THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

GRAPH MODELS AND SYSTEM DIAGNOSIS

A Dissertation
SUBMITTED TO THE GRADUATE FACULTY
in partial fulfillment of the requirements for the
degree of
DOCTOR OF PHILOSOPHY

By
Chad C. Lamb
Norman, Oklahoma
1997

UMI Number: 9812257

GRAPH MODELS AND SYSTEM DIAGNOSIS

A Dissertation APPROVED FOR THE

SCHOOL OF ELECTRICAL AND COMPUTER ENGINEERING

BY

Linda S. DeBrunner, Co-Chair

K. Thulasiraman, Co-Chair

Anindya Das

John Fagan

John Y. Cheung

Sridhar Radhakrishnan

Acknowledgments

I would like to thank all my committee members for their help. Special thanks go to Dr. L. DeBrunner, Dr. Thulasiraman, and Dr. Das for their extra effort and guidance throughout this research. And special thanks to Dr. Fagan whose contributions were greatly appreciated. Finally, I would like to thank my family: Suzette, Timmy, and my mother and father, Margaret and Fred, for helping me through this.

Table of Contents

List of Tables

List of Figures

Abstract

The connectivity of a system is an important measure of fault tolerance. Typical interconnection networks are sparse and symmetrical. The classical diagnosis theories, such as t-diagnosability, are not appropriate for these systems because the number of faulty elements is limited by the minimum node degree, which is very small for sparse systems. Another disadvantage of the classical approach is the need for a supervisory processor. This processor is a bottleneck in the system, unnecessarily burdening the communication paths. The supervisor processor is also a single-point failure in the system. The theory of local diagnosis overcomes these limitations.

In this research, we investigate the local diagnosability of multiprocessor systems for permanently faulty processing elements (PEs). There are three system-level diagnosis problems: the characterization problem is to determine the necessary and sufficient conditions for the diagnosability of a system; the diagnosability problem is to determine the maximum value of certain parameters for which the system is diagnosable; and the diagnosis problem is to find an algorithm that identifies all faulty PEs based on a given syndrome. It is assumed that each set of faulty PEs is from some *allowable fault set*, which is a restricted set of faults.

We show a solution to the characterization, diagnosability, and diagnosis problems with the *t-in-L_2 diagnosability* theory. In this theory, if there are at most t faulty PEs in the local neighborhood set of x, and at most t(m)

faulty PEs in the extended-local neighborhood set of x, then we can uniquely diagnose any graph. The extended-local fault constraint, t(m), is a function of the local fault constraint, t, and the degree of node x.

We develop a general approach for the t-in-L_2 diagnosis of an arbitrary graph. We apply the approach to the special cases of mesh and hypercube graphs. We also show the applicability of our theory to the diagnosis of the GNSS system.

1. Introduction

The idea of a computer system consisting of multiple processors is not new. Parallel computing machines are found in the literature as far back as the 1920s [DEN86]. A concise history of parallel processing is presented in [HOC81].

Sequential computers are approaching a fundamental physical limit on their computational power, the speed of light (3×10^8 m/sec in a vacuum). This results in a signal transmission speed for silicon of at most 3×10^7 m/sec [DEC89]. A chip of 3 cm in diameter can propagate a signal in approximately 1 nanosecond. If such a chip can support one floating-point operation in the time of one signal propagation, then a computer equipped with such a chip can handle a maximum of 1000 MFLOPS (assuming the processor has a single ALU unit, one memory unit, etc.). A similar limit can be estimated for other materials, such as gallium arsenide (GaAs). Therefore, sequential processors will reach the maximum speed physically possible in the near future.

There are many problems that require over 1,000 times the maximum computing power of uni-processors. These include three-dimensional fluid flow calculations, real-time simulations of complex systems, and intelligent robots [DEC89]. Thus, the need for parallel computers is evident. Research is succeeding in developing sophisticated techniques to penetrate the software barrier. Compilers that convert existing sequential high-level code into parallel machine code that will run efficiently on new parallel architectures exist. Parallel programming languages have been defined and implemented that take advantage of the parallelism of algorithms to solve specific problems [DEC89].

1.1. The Need for Fault Tolerance

Some massively parallel processing grand challenge applications can be solved now thanks to the development of multiprocessor systems like the Cray MPP, the Fujitsu VPP500, Intel's Paragon, and Thinking Machines' CM-5, to name a few [HWA93]. These machines can be configured to have thousands of processing elements (PEs) connected in a mesh array, hypercube, or some other topology. However, the introduction of these machines to solve these problems has created another problem for the system designer -- the need for fault tolerance.

Assume each PE (or node) in the system has a small probability of failure, λ. Also assume the failure of each PE is independent of the failure of any other. Then, the probability of system failure will be $m\lambda$ for a system with m PEs. For a uni-processor system, this probability will be relatively insignificant since m=1. However, when m = 1000, as in the multiprocessor systems described above, the probability of system failure becomes significant. Therefore, fault tolerance has become a necessity, rather than a luxury. If we design our system such that it can tolerate one or more PE failures and still perform the desired function, then we can considerably reduce this $m\lambda$ probability of failure.

Fault tolerance is applied to other systems as well, not just multiprocessor systems. Networks, critical applications, like space systems or medical systems, and aircraft navigation systems are a few examples where fault tolerance is designed into the system.

1.2. System-Level Diagnosis

In a distributed multiprocessor system, identification of faulty PEs is essential for the continued life of the system. A system can withstand faults by logically removing the faulty nodes. The system experiences *graceful degradation* if it is designed with this capability. Graceful degradation is defined as the ability of a system to decrease its level of performance, while continuing to perform its operations. The system continues to function properly, but the performance is reduced.

Identification of faulty PEs enables us to swap-in fault-free PEs to replace the faulty ones. This is called reconfiguration. The system tolerates faults, and does not experience graceful degradation since the faulty PEs are replaced with good ones. The system will perform at a performance level that is similar to the original performance level.

System-level diagnosis (SLD) is a method by which faulty PEs in a system are identified. Preparata et al. [PRE67] describe such a method. In SLD, all PEs in the system will perform a task or "test." The outcome of this task is compared with the outcome of a neighbor. If the outcomes do not match, then at least one of the PEs is faulty. Every pair of adjacent PEs in the system performs this comparison (this is the comparison model, see Section 2.3). Once the faulty units are identified, a reconfiguration scheme can remove them from the system.

A problem with traditional SLD methods, also shown in [PRE67], is that they approach the problem of diagnosis globally [DAS89]. That is, the diagnosis of the system is controlled by a supervisory node. The supervisory node distributes the necessary information to each node and collects all data, e.g., test results, from every node. While this global approach may appear to simplify the process, it creates an unwanted bottleneck in the system. Also, traditional approaches, such as t-diagnosable theory, allow a maximum of t faulty PEs in the system. For a mesh structure, t = 3, which is the degree of the graph (see Section 2.1 on graph theory basics). So a mesh with n = 50 PEs is 3-diagnosable, as is a mesh with n = 1000 PEs. The drawbacks of global diagnosis are further examined in Chapter 2.

To overcome these problems, probabilistic approaches and methods based on local fault constraints have been considered. Probabilistic approaches use simple algorithms that identify faulty PEs with a very high probability. In [BLO88] it was shown that each PE in the system must be tested by (adjacent to) more than log n other PEs, where n is the number of PEs in the system, for a high probability of diagnosis. Others showed that as $n \rightarrow \infty$, the probability of diagnosis$\rightarrow 1$ [SCH87][BLO89][RAG89][DAH87][FUS89]. Another approach is to perform more than one test, where a test is defined to be the execution of the same task by all PEs, then the outcomes of neighbors are compared [FUS89]. In all these approaches, there is still some probability of not being able to diagnose the system, i.e., these approaches are not deterministic.

These shortcomings are mitigated with a deterministic, local diagnosis approach. Each node is responsible for testing its immediate neighbors. All test results are stored locally, and the bottleneck of a supervisory node is eliminated. Local diagnosis is discussed in detail in Section 2.6.

In this research, we investigate local diagnosis approaches to system-level diagnosis. Each node uses information from its immediate neighbors and from their neighbors. We refer to this as the local neighborhood and the extended-local neighborhood.

1.3. Summary

As computer systems become more complex, there is an increasing need for fault tolerance. A multiprocessor system must have the ability to perform self tests and to isolate the faults in the system as they occur. Once a fault has been diagnosed, a reconfiguration scheme is used to either remove the faulty node by logically replacing it with a spare node or allow the system to perform without the faulty node. However, the first step is to recognize a fault in the system, which is the goal of this research. In the chapters that follow, we develop a new system-level diagnosis theory called t-in-L_2 diagnosis.

2. System-Level Diagnosis: Background

A classic paper by Preparata, Metze, and Chien was written over thirty years ago on system-level fault diagnosis [PRE67]. In this paper, the authors describe a general procedure to identify faulty and fault-free PEs in a multiprocessor system. Each PE is tested by other PEs in the system. The authors make no assumptions about the PEs, i.e., each PE can be either faulty or fault-free. However, the specific model used to determine the test results and the interpretation of those test results is not specified.

There are several models in the literature [SOM87] for the interpretation of the test results, the most common of which are described in Section 2.3. Before describing these models, and the basic diagnosis problems (Section 2.2), we review some graph theory and set theory basics in Section 2.1.

2.1. Background

In this section, we provide definitions for the terms we use in this research. This includes graph theory, set theory, and system-level diagnosis. Some basic knowledge of graph and set theory is assumed.

