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THE RELATIVISTIC QUANTUM-MECHANICAL PROBLEM

OF THE TWO-BODY SYSTEM

APPROVED BY:

DISSERTATION COMMITTEE

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CHAPTER I

INTRODUCTION

The relativistic quantum-mechanical problem of the two-body system has been the topic of many articles. Attempts were first made to set up an equation describing two-electron atoms, taking into account the interaction of the two electrons. Starting from the Hamiltonian function derived from the classical theory by Darwin,(1) a wave equation for two electrons was set up by Breit.(2) The wave equation consists of the sum of the Dirac Hamiltonians of two free electrons plus an interaction term. The Breit interaction, however, is not Lorentz invariant and the equation is applicable only to two electrons. No derivation of the Breit equation for two bodies with an arbitrary interaction has been made and discussed and there has been no attempt to derive equations similar to Breit's for particles of higher spin. For two particles of arbitrary spin, a wave equation is still lacking.

A four-dimensional, relativistic, integral wave equation for a two-body system was obtained from the Feynman(3) formalism by Salpeter and Bethe(4) in 1951.

This equation, which was given the name Bethe-Salpeter equation, was first proposed without derivation by Nambu[5] in 1950. A field-theoretical derivation was given by Gell-Mann and Low[6] (1951). The differential form of the B-S equation is that which most authors discuss. Yet the integral and differential forms are not equivalent because although solutions to the B-S integral equations are solutions to the B-S differential equation the reverse is not true. Since the B-S differential equation is obtained from the integral form by the operation of two differential operators, therefore only part of the set of solutions to the B-S differential equation which satisfy the integral equation are acceptable solutions. Another obvious way of showing that the B-S differential equation contains extra solutions is by putting the interaction equal to zero. The resulting equation has more solutions than just the two free particle wave function. The trouble with the B-S differential equation may be due to the fact that it is a higher order differential equation than the single particle wave equations. Another feature of the B-S equation is the appearance of the relative time variable, whose physical role is not clear.

The B-S differential equation has been solved for the ladder approximation by Goldstein[7] for spin-0 particles and by Wick [8] for spin-$\frac{1}{2}$ particles. The solutions were obtained in momentum space with certain approximations, but no general solution of the B-S equation has been found for the ladder approximation.

Another attempt to solve the two-body problem is the quasipotential approach developed by Logunov and Tavkhelidze.[9] The wave function which they introduced is a generalization of the non-relativistic wave function. It is a function of only one time variable and satisfies a Schrodinger type equation. However, the potential in this equation is complicated and non-local and can only be determined by the matrix element of the scattering amplitude. It is difficult to solve for realistic potentials.

The Logunov and Tavkhelidze equation is not the only quasipotential equation that is used to describe the two-body system. Some authors[10] postulate equations starting from general assumptions. Others[11] try to generalize the Logunov and Tavkhelidze equation for two particles of unequal mass. The essential difference between all these approaches is the way the potential V between the two bodies is included in the equation. The main defect of these three-dimensional quasipotential equations is that they do not have a firm theoretical basis.

It is the objective of the present work to derive wave equations describing two particles interacting through an arbitrary potential V. Instead of considering only spin-0 and spin-$\frac{1}{2}$ particles, our method applies to particles of arbitrary spin. However, we treat our problem purely as a quantum-mechanical one. There is no claim that our equations will describe any field-theoretic processes such as pair creation and annihilation of particles and antiparticles. In the derivation and, finally, the solution of the equation, we try to be as general as possible in that, where possible, we consider particles of unequal mass and unequal energies. Besides deriving the wave equations, we also show the connection between the B-S equation, the quasipotential equation, the Breit equation and the Bogolubov equation.(11) Then we consider the two-particle equations for spin-0, spin-$\frac{1}{2}$ and spin-1 separately. We reduce these matrix equations to their simplest form and solve these equations in the spin-0 and spin-$\frac{1}{2}$ cases for square well and Coulomb potentials.

Throughout this paper, natural units are used in which $\hbar$, Planck's constant divided by 2π and c, the speed of light, are taken as unity, i.e.

$$\hbar = c = 1$$

We use the imaginary fourth component convention in which the space-time four-vectors are $x_\mu = (x, y, z, it)$ and the invariant is $x_\mu x_\mu = x^2 + y^2 + z^2 - t^2$. The four derivatives are $\partial_\mu = \frac{\partial}{\partial x_\mu} = (\nabla, \frac{1}{i}\frac{\partial}{\partial t})$. The scalar product of two real four-vectors a and b is denoted by

$$ab = a_\mu b_\mu = a_1 b_1 + a_2 b_2 + a_3 b_3 - a_0 b_0$$

where $a_4 = ia_0$, $b_4 = ib_0$ and a_0, b_0 are real.

Indices representing summation over all four components 1,...4 are denoted by Greek letters μ, ν, α... while indices representing summation over three space components 1, 2, 3 are denoted by Latin letters i, j, k ....,

CHAPTER II

TWO-PARTICLE WAVE EQUATION

A. A Two-particle Wave Equation from the B-S Integral Equation

The non-relativistic Schrodinger equation

$$\left(\frac{\nabla^2}{2m} - V + i\frac{\partial}{\partial t}\right)\psi = 0\,, \qquad (2.1)$$

can be cast into an integral form by means of the non-relativistic free particle Green's function which satisfies

$$\left(\frac{\nabla^2}{2m} + i\frac{\partial}{\partial t}\right)G(\vec{x}-\vec{x}',\ t-t') = i\,\delta(\vec{x}-\vec{x}')\delta(t-t').$$

This Green's function can be written as an integral

$$G(\vec{x}-\vec{x}',\ t-t') = \frac{i}{(2\pi)^4}\int \frac{\exp(ip(x-x'))}{-\frac{\vec{p}^2}{2m} + E}d^3p\,dE \quad .$$

With this Green's function, Eq. (1.1) can be put into the following integral form

$$\psi(\vec{x},\ t) = \psi_0(\vec{x},\ t) - i\int G(\vec{x} - \vec{x}',\ t-t')V(\vec{x}')\psi(\vec{x}',t')d^3x'dt' \qquad (2.2)$$

where $\psi_0(\vec{x},\ t)$ is the free particle wave function. On a more intuitive basis, the B-S equation is a two-particle relativistic generalization of the Schrodinger integral equation. It can also be derived using field theory. The B-S equation is

$$\psi(x_1, x_2) = \psi_0(x_1, x_2) - i\int G_1(x_1-x_1')G_2(x_2-x_2') V(x_1', x_2')\psi(x_1', x_2')d^4x_1'd^4x_2' \quad (2.3)$$

where $\psi_0(x_1, x_2)$ is the wave function for two free particles; $G_1(x_1-x_1')$ and $G_2(x_2-x_2')$ are the free particle Green's function for particles one and two; and $d^4x_1' = d\vec{x}_1'dt_1'$; $d^4x_2' = d\vec{x}_2'dt_2'$.

Our aim is to derive from the integral B-S equation (2.3) a differential equation such that the following basic requirements are satisfied:-

(i) If the interaction term is neglected, the solution should be the product of two single particle wave functions, one for each particle.

(ii) On taking the non-relativistic limit, the equation should reduce to the two-particle Schrodinger wave equation.

(iii) The equation should contain a single time variable.

We assume that the B-S integral equation is valid for arbitrary instantaneous potentials $V(\vec{x}_1, \vec{x}_2)\delta(t_1-t_2)$. The two Green's functions in the B-S integral equation are taken to be retarded functions like the Green's function used in Eq. (2.2).

Consider a general single-particle Hamiltonian wave equation

$$H(\nabla)\psi = i\frac{\partial\psi}{\partial t}\,, \qquad (2.4)$$

where $H(\nabla)$ is the Hamiltonian operator of a particle of arbitrary spin. For a spin-0 particle,[12]

$$H = \frac{\nabla^2}{2m}\sigma - m\rho^{*},$$

with $\sigma = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and $\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For a spin-$\frac{1}{2}$ particle,

$$H = -i\vec{\alpha}\cdot\nabla + \beta m$$

where $\vec{\alpha}$ and β are the usual matrices in Dirac Theory.

For a spin-1 particle,[13]

$$H = \beta E - i\frac{2E^2(\vec{\alpha}\cdot\nabla)}{2E^2 - m^2} + \frac{2E\beta(\vec{\alpha}\cdot\nabla)^2}{2E^2 - m^2}\,,$$

with $\alpha_i = \begin{pmatrix} 0 & -iS_i \\ iS_i & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $E = \sqrt{-\nabla^2 + m^2}$.

The S_i are the 3x3 spin-1 matrices.

The Green's function for Eq. (2.4) satisfies the equation

$$\left(H(\nabla') - i\frac{\partial}{\partial t'}\right)G(x'-x) = i\delta^4(x'-x)$$

and can be written as

---

* H can be obtained from Duffin's five component wave equation for spin-0 particles, and will be discussed in Chapter III-A of this paper.

$$G(x'-x) = \frac{1}{(2\pi)^4}\int \frac{H(i\vec{p}) + E}{\vec{p}^2+m^2-E^2}\exp(ip(x'-x))d^3p\,dE \,. \qquad (2.5)$$

where we use the fact that $H^2(i\vec{p}) = \vec{p}^2 + m^2$ for any relativistic Hamiltonian $H(i\vec{p})$. The contour of integration on the E plane is given in Figure 1

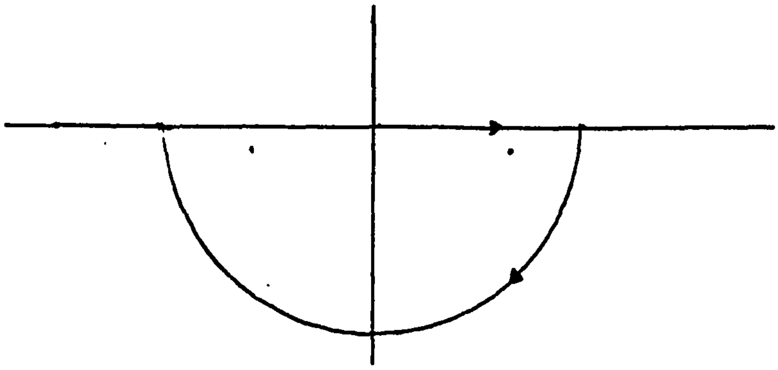

Figure 1.

in which we displace the two poles an infinitesimal distance below the real axis. This corresponds to choosing $G(x'-x)$ to be retarded so that, for $t'<t$, $G(x'-x) = 0$. The Green's function, therefore, relates the wave function at a later time to its value at an earlier time. After evaluating the E integral, we obtain from (2.5)

$$G(x'-x) = \begin{cases} 0 & t'<t \\ -\frac{1}{2}\delta^3(\vec{x}'-\vec{x}) & t'=t \\ -\frac{1}{2(2\pi)^3}\int d^3p\, \exp(i\vec{p}\cdot(\vec{x}'-\vec{x}))\Big\{\exp(iE(t'-t)) + \exp(-iE(t'-t)) & \\ \qquad + \frac{H(i\vec{p})}{E}\Big[\exp(-iE(t'-t)) - \exp(iE(t'-t))\Big]\Big\} & t'>t\,; \end{cases}$$

$$\text{and} \lim_{t'\to t^+} G(x'-x) = -\delta^3(\vec{x}'-\vec{x})\,, \qquad (2.6)$$

where $E = \sqrt{\vec{p}^2+m^2}$.

For a general instantaneous potential, the B-S equation (2.3) becomes

$$\psi(x_1,x_2) = \psi_0(x_1,x_2) - i\int G_1(x_1-x_1')G_2(x_2-x_2')V(\vec{x}_1',\vec{x}_2')\delta(t_1'-t_2')\,\psi(x_1',x_2')d^4x_1'd^4x_2' , \tag{2.7}$$

with the Green's functions given by (2.6). Eq. (2.7) is the B-S equation for two particles, each being described by the Hamiltonian equation $H(\nabla)\psi = i\frac{\partial\psi}{\partial t}$. Operating on both sides of Eq. (2.7) with $H_1(\nabla_1) - i\frac{\partial}{\partial t_1}$, we obtain

$$\begin{aligned}(H_1(\nabla_1) - i\tfrac{\partial}{\partial t_1})\psi(x_1,x_2) &= \int\delta(x_1-x_1')G_2(x_2-x_2')V(\vec{x}_1',\vec{x}_2')\delta(t_1'-t_2')\,\psi(x_1',x_2')d^4x_1'd^4x_2' \\ &= \int G_2(x_2-x_2')V(\vec{x}_1,\vec{x}_2')\delta(t_1-t_2')\psi(x_1,x_2')d^4x_2' \\ &= \int G_2(x_2-x_2')\Big|_{t_2'=t_1} V(\vec{X}_1,\vec{x}_2')\psi(x_1,x_2')\Big|_{t_2'=t_1} d^3x_2' .\end{aligned} \tag{2.8}$$

A second equation can be obtained from Eq. (2.7) by operating on its both sides by $(H_2(\nabla_2) - i\frac{\partial}{\partial t_2})$. The equation is

$$(H_2(\nabla_2) - i\tfrac{\partial}{\partial t_2})\psi(x_1,x_2) = \int G_1(x_1-x_1')\Big|_{t_1'=t_2} V(\vec{x}_1',\vec{x}_2)\psi(x_1', x_2)\Big|_{t_1'=t_2} d^3x_1' . \tag{2.9}$$

We add Eqs. (2.8) and (2.9) and use (2.6) and the fact that

$$\lim_{\substack{t_1 \to t \\ t_2 \to t}} \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right)\psi(\vec{x}_1, t_1, \vec{x}_2, t_2) = \frac{\partial}{\partial t}\psi(\vec{x}_1, \vec{x}_2, t).$$

Then we obtain

$$\left[H_1(\nabla_1) + H_2(\nabla_2) - i\frac{\partial}{\partial t}\right]\psi(\vec{x}_1, \vec{x}_2, t) = -V(\vec{x}_1, \vec{x}_2)\psi(\vec{x}_1, \vec{x}_2, t). \tag{2.10}$$

Eq. (2.10) is the result we obtain from Eqs. (2.8) and (2.9) for all possibilities $t_1>t_2$, $t_1=t_2$ and $t_1<t_2$. For stationary state solutions, we let $\psi(\vec{x}_1, \vec{x}_2, t) = \psi(\vec{x}_1, \vec{x}_2) \exp(-iEt)$. Eq. (2.10) becomes

$$\left[H_1(\nabla_1) + H_2(\nabla_2) + V(\vec{x}_1, \vec{x}_2)\right]\psi(\vec{x}_1, \vec{x}_2) = E\psi(\vec{x}_1, \vec{x}_2), \tag{2.11}$$

where E is the total energy of the system of two particles. Eq. (2.11) contains the sum of two Hamiltonians and has the same form as the Breit equation for two spin-$\frac{1}{2}$ particles.

When the interaction $V(\vec{x}_1, \vec{x}_2)$ is put to zero, Eq. (2.11) becomes

$$\left[H_1(\nabla_1) + H_2(\nabla_2)\right]\psi(\vec{x}_1, \vec{x}_2) = E\psi(\vec{x}_1, \vec{x}_2).$$

The solutions are $\psi(\vec{x}_1, \vec{x}_2) = \psi_1(\vec{x}_1)\psi_2(\vec{x}_2)$ with

$$H_1(\nabla_1)\psi_1(\vec{x}_1) = E_1\psi_1(\vec{x}_1),$$

$$H_2(\nabla_2)\psi_2(\vec{x}_2) = E_2\psi_2(\vec{x}_2),$$

and $E_1 + E_2 = E$.

For the non-relativistic limit, we assume

$$E_i - m_i = \varepsilon_i \sim O(v^2), \qquad i = 1, 2$$

$$V(\vec{x}_1, \vec{x}_2) \sim O(v^2),$$

$$\text{and } \nabla V(\vec{x}_1, \vec{x}_2) \sim \frac{V(\vec{x}_1, \vec{x}_2)}{a} \sim O(v^3),^*$$

where a represents the linear dimensions of the system, and

$$\frac{1}{a} \sim p \sim mv.$$

Putting $t_2 = t_1$ in Eq. (2.8), $t_1 = t_2$ in Eq. (2.9) and writing $\psi(x_1, x_2) = \psi(\vec{x}_1, \vec{x}_2) \exp(-i(E_1 t_1 + E_2 t_2))$, we have

$$H_i(\nabla_i)\psi(\vec{x}_1, \vec{x}_2) = (E_i - \frac{V(\vec{x}_1, \vec{x}_2)}{2})\psi(\vec{x}_1, \vec{x}_2). \qquad (2.12)$$

Operating on both sides of Eq. (2.12) with $H_i(\nabla_i)$, we obtain

$$H_i^2(\nabla_i)\psi(\vec{x}_1, \vec{x}_i) = (E_i - \frac{V(\vec{x}_1, \vec{x}_2)}{2})^2\psi(\vec{x}_1, \vec{x}_2) + \text{Extra terms.} \qquad (2.13)$$

The order of magnitude of the extra terms is $\nabla V(\vec{x}_1, \vec{x}_2) \sim O(v^3)$ or higher. Simplifying (2.13) and collecting terms up to the order of v^2, we obtain

$$-\frac{\nabla_i^2}{2m_i}\psi(\vec{x}_1, \vec{x}_2) = \left[\varepsilon_i - \frac{V(\vec{x}_1, \vec{x}_2)}{2}\right]\psi(\vec{x}_1, \vec{x}_2), \qquad (2.14)$$

where $i = 1, 2$.

*We use the same approximation as Schiff(3rd Ed.) p. 481.

Adding the two equations of (2.14), we have the two-particle Schrodinger equation,

$$\left[-\frac{\nabla_1^2}{2m_1} - \frac{\nabla_2^2}{2m_2} + V(\vec{x}_1, \vec{x}_2)\right]\psi(\vec{x}_1, \vec{x}_2) = \mathcal{E}\psi(\vec{x}_1, \vec{x}_2),$$

with $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$.

Hence Eq. (2.10) satisfies our requirements.

B. Semi-field Theoretic Derivation of the Two-particle Equation

In this section, the two-particle equation (2.10) is derived again using a semi-field theoretic method. This formalism is a generalization of the non-relativistic quantum field theory which assumes a Hamiltonian operator for two particles interacting through a potential $V(\vec{x}_1, \vec{x}_2)$ to be [14]

$$\mathbb{H} = \int\bar{\psi}(\vec{x}, t)\left(-\frac{\nabla^2}{2m}\right)\psi(\vec{x}, t)d^3x + \frac{1}{2}\int d^3x d^3x'\bar{\psi}(\vec{x}, t)\bar{\psi}(\vec{x}', t)V(\vec{x}, \vec{x}')\psi(\vec{x}', t)\psi(\vec{x}, t), \quad (2.15)$$

where $\bar{\psi}$ and ψ are creation and destruction operators. To generalize (2.15) to two particles, each described by the Hamiltonian wave equation $i\frac{\partial\psi}{\partial t} = H\psi$ with interaction $V(\vec{x}_1, \vec{x}_2)$, we assume that the Hamiltonian operator is given by

$$\mathbb{H} = \int\bar{\psi}(\vec{x}, t)H\psi(\vec{x}, t)d^3x + \frac{1}{2}\int d^3x d^3x'\bar{\psi}(\vec{x}, t)\bar{\psi}(\vec{x}', t)V(\vec{x}, \vec{x}')\psi(\vec{x}', t)\psi(\vec{x}, t). \quad (2.16)$$

This is the Hamiltonian operator we will use to derive wave equations describing two interacting particles.

