $\mathrm{SL}(n)$

Let F be a non-Archimedean local field and let $G = \mathrm{SL}(n, F)$.

Definition 11. *A Langlands triple (or data) is a set of three objects (P, σ, ν) where $P = MN$ is a standard parabolic subgroup of G, σ is an irreducible tempered representation of the Levi factor*

$$M \;=\; \Big\{\mathrm{diag}(g_1, \ldots, g_r) \in \mathrm{GL}_n(F) \;:\; g_i \in \mathrm{GL}_{n_i}(F)\Big\} \cap G,$$

and $\nu : M \to \mathbb{R}_{>0}$ is a character of the form

$$\nu\big(\mathrm{diag}(g_1, \ldots, g_r)\big) \;=\; \prod_{i=1}^{r} |\det g_i|_F^{\nu_i}, \qquad \nu_i \in \mathbb{R},$$

satisfying the condition

$$\nu_1 \;>\; \nu_2 \;>\; \cdots \;>\; \nu_r.$$

Here $|\cdot|_F$ is the normalized absolute value on F.

Theorem 2.9.1 (Langlands Classification)**.** *(i) If (P, σ, ν) is a Langlands triple then the normalized parabolically induced representation $i_P^G(\sigma\nu)$ admits a unique irreducible quotient, which we call Langlands quotient and denote $J(P, \sigma, \nu)$.*

(ii) For each irreducible smooth representation π of G, there exists a Langlands triple (P, σ, ν) such that $\pi \simeq J(P, \sigma, \nu)$.

(iii) For any Langlands triples (P_1, σ_1, ν_1) and (P_2, σ_2, ν_2), if $J(P_1, \sigma_1, \nu_1) \simeq J(P_2, \sigma_2, \nu_2)$ then $P_1 = P_2$, $\sigma_1 \simeq \sigma_2$, and $\nu_1 = \nu_2$.

Proof. We refer to [Kon03, Theorem 3.5].

□

Central characters and Langlands quotients

We record a simple fact about the behavior of central characters under passage to a Langlands quotient.

Lemma 2.9.2. *Let $P = MN$ be a parabolic subgroup of G and let τ be an irreducible smooth representation of M. If π is an irreducible quotient of the normalized induced representation $i_P^G(\tau)$, then for every $z \in Z(G)$ one has*

$$\omega_\pi(z) = \omega_\tau(z).$$

In particular, if π is the Langlands quotient of $i_P^G(\sigma\nu)$, then

$$\omega_\pi(z) = \omega_{\sigma\nu}(z).$$

Proof. Recall that $i_P^G(\tau)$ is realized on the space of smooth functions $f\colon G \to V_\tau$ satisfying

$$f(mng) = \delta_P(m)^{1/2}\tau(m)f(g), \qquad m \in M,\ n \in N,\ g \in G,$$

and that G acts by right translation:

$$(g_0 \cdot f)(g) = f(gg_0), \qquad \text{for } g_0, g \in G.$$

Let $z \in Z(G)$. Then $z \in M$ and $\mathrm{Ad}(z)$ is trivial on $\mathrm{Lie}(G)$, hence also on $\mathrm{Lie}(N)$. Therefore

$$\delta_P(z) = \left|\det\left(\mathrm{Ad}(z)\mid_{\mathrm{Lie}(N)}\right)\right|_F = 1.$$

Hence, for every $f \in i_P^G(\tau)$ and every $g \in G$, we have

$$(z \cdot f)(g) = f(gz) = f(zg) = \delta_P(z)^{1/2}\tau(z)f(g) = \tau(z)f(g).$$

Therefore z acts on $i_P^G(\tau)$ through the operator $\tau(z)$. It follows that z acts on $i_P^G(\tau)$ by the scalar $\omega_\tau(z)$. Passing to the irreducible quotient π, we conclude that z acts on π by the same scalar. Hence

$$\omega_\pi(z) = \omega_\tau(z).$$

The final statement is obtained by applying the first part with $\tau = \sigma\nu$.

$\square$

Corollary 2.9.3. *With notation as above, if $z = -I_n \in Z(G)$, then*

$$\omega_\pi(-I_n) = \omega_{\sigma\nu}(-I_n) = \omega_\sigma(-I_n).$$

Proof. By Lemma 2.9.2, applied to the Langlands quotient π of $i_P^G(\sigma\nu)$, we have

$$\omega_\pi(-I_n) = \omega_{\sigma\nu}(-I_n).$$

Note that

$$\omega_{\sigma\nu}(-I_n) = \omega_\sigma(-I_n)\,\nu(-I_n),$$

and ν has the form

$$\nu(\mathrm{diag}(g_1, \ldots, g_r)) = \prod_{i=1}^{r} |\det(g_i)|_F^{s_i}$$

for some real numbers s_i. Evaluating at $-I_n$, each block is $-I_{n_i}$, so

$$\nu(-I_n) = \prod_{i=1}^{r} |\det(-I_{n_i})|_F^{s_i}.$$

Since $|\det(-I_{n_i})|_F = |(-1)^{n_i}|_F = 1$ for every i, we get $\nu(-I_n) = 1$. It follows that

$$\omega_{\sigma\nu}(-I_n) = \omega_\sigma(-I_n),$$

and so

$$\omega_\pi(-I_n) = \omega_{\sigma\nu}(-I_n) = \omega_\sigma(-I_n).$$

$\square$

Contragredients of Langlands quotients

We next record the behavior of Langlands data under passage to the contragredient.

Proposition 2.9.4. *Let π be an irreducible smooth representation of G such that $\pi \simeq J(P,\sigma,\nu)$. Let w_0 be the matrix in G defined by*

$$w_0 \;=\; \begin{cases} J, & \text{if } n \equiv 0,1 \pmod 4, \\ \\ D\,J, & \text{if } n \equiv 2,3 \pmod 4, \end{cases}$$

where J is the anti-diagonal matrix consisting of 1 and $D = \mathrm{diag}(-1,1,\dots,1)$. Then we have

$$\pi^\vee \simeq J(P',\sigma',\nu')$$

where $P' = w_0\overline{P}w_0^{-1}$, $\sigma' = (\sigma^\vee)^{w_0}$ and $\nu' = (\nu^{-1})^{w_0}$. Here, $\overline{P}$ is the opposite parabolic to P.

Proof. Since π is a quotient of $i_P^G(\sigma\nu)$, there is a surjection

$$i_P^G(\sigma\nu) \twoheadrightarrow \pi.$$

Taking contragredients yields an injection

$$\pi^\vee \hookrightarrow \big(i_P^G(\sigma\nu)\big)^\vee.$$

By Lemma 2.5.2, we have

$$\big(i_P^G(\sigma\nu)\big)^\vee \;\simeq\; i_P^G(\sigma^\vee\nu^{-1}).$$

Hence, the dual representation $\pi^\vee$ embeds in $i_P^G(\sigma^\vee\nu^{-1})$. From Langlands theory,

$$\mathrm{Hom}_G\big(i_{\overline{P}}^G(\sigma^\vee\nu^{-1}),\; i_P^G(\sigma^\vee\nu^{-1})\big)$$

is 1-dimensional and $\pi^\vee$ is the image of any nonzero element in this space. Therefore, $\pi^\vee$ is the unique irreducible quotient of $i_{\overline{P}}^G(\sigma^\vee\nu^{-1})$. Note that $P' = w_0\,\overline{P}\,w_0^{-1}$ is a standard parabolic subgroup and we have

$$i_{\overline{P}}^G(\sigma^\vee\nu^{-1}) \simeq i_{P'}^G\big((\sigma^\vee)^{w_0}(\nu^{-1})^{w_0}\big).$$

Therefore, $\pi^\vee$ is the unique irreducible quotient of $i_{P'}^G\big((\sigma^\vee)^{w_0}(\nu^{-1})^{w_0}\big)$. We now check that $\big(P', \sigma', \nu'\big)$ is a Langlands triple for G, where $\sigma' = (\sigma^\vee)^{w_0}$ and $\nu' = (\nu^{-1})^{w_0}$. Note that temperedness is preserved under dual operation $\vee$ and under conjugation, so $(\sigma^\vee)^{w_0}$ is tempered on

$$w_0 M w_0^{-1} = \Big\{ \operatorname{diag}(g_r, \ldots, g_1) : g_i \in \mathrm{GL}_{n_i}(F) \Big\} \cap G.$$

Also, ν' is a character positive character on $w_0 M w_0^{-1}$ satisfying the relevant condition since for each $g = \operatorname{diag}(g_r, g_{r-1}, \ldots, g_1) \in w_0 M w_0^{-1}$ we have

$$\begin{aligned}
\nu'(g) &= (\nu^{-1})^{w_0}(g) = \nu^{-1}(w_0 g w_0^{-1}) = \nu^{-1}\big(\operatorname{diag}(g_1, g_2, \ldots, g_r)\big) = \frac{1}{\nu(g_1, g_2, \ldots, g_r)} \\
&= \frac{1}{\prod_{i=1}^r |\det g_i|_F^{\nu_i}}, \quad \nu_i \in \mathbb{R} \text{ with } \nu_1 > \nu_2 > \cdots > \nu_r \\
&= \prod_{i=1}^r |\det g_{r+1-i}|_F^{-\nu_{r+1-i}} \quad \text{with } -\nu_r > -\nu_{r-1} > \cdots > -\nu_1.
\end{aligned}$$

Therefore (P', σ', ν') is a Langlands triple for G, and

$$\pi^\vee \simeq J(P', \sigma', \nu').$$

□

Corollary 2.9.5. *Let π be an irreducible self-dual representation of G, and let (P, σ, ν) be the Langlands triple of π. Then the tuple of diagonal block sizes $(n_1, n_2, \ldots, n_r)$ of P is palindromic; that is, $(n_1, n_2, \ldots, n_r) = (n_r, n_{r-1}, \ldots, n_1)$.*

Proof. Since $\pi \simeq \pi^\vee$, by Proposition 2.9.4 and Theorem 2.9.1, we have

$$J(P,\sigma,\nu) \simeq J(P',\sigma',\nu') \implies P = P', \sigma \simeq \sigma' \text{ and } \nu = \nu'$$

together with the same notation P', σ' and ν' in Proposition 2.9.4. Since the conjugation of $\overline{P}$ by the element w_0 reverses the diagonal block sizes of P, this proves the palindromic property. $\square$

Chapter 3

Self-Dual Representations and Signs

Let F be a non-Archimedean local field, and let G be the group of F-points of a connected reductive algebraic group defined over F. Let (π, V) be a smooth irreducible complex representation of G. We write $(\pi^\vee, V^\vee)$ for the smooth dual or contragredient of (π, V). Let θ be a continuous automorphism of G of order at most two. Let (π^θ, V) be the θ-twist of π defined by

$$\pi^\theta(g)v = \pi(\theta(g))v \text{ for all } g \in G, v \in V.$$

Suppose that $\pi^\theta \simeq \pi^\vee\chi$ for some character χ of G. Note that

$$\mathrm{Hom}(\pi^\theta, \pi^\vee\chi) \simeq \mathrm{Bil}_{\pi\chi^{-1},\pi^\theta}(V)$$

via $s \mapsto b_s : V \times V \to \mathbb{C}$ where $b_s(v_1, v_2) = \langle v_1, s(v_2)\rangle$ for all $v_1, v_2 \in V$, and $\mathrm{Bil}_{\pi\chi^{-1},\pi^\theta}(V)$ is the space of all bilinear forms b on V such that

$$b(\pi(g)\chi^{-1}(g)v_1, \pi^\theta(g)v_2) = b(v_1, v_2)$$

for all $g \in G, v_1, v_2 \in V$. For simplicity, we will say that such forms are $(\pi\chi^{-1}, \pi^\theta)$-invariant.