We start by defining the *Principle of Mathematical Induction.* The Principle of Mathematical Induction is used to prove a statement, A_n, involving an integer n. We start with A_1 being true, where A_1 is a statement. Then we want to show A_{c+1} is true, assuming A_c is true for any positive integer c. If we show this, then we can say A_n is true for any positive integer n. This is the Principle of Mathematical Induction.

A *graph* is a set of nodes joined by a set of edges. We denote this by $G=(U, E)$, where U is the set of nodes in G and E is the set of edges of G. A *path* in a graph is a set of distinct nodes that are connected by distinct edges (if the nodes are distinct then necessarily the edges are distinct).

A graph is *connected* if all pairs of nodes are joined by a path. A *cycle* in a graph is a path that begins and ends at the same node, i.e., $x_0 = x_n$. The *cycle length* is the distance between x_0 and x_n, namely $d(x_0, x_n)= n$.

The *degree* of a node x_i, $\deg(x_i)$ or d, in a graph is the number of edges incident on x_i. A graph is called *regular* of degree n if all nodes have the same degree, i.e., if $\deg(x_i) = n, \forall x_i \in U$.

The *distance* between two nodes in a graph, $d(x_i, x_j)$, is the length of the shortest path between x_i and x_j in the graph. A shortest x_i - x_j path is called a *geodesic*. A *closed ring* or *cutset*, R, in a graph G is a set of nodes in G whose removal disconnects the graph, i.e., there are at least two components in G-R. A *minimum cutset* or *proper cutset* is a set of nodes whose removal reduces G to a graph with exactly two components, i.e., G-R has exactly two components. If $x_i \in R$ then the graph $G-[R-\{x_i\}], \forall x_i \in R$, is connected if R is a proper cutset.

A multiprocessor system M can be modeled as a graph G by representing each processor in M by a node in G. The interconnects in M may be unidirectional (*directed graph*) or bi-directional (*undirected graph*). Two nodes, x_i and x_j, in G are adjacent if $d(x_i, x_j) = 1$, where $x_i, x_j \in U$.

A *syndrome* is the collection of test results between all adjacent processors in a system. All processors are assigned to perform identical tasks, after which the outputs of adjacent nodes are compared. Two nodes, x_i and x_j, are associated with an outcome a_{ij} in a syndrome S if $d(x_i, x_j) = 1$. For the comparison model, for example, $a_{ij} = 0$ if the outputs of x_i and x_j match (either x_i and x_j are both fault-free or both faulty), $a_{ij} = 1$ if the outputs of x_i and x_j do not match (at least one of the two nodes is faulty). If $a_{ij} = 0(1)$ then we say x_i and x_j are joined by a 0-link (1-link). See Section 2.3 on diagnosis models for more details on the different models.

A *fault set*, F, is a set of nodes in G, all of which are faulty. A *permissible fault set* is one that does not violate the fault constraints. An *allowable fault set*, F, is a permissible fault set that is consistent with the given syndrome, that is, the nodes in F are faulty, the nodes in *U* - F are fault-free. *Syndrome decoding* is the interpretation of the syndrome that is consistent with the requirements of the fault constraints and consistent with the given syndrome S.

There may be multiple allowable fault sets, $S(F_i)$, that are consistent with a given syndrome S. We can call two such sets F_i and F_j, where $F_i \neq F_j$. The union of F_i and F_j is denoted by $F_i \cup F_j$, the intersection is denoted by $F_i \cap F_j$, and the symmetrical difference is denoted by $F_i \oplus F_j$. These sets are shown in Figure 2.1.

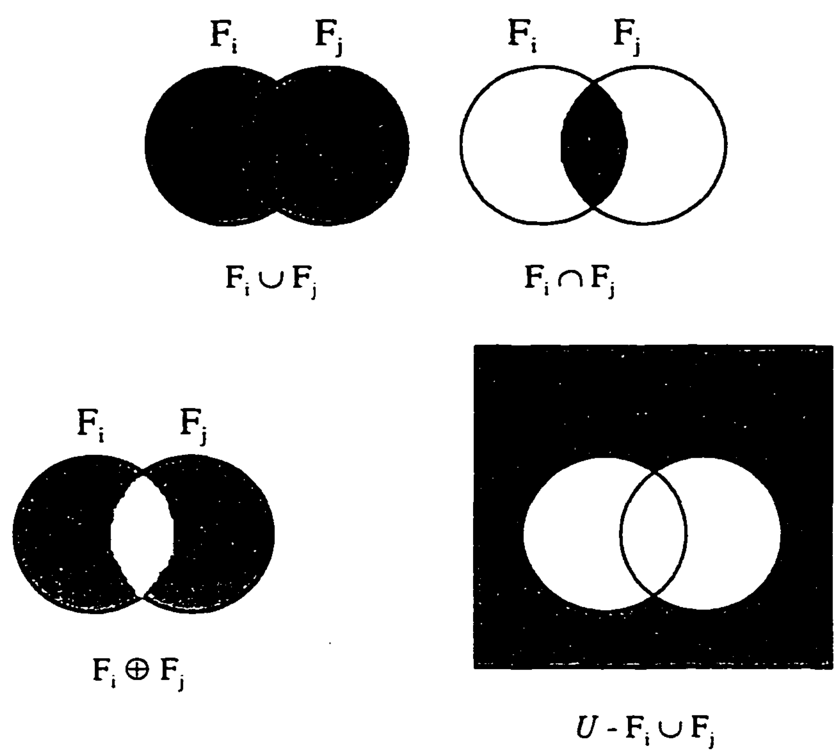


Figure 2.1 Set Theory

The nodes in the set $F_i \cap F_j$ are faulty with respect to either fault set, and the nodes in U - $(F_i \cup F_j)$ are fault-free with respect to either fault set. The nodes in F_i - F_j are faulty with respect to F_i but fault-free with respect to F_j. Likewise, the nodes in F_j - F_i are faulty with respect to F_j and fault-free with respect to F_i. So, if $z \in F_i \oplus F_j$ then z cannot be labeled as faulty or fault-free and the graph is *not diagnosable*, since we cannot uniquely identify every node in the graph. A graph is *diagnosable* if and only if $F_i = F_j$ $\forall$i,j, and i$\neq$j, i.e., there is only one unique fault set for any given syndrome.

2.2. Diagnosis Problems

There are three basic problems addressed in the literature regarding system-level diagnosis: the characterization problem, the diagnosability problem, and the diagnosis problem [SOM87].

The *Characterization Problem*

The characterization problem is to determine the necessary and sufficient conditions for the diagnosability of a system.

The *Diagnosability Problem*

The diagnosability problem is to determine the maximum value of certain parameters for which the system is diagnosable.

The *Diagnosis Problem*

The diagnosis problem is to find an algorithm that identifies all faulty PEs based on a given syndrome. It is assumed that each set of faulty PEs is from some *allowable fault set*, which is a restricted set of faults.

2.3. Diagnosis Models

There are three popular models in the literature used for system-level diagnosis: the *symmetric invalidation model* (PMC model) [PRE67], *asymmetric invalidation model* (BGM model) [BAR76], and the *comparison model* [MAL80, SOM87].

In the PMC model, the result of a test is reliable only if the testing PE is fault-free. If a faulty PE is testing another PE in the system, the result will be unreliable. This model is shown in Figure 2.2a. An unreliable test result is shown as a 1\0. The syndrome must be interpreted to determine the faulty PEs.

In the BGM model, when a faulty PE tests another faulty PE, the test outcome will be a 1. So, unlike the PMC model, the testing PE can be faulty, and the test result will be reliable if the other PE is also faulty. This model is shown in Figure 2.2b. If a faulty PE tests a fault-free PE, the test outcome is unreliable.

In the PMC and BGM models, a PE will test another PE (or PEs) in the system. The test links are uni-directional. In the comparison model, the test links are bi-directional. To implement this, the same task is assigned to every PE in the system, and the outputs are compared. A test outcome (comparison) is unreliable only if both PEs are faulty. This model is shown in Figure 2.2c. Similar to the other two models, the syndrome must be interpreted to determine which nodes are actually faulty.

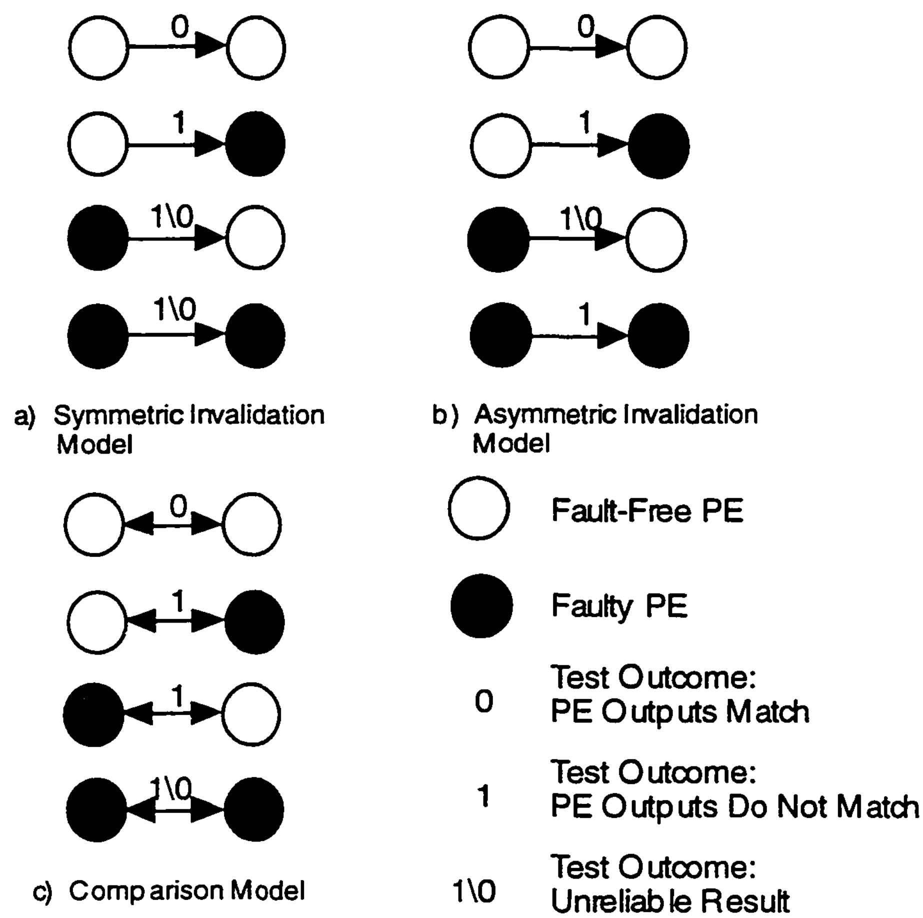


Figure 2.2 Classic Models for System-Level Diagnosis

2.4. Measures of Diagnosability

In this section we describe the existing theories used in system-level diagnosis. The classical approach is based on the t-diagnosis theory. We also summarize work done on the t-characterization problem and t-diagnosis algorithms. An improvement on this method is the t/s-diagnosability theory, whereby fewer tests are required to identify the faulty PEs. Then we look at a simplified theory called sequential diagnosis. The distributed and probabilistic approaches complete our review.