The equations of motion for the field operators ψ and $\bar{\psi}$ are the same as in usual quantum mechanics

$$i\frac{\partial\psi(\vec{x}, t)}{\partial t} = \left[\psi(\vec{x}, t), \mathbb{H}\right]_- , \quad \text{and} \quad i\frac{\partial\bar{\psi}(\vec{x}, t)}{\partial t} = \left[\bar{\psi}(\vec{x}, t), \mathbb{H}\right]_- . \tag{2.17}$$

The commutation rules for ψ and $\bar{\psi}$ are

$$\left[\psi(\vec{x}, t), \bar{\psi}(\vec{x}', t)\right]_- = \delta^3(\vec{x} - \vec{x}')$$
$$\left[\psi(\vec{x}, t), \psi(\vec{x}', t)\right]_- = \left[\bar{\psi}(\vec{x}, t), \bar{\psi}(\vec{x}', t)\right]_- = 0 .$$

Using the Hamiltonian operator from (2.16), the equations of motion (2.17) become

$$i\frac{\partial\psi(\vec{x}, t)}{\partial t} = \left[\psi(\vec{x}, t), \mathbb{H}\right]_-$$
$$= H\psi(\vec{x}, t) + \int d^3x'\bar{\psi}(\vec{x}',t)V(\vec{x},\vec{x}')\psi(\vec{x}', t)\psi(\vec{x}, t) ,$$

and (2.18)

$$i\frac{\partial\bar{\psi}(\vec{x}, t)}{\partial t} = \left[\bar{\psi}(\vec{x}, t), \mathbb{H}\right]_-$$
$$= -\bar{\psi}(\vec{x}, t)H - \int d^3x'\bar{\psi}(\vec{x},t)\bar{\psi}(\vec{x}', t)V(\vec{x}, \vec{x}')\psi(\vec{x}', t).$$

A vacuum state $|0\rangle$ is introduced by requiring $\psi(\vec{x}, t)|0\rangle = \langle 0|\bar{\psi} = 0$. From the vacuum state, a state of two particles $|2, E\rangle$ can be constructed by operating on $|0\rangle$ with $\bar{\psi}$ i.e.

$$|2, E\rangle = \bar{\psi}(\vec{x}, t)\bar{\psi}(\vec{x}', t)\,|0\rangle .$$

We now define a two-time wave function for a system of two particles $\phi(\vec{x}_1, \vec{x}_2, t_1, t_2)$ as

$$\phi(\vec{x}_1, \vec{x}_2, t_1, t_2) = \langle 0 | T\psi(\vec{x}_1, t_1)\psi(\vec{x}_2, t_2) | 2, E \rangle,$$

where T is the time ordering operator defined by

$$TA(\vec{x}_1, t_1)B(\vec{x}_2, t_2) = \frac{1}{2}\left[A(\vec{x}_1, t_1), B(\vec{x}_2, t_2)\right]_+ + \frac{1}{2}\varepsilon(t_1 - t_2)\left[A(\vec{x}_1, t_1), B(\vec{x}_2, t_2)\right]_- ,$$

with
$$\varepsilon(t) = 1 \qquad t > 0$$
$$= -1 \qquad t < 0 .$$

The equal time wave function is taken to be the limit of the two time wave function as t_1 and t_2 approach a common t. The propagator or Green's function is defined as

$$G(x_1 - x_2) = \langle 0 | T\psi(\vec{x}_1, t_1)\bar{\psi}(\vec{x}_2, t_2) | 0 \rangle.$$

The equation satisfied by G can be obtained as follows:-

$$i\frac{\partial G(x_1-x_2)}{\partial t_1} = i\left[\frac{1}{2}\langle 0 | \frac{\partial \psi(\vec{x}_1, t_1)}{\partial t_1}\bar{\psi}(\vec{x}_2, t_2) + \bar{\psi}(\vec{x}_2, t_2)\frac{\partial \psi(\vec{x}_1, t_1)}{\partial t_1} | 0 \rangle\right]$$
$$+ \frac{1}{2}\frac{\partial \varepsilon(t_1-t_2)}{\partial t_1}\langle 0 | \psi(\vec{x}_1, t_1)\bar{\psi}(\vec{x}_2, t_2) - \bar{\psi}(\vec{x}_2, t_2)\psi(\vec{x}_1, t_1) | 0 \rangle$$
$$+ \frac{1}{2}\varepsilon(t_1-t_2)\langle 0 | \frac{\partial \psi(\vec{x}_1, t_1)}{\partial t_1}\bar{\psi}(\vec{x}_2, t_2) - \bar{\psi}(\vec{x}_2, t_2)\frac{\partial \psi(\vec{x}_1, t_1)}{\partial t_1} | 0 \rangle$$
$$= \langle 0 | T\left[i\frac{\partial \psi(\vec{x}_1, t_1)}{\partial t_1}\bar{\psi}(\vec{x}_2, t_2)\right] | 0 \rangle$$
$$+ \frac{1}{2}\frac{\partial \varepsilon(t_1-t_2)}{\partial t_1}\langle 0 | \psi(\vec{x}_1, t_1)\bar{\psi}(\vec{x}_2, t_2) - \bar{\psi}(\vec{x}_2, t_2)\psi(\vec{x}_1, t_1) | 0 \rangle.$$

Using (2.18) for $i\frac{\partial\psi}{\partial t_1}$ and noting that $\psi|0\rangle=\langle 0|\bar{\psi}= 0$, we have

$$(i\frac{\partial}{\partial t_1} - H_1)G(x_1-x_2) = i\delta^4(x_1-x_2), \tag{2.19}$$

where we also make use of the fact that $\frac{d\varepsilon(t)}{dt} = 2\delta(t)$.*

Our next job is to find an equation that the two-particle wave function satisfies. We first compute

$$i\frac{\partial\phi(\vec{x}_1,\vec{x}_2,t_1,t_2)}{\partial t_1}=\frac{1}{2}\langle 0|i\frac{\partial\psi(\vec{x}_1,t_1)}{\partial t_1}\psi(\vec{x}_2,t_2)+\psi(\vec{x}_2,t_2)i\frac{\partial\psi(\vec{x}_1,t_1)}{\partial t_1}|2,E\rangle$$

$$+\frac{i}{2}\frac{\partial\varepsilon(t_1-t_2)}{\partial t_1}\langle 0|\psi(\vec{x}_1,t_1)\psi(\vec{x}_2,t_2)-\psi(\vec{x}_2,t_2)\psi(\vec{x}_1,t_1)|2,E\rangle$$

$$+\frac{1}{2}\varepsilon(t_1-t_2)\langle 0|i\frac{\partial\psi(\vec{x}_1,t_1)}{\partial t_1}\psi(\vec{x}_2,t_2)-\psi(\vec{x}_2,t_2)i\frac{\partial\psi(\vec{x}_1,t_1)}{\partial t_1}|2,E\rangle$$

$$=\langle 0|T(i\frac{\partial\psi(\vec{x}_1,t_1)}{\partial t_1}\psi(\vec{x}_2,t_2))|2,E\rangle$$

$$=\langle 0|T(H_1\psi(\vec{x}_1,t_1)+\int d^3x'\bar{\psi}(\vec{x}',t_1)V(\vec{x}',\vec{x}_1)$$

$$\psi(\vec{x}',t_1)\psi(\vec{x}_1,t_1))\psi(\vec{x}_2,t_2)|2,E\rangle,$$

so that

$$(i\frac{\partial}{\partial t_1} - H_1)\phi(\vec{x}_1,\vec{x}_2,t_1,t_2)$$

$$=\int d^3x'\langle 0|T\psi(\vec{x}_2,t_2)\bar{\psi}(\vec{x}',t_1)|0\rangle\langle 0|T\psi(\vec{x}',t_1)\psi(\vec{x}_1,t_1)|2,E\rangle V(\vec{x}',\vec{x}_1)$$

$$=\int d^3x'G(x_2-x')^{t'=t_1}\phi(\vec{x}_1,\vec{x}',t_1,t_1)V(\vec{x}',\vec{x}_1). \tag{2.20}$$

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* See Appendix.

Following the same procedure, we obtain another equation

$$(i\frac{\partial}{\partial t_2} - H_2)\phi(\vec{x}_1, \vec{x}_2, t_1, t_2)$$

$$= \int d^3x' G(x_1 - x')\, \phi(\vec{x}_2, \vec{x}', t_2, t_2)\Big|_{t'=t_2} V(\vec{x}', \vec{x}_2) \,. \qquad (2.21)$$

The Green's function in (2.19) differs from that in the previous section only by a minus sign. Therefore, Eqs. (2.20) and (2.21) are identical with Eqs. (2.8) and (2.9). Thus they give the same wave equation (2.10) for the two-particle wave function ϕ.

C. <u>Connections between Different Two-particle Equations</u>

In section one, a two-particle wave equation was derived from the B-S integral equation for particles which are described by the single particle Hamiltonian wave equation. In this section, the same formalism will be applied to a spin-0 wave equation, the Klein-Gordon equation, which is not in Hamiltonian form. The Klein-Gordon equation which contains a second-order time derivative differs from the Hamiltonian type of wave equation which contains only a first-order time derivative. The Klein-Gordon equation is

$$(-\Box^2 + m^2)\psi(x) = 0 \,,$$

where $\Box^2 = \nabla^2 - \frac{\partial^2}{\partial t^2}$.

The B-S equation for two Klein-Gordon particles interacting through an instantaneous potential is

$$\psi(x_1,x_2) = \psi_0(x_1,x_2) - i\int G_1(x_1-x_1')G_2(x_2-x_2')V(\vec{x}_1',\vec{x}_2')$$

$$\delta(t_1'-t_2')\psi(x_1',x_2')d^4x_1'd^4x_2', \quad (2.22)$$

where the Green's function satisfies the equation,

$$(-\Box'^2 + m^2)G(x'-x) = \delta^4(x'-x),$$

and can be expressed in the integral form

$$G(x'-x) = \frac{1}{(2\pi)^4}\int \frac{\exp(ip(x'-x))}{-E^2+\vec{p}^2+m^2}d^3pdE.$$

As in section one, we displace the two poles $E = \pm\sqrt{\vec{p}^2+m^2}$ an infinitesimal distance below the real axis and take the contour of integration for the variable E to be the same as in Figure 1. So the Green's function is again retarded. After evaluating the E integral, the Green's function can be shown to be

$$G(x'-x) = \frac{i}{2(2\pi)^2}\int d^3p \exp(i\vec{p}(\vec{x}'-\vec{x}))\left[\frac{\exp(-iE(t'-t))}{E} - \frac{\exp(iE(t'-t))}{E}\right],$$

with $E = \sqrt{\vec{p}^2+m^2}$.

The time limit of the time derivative is

$$\lim_{t'\to t^+} \frac{\partial G(x'-x)}{\partial t'} = \delta^3(\vec{x}'-\vec{x}). \quad (2.23)$$

We return to the B-S equation (2.22) for Klein-Gordon particles. Operating on both sides of Eq. (2.22) with $(-\Box_1^2 + M_1^2)$ gives

$$(-\Box_1^2 + m_1^2)\psi(x_1,x_2) = -i\int G_2(x_2-x_2')\Big|^{t_2'=t_1} V(\vec{x}_1,\vec{x}_2')\psi(x_1,x_2')\Big|^{t_2'=t_1} d^3x_2'. \quad (2.24)$$

For stationary states,

$$\psi(x_1,x_2) = \psi(\vec{x}_1,\vec{x}_2)\ \exp\ (-i(E_1t_1+E_2t_2))$$

Eq. (2.24) becomes

$$(-\nabla_1^2 - E_1^2 + m_1^2)\psi(x_1,x_2) = -i\int G_2(x_2-x_2')\Big|_{t_2'=t_1} V(\vec{x}_1,\vec{x}_2')\psi(x_1,x_2')\Big|_{t_2'=t_1} d^3x_2'. \qquad (2.25)$$

Differentiating both sides of (2.25) with respect to t_2 and taking limit $t_2 \to t_1^+$, we obtain, using (2.23)

$$(-\nabla_1^2 - E_1^2 + m_1^2)\psi(\vec{x}_1,\vec{x}_2) = \frac{V(\vec{x}_1,\vec{x}_2)}{E_2}\psi(\vec{x}_1,\vec{x}_2). \qquad (2.26)$$

Following similar steps, we obtain a second equation

$$(-\nabla_2^2 - E_2^2 + m_2^2)\psi(\vec{x}_1,\vec{x}_2) = \frac{V(\vec{x}_1,\vec{x}_2)}{E_1}\psi(\vec{x}_1,\vec{x}_2). \qquad (2.27)$$

Eqs. (2.26) and (2.27) are similar but not exactly the same as Bogolubov's[11] Eqs. (2.9) and (2.10) which are

$$(-\nabla_1^2 - E_1^2 + m_1^2)\psi = V'\psi$$

and

$$(-\nabla_2^2 - E_2^2 + m_2^2)\psi = V'\psi.$$

These two equations which Bogolubov obtained by generalizing the quasipotential equation for two particles of unequal masses hold only when the potential V' is a function of energy E and relative position of the two particles. So for particles which have approximately equal energy, that is, $E_1 \doteq E_2 \doteq \frac{E}{2}$, our equations (2.26) and (2.27) reduce to Bogolubov's if we define $V' = \frac{2V}{E}$. For the case of

$\left| E_1 - E_2 \right| \ll 1$, we obtain Bogolubov's subsidiary condition on ψ by equating the left hand sides of Eqs. (2.26) and (2.27). The condition is

$$(-\nabla_1^2 - E_1^2 + m_1^2)\psi = (-\nabla_2^2 - E_2^2 + m_2^2)\psi \,. \qquad (2.28)$$

In the CM frame, (2.28) reduces to

$$E_1^2 - E_2^2 = m_1^2 - m_2^2 ,$$

$$\text{or} \quad E(E_1 - E_2) = m_1^2 - m_2^2 ,$$

with $E = E_1 + E_2$.

Solving for E_1 and E_2, we have

$$\begin{aligned} E_1 &= \frac{E}{2} + \frac{m_1^2 - m_2^2}{2E} , \\ E_2 &= \frac{E}{2} - \frac{m_1^2 - m_2^2}{2E} . \end{aligned} \qquad (2.29)$$

The subsidiary condition (2.28) and E_1 and E_2 as given by (2.29) are identical with those obtained by Bogolubov. Adding Eqs. (2.26) and (2.27) and using (2.29) we have a single equation for ψ . In the CM frame, the equation is

$$\left[-2\nabla_{\vec{x}}^2 - \frac{E^2}{2} - \frac{(m_1^2-m_2^2)^2}{2E^2} + m_1^2 + m_2^2\right]\psi$$

$$= 2EV\left[\frac{1}{E^2+m_1^2-m_2^2} + \frac{1}{E^2-m_1^2+m_2^2}\right]\psi \,, \qquad (2.30)$$

where $\vec{x} = \vec{x}_1 - \vec{x}_2$.

For two particles of equal masses, we have $m_1 = m_2 = m$ and $E_1 = E_2 = \frac{E}{2}$. Eq. (2.30) thus reduces to

$$(-\nabla_{\vec{x}}^2 - \frac{E^2}{4} + m^2)\psi = \frac{2V}{E}\psi . \tag{2.31}$$

Eq. (2.31) is exactly the Logunov and Tavkhelidze[9] quasipotential equation for two spin-0 particles of equal mass if we replace $\frac{E}{2}$, the energy of one particle, on the right hand side of the equation by $\sqrt{-\nabla_{\vec{x}}^2+m^2}$.

CHAPTER III

TWO SPIN-ZERO PARTICLES

A. Spin-zero Particle Hamiltonian

A spin-0 particle can be described by the equations

$$\frac{\partial \psi}{\partial x_\mu} = m\phi_\mu \;, \qquad \text{and} \qquad \frac{\partial \phi_\mu}{\partial x_\mu} = m\psi,^{(12)} \tag{3.1}$$

which imply that ψ satisfies the Klein-Gordon equation,

$$\frac{\partial^2 \psi}{\partial x_\mu \partial x_\mu} = m\frac{\partial \phi_\mu}{\partial x_\mu} = m^2\psi.$$

Using Eq. (3.1), a Hamiltonian equation of the form

$$\frac{\partial}{\partial x_4}\begin{pmatrix} \psi \\ \phi_4 \end{pmatrix} = (m\rho - \frac{\nabla^2}{2m}\sigma)\begin{pmatrix} \psi \\ \phi_4 \end{pmatrix}, \tag{3.2}$$

where

$$\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$

can be obtained for a spin-0 particle. Defining

$$\Psi = \begin{pmatrix} \psi \\ \phi_4 \end{pmatrix},$$

Eq. (3.2) has the form

$$\frac{\partial \Psi}{\partial x_4} = (m\rho - \frac{\nabla^2}{2m}\sigma)\Psi,$$

or

$$i\frac{\partial \Psi}{\partial t} = H\Psi \;,$$

where $H = \frac{\nabla^2}{2m}\sigma - m\rho$ is the Hamiltonian operator and

$$H^2 = (-\nabla^2 + m^2)I.$$

The Ψ as defined here has complicated tranformation properties.

B. Two Spin-zero Particles

The wave equation for two particles is

$$(H_1 + H_2 + V)\Psi = E\Psi$$

which, for two spin-0 particles, becomes, symbolically

$$\left[\frac{\nabla_1^2}{2m_1}\sigma_1 + \frac{\nabla_2^2}{2m_2}\sigma_2 - (m_1\rho_1 + m_2\rho_2) + V\right]\Psi(\vec{x}_1,\vec{x}_2)$$
$$= E\Psi(\vec{x}_1,\vec{x}_2), \tag{3.3}$$

where the two-particle wave function

$$\Psi(\vec{x}_1,\vec{x}_2) = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} ,$$

for a single particle $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi \\ \phi_4 \end{pmatrix}$,

with the first index associated with the first particle and the second index with the second particle. Eq. (3.3) has the following matrix form

$$\frac{\nabla_1^2}{2m_1}\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix}\begin{pmatrix}\Psi_{11} & \Psi_{12}\\ \Psi_{21} & \Psi_{22}\end{pmatrix} + \frac{\nabla_2^2}{2m_2}\begin{pmatrix}\Psi_{11} & \Psi_{12}\\ \Psi_{21} & \Psi_{22}\end{pmatrix}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}$$

$$-\left[m_1\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\Psi_{11} & \Psi_{12}\\ \Psi_{21} & \Psi_{22}\end{pmatrix} + m_2\begin{pmatrix}\Psi_{11} & \Psi_{12}\\ \Psi_{21} & \Psi_{22}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] + V\begin{pmatrix}\Psi_{11} & \Psi_{12}\\ \Psi_{21} & \Psi_{22}\end{pmatrix}$$

$$= E\begin{pmatrix}\Psi_{11} & \Psi_{12}\\ \Psi_{21} & \Psi_{22}\end{pmatrix}.$$

Equating each component of the matrix equation, we have

$$(E - V)\Psi_{11} + m_1\Psi_{21} + m_2\Psi_{12} = 0,$$

$$(E - V)\Psi_{22} - \frac{\nabla_1^2}{m_1}\Psi_{12} - \frac{\nabla_2^2}{m_2}\Psi_{21} + m_1\Psi_{12} + m_2\Psi_{21} = 0,$$

$$(E - V)\Psi_{12} - \frac{\nabla_2^2}{m_2}\Psi_{11} + m_1\Psi_{22} + m_2\Psi_{11} = 0,$$

$$(E - V)\Psi_{21} - \frac{\nabla_1^2}{m_1}\Psi_{11} + m_1\Psi_{11} + m_2\Psi_{22} = 0.$$

These four equations can be written in the following way:

$$\begin{pmatrix} E - V & m_2 & m_1 & 0 \\ -\frac{\nabla_2^2}{m_2} + m_2 & E - V & 0 & m_1 \\ -\frac{\nabla_1^2}{m_1} + m_1 & 0 & E - V & m_2 \\ 0 & -\frac{\nabla_1^2}{m_1} + m_1 & -\frac{\nabla_2^2}{m_2} + m_2 & E - V \end{pmatrix}\begin{pmatrix}\Psi_{11}\\ \Psi_{12}\\ \Psi_{21}\\ \Psi_{22}\end{pmatrix} = 0. \qquad (3.4)$$

C. Two Free Spin-zero Particles

We consider first the case of two free particles in which we put V = 0. Going to momentum space, we replace ∇_1^2 by $-p_1^2$ and ∇_2^2 by $-p_2^2$. In order that the solution be nontrivial, the determinant of the matrix must be zero, i.e., we must have

$$\begin{vmatrix} E & m_2 & m_1 & 0 \\ \frac{p_2^2}{m_2} + m_2 & E & 0 & m_1 \\ \frac{p_1^2}{m_1} + m_1 & 0 & E & m_2 \\ 0 & \frac{p_1^2}{m_1} + m_1 & \frac{p_2^2}{m_2} + m_2 & E \end{vmatrix} = 0.$$

Expanding the determinant and solving for E gives

$$E = \pm(\sqrt{p_1^2 + m_1^2} \pm \sqrt{p_2^2 + m_2^2}).$$

D. Two Spin-zero Particles with an Arbitrary Interaction

Next, we consider Eq. (3.4) in the CM frame in which $\nabla_1^2 = \nabla_2^2 = \nabla_x^2$ where $\vec{x} = \vec{x}_1 - \vec{x}_2$ is the relative coordinate. We suppress the letter x in ∇_x^2 in the following.