Now let $b \in \mathrm{Bil}_{\pi\chi^{-1},\pi^{\theta}}(V)$ be a nonzero form. Then we may define another bilinear form b' on V by $b'(v_1, v_2) = b(v_2, v_1)$ for all $v_1, v_2 \in V$. Assume $\chi \circ \theta = \chi$. By Lemma 3.0.1, b' is $(\pi\chi^{-1}, \pi^{\theta})-$invariant. Since we have

$$\dim(\mathrm{Bil}_{\pi\chi^{-1},\pi^{\theta}}(V)) = 1$$

by Schur's lemma, we must have

$$b' = \lambda b$$

for some $\lambda \in \mathbb{C}$. Therefore, there exists $v_1, v_2 \in V$ such that

$$0 \neq b(v_1, v_2) = b'(v_2, v_1) = \lambda b(v_2, v_1) = \lambda b'(v_1, v_2) = \lambda^2 b(v_1, v_2)$$

So, b is either symmetric or skew-symmetric. We set $\lambda = \epsilon_{\theta,\chi}(\pi)$ and call it **twisted sign** of π. It clearly depends only on the equivalence class of π. If θ is the trivial automorphism of G and χ is the trivial character of G, we simply write $\epsilon(\pi)$ and call it the **ordinary sign**. Similarly, if θ is trivial, we write $\epsilon_{\chi}(\pi)$, and if χ is trivial, we write $\epsilon_{\theta}(\pi)$.

Lemma 3.0.1. *If $\chi \circ \theta = \chi$, then $b' \in \mathrm{Bil}_{\pi\chi^{-1},\pi^{\theta}}(V)$.*

Proof. Let $g \in G$. Note that there exists an element $h \in G$ such that $\theta(h) = g$. Then, for any $u, v \in V$, we have

$$\begin{aligned}
b'(\pi(g)\chi^{-1}(g)u, \pi^{\theta}(g)v) &= b(\pi^{\theta}(g)v, \pi(g)\chi^{-1}(g)u) \\
&= b(\pi^{\theta}(\theta(h))v, \pi(\theta(h))\chi^{-1}(\theta(h))u) \\
&= b(\pi(h)\chi^{-1}(\theta(h))v, \pi^{\theta}(h)u) \\
&= b(\pi(h)\chi^{-1}(h)v, \pi^{\theta}(h)u) \quad \text{since } \chi \circ \theta = \chi \\
&= b(v, u) \quad \text{since } b \text{ is } (\pi\chi^{-1}, \pi^{\theta})-\text{invariant} \\
&= b'(u, v).
\end{aligned}$$

□

Proposition 3.0.2. *Let (π, V) be an irreducible self-dual representation of G. Let $x \in G$ such that $x^2 \in Z(G)$. Let θ be the inner automorphism of G defined by $\theta(g) = xgx^{-1}$ for all $g \in G$. Then we have*

$$\epsilon(\pi) = \epsilon_\theta(\pi)\omega_\pi(x^2)$$

where ω_π is the central character of π.

Proof. Note that $\mathrm{Bil}_G(V) \simeq \mathrm{Hom}(\pi, \pi^\vee) \simeq \mathrm{Hom}(\pi^\theta, \pi^\vee) \simeq \mathrm{Bil}_{\pi,\pi^\theta}(V)$. Let b be a nonzero $G-$invariant bilinear form on V. Then we may define the nonzero form $b_{F_b \circ \phi} \in \mathrm{Bil}_{\pi,\pi^\theta}(V)$ by $b_{F_b\circ\phi}(u, v) = \langle u, (F_b \circ \phi)v\rangle$ for all $u, v \in V$, where $\phi = \pi(x)^{-1}$ and $F_b \in \mathrm{Hom}(\pi, \pi^\vee)$ defined by $F_b(v) = b(-, v)$ for all $v \in V$. Since $b_{F_b\circ\phi} \neq 0$, there exist $u, v \in V$ such that

$$\begin{aligned} 0 \neq b_{F_b\circ\phi}(u, v) &= \langle u, (F_b \circ \phi)v\rangle = b(u, \phi(v)) \\ &= \epsilon(\pi)b(\phi(v), u) \\ &= \epsilon(\pi)b(\phi^2(v), \phi(u)) \\ &= \epsilon(\pi)\omega_\pi(x^{-2})b(v, \phi(u)) \\ &= \epsilon(\pi)\omega_\pi(x^{-2})b_{F_b\circ\phi}(v, u) \\ &= \epsilon(\pi)\omega_\pi(x^{-2})\epsilon_\theta(\pi)b_{F_b\circ\phi}(u, v). \end{aligned}$$

Therefore, $\epsilon(\pi)\omega_\pi(x^{-2})\epsilon_\theta(\pi) = 1$, equivalently,

$$\epsilon(\pi) = \epsilon_\theta(\pi)\omega_\pi(x^2).$$

□

Proposition 3.0.3. *Let σ_0 be a smooth irreducible representation of G such that $\sigma^\theta \simeq \sigma\chi$ for some character χ of G and some inner automorphism θ of G given by an involution*

$x \in G$. Then we have

$$\epsilon_{\theta,\chi}(\sigma_0) = \epsilon_\chi(\sigma_0)\chi(x).$$

Proof. Note that $\mathrm{Bil}_{\sigma\chi^{-1},\sigma}(V) \simeq \mathrm{Hom}(\sigma, \sigma^\vee\chi) \simeq \mathrm{Hom}(\sigma^\theta, \sigma^\vee\chi) \simeq \mathrm{Bil}_{\sigma\chi^{-1},\sigma^\theta}(V)$. Let b be a nonzero $(\sigma\chi^{-1}, \sigma)-$invariant bilinear form on V. Then we may define the nonzero form $b_{F_b\circ\phi} \in \mathrm{Bil}_{\sigma\chi^{-1},\sigma^\theta}(V)$ by $b_{F_b\circ\phi}(u, v) = \langle u, (F_b \circ \phi)v\rangle$ for all $u, v \in V$, where $\phi = \sigma(x)^{-1}$ and $F_b \in \mathrm{Hom}(\sigma, \sigma^\vee)$ defined by $F_b(v) = b(-, v)$ for all $v \in V$. Since $b_{F_b\circ\phi} \neq 0$, there exist $u, v \in V$ such that

$$\begin{aligned}
0 \neq b_{F_b\circ\phi}(u, v) &= \langle u, (F_b \circ \phi)v\rangle = b(u, \phi(v)) \\
&= \epsilon_\chi(\sigma)b(\phi(v), u) \\
&= \epsilon_\chi(\sigma)b(\sigma(x^{-1})\chi^{-1}(x^{-1})\phi(v), \phi(u)) \\
&= \epsilon_\chi(\sigma)\chi(x)b(v, \phi(u)) \\
&= \epsilon_\chi(\sigma)\chi(x)b_{F_b\circ\phi}(v, u) \\
&= \epsilon_\chi(\sigma)\chi(x)\epsilon_{\theta,\chi}(\sigma)b_{F_b\circ\phi}(u, v)
\end{aligned}$$

Therefore, $\epsilon_\chi(\sigma)\chi(x)\epsilon_{\theta,\chi}(\sigma) = 1$, equivalently,

$$\epsilon_{\theta,\chi}(\sigma) = \epsilon_\chi(\sigma)\chi(x).$$

□

3.1 Reduction to Tempered Case

In [RS12], Roche and Spallone reduce the problem of computing the twisted sign to the case of tempered representations. Let F be a non-Archimedean local field, and let G be the group of F-points of a connected reductive algebraic group defined over F. Let θ be an involutory automorphism of G and suppose that $\pi^\theta \simeq \pi^\vee$. Let

(P, σ, ν) be the triple associated to π via the Langlands classification. Suppose that P has Levi decomposition $P = MN$. Under certain assumptions on the involution θ, they prove the following:

Theorem 3.1.1 (Roche-Spallone)**.** *Let π be an irreducible smooth representation of G such that $\pi^\theta \simeq \pi^\vee$. Suppose the Langlands classification attaches the triple (P, σ, ν) to π. Then $\sigma^\theta \simeq \sigma^\vee$ and $\epsilon_\theta(\pi) = \epsilon_\theta(\sigma)$.*

Remark. In the above theorem, for σ^θ to make sense, we need that $\theta(M) = M$. This follows from the assumptions on the involution θ.

Chapter 4

Results Used In the Main Theorem

4.1 Prasad's idea for computing the sign

In [Pra99], Prasad gives a criterion to compute the sign for an irreducible self-dual generic representation of a p-adic group G. He shows that for generic representations the sign is determined by the value of the central character at a special central element. We modify his result below by incorporating the character χ in Proposition 5.0.2

Lemma 4.1.1 (Finite Group Case). *Let H be a subgroup of a finite group G. Let $s \in N_G(H)$ be such that $s^2 \in Z(G)$. Let*

$$\psi : H \longrightarrow \mathbb{C}^\times$$

be a (one-dimensional) character of H such that

$$\psi(h)^{-1} = \psi(shs^{-1}) \qquad \textit{for all } h \in H,$$

equivalently, $\psi^\vee \simeq \psi \circ \mathrm{Int}(s)$.

Let (π, V) be an irreducible self-dual representation of G such that ψ appears with multiplicity 1 in $\pi|_H$. Then

$$\epsilon(\pi) = \omega_\pi(s^2),$$

where ω_π is the central character of π.

Proof. Let B be a non-zero form in $\mathrm{Bil}_G(V)$. Let

$$V^\psi := \{v \in V : \pi(h)v = \psi(h)v \text{ for all } h \in H\}$$

denote the ψ-isotypic subspace of $\pi|_H$, that is, the direct sum of copies of ψ in $\pi|_H$. By hypothesis, ψ has multiplicity 1 in $\pi|_H$, and since ψ is one-dimensional, we have

$$\dim V^\psi = 1.$$

Choose $0 \neq v_0 \in V^\psi$. We claim:

$$B\big(v_0, \pi(s)v_0\big) \neq 0. \tag{4.1}$$

Assuming this, we have

$$0 \neq B(v_0, \pi(s)v_0) = B(\pi(s)v_0, \pi(s^2)v_0) = \omega_\pi(s^2)B(\pi(s)v_0, v_0) = \omega_\pi(s^2)\epsilon(\pi)B(v_0, \pi(s)v_0),$$

so $\omega_\pi(s^2)\epsilon(\pi) = 1$ which implies that $\omega_\pi(s^2)\epsilon(\pi)^2 = \epsilon(\pi)$, that is,

$$\epsilon(\pi) = \omega_\pi(s^2).$$

Thus it only remains to prove the claim (4.1).