2.4.1. t-Diagnosis

Given a syndrome and a system with no more than t faults, the system is said to be *t-diagnosable* if every PE in the system can be correctly identified as faulty or fault-free. Relating this to the SLD problems defined in Section 2.2, the t-characterization problem is to determine the necessary and sufficient conditions for the system to be t-diagnosable. The t-diagnosability problem determines the largest value of t for which the system is t-diagnosable. The t-diagnosis problem addresses the location of faults in the system, given that the system is t-diagnosable. t-diagnosis is referred to as *diagnosis without repair*, which means any fault set is uniquely and correctly identified.

The t-characterization problem for the PMC model was considered in [HAK74][ALL75][KOH78]. The t-characterization problem for the BGM model was considered in [BAR76]. The t-characterization problem for the comparison model was considered in [CHW81].

Many algorithms exist in the literature addressing the t-diagnosability problem [SUL84][NAR86][RAG89][SOM89-1]. For example, Sullivan [SUL84] developed an algorithm based on the PMC model to determine the t-diagnosability of a system in $O\,(mp^{1.5})$, where m is the number of tests and p is the number of PEs in the system. Raghavan [RAG89] developed a better algorithm that runs in $O\,(pt^{2.5})$ (assuming a large number of PEs and a correspondingly small number for t). Somani [SOM89] developed an algorithm based on the BGM model to determine the t-diagnosability of a system in $O\,(p^3)$.

Kameda et al. described a polynomial time t-diagnosis algorithm in [KAM75]. Dahbura and Masson [DAH84] developed an algorithm based on the PMC model for t-diagnosis with complexity $O\,(n^{2.5})$. Sullivan [SUL88] presented a branch and bound technique based on the algorithm in [KAM75] with complexity $O\,(|E|+t^3)$, where $|E|$ is the number of tests. Meyer [MEY84] presented an $O\,(m)$ algorithm for the BGM model.

2.4.2. t/s-Diagnosis

Friedman introduced the concept of t/s-diagnosability in [FRI75]. A system is said to be *t/s-diagnosable* if there are no more than t faults in the system, all of which can be isolated within a group of s PEs, where $s \geq t$. This group of PEs may also contain fault-free PEs. Thus, this method relaxes the constraint of all faulty and fault-free PEs being correctly identified. Therefore, a system that is t/s-diagnosable requires fewer tests than a t-diagnosable system.

Sullivan [SUL86] developed algorithms for t/t-diagnosability and t/t+k-diagnosability that run in polynomial time. Das and Thulasiraman [DAS89][DAS93] presented algorithms for the t/s-diagnosis and t/t+1 diagnosis problems. Yang et al. considered the t/t-diagnosis problem in [YAN86].

2.4.3. Sequential Diagnosis

In *sequential diagnosis*, the objective is to correctly identify at least one faulty PE in the system [SOM87][MAL80]. This approach is called an *incomplete diagnosis*, since all faulty units may not be identified by a given syndrome. Using a reconfiguration method, the faulty PE is replaced and the diagnosis is performed again to identify another faulty PE. Sequential diagnosis is also referred to as *diagnosis with repair.* An algorithm that performs diagnosis with repair is considered a low-quality diagnosis algorithm, whereas diagnosis without repair is considered high-quality [SOM87][RAG89].

2.4.4. Distributed Diagnosis

Kuhl and Reddy introduced *distributed diagnosis* in [KUH80]. With this method, each PE tests its neighbors and forwards the results, as well as the test results from its neighbors that are fault-free, to other neighbors. In doing so, each fault-free PE creates a table of the test outcomes for the system.

A system is *t-fault diagnosable with distributed control* if every fault-free PE in the system can correctly identify every other PE in the system as faulty or fault-free, provided there are at most t faults. A distributed diagnosis algorithm for regular structures was proposed by Somani and Agarwal in [SOM89-2]. The authors also describe the hardware implementation for local syndrome decoding. At the University of Erlangen – Nürnberg [MAE86] and at Carnegie-Mellon University [BIA90][BIA91], distributed system-level diagnosis algorithms have been implemented.

2.5. Probabilistic Diagnosis

With probabilistic diagnosis, each PE is associated with it a probability of being faulty. The idea behind probabilistic diagnosis is to determine the most likely fault set. A test can also have a probability of being faulty, since a test may not have complete fault coverage, as shown in [BLO77]. Dahbura et al. considered the probability of failure for PEs and a probability of correctness for the tests [DAH87]. A system is *p-t diagnosable* if there is a unique, consistent fault set whose probability of occurrence is greater than p for every syndrome [MAH76].

2.6. Local Diagnosis

The above diagnosis algorithms are performed using a supervisory PE that collects the test results. If this supervisory PE is faulty, then the entire system cannot be diagnosed. Thus the supervisory PE is a bottleneck in the system. For local fault diagnosis, a PE and its immediate neighbors handle the fault diagnosis. Also, there is a maximum value for t in a t-diagnosable system. This bound on t is very small compared to the number of PEs in the system (it is generally the degree of the smallest degree PE in the system). The reason for this is the limited interconnection between processors in many popular systems, e.g., rectangular grids, hypercubes, and binary trees. So, in a rectangular grid, for example, there can be no more than $t = 4$ faulty nodes. This limitation motivates the need for local diagnosis.

2.6.1. t-in-L_1 Diagnosis

If L(x) denotes the set of PEs adjacent to x in a multiprocessor system, then a system is said to be *t-in-L_1 diagnosable* if there are no more than $t \leq \left\lfloor \frac{|L(x)|}{2} \right\rfloor + 1$ faults in $L(x) \cup \{x\}$, and the system exhibits certain interconnection characteristics [DAS93]. Determining if a system is locally diagnosable is more practical than the global diagnosis methods. One advantage is that nodes in the system can be tested simultaneously. This can reduce test time dramatically. Also, the fault constraints for global diagnosis are impractical, since there are relatively few allowable faults compared to the number of PEs in a multiprocessor system. For example, a 1024 node system configured as a mesh is only 3-diagnosable in the traditional global diagnosis approach, since it can tolerate at most 1 less than the connectivity of the entire system. The local diagnosis approach for the same 1024 node system can tolerate 511 faults.

In [DAS93], sufficient conditions for the unique local diagnosis of a system were derived. They also examined several regular interconnected systems for t-in-L_1 diagnosability. Using the comparison model, the authors investigated the t-in-L_1 diagnosability of the closed rectangular grid, hexagonal and octagonal grid systems, and hypercubes.

2.7. Summary

In this chapter, we have summarized the work done in system-level diagnosis. We described the three system-level diagnosis problems: the characterization problem, the diagnosability problem, and the diagnosis problem. We described the various models used in solving the system-level diagnosis problems: the symmetric invalidation model, the asymmetric invalidation model, and the comparison model.

We also reviewed the measures of diagnosability used in system-level diagnosis. The measures include deterministic and probabilistic approaches, sequential and distributed approaches, local and global approaches. We concluded the chapter summarizing the local diagnosis approach t-in-L_1 diagnosis. This approach was the inspiration for the diagnosis theories we develop in this research.

3. System-Level Diagnosis: Local Diagnosis

In [DAS93], the t-in-L_1 diagnosis theory is developed. In this theory, a graph $G(U, E)$ is defined to be t-in-L_1 diagnosable if all faulty nodes can be uniquely diagnosed, assuming that there are fewer than $|U|/2$ faulty nodes in G, and there are no more than $\left\lfloor \frac{\deg(x)}{2} \right\rfloor + 1$ faults in the local neighborhood of x, $\forall x \in U$. For this chapter, we develop a new approach to local diagnosis: the t-in-L_2 diagnosis theory. We first define our notation.

Let the nodes in L(x) be identified as the set $\{z_1, z_2, ..., z_{\deg(x)}\}$ and the nodes in $L(z_i)$ be identified as the set $\{x, z_{i1}, z_{i2}, ..., z_{i(\deg(x)-1)}\}$, where i=1, 2, ..., deg(x). This is shown in Figure 3.1 for a regular graph of degree 4. We call the set $L(x) \cup \{x\}$, the *local set of x*, and the set $L(L(x)) \cup L(x) \cup \{x\}$, the *extended-local set of x* (we have simplified the notation by letting $L(\{L(x)\}) = L(L(x))$, which represents the set of nodes that are distance 2 from x). The local set of x and extended-local set of x are shown by the shaded nodes in Figure 3.2a and Figure 3.2b, respectively.