In the CM frame, Eq. (3.4) becomes

$$\begin{pmatrix} E - V & m_2 & m_1 & 0 \\ -\frac{\nabla^2}{m_2} + m_2 & E - V & 0 & m_1 \\ -\frac{\nabla^2}{m_1} + m_1 & 0 & E - V & m_2 \\ 0 & -\frac{\nabla^2}{m_1} + m_1 & -\frac{\nabla^2}{m_2} + m_2 & E - V \end{pmatrix} \begin{pmatrix} \Psi_{11} \\ \Psi_{12} \\ \Psi_{21} \\ \Psi_{22} \end{pmatrix} = 0 . \tag{3.5}$$

Eq. (3.5) can be simplified to obtain a single equation involving only Ψ_{11}. Multiplying the second equation by m_2 and the third equation by m_1 and adding, we have

$$(-2\nabla^2 + m_1^2 + m_2^2)\Psi_{11} + m_2(E - V)\Psi_{12} + m_1(E - V)\Psi_{21} + 2m_1m_2\Psi_{22} = 0 . \tag{3.6}$$

Substituting $m_2\Psi_{12} + m_1\Psi_{21}$ from the first of Eq. (3.5) into Eq. (3.6) gives

$$\left[-2\nabla^2 + m_1^2 + m_2^2 - (E - V)^2\right]\Psi_{11} + 2m_1m_2\Psi_{22} = 0 . \tag{3.7}$$

The first equation of (3.5) when added to the last gives

$$(E - V)(\Psi_{11} + \Psi_{22}) - \nabla^2\left(\frac{1}{m_1}\Psi_{12} + \frac{1}{m_2}\Psi_{21}\right) + (m_1 + m_2)\Psi_{12} + (m_1 + m_2)\Psi_{21} = 0 . \tag{3.8}$$

From the first, second and third equations of (3.5), we have

$$\frac{1}{m_1}\Psi_{12} + \frac{1}{m_2}\Psi_{21} = -\frac{(E - V)}{m_1 m_2}\Psi_{11},$$

$$\Psi_{12} = \frac{1}{(E - V)}\left(\left(\frac{\nabla^2}{m_2} - m_2\right)\Psi_{11} - m_1\Psi_{22}\right), \qquad (3.9)$$

and $$\Psi_{21} = \frac{1}{E - V}\left(\left(\frac{\nabla^2}{m_1} - m_1\right)\Psi_{11} - m_2\Psi_{22}\right).$$

Substituting (3.9) into (3.8), we obtain

$$\left[(E - V) + \frac{1}{m_1 m_2}\nabla^2(E - V) + \frac{m_1+m_2}{E - V}\left(\frac{\nabla^2}{m_2} - m_2 + \frac{\nabla^2}{m_1} - m_1\right)\right]\Psi_{11}$$

$$+ \left[(E - V) - \frac{(m_1+m_2)^2}{E - V}\right]\Psi_{22} = 0. \qquad (3.10)$$

Substituting Ψ_{22} from (3.7) into (3.10) gives

$$\left[(E - V) + \frac{1}{m_1 m_2}\nabla^2(E - V) + \frac{m_1+m_2}{E - V}\left(\frac{\nabla^2}{m_2} - m_2 + \frac{\nabla^2}{m_1} - m_1\right)\right.$$

$$\left.+ \frac{(E - V)^2 - (m_1 + m_2)^2}{2m_1 m_2(E - V)}\left(2\nabla^2 - (m_1^2 + m_2^2) + (E - V)^2\right)\right]\Psi_{11} = 0,$$

which reduces to

$$\left[\nabla^2 - (m_1^2 + m_2^2) + \frac{(E - V)^2}{2} + \frac{(m_1 + m_2)^2(m_1 - m_2)^2}{2(E - V)^2}\right.$$

$$\left.+ \frac{1}{E - V}\nabla^2(E - V)\right]\Psi_{11} = 0. \qquad (3.11)$$

Ψ_{11} corresponds to the usual Klein-Gordon wave function for two particles.

E. Two Spin-zero Particles with a Square Well Interaction

For a square well potential,

$$V(r) = -V_0 \qquad \text{for } r \leq a$$

$$= 0 \qquad \text{for } r > a \,.$$

For $r \leq a$, Eq. (3.11) becomes

$$\left(\nabla^2 + \frac{(m_1 + m_2)^2(m_1 - m_2)^2}{4(E + V_0)^2} - \frac{(m_1^2 + m_2^2)}{2} + \frac{(E + V_0)^2}{4}\right)\Psi_{11} = 0 \,. \tag{3.12}$$

Putting $\Psi_{11} = f_{11}(r)Y_{\ell m}(\theta,\phi)$ in Eq. (3.12), the radial equation is

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + K^2 - \frac{\ell(\ell+1)}{r^2}\right)f_{11}(r) = 0 \tag{3.13}$$

where

$$K^2 = \frac{(m_1 + m_2)^2(m_1 - m_2)^2}{4(E + V_0)^2} - \frac{(m_1^2 + m_2^2)}{2} + \frac{(E + V_0)^2}{4} \,.$$

The solution of Eq. (3.13) is

$$f_{11}(r) = A\, j_\lambda(Kr) \,.$$

The other components of the wave function can be obtained from (3.7) and (3.9)

$$\Psi_{22} = \frac{A}{m_1m_2}\left(\frac{(E+V_0)^2}{4} - \frac{(m_1+m_2)^2(m_1-m_2)^2}{4(E+V_0)^2}\right)j_\ell(Kr)Y_{\ell m}(\theta,\phi),$$

$$\Psi_{12} = \frac{-A}{m_2}\left(\frac{(E+V_0)}{2} - \frac{(m_1^2-m_2^2)}{2(E+V_0)}\right)j_\ell(Kr)Y_{\ell m}(\theta,\phi),$$

$$\Psi_{21} = \frac{-A}{m_1}\left(\frac{(E+V_0)}{2} - \frac{(m_2^2-m_1^2)}{2(E+V_0)}\right)j_\ell(Kr)Y_{\ell m}(\theta,\phi).$$

For $r > a$, Eq. (3.11) reduces to

$$\left(\nabla^2 + \frac{(m_1+m_2)^2(m_1-m_1)^2}{4E^2} - \frac{(m_1^2+m_2^2)}{2} + \frac{E^2}{4}\right)\Psi_{11} = 0. \quad (3.14)$$

Putting $\Psi_{11} = g_{11}(r)Y_{\ell m}(\theta,\phi)$, the radial equation becomes

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \alpha^2 - \frac{\ell(\ell+1)}{r^2}\right)g_{11}(r) = 0 \quad (3.15)$$

where $\alpha^2 = \dfrac{(m_1+m_2)^2(m_1-m_2)^2}{4E^2} - \dfrac{(m_1^2+m_2^2)}{2} + \dfrac{E^2}{4}$.

The solution of Eq. (3.15) is

$$g_{11}(r) = A'h_\ell^{(1)}(\alpha r).$$

Again, other components of the wave function can be obtained from (3.7) and (3.9)

$$\Psi_{22} = \frac{A'}{m_1m_2}\left(\frac{E^2}{4} - \frac{(m_1+m_2)^2(m_1-m_2)^2}{4E^2}\right)h_\ell^{(1)}(\alpha r)Y_{\ell m}(\theta,\phi),$$

$$\Psi_{12} = \frac{-A'}{m_2}\left(\frac{E}{4} - \frac{(m_1^2-m_2^2)}{2E}\right)h_\ell^{(1)}(\alpha r)Y_{\ell m}(\theta,\phi),$$

$$\Psi_{21} = \frac{-A'}{m_1}\left(\frac{E}{4} - \frac{(m_2^2-m_1^2)}{2E}\right)h_\ell^{(1)}(\alpha r)Y_{\ell m}(\theta,\phi).$$

To determine the energy levels, we match the solution inside and outside the well at r = a. If we match Ψ_{11} and the first derivative of Ψ_{11} at r = a, we have

$$A j_l(Ka) = A' h_l^{(1)}(\alpha a),$$

and

$$K A j_l'(Ka) = \alpha A' h_l^{(1)'}(\alpha a) .$$

Dividing one by the other, we obtain

$$\frac{j_l(Ka)}{K j_l'(Ka)} = \frac{h_l^{(1)}(\alpha a)}{\alpha h_l^{(1)'}(\alpha a)}$$

which will give the energy levels. If we match other components of the wave function at r = a, then, for $r \leqslant a$, the wave function contains in the coefficient of the spherical Bessel function a term involving $E + V_0$ whereas for $r > a$ a term involving E. This is due to the fact that the potential changes discontinuously at r = a. For a more realistic potential given by Figure 2,

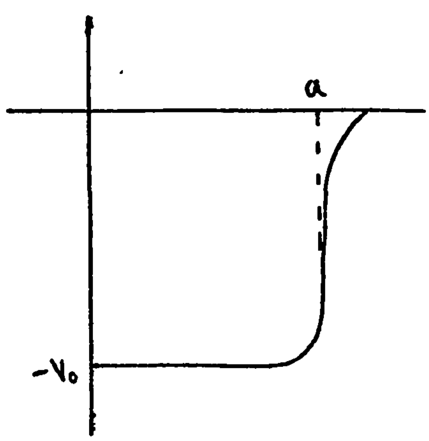


Figure 2

the potential is continuous at $r = a$. The value of the limit of the potential from both sides is the same and so matching any component of the wave function will give the same result.

F. Two Spin-zero Particles with a Coulomb Interaction

In this section, we apply our two spin-0 particle wave equation to describe a pionic atom which consists of a negatively charged pion circulating around a positively charged pion. For this system, $m_1 = m_2 = m$ and the potential between the two pions is Coulomb given by $V(r) = -\frac{e^2}{r}$.

Eq. (3.11) becomes in this case

$$\left(\nabla^2 - 2m^2 + \frac{(E + \frac{e^2}{r})^2}{2} + \frac{E}{E + \frac{\ell^2}{r}}\nabla^2\right)\Psi_{11} + \frac{\nabla^2(\frac{\ell^2}{r}\Psi_{11})}{E + \frac{\ell^2}{r}} = 0. \tag{3.16}$$

The last term in Eq. (3.16) can be simplified. We consider the region $r \neq 0$, because the potential is singular at the origin. For this region,

$$\nabla^2\left(\frac{1}{r}\Psi_{11}\right) = \frac{1}{r}\nabla^2\Psi_{11} - \frac{2}{r^2}\frac{\partial}{\partial r}\Psi_{11}.$$

Therefore, Eq. (3.16) becomes

$$\left(\nabla^2 - m^2 + \frac{(E + \frac{\ell^2}{r})^2}{4} - \frac{\frac{\ell^2}{r^2}\frac{\partial}{\partial r}}{E + \frac{\ell^2}{r}}\right)\Psi_{11} = 0.$$

Putting $\Psi_{11} = f_{11}(r)Y_{\ell m}(\theta,\phi)$, we have, for the radial equation,

$$\left\{\frac{d^2}{dr^2} + \frac{1}{r}\left(2 - \frac{\frac{e^2}{r}}{E + \frac{e^2}{r}}\right)\frac{d}{dr} + \frac{(E + \frac{e^2}{r})^2}{4} - \frac{\ell(\ell+1)}{r^2} - m^2\right\}f_{11}(r) = 0; \tag{3.17}$$

but $$-\frac{e^2}{r(Er + e^2)} = -\frac{1}{r} + \frac{1}{r + \frac{e^2}{E}}.$$

Therefore, Eq. (3.17) becomes

$$\left\{\frac{d^2}{dr^2} + \left(\frac{1}{r} + \frac{1}{r + \frac{e^2}{E}}\right)\frac{d}{dr} + \left(\frac{E^2}{4} - m^2\right) + \frac{Ee^2}{2r} + \frac{\frac{e^4}{4} - \ell(\ell+1)}{r^2}\right\}f_{11}(r)$$

$$= 0. \tag{3.18}$$

Now we consider the ratio $\frac{e^2}{E}$. e^2 is of the order of 10^{-20} statcoulomb^2; E is of the order of the mass of a pion and is 10^2 Mev or 10^{-4} erg. Therefore, the ratio $\frac{e^2}{E}$ is of the order of 10^{-16} cm, i.e.

$$\frac{e^2}{E} \sim 10^{-16} << 10^{-13} = \text{radius of electron}.$$

Eq. (3.18) has no exact solution because of the presence of the term $\frac{1}{r + e^2/E}$. However, since $\frac{e^2}{E}$ << radius of electron, we can neglect $\frac{e^2}{E}$ in the term $\frac{1}{r + e^2/E}$ and approximate it to be $\frac{1}{r}$. With this approximation, Eq. (3.18) takes the form

$$\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + (-K^2 + \frac{Ee^2}{2r} + \frac{\alpha^2}{r^2}) f_{11}(r) = 0, \quad (3.19)$$

where $K^2 = m^2 - \frac{E^2}{4}$

and $\alpha^2 = \frac{e^4}{4} - \ell(\ell+1)$.

Putting $f_{11}(r) = g_{11}(r)e^{-Kr}$, the equation for $g_{11}(r)$ is

$$\left[\frac{d^2}{dr^2} + (\frac{2}{r} - 2K)\frac{d}{dr} + (\frac{Ee^2}{2r} - \frac{2K}{r}) + \frac{\alpha^2}{r^2} \right] g_{11}(r) = 0. \quad (3.20)$$

Solutions of $g_{11}(r)$ can be obtained using power series method. We put

$$g_{11}(r) = r^s \sum_{n=0}^{\infty} a_n r^n$$

$$= r^s L(r). \quad (3.21)$$

Substituting (3.21) into (3.20) results in an equation for $L(r)$,

$$L''(r) + (-2K + \frac{2s + 2}{r})L'(r) + \left(\frac{\frac{Ee^2}{2} - 2K(s+1)}{r} + \frac{s(s+1) + \alpha^2}{r^2} \right) L(r)$$

$$= 0. \quad (3.22)$$

The indicial equation for s from (3.20) and (3.21) is

$$s(s-1) + 2s + \alpha^2 = 0. \quad (3.23)$$

The solution is

$$s = \frac{-1 \pm \sqrt{1 - 4\alpha^2}}{2}$$

$$= \frac{-1 \pm \sqrt{(2\ell+1)^2 - e^4}}{2} .$$

For $\ell > 0$, s is positive or negative according to our choice of the sign of the radical. For $\ell = 0$, since $e^4 = \frac{1}{(137)^2} \ll 1$, the choice of the upper sign gives a value of s close to zero. Therefore, for all values of ℓ, we choose the upper sign, so that $g_{11}(r)$ will not diverge when r is near the origin.

From (3.20) and (3.21), the recursion relation between the a_n is

$$a_{n+1} = \frac{2\kappa(n+s+1) - \frac{Ee^2}{2}}{(n+s+1)(n+s+2)+\alpha^2} a_n .$$

As $n \rightarrow \infty$, the ratio $\frac{a_{n+1}}{a_n} \rightarrow \frac{2\kappa}{n}$ and $L(r) \rightarrow e^{2\kappa r}$.

Therefore, the wave function will diverge for large r. The series must be made to terminate. So we have

$$2\kappa(n+s+1) = \frac{Ee^2}{2};$$

or

$$\kappa^2 = m^2 - \frac{E^2}{4} = \frac{E^2 e^4}{16}(n+s+1)^{-2} .$$

This gives $E = 2m\left[1 + \frac{e^4}{4}(n+s+1)^{-2}\right]^{-\frac{1}{2}}$

$$= 2m\left[1 + \frac{e^4}{4}\left(n+\tfrac{1}{2}+\tfrac{1}{2}(2\ell+1) - \frac{e^4}{4(2\ell+1)}\right)^{-2}\right]^{-\frac{1}{2}}$$

$$= 2m\left[1 - \frac{e^4}{8n'^2} + \frac{e^8}{16n'^3}\left(\frac{3}{8n'} - \frac{1}{(2\ell+1)}\right) + \cdots\right] \qquad (3.24)$$

to terms of order e^8, where $n' = n + \ell + 1$ is the principal quantum number and can take on positive integral values. The reduced mass of the system is $\mu = \frac{m}{2}$ in terms of which (3.24) becomes

$$E = 2m - \frac{\mu e^4}{2n^2} + \frac{\mu e^8}{4n^3}\left(\frac{3}{8n} - \frac{1}{(2\ell+1)}\right) \quad \cdots$$

in which we omit the prime in the n. The first term of the right side is the rest mass of the system. The second term is identical with the one for nonrelativistic particles. The third term is the fine-structure energy which differs from usual K-G and Dirac one particle levels.

Eq. (3.22) can be solved exactly. Using (3.23) and putting $\rho = 2\chi r$, Eq. (3.22) becomes

$$L''(\rho) + \left(-1 + \frac{2s+2}{\rho}\right)L'(\rho) + \frac{\frac{Ee^2}{4\chi} - (s+1)}{\rho}L(\rho) = 0. \qquad (3.25)$$

The associated Laquerre polynomial $L_q^p(\rho)$ satisfies the associated Laquerre equation

$$L_q^{p\,''}(\rho) + \left(-1 + \frac{p+1}{\rho}\right)L_q^{p\,'}(\rho) + \frac{(q-p)}{\rho}L_q^p(\rho) = 0. \qquad (3.26)$$

Comparing Eqs. (3.25) and (3.26), we have

$$p = 2s + 1$$

$$q = \frac{Ee^2}{4\chi} + s\,,$$

therefore $\Psi_{11} = \exp(-\chi r) r^s L^{2s+1}_{\frac{Ee^2}{4\chi}+s}(\rho) Y_{\ell m}(\theta,\phi)$. Other components of the wave function can be found from (3.7), (3.9) and (3.19). The complete wave function is

$$\Psi = C \begin{pmatrix} \exp(-\chi r) r^s L^{2s+1}_{\frac{Ee^2}{4\chi}+s}(\rho) Y_{\ell m}(\theta,\phi) \\ -\frac{1}{2m}(E+\frac{e^2}{r}) \exp(-\chi r) r^s L^{2s+1}_{\frac{Ee^2}{4\chi}+s}(\rho) Y_{\ell m}(\theta,\phi) \\ -\frac{1}{2m}(E+\frac{e^2}{r}) \exp(-\chi r) r^s L^{2s+1}_{\frac{Ee^2}{4\chi}+s}(\rho) Y_{\ell m}(\theta,\phi) \\ \frac{(E+\frac{e^2}{r})^2}{4m^2} \exp(-\chi r) r^s L^{2s+1}_{\frac{Ee^2}{4\chi}+s}(\rho) Y_{\ell m}(\theta,\phi) \end{pmatrix}.$$

G. Scattering of Two Spin-zero Particles

In this section, we consider the scattering of two spin-0 particles using a repulsive square well

$$V = V_0\,, \qquad \text{for } r \leq a\,,$$

$$= 0\,, \qquad \text{for } r > a\,.$$

The equation for Ψ_{11} for the two regions can be written similarly to Eqs. (3.13) and (3.14). The solutions are

$$A j_\ell(K'r), \qquad \text{for } r \leq a\,,$$

$$B\big(j_\ell(\alpha r) - \tan\delta_\ell\, \eta_\ell(\alpha r)\big), \quad \text{for } r > a\,,$$

$$\text{where } K'^2 = \frac{(m_1 + m_2)^2(m_1 - m_2)^2}{4(E - V_0)^2} - \frac{(m_1^2 + m_2^2)}{2} + \frac{(E - V_0)^2}{4}$$

$$\text{and} \quad \alpha^2 = \frac{(m_1 + m_2)^2(m_1 - m_2)^2}{4E^2} - \frac{(m_1^2 + m_2^2)}{2} + \frac{E^2}{4}.$$

Imposing the continuity conditions gives

$$A j_\ell(K'a) = B\left[j_\ell(\alpha a) - \tan\delta_\ell\, \eta_\ell(\alpha a)\right];$$

$$\text{and} \quad AK' j_\ell'(K'a) = \alpha B\left[j_\ell'(\alpha a) - \tan\delta_\ell\, \eta_\ell'(\alpha a)\right].$$

On dividing, we obtain

$$\frac{K' j_\ell'(K'a)}{j_\ell(K'a)} = \frac{\alpha\left(j_\ell'(\alpha a) - \tan\delta_\ell\, \eta_\ell'(\alpha a)\right)}{j_\ell(\alpha a) - \tan\delta_\ell\, \eta_\ell(\alpha a)}.$$

Solving for $\tan\delta_\ell$ gives

$$\tan\delta_\ell = \frac{\alpha j_\ell(K'a) j_\ell'(\alpha a) - K' j_\ell'(K'a) j_\ell(\alpha a)}{\alpha \eta_\ell'(\alpha a) j_\ell(K'a) - K' j_\ell'(K'a)\eta_\ell(\alpha a)}. \qquad (3.27)$$

Using the identity

$$\frac{d}{d\rho} j_\ell(\rho) = \frac{1}{2\ell + 1}\left[\ell j_{\ell-1}(\rho) - (\ell + 1) j_{\ell+1}(\rho)\right],$$

which is also true for $\eta_\ell(\rho)$, (3.27) becomes

$$\tan\delta_\ell = \frac{\alpha j_\ell(K'a)\left[\ell j_{\ell-1}(\alpha a) - (\ell+1) j_{\ell+1}(\alpha a)\right] - K' j_\ell'(K'a) j_\ell(\alpha a)(2\ell+1)}{\alpha j_\ell(K'a)\left[\ell \eta_{\ell-1}(\alpha a) - (\ell+1)\eta_{\ell+1}(\alpha a)\right] - K' j_\ell'(K'a)\eta_\ell(\alpha a)(2\ell+1)}.$$

We assume that $|m_1 - m_2| \ll m$ and a is of the order of 0.1 fermi so that $\alpha a \ll 1$ (check: for E = 600 Mev, a = 0.1 fermi, $\alpha a = 0.091$) We use the following properties

$$j_\ell(\rho) \xrightarrow[\rho\to 0]{} \frac{\rho^\ell}{1.3.5 \ldots (2\ell + 1)},$$

$$\eta_\ell(\rho) \xrightarrow[\rho\to 0]{} \frac{(-1).\ 1.3.5. \ldots (2\ell-1)}{\rho^{\ell+1}},$$

$$\text{and} \quad j_{\ell+1}(\rho) = -j_\ell'(\rho) + \frac{\ell}{\rho} j_\ell(\rho)$$

to get

$$\tan \delta_\ell = \frac{(\alpha a)^{2\ell+1}}{((-1)\ 1.3.5.\ .\ .(2\ell+1))(1.3.5.\ .\ .(2\ell-1))}$$

$$\times \frac{j_{\ell+1}(K'a) - \frac{(\ell+1)(\alpha a)^2 j_\ell(K'a)}{(2\ell+3)(2\ell+1)K'a}}{j_{\ell+1}(K'a) + \left(\frac{\ell(\alpha a)^2}{(2\ell-1)(2\ell+1)K'a} - \frac{2\ell+1}{K'a}\right) j_\ell(K'a)} . \qquad (3.28)$$