***Proof of the claim* $B(v_0, \pi(s)v_0) \neq 0$.**

Since B is non-degenerate and $v_0 \neq 0$, there exists a nonzero element $w \in V$ such that $B(v_0, w) \neq 0$. Since V is finite dimensional vector space, there exists an irreducible subrepresentation (ρ_2, W_2) of $(\pi|_H, V)$ such that $w \in W_2$. Let $(\rho_1, W_1) = (\psi, V^\psi)$. Since we have

$$\mathrm{Bil}_H(W_1, W_2) \simeq \mathrm{Hom}_H(\rho_2, \rho_1^\vee),$$

the nonzero form $B|_{W_1 \times W_2}$ in $\mathrm{Bil}_H(W_1, W_2)$ corresponds to an isomorphism

$$W_2 \simeq W_1^\vee,$$

as $H-$representations, by Schur's lemma.

We know that $\psi^\vee \simeq \psi^{-1}$. So we have $W_1^\vee \simeq V_{\psi^{-1}}$, where $V_{\psi^{-1}}$ is the underlying one-dimensional (abstract) vector space of ψ^{-1}. Note that ψ^{-1} has multiplicity 1 in $\pi|_H$ since we have

$$\mathrm{Hom}_H(\psi^{-1}, \pi|_H) \simeq \mathrm{Hom}_H(\psi \circ Int(s), \pi|_H) \simeq \mathrm{Hom}_H(\psi, \pi|_H)$$

together with the multiplicity property of ψ. Therefore, we must have

$$V_{\psi^{-1}} \simeq V^{\psi^{-1}} := \{v \in V : \pi(h)v = \psi^{-1}(h)v \text{ for all } h \in H\}.$$

So, $V^{\psi^{-1}}$ is one-dimensional since ψ^{-1} is one-dimensional.

Note that $\pi(s)v_0 \in V^{\psi^{-1}}$ because for any $h \in H$ we have:

$$\begin{aligned}
\pi(h)\pi(s)v_0 &= \pi(hs)v_0 = \pi(s^2 s^{-2} hs)v_0 \\
&= \pi(s^2 hs^{-1})v_0 \qquad \text{since } s^2 \in Z(G) \\
&= \pi(s)\psi(shs^{-1})v_0 \qquad \text{since } v_0 \in V^\psi \\
&= \psi(shs^{-1})\pi(s)v_0 \\
&= \psi^{-1}(h)\pi(s)v_0.
\end{aligned}$$

So, the vector $\pi(s)v_0$ spans $V^{\psi^{-1}}$. Hence, we get

$$W_2 \simeq W_1^\vee \simeq V_{\psi^{-1}} \simeq V^{\psi^{-1}} = \mathbb{C}\pi(s)v_0$$

Since ψ^{-1} has multiplicity 1 in $\pi|_H$, we have $W_2 = \mathbb{C}\pi(s)v_0$. Therefore, $w = \lambda\pi(s)v_0$ for some $\lambda \in \mathbb{C}^\times$. Finally,

$$0 \neq B(v_0, w) = B\big(v_0, \lambda\, \pi(s)v_0\big) = \lambda\, B(v_0, \pi(s)v_0),$$

so $B(v_0, \pi(s)v_0) \neq 0$, as claimed.

$\square$

Lemma 4.1.2 (Local Field Case with Twist)**.** *Let G be the group of F-points of a connected reductive algebraic group defined over F. Let H be a compact open subgroup of G. Let $s \in N_G(H)$ be such that $s^2 \in Z(G)$. Let*

$$\psi : H \longrightarrow \mathbb{C}^\times$$

be a one-dimensional representation of H such that

$$\psi^{-1}(h) = \psi(shs^{-1}) \qquad \text{for all } h \in H,$$

equivalently, $\psi^\vee \simeq \psi \circ \mathrm{Int}(s)$. Let (σ_0, V) be an irreducible smooth representation of G such that ψ appears with multiplicity 1 in $\sigma_0|_H$. Suppose that $\sigma_0 \simeq \sigma_0^\vee\chi$ for some character χ of G which is trivial on H. Then

$$\epsilon_\chi(\sigma_0) = \omega_{\sigma_0}(s^2)\chi^{-1}(s),$$

where ω_{σ_0} is the central character of σ_0.

Proof. By hypothesis $\sigma_0 \simeq \sigma_0^\vee\chi$, there exists a nonzero

$$B \in \mathrm{Bil}_{\sigma_0\chi^{-1},\sigma_0}(V).$$

Define the ψ-isotypic subspace

$$V^\psi := \{v \in V : \sigma_0(h)v = \psi(h)v \text{ for all } h \in H\}.$$

By hypothesis, ψ appears with multiplicity 1 in $\sigma_0|_H$, and since ψ is one-dimensional we have

$$\dim V^{\psi} = 1.$$

Choose a nonzero $v_0 \in V^{\psi}$. We claim:

$$B(v_0, \sigma_0(s)v_0) \neq 0. \tag{4.2}$$

Assuming this, we have

$$\begin{aligned} 0 \neq B(v_0, \sigma_0(s)v_0) &= B(\sigma_0(s)\chi^{-1}(s)v_0, \sigma_0(s^2)v_0) \\ &= \omega_{\sigma_0}(s^2)\chi^{-1}(s)B(\sigma_0(s)v_0, v_0) \\ &= \omega_{\sigma_0}(s^2)\chi^{-1}(s)\epsilon_\chi(\sigma_0)B(v_0, \sigma_0(s)v_0), \end{aligned}$$

so $\omega_{\sigma_0}(s^2)\chi^{-1}(s)\epsilon_\chi(\sigma_0) = 1$, that is,

$$\epsilon_\chi(\sigma_0) = \omega_{\sigma_0}(s^2)\chi^{-1}(s).$$

Thus it only remains to prove the claim (4.2).

***Proof of the claim* $B(v_0, \sigma_0(s)v_0) \neq 0$.**

Since B is non-degenerate and $v_0 \neq 0$, there exists a nonzero element $w \in V$ such that $B(v_0, w) \neq 0$. Let W be the H-subrepresentation of V generated by w:

$$W := \mathrm{span}\{\sigma_0(h)w : h \in H\}.$$

Because H is compact and σ_0 is smooth, W is finite-dimensional. Choose an irreducible subrepresentation (ρ_2, W_2) of $(\sigma_0|_H, W)$ such that $w \in W_2$. Let $(\rho_1, W_1) = (\psi, V^{\psi})$. Using the standard identification

$$\mathrm{Bil}_H(W_1, W_2) \simeq \mathrm{Hom}_H(\rho_2, \rho_1^{\vee}),$$

and Schur's lemma, the nonzero form $B|_{W_1\times W_2}$ in $\mathrm{Bil}_H(W_1,W_2)$ corresponds to an isomorphism

$$W_2 \simeq W_1^\vee,$$

as $H-$representations.

We know that $\psi^\vee \simeq \psi^{-1}$. So we have $W_1^\vee \simeq V_{\psi^{-1}}$, where $V_{\psi^{-1}}$ is the underlying one-dimensional (abstract) vector space of ψ^{-1}. Note that ψ^{-1} has multiplicity 1 in $\sigma_0|_H$ since we have

$$\mathrm{Hom}_H(\psi^{-1},\sigma_0|_H) \simeq \mathrm{Hom}_H(\psi\circ Int(s),\sigma_0|_H) \simeq \mathrm{Hom}_H(\psi,\sigma_0|_H)$$

together with the multiplicity property of ψ. Therefore, we must have

$$V_{\psi^{-1}} \simeq V^{\psi^{-1}} := \{v\in V : \sigma_0(h)v = \psi^{-1}(h)v \text{ for all } h\in H\}.$$

So, $V^{\psi^{-1}}$ is one-dimensional since ψ^{-1} is one-dimensional.

Note that $\sigma_0(s)v_0 \in V^{\psi^{-1}}$ because for any $h\in H$ we have:

$$\begin{aligned}
\sigma_0(h)\sigma_0(s)v_0 &= \sigma_0(hs)v_0 = \sigma_0(s^2s^{-2}hs)v_0 \\
&= \sigma_0(s^2hs^{-1})v_0 \qquad \text{since } s^2\in Z(G) \\
&= \sigma_0(s)\psi(shs^{-1})v_0 \qquad \text{since } v_0\in V^\psi \\
&= \psi(shs^{-1})\sigma_0(s)v_0 \\
&= \psi^{-1}(h)\sigma_0(s)v_0.
\end{aligned}$$

So, the vector $\sigma_0(s)v_0$ spans $V^{\psi^{-1}}$. Hence, we get

$$W_2 \simeq W_1^\vee \simeq V_{\psi^{-1}} \simeq V^{\psi^{-1}} = \mathbb{C}\sigma_0(s)v_0$$

Since ψ^{-1} has multiplicity 1 in $\sigma_0|_H$, we have $W_2 = \mathbb{C}\sigma_0(s)v_0$. Therefore, $w = \lambda\sigma_0(s)v_0$ for some $\lambda\in\mathbb{C}^\times$. Finally,

$$0 \neq B(v_0,w) = B\big(v_0,\lambda\,\sigma_0(s)v_0\big) = \lambda\,B(v_0,\sigma_0(s)v_0),$$

so $B(v_0, \sigma_0(s)v_0) \neq 0$, as claimed. $\square$

By Rodier's Theorem 2.8.1 and Lemma 4.1.2, we have the following corollary.

Corollary 4.1.3. *Assume that F is a non-archimedean local field of characteristic 0 whose residual characteristic p is sufficiently large in the sense of Rodier (in particular $p \geq 2h+1$, where h is the height of the highest root). Let G be the group of $F-$points of a connected reductive group defined and split over F. Let $B = TU$ be a Levi decomposition of a Borel subgroup B of G. Assume that T has an element s which acts by -1 on each simple root subgroup of U. Then*

(1) $s^2 \in Z(G)$

(2) If σ_0 is an irreducible generic representation of G with $\sigma_0 \simeq \sigma_0^\vee \chi$ for some character χ of G, then $\epsilon_\chi(\sigma_0) = \omega_{\sigma_0}(s^2)\chi^{-1}(s)$.

Proof. (1) Let $\Phi = \Phi(T, G)$ be the root system. By assumption, for every simple root α (with respect to (G, T, B)) we have $\alpha(s) = -1$. For any root $\beta \in \Phi$ we can write

$$\beta = \sum_i n_i \alpha_i, \qquad n_i \in \mathbb{Z},$$

hence

$$\beta(s) = \prod_i \alpha_i(s)^{n_i} = \prod_i (-1)^{n_i} \in \{\pm 1\},$$

and therefore $\beta(s^2) = \beta(s)^2 = 1$ for all $\beta \in \Phi$. Since $s \in T$, this shows that s^2 lies in

$$Z(G) \cap T \;=\; \bigcap_{\beta \in \Phi} \ker(\beta),$$

and hence $s^2 \in Z(G)$.