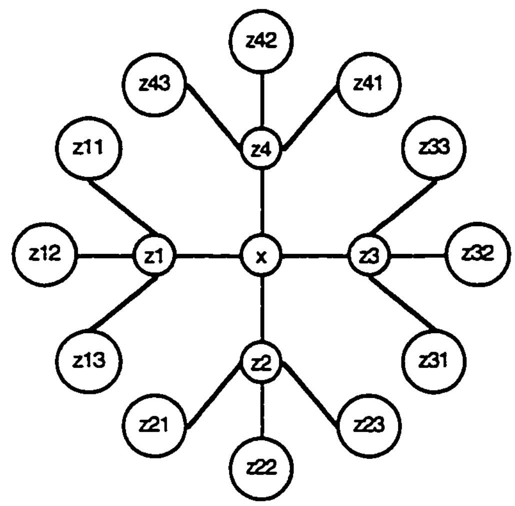
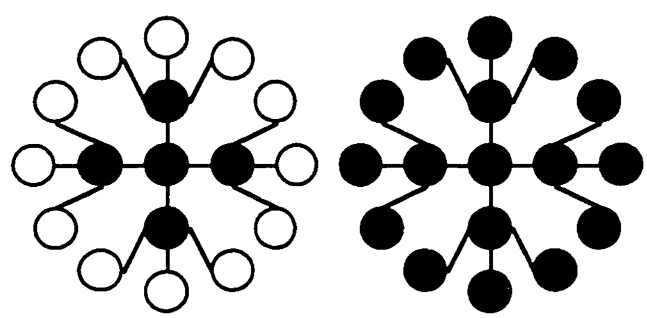


Figure 3.1 Regular Graph with Minimum Cycle Length 5



Local Set of x (a) Extended-Local Set of x (b)

Figure 3.2 Local and Extended-Local Set of x

Let $\overline{N}(\{x\})$ be the set of nodes in L(x) that share a 0-link with x and N({x}) be the set of nodes in L(x) that share a 1-link with x. To simplify the notation, we call $\overline{N}(\{x\}) \rightarrow \overline{N}$ and $N(\{x\}) \rightarrow N$. Similarly, $\overline{N}(\overline{N})$ is the set of nodes that share a 0-link with the nodes in $\overline{N}$ (not including the nodes in $\overline{N}$), $\overline{N}(N)$ is the set of nodes that share a 0-link with the nodes in N (not including the nodes in

N), $N(\overline{N})$ is the set of nodes that share a 1-link with the nodes $\overline{N}$ (not including the nodes in $\overline{N}$), and N(N) is the set of nodes that share a 1-link with the nodes in N (not including the nodes in N).

Let N_i be the node $z_i \in$ N that shares a 1-link with x and $N(N_i)$ be the set of nodes that share a 1-link with z_i, not including the node z_i. Similarly $\overline{N}(N_i)$ is the set of nodes that share a 0-link with z_i. Referring to Figure 3.3, $N = \{z_2, z_3\}$, $\overline{N} = \{z_1, z_4\}$, $\overline{N}(\overline{N}) = \{z_{42}\}$, $\overline{N}(N) = \{z_{22}, z_{32}\}$, $N(\overline{N}) = \{z_{11}, z_{12}, z_{13}, z_{41}, z_{43}\}$, $N(N) = \{z_{21}, z_{23}, z_{31}, z_{33}\}$, $\overline{N}(N_3) = \{z_{32}\}$, $N(N_3) = \{z_{31}, z_{33}\}$.

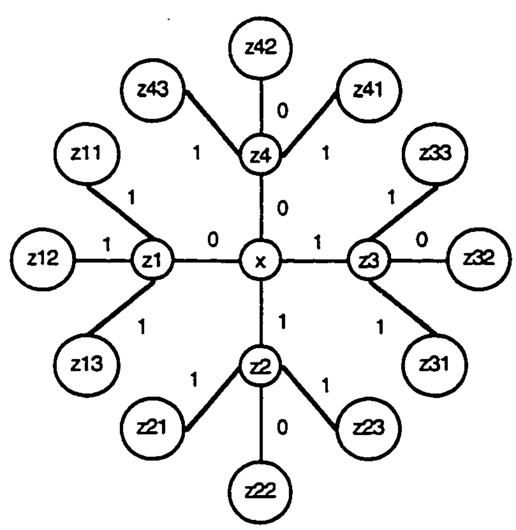


Figure 3.3 Graph with a Syndrome

In the following arguments, the local fault constraint (lfc) is the number of faulty nodes in the local set of x that are allowed. The extended-local fault constraint (elfc) is the number of faulty nodes in the extended-local set of x that are allowed. Also, in the figures that follow, the faulty nodes are depicted by dark circles, fault-free nodes by light circles, and nodes labeled using the notation z_{ij} are undetermined.

3.1. Asymmetric-Comparison Model

We introduce the asymmetric-comparison model, shown in Figure 3.4c. It combines the comparison model (Figure 3.4a) and the asymmetric invalidation model (Figure 3.4b). In the asymmetric-comparison model, a fault-free node and a faulty node generate a test outcome of 1, as in the comparison model. However, the comparison model assumes an unreliable test outcome when both nodes are faulty. The asymmetric invalidation model assumes two faulty nodes will not produce the same result and therefore cannot produce a test outcome of 0. The asymmetric-comparison model gives a test outcome of 1 when either node is faulty or both nodes are faulty.

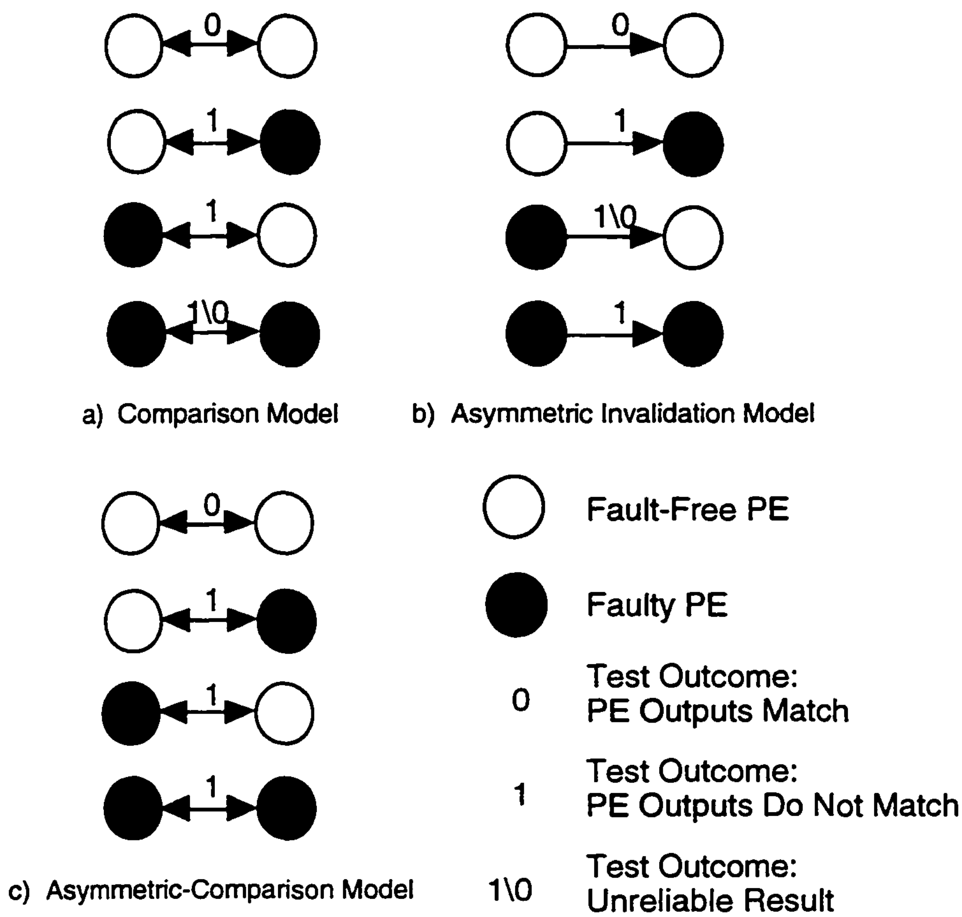


Figure 3.4 Asymmetric Comparison Model

Now that we have introduced a new diagnosis model, the asymmetric-comparison model, we can use this model to determine the diagnosability of a graph. The asymmetric-comparison model is an optimistic model in that it assumes the test outcome between any pair of nodes, with at least one of those nodes being faulty, will always result in a 1-link. In both the asymmetric invalidation model and the comparison model, there is an unreliable test result, as shown in Figure 3.4. First we develop the theory.

Lemma 3.1

Given a system G with syndrome S, let F_1 and F_2 be two allowable fault sets, where $F_1 \cup F_2 \neq U$. Let $x \in F_1 \oplus F_2$ and let $y \in L(x)$. Using the asymmetric-comparison model:

1. if $x \in F_1 - F_2$ then $y \in F_2$
2. if $x \in F_2 - F_1$ then $y \in F_1$.

Proof:

Condition 1: $x \in F_1 - F_2$.

Since $x \in F_1 - F_2$, x is faulty with respect to F_1 and fault-free with respect to F_2. Using the asymmetric-comparison model, there is a 1-link between x and y, since x is faulty with respect to F_1. Now, x is fault-free with respect to F_2, but there is a 1-link between x and y, so $y \in F_2$.

Condition 1: $x \in F_2 - F_1$.