The resonance condition is $\delta_\ell = \frac{\pi}{2}$ or $\tan \delta_\ell = \infty$. This occurs when the denominator of (3.28) vanishes, i.e.

$$\frac{j_{\ell+1}(K'a)}{j_\ell(K'a)} = \frac{2\ell+1}{K'a} - \frac{\ell}{(2\ell-1)(2\ell+1)} \frac{(\alpha a)^2}{K'a} .$$

For an arbitrary central potential, the condition of resonant scattering can be obtained through phase shift analysis. Just as in the nonrelativistic case, we require that $f_{11}(\alpha r)$ behaves asymptotically as a free particle, i.e.

$$f_{11}(\alpha r) \xrightarrow[r\to\infty]{} j_\ell(\alpha r) + \tan \delta_\ell\, \eta_\ell(\alpha r)$$

$$\sim \frac{1}{\alpha r} \sin(\alpha r - \frac{\pi \ell}{2} - \delta_\ell), \qquad (3.29)$$

so that the main effect of the potential is to shift the phase. By putting $\Psi_{11} = f_{11}(r) Y_{1m}(\theta,\phi)$ in Eq. (3.11), the equation that $f_{11}(\alpha r)$ satisfies is

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \alpha^2 + \frac{(m_1+m_2)^2(m_1-m_2)^2}{4(E - V)^2} - \frac{(m_1+m_2)^2(m_1-m_2)^2}{4E^2} \right.$$

$$\left. + \frac{V^2}{4} - \frac{EV}{2} + \mathcal{U}(r) - \frac{\ell(\ell+1)}{r^2} \right] f_{11}(r) = 0 , \qquad (3.30)$$

where we put

$$\frac{1}{E - V}\nabla^2(E - V)\Psi_{11} = \nabla^2\Psi_{11} + 2\mathcal{U}(r)\Psi_{11}.$$

The $\mathcal{U}(r)$ here is an operator on Ψ_{11} and is not only a function. We define another operator,

$$-U(r) = \frac{(m_1+m_2)^2(m_1-m_2)^2}{4(E - V)^2} - \frac{(m_1+m_2)^2(m_1-m_2)^2}{4E^2} + \frac{V^2}{4} - \frac{EV}{2} + \mathcal{U}(r),$$

so that Eq. (3.30) becomes

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \alpha^2 - \frac{\ell(\ell+1)}{r^2} - U(r)\right] f_{11}(\alpha r) = 0. \tag{3.31}$$

We let the solution be

$$f_{11}(\alpha r) = j_\ell(\alpha r) + \chi_\ell(\alpha r), \tag{3.32}$$

where $\chi_\ell(\alpha r)$ is assumed to be small compared with $j_\ell(\alpha r)$. Substituting (3.32) into (3.31) gives

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \alpha^2 - \frac{\ell(\ell+1)}{r^2}\right] \chi_\ell(\alpha r) = U(r) j_\ell(\alpha r),$$

where we drop the small term $U(r)\chi_\ell(\alpha r)$. This corresponds to Born's Approximation. The Green's function for the operator

$$\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \alpha^2 - \frac{\ell(\ell+1)}{r^2},$$

is $g_\ell(r,r') = \alpha j_\ell(\alpha r_<)\eta_\ell(\alpha r_>)$ where $r_<(r_>)$ is the smaller (larger) of r and r' so that

$$\chi_\ell(\alpha r) = \alpha \int_0^\infty j_\ell(\alpha r_<)\eta_\ell(\alpha r_>)U(r')j_\ell(\alpha r')r'^2 dr'$$

$$\underset{r\to\infty}{=} \alpha\, \eta_\ell(\alpha r)\int_0^\infty j_\ell(\alpha r')U(r')j_\ell(\alpha r')r'^2 dr'. \tag{3.33}$$

Comparing (3.29) with (3.32) and (3.33), we have

$$\tan \delta_\ell = \alpha \int_0^\infty j_\ell(\alpha r) U(r) j_\ell(\alpha r) r^2 dr \ . \tag{3.34}$$

The condition for resonant scattering is obtained when $\delta_\ell = \frac{\pi}{2}$. Eq. (3.34) is similar to the nonrelativistic case except that here $U(r)$ is an operator and is completely different from the corresponding $U(r)$ in the nonrelativistic case.

CHAPTER IV

TWO SPIN-HALF PARTICLES

A. Wave Equation for Two Spin-half Particles

The single spin-half particle wave equation is the Dirac Equation

$$i\frac{\partial\psi}{\partial t} = (-i\nabla.\vec{\alpha} + \beta m)\psi ,$$

in which

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \text{and} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

where the $\vec{\sigma}$ are the 2x2 Pauli matrices and I is the 2x2 unit matrix. The Dirac Equation is already in Hamiltonian form. The two-particle wave equation for spin-$\frac{1}{2}$ particles is

$$\left[-i(\nabla_1 \cdot \vec{\alpha}_1 + \nabla_2 \cdot \vec{\alpha}_2) + (m_1\beta_1 + m_2\beta_2)\right]_{ab,cd} \psi_{cd}(\vec{x}_1,\vec{x}_2)$$

$$+ V_{ab,cd}(|\vec{x}_1-\vec{x}_2|)\psi_{cd}(\vec{x}_1,\vec{x}_2) = E\psi_{ab}(\vec{x}_1,\vec{x}_2) \;. \;. \;. , \qquad (4.1)$$

which is sometimes called the Breit Equation.[2] One deviation is that in Breit's original paper, the interaction V is replaced by the so-called Breit interaction.

In the CM frame, Eq. (4.1) becomes

$$\left[-i\nabla\cdot(\vec{\alpha}_1 - \vec{\alpha}_2) + m_1\beta_1 + m_2\beta_2\right]_{ab,cd}\psi_{cd}(\vec{x}) + V_{ab,cd}(x)\psi_{cd}(\vec{x})$$

$$= E\psi_{ab}(\vec{x}). \tag{4.2}$$

The wave function ψ in spinor form can be written as

$$\psi = \begin{bmatrix} \chi_{\dot{\alpha}_1\dot{\alpha}_2} \\ \xi_{\dot{\alpha}_1}{}^{\alpha_2} \\ \eta^{\alpha_1}{}_{\dot{\alpha}_2} \\ \phi^{\alpha_1\alpha_2} \end{bmatrix}. \tag{4.3}$$

The interaction is taken to be

$$V_{ab,cd} = V(I)_{ab,cd}.$$

Putting (4.3) into (4.2), we obtain four equations for the four components of ψ,

$$E\chi_{\dot{\alpha}_1\dot{\alpha}_2} = -i\nabla\cdot(\vec{\sigma})_{\alpha_1\beta_1}\eta^{\beta_1}{}_{\dot{\alpha}_2} + i\nabla\cdot(\vec{\sigma})_{\alpha_2\beta_2}\xi_{\dot{\alpha}_1}{}^{\beta_2} + (m_1 + m_2 + V)\chi_{\dot{\alpha}_1\dot{\alpha}_2},$$

$$E\xi_{\dot{\alpha}_1}{}^{\alpha_2} = -i\nabla\cdot(\vec{\sigma})_{\alpha_1\beta_1}\phi^{\beta_1\alpha_2} + i\nabla\cdot(\vec{\sigma})_{\alpha_2\beta_2}\chi_{\dot{\alpha}_1\dot{\beta}_2} + (m_1 - m_2 + V)\xi_{\dot{\alpha}_1}{}^{\alpha_2},$$

$$E\eta^{\alpha_1}{}_{\dot{\alpha}_2} = -i\nabla\cdot(\vec{\sigma})_{\alpha_1\beta_1}\chi_{\dot{\beta}_1\dot{\alpha}_2} + i\nabla\cdot(\vec{\sigma})_{\alpha_2\beta_2}\phi^{\alpha_1\beta_2} + (-m_1 + m_2 + V)\eta^{\alpha_1}{}_{\dot{\alpha}_2},$$

$$E\phi^{\alpha_1\alpha_2} = -i\nabla\cdot(\vec{\sigma})_{\alpha_1\beta_1}\xi_{\dot{\beta}_1}{}^{\alpha_2} + i\nabla\cdot(\vec{\sigma})_{\alpha_2\beta_2}\eta^{\alpha_1}{}_{\dot{\beta}_2} + (-m_1 - m_2 + V)\phi^{\alpha_1\alpha_2}.$$

If we let $M = m_1 + m_2$ and $\Delta = m_1 - m_2$, we have

$$
\begin{aligned}
(E - M - V)\chi_{\dot\alpha_1\dot\alpha_2} &= i\nabla\cdot\left[(\vec\sigma)_{\alpha_2\beta_2}\xi_{\dot\alpha_1}{}^{\beta_2} - (\vec\sigma)_{\alpha_1\beta_1}\eta^{\beta_1}{}_{\dot\alpha_2}\right],\\
(E - \Delta - V)\xi_{\dot\alpha_1}{}^{\alpha_2} &= i\nabla\cdot\left[(\vec\sigma)_{\alpha_2\beta_2}\chi_{\dot\alpha_1\dot\beta_2} - (\vec\sigma)_{\alpha_1\beta_1}\phi^{\beta_1\alpha_2}\right],\\
(E + \Delta - V)\eta^{\alpha_1}{}_{\dot\alpha_2} &= i\nabla\cdot\left[(\vec\sigma)_{\alpha_2\beta_2}\phi^{\alpha_1\beta_2} - (\vec\sigma)_{\alpha_1\beta_1}\chi_{\dot\beta_1\dot\alpha_2}\right],\\
(E + M - V)\phi^{\alpha_1\alpha_2} &= i\nabla\cdot\left[(\vec\sigma)_{\alpha_2\beta_2}\eta^{\alpha_1}{}_{\dot\beta_2} - (\vec\sigma)_{\alpha_1\beta_1}\xi_{\beta_1}{}^{\alpha_2}\right].
\end{aligned}
\tag{4.4}
$$

Eqs. (4.4) are written in the CM frame, so the states may be classified by their transformation properties under rotations. For purposes of rotations, the lower $\dot\alpha$ indices are equivalent to the upper α indices. So we let

$$\chi_{\dot\alpha_1\dot\alpha_2} \longrightarrow \chi^{\alpha_1\alpha_2},\ \xi_{\dot\alpha_1}{}^{\alpha_2} \longrightarrow \xi^{\alpha_1\alpha_2},\ \eta^{\alpha_1}{}_{\dot\alpha_2} \longrightarrow \eta^{\alpha_1\alpha_2} \text{ and } \phi^{\alpha_1\alpha_2} \longrightarrow \phi^{\alpha_1\alpha_2}.$$

Each of the above spinors has a symmetric and an antisymmetric part, that is,

$$\chi^{\alpha_1\alpha_2} = \chi_s^{\alpha_1\alpha_2} + \epsilon^{\alpha_1\alpha_2}\chi,$$

where $\epsilon^{\alpha_1\alpha_2} = (i\sigma_2)_{\alpha_1\alpha_2}$, etc..

Substituting these into(4.4), we have

$$(E - M - V)(\chi_s^{\alpha_1\alpha_2} + \epsilon^{\alpha_1\alpha_2}\chi) = i\nabla\cdot((\vec{\sigma})_{\alpha_2\beta_2}\xi_s^{\alpha_1\beta_2} - (\vec{\sigma})_{\alpha_1\beta_1}\eta_s^{\beta_1\alpha_2})$$
$$+ i\nabla\cdot(-i(\vec{\sigma}\sigma_2)_{\alpha_2\alpha_1}\xi - i(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2}\eta),$$

$$(E - \Delta - V)(\xi_s^{\alpha_1\alpha_2} + \epsilon^{\alpha_1\alpha_2}\xi) = i\nabla\cdot((\vec{\sigma})_{\alpha_2\beta_2}\chi_s^{\alpha_1\beta_2} - (\vec{\sigma})_{\alpha_1\beta_1}\phi_s^{\beta_1\alpha_2})$$
$$+ i\nabla\cdot(-i(\vec{\sigma}\sigma_2)_{\alpha_2\alpha_1}\chi - i(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2}\phi), \qquad (4.5)$$

$$(E + \Delta - V)(\eta_s^{\alpha_1\alpha_2} + \epsilon^{\alpha_1\alpha_2}\eta) = i\nabla\cdot((\vec{\sigma})_{\alpha_2\beta_2}\phi_s^{\alpha_1\beta_2} - (\vec{\sigma})_{\alpha_1\beta_1}\chi_s^{\beta_1\alpha_2})$$
$$+ i\nabla\cdot(-i(\vec{\sigma}\sigma_2)_{\alpha_2\alpha_1}\phi - i(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2}\chi),$$

$$(E + M - V)(\phi_s^{\alpha_1\alpha_2} + \epsilon^{\alpha_1\alpha_2}\phi) = i\nabla\cdot((\vec{\sigma})_{\alpha_2\beta_2}\eta_s^{\alpha_1\beta_2} - (\vec{\sigma})_{\alpha_1\beta_1}\xi_s^{\beta_1\alpha_2})$$
$$+ i\nabla\cdot(-i(\vec{\sigma}\sigma_2)_{\alpha_2\alpha_1}\eta - i(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2}\xi).$$

By separating the antisymmetric part from the symmetric part, the four equations in (4.5) reduce to eight. We will drop the subscript s in the following.

The symmetric part of the first equation in (4.5) is

$$(E - M - V)\chi^{\alpha_1\alpha_2} = \tfrac{1}{2}\nabla\cdot((\vec{\sigma})_{\alpha_2\beta_2}\delta_{\alpha_1\beta_1} + (\vec{\sigma})_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2})(\xi^{\beta_1\beta_2} - \eta^{\beta_1\beta_2})$$
$$+ \nabla\cdot(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2}(\xi + \eta), \qquad (4.6)$$

where we make use of the fact that

$$(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2} = (\vec{\sigma}\sigma_2)_{\alpha_2\alpha_1}.$$

The antisymmetric part is

$$(E - M - V)\epsilon^{\alpha_1\alpha_2}\chi = \tfrac{1}{2}\nabla\cdot\left[\delta_{\alpha_1\beta_1}(\vec{\sigma})_{\alpha_2\beta_2} - \delta_{\alpha_2\beta_2}(\vec{\sigma})_{\alpha_1\beta_1}\right]\left[\xi^{\beta_1\beta_2} + \eta^{\beta_1\beta_2}\right]. \tag{4.7}$$

Multiplying Eq. (4.7) on the left by $(i\sigma_2)_{\alpha_1\alpha_2}$, we have

$$(E - M - V)\chi = -\tfrac{1}{4}\nabla\cdot\left[(\sigma_2\vec{\sigma})_{\beta_1\beta_2} + (\sigma_2\vec{\sigma})_{\beta_2\beta_1}\right]\left[\xi^{\beta_1\beta_2} + \eta^{\beta_1\beta_2}\right]$$

$$= -\tfrac{1}{2}\nabla\cdot(\sigma_2\vec{\sigma})_{\beta_1\beta_2}(\xi^{\beta_1\beta_2} + \eta^{\beta_1\beta_2}).$$

We can obtain a pair of equation from each of the second, third and fourth equations of (4.5) by making the following substitutions into Eqs. (4.6) and (4.7):

Equation	Substitution
second	$M \longrightarrow \Delta$, $\chi \longrightarrow \xi$, $\xi \longrightarrow \chi$, $\eta \longrightarrow \phi$
third	$M \longrightarrow -\Delta$, $\chi \longrightarrow \eta$, $\xi \longrightarrow \phi$, $\eta \longrightarrow \chi$
fourth	$M \longrightarrow -M$, $\chi \longrightarrow \phi$, $\xi \longrightarrow \eta$, $\eta \longrightarrow \xi$

If we let $\vec{S}_{\alpha_1\alpha_2,\beta_1\beta_2} = \frac{1}{2}\left((\vec{\sigma})_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2} + \delta_{\alpha_1\beta_1}(\vec{\sigma})_{\alpha_2\beta_2}\right)$, then the eight equations from (4.5) are

$$(E - M - V)\chi^{\alpha_1\alpha_2} = i\nabla\cdot\vec{S}_{\alpha_1\alpha_2,\beta_1\beta_2}(\xi^{\beta_1\beta_2} - \eta^{\beta_1\beta_2}) + \nabla\cdot(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2}(\xi+\eta),$$

$$(E - \Delta - V)\xi^{\alpha_1\alpha_2} = i\nabla\cdot\vec{S}_{\alpha_1\alpha_2,\beta_1\beta_2}(\chi^{\beta_1\beta_2} - \phi^{\beta_1\beta_2}) + \nabla\cdot(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2}(\chi+\phi),$$

$$(E + \Delta - V)\eta^{\alpha_1\alpha_2} = i\nabla\cdot\vec{S}_{\alpha_1\alpha_2,\beta_1\beta_2}(\phi^{\beta_1\beta_2} - \chi^{\beta_1\beta_2}) + \nabla\cdot(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2}(\chi+\phi),$$

$$(E + M - V)\phi^{\alpha_1\alpha_2} = i\nabla\cdot\vec{S}_{\alpha_1\alpha_2,\beta_1\beta_2}(\eta^{\beta_1\beta_2} - \xi^{\beta_1\beta_2}) + \nabla\cdot(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2}(\xi+\eta),$$

$$(E - M - V)\chi = -\tfrac{1}{2}\nabla\cdot(\sigma_2\vec{\sigma})_{\beta_1\beta_2}(\xi^{\beta_1\beta_2} + \eta^{\beta_1\beta_2}),$$

$$(E - \Delta - V)\xi = -\tfrac{1}{2}\nabla\cdot(\sigma_2\vec{\sigma})_{\beta_1\beta_2}(\chi^{\beta_1\beta_2} + \phi^{\beta_1\beta_2}), \tag{4.8}$$

$$(E + \Delta - V)\eta = -\tfrac{1}{2}\nabla\cdot(\sigma_2\vec{\sigma})_{\beta_1\beta_2}(\chi^{\beta_1\beta_2} + \phi^{\beta_1\beta_2}),$$

$$(E + M - V)\phi = -\tfrac{1}{2}\nabla\cdot(\sigma_2\vec{\sigma})_{\beta_1\beta_2}(\xi^{\beta_1\beta_2} + \eta^{\beta_1\beta_2}).$$

Adding and subtracting the first and the fourth, the second and the third, the fifth and the eighth and the sixth and the seventh equations in (4.8), we have

$$(E - V)(\chi^{\alpha_1\alpha_2}+\phi^{\alpha_1\alpha_2})-M(\chi^{\alpha_1\alpha_2}-\phi^{\alpha_1\alpha_2})= 2\nabla\cdot(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2}(\xi+\eta),$$

$$(E - V)(\xi^{\alpha_1\alpha_2}+\eta^{\alpha_1\alpha_2})-\Delta(\xi^{\alpha_1\alpha_2}-\eta^{\alpha_1\alpha_2})= 2\nabla\cdot(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2}(\chi+\phi),$$

$$(E - V)(\chi^{\alpha_1\alpha_2}-\phi^{\alpha_1\alpha_2})-M(\chi^{\alpha_1\alpha_2}+\phi^{\alpha_1\alpha_2})= 2i\nabla\cdot\vec{S}_{\alpha_1\alpha_2,\beta_1\beta_2}(\xi^{\beta_1\beta_2}-\eta^{\beta_1\beta_2}),$$

$$(E - V)(\xi^{\alpha_1\alpha_2}-\eta^{\alpha_1\alpha_2})-\Delta(\xi^{\alpha_1\alpha_2}+\eta^{\alpha_1\alpha_2})= 2i\nabla\cdot\vec{S}_{\alpha_1\alpha_2,\beta_1\beta_2}(\chi^{\beta_1\beta_2}-\phi^{\beta_1\beta_2}),$$

$$\begin{aligned}
&(E - V)(\chi-\phi) = M(\chi+\phi),\\
&(E - V)(\xi-\eta) = \Delta(\xi+\eta), \qquad\qquad (4.9)\\
&(E - V)(\chi+\phi) - M(\chi-\phi) = -\nabla\cdot(\sigma_2\vec{\sigma})_{\beta_1\beta_2}(\xi^{\beta_1\beta_2}+\eta^{\beta_1\beta_2}),\\
&(E - V)(\xi+\eta) - \Delta(\xi-\eta) = -\nabla\cdot(\sigma_2\vec{\sigma})_{\beta_1\beta_2}(\chi^{\beta_1\beta_2}+\phi^{\beta_1\beta_2}).
\end{aligned}$$

It is more convenient to rewrite Eqs. (4.9) in vector form. We define a vector $\vec{A}$ corresponding to a spinor $A^{\alpha\dot{\beta}}$ as

$$\begin{aligned}
\vec{A} &= +\tfrac{i}{2}(\vec{\sigma})_{\beta\alpha}A^{\alpha\dot{\beta}}\\
&= +\tfrac{i}{2}(\vec{\sigma})_{\beta\alpha}(i\sigma_2)_{\beta\gamma}A^{\alpha}{}_{\dot{\gamma}}\,.
\end{aligned}$$