(2) Let σ_0 be an irreducible generic representation of G with $\sigma_0 \simeq \sigma_0^\vee \chi$ for a character χ of G. By genericity, there exists a nondegenerate character $\psi : U \to \mathbb{C}^\times$ such that $\mathrm{Hom}_U(\sigma_0, \psi) \neq 0$.

By the multiplicity–one theorem 2.7.1, we have $\dim_{\mathbb{C}} \mathrm{Hom}_U(\sigma_0, \psi) \leq 1$, hence

$$\dim_{\mathbb{C}} \mathrm{Hom}_U(\sigma_0, \psi) = 1.$$

By Rodier's Theorem 2.8.1, there exists a compact open subgroup $K \subset G$ and a character $\psi_K : K \to \mathbb{C}^\times$ such that

$$\dim_{\mathbb{C}} \mathrm{Hom}_K(\sigma_0, \psi_K) = \dim_{\mathbb{C}} \mathrm{Hom}_U(\sigma_0, \psi) = 1.$$

In particular, ψ_K occurs with multiplicity one in $\sigma_0|_K$. Moreover, the construction of Rodier allows us to choose K so that:

- $K \subset \ker(\chi)$
- K is normalized by s
- $\psi_K^{-1}(k) = \psi_K(sks^{-1})$ for all $k \in K$.

Now set $H := K$ and $\psi := \psi_K$. By part (1), $s^2 \in Z(G)$. Since the tuple $(G, H, s, \psi, \sigma_0, \chi)$ satisfies all hypotheses of Lemma 4.1.2, we conclude

$$\epsilon_\chi(\sigma_0) = \omega_{\sigma_0}(s^2)\chi^{-1}(s).$$

$\square$

Remark 4.1.4. *For $G = \mathrm{GL}(n, F)$, Rodier observes that one can avoid the assumption $\mathrm{char}(F) = 0$ and the bound $p \geq 2h + 1$ (where h is the height of the highest root); see [Rod75, §VI].*

We now want to illustrate how to choose such a compact open subgroup K in the proof of Corollary 4.1.3 for the case of $G = GL(n, F)$.

Corollary 4.1.5. *Let F be a non-archimedean local field. Let $G = \mathrm{GL}(n, F)$. Let $B = TU$ be a Levi decomposition of a Borel subgroup B of G. Assume that T has an element s which acts by -1 on each simple root subgroup of U. Then*

(1) $s^2 \in Z(G)$

(2) If σ_0 *is an irreducible generic representation of* G *with* $\sigma_0 \simeq \sigma_0^\vee \chi$ *for some character* χ *of* G*, then* $\epsilon_\chi(\sigma_0) = \omega_{\sigma_0}(s^2)\chi^{-1}(s)$.

Proof. By Remark 2.6.1, we have

$$s = a \ \mathrm{diag}(1, -1, 1, -1, \dots, (-1)^{n-1}) \qquad \text{for some } a \in F^\times.$$

Therefore

$$s^2 = a^2 I_n \in Z(G).$$

This proves (1). To prove (2), let (K'_m, ψ'_m) be the pairs defined in Section 2.8. We show that for all sufficiently large m, the subgroup K'_m and the character ψ'_m satisfy the hypotheses of Lemma 4.1.2.

Step 1: $\dim \mathrm{Hom}_{K'_m}(\sigma_0, \psi'_m) = 1$ *for all sufficiently large* m. Since σ_0 is generic, Corollary 2.8.6 implies that

$$\dim \mathrm{Hom}_{K'_m}(\sigma_0, \psi'_m) = 1$$

for all sufficiently large m.

Step 2: $K'_m \subset \ker(\chi)$ *for all sufficiently large* m. Since χ is a smooth character of G, its kernel is an open subgroup of G. The principal congruence subgroups

$$K_m = 1 + \varpi^m M_n(\mathcal{O}) \qquad \text{for } m \geq 1$$

form a neighborhood basis of the identity, so for all sufficiently large m we have

$$K_m \subset \ker(\chi).$$

Now let $k \in K'_m$. By definition,

$$K'_m = d^m K_m d^{-m},$$

so we may write

$$k = d^m k_0 d^{-m} \qquad \text{for some } k_0 \in K_m.$$

Since χ is a character,

$$\chi(k) = \chi(d^m)\chi(k_0)\chi(d^{-m}).$$

Because $k_0 \in K_m \subset \ker(\chi)$, we have $\chi(k_0) = 1$, and

$$\chi(d^m)\chi(d^{-m}) = \chi(d^m d^{-m}) = \chi(I_n) = 1.$$

Hence

$$\chi(k) = 1.$$

Therefore $k \in \ker(\chi)$, and thus

$$K'_m \subset \ker(\chi)$$

for all sufficiently large m.

Step 3: s normalizes K'_m. Since

$$s = a \ \mathrm{diag}(1, -1, 1, -1, \ldots, (-1)^{n-1}),$$

all diagonal ratios s_i/s_j lie in $\{\pm 1\}$. Hence if $X = (x_{ij}) \in \varpi^m M_n(\mathcal{O})$, then

$$(sXs^{-1})_{ij} = \frac{s_i}{s_j} x_{ij} \in \varpi^m \mathcal{O},$$

so $sXs^{-1} \in \varpi^m M_n(\mathcal{O})$. Therefore

$$sK_m s^{-1} = K_m.$$

Also, s and d are both diagonal, hence commute. Thus

$$sK'_m s^{-1} = sd^m K_m d^{-m} s^{-1} = d^m (sK_m s^{-1}) d^{-m} = d^m K_m d^{-m} = K'_m.$$

Hence $s \in N_G(K'_m)$.

Step 4: $\psi'_m(sks^{-1}) = \psi'_m(k)^{-1}$ *for all* $k \in K'_m$. Let $k \in K'_m$. Write

$$k = d^m(1+X)d^{-m} \qquad \text{with } X \in \varpi^m M_n(\mathcal{O}).$$

Since s commutes with d, we have

$$d^{-m}(sks^{-1})d^m = s(1+X)s^{-1} = 1 + sXs^{-1}.$$

Hence

$$\begin{aligned} \psi'_m(sks^{-1}) &= \psi_m(1+sXs^{-1}) \\ &= \psi_F\left(\varpi^{-2m}\sum_{i=1}^{n-1}(sXs^{-1})_{i,i+1}\right). \end{aligned}$$

Now

$$(sXs^{-1})_{i,i+1} = \frac{s_i}{s_{i+1}}X_{i,i+1} = \alpha_i(s)X_{i,i+1} = -\,X_{i,i+1},$$

because $\alpha_i(s) = -1$ for every simple root α_i. Therefore

$$\begin{aligned} \psi'_m(sks^{-1}) &= \psi_F\left(\varpi^{-2m}\sum_{i=1}^{n-1}(-X_{i,i+1})\right) \\ &= \psi_F\left(\varpi^{-2m}\sum_{i=1}^{n-1}X_{i,i+1}\right)^{-1} \\ &= \psi_m(1+X)^{-1} \\ &= \psi'_m(k)^{-1}. \end{aligned}$$

Thus

$$\psi'_m(sks^{-1}) = \psi'_m(k)^{-1} \qquad \text{for all } k \in K'_m.$$

By Steps 1–4, for all sufficiently large m, the compact open subgroup K'_m and the character ψ'_m satisfy the hypotheses of Lemma 4.1.2. Therefore,

$$\epsilon_\chi(\sigma_0) = \omega_{\sigma_0}(s^2)\chi^{-1}(s).$$

This proves (2), and completes the proof. $\square$

Chapter 5

Main Theorem

Throughout this section, we set $G = \mathrm{SL}(n, F)$. Let π be an irreducible self-dual (smooth) representation of G. By Langlands theory, there exists a unique triple (P, σ, ν) such that π is the unique irreducible quotient of $i_P^G(\sigma\nu)$. This consists of a standard parabolic subgroup P of G, an irreducible tempered representation σ of the standard Levi component M of P (σ is unique up to equivalence). Note that

$$M = \begin{bmatrix} \mathrm{GL}_{n_1}(F) & & \\ & \ddots & \\ & & \mathrm{GL}_{n_r}(F) \end{bmatrix}_{n\times n} \cap G.$$

Note that the self duality of π implies that $(n_1, n_2, \ldots, n_r) = (n_r, n_{r-1}, \ldots, n_1)$ by Corollary 2.9.5. So we may write

$$M = (\mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_l}(F) \times \mathrm{GL}_{n_0}(F) \times \mathrm{GL}_{n_l}(F) \times \cdots \times \mathrm{GL}_{n_1}(F)) \cap G$$

and say $\mathrm{GL}_{n_0}(F)$ is the trivial group if $n_0 = 0$. Let θ be the inner automorphism of G given by the matrix $w_\theta \in G$, where w_θ is equal to one of the three matrices:

- If $n_0 = 0$ then $w_\theta = \begin{bmatrix} & & & & & & & -I_{n_1} \\ & & & & & & -I_{n_2} & \\ & & & & & \iddots & & \\ & & & & -I_{n_\ell} & & & \\ & & & I_{n_\ell} & & & & \\ & & \iddots & & & & & \\ & I_{n_2} & & & & & & \\ I_{n_1} & & & & & & & \end{bmatrix}_{n\times n}$

- If $n_1+n_2+\cdots+n_l$ is even and $n_0 \neq 0$ then $w_\theta = \begin{bmatrix} & & & & & & I_{n_1} \\ & & & & & I_{n_2} & \\ & & & & \iddots & & \\ & & & I_{n_0} & & & \\ & & \iddots & & & & \\ & I_{n_2} & & & & & \\ I_{n_1} & & & & & & \end{bmatrix}_{n\times n}$

- If $n_1 + n_2 + \cdots + n_l$ is odd and $n_0 \neq 0$ then

$$w_\theta = \begin{bmatrix} & & & & & & I_{n_1} \\ & & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & I_{n_\ell} & & \\ & & & \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{n_0 \times n_0} & & & \\ & & I_{n_\ell} & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & & \\ I_{n_1} & & & & & & \end{bmatrix}_{n \times n}$$

Here I_{n_i}'s denote the n_i by n_i identity matrices. By Theorem 3.1.1 and Proposition 3.0.2, we have $\epsilon(\pi) = \begin{cases} \epsilon_\theta(\pi)\omega_\pi(-1) & \text{if } n_0 = 0, \\ \epsilon_\theta(\pi) & \text{if } n_0 \neq 0 \end{cases}$ and $\epsilon_\theta(\pi) = \epsilon_\theta(\sigma)$. Therefore, we want to understand the twisted sign $\epsilon_\theta(\sigma)$. Since the subgroup M is difficult to work with directly, we transfer the problem to

$$\widetilde{M} = \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_\ell}(F) \times \mathrm{GL}_{n_0}(F) \times \mathrm{GL}_{n_\ell}(F) \times \cdots \times \mathrm{GL}_{n_1}(F).$$

Proposition 5.0.1. *With notation as above, there exists an irreducible tempered representation $\widetilde{\sigma}$ of $\widetilde{M}$ such that σ occurs in $\widetilde{\sigma}|_M$.*

Proof. Set $G = \widetilde{M}$ and $G_1 = M$. Since σ is tempered, its central character is unitary. By [Tad92, Proposition 2.2], there exists an irreducible representation $\widetilde{\sigma}$ of G such that σ is isomorphic to a subrepresentation of $\widetilde{\sigma}|_{G_1}$, and moreover $\widetilde{\sigma}$ may be chosen to have unitary central character.