Since $x \in F_2 - F_1$, x is faulty with respect to F_2 and fault-free with respect to F_1. Using the asymmetric-comparison model, there is a 1-link between x and y, since x is faulty with respect to F_2. Now, x is fault-free with respect to F_1, but there is a 1-link between x and y, so $y \in F_1$. ❑

Theorem 3.1

Given a system G with syndrome S. If there are no more than $t = |L(x)| - 1$ faults in $L(x) \cup \{x\}$, $\forall x \in U$, then G is uniquely diagnosable, under the asymmetric-comparison model.

Proof:

Assume the system G is not uniquely diagnosable. There are two allowable fault sets, F_1 and F_2, where $F_1 \neq F_2$. Then there exists $x \in F_1 \oplus F_2$. Without loss of generality, let $x \in F_1 - F_2$. Then, by Lemma 3.1, $y \in F_2$ $\forall y \in L(x)$. However, then there are $|F_2| = |L(x)|$ faults in $L(x) \cup \{x\}$, which is impossible. Therefore, by contradiction G is uniquely diagnosable. ❑

3.2. Local Diagnosis - Comparison Model

We have shown in the previous chapter that local diagnosis has many advantages over global diagnosis. In this section, we expand on the local diagnosis theory. We use the comparison model throughout the remainder of this research. We start our analysis by developing several lemmas and theorems to support our theory.

Lemma 3.2

From [DAS93]: Given a graph G and syndrome S. Let F_1 and F_2 be two distinct allowable fault sets for a given syndrome such that $F_1 \cup F_2 \neq U$. The $L(z) - F_1$ and $L(z) - F_2$ are both nonempty, $\forall z \in U$. Under the comparison model, there exist x, $y \in U$ such that:

1. $y \in U - (F_1 \cup F_2)$
2. $x \in (F_1 \oplus F_2)$
3. $2 \leq d(x, y) \leq 3$ and $d(x, y)$ is minimum among all x and y satisfying conditions 1 and 2.

Proof:

See [DAS93].

Lemma 3.3

Given a graph G which satisfies the local fault constraint of no more than $k = deg(x) - 1$ faulty nodes in $L(x) \cup \{x\}$, $\forall x \in U$. Let F_1 and F_2 be two distinct allowable fault sets for a given syndrome S, where $F_1 \cup F_2 \neq U$, and $L(x) - F_1$ and $L(x) - F_2$ are both nonempty. So $\forall x \in F_1 \oplus F_2$, if $y \in L(x)$ then under the comparison model:

1. If $|L(x)| > 3$ then:

(a) $|F_1 \cap F_2 \cap L(x)| \leq |L(x)| - 3$

(b) $|(F_1 \oplus F_2) \cap L(x)| \geq 3$.

2. If $|L(x) \leq 3$ then:

(a) $y \in F_1 \oplus F_2$.

<u>Proof:</u>

Condition 1a:

Without loss of generality, assume $x \in F_1 - F_2$. Then, x is faulty with respect to F_1 and fault-free with respect to F_2 (see Figure 3.5). Let *Q2* denote the set of nodes in L(x) that are fault-free with respect to F_2. Every node in *Q2* must have a 0-link with x since they are all fault-free with respect to F_2 and fault-free pairs of nodes must be connected by a 0-link. Since every node in *Q2* has a 0-link with x, and since x is faulty with respect to F_1, every node in *Q2* is faulty with respect to F_1.

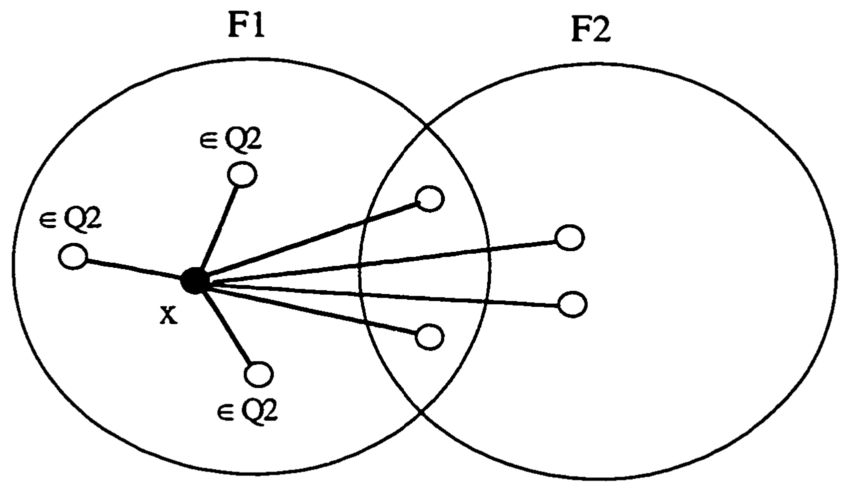


Figure 3.5 Sets F1, F2, F1$\cap$F2

Since there are at most k faulty nodes in L(x)$\cup$\{x\} and since x is faulty with respect to F_1, then $|Q2| \leq k - 1$. Furthermore, each node in L(x)-*Q2* is faulty with respect to F_2, since *Q2* denote the set of nodes in L(x) that are fault-free with respect to F_2. Therefore, $|L(x)-Q2| \leq k$ (by the lfc), and $|Q2| \geq |L(x)| - k$, which simplifies to $|Q2| \geq 1$. Thus:

$$1 \leq |Q2| \leq k - 1.$$

If more than k - 2 nodes in L(x)-*Q2* are faulty with respect to F_1, then the number of faulty nodes with respect to F_1 exceeds the lfc of k in L(x)$\cup$\{x\}, because $|Q2| \geq 1$, $Q2 \subset F1$, and x is faulty with respect to F_1.

Condition 1b:

Since all nodes in *Q2* are contained in F_1-F_2, and all nodes except at most k - 2 in L(x) - *Q2* belong to F_2-F_1, then there are at least $|Q2| + [|L(x)| - |Q2| - (k - 2)] = |L(x)| - (k - 2) = 3$ nodes which belong to $F_1 \oplus F_2$. So $|(F_1 \oplus F_2) \cap L(x)| \geq 3$.

Condition 2a.

We know $y \notin F_1 \cap F_2$, otherwise the local fault constraint is violated. Also, $y \notin U - (F_1 \cup F_2)$ by Lemma 3.2. Therefore, $y \in F_1 \oplus F_2$, $\forall y \in L(x)$. ❑

Lemma 3.4

Let G denote a graph that satisfies the local fault constraint of no more than k = deg(x) - 1 faulty nodes in $L(x) \cup \{x\}$, $\forall x \in U$. Given a syndrome S under the comparison model, let w be a node in G, with at least deg(w) - 2 nodes in L(w) that have been correctly diagnosed as faulty or fault-free. Then, w can also be correctly diagnosed.

<u>Proof:</u>

Let *CL* denote the set of correctly diagnosed nodes in L(w), and let $y \in CL$. Then $|CL| \geq$ deg(w) - 2.

Case 1: y is fault-free, for some $y \in CL$.

If y is fault-free for some $y \in CL$, then w is uniquely diagnosable.

Case 2: y is faulty, for all $y \in CL$.

If y is faulty for all $y \in CL$, then there are at least deg(w) - 2 faulty nodes in L(w). If w has a 1-link with each of the unlabeled nodes, then w must be faulty and the unlabeled nodes in L(w) are fault-free. If w has at least one 0-link with an unlabeled node in L(w), then w is fault-free. So w and all nodes in L(w) can be correctly diagnosed as faulty or fault-free. ❑

Theorem 3.2

Given a graph G that satisfies the local fault constraint (lfc) of no more than deg(x) - 1 faulty nodes in $L(x) \cup \{x\}$, $\forall x \in U$. Given a syndrome S, let F_1 and F_2 be two distinct allowable fault sets for S, with $F_1 \cup F_2 \neq U$. If there is no vertex cut set R in G such that $y \in F_1 \cap F_2$, $\forall y \in R$, then the graph is uniquely diagnosable.

<u>Proof:</u>

Since $F_1 \cup F_2 \neq U$, then $\exists x_n \in U - (F_1 \cup F_2)$. Assume there exists $z \in F_1 \oplus F_2$. There is no vertex cut set R in G such that $y \in F_1 \cap F_2$, $\forall y \in R$. Then, $\forall x_i, x_j \in U$, there exists at least one open path P from x_i to x_j such that there does not exist $y \in F_1 \cap F_2$ in P [HAR69]. Letting $x_i = z$ and $x_j = x_n$, we define the path P as $zx_1 \ldots x_{n-1}x_n$.

Since $z \in F_1 \oplus F_2$, then by Lemma 3.3, any node in L(z) must belong to $F_1 \cap F_2$ or $F_1 \oplus F_2$ (if $h \in F_1 \cup F_2$, then $h \in F_1 \cap F_2$ or $h \in F_1 \oplus F_2$). Since $P \cap (F_1 \cap F_2) = \varnothing$, then $x_1 \in P \cap L(z)$ must be a member of the set $F_1 \oplus F_2$. Similarly, since $x_2 \in P \cap L(x_1)$, then x_2 must also be a member of set $F_1 \oplus F_2$

So, $x_k \in [P \cap L(x_{k-1}) \cap F_1 \oplus F_2]$ for k=1,2. Now, suppose $x_k \in [P \cap L(x_{k-1}) \cap F_1 \oplus F_2]$ for some integer k>2. We want to show that $x_{k+1} \in [P \cap L(x_k) \cap F_1 \oplus F_2]$, and thereby prove by mathematical induction that if $x_k \in F_1 \oplus F_2$, then $x_{k+1} \in F_1 \oplus F_2$.