For rotations, $A^{\alpha}{}_{\dot{\gamma}} = A^{\alpha\gamma}$. Therefore, $\vec{A} = \frac{1}{2}(\sigma_2\vec{\sigma})_{\gamma\alpha}A^{\alpha\gamma}$. We can change Eqs. (4.9) to vector form by multiplying all equations from the left with $\frac{1}{2}(\sigma_2\vec{\sigma})_{\alpha_2\alpha_1}$. When we do that, we need the following two identities

(A)
$$\frac{1}{2}(\sigma_2\vec{\sigma})_{\alpha_2\alpha_1}\nabla\cdot(\vec{\sigma}\sigma_2)_{\alpha_1\alpha_2}$$
$$= \frac{1}{2}(\sigma_2\vec{\sigma})_{\alpha_2\alpha_1}\partial_i(\sigma_i\sigma_2)_{\alpha_1\alpha_2}$$
$$= \frac{1}{2}\partial_i(\sigma_i\sigma_2\sigma_2\vec{\sigma})_{\alpha_1\alpha_1}$$
$$= \frac{1}{2}\partial_i(\sigma_i\vec{\sigma})_{\alpha_1\alpha_1}$$
$$= \nabla\ ;$$

(B)
$$\frac{1}{2}(\sigma_2\vec{\sigma})_{\alpha_2\alpha_1}\nabla\cdot\vec{S}_{\alpha_1\alpha_2,\beta_1\beta_2}$$
$$= \frac{1}{4}(\sigma_2\vec{\sigma})_{\alpha_2\alpha_1}\nabla\cdot((\vec{\sigma})_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2} + \delta_{\alpha_1\beta_1}(\vec{\sigma})_{\alpha_2\beta_2})$$
$$= \frac{1}{4}\partial_i(\sigma_2\vec{\sigma}\sigma_i)_{\beta_2\beta_1} + \frac{1}{4}\partial_i(\sigma_2\vec{\sigma}\sigma_i)_{\beta_1\beta_2}$$
$$=-\frac{1}{4}\partial_i(\tilde{\sigma}_i\tilde{\vec{\sigma}}\sigma_2)_{\beta_1\beta_2} + \frac{1}{4}\partial_i(\sigma_2\vec{\sigma}\sigma_i)_{\beta_1\beta_2}$$
$$=-\frac{1}{4}\partial_i(\sigma_2\sigma_i\vec{\sigma})_{\beta_1\beta_2} + \frac{1}{4}\partial_i(\sigma_2\vec{\sigma}\sigma_i)_{\beta_1\beta_2}$$
$$=-\frac{1}{4}\partial_i(\sigma_2\sigma_i\sigma_j\ \hat{j})_{\beta_1\beta_2} + \frac{1}{4}\partial_i(\sigma_2\sigma_j\hat{j}\sigma_i)_{\beta_1\beta_2}$$
$$=-\frac{i}{4}\partial_i(\sigma_2\epsilon_{ijk}\sigma_k\hat{j})_{\beta_1\beta_2} + \frac{i}{4}\partial_i(\sigma_2\epsilon_{jik}\sigma_k\hat{j})_{\beta_1\beta_2}$$
$$= \frac{i}{2}\epsilon_{jik}\partial_i\hat{j}(\sigma_2\sigma_k)_{\beta_1\beta_2}$$
$$= \frac{i}{2}\epsilon_{ijk}\partial_j\underline{i}(\sigma_2\sigma_k)_{\beta_1\beta_2}\ .$$

Making use of (A) and (B), Eqs. (4.9) become, after multiplying with $\frac{1}{2}(\sigma_2\vec{\sigma})_{\alpha_2\alpha_1}$ from the left for the first four equations,

$$
\begin{aligned}
&(E - V)(\vec{\chi}+\vec{\phi}) - M(\vec{\chi}-\vec{\phi}) = 2\nabla(\xi+\eta), \\
&(E - V)(\vec{\xi}+\vec{\eta}) - \Delta(\vec{\xi}-\vec{\eta}) = 2\nabla(\chi+\phi), \\
&(E - V)(\vec{\chi}-\vec{\phi}) - M(\vec{\chi}+\vec{\phi}) = -2\nabla\times(\vec{\xi}-\vec{\eta}), \\
&(E - V)(\vec{\xi}-\vec{\eta}) - \Delta(\vec{\xi}+\vec{\eta}) = -2\nabla\times(\vec{\chi}-\vec{\phi}), \\
&(E - V)(\chi-\phi) = M(\chi+\phi), \\
&(E - V)(\xi-\eta) = \Delta(\xi+\eta), \\
&(E - V)(\chi+\phi) - M(\chi-\phi) = -2\nabla\cdot(\vec{\xi}+\vec{\eta}), \\
&(E - V)(\xi+\eta) - \Delta(\xi-\eta) = -2\nabla\cdot(\vec{\chi}+\vec{\phi}).
\end{aligned}
\tag{4.10}
$$

A set of equations similar to Eqs. (4.10) was derived by Moseley and Rosen(15) by equating each one of the sixteen components of Eq. (4.1).

From Eqs. (4.10), it can be seen that $\frac{1}{i}\nabla\psi=\vec{P}\psi\longrightarrow 0$ and $E\longrightarrow M$ as velocity $v\longrightarrow 0$; and all components vanish except $\vec{\chi}$ and χ which, therefore, are the large components for two particles. The state of the system is labelled according to the properties of the large components which will survive in the non-relativistic limit. It is only in the non-relativistic limit that a state has a definite L, orbital angular momentum, a definite J, total angular momentum, and parity.

B. The J^+ Triplet Solution with a Square Well Interaction

In this section, we seek triplet solutions of Eqs. (4.10) for a state of arbitrary total angular momentum J and orbital angular momentum L where L = J or J $\pm$ 1. We consider the J^+ state and assume $J \neq 0$ to be odd. The case of even J can be treated in similar manner. It can be seen from Eqs. (4.10) that $\vec{\chi} \pm \vec{\phi}$ and $\chi \pm \phi$ have the same parity which is opposite to the parity of $\vec{\xi} \pm \vec{\eta}$ and $\xi \pm \eta$. The components of the wave function can be written as

$$\begin{aligned} \chi + \phi &= \chi - \phi = 0, \\ \vec{\chi} \pm \vec{\phi} &= F_{\pm}^{-1}(r)\, \vec{Y}_{JJ-1M}(\theta,\phi) + F_{\pm}^{1}(r)\vec{Y}_{JJ+1M}(\theta,\phi), \\ \xi \pm \eta &= H_{\pm}(r)Y_{JM}(\theta,\phi), \\ \vec{\xi} \pm \vec{\eta} &= G_{\pm}(r)\vec{Y}_{JJM}(\theta,\phi), \end{aligned} \tag{4.11}$$

where $\vec{Y}_{JLM}(\theta,\phi)$(16) are the vector spherical harmonics defined by

$$\vec{Y}_{JLM}(\theta,\phi) = \sum_{m,s} \langle Lm1s | L1JM\rangle Y_{Lm}(\theta,\phi)V_s ,$$

with $s = \pm 1,\ 0$.

The vector V_s is given by

$$V_{+1} = -\frac{1}{\sqrt{2}}(V_x + iV_y),$$

$$V_0 = V_z,$$

$$V_{-1} = \frac{1}{\sqrt{2}}(V_x - iV_y),$$

where V_x, V_y and V_z are unit vectors along the x, y and z axes respectively. We are going to make use of the following properties of the vector spherical harmonics

$$\nabla\Phi(r)Y_{\ell m} = -\left(\frac{\ell+1}{2\ell+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} - \frac{\ell}{r}\right)\Phi\vec{Y}_{\ell,\ell+1,m}(\theta,\phi) + \left(\frac{\ell}{2\ell+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} + \frac{\ell+1}{r}\right)\Phi\vec{Y}_{\ell,\ell-1,m},$$

$$\nabla\cdot(\Phi(r)\vec{Y}_{\ell,\ell+1,m}(\theta,\phi)) = -\left(\frac{\ell+1}{2\ell+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} + \frac{\ell+2}{r}\right)\Phi\cdot Y_{\ell m},$$

$$\nabla\cdot(\Phi(r)\vec{Y}_{\ell,\ell,m}(\theta,\phi)) = 0 \text{ for any } \Phi(r), \tag{4.12}$$

$$\nabla\cdot(\Phi(r)\vec{Y}_{\ell,\ell-1,m}(\theta,\phi)) = \left(\frac{\ell}{2\ell+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} - \frac{\ell-1}{r}\right)\Phi Y_{\ell m},$$

$$\nabla\times(\Phi(r)\vec{Y}_{\ell,\ell+1,m}(\theta,\phi)) = i\left(\frac{d}{dr} + \frac{\ell+2}{r}\right)\Phi\left(\frac{\ell}{2\ell+1}\right)^{\frac{1}{2}}\vec{Y}_{\ell,\ell,m},$$

$$\nabla\times(\Phi(r)\vec{Y}_{\ell,\ell,m}(\theta,\phi)) = i\left(\frac{d}{dr} - \frac{\ell}{r}\right)\Phi\left(\frac{\ell}{2\ell+1}\right)^{\frac{1}{2}}\vec{Y}_{\ell,\ell+1,m} + i\left(\frac{d}{dr} + \frac{\ell+1}{r}\right)\Phi\left(\frac{\ell+1}{2\ell+1}\right)^{\frac{1}{2}}\vec{Y}_{\ell,\ell-1,m},$$

$$\nabla\times(\Phi(r)\vec{Y}_{\ell,\ell-1,m}(\theta,\phi)) = i\left(\frac{d}{dr} - \frac{\ell-1}{r}\right)\left(\frac{\ell+1}{2\ell+1}\right)^{\frac{1}{2}}\vec{Y}_{\ell,\ell,m}.$$

Substituting (4.11) into Eqs. (4.10) and making use of the above properties of vector spherical harmonics, it can be shown after some manipulation that Eqs. (4.10) become

$$(E - V)F_+^{(-1)}(r) - MF_-^{(-1)}(r) = 2\left(\frac{J}{2J+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} + \frac{J+1}{r}\right)H_+(r),$$

$$(E - V)F_+^{(1)}(r) - MF_-^{(1)}(r) = -2\left(\frac{J+1}{2J+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} - \frac{J}{r}\right)H_+(r),$$

$$(E - V)F_-^{(-1)}(r) - MF_+^{(-1)}(r) = -2i\left(\frac{J+1}{2J+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} + \frac{J+1}{r}\right)G_-(r),$$

$$(E - V)F_-^{(1)}(r) - MF_+^{(1)}(r) = -2i\left(\frac{J}{2J+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} - \frac{J}{r}\right)G_-(r),$$

$$(E - V)G_-(r) - \Delta G_+(r) = -2i\left(\frac{J+1}{2J+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} - \frac{J-1}{r}\right)F_-^{(-1)}(r)$$

$$- 2i\left(\frac{J}{2J+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} + \frac{J+2}{r}\right)F_-^{(1)}(r),$$

$$(E - V)H_+(r) - \Delta H_-(r) = -2\left(\frac{J}{2J+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} - \frac{J-1}{r}\right)F_+^{(-1)}(r)$$

$$+ 2\left(\frac{J+1}{2J+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} + \frac{J+2}{r}\right)F_+^{(1)}(r),$$

$$(E - V)G_+(r) = \Delta G_-(r),$$

$$(E - V)H_-(r) = \Delta H_+(r). \tag{4.13}$$

We consider the case of square well potential, so that

$$V = -V_0 \qquad \text{for } r \leq a,$$

$$= 0 \qquad \text{for } r > a.$$

We deal with the region $r \leqslant a$ first so that $V = -V_0$. From the eight equations of (4.13), we can eliminate four unknowns $G_\pm(r)$ and $H_\pm(r)$ giving four equations with only $F_\pm^{\pm1}(r)$. The four equations are

$$((E+V_0)^2 - \Delta^2)((E+V_0)F_+^{(-1)}(r) - MF_-^{(-1)}(r))$$

$$= 2(\tfrac{J}{2J+1})^{\frac{1}{2}}(\tfrac{d}{dr} + \tfrac{J+1}{r})\Big[(E+V_0)(-2(\tfrac{J}{2J+1})^{\frac{1}{2}}(\tfrac{d}{dr} - \tfrac{J-1}{r})F_+^{(-1)}(r)$$

$$+ 2(\tfrac{J+1}{2J+1})^{\frac{1}{2}}(\tfrac{d}{dr} + \tfrac{J+2}{r})F_+^{(1)}(r))\Big] , \tag{4.14}$$

$$((E+V_0)^2 - \Delta^2)((E+V_0)F_+^{(1)}(r) - MF_-^{(1)}(r))$$

$$= -2(\tfrac{J+1}{2J+1})^{\frac{1}{2}}(\tfrac{d}{dr} - \tfrac{J}{r})\Big[(E+V_0)(-2(\tfrac{J}{2J+1})^{\frac{1}{2}}(\tfrac{d}{dr} - \tfrac{J-1}{r})F_+^{(-1)}(r)$$

$$+ 2(\tfrac{J+1}{2J+1})^{\frac{1}{2}}(\tfrac{d}{dr} + \tfrac{J+2}{r})F_+^{(1)}(r))\Big], \tag{4.15}$$

$$((E+V_0)^2 - \Delta^2)((E+V_0)F_-^{(-1)}(r) - MF_+^{(-1)}(r))$$

$$= (-2i)(\tfrac{J+1}{2J+1})^{\frac{1}{2}}(\tfrac{d}{dr} + \tfrac{J+1}{r})\Big[(E+V_0)(-2i(\tfrac{J+1}{2J+1})^{\frac{1}{2}}(\tfrac{d}{dr} - \tfrac{J-1}{r})F_-^{(-1)}(r)$$

$$- 2i(\tfrac{J}{2J+1})^{\frac{1}{2}}(\tfrac{d}{dr} + \tfrac{J+2}{r})F_-^{(1)}(r))\Big], \tag{4.16}$$

$$((E+V_0)^2 - \Delta^2)((E+V_0)F_-^{(1)}(r) - MF_+^{(1)}(r))$$

$$= (-2i)\left(\frac{J}{2J+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} - \frac{J}{r}\right)\left\{(E + V_0)\left[-2i\left(\frac{J+1}{2J+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} - \frac{J-1}{r}\right)F_-^{(-1)}(r)\right.\right.$$

$$\left.\left. -2i\left(\frac{J}{2J+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} + \frac{J+2}{r}\right)F_-^{(1)}(r)\right]\right\}. \qquad (4.17)$$

It is difficult to reduce these four equations in four unknowns to a single equation in one unknown and then to solve the resulting equation. This is because decreasing the number of equation will increase the order of the differential equation. Therefore, we will use a different approach to solve these four equations. Each of the four equations (4.14) to (4.17) contains only three of the four unknowns. Combining these four equations, we can obtain another set of four equations each of which contains all the four unknowns. The equations are

$$\left\{\left[(E+V_0)^2 - \Delta^2\right]\left[(E+V_0)^2 - M^2\right] + \frac{4J}{2J+1}(E+V_0)^2\left(\frac{d}{dr} + \frac{J+1}{r}\right)\right.$$

$$\left.\left(\frac{d}{dr} - \frac{J-1}{r}\right)\right\}F_+^{(-1)}(r) - 4\left(\frac{J+1}{2J+1}\right)^{\frac{1}{2}}\left(\frac{J}{2J+1}\right)^{\frac{1}{2}}(E+V_0)^2\left(\frac{d}{dr} + \frac{J+1}{r}\right)$$

$$\left(\frac{d}{dr} + \frac{J+2}{r}\right)F_+^{(1)}(r) + 4M\left(\frac{J+1}{2J+1}\right)(E+V_0)\left(\frac{d}{dr} + \frac{J+1}{r}\right)\left(\frac{d}{dr} - \frac{J-1}{r}\right)F_-^{(-1)}(r)$$

$$+ 4M\frac{(J(J+1))^{\frac{1}{2}}}{2J+1}(E+V_0)\left(\frac{d}{dr} + \frac{J+1}{r}\right)\left(\frac{d}{dr} + \frac{J+2}{r}\right)F_-^{(1)}(r) = 0, \qquad (4.18)$$

$$\Big\{\Big[(E+V_0)^2 - \Delta^2\Big]\Big[(E+V_0)^2 - M^2\Big] + 4\Big(\frac{J+1}{2J+1}\Big)(E+V_0)^2\Big(\frac{d}{dr} + \frac{J+1}{r}\Big)$$

$$\Big(\frac{d}{dr} - \frac{J-1}{r}\Big)\Big\}F_-^{(-1)}(r) + \frac{4(J(J+1))^{\frac{1}{2}}}{2J+1}(E+V_0)^2\Big(\frac{d}{dr} + \frac{J+1}{r}\Big)$$

$$\Big(\frac{d}{dr} + \frac{J+2}{r}\Big)F_-^{(1)}(r) + \frac{4J}{2J+1}M(E+V_0)\Big(\frac{d}{dr} + \frac{J+1}{r}\Big)\Big(\frac{d}{dr} - \frac{J-1}{r}\Big)F_+^{(-1)}(r)$$

$$- \frac{4(J(J+1))^{\frac{1}{2}}}{2J+1}M(E+V_0)\Big(\frac{d}{dr} + \frac{J+1}{r}\Big)\Big(\frac{d}{dr} + \frac{J+2}{r}\Big)F_+^{(1)}(r) = 0 \,, \tag{4.19}$$

$$\Big\{\Big[(E+V_0)^2 - \Delta^2\Big]\Big[(E+V_0)^2 - M^2\Big] + 4\Big(\frac{J+1}{2J+1}\Big)(E+V_0)^2\Big(\frac{d}{dr} - \frac{J}{r}\Big)$$

$$\Big(\frac{d}{dr} + \frac{J+2}{r}\Big)\Big\}F_+^{(1)}(r) - \frac{4(J(J+1))^{\frac{1}{2}}}{2J+1}(E+V_0)^2\Big(\frac{d}{dr}-\frac{J}{r}\Big)\Big(\frac{d}{dr}-\frac{J-1}{r}\Big)F_+^{(-1)}(r)$$

$$+ \frac{4(J(J+1))^{\frac{1}{2}}}{2J+1}M(E+V_0)\Big(\frac{d}{dr} - \frac{J}{r}\Big)\Big(\frac{d}{dr} - \frac{J-1}{r}\Big)F_-^{(-1)}(r)$$

$$+ \frac{4J}{2J+1}M(E+V_0)\Big(\frac{d}{dr} - \frac{J}{r}\Big)\Big(\frac{d}{dr} + \frac{J+2}{r}\Big)F_-^{(1)}(r) = 0 \,, \tag{4.20}$$

$$\Big\{\Big[(E+V_0)^2 - \Delta^2\Big]\Big[(E+V_0)^2 - M^2\Big] + \frac{4J}{2J+1}(E+V_0)^2\Big(\frac{d}{dr}-\frac{J}{r}\Big)\Big(\frac{d}{dr}+\frac{J+2}{r}\Big)\Big\}F_-^{(1)}(r)$$

$$+ 4\frac{(J(J+1))^{\frac{1}{2}}}{2J+1}(E+V_0)^2\Big(\frac{d}{dr} - \frac{J}{r}\Big)\Big(\frac{d}{dr} - \frac{J-1}{r}\Big)F_-^{(-1)}(r)$$

$$-4\frac{(J(J+1))^{\frac{1}{2}}}{2J+1}M(E+V_0)\Big(\frac{d}{dr} - \frac{J}{r}\Big)\Big(\frac{d}{dr} - \frac{J-1}{r}\Big)F_+^{(-1)}(r)$$

$$+ 4\frac{J+1}{2J+1}M(E+V_0)\Big(\frac{d}{dr} - \frac{J}{r}\Big)\Big(\frac{d}{dr} + \frac{J+2}{r}\Big)F_+^{(1)}(r) = 0 \,. \tag{4.21}$$

In the four equations (4.18) to (4.21), there are four different differential operators which possess special properties that we can use to simplify the equations. We define

$$\mathbb{A}_1 = \left(\frac{d}{dr} + \frac{J+1}{r}\right)\left(\frac{d}{dr} - \frac{J-1}{r}\right)$$

$$= \frac{d^2}{dr^2} + \frac{2d}{rdr} - \frac{J(J-1)}{r^2} \quad ,$$

$$\mathbb{A}_2 = \left(\frac{d}{dr} + \frac{J+1}{r}\right)\left(\frac{d}{dr} + \frac{J+2}{r}\right)$$

$$= \frac{d^2}{dr^2} + \frac{2J+3}{r}\frac{d}{dr} + \frac{J(J+2)}{r^2} \quad ,$$

$$\mathbb{A}_3 = \left(\frac{d}{dr} - \frac{J}{r}\right)\left(\frac{d}{dr} + \frac{J+2}{r}\right)$$

$$= \frac{d^2}{dr^2} + \frac{2d}{rdr} - \frac{(J+2)(J+1)}{r^2} \, ,$$

$$\mathbb{A}_4 = \left(\frac{d}{dr} - \frac{J}{r}\right)\left(\frac{d}{dr} - \frac{J-1}{r}\right)$$

$$= \frac{d^2}{dr^2} - \frac{2J-1}{r}\frac{d}{dr} + \frac{(J-1)(J+1)}{r^2}$$

and $\quad f(x) = (E+V_0)^2 - X^2.$

Eqs. (4.18) to (4.21) can be rewritten in terms of $\mathbb{A}_1$ to $\mathbb{A}_4$ and f(x).