Now apply [Tad92, Proposition 2.7(iv)]: for an irreducible G-representation with unitary central character, $\widetilde{\sigma}$ is tempered if and only if $\widetilde{\sigma}|_{G_1}$ has a tempered irreducible subrepresentation. Hence, $\widetilde{\sigma}$ is tempered.

$\square$

Note that for each element $m = \mathrm{diag}(g_1, \dots g_\ell, g_0, g'_\ell, \dots, g'_1) \in \widetilde{M}$ we have

$$w_\theta m w_\theta^{-1} = \mathrm{diag}(g'_1, \dots g'_\ell, \Sigma_\theta g_0 \Sigma_\theta^{-1}, g_\ell, \dots, g_1) \in \widetilde{M},$$

where Σ_θ is the middle block matrix of w_θ. Hence $\theta(\widetilde{M}) = \widetilde{M}$ and $\theta(M) = M$. Hence the twists σ^θ and $\widetilde{\sigma}^\theta$ are well-defined.

Proposition 5.0.2. *With notation as above, there exists a smooth character $\chi : \widetilde{M} \to \mathbb{C}^\times$ which is trivial on M such that*

$$\widetilde{\sigma}^\theta \;\simeq\; \widetilde{\sigma}^\vee \chi.$$

Proof. By Theorem 3.1.1, we have $\sigma^\theta \simeq \sigma^\vee$. Set $\pi_1 = \widetilde{\sigma}^\vee$ and $\pi_2 = \widetilde{\sigma}^\theta$. Since $\widetilde{M}_{\mathrm{der}} \subseteq M$, Tadić shows that for any irreducible smooth representation π of $\widetilde{M}$, the restriction $\pi|_M$ is a finite direct sum of irreducible M-representations ([Tad92, Lemma 2.1]). In particular, $\pi_1|_M$ and $\pi_2|_M$ are semisimple.

Since σ occurs in $\widetilde{\sigma}|_M$, it follows that $\sigma^\vee$ occurs in $(\widetilde{\sigma}|_M)^\vee \simeq \pi_1|_M$, and σ^θ occurs in $(\widetilde{\sigma}|_M)^\theta \simeq \widetilde{\sigma}^\theta|_M = \pi_2|_M$. Since $\sigma^\theta \simeq \sigma^\vee$, the semisimple representations $\pi_1|_M$ and $\pi_2|_M$ share an isomorphic irreducible summand, hence

$$\mathrm{Hom}_M(\pi_1, \pi_2) \neq 0.$$

Let $A = \widetilde{M}/M$ and let $\widehat{A} = \mathrm{Hom}(A, \mathbb{C}^\times)$. Via inflation along the quotient map $\widetilde{M} \twoheadrightarrow A$, we identify $\widehat{A}$ with the subgroup of $\mathrm{Hom}(\widetilde{M}, \mathbb{C}^\times)$ consisting of smooth characters trivial on M. Now apply [Tad92, Proposition 2.4], which states that

$$\#\{\chi \in \widehat{A} : \; \pi_2 \simeq \pi_1 \chi\} \;=\; \dim \mathrm{Hom}_M(\pi_1, \pi_2).$$

Since the right-hand side is nonzero, there exists a character $\chi \in \operatorname{Hom}(\widetilde{M}, \mathbb{C}^\times)$ trivial on M such that $\pi_2 \simeq \pi_1\chi$. $\square$

Proposition 5.0.3. *With notation as above, we have $\epsilon_\theta(\sigma) = \epsilon_{\theta,\chi}(\widetilde{\sigma})$ since $\widetilde{\sigma}|_M$ is multiplicity free.*

Proof. Let $B \in \mathrm{Bil}_{\widetilde{\sigma}\chi^{-1},\widetilde{\sigma}^\theta}(W) \setminus \{0\}$, that is, nondegenerate. Note that $B|_{V\times V} \in \mathrm{Bil}_{\sigma,\sigma^\theta}(V)$. So, we want to show that $B|_{V\times V} \neq 0$, that is, nondegenerate. Let $u \in V$. Suppose $B|_{V\times V}(u,v) = 0$ for all $v \in V$. By the nondegeneracy of B, it is enough to prove that $B(u,w) = 0$ for all $w \in W$.

Let $W = V \oplus V_2 \oplus V_3 \oplus \cdots \oplus V_e$ for some irreducible subrepresentations $V_2, V_3, \ldots, V_e$ of $(\widetilde{\sigma}|_M, W)$. Let $w \in W$. Then $w = v + v_2 + v_3 + \cdots + v_e$ for some $v \in V, v_2 \in V_2, v_3 \in V_3, \ldots, v_e \in V_e$. So, we have

$$B(u,w) = B(u,v) + \sum_{i=2}^{e} B(u,v_i) = \sum_{i=2}^{e} B(u,v_i)$$

But $B(u,v_i) = 0$ for all $i \in \{2,3,\ldots,e\}$. Otherwise, there exists an element $i \in \{2,3,\ldots,e\}$ such that

$$0 \neq B|_{V\times V_i} \in \mathrm{Bil}_{\widetilde{\sigma}|_M,\widetilde{\sigma}|_M^\theta}(V,V_i) \simeq \operatorname{Hom}_M(V_i, V^\vee)$$

So, $V_i \simeq V^\vee \simeq V$ as representations. This is a contradiction by fact that $(\widetilde{\sigma}|_M, W)$ is multiplicity free. So, we must havea $B(u,w) = 0$ for all $w \in W$. So, $u = 0$. So, $B|_{V\times V}$ is nondegenerate. $\square$

Assume that $n_0 > 0$. Then we have

$$\widetilde{\sigma} = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_\ell \otimes \sigma_0 \otimes \sigma'_\ell \otimes \cdots \otimes \sigma'_2 \otimes \sigma'_1$$

for some irreducible representations σ_1, σ'_1 of $\mathrm{GL}_{n_1}(F), \sigma_2, \sigma'_2$ of $\mathrm{GL}_{n_2}(F), \ldots, \sigma_\ell, \sigma'_\ell$ of $\mathrm{GL}_{n_\ell}(F)$, and σ_0 of $\mathrm{GL}_{n_0}(F)$. Since $\widetilde{\sigma}^\theta \simeq \widetilde{\sigma}^\vee\chi$ for some linear character

$$\chi : \widetilde{M} \to \mathbb{C}^\times$$

which is trivial on M by Proposition $5.0.2$, we have

$$(\sigma_1\otimes\sigma_2\otimes\cdots\otimes\sigma_\ell\otimes\sigma_0\otimes\sigma'_\ell\otimes\cdots\otimes\sigma'_2\otimes\sigma'_1)^\theta \simeq (\sigma_1\otimes\sigma_2\otimes\cdots\otimes\sigma_\ell\otimes\sigma_0\otimes\sigma'_\ell\otimes\cdots\otimes\sigma'_2\otimes\sigma'_1)^\vee\chi$$

which implies that

$$\sigma'_1 \otimes \sigma'_2 \otimes \cdots \otimes \sigma'_\ell \otimes \sigma_0^{\theta_0} \otimes \sigma_\ell \otimes \cdots \otimes \sigma_2 \otimes \sigma_1$$

is isomorphic to

$$\sigma_1^\vee\chi_1 \otimes \sigma_2^\vee\chi_2 \otimes \cdots \otimes \sigma_\ell^\vee\chi_\ell \otimes \sigma_0^\vee\chi_0 \otimes \sigma_\ell'^\vee\chi_\ell \otimes \cdots \otimes \sigma_2'^\vee\chi_2 \otimes \sigma_1'^\vee\chi_1,$$

where θ_0 is the inner automorphism of $\mathrm{GL}_{n_0}(F)$ given by the middle block matrix Σ_θ of w_θ, and $\chi_i = \chi|_{GL_{n_i}(F)}$ for all $i = 1, 2, \ldots, \ell$.

So,

$$\sigma'_1 \simeq \sigma_1^\vee\chi_1 \text{ , } \sigma'_2 \simeq \sigma_2^\vee\chi_2 \text{ , } \ldots, \sigma'_\ell \simeq \sigma_\ell^\vee\chi_\ell \text{ , and } \sigma_0^{\theta_0} \simeq \sigma_0^\vee\chi_0$$

So,

$$\widetilde{\sigma} \simeq \sigma_1 \otimes \sigma_2 \otimes \ldots \sigma_\ell \otimes \sigma_0 \otimes \sigma_\ell^\vee\chi \otimes \cdots \otimes \sigma_2^\vee\chi \otimes \sigma_1^\vee\chi$$

Proposition 5.0.4. *With notation as aboove, we have* $\epsilon_{\theta,\chi}(\widetilde{\sigma}) = \epsilon_{\theta_0,\chi_0}(\sigma_0)$.

Proof. When $n_0 = 0$, one may regard $\mathrm{GL}_{n_0}(F)$ as the trivial group. In this case there is no middle-block contribution, and hence

$$\epsilon_{\theta,\chi}(\widetilde{\sigma}) = 1.$$

Assume $n_0 > 0$. Let V_i be the space of σ_i for each i. Then the representation space of $\widetilde{\sigma}$ is

$$V = V_1 \otimes \cdots \otimes V_\ell \otimes V_0 \otimes V_\ell^\vee \otimes \cdots \otimes V_1^\vee.$$

Let b_0 be a nonzero $(\sigma_0\chi_0^{-1}, \sigma_0^{\theta_0})$-invariant bilinear form on V_0. We now define a bilinear form B on V. For pure tensors

$$u = u_1 \otimes \cdots \otimes u_\ell \otimes u_0 \otimes u_\ell^\vee \otimes \cdots \otimes u_1^\vee$$

and

$$v = v_1 \otimes \cdots \otimes v_\ell \otimes v_0 \otimes v_\ell^\vee \otimes \cdots \otimes v_1^\vee,$$

set

$$B(u,v) = \left(\prod_{i=1}^{\ell} \langle u_i, v_i^\vee \rangle \langle v_i, u_i^\vee \rangle \right) b_0(u_0, v_0),$$

where

$$\langle u_i, v_i^\vee \rangle := v_i^\vee(u_i).$$

We first show that B is nonzero. Since $b_0 \neq 0$, there exist $u_0, v_0 \in V_0$ such that

$$b_0(u_0, v_0) \neq 0.$$

Note that we may choose $u_i, v_i \in V_i$ and $u_i^\vee, v_i^\vee \in V_i^\vee$ such that

$$\langle u_i, v_i^\vee \rangle \neq 0 \qquad \text{and} \qquad \langle v_i, u_i^\vee \rangle \neq 0.$$