Since $x_k \in F_1 \oplus F_2$, then by Lemma 3.3, any node in $L(x_k)$ must belong to $F_1 \cap F_2$ or $F_1 \oplus F_2$. Then $x_{k+1} \in P \cap L(x_k)$ must be a member of the set $F_1 \oplus F_2$ since $P \cap (F_1 \cap F_2) = \varnothing$.

So, we have shown that $x_1 \in F_1 \oplus F_2$, and if $x_k \in F_1 \oplus F_2$, then $x_{k+1} \in F_1 \oplus F_2$. Therefore, by the principal of mathematical induction, $x_k \in F_1 \oplus F_2$ for all integers k. But for k=n, $x_n \in F_1 \oplus F_2$, which contradicts our assumption that $x_n \in \mathcal{U} - (F_1 \cup F_2)$. ❑

3.2.1. t-in-L_2 Diagnosis

In Chapter 2 we defined the diagnosability problem as the ability to determine a uniquely diagnosable family of fault sets in a system, given a singly-producible syndrome. In [DAS93], t-in-L_1 diagnosis was introduced. This theory requires that there be no more than $\left\lfloor \frac{deg(x)}{2} \right\rfloor + 1$ faults in the local set of x. For unique diagnosability, certain structural constraints need to be satisfied.

To improve on this, we relax the local fault constraint restriction by defining a new diagnosability criterion, t-in-L_2 diagnosis, where we consider the nodes in the extended-local set of x.

We look at the general case first. Then we show how this general case applies to specific graphs like the mesh, hypercube, and hexagon structures. As in [DAS93], we use the comparison model for our diagnosis.

3.2.1.1. General Case

We develop the t-in-L_2 diagnosis theory for a general graph in this section. First we develop supporting lemmas for the theory. A general graph is defined to be regular of degree deg(x), with minimum cycle length of 5.

Lemma 3.5

Given a graph G with a local fault constraint (lfc) of no more than t faults in $L(x) \cup \{x\}$, $\forall x \in U$, where $t = d - 1$ and F is the set of faulty nodes in G, then

1. If $|N| > t$, then $x \in F$

2. If $|N| < d - t + 1$, then $x \in U - F$.

<u>Proof:</u>

Condition 1: $|N| > t$.

Suppose $x \notin F$ (i.e., $x \in U - F$). Then, there are at least t + 1 1-links incident on x, and therefore at least t + 1 faulty nodes in L(x), which violates the local fault constraint, therefore $x \in F$. (This is shown in Figure 3.6 for a graph with d = 4 and t = d - 1.)

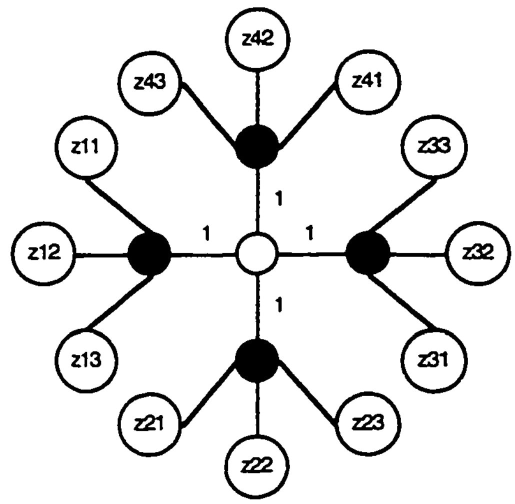


Figure 3.6 Graph with $|N|>t$, $t=d-1$, $x\in U$-F

Condition 2: $|N| < d - t + 1$.

Suppose $x \notin U - F$ (i.e., $x \in F$). There are at most

$|N| = [d - t]$

1-links incident on x, and there are at least

$|\overline{N}| = [d - |N|]$

0-links incident on x since $d = |\overline{N}| + |N|$. So, there are at least

$d - [d - t] = t$

0-links incident on x. Since $x \in F$, all nodes that share a 0-link with x are also in the set F. (This is shown in Figure 3.7 for a graph with $d = 4$ and $t = d - 1$.) Then, there are at least $t + 1$ nodes in the local set of x, which violates the local fault constraint, so $x \in U - F$. ❑

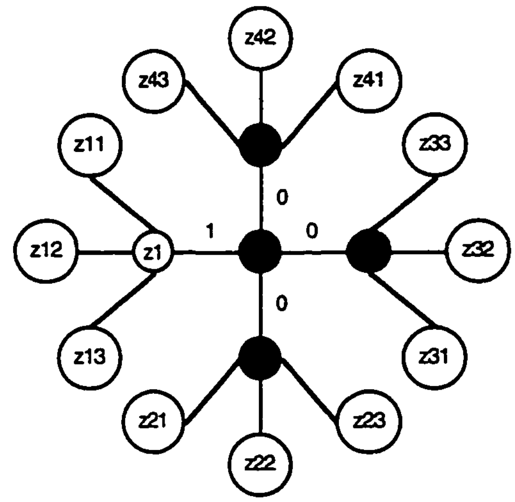


Figure 3.7 Graph with $|N|<d-t+1$, $t=d-1$, $x \in F$

Lemma 3.6

Given a regular graph G of degree d, with a local fault constraint (lfc) of no more than t faults in $L(x) \cup \{x\}$, $\forall x \in U$. Let F be the set of faulty nodes in G, then

1. If $|N| \le d - t + 1$ and $\exists z_i$, such that $|N(N_i)| \ge t$, then $x \in U - F$.

2. If $d - t + 2 \le |N| \le t$ and $|N(N_i)| \ge t$, for $i = 1$ to $[|N| - (d - t)]$, then $x \in U - F$.

3. If $\exists z_i$, such that $|N(N_i)| < [d - t]$, then $x \in F$.

Proof:

Condition 1: $|N| \leq d - t + 1$ and $\exists z_i$, such that $|N(N_i)| \geq t$.

Suppose $x \notin U - F$ (i.e., $x \in F$). Then, clearly $z_i \in F$ by Lemma 3.5. We know the nodes in $\overline{N}$ are faulty, since x is faulty, and $|\overline{N}| = d - |N|$. So there are at least

$$1 + 1 + (d - |N|) = 2 + d - (d - t + 1) = t + 1$$

faulty nodes in the local set of x, a contradiction. So $x \in U - F$. (This is shown in Figure 3.8 for an example with $d = 4$, $t = d - 1$, and $|N| = t$.)

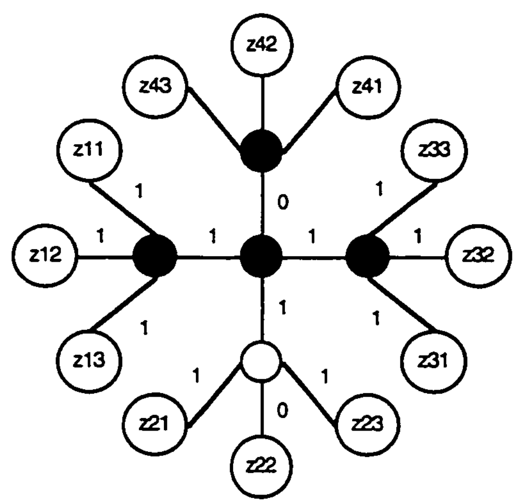


Figure 3.8 Regular Graph with $|N|=t$, $|N(N_1)|=t$, $|N(N_3)|=d-1$

Condition 2: $d - t + 2 \leq |N| \leq t$ and $|N(N_i)| \geq t$, for $i = 1$ to $[|N| - (d - t)]$.

Suppose $x \notin U - F$ (i.e., $x \in F$). Then, clearly $z_i \in F$ by Lemma 3.5, for $i = 1$ to $[|N| - (d - t)]$. The nodes in $\overline{N}$ are faulty, since x is faulty, and $|\overline{N}| = d - |N|$. So there are at least

$$1 + [|N| - (d - t)] + (d - |N|) = t + 1$$

faulty nodes in the local set of x, a contradiction. So $x \in U - F$.

Condition 3: $\exists z_i$, $|N(N_i)| < [d - t]$.

Suppose $x \notin F$ (i.e., $x \in U - F$). So x and z_i share a 1-link. The number of 0-links incident on z_i is at least

$$(d - 1) - (d - t - 1) = t.$$

We know $z_i \in F$ since $x \in U - F$, and so the nodes that share a 0-link with z_i are also faulty. But then at least $t + 1$ nodes in the local set of z_i are faulty, a contradiction. So $x \in F$. (This is shown in Figure 3.9, with $t = d - 1$.) ❑

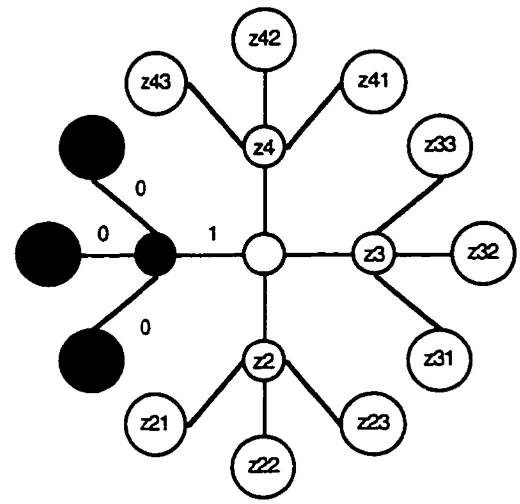


Figure 3.9 Regular Graph with $|N(N_1)|<d-t$, $t=d-1$, $x\in U-F$, $z1\in F$

Lemma 3.7

Given a regular graph G of degree d, with minimum cycle length of 5, and syndrome S. Let F be an allowable fault set for S. The number of faulty nodes in the extended-local set of x, $\forall x\in U$ is given by:

1. $|F(x)| \geq |N|+|\overline{N}(N)|+|N(\overline{N})|$, if $x\in U - F$

2. $|F'(x)| \geq 1+|\overline{N}|+|\overline{N}(\overline{N})|+\sum_{i=1}^{|N|}\min\left[|N(N_i)|, d-|N(N_i)|\right]$, if $x\in F$.