$$\left[f(\Delta)f(M) + \frac{4J}{2J+1}(E+V_0)^2\mathbb{A}_1\right]F_+^{(-1)}(r) - \frac{4(J(J+1))^{\frac{1}{2}}}{2J+1}(E+V_0)^2\mathbb{A}_2F_+^{(1)}(r)$$

$$+ \frac{4(J+1)}{2J+1}M(E+V_0)\mathbb{A}_1F_-^{(-1)}(r) + \frac{4(J(J+1))^{\frac{1}{2}}}{2J+1}M(E+V_0)\mathbb{A}_2F_-^{(1)}(r) = 0\,,$$

$$\left[f(\Delta)f(M) + \frac{4(J+1)}{2J+1}(E+V_0)^2\mathbb{A}_1\right]F_-^{(-1)}(r) + \frac{4(J(J+1))^{\frac{1}{2}}}{2J+1}(E+V_0)^2\mathbb{A}_2F_-^{(1)}(r)$$

$$+ \frac{4J}{2J+1}M(E+V_0)\mathbb{A}_1F_+^{(-1)}(r) - \frac{4(J(J+1))^{\frac{1}{2}}}{2J+1}M(E+V_0)\mathbb{A}_2F_+^{(1)}(r) = 0, \quad (4.22)$$

$$\left[f(\Delta)f(M) + \frac{4(J+1)}{2J+1}(E+V_0)^2\mathbb{A}_3\right]F_+^{(1)}(r) - \frac{4(J(J+1))^{\frac{1}{2}}}{2J+1}(E+V_0)^2\mathbb{A}_4F_+^{(-1)}(r)$$

$$+ \frac{4\ J(J+1)^{\frac{1}{2}}}{2J+1}M(E+V_0)\mathbb{A}_4F_-^{(-1)}(r) + \frac{4J}{2J+1}M(E+V_0)\mathbb{A}_3F_-^{(1)}(r) = 0\,,$$

$$\left[f(\Delta)f(M) + \frac{4J}{2J+1}(E+V_0)^2\mathbb{A}_3\right]F_-^{(1)}(r) + \frac{4(J(J+1))^{\frac{1}{2}}}{2J+1}(E+V_0)^2\mathbb{A}_4F_-^{(-1)}(r)$$

$$- \frac{4(J(J+1))^{\frac{1}{2}}}{2J+1}M(E+V_0)\mathbb{A}_4F_+^{(-1)}(r) + \frac{4(J+1)}{2J+1}M(E+V_0)\mathbb{A}_3F_+^{(1)}(r) = 0.$$

The special properties of the four operators $\mathbb{A}_1$—$\mathbb{A}_4$ are discussed here.

$$\mathbb{A}_1 j_{J-1}(Kr) = \left(\frac{d^2}{dr^2} + \frac{2d}{rdr} - \frac{J(J-1)}{r^2}\right) j_{J-1}(Kr)$$

$$= -K^2 j_{J-1}(Kr), \qquad (4.23)$$

$$\mathbb{A}_3 j_{J+1}(Kr) = \left(\frac{d^2}{dr^2} + \frac{2d}{rdr} - \frac{(J+1)(J+2)}{r^2}\right) j_{J+1}(Kr)$$

$$= -K^2 j_{J+1}(Kr). \qquad (4.24)$$

Since $\frac{d}{d\rho}\rho^{\ell+1}j_\ell(\rho) = \rho^{\ell+1}j_{\ell-1}(\rho)$, we have

$$\frac{d}{d\rho}\left[\frac{1d}{\rho d\rho}(\rho^{J+2}j_{J+1}(\rho)\ )\right] = \frac{d}{d\rho}\left[\rho^{J+1}j_J(\rho)\right]$$

$$= \rho^{J+1}j_{J-1}(\rho); \qquad (4.25)$$

but

$$\frac{d}{d\rho}\left[\frac{1d}{\rho d}(\rho^{J+2}j_{J+1}(\rho))\right] = J(J+2)\rho^{J-1}j_{J+1}(\rho) + (2J+3)\rho^J\frac{d}{d\rho}j_{J+1}(\rho)$$

$$+ \rho^{J+1}\frac{d^2}{d\rho^2}j_{J+1}(\rho). \qquad (4.26)$$

From (4.25) and (4.26), we have

$$\left(\frac{d^2}{d\rho^2} + \frac{2J+3}{\rho}\frac{d}{d\rho} + \frac{J(J+2)}{\rho^2}\right)j_{J+1}(\rho) = j_{J-1}(\rho).$$

Putting $\rho = Kr$, we obtain

$$\left[\frac{d^2}{dr^2} + \frac{2J+3}{r}\frac{d}{dr} + \frac{J(J+2)}{r^2}\right]j_{J+1}(Kr) = K^2 j_{J-1}(Kr),$$

or $\quad \mathbb{A}_2 j_{J+1}(Kr) = K^2 j_{J-1}(Kr). \qquad (4.27)$

Again, since $\frac{d}{d\rho}\rho^{-\ell} j_\ell(\rho) = -\rho^{-\ell} j_{\ell+1}(\rho)$, we have

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}\rho^{-J+1} j_{J-1}(\rho)\right] = \rho^{-J} j_{J+1}(\rho), \tag{4.28}$$

but

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}\rho^{-J+1} j_{J-1}(\rho)\right]$$

$$= \rho^{-J}\frac{d^2}{d\rho^2} j_{J-1}(\rho) + (-2J+1)\rho^{-J-1}\frac{d}{d\rho} j_{J-1}(\rho)$$

$$+ (J-1)(J+1)\rho^{-J-2} j_{J-1}(\rho). \tag{4.29}$$

Therefore, from (4.28) and (4.29), we have

$$\left(\frac{d^2}{d\rho^2} - \frac{2J-1}{\rho}\frac{d}{d\rho} + \frac{(J-1)(J+1)}{\rho^2}\right) j_{J-1}(\rho) = j_{J+1}(\rho).$$

Putting $\rho = Kr$, we obtain

$$\left[\frac{d^2}{dr^2} - \frac{2J-1}{r}\frac{d}{dr} + \frac{(J-1)(J+1)}{r^2}\right] j_{J-1}(Kr) = K^2 j_{J+1}(Kr),$$

or $$\mathbb{A}_4 j_{J-1}(Kr) = K^2 j_{J+1}(Kr). \tag{4.30}$$

We are going to make use of (4.23), (4.24), (4.27) and (4.30) for the solution of Eqs. (4.22). We assume that the solution of Eqs. (4.22) is given by

$$
\begin{aligned}
F_{+}^{(-1)}(r) &= A_{+} j_{J-1}(Kr)\,, \\
F_{-}^{(-1)}(r) &= A_{-} j_{J-1}(Kr)\,, \\
F_{+}^{(1)}(r) &= B_{+} j_{J+1}(Kr)\,, \\
F_{-}^{(1)}(r) &= B_{-} j_{J+1}(Kr)\,,
\end{aligned}
\qquad (4.31)
$$

where $A_{\pm}$ and $B_{\pm}$ are constants to be evaluated. Substituting (4.31) into Eqs. (4.22) and using (4.23), (4.24), (4.27) and (4.30), we have four equations for the constants $A_{\pm}$ and $B_{\pm}$. The equations can be written in the following matrix form

$$
\begin{bmatrix}
\frac{f(\Delta)f(M)}{K^2} - \frac{4J}{2J+1}(E+V_0)^2 & -\frac{4(J+1)}{2J+1}M(E+V_0) & \frac{-4[J(J+1)]^{1/2}}{2J+1}(E+V_0)^2 & \frac{4[J(J+1)]^{1/2}}{2J+1}M(E+V_0) \\
\frac{-4J}{2J+1}M(E+V_0) & \frac{f(\Delta)f(M)}{K^2} - \frac{4(J+1)}{2J+1}(E+V_0)^2 & \frac{-4[J(J+1)]^{1/2}}{2J+1}M(E+V_0) & \frac{4[J(J+1)]^{1/2}}{2J+1}(E+V_0)^2 \\
\frac{-4[J(J+1)]^{1/2}}{2J+1}(E+V_0) & \frac{4[J(J+1)]^{1/2}}{2J+1}M(E+V_0) & \frac{f(\Delta)f(M)}{K^2} - \frac{4(J+1)}{2J+1}(E+V_0)^2 & \frac{-4J}{2J+1}M(E+V_0) \\
\frac{-4[J(J+1)]^{1/2}}{2J+1}M(E+V_0) & \frac{4[J(J+1)]^{1/2}}{2J+1}(E+V_0)^2 & \frac{-4(J+1)}{2J+1}M(E+V_0) & \frac{f(\Delta)f(M)}{K^2} - \frac{4J}{2J+1}(E+V_0)^2
\end{bmatrix}
\begin{bmatrix} A_{+} \\ A_{-} \\ B_{+} \\ B_{-} \end{bmatrix} = 0.
$$

For nontrivial solutions, the determinant of the matrix must be zero. This allows us to evaluate K. Evaluating the determinant and putting it equal to zero gives

$$K^2 = \frac{f(\Delta)f(M)}{4(E+V_0)^2} .$$

Substituting the value of K^2 into (4.32), the equations for $A_\pm$ and $B_\pm$ are

$$\begin{bmatrix} \frac{4(J+1)}{2J+1}(E+V_0)^2 & \frac{-4(J+1)}{2J+1}M(E+V_0) & \frac{-4[J(J+1)]^{1/2}}{2J+1}(E+V_0)^2 & \frac{4[J(J+1)]^{1/2}}{2J+1}M(E+V_0) \\ \frac{-4J}{2J+1}M(E+V_0) & \frac{4J}{2J+1}(E+V_0)^2 & \frac{-4[J(J+1)]^{1/2}}{2J+1}M(E+V_0) & \frac{4[J(J+1)]^{1/2}}{2J+1}(E+V_0)^2 \\ \frac{-4[J(J+1)]^{1/2}}{2J+1}(E+V_0)^2 & \frac{4[J(J+1)]^{1/2}}{2J+1}M(E+V_0) & \frac{4J}{2J+1}(E+V_0)^2 & \frac{-4J}{2J+1}M(E+V_0) \\ \frac{-4[J(J+1)]^{1/2}}{2J+1}M(E+V_0) & \frac{4[J(J+1)]^{1/2}}{2J+1}(E+V_0)^2 & \frac{-4(J+1)}{2J+1}M(E+V_0) & \frac{4(J+1)}{2J+1}(E+V_0)^2 \end{bmatrix} \begin{bmatrix} A_+ \\ A_- \\ B_+ \\ B_- \end{bmatrix} = 0.$$

The first and third equations are equivalent and so are the second and the fourth. So we have only two independent equations

$$\begin{aligned} &(E+V_0)\left(A_+ - \left(\tfrac{J}{J+1}\right)^{\frac{1}{2}}B_+\right) - M\left(A_- - \left(\tfrac{J}{J+1}\right)^{\frac{1}{2}}B_-\right) = 0, \\ &(E+V_0)\left(A_- + \left(\tfrac{J+1}{J}\right)^{\frac{1}{2}}B_-\right) - M\left(A_+ + \left(\tfrac{J+1}{J}\right)^{\frac{1}{2}}B_+\right) = 0. \end{aligned} \qquad (4.33)$$

Eqs. (4.33) are the two conditions on the four constants $A_{\pm}$ and $B_{\pm}$. We need one more condition, so that there is only one arbitrary constant for the wave function. This last condition can be found as follows. From (4.11) and (4.31) we have

$$\vec{\chi} \pm \vec{\phi} = A_{\pm} j_{J-1}(Kr)\vec{Y}_{JJ-1M}(\theta,\phi) + B_{\pm} j_{J+1}(Kr)\vec{Y}_{JJ+1M}(\theta,\phi)$$

so that

$$\vec{\chi} = \tfrac{1}{2}(A_+ + A_-) j_{J-1}(Kr)\vec{Y}_{JJ-1M}(\theta,\phi) + \tfrac{1}{2}(B_+ + B_-) j_{J+1}(Kr)\vec{Y}_{JJ+1M}(\theta,\phi). \tag{4.34}$$

$\vec{\chi}$ is the largest component of the wave function and will remain in the non-relativistic limit. The state is named after $\vec{\chi}$. The orbital angular momentum L of $\vec{\chi}$ is either J-1 or J+1 and cannot be a mixture of both. Therefore, from (4.34), we have either $B_+ = -B_-$ or $A_+ = -A_-$, corresponding to $L = J-1$ and $L = J+1$ respectively.

For the case of $B_+ = -B_-$, (4.33) can be solved to give

$$A_- = \frac{(J+1)(E+V_0) + JM}{J(E+V_0) + (J+1)M} A_+$$

and

$$B_+ = -B_- = \frac{\left(J(J+1)\right)^{\frac{1}{2}}(E+V_0-M)}{J(E+V_0) + (J+1)M} A_+ .$$

Other components of the wave function $G_{\pm}(r)$ and $H_{\pm}(r)$ can be solved from the last four equations of (4.13). We use the following properties of the spherical Bessel Functions:

$$(\frac{d}{d\rho} - \frac{J-1}{\rho}) j_{J-1}(\rho) = \rho^{J-1}\frac{d}{d\rho}\left[\rho^{-(J-1)} j_{J-1}(\rho)\right]$$

$$= -j_J(\rho),$$

$$\text{or } (\frac{d}{dr} - \frac{J-1}{r}) j_{J-1}(Kr) = -Kj_J(Kr), \tag{4.35}$$

$$(\frac{d}{d\rho} + \frac{J+2}{\rho}) j_{J+1}(\rho) = \rho^{-J-2}\frac{d}{d\rho}\, \rho^{J+2} j_{J+1}(\rho)$$

$$= j_J(\rho),$$

$$\text{or } (\frac{d}{dr} + \frac{J+2}{r}) j_{J+1}(Kr) = Kj_J(Kr). \tag{4.36}$$

We assume $G_{\pm}(r) = C_{\pm} j_J(Kr)$,

(4.37)

and $H_{\pm}(r) = D_{\pm} j_J(Kr)$.

Substituting (4.37) into the last four equations of (4.13) and using (4.35) and (4.36), we obtain

$$(E+V_0)C_- - \Delta C_+ = \frac{2iK}{(2J+1)^{\frac{1}{2}}}\left((J+1)^{\frac{1}{2}}A_- - J^{\frac{1}{2}}B_-\right),$$

$$(E+V_0)D_+ - \Delta D_- = \frac{2K}{(2J+1)^{\frac{1}{2}}}\left(J^{\frac{1}{2}}A_+ + (J+1)^{\frac{1}{2}}B_+\right),$$

$$(E+V_0)C_+ = \Delta C_-, \tag{4.38}$$

$$(E+V_0)D_- = \Delta D_+,$$

from which we find

$$C_- = \frac{E+V_0}{\Delta}C_+ = \frac{2iK(J+1)^{\frac{1}{2}}(2J+1)^{\frac{1}{2}}(E+V_0)^2}{\left((E+V_0)^2-\Delta^2\right)\left(J(E+V_0)+(J+1)M\right)}A_+,$$

$$D_+ = \frac{E+V_0}{\Delta}D_- = \frac{2K\left(J(J+1)\right)^{\frac{1}{2}}(E+V_0)^2}{\left((E+V_0)^2-\Delta^2\right)\left(J(E+V_0)+(J+1)M\right)}A_+,$$

so that the complete wave function is

$$\vec{\chi} = A_+\frac{(2J+1)(E+V_0+M)}{2\left((J+1)M + J(E+V_0)\right)}\, j_{J-1}(Kr)\vec{Y}_{JJ-1M}(\theta,\phi),$$

$$\vec{\phi} = A_+\frac{\left(J(J+1)\right)^{\frac{1}{2}}(E+V_0-M)}{J(E+V_0) + (J+1)M}\, j_{J+1}(Kr)\vec{Y}_{JJ+1M}(\theta,\phi)$$

$$+ \frac{A_\pm}{2}\frac{M - (E+V_0)}{J(E+V_0) + (J+1)M}\, j_{J-1}(Kr)\vec{Y}_{JJ-1M}(\theta,\phi),$$

$$\vec{\xi} = A_+\frac{iK(J+1)^{\frac{1}{2}}(2J+1)^{\frac{1}{2}}(E+V_0)}{(E+V_0-\Delta)\left(J(E+V_0) + (J+1)M\right)}j_J(Kr)\vec{Y}_{JJM}(\theta,\phi),$$

$$\vec{\eta} = A_+ \frac{-iK(J+1)^{\frac{1}{2}}(2J+1)^{\frac{1}{2}}(E+V_0)}{(E+V_0+\Delta)(J(E+V_0) + (J+1)M)} j_J(Kr)\vec{Y}_{JJM}(\theta,\phi) ,$$

$$\xi = A_+ \frac{K(J(J+1))^{\frac{1}{2}}(E+V_0)}{(E+V_0-\Delta)(J(E+V_0) + (J+1)M)} j_J(Kr)Y_{JM}(\theta,\phi) ,$$

$$\eta = A_+ \frac{-K(J(J+1))^{\frac{1}{2}}(E+V_0)}{(E+V_0+\Delta)(J(E+V_0)+(J+1)M)} j_J(Kr)Y_{JM}(\theta,\phi) .$$

For the case of $A_+ = -A_-$, Eqs. (4.33) and (4.38) give

$$B_- = \frac{J(E+V_0) + (J+1)M}{(J(J+1))^{\frac{1}{2}}(E+V_0-M)} A_+ ,$$

$$B_+ = \frac{(J+1)(E+V_0) + JM}{(J(J+L))^{\frac{1}{2}}(E+V_0-M)} A_+ ,$$

$$C_- = \frac{E+V_0}{\Delta} C_+ = \frac{-2iK(2J+1)^{\frac{1}{2}}(E+V_0)^2}{(J+1)^{\frac{1}{2}}(E+V_0-M)f(\Delta)} A_+ ,$$

$$D_+ = \frac{E+V_0}{\Delta} D_- = \frac{2K(J+1)^{\frac{1}{2}}(E+V_0)^2}{(J(2J+1))^{\frac{1}{2}}(E+V_0-M)f(\Delta)} A_+ ,$$

so that the complete wave function is

$$\vec{\chi} = \frac{A_+(2J+1)(E+V_0+M)}{2(J(J+1))^{\frac{1}{2}}(E+V_0-M)} j_{J+1}(Kr)\vec{Y}_{JJ+1M}(\theta,\phi) ,$$

$$\vec{\phi} = A_+ j_{J-1}(Kr)\vec{Y}_{JJ-1M}(\theta,\phi) + A_+ \frac{E+V_0-M}{2(J(J+1))^{\frac{1}{2}}(E+V_0-M)} j_{J+1}(Kr)\vec{Y}_{JJ+1M}(\theta,\phi),$$

$$\vec{\xi} = A_+ \frac{iK(2J+1)^{\frac{1}{2}}(E+V_0)}{(J+1)^{\frac{1}{2}}(E+V_0-M)(E+V_0-\Delta)} j_J(Kr)\vec{Y}_{JJM}(\theta,\phi) ,$$

$$\vec{\eta} = A_{+}\frac{iK(2J+1)^{\frac{1}{2}}(E+V_0)}{(J+1)^{\frac{1}{2}}(E+V_0-M)(E+V_0+\Delta)}\ j_J(Kr)\vec{Y}_{JJM}(\theta,\phi),$$

$$\xi = A_{+}\frac{K(J+1)^{\frac{1}{2}}(E+V_0)}{(J(2J+1))^{\frac{1}{2}}(E+V_0-M)(E+V_0-\Delta)}\ j_J(Kr)\vec{Y}_{JJM}(\theta,\phi),$$

$$\eta = A_{+}\frac{K(J+1)^{\frac{1}{2}}(E+V_0)}{(J(2J+1))^{\frac{1}{2}}(E+V_0-M)(E+V_0+\Delta)}\ j_J(Kr)\vec{Y}_{JJM}(\theta,\phi).$$

The wave functions which we obtained for the two cases of $L = J\pm 1$ are solutions of the wave equation inside the square well. Outside the well, the solutions will consist of spherical Hankel functions instead of spherical Bessel function. For $V_0 > M$, the value of K^2 is positive inside the well but negative outside if $E^2 > \Delta^2$, so that the solution for $\vec{\chi}$ outside the well is

$$\vec{\chi} = B\frac{(2J+1)(E+M)}{(J+1)M+JE}\ h^{(1)}_{J-1}(i\alpha r)\vec{Y}_{JJ-1M}(\theta,\phi) \qquad \text{for } L = J-1,$$

where $\alpha = \sqrt{\dfrac{(M^2-E^2)(E^2-\Delta^2)}{4E^2}}$.