For these choices of u and v, we obtain

$$B(u,v) = \left(\prod_{i=1}^{\ell} \langle u_i, v_i^\vee \rangle \langle v_i, u_i^\vee \rangle \right) b_0(u_0, v_0) \neq 0.$$

Thus B is nonzero. Next, we show that B is $(\widetilde{\sigma}\chi^{-1}, \widetilde{\sigma}^\theta)$ -invariant. Let

$$m = (a_1, \ldots, a_\ell, m_0, b_\ell, \ldots, b_1) \in \widetilde{M}.$$

For each pure tensors $u, v \in V$ we have

$$B\big(\widetilde{\sigma}(m)\chi^{-1}(m)u, \widetilde{\sigma}^\theta(m)v\big) = \chi^{-1}(m) B\big(\widetilde{\sigma}(m)u, \widetilde{\sigma}^\theta(m)v\big). \tag{5.1}$$

Note that

$$\widetilde{\sigma}(m)u = \sigma_1(a_1)u_1 \otimes \cdots \otimes \sigma_\ell(a_\ell)u_\ell \otimes \sigma_0(m_0)u_0 \otimes (\sigma_\ell^\vee \chi_\ell)(b_\ell)u_\ell^\vee \otimes \cdots \otimes (\sigma_1^\vee \chi_1)(b_1)u_1^\vee$$

and

$$\begin{aligned} &\widetilde{\sigma}^\theta(m)v = \widetilde{\sigma}(\theta(m))v \\ &= \sigma_1(b_1)v_1 \otimes \cdots \otimes \sigma_\ell(b_\ell)v_\ell \otimes \sigma_0(\theta_0(m_0))v_0 \otimes (\sigma_\ell^\vee \chi_\ell)(a_\ell)v_\ell^\vee \otimes \cdots \otimes (\sigma_1^\vee \chi_1)(a_1)v_1^\vee. \end{aligned}$$

Substituting these expressions into (5.1), we obtain

$$B\big(\widetilde{\sigma}(m)\chi^{-1}(m)u, \widetilde{\sigma}^{\theta}(m)v\big)$$
$$= \chi^{-1}(m)\left(\prod_{i=1}^{\ell}\langle\sigma_i(a_i)u_i, (\sigma_i^{\vee}\chi_i)(a_i)v_i^{\vee}\rangle\,\langle\sigma_i(b_i)v_i, (\sigma_i^{\vee}\chi_i)(b_i)u_i^{\vee}\rangle\right)$$
$$.b_0\big(\sigma_0(m_0)u_0, \sigma_0(\theta_0(m_0))v_0\big). \tag{5.2}$$

By the definition of the contragradient action,

$$\begin{aligned}\langle\sigma_i(a_i)u_i, (\sigma_i^{\vee}\chi_i)(a_i)v_i^{\vee}\rangle &= \chi_i(a_i)\,\langle\sigma_i(a_i)u_i, \sigma_i^{\vee}(a_i)v_i^{\vee}\rangle\\ &= \chi_i(a_i)v_i^{\vee}\big(\sigma_i(a_i^{-1})\sigma_i(a_i)u_i\big)\\ &= \chi_i(a_i)v_i^{\vee}(u_i)\\ &= \chi_i(a_i)\langle u_i, v_i^{\vee}\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle\sigma_i(b_i)v_i, (\sigma_i^{\vee}\chi_i)(b_i)u_i^{\vee}\rangle &= \chi_i(b_i)\,\langle\sigma_i(b_i)v_i, \sigma_i^{\vee}(b_i)u_i^{\vee}\rangle\\ &= \chi_i(b_i)u_i^{\vee}\big(\sigma_i(b_i^{-1})\sigma_i(b_i)v_i\big)\\ &= \chi_i(b_i)u_i^{\vee}(v_i)\\ &= \chi_i(b_i)\langle v_i, u_i^{\vee}\rangle.\end{aligned}$$

Substituting these expressions into (5.2) and using the invariance property of b_0, we obtain

$$B\big(\widetilde{\sigma}(m)\chi^{-1}(m)u, \widetilde{\sigma}^{\theta}(m)v\big)$$
$$= \chi^{-1}(m)\left(\prod_{i=1}^{\ell}\chi_i(a_i)\chi_i(b_i)\right)\chi_0(m_0)\left(\prod_{i=1}^{\ell}\langle u_i, v_i^{\vee}\rangle\langle v_i, u_i^{\vee}\rangle\right)b_0(u_0, v_0).$$

But

$$\chi(m) = \left(\prod_{i=1}^{\ell}\chi_i(a_i)\chi_i(b_i)\right)\chi_0(m_0).$$

It follows that

$$B\big(\widetilde{\sigma}(m)\chi^{-1}(m)u, \widetilde{\sigma}^{\theta}(m)v\big) = B(u, v).$$

Thus B is $(\widetilde{\sigma}\chi^{-1}, \widetilde{\sigma}^{\theta})$-invariant. Since

$$\begin{aligned} B(v,u) &= \left(\prod_{i=1}^{\ell} \langle u_i, v_i^{\vee}\rangle \langle v_i, u_i^{\vee}\rangle\right) b_0(v_0,u_0) \\ &= \epsilon_{\theta_0,\chi_0}(\sigma_0) \left(\prod_{i=1}^{\ell} \langle u_i, v_i^{\vee}\rangle \langle v_i, u_i^{\vee}\rangle\right) b_0(u_0,v_0) \\ &= \epsilon_{\theta_0,\chi_0}(\sigma_0) B(u,v) \end{aligned}$$

we get

$$\epsilon_{\theta,\chi}(\widetilde{\sigma}) = \epsilon_{\theta_0,\chi_0}(\sigma_0).$$

□

Proposition 5.0.5. *With notation as above, we have $\epsilon_{\theta_0,\chi_0}(\sigma_0) = \epsilon_{\chi_0}(\sigma_0)\chi_0(\Sigma_\theta)$.*

Proof. This is Proposition 3.0.3. □

Lemma 5.0.6. *With notation as above, σ_0 is generic.*

Proof. Recall that $\widetilde{\sigma}$ is tempered by Proposition 5.0.1, and that

$$\widetilde{\sigma} \simeq \sigma_1 \otimes \sigma_2 \otimes \ldots \sigma_\ell \otimes \sigma_0 \otimes \sigma_\ell^{\vee}\chi \otimes \cdots \otimes \sigma_2^{\vee}\chi \otimes \sigma_1^{\vee}\chi,$$

By Lemma 2.3.2, temperedness of $\widetilde{\sigma}$ implies that each factor in the above decomposition is tempered. In particular, σ_0 is tempered. Finally, by [PR, Theorems 9.3 and 9.6], every tempered representation of $\mathrm{GL}_{n_0}(F)$ is generic. Hence σ_0 is generic.

□

Remark 5.0.7 (Restriction of χ to the middle block)**.** *Recall that*

$$M = \widetilde{M} \cap \mathrm{SL}_n(F) = \ker\bigl(\widetilde{\det} : \widetilde{M} \to F^{\times}\bigr),$$

where for a block diagonal element $m = diag(g_1, \ldots, g_\ell, g_0, g_\ell', \ldots, g_1') \in \widetilde{M}$ we set $\widetilde{\det}(m) = \Bigl(\prod_{i=1}^{\ell} \det(g_i)\Bigr) \det(g_0) \Bigl(\prod_{i=1}^{\ell} \det(g_i')\Bigr)$. Since χ is trivial on M, it factors *through the quotient* $\widetilde{M}/M \simeq F^\times$*. Hence there exists a (unique) smooth character*

$$\xi : F^\times \to \mathbb{C}^\times$$

such that

$$\chi = \xi \circ \widetilde{\det}.$$

Let $\iota_0 : \mathrm{GL}_{n_0}(F) \hookrightarrow \widetilde{M}$ *be the natural embedding into the middle block,*

$$\iota_0(g_0) = diag(I, \ldots, I, g_0, I, \ldots, I).$$

Then $\widetilde{\det}(\iota_0(g_0)) = \det(g_0)$*, and therefore the restriction* $\chi_0 := \chi|_{\mathrm{GL}_{n_0}(F)}$ *satisfies*

$$\chi_0(g_0) = \chi(\iota_0(g_0)) = (\xi \circ \widetilde{\det})(\iota_0(g_0)) = \xi(\det(g_0)).$$

In particular, we may regard ξ *as a character of* $F^\times$ *and, by abuse of notation, denote it again by* χ*. Then*

$$\chi_0 = \chi \circ \det \qquad \textit{on } \mathrm{GL}_{n_0}(F).$$

Also, we will freely use the notation $\chi(-1)$*. It means* $\xi(-1)$ *(a value in* $\{\pm 1\}$*). Equivalently, one may choose any* $m \in \widetilde{M}$ *with* $\widetilde{\det}(m) = -1$ *and then* $\chi(-1) = \chi(m)$*, which is independent of the choice of* m *because* $\ker(\widetilde{\det}) = M$ *and* χ *is trivial on* M*. Additionally, for* $s \in \mathrm{GL}_{n_0}(F)$ *one has*

$$\chi_0^{-1}(s) = \chi_0(s)^{-1} = \xi(\det(s))^{-1}.$$

In particular, if $\det(s) \in \{\pm 1\}$*, then* $(\det(s))^2 = 1$*, so*

$$\chi_0(s)^2 = \xi\big((\det(s))^2\big) = \xi(1) = 1,$$

and therefore $\chi_0(s) \in \{\pm 1\}$ *and* $\chi_0^{-1}(s) = \chi_0(s)$*.*

Corollary 5.0.8. *Let π be an irreducible self-dual representation of $\mathrm{SL}(n, F)$, and let $(n_1, \ldots, n_\ell, n_0, n_\ell, \ldots, n_1)$ be the block sizes in the Levi attached to the Langlands data for π. Then*

$$\epsilon(\pi) = \begin{cases} \omega_\pi(-1), & \text{if } n_0 = 0, \\ \\ \omega_{\sigma_0}(s^2)\, \chi^{-1}(\det s)\, \chi(-1)^{n_1+\cdots+n_\ell}, & \text{if } n_0 > 0, \end{cases}$$

for any $s \in T_0 \subset \mathrm{GL}_{n_0}(F)$ that acts by -1 on all simple roots in U_0 with respect to the standard Borel $B_0 = T_0U_0$ of $\mathrm{GL}_{n_0}(F)$ (equivalently, $s \in \{a \cdot \mathrm{diag}(1, -1, 1, -1, \ldots, (-1)^{n_0-1}) : a \in F^\times\}$). Clearly, if $\det(s) \in \{\pm 1\}$, then $\chi^{-1}(\det s) = \chi(\det s)$.

Proof. Let $\theta = \mathrm{Int}(w_\theta)$ be the inner automorphism used in Theorem 3.1.1. Note that

- if $n_0 = 0$, then $w_\theta^2 = -I_n$, hence $\omega_\pi(w_\theta^2) = \omega_\pi(-1)$;
- if $n_0 \neq 0$, then $w_\theta^2 = I_n$, hence $\omega_\pi(w_\theta^2) = 1$.