<u>Proof:</u>

Condition 1: $|F(x)| \geq |N|+|\overline{N}(N)|+|N(\overline{N})|$.

The derivation of $|F(x)|$ follows directly from the definition of syndrome decoding using the comparison model. (An example of this is shown in Figure

3.10). Since $x \in U - F$, the nodes that share a 1-link with x are faulty (this is the set N), and all nodes that share a 0-link with the nodes in N are also faulty (this is the set $\overline{N}(N)$). We know the nodes that share a 0-link with x are fault-free (this is the set $\overline{N}$), so all nodes that share a 1-link with $\overline{N}$ are also faulty (this is the set $N(\overline{N})$).

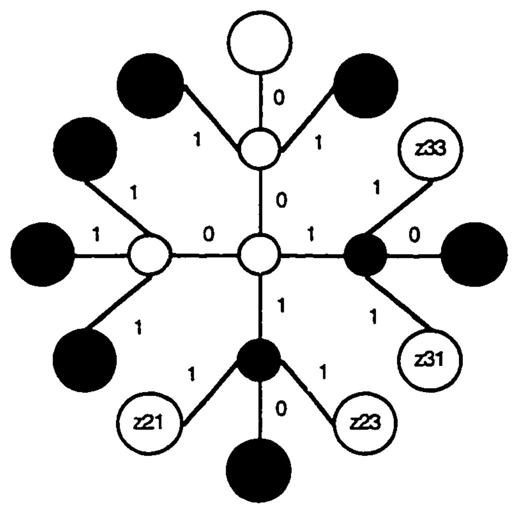


Figure 3.10 Example F(x) Set

Condition 2: $|F'(x)| \geq 1+|\overline{N}|+|\overline{N}(\overline{N})|+\sum_{i=1}^{|N|}\min\left[|N(N_i)|, d-|N(N_i)|\right]$.

In addition to $x \in F$, the nodes in $\overline{N}$ and $\overline{N}(\overline{N})$ are also faulty. This gives the term $1 + |\overline{N}| + |\overline{N}(\overline{N})|$. All of the nodes that share a 1-link with x cannot be labeled as faulty or fault-free, since $x \in F$. However, if $z_i \in N$ is fault-free then all nodes that share a 1-link with it are faulty (this is the term $|N(N_i)|$ inside the 'min' function). If $z_i \in N$ is faulty then all nodes that share a 0-link with it are faulty (this is the term [d - $|N(N_i)|$] inside the 'min' function). Therefore, at least min[$|N(N_i)|$, d - $|N(N_i)|$] nodes in $L(z_i)$ are faulty. This is true for every $z_i \in N$. (Figure 3.11 is an example of this.) ❑

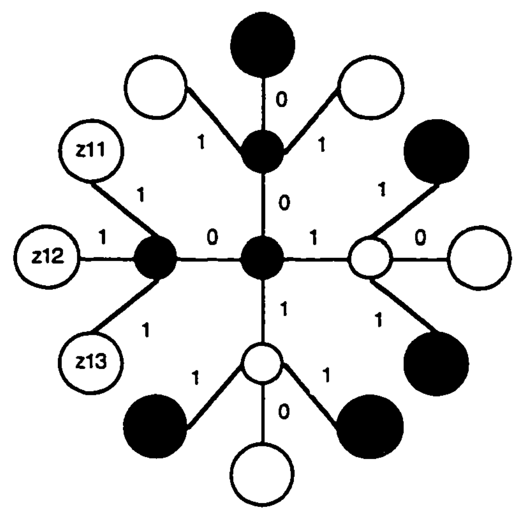


Figure 3.11 Example F′(x) Set

We now define the t-in-L_2 diagnosis theory. A system is t-in-L_2 diagnosable if there is a unique fault set for any given syndrome, provided that there are at most t faults in the local set of each node x, and at most elfc faults in the extended-local set of each node x, where the elfc (extended-local fault set) is a function of t and the degree of node x.

We use Lemma 3.5, Lemma 3.6, and Lemma 3.7 to support our theory. Theorem 3.3 defines the t-in-L_2 diagnosability of a general graph. Theorem 3.4 defines the t-in-L_2 diagnosability of a mesh graph, and Theorem 3.5 defines the t-in-L_2 diagnosability of a hypercube graph.

Theorem 3.3

Given a regular graph G of degree d, with minimum cycle length of 5, and syndrome S, let F be an allowable fault set for S such that $F \neq U$. If there are no more than $t = d - m$ faults in $L(x) \cup \{x\}$, $\forall x \in U$, where $1 \leq m \leq \frac{d}{2} - 1$, and no more than $t(m) = \left\lfloor \frac{(2+m) \cdot d}{2} \right\rfloor + m^2$ faults in $L(L(x)) \cup L(x) \cup \{x\}$, $\forall x \in U$, then G is t-in-L_2 diagnosable.

Proof:

To prove the t-in-L_2 diagnosability of a graph G, we must show that if the local and extended-local fault constraints are not violated, then the fault set F is unique for any given syndrome. We show that $x \in F$ if and only if at least one of the following conditions are met:

1. $|N| > t$
2. $\exists z_i$, such that $|N(N_i)| < [d - t]$
3. $|F(x)| > t(m)$, where $F(x) = N \cup \overline{N}(N) \cup N(\overline{N})$.

First, we show that if any one of the conditions is true, then $x \in F$. If condition 1 or 2 is true, then $x \in F$ by Lemma 3.5 or Lemma 3.6, respectively. If condition 3 is true, and $x \in U - F$, then there are more than t(m) faulty nodes in the extended-local set of x. But this is a contradiction since the elfc requires no more than t(m) faulty nodes in the extended-local set of x, so $x \in F$.

Next we show by contradiction that if none of the conditions is satisfied, then $x \notin F$. Since G is regular with minimum cycle length of 5, the number of nodes in the extended-local set of x is

$$d^2 + 1 = |N| + |\overline{N}(N)| + |N(\overline{N})| + 1 + |\overline{N}| + |\overline{N}(\overline{N})| + |N(N)|. \quad (1)$$

Substituting |F(x)| and |F′(x)|, from Lemma 3.7, into equation 1 yields

$$d^2 + 1 \le |F(x)| + |F'(x)| + |N(N)| - \sum_{i=1}^{|N|} \min\left[|N(N_i)|, d - |N(N_i)|\right]. \quad (2)$$

Since condition 3 is not true, $|F(x)| \le t(m)$. We also know that $|F'(x)| \le t(m)$, otherwise the extended-local fault constraint is violated. Substituting $|F'(x)| \le t(m)$ and $|F(x)| \le t(m)$ into equation 2 yields

$$d^2 + 1 \le t(m) + t(m) + |N(N)| - \sum_{i=1}^{|N|} \min\left[|N(N_i)|, d - |N(N_i)|\right].$$

Simplifying and rearranging this equation yields

$$d^2 - 2 \bullet t(m) + 1 \le |N(N)| - \sum_{i=1}^{|N|} \min\left[|N(N_i)|, d - |N(N_i)|\right].$$

Rewriting |N(N)| as $\sum_{i=1}^{|N|} |N(N_i)|$, we can rewrite this equation as

$$d^2 - 2 \bullet t(m) + 1 \le \sum_{i=1}^{|N|} \left[|N(N_i)| - \min\left\{|N(N_i)|, d - |N(N_i)|\right\}\right]. \quad (3)$$

To maximize the summation on the right side of the inequality, $|N(N_i)| > \frac{d}{2}$,

since if $|N(N_i)| \leq \frac{d}{2}$, then the 'min' function returns $|N(N_i)|$, which will cancel with

the first term in the summation. So $\forall i$, the 'min' function returns $d - |N(N_i)|$

since

$$\min\left\{\frac{d}{2}+1, d-\left(\frac{d}{2}+1\right)\right\} = d-\left(\frac{d}{2}+1\right) = d-|N(N_i)|.$$

The summation is reduced to

$$\sum_{i=1}^{|N|}\left[|N(N_i)| - (d - |N(N_i)|)\right]$$

and so equation 3 is reduced to

$$d^2 - 2 \bullet t(m) + 1 \leq 2 \cdot \left[\sum_{i=1}^{|N|}|N(N_i)|\right] - \sum_{i=1}^{|N|} d$$

which further reduces to

$$d^2 - 2 \bullet t(m) + 1 \leq 2 \cdot \left[\sum_{i=1}^{|N|}|N(N_i)|\right] - d \cdot |N|. \qquad (4)$$

Consider two cases: $|N| \geq d - t + 2$ and $|N| < d - t + 2$. From Lemma 3.6, if $|N| \geq$ $d - t + 2$, then

$|N(N_i)| = t - 1$, for $i = 1$ to $[d - t + 1]$ and

$|N(N_i)| = d - 1$, for $i = [d - t + 2]$ to $|N|$,

so

$$\sum_{i=1}^{|N|} |N(N_i)|_{case\ 1} = (t - 1)\bullet(d - t + 1) + (d - 1)\bullet(|N| - [d - t + 1]). \tag{5}$$

From Lemma 3.6, if $|N| < d - t + 2$ then we have

$$|N(N_i)| = t - 1, \text{ for } i = 1 \text{ to } |N|$$

so

$$\sum_{i=1}^{|N|} |N(N_i)|_{case\ 2} = (t - 1)\bullet|N|. \tag{6}$$

Case 1: $|N| \geq d - t + 2$.

Equation 5 simplifies to

$$\sum_{i=1}^{|N|} |N(N_i)| = -d^2 - d + 2td - t^2 + t - |N| + d|N|.$$

Substituting this into equation 4, letting $t = d - m$ and $|N| = t$ (from Lemma 3.5, $|N| \leq t$, so the maximum value for $|N|$ is t) yields

$$d^2 - 2\bullet t(m) + 1 \leq d^2 - 2d - md - 2m^2,$$

which simplifies to

$$t(m) \geq d + \frac{md}{2} + m^2 + \frac{1}{2}. \tag{7}$$

We know

$$t(m) = \left\lfloor \frac{(2+m)\cdot d}{2} \right\rfloor + m^2.$$

Substituting this into equation 7, and simplifying yields

$$\left\lfloor \frac{(2+m)\cdot d}{2} \right\rfloor \geq d + \frac{md}{2} + \frac{1}{2}.$$

If the numerator inside the floor function is even, then we have

$$\frac{(2+m)\cdot d}{2} \geq d + \frac{md}{2} + \frac{1}{2},$$

which simplifies to

$$d + \frac{md}{2} \geq d + \frac{md}{2} + \frac{1}{2},$$

which is a contradiction. If the numerator inside the floor function is odd, then we have

$$\frac{(2+m)\cdot d-1}{2} \geq d + \frac{md}{2} + \frac{1}{2},$$

which simplifies to

$$d + \frac{md}{2} - \frac{1}{2} \geq d + \frac{md}{2} + \frac{1}{2},$$

which is also a contradiction.

Case 2: $|N| < d - t + 2$.

Substituting equation 6 into equation 4, letting $t = d - m$ and $|N| = d - t + 1$ (which is the maximum value for $|N|$) yields

$$d^2 - 2\bullet t(m) + 1 \leq d + m - 2m^2 - 4m - 2,$$

which simplifies to

$$t(m) \geq \frac{d^2}{2} - \frac{d}{2} - \frac{md}{2} + m^2 + 2m + \frac{3}{2}. \tag{8}$$

But,

$$t(m) = \left\lfloor \frac{(2+m)\cdot d}{2} \right\rfloor + m^2.$$

Substituting this into equation 8, and simplifying yields

$$\left\lfloor \frac{(2+m)\cdot d}{2} \right\rfloor \geq \frac{d^2}{2} - \frac{d}{2} - \frac{md}{2} + 2m + \frac{3}{2}$$

If the numerator inside the floor function is even, then we have

$$\frac{(2+m)\cdot d}{2} \geq \frac{d^2}{2} - \frac{d}{2} - \frac{md}{2} + 2m + \frac{3}{2},$$

which simplifies to

$$m \geq \frac{d^2 - 3d + 3}{2d - 4}.$$

But, $m \leq \frac{d}{2} - 1$. Substituting this for m and simplifying yields

$$d^2 - 4d + 4 \geq d^2 - 3d + 3$$

which is a contradiction. If the numerator inside the floor function is odd, then we have

$$\frac{(2+m)\cdot d - 1}{2} \geq \frac{d^2}{2} - \frac{d}{2} - \frac{md}{2} + 2m + \frac{3}{2},$$

which simplifies to

$$m \geq \frac{d^2 - 3d + 4}{2d - 4}.$$

But, $m \leq \frac{d}{2} - 1$. Substituting this for m and simplifying yields

$$d^2 - 4d + 4 \geq d^2 - 3d + 4$$

which is also a contradiction. ❑

3.2.1.2. Linear Programming Model

In this section, we use a linear programming model to prove the t-in-L_2 diagnosis theory. The results of this analysis are compared to Theorem 3.3. We show that the linear programming model is consistent with the theorem. The derivation of the constraint equations for the linear model ultimately led us to the t-in-L_2 diagnosis theory proof in Theorem 3.3. So the exercise of developing a linear programming model proved to be very helpful in the mathematical proof of the theorem.

We can compare Theorem 3.3 for t-in-L_2 diagnosability with a linear programming model. In generating the set of constraint equations for this model, our objective is to find the range of values for the variables N, $\overline{N}$, $\overline{N}(\overline{N})$, $\overline{N}(N)$, $N(\overline{N})$, and N(N), such that the status of the node x remains undetermined.

For example, if constraint 1 in Listing 3.1 is violated, then $|N| > t$ or $|N| < d - t + 1$. If $|N| > t$ and $x \in \mathcal{U} - F$, then there are more than t 1-links incident on node x, and therefore more than t faults, which is impossible, so $x \in F$.

Similarly, if $|N| < d - t + 1$ and $x \in F$ then there are at least t 0-links incident on x, and therefore at least t + 1 faults, which is impossible, so $x \in \mathcal{U} - F$.

The remaining constraints are similarly derived (see Appendix A.1). Lindo, a linear program problem solver, is the software program we used to solve the set of equations [BRA77]. It uses the simplex method [BRA77]. The set of constraint equations, along with the objective function, are given in Listing 3.1.

1	$d - t + 1$	$\leq |N| \quad \leq t$	
2	$d - |N|$	$= |\overline{N}|$	
3	$d - t + 1$	$\leq |N(\overline{N}_i)| \leq t,$	$\forall i$
4	$(d - 1) - |N(\overline{N}_i)|$	$= |\overline{N}(\overline{N}_i)|$	
5	$d - t$	$\leq |N(N_i)| \leq t - 1,$	for $i = 1, \ldots, [d - t + 1]$
6	$d - t$	$\leq |N(N_i)| \leq d - 1,$	for $i = [d - t + 2], \ldots, |N|$
7	$(d - 1) - |N(N_i)|$	$= |\overline{N}(N_i)|$	

Objective: Minimum of [Maximum of (8, 9)]

8 $$|N| + |N(\overline{N})| + \sum_{i=1}^{|N|} \left|\overline{N}(N_i)\right|$$

9 $$1 + |\overline{N}| + |\overline{N}(\overline{N})| + \sum_{i=1}^{|N|} \min\left[\left|N(N_i)\right|, d - \left|N(N_i)\right|\right]$$

Listing 3.1 Constraint Equations

Using equations 1 - 7 as constraints, we use linear programming to minimize the maximum of equations 8 and 9. Equation 8 represents the number of faulty nodes in the extended-local set of x if $x \in U$ - F. Equation 9 represents the number of faulty nodes in the extended-local set of x if $x \in$ F. Equations 8 and 9 are inversely proportional; as constraint 8 increases, constraint 9 decreases, and as constraint 9 increases, constraint 8 decreases. The objective is to find the maximum number of faulty nodes in the extended-local set of x (the elfc). So, we want to find the value of the elfc such that both constraint equations 8 and 9 are maximized.

If t = d - 1, we determine the elfc t(m) = $\left\lfloor \frac{3 \cdot d}{2} \right\rfloor + 1$. Therefore, if the lfc and elfc are not violated, i.e., if there are no more than t faults in $L(x) \cup \{x\}$ and no more than t(m) faults in $L(L(x)) \cup L(x) \cup \{x\}$, then x is correctly identified as faulty or fault-free, $\forall x$. Additional values for the lfc, and the corresponding elfc, are shown in Appendix A.1, as well as additional information on the linear program.

3.2.1.3. Special Case - Mesh

As with the t-in-L_2 diagnosability of a general graph, we first provide supporting lemmas for the t-in-L_2 diagnosability of a mesh graph. The analysis of the t-in-L_2 diagnosability of a mesh follows directly from the general case. We provide a detailed proof here for completeness.

Lemma 3.8

Given a mesh graph and degree $d = 4$, and a local fault constraint (lfc) of no more than $t = 3$ faults in $L(x) \cup \{x\}$, $\forall x \in U$, and an extended-local fault constraint (elfc) of no more than $t(m) = 6$ faults in $L(L(x)) \cup L(x) \cup \{x\}$, $\forall x \in U$. If $\exists z_i$, such that $|N(N_i)| \geq t$, then x is uniquely diagnosable.

<u>Proof:</u>

In the following, we assume $\exists z_i$, such that $|N(N_i)| \geq t$. There are two cases: $N = \{z_1, z_2, z_4\}$ and $N = \{z_1, z_2, z_3\}$.

We know $t \leq d - 1 = 3$. If $|N| \leq 2$, then $x \in U - F$, otherwise the local fault constraint in the local set of x is violated. Since $|N| = 3$, $|\overline{N}| = 1$, and assume $|N(N_1)| \geq 3$. (See Figure 3.12.) Then $z_1 \in F$.

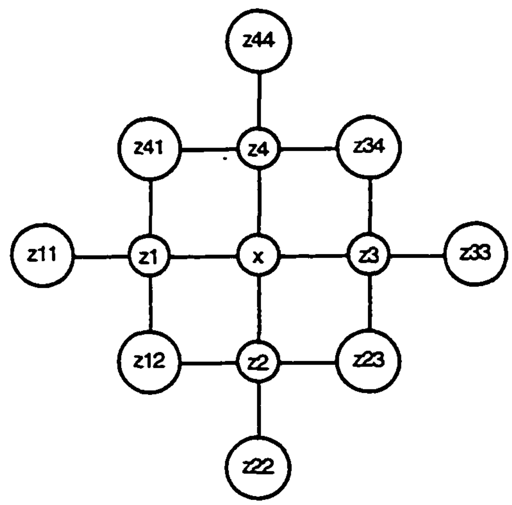


Figure 3.12 Mesh Subgraph

Case 1: $N = \{z_1, z_2, z_4\}$.

We know that $|N(N_i)| \geq [d - t] = 1$, for $