The energy levels can be obtained by matching the largest components of the solutions inside and outside the well at the boundary. For states with $L = J\pm 1$, the condition is

$$\frac{Kj'_{J\pm1}(Ka)}{j_{J\pm1}(Ka)} = \frac{i\alpha h^{(1)\prime}_{J\pm1}(i\alpha a)}{h^{(1)}_{J\pm1}(i\alpha a)} \quad . \qquad (4.39)$$

This condition is the same as for two spin-0 particles, indicating that the energy levels for a square well potential depends only on L and do not depend on the spin of the particles. In fact, (4.39) has the same form as non-relativistic particles,except in the present case, the K and α are all different from the non-relativistic theory.

C. The L^+ Singlet Solutions

In this section we consider the L^+ singlet solution to Eqs. (4.10). We assume L to be even. The case of odd L can be treated in similar manner. The state is named by the large components $\vec{\chi}$ and χ. For L^+ scalar solution, we have

$$\vec{\chi} = 0 \qquad \text{and} \qquad \chi = g_\chi(r)Y_{LM}(\theta,\phi) .$$

Since $\vec{\chi}$, χ, $\vec{\phi}$ and ϕ are of the same parity, which is opposite to that of $\vec{\xi}$, ξ, $\vec{\eta}$ and η, the components of the wave function can be written in the following way:

$$\chi \pm \phi = f_\pm(r)Y_{LM}(\theta,\phi) ,$$

$$\vec{\xi} \pm \vec{\eta} = g_\pm^{+1}(r)\vec{Y}_{LL+1M}(\theta,\phi) + g_\pm^{-1}(r)\vec{Y}_{LL-1M}(\theta,\phi) , \tag{4.40}$$

$$\xi \pm \eta = 0,$$

$$\vec{\phi} = f_\phi(r)\vec{Y}_{LLM}(\theta,\phi) = 0 ,$$

That $\xi = \eta = \vec{\phi} = 0$ can be seen from the sixth, eighth and the first equations of (4.10). Putting (4.40) into (4.10), we obtain

$$(E-V)g_+^{+1}(r) - \Delta g_-^{+1}(r) = -2\left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} - \frac{L}{r}\right)f_+(r),$$

$$(E-V)g_+^{-1}(r) - \Delta g_-^{-1}(r) = 2\left(\frac{L}{2L+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} + \frac{L+1}{r}\right)f_+(r),$$

$$(E-V)f_+(r) - Mf_-(r) = 2\left(\frac{L+1}{2L+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} + \frac{L+2}{r}\right)g_+^{+1}(r)$$

$$-2\left(\frac{L}{2L+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr} - \frac{L-1}{r}\right)g_+^{-1}(r),$$

$$(E-V)g_-^{+1}(r) = \Delta g_+^{+1}(r),$$

$$(E-V)g_-^{-1}(r) = \Delta g_+^{-1}(r), \qquad (4.41)$$

$$(E-V)f_-(r) = Mf_+(r),$$

where we have used the properties of the vector spherical harmonics. From Eqs. (4.41) we can obtain after some calculation a single equation for $f_+(r)$

$$\left[\frac{d^2}{dr^2} + \left(\frac{2}{r} + \frac{(E-V)^2 + \Delta^2}{(E-V)^2 - \Delta^2}\,\frac{\frac{dV}{dr}}{E-V}\right)\frac{d}{dr} + \frac{((E-V)^2 - M^2)((E-V)^2 - \Delta^2)}{4(E-V)^2}\right.$$

$$\left. - \frac{L(L+1)}{r^2}\right] f_+(r) = 0. \qquad (4.42)$$

Eq. (4.42) is the singlet two spin-$\frac{1}{2}$ particle wave equation for the radial wave function $f_+(r)$. Except for the third term, it is identical with the corresponding equation for two spin-0 particles. For square well potential, $\frac{dV}{dr}=0$, so Eq. (4.42) reduces to an equation exactly the same as Eqs. (3.13) and (3.15) for two spin-0 particles. These results are consistent because the scalar solutions correspond to states with total spin zero. For two particles of equal mass with Coulomb potential $V(r) = -\frac{e^2}{r}$, Eq. (4.42) becomes

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\left(2 + \frac{\frac{e^2}{r}}{E + \frac{e^2}{r}}\right)\frac{d}{dr} + \frac{\left(E + \frac{e^2}{r}\right)^2 - M^2}{4} - \frac{L(L+1)}{r^2}\right] f_+(r) = 0 . \tag{4.43}$$

This equation differs from Eq. (3.17) only by the sign of the term $\frac{\frac{e^2}{r}}{E + \frac{e^2}{r}} = \frac{1}{r} - \frac{1}{r + \frac{e^2}{E}}$. For a system of an electron and a positron $\frac{e^2}{E} \ll 10^{-13}$ = radius of electron. If we neglect $\frac{e^2}{E}$ compared with r, then Eq. (4.43) is the same as Eq. (3.19).

Because of the presence of the third term in Eq. (4.42), exact solution is difficult to obtain for most potentials V except a square well. However, we can still discuss some characteristics of this equation by looking at solutions asymphotically. We consider a harmonic potential of the form $V(r) = \frac{1}{2}kr^2$. Examining the solutions at $r \to \infty$, Eq. (4.42) yields

$$\left(\frac{d^2}{dr^2} + \frac{k^2r^4}{16}\right)f_+(r) = 0 \; . \tag{4.44}$$

We let the leading term of the solution be $r^s\exp(-\alpha r^t)$ with s, α and t to be determined. The leading term of $\frac{d^2}{dr^2}(r^s\exp(-\alpha r^t))$ is $\alpha^2 t^2 r^{s+2t-2}\exp(-\alpha r^t)$. Therefore, comparing the leading terms, we have

$$\alpha^2 t^2 r^{s+2t-2}\exp(-\alpha r^t) = -\frac{k^2 r^{s+4}}{16}\exp(-\alpha r^t) \, ,$$

which implies that $t = 3$ and $\alpha = \frac{\pm ik}{12}$. This leads to an unbounded, oscillating solution at $r \to \infty$ and resembles the Klein Paradox of the single spin-$\frac{1}{2}$ particle. For the region $r \to 0$, Eq. (4.42) becomes

$$\left[\frac{d^2}{dr^2} + \frac{2d}{rdr} + \frac{(E^2-M^2)(E^2-\Delta^2)}{4E^2} - \frac{L(L+1)}{r^2}\right]f_+(r) = 0 \, ,$$

which is the same equation for no potential. We cannot find the exact solution for general r.

CHAPTER V

TWO SPIN-ONE PARTICLES

A. Wave Equation for Single Spin-one Particle

There are several wave equations describing a single spin-one particle. They are all equivalent when the potential is put to be zero. The spin-1 Hamiltonian equation which we mentioned in Chapter I is difficult to handle. Instead of using it, we choose Weinberg's[17] equation which is a generalization of Dirac's spin-$\frac{1}{2}$ equation. Weinberg's equation is

$$(p_\mu p_\nu \gamma_{\mu\nu} + m^2)\psi = 0 . \tag{5.1}$$

The wave function ψ is a six-component column matrix; p_μ is $-i\partial_\mu$ and

$$\gamma_{\mu\nu} = \begin{pmatrix} 0 & -S_{\mu\nu}{}^{+} \\ & \\ -S_{\mu\nu} & 0 \end{pmatrix} , \tag{5.2}$$

with $S_{\mu\nu}$ defined by

$$\begin{aligned} S_{ij} &= S_i S_j + S_j S_i - \delta_{ij} , \\ S_{i4} &= S_{4i} = iS_i , \\ S_{44} &= -I , \end{aligned} \tag{5.3}$$

where the S_i are the 3x3 spin one matrices. We use the same matrix representation as Sankaranarayanan and Good[18] for $\gamma_{\mu\nu}$.

The Green's function for Eq. (5.1) satisfies the equation

$$(p'_\mu p'_\nu \gamma_{\mu\nu} + m^2)G(x',x) = i\delta(x'-x)$$

and can be expressed as an integral

$$G(x',x) = \frac{i}{(2\pi)^4}\int d^3p \exp(i\vec{p}.(\vec{x}'-\vec{x}))\int \frac{(p_\mu p_\nu \gamma_{\mu\nu} - m^2)\exp(-iE(t'-t))}{2m^2}$$

$$\left(\frac{1}{E^2 - \vec{p}^2 - m^2} - \frac{1}{E^2 - \vec{p}^2 + m^2}\right) dE ,$$

where we use the fact that $(p_\mu p_\nu \gamma_{\mu\nu})^2 = p^4$.

There are four singularities along the line of integration. We displace them an infinitesimal distance below the real axis. So the contour of the E integration is the same as in Figure I. The two poles $E = \pm\sqrt{\vec{p}^2 + m^2}$ correspond to particles with real mass, whereas the other two poles $E = \pm\sqrt{\vec{p}^2 - m^2}$ correspond to particles with imaginary mass (tachyons). For our purpose, we consider only particles with real masses. Therefore, we drop the two poles $E = \pm\sqrt{\vec{p}^2 - m^2}$. Thus the Green's function becomes

$$G(x',x) = \frac{i}{(2\pi)^4}\int d^3p \exp(i\vec{p}.(\vec{x}'-\vec{x}))\int \frac{(p_\mu p_\nu \gamma_{\mu\nu} - m^2)\exp(-iE(t'-t))}{2m^2(E^2-\vec{p}^2-m^2)} dE,$$

which is similar to the one obtained by Weinberg.[17] After evaluating the E integral, it can be shown that

$$\lim_{t'\to t^+} G(x',x) = \frac{i}{(2\pi)^3 m^2}\int d^3p \exp(i\vec{p}\cdot(\vec{x}'-\vec{x}))p_i\gamma_{4i}$$

$$= \frac{1}{m^2}\gamma_{4i}\partial_i'\delta(\vec{x}'-\vec{x}), \qquad (5.4)$$

where from (5.2) and (5.3)

$$\gamma_{4i} = i\begin{pmatrix} 0 & S_i \\ -S_i & 0 \end{pmatrix}.$$

B. Wave Equation for Two Spin-one Particles

To obtain a wave equation for spin-1 particles, we start with the B-S integral equation and follow the same formalism as (1-A) and (1-C). In (1-A), the single particle wave equation is of Hamiltonian type and contains only a first order time derivative. In (1-C), the single particle wave equation contains a second order time derivative. In this section, the single particle wave equation (5.1) contains both first and second order time derivatives. Our formalism works for all these cases. Applying the operator $(p_\mu^{(1)}p_\nu^{(1)}\gamma_{\mu\nu}^{(1)} + m_1^2)$ to the B-S integral equation with instantaneous interaction, we obtain

$$(p_\mu^{(1)}p_\nu^{(1)}\gamma_{\mu\nu}^{(1)} + m_1^2)\psi(x_1,x_2) = \int G_2(x_2,x_2')\Big|_{t_1'=t_1} V(x_1,x_2')\psi(x_1,x_2')\Big|_{t_2'=t_1} d^3x_2'.$$

We take limit of the above equation as $t_2 \to t_1^+$. Using (5.4) we have

$$(p_\mu^{(1)} p_\nu^{(1)} \gamma_{\mu\nu}^{(1)} + m_{(1)}^2)\psi(\vec{x}_1, t_1, \vec{x}_2, t_1)$$

$$= -\frac{1}{m_2^2}\gamma_{4i}^{(2)} \partial_i^{(2)} (V(\vec{x}_1, \vec{x}_2)\psi(\vec{x}_1, t_1, \vec{x}_2, t_1)). \tag{5.5}$$

For steady state solutions, $\psi(x_1, x_2) = \psi(\vec{x}_1, \vec{x}_2)\exp(-i(E_1 t_1 + E_2 t_2))$,

Eq. (5.5) becomes $(p_\mu^{(1)} p_\nu^{(1)} \gamma_{\mu\nu}^{(1)} + m_1^2)\psi(\vec{x}_1, \vec{x}_2)$

$= -\frac{1}{m_2^2}\gamma_{4i}^{(2)} \partial_i^{(2)} (V(\vec{x}_1, \vec{x}_2)\psi(\vec{x}_1, \vec{x}_2))$ with $p_4^{(1)} = iE_1$.

Another equation can be obtained by operating on the B-S equation with $(p_\mu^{(2)} p_\nu^{(2)} \gamma_{\mu\nu}^{(2)} + m_{(2)}^2)$. Adding the two equations, we have the wave equation for two spin-1 particles, symbolically,

$$(p_\mu^{(1)} p_\nu^{(1)} \gamma_{\mu\nu}^{(1)} + m_{(1)}^2 + p_\mu^{(2)} p_\nu^{(2)} \gamma_{\mu\nu}^{(2)} + m_{(2)}^2)\psi(\vec{x}_1, \vec{x}_2)$$

$$= -\left(\frac{\gamma_{4i}^{(1)} \partial_i^{(1)}}{m_{(1)}^2} + \frac{\gamma_{4i}^{(2)} \partial_i^{(2)}}{m_{(2)}^2}\right)(V(\vec{x}_1, \vec{x}_2)\psi(\vec{x}_1, \vec{x}_2)). \tag{5.6}$$

C. <u>Two Spin-one Particles with Square Well Interaction</u>

For single spin-one particles, the wave function is

$$\psi = \begin{pmatrix} \chi \\ \phi \end{pmatrix},$$

where χ and ϕ have three components and are related to spinors in the following way:

$$\chi_1 = \chi_{\dot{1}\dot{1}} \;, \qquad \chi_2 = \sqrt{2}\chi_{\dot{1}\dot{2}} \;, \qquad \chi_3 = \chi_{\dot{2}\dot{2}} \;,$$

$$\phi_1 = \phi^{11} \;, \qquad \phi_2 = \sqrt{2}\phi^{12} \;, \qquad \phi_3 = \phi^{22} \;.$$

For two spin-1 particles, the wave function can be written as

$$\psi = \begin{bmatrix} \chi_1\chi_2 & \chi_1\phi_2 \\ \phi_1\chi_2 & \phi_1\phi_2 \end{bmatrix}$$

$$= \begin{bmatrix} \chi & \eta \\ \xi & \phi \end{bmatrix} . \tag{5.7}$$

Using (5.7), (5.2) and (5.3), Eqs. (5.6) reduce to the following four equations

$$-p_\mu^{(1)} p_\nu^{(1)} (S_{\mu\nu}^+)_{\ell n} \xi^{nm} - p_\mu^{(2)} p_\nu^{(2)} \eta^{\ell n} (\widetilde{S_{\mu\nu}^+})_{nm} + (m_{(1)}^2 + m_{(2)}^2) \chi^{\ell m}$$

$$= (-i)\left(\frac{\vec{S} \cdot \nabla_{(1)}}{m_{(1)}^2}\right)_{\ell n} (V\xi^{nm}) - i\left(\frac{\nabla_{(2)} (V\eta)}{m_{(2)}^2}\right)^{\ell n} \cdot (\widetilde{\vec{S}})_{nm} ,$$

$$-p_\mu^{(1)} p_\nu^{(1)} (S_{\mu\nu})_{\ell n} \eta^{nm} - p_\mu^{(2)} p_\nu^{(2)} \xi^{\ell n} (\widetilde{S}_{\mu\nu})_{nm} + (m_{(1)}^2 + m_{(2)}^2) \phi^{\ell m}$$

$$= i\left(\frac{\vec{S} \cdot \nabla_{(1)}}{m_{(1)}^2}\right)_{\ell n} (V\eta^{nm}) + i\left(\frac{\nabla_{(2)} (V\xi)}{m_{(2)}^2}\right)^{\ell n} \cdot (\widetilde{\vec{S}})_{nm} ,$$

$$-p_\mu^{(1)} p_\nu^{(1)} (S_{\mu\nu}^+)_{\ell n} \phi^{nm} - p_\mu^{(2)} p_\nu^{(2)} \chi^{\ell n} (\widetilde{S}_{\mu\nu})_{nm} + (m_{(1)}^2 + m_{(2)}^2) \eta^{\ell m}$$

$$= -i\left(\frac{\vec{S} \cdot \nabla_{(1)}}{m_{(1)}^2}\right)_{\ell n} (V\phi^{nm}) + i\left(\frac{\nabla_{(2)} (V\chi)}{m_{(2)}^2}\right)^{\ell n} \cdot (\widetilde{\vec{S}})_{nm} ,$$

$$-p_\mu^{(1)} p_\nu^{(1)} (S_{\mu\nu})_{\ell n} \chi^{nm} - p_\mu^{(2)} p_\nu^{(2)} \phi^{\ell n} (\widetilde{S_{\mu\nu}^+})_{nm} + (m_{(1)}^2 + m_{(2)}^2) \xi^{\ell m}$$

$$= i\left(\frac{\vec{S} \cdot \nabla_{(1)}}{m_{(1)}^2}\right)_{\ell n} (V\chi^{nm}) - i\left(\frac{\nabla_{(2)} (V\phi)}{m_{(2)}^2}\right)^{\ell n} \cdot (\widetilde{\vec{S}})_{nm} .$$

These four equations can be rearranged in the following way to give

$$
\begin{aligned}
&(m^2_{(1)} + m^2_{(2)})\chi^{\ell m} - (p^{(1)}_\mu p^{(1)}_\nu (S^+_{\mu\nu}) - i\frac{\vec{S}\cdot\nabla_{(1)}}{m^2_{(1)}}V)_{\ell n}\xi^{nm} \\
&-(p^{(2)}_\mu p^{(2)}_\nu \tilde{S}^+_{\mu\nu} - i\frac{\tilde{\vec{S}}\cdot\nabla_{(2)}}{m^2_{(2)}}V)_{nm}\eta^{\ell n} = 0 , \\
&(m^2_{(1)} + m^2_{(2)})\phi^{\ell m} - (p^{(1)}_\mu p^{(1)}_\nu S_{\mu\nu} + i\frac{\vec{S}\cdot\nabla_{(1)}}{m^2_{(1)}}V)_{\ell n}\eta^{nm} \\
&- (p^{(2)}_\mu p^{(2)}_\nu \tilde{S}_{\mu\nu} + i\frac{\tilde{\vec{S}}\cdot\nabla_{(2)}}{m^2_{(2)}} V)_{nm}\xi^{\ell n} = 0 , \\
&(m^2_{(1)} + m^2_{(2)})\eta^{\ell m} - (p^{(1)}_\mu p^{(1)}_\nu S^+_{\mu\nu} - i\frac{\vec{S}\cdot\nabla_{(1)}}{m^2_{(1)}}V)_{\ell n}\phi^{nm} \\
&-(p^{(2)}_\mu p^{(2)}_\nu \tilde{S}_{\mu\nu} + i\frac{\tilde{\vec{S}}\cdot\nabla_{(2)}}{m^2_{(2)}}V)_{nm}\chi^{\ell n} = 0 , \\
&(m^2_{(1)} + m^2_{(2)})\xi^{\ell m} - (p^{(1)}_\mu p^{(1)}_\nu S_{\mu\nu} + i\frac{\vec{S}\cdot\nabla_{(1)}}{m^2_{(1)}}V)_{\ell n}\chi^{nm} \\
&-(p^{(2)}_\mu p^{(2)}_\nu \tilde{S}^+_{\mu\nu} - i\frac{\tilde{\vec{S}}\cdot\nabla_{(2)}}{m^2_{(2)}}V)_{nm}\phi^{\ell n} = 0 .
\end{aligned}
\tag{5.8}
$$

We consider the case of two particles of equal mass, so that $m_{(1)} = m_{(2)} = m$ and $E_1 = E_2 = \frac{E}{2}$. In the CM frame, we have

$$p^{(1)}_i p^{(1)}_i = p^{(2)}_i p^{(2)}_i = -\nabla^2_x$$

and

$$\nabla_{(1)} = \nabla_x \quad , \quad \nabla_{(2)} = -\nabla_x ,$$

where x is the relative coordinate. For the above case, Eqs. (5.8) become

$$2m^2\chi^{\ell m} - (-\partial_i\partial_j(S_{ij}) - E\partial_i(S_{i4}) - \frac{E^2}{4}S_{44} - i\frac{\partial_i S_i}{m^2}V)_{\ell n}\xi^{nm}$$

$$-(-\partial_i\partial_j(S_{ij}) - E\partial_i(S_{i4}) - \frac{E^2}{4}S_{44} - i\frac{\partial_i S_i}{m^2}V)_{nm}\eta^{\ell n} = 0\,,$$

$$2m^2\phi^{\ell m} - (-\partial_i\partial_j(S_{ij}) + E\partial_i(S_{i4}) - \frac{E^2}{4}S_{44} + i\frac{\partial_i S_i}{m^2}V)_{\ell n}\eta^{nm}$$

$$-(-\partial_i\partial_j(S_{ij}) + E\partial_i(S_{i4}) - \frac{E^2}{4}S_{44} + i\frac{\partial_i S_i}{m^2})_{nm}\xi^{\ell n} = 0\,,$$

$$2m^2\eta^{\ell m} - (-\partial_i\partial_j(S_{ij}) - E\partial_i(S_{i4}) - \frac{E^2}{4}S_{44} - i\frac{\partial_i S_i}{m^2}V)_{\ell n}\phi^{nm}$$

$$-(-\partial_i\partial_j(S_{ij}) + E\partial_i(S_{i4}) - \frac{E^2}{4}S_{44} + i\frac{\partial_i S_i}{m^2}V)_{nm}\chi^{\ell n} = 0\,, \qquad (5.9)$$

$$2m^2\xi^{\ell m} - (-\partial_i\partial_j(S_{ij}) + E\partial_i S_{i4} - \frac{E^2}{4}(S_{44}) + i\frac{\partial_i S_i}{m^2}V)_{\ell n}\chi^{nm}$$

$$-(-\partial_i\partial_j S_{ij} - E\partial_i S_{i4} - \frac{E^2}{4}S_{44} - i\frac{\partial_i S_i}{m^2}V)_{nm}\phi^{\ell n} = 0\,.$$