By Proposition 3.0.2, we have

$$\epsilon(\pi) = \begin{cases} \epsilon_\theta(\pi)\, \omega_\pi(-1), & \text{if } n_0 = 0, \\ \\ \epsilon_\theta(\pi), & \text{if } n_0 > 0. \end{cases} \tag{5.3}$$

By Theorem 3.1.1 we have

$$\epsilon_\theta(\pi) = \epsilon_\theta(\sigma), \tag{5.4}$$

where σ is the tempered representation of the Levi M appearing in the Langlands data for π. By Proposition 5.0.1 there exists an irreducible tempered representation $\widetilde{\sigma}$ on

$$\widetilde{M} = \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_\ell}(F) \times \mathrm{GL}_{n_0}(F) \times \mathrm{GL}_{n_\ell}(F) \times \cdots \times \mathrm{GL}_{n_1}(F)$$

so that σ occurs in $\widetilde{\sigma}|_M$. By Proposition 5.0.3,

$$\epsilon_\theta(\sigma) = \epsilon_{\theta,\chi}(\widetilde{\sigma}) \tag{5.5}$$

By 5.4, and 5.5,

$$\epsilon_\theta(\pi) = \epsilon_{\theta,\chi}(\widetilde{\sigma}) \tag{5.6}$$

By Proposition 5.0.4 we get

$$\epsilon_\theta(\pi) = 1 \qquad \text{if } n_0 = 0,$$

which is the first line of the corollary together with (5.3). If $n_0 > 0$, then the chain of equalities proved and Proposition 5.0.4 gives

$$\epsilon(\pi) = \epsilon_\theta(\pi) = \epsilon_{\theta,\chi}(\widetilde{\sigma}) = \epsilon_{\theta_0,\chi_0}(\sigma_0), \tag{5.7}$$

where σ_0 is the middle-block representation of $\mathrm{GL}_{n_0}(F)$ and $\chi_0 = \chi|_{\mathrm{GL}_{n_0}(F)}$. Proposition 5.0.5 yields

$$\epsilon_{\theta_0,\chi_0}(\sigma_0) = \epsilon_{\chi_0}(\sigma_0)\,\chi_0(\Sigma_\theta), \tag{5.8}$$

where Σ_θ is the middle block matrix of w_θ. By Corollary 4.1.3(2), for any $s \in T_0$ acting by -1 on all simple roots in U_0 we have

$$\epsilon_{\chi_0}(\sigma_0) = \omega_{\sigma_0}(s^2)\,\chi_0^{-1}(s). \tag{5.9}$$

Combining (5.7), (5.8), and (5.9),

$$\epsilon(\pi) = \omega_{\sigma_0}(s^2)\,\chi_0^{-1}(s)\,\chi_0(\Sigma_\theta) \qquad \text{if } n_0 > 0.$$

So if $n_0 > 0$ then

$$\begin{aligned} \epsilon(\pi) &= \omega_{\sigma_0}(s^2)\,\chi^{-1}(\det s)\,\chi(\det \Sigma_\theta) \qquad \text{by Remark 5.0.7} \\ &= \omega_{\sigma_0}(s^2)\,\chi^{-1}(\det s)\,\chi(-1)^{n_1+\cdots+n_\ell}, \end{aligned}$$

Finally, if $\det(s) \in \{\pm 1\}$ then $(\det s)^2 = 1$, so $\chi(\det s)^2 = \chi(1) = 1$, hence $\chi^{-1}(\det s) = \chi(\det s)$. $\square$

Remark 5.0.9. *For practical reasons, we choose* $s = \operatorname{diag}(1, -1, \ldots, (-1)^{n_0-1})$. *Then* $s^2 = 1$, $\det s = (-1)^{\lfloor n_0/2 \rfloor}$ *and we derive:*

$$\epsilon(\pi) = \begin{cases} \omega_\pi(-1), & \text{if } n_0 = 0, \\ \chi(-1)^{\lfloor n_0/2 \rfloor}\, \chi(-1)^{\,n_1+\cdots+n_\ell}, & \text{if } n_0 > 0, \end{cases}$$

We now have the following cases.

Case 1. Suppose that n is odd. Then n_0 is odd. Pick $s = \operatorname{diag}(1, -1, 1, -1, \ldots, 1)$. Since $n_0 > 0$, Remark 5.0.9 yields

$$\epsilon(\pi) = \chi(-1)^{\lfloor n_0/2 \rfloor}\, \chi(-1)^{\,n_1+\cdots+n_\ell} = \chi(-1)^{\lfloor n_0/2 \rfloor + n_1 + \cdots + n_\ell}.$$

Note that

$$\begin{aligned} \sigma_0^\vee \simeq \sigma_0 \chi &\Rightarrow \omega_{\sigma_0}^{-1} = \omega_{\sigma_0} \chi^{n_0} \\ &\Rightarrow \omega_{\sigma_0}(-1) = \omega_{\sigma_0}(-1)\chi(-1)^{n_0} \\ &\Rightarrow 1 = \chi(-1)^{n_0} = \chi(-1) \end{aligned}$$

So, we conclude that $\epsilon(\pi) = 1$ if n is odd.

Case 2. Suppose that $n \equiv 0 \mod 4$. Then n_0 must be even since $n = n_0 + 2\sum_{i=1}^{\ell} n_i$. Pick $s = \operatorname{diag}(1, -1, 1, -1, \ldots, 1, -1) \in \mathrm{GL}_{n_0}(F)$.

Subcase $n_0 > 0$. By Remark 5.0.9, we have

$$\epsilon(\pi) = \chi(-1)^{\lfloor n_0/2 \rfloor}\, \chi(-1)^{\,n_1+n_2+\cdots+n_\ell}$$

If $\sum_{i=1}^{\ell} n_i$ is even, then $4|n_0$, so $\epsilon(\pi) = 1$. Similarly, if $\sum_{i=1}^{\ell} n_i$ is odd, then $n_0 \equiv 2 \mod 4$, so $n_0/2$ is odd. So $\epsilon(\pi) = 1$.

Subcase $n_0 = 0$. In this subcase, we have

$$\begin{aligned}
\epsilon(\pi) &= \omega_\pi(-1) && \text{by Remark 5.0.9}\\
&= \omega_\sigma(-1) && \text{by Corollary 2.9.3}\\
&= \omega_{\widetilde{\sigma}}(-1)\\
&= \omega_{\sigma_1}(-1)\cdots\omega_{\sigma_l}(-1)\cdot\omega_{\sigma_l}^{-1}(-1)\cdots\omega_{\sigma_1}^{-1}(-1)\cdot\chi(-1)^{n_l}\cdots\chi(-1)^{n_1}\\
&= \chi(-1)^{n_1+n_2+\cdots+n_l} = 1 && \text{since } \sum_{i=1}^{\ell} n_i \text{ is even.}
\end{aligned}$$

We conclude that $\epsilon(\pi) = 1$ if $n \equiv 0 \mod 4$.

Case 3a. Suppose that $n \equiv 2 \pmod 4$. Then n_0 must be even. Pick $s = \mathrm{diag}(1, -1, 1, -1 \ldots, -1, 1)_{n_0\times n_0}$. Note that if $n_0 = 0$ then

$$\epsilon(\pi) = \omega_\pi(-1) = \chi(-1)^{n_1+n_2+\cdots+n_\ell} = \chi(-1),$$

since $\sum_{i=1}^{\ell} n_i$ is odd. Suppose $n_0 > 0$. By Remark 5.0.9, we have

$$\epsilon(\pi) = \chi(-1)^{n_0/2}\,\chi(-1)^{\,n_1+n_2+\cdots+n_\ell} \tag{5.10}$$

If $\sum_{i=1}^{\ell} n_i$ is even then $n_0 \equiv 2 \pmod 4$. So $n_0/2$ is odd. So, Equation (5.10) becomes

$$\epsilon(\pi) = \chi(-1).$$

Similary, if $\sum_{i=1}^{\ell} n_i$ is odd, then $4|n_0$. So, Equation (5.10) becomes

$$\epsilon(\pi) = \chi(-1).$$

We conclude that $\epsilon(\pi) = \chi(-1)$ if $n \equiv 2 \mod 4$.

Case 3b. Suppose that $n \equiv 2 \pmod 4$ and assume that F contains an element i with $i^2 = -1$. If $n_0 = 0$ then $\epsilon(\pi) = \omega_\pi(-1)$ by Corollary 5.0.8. Suppose $n_0 > 0$. Then n_0 must be even. Pick

$$s \;=\; \mathrm{diag}(i, -i, i, -i, \ldots, i, -i) \in \mathrm{GL}_{n_0}(F).$$

Then we have

$$s^2 = -I_{n_0}, \qquad \det(s) = 1.$$

and Corollary 5.0.8 gives

$$\begin{aligned} \epsilon(\pi) &= \omega_{\sigma_0}(s^2)\,\chi^{-1}(\det s)\,\chi(-1)^{\,n_1+\cdots+n_\ell} \\ &= \omega_{\sigma_0}(-1)\chi(-1)^{\,n_1+\cdots+n_\ell} \qquad (5.11) \end{aligned}$$

On the other hand,

$$\begin{aligned} \omega_\pi(-1) &= \omega_\sigma(-1) = \omega_{\widetilde{\sigma}}(-1) \\ &= \omega_{\sigma_1}(-1)\cdots\omega_{\sigma_\ell}(-1)\cdot\omega_{\sigma_0}(-1)\cdot\omega_{\sigma_\ell}^{-1}(-1)\cdots\omega_{\sigma_1}^{-1}(-1)\cdot\chi(-1)^{n_l}\cdots\chi(-1)^{n_1} \\ &= \omega_{\sigma_0}(-1)\chi(-1)^{n_1+n_2+\cdots+n_\ell} \qquad (5.12) \end{aligned}$$

By (5.11) and (5.12), we conclude that $\epsilon(\pi) = \omega_\pi(-1)$ if $n \equiv 2 \pmod 4$ and F contains an element i with $i^2 = -1$.

5.1 When $n \equiv 2 \pmod 4$ and F Does Not Contain a Square Root of -1

The situation becomes considerably more subtle when F lacks a square root of -1. In this case, Prasad [Pra99] has constructed explicit counterexamples showing that no simple criterion based solely on the central character can exist.