We define two operators B_1 and B_2 as

$$B_1 = \frac{1}{2m^2}(-\partial_i\partial_j(S_{ij}) - E\partial_i(S_{i4}) - \frac{E^2}{4}S_{44} - iV\frac{\partial_i S_i}{m^2} - i\frac{S_i(\partial_i V)}{m^2})\,,$$

$$B_2 = \frac{1}{2m^2}(-\partial_i\partial_j(S_{ij}) + E\partial_i(S_{i4}) - \frac{E^2}{4}S_{44} + iV\frac{\partial_i S_i}{m^2} \div i\frac{S_i(\partial_i V)}{m^2})\,.$$

In terms of the operators B_1 and B_2, Eqs. (5.9) become

$$\begin{aligned}
&\chi^{\ell m} - (B_1)_{\ell n}\xi^{nm} - (B_1)_{nm}\eta^{\ell n} = 0 \,, \\
&\phi^{\ell m} - (B_2)_{\ell n}\eta^{nm} - (B_2)_{nm}\xi^{\ell n} = 0 \,, \\
&\eta^{\ell m} - (B_1)_{\ell n}\phi^{nm} - (B_2)_{nm}\chi^{\ell n} = 0 \,, \\
&\xi^{\ell m} - (B_2)_{\ell n}\chi^{nm} - (B_1)_{nm}\phi^{\ell n} = 0 .
\end{aligned} \tag{5.10}$$

The two operators B_1 and B_2 can be separated into symmetric and antisymmetric parts. Let

$$B_s = \frac{1}{2m^2}\left(-\partial_i\partial_j(S_{ij}) - \frac{E^2}{4}S_{44}\right)$$

and

$$B_A = \frac{1}{2m^2}\left(E\,\partial_i(S_{i4}) + iV\frac{\partial_i S_i}{m^2} + i\frac{S_i(\partial_i V)}{m^2}\right) .$$

Then B_1 and B_2 can be written as

$$B_1 = B_s - B_A \,,$$

$$B_2 = B_s + B_A \,.$$

The different components of the wave functions χ, ϕ, ξ and η can be separated into symmetric and antisymmetric parts, for example

$$\chi_s^{\ell m} = \frac{\chi^{\ell m} + \chi^{m\ell}}{2} ,$$

$$\chi_A^{\ell m} = \frac{\chi^{\ell m} - \chi^{m\ell}}{2} ,$$

so that

$$\chi^{\ell m} = \chi_s^{\ell m} + \chi_A^{\ell m} .$$

Making use of the fact that B_1 and B_2 are the difference and the sum of symmetric and antisymmetric operators, we can simplify (5.10) by taking the trace of the first two equations, that is, we put $\ell = m$. Since the trace of the matrix product of two tensors matrices is zero, if one is symmetric and the other is antisymmetric, we have

$$\chi^{\ell\ell} - (B_s)_{\ell n} \xi_s^{n\ell} + (B_A)_{\ell n} \xi_A^{n\ell} - (B_s)_{n\ell} \eta_s^{\ell n} + (B_A)_{n\ell} \eta_A^{\ell n} = 0 ,$$

and

$$\phi^{\ell\ell} - (B_s)_{\ell n} \eta_s^{n\ell} - (B_A)_{\ell n} \eta_A^{n\ell} - (B_s)_{n\ell} \xi_s^{\ell n} - (B_A)_{n\ell} \xi_A^{\ell n} = 0 .$$

Subtracting and adding these two equations, we obtain

$$(\chi^{\ell\ell} - \phi^{\ell\ell}) + 2\mathrm{Tr}(B_A \xi_A) + 2\mathrm{Tr}(B_A \eta_A) = 0 ,$$

and (5.11)

$$(\chi^{\ell\ell} + \phi^{\ell\ell}) - 2\mathrm{Tr}(B_s \xi_s) - 2\mathrm{Tr}(B_s \eta_s) = 0 .$$

We can compute $Tr(B_A \xi_A)$, $Tr(B_A \eta_A)$, $Tr(B_s \xi_s)$ and $Tr(B_s \eta_s)$ from the third and fourth equations of (5.10)

$$
\begin{aligned}
Tr(B_A \xi_A) &= Tr(B_A B_2 \chi) + Tr(B_1 B_A \phi), \\
Tr(B_A \eta_A) &= Tr(B_A B_1 \phi) + Tr(B_2 B_A \chi), \\
Tr(B_s \eta_s) &= Tr(B_s B_1 \phi) + Tr(B_2 B_s \chi), \\
Tr(B_s \xi_s) &= Tr(B_1 B_s \phi) + Tr(B_s B_2 \chi).
\end{aligned} \tag{5.12}
$$

Substituting (5.12) into (5.11), we obtain

$$(\chi'' - \phi'') + 2(Tr(B_A B_2 \chi) + Tr(B_2 B_A \chi) + Tr(B_A B_1 \phi) + Tr(B_1 B_A \phi))$$

$$= 0,$$

and

$$(\chi'' + \phi'') - 2(Tr(B_s B_2 \chi) + Tr(B_2 B_s \chi) + Tr(B_s B_1 \phi) + Tr(B_1 B_s \phi))$$

$$= 0. \tag{5.13}$$

Before we simplify Eqs. (5.13), we discuss some properties of the operators B_s, B_A, B_1 and B_2 and the transformation properties of the wave functions. It can be easily shown that

$$
\begin{aligned}
B_A B_2 + B_2 B_A &= B_A B_s + B_s B_A + 2B_A B_A, \\
B_A B_1 + B_1 B_A &= B_A B_s + B_s B_A - 2B_A B_A, \\
B_s B_2 + B_2 B_s &= B_A B_s + B_s B_A + 2B_s B_s, \\
B_s B_1 + B_1 B_s &= -(B_A B_s + B_s B_A) + 2B_s B_s,
\end{aligned} \tag{5.14}
$$

and

$$\mathrm{Tr}((B_A B_s + B_s B_A)\phi_s) = 0,$$
$$\mathrm{Tr}(B_A B_A \phi_A) = \mathrm{Tr}(B_s B_s \phi_A) = 0, \tag{5.15}$$

where ϕ_s and ϕ_A are symmetric and antisymmetric respectively. For the transformation properties of the wave function, we note that the single particle wave function can be expressed as spinors in the following way:

$$\chi_1 = \chi_{\dot{1}\dot{1}},$$
$$\chi_2 = \sqrt{2}\,\chi_{\dot{1}\dot{2}},$$
$$\chi_3 = \chi_{\dot{2}\dot{2}}.$$

The spinors can be raised by the raising operator $i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ so that

$$\chi^1 = \chi^{\dot{1}\dot{1}} = \chi_{\dot{2}\dot{2}} = \chi_3,$$
$$\chi^2 = \sqrt{2}\chi^{\dot{1}\dot{2}} = -\sqrt{2}\chi_{\dot{1}\dot{2}} = -\chi_2,$$
$$\chi^3 = \chi^{\dot{2}\dot{2}} = \chi_{\dot{1}\dot{1}} = \chi_1,$$

or

$$\begin{bmatrix} \chi^1 \\ \chi^2 \\ \chi^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix}.$$

We define $g^{\ell m}$ such that

$$\chi^{\ell} = g^{\ell m}\chi_m ,$$

where

$$g^{\ell m} = g_{\ell m} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The scalar part of the wave function is given by

$$g_{\ell m}\chi^{\ell m} = \chi^{\ell}{}_{\ell} .$$

Now the tensor wave function $\chi^{\ell m}$ can be separated into three parts:- symmetric traceless, antisymmetric and scalar parts in the following way

$$\chi^{\ell m} = \left(\frac{\chi^{\ell m} + \chi^{m\ell}}{2} - \frac{1}{3} g^{\ell m}\chi^{\ell}{}_{\ell}\right) + \left(\frac{\chi^{\ell m} - \chi^{m\ell}}{2}\right) + \frac{1}{3}g^{\ell m}\chi^{\ell}{}_{\ell} .$$

The symmetric traceless part, $\frac{\chi^{\ell m} + \chi^{m\ell}}{2} - \frac{1}{3}g^{\ell m}\chi^{\ell}{}_{\ell}$, which has five components represents a particle of spin two. The antisymmetric part $\frac{\chi^{\ell m} - \chi^{m\ell}}{2}$ which has three components represents a particle of spin one and the scalar $\frac{1}{3}g^{\ell m}\chi^{\ell}{}_{\ell}$ represents a particle of spin zero.

From Eq. (5.9), it can be seen that the four components of the wave function χ, ϕ, η and ξ are the same order of magnitude in units of the velocity v. Therefore, there is no large or small component. The state of the system of two particles can be represented in the non-relativistic limit by any one of the four components. We choose χ to represent the state. Now we want to consider the simplest state of the system, namely the spin-0 state. For such a state, χ^{lm} is taken to be $\frac{1}{3}g^{lm}\chi^{l}{}_{l}$ or $g^{lm}\bar{\chi}$ where $\bar{\chi} = \frac{1}{3}\chi^{l}{}_{l}$. Using (5.14) and (5.15), we have

$$\mathrm{Tr}(B_A B_2 \chi) + \mathrm{Tr}(B_2 B_A \chi) = 2\mathrm{Tr}(B_A B_A g)\bar{\chi} ,$$

and

$$\mathrm{Tr}(B_s B_2 \chi) + \mathrm{Tr}(B_2 B_s \chi) = 2\mathrm{Tr}(B_s B_s g)\bar{\chi} ,$$

so that Eqs. (5.13) reduce to

$$-(\bar{\chi} + \phi^{ll}) + 2(2\mathrm{Tr}(B_A B_A g)\bar{\chi} + \mathrm{Tr}(B_A B_1 \phi) + \mathrm{Tr}(B_1 B_A \phi)) = 0,$$

and

$$(-\bar{\chi} + \phi^{ll}) \quad 2(2\mathrm{Tr}(B_s B_s g)\bar{\chi} + \mathrm{Tr}(B_s B_1 \phi) + \mathrm{Tr}(B_1 B_s \phi)) = 0. \tag{5.16}$$

From Eqs. (5.10), it can be seen that the four components of the wave function are grouped into pairs. That is, χ^{lm}, ϕ^{lm} appear in pairs; so are ξ^{lm} and η^{lm}. Eqs. (5.16) contain both χ^{lm} and ϕ^{lm}. χ^{lm} has already been used to specify the state. The remaining ϕ^{lm} can take one of the three possibilities

(1) symmetric traceless $\frac{\phi^{\ell m}+\phi^{m\ell}}{2} - g^{\ell m}\bar{\phi}$ where $\bar{\phi} = \frac{1}{3}\phi^{\ell}{}_{\ell}$,

(2) antisymmetric $\frac{\phi^{\ell m}-\phi^{m\ell}}{2}$,

(3) scalar $g^{\ell m}\bar{\phi}$.

We consider the simplest case. We assume $\phi = g^{\ell m}\bar{\phi}$ so that

$$\mathrm{Tr}(B_A B_1 \phi) + \mathrm{Tr}(B_1 B_A \phi) = -2\mathrm{Tr}(B_A B_A g)\bar{\phi},$$

and

$$\mathrm{Tr}(B_1 B_s \phi) + \mathrm{Tr}(B_s B_1 \phi) = 2\mathrm{Tr}(B_s B_s g)\bar{\phi}.$$

Eq. (5.16) becomes

$$(\bar{\chi} - \bar{\phi}) - 4\mathrm{Tr}(B_A B_A g)(\bar{\chi} - \bar{\phi}) = 0\,,$$

and (5.17)

$$(\bar{\chi} + \bar{\phi}) + 4\mathrm{Tr}(B_s B_s g)(\bar{\chi} + \bar{\phi}) = 0\,,$$

which can be simplified after we compute $\mathrm{Tr}(B_A B_A g)$ and $\mathrm{Tr}(B_s B_s g)$;

$$B_A = \frac{i}{2m^2}((E + \frac{V}{m^2})\partial_i S_i + \frac{1}{m^2} S_i(\partial_i V))\,,$$

or

$$(B_A)_{jk} = \frac{1}{2m^2}((E + \frac{V}{m^2})\partial_i + \frac{(\partial_i V)}{m^2})\epsilon_{ijk}\,,$$

where we use $(S_i)_{jk} = -i\epsilon_{ijk}$.

In matrix form,

$$B_A = \frac{1}{2m^2}\begin{pmatrix} 0 & (E+\frac{V}{m^2})\partial_z+\frac{(\partial_z V)}{m^2} & -(E+\frac{V}{m^2})\partial_y-\frac{(\partial_y V)}{m^2} \\ -(E+\frac{V}{m^2})\partial_z-\frac{\partial_z V}{m^2} & 0 & (E+\frac{V}{m^2})\partial_x+\frac{(\partial_x V)}{m^2} \\ (E+\frac{V}{m^2})\partial_y+\frac{(\partial_y V)}{m^2} & -(E+\frac{V}{m^2})\partial_x-\frac{(\partial_x V)}{m^2} & 0 \end{pmatrix}.$$

With the matrix represnetation of $(S_i)_{jk}$ by $-i\epsilon_{ijk}$, it can be shown that the metric g_{ij} becomes the identity δ_{ij}. Thus

$$\begin{aligned} Tr(B_A B_A g) &= (B_A B_A)_{ii} \\ &= \frac{-1}{2m^4}\Big[((E+\frac{V}{m^2})\partial_x+\frac{(\partial_x V)}{m^2})^2+((E+\frac{V}{m^2})\partial_y+\frac{(\partial_y V)}{m^2})^2 \\ &\quad +((E+\frac{V}{m^2})\partial_z+\frac{(\partial_z V)}{m^2})^2\Big]. \end{aligned}$$

For constant potential, $V = V_0$,

$$Tr(B_A B_A g) = \frac{-1}{2m^4}(E+\frac{V_0}{m^2})^2\nabla^2 \quad . \tag{5.18}$$

Next we compute $Tr(B_s B_s g)$,

$$B_s = \frac{-1}{2m^2}(\partial_i \partial_j (S_{ij}) + \frac{E^2}{4}S_{44}).$$

In matrix form, we have

$$B_s=\frac{1}{2m^2}\begin{pmatrix} \frac{E^2}{4}+\partial_x^2-\partial_y^2-\partial_z^2 & 2\partial_x\partial_y & 2\partial_x\partial_z \\ 2\partial_x\partial_y & \frac{E^2}{4}-\partial_x^2+\partial_y^2-\partial_z^2 & 2\partial_y\partial_z \\ 2\partial_x\partial_z & 2\partial_y\partial_z & \frac{E^2}{4}-\partial_x^2-\partial_y^2+\partial_z^2 \end{pmatrix},$$

$$Tr(B_sB_sg) = (B_sB_s)_{ii} .$$

Using the above matrix of B_s, it can be shown that

$$Tr(B_sB_sg) = \frac{1}{4m^4}(3(\nabla^2)^2 - \frac{E^2}{2}(\nabla^2 - \frac{3E^2}{8})). \qquad (5.19)$$

With (5.18) and (5.19), Eqs. (5.17) become

$$(2(E + \frac{V_0}{m^2})^2\nabla^2 + m^4)(\bar{\chi} - \bar{\phi}) = 0,$$

and

$$(3(\nabla^2)^2 - \frac{E^2}{2}(\nabla^2 - \frac{3E^2}{8}) + m^4)(\bar{\chi} + \bar{\phi}) = 0.$$

which are the equations describing two scalar spin-1 particles with a constant potential V_0.

CHAPTER VI

CONCLUSION

A wave equation for two particles of arbitrary spin was derived using the B-S integral equation with an arbitrary instantaneous interaction. The equation contains the sum of Hamiltonians for the two particles, and has the same form as the Breit equation which was set up for two spin-half particles. The two-particle wave equation was derived again using field theory. The advantages of this equation are that it contains a single time variable and that it is of Hamiltonian form. Moreover, when the interaction is set to be zero, the solution of the equation is just the product of two free particle wave function with no extra solutions. It also reduces to the two-particle Schrodinger equation on taking the non-relativistic limit. Applying the same formalism to Klein-Gordon particles we can show the connection between the B-S equation, the quasipotential two-particle equation and the Bogolubov equation.

The wave function for two spin-zero particles in the Hamiltonian formalism has four components. Simplifying the wave equation for two spin-zero particles gives four equations for the four components of the wave function. The four equations can be combined to yield a single equation for one component. This equation holds for two particles of unequal mass and arbitrary potential which is a function of the magnitude of the relative position of the two particles. For a square well interaction, the solutions are the product of spherical Harmonics and spherical Bessel functions. For Coulomb interaction, an approximate solution was found and energy levels were determined. In the expression for the energy levels, the first two terms are identical with the corresponding terms for non-relativistic particles. The third term gives a fine-structure energy which is of order of e^8. Our result, thus, gives the reason why one can use the energy level formula of non-relativistic hydrogen atom for a system of two pions and still obtain the result which agrees with experiment.

For the system of two spin-one-half particles, the wave function has sixteen components which can be classified into four scalar components and four 3-vector components. The wave equation can be reduced to eight equations containing both the scalar and 3-vector components. For the triplet state solution with a square well interaction, a single equation involving one unknown function is difficult to obtain. Instead, we solve a system of four simultaneous equations, making use of the special properties of the operators involved in the equations. The different components of the wave functions were found to be the product of vector Harmonics and spherical Bessel function, but in this case, the order of the spherical Bessel function is not the same for all the components. However, the condition to determine the energy levels was found to be the same as for the case of two spin-zero particles, indicating that for a square well potential the energy is independent of the total spin and depends only on the angular momentum L. For the singlet state solution, it is possible to obtain a single equation for one of the components of the wave function. This equation resembles the one we has for two spin-zero particles. The only difference is in the term involving the potential V. However, the two equations are exactly identical for the case of square well potential and for the approximate equation for Coulomb potential. With a spin independent interaction this is not hard to understand because the singlet states of the system of two spin-one-half particles have zero spin and should correspond to a system of two spin-zero particles.

The wave function of a system of two spin-one particles has thirty-six components which can be classified into four second-order tensor functions. The wave equation can be separated into four tensor equations each of which is, itself, a set of nine equations. Because of the transformation properties of these tensor functions, the metric is antidiagonal. It is only for the simplest state, the scalar-scalar state, that we can reduce the complete set of equation to two equations involving two unknown functions. These two equations holds for arbitrary interaction, and for square well interaction they can be further simplified.

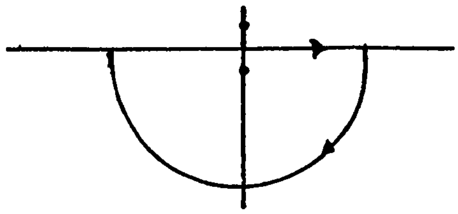
APPENDIX

DERIVATIVE OF THE STEP FUNCTION $\varepsilon(t)$

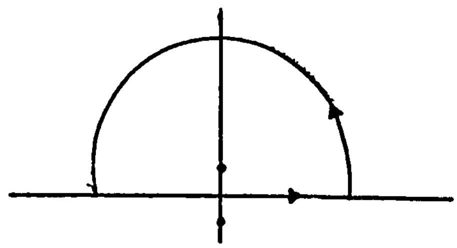
$\varepsilon(t)$ can be shown to have the form

$$\varepsilon(t) = \lim_{\delta \to 0} \left(-\frac{1}{2\pi i}\right) \int_{-\infty}^{\infty} \exp(-iwt) \frac{2w}{(w+i\delta)(w-i\delta)}\, dw. \qquad (1)$$

The two poles are $w = \pm i\delta$



(a) Contour of integration for $t > 0$.



(b) Contour of integration for $t < 0$.

For $t > 0$, the contour of integration is the lower semicircle (a) so that

$$\varepsilon(t) = \lim_{\delta \to 0} \left(-\frac{1}{2\pi i}\right)(-2\pi i)\left(\frac{\exp(-i(-i\delta))2(-i\delta)}{-2i\delta}\right) = 1 .$$

For $t < 0$, the contour of integration is the upper semicircle (b) so that

$$\varepsilon(t) = \lim_{\delta \to 0} \left(-\frac{1}{2\pi i}\right)(2\pi i)\left(\frac{\exp(-i(i\delta))2(i\delta)}{2i\delta}\right) = -1 .$$

From (1)

$$\frac{d\varepsilon(t)}{dt} = \lim_{\delta \to 0} \left(-\frac{1}{2\pi i}\right) \int_{-\infty}^{\infty} \frac{(-iw)(2w)\exp(-iwt)}{(w+i\delta)(w-i\delta)}\, dw$$

$$= 2\delta(t).$$

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