Theorem 5.1.1 (Prasad)**.** *Let F be a non-Archimedean local field whose residue field has cardinality $q \equiv 3 \pmod 4$. Then for $n = 4m+2$ with $m \geq 1$, there exist irreducible, self-dual, generic representations π of $\mathrm{SL}(n, F)$ such that:*

1. *π is symplectic ($\epsilon(\pi) = -1$) even though $\omega_\pi(-I) = 1$*
2. *π is orthogonal ($\epsilon(\pi) = 1$) even though $\omega_\pi(-I) = -1$*

5.2 Example of a symplectic self-dual representation π of $\mathrm{SL}(6,F)$

Let F be a non-archimedean local field with ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{p}$, and residue field $k = \mathcal{O}_F/\mathfrak{p} \simeq \mathbb{F}_q$ of cardinality $q \equiv 3 \pmod 4$. Put $G = \mathrm{SL}(6,F)$, $K = \mathrm{SL}(6,\mathcal{O}_F)$, and $K_1 = \ker\big(K \to \mathrm{SL}(6,k)\big)$. Then $K/K_1 \simeq \mathrm{SL}(6,k)$. In [Pra98, §9], Prasad constructs an irreducible self-dual representation ρ of $\mathrm{SL}(6,k)$ which is *symplectic*, i.e. $\epsilon(\rho) = -1$, and for which one can arrange that $-I_6$ acts trivially on ρ, i.e. $\rho(-I_6) = \mathrm{Id}$. Let $\widetilde{\rho}$ denote the inflation of ρ to K, i.e. $\widetilde{\rho}$ is the representation of K whose restriction to K_1 is trivial and whose action factors through $K/K_1 \simeq \mathrm{SL}(6,k)$. Clearly, $\widetilde{\rho}$ remains irreducible and self-dual, and still satisfies $\epsilon(\widetilde{\rho}) = \epsilon(\rho) = -1$ and $\widetilde{\rho}(-I_6) = \mathrm{Id}$. Define

$$\pi \;:=\; \text{c-Ind}_K^G(\widetilde{\rho}).$$

Note that π is self-dual since

$$\pi^\vee \;\simeq\; \text{c-Ind}_K^G(\widetilde{\rho}^\vee) \;\simeq\; \text{c-Ind}_K^G(\widetilde{\rho}) \;=\; \pi.$$

Using Mackey's irreducibility criterion, one can show that π is irreducible.

By Lemma 5.2.1, we get $\epsilon(\pi) = \epsilon(\widetilde{\rho})$. Clearly, $\epsilon(\widetilde{\rho}) = \epsilon(\rho) = -1$. So, $\epsilon(\pi) = -1$. Finally, since $-I_6 \in K$ acts trivially on $\widetilde{\rho}$, it also acts trivially on π, so $-I_6$ acts trivially on π.

Lemma 5.2.1. *With the notation as above, we have $\epsilon(\pi) = \epsilon(\widetilde{\rho})$.*

Proof. Let W be the underlying vector space of π, and let V be the underlying vector space of $\widetilde{\rho}$.

We first show that $\widetilde{\rho}$ occurs in $\pi|_K$ with multiplicity one. By Frobenius Reciprocity theorem (See [Cas95, §2.4]),

$$\mathrm{Hom}_K(\widetilde{\rho}, \pi|_K) \cong \mathrm{Hom}_G(\text{c-Ind}_K^G \widetilde{\rho}, \pi).$$

Since π is irreducible, Schur's lemma gives

$$\mathrm{Hom}_G(\text{c-Ind}_K^G \widetilde{\rho}, \pi) = \mathrm{End}_G(\pi) \cong \mathbb{C}.$$

Hence $\widetilde{\rho}$ occurs in $\pi|_K$ with multiplicity one.

Let $V_0 \subseteq W$ be the unique K-subrepresentation of $\pi|_K$ isomorphic to $\widetilde{\rho}$. Since both π and $\widetilde{\rho}$ are irreducible self-dual, Schur's lemma gives

$$\dim \mathrm{Bil}_G(W) = 1, \qquad \dim \mathrm{Bil}_K(V_0) = 1.$$

Consider the restriction map

$$\phi : \mathrm{Bil}_G(W) \longrightarrow \mathrm{Bil}_K(V_0), \qquad B \longmapsto B|_{V_0 \times V_0}.$$

It is enough to show that ϕ is nonzero.

Let $0 \neq B_\pi \in \mathrm{Bil}_G(W)$, and choose $0 \neq w \in V_0$. Since B_π is nondegenerate, there exists $y \in W$ such that

$$B_\pi(w, y) \neq 0.$$

Let Y be the K-subrepresentation of W generated by y. Because K is compact and π is smooth, Y is finite-dimensional. Choose an irreducible K-subrepresentation $Y_0 \subseteq Y$ such that

$$B_\pi|_{V_0 \times Y_0} \neq 0.$$

Then

$$B_\pi|_{V_0 \times Y_0} \in \mathrm{Bil}_K(V_0, Y_0) \cong \mathrm{Hom}_K(Y_0, V_0^\vee).$$

Thus $Y_0 \cong V_0^\vee$. Since $\widetilde{\rho}$ is self-dual, we have $V_0^\vee \cong V_0$, so $Y_0 \cong V_0$. By multiplicity one, this forces $Y_0 = V_0$. Therefore

$$B_\pi|_{V_0 \times V_0} \neq 0.$$

So ϕ is nonzero.

Let $0 \neq B_{\widetilde{\rho}} \in \mathrm{Bil}_K(V_0)$. Since ϕ is an isomorphism, $B_\pi|_{V_0 \times V_0}$ is a nonzero scalar multiple of $B_{\widetilde{\rho}}$, so these two forms have the same symmetry type. Hence

$$\epsilon(\pi) = \epsilon(\widetilde{\rho}).$$

□

Proposition 5.2.2 (The sign for $\mathrm{SL}(2, F)$)**.** *Let F be a non-Archimedean local field and let π be an irreducible smooth* self-dual *representation of $G = \mathrm{SL}(2, F)$. Then*

$$\epsilon(\pi) = \omega_\pi(-1).$$

Proof. Let (P, σ, ν) be the Langlands triple attached to π. By Theorem 2.9.1, π is the unique irreducible quotient of $i_P^G(\sigma\nu)$, where σ is an irreducible tempered representation of the standard Levi component M of P. If π is not tempered, then we must have $P \neq G$. Since $G = \mathrm{SL}(2, F)$ has only one proper standard parabolic, namely the Borel subgroup B, it follows that $P = B$ and $M \simeq F^\times$ is a (split) torus. Equivalently, in the notation of the Levi attached to Langlands data, there is no middle block, so $n_0 = 0$. By Case 3a, we have

$$\epsilon(\pi) = \omega_\pi(-1).$$

Suppose π is tempered. Then by the uniqueness of Langlands classification, $P = G = M$, $\sigma \simeq \pi$, and $\nu = 1$. By Proposition 5.0.1, there exists an irreducible representation $\widetilde{\sigma}$ of $\widetilde{M} := \mathrm{GL}(2, F)$ such that π occurs in $\widetilde{\sigma}|_M$. Let $\theta \in \mathrm{Aut}(G)$ defined by $\theta(g) = {}^t g^{-1}$ for all $g \in G$.

For 2×2 matrices we have the identity

$$ {}^t g^{-1} = (\det g)^{-1} J g J^{-1}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

Since $\widetilde{\sigma}$ is admissible (see [BZ77, §3.6]), we may apply Theorem 4.2.2 (i) from [Bum97] , which gives an isomorphism

$$ \widetilde{\sigma}^\theta \simeq \widetilde{\sigma}^\vee. \tag{5.13} $$

Claim 1: $\widetilde{\sigma} \simeq \widetilde{\sigma}^\vee \otimes (\omega_{\widetilde{\sigma}} \circ \det)$**.**

Note that $\widetilde{\sigma} \circ \mathrm{Int}(J) \simeq \widetilde{\sigma}$ with $(\widetilde{\sigma} \circ \mathrm{Int}(J))(g) = T^{-1} \circ \widetilde{\sigma}(g) \circ T$ for all $g \in \widetilde{M}$, where $T = \widetilde{\sigma}(J)^{-1}$. So for each $g \in \widetilde{M}$ we have

$$ \begin{aligned} \widetilde{\sigma}^\theta(g) &= \widetilde{\sigma}((\det g)^{-1} \mathrm{Int}(J)(g)) \\ &= \omega_{\widetilde{\sigma}}((\det g)^{-1} I)(\widetilde{\sigma} \circ \mathrm{Int}(J))(g) \\ &= \omega_{\widetilde{\sigma}}((\det g)^{-1} I) T^{-1} \circ \widetilde{\sigma}(g) \circ T \end{aligned} $$

It follows that $\widetilde{\sigma}^\theta \simeq \widetilde{\sigma} \otimes (\omega_{\widetilde{\sigma}} \circ \det)^{-1}$. By (5.13), we get $\widetilde{\sigma}^\vee \simeq \widetilde{\sigma} \otimes (\omega_{\widetilde{\sigma}} \circ \det)^{-1}$, equivalently,

$$ \widetilde{\sigma} \simeq \widetilde{\sigma}^\vee \otimes (\omega_{\widetilde{\sigma}} \circ \det), $$

which proves Claim 1.

Claim 2: $\epsilon(\pi) = \omega_{\widetilde{\sigma}}(-1)$**.**

Let $\psi := \omega_{\widetilde{\sigma}} \circ \det$ and $s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By Corollary 4.1.3, we have

$$ \epsilon_\psi(\widetilde{\sigma}) = \omega_{\widetilde{\sigma}}(s^2)\psi^{-1}(s) = \psi^{-1}(s) = \omega_{\widetilde{\sigma}}(\det s)^{-1} = \omega_{\widetilde{\sigma}}(-1), $$

which implies that

$$\epsilon_\psi(\widetilde{\sigma}) = \omega_{\widetilde{\sigma}}(-1) \tag{5.14}$$

Also, we have

$$\epsilon(\pi) = \epsilon_\psi(\widetilde{\sigma}) \tag{5.15}$$

by Proposition 5.0.3. By (5.14) and (5.15), we get

$$\epsilon(\pi) = \omega_{\widetilde{\sigma}}(-1). \tag{5.16}$$

This proves Claim 2. We now show that

$$\omega_{\widetilde{\sigma}}(-1) = \omega_\pi(-1). \tag{5.17}$$

Note that $\widetilde{\sigma}(-I_2) = \omega_{\widetilde{\sigma}}(-1)\mathrm{Id}$ and $\pi(-I_2) = \omega_\pi(-1)\mathrm{Id}$ since $-I_2$ is central in both groups M and $\widetilde{M}$. Since π occurs as an irreducible constituent of $\widetilde{\sigma}|_M$, there exists a nonzero M-invariant subspace $W \subseteq \widetilde{\sigma}$ on which M acts via π. For any $0 \neq v \in W$ we then have, $\widetilde{\sigma}(-I_2)v = \omega_{\widetilde{\sigma}}(-1)\, v$, and $\widetilde{\sigma}(-I_2)v = \pi(-I_2)v = \omega_\pi(-1)\, v$. Therefore, $0 = \omega_{\widetilde{\sigma}}(-1)v - \omega_\pi(-1)v = (\omega_{\widetilde{\sigma}}(-1) - \omega_\pi(-1))v$. Since $v \neq 0$, we must have $\omega_{\widetilde{\sigma}}(-1) = \omega_\pi(-1)$, as desired. By (5.16) and (5.17), we conclude that

$$\epsilon(\pi) = \omega_\pi(-1).$$

□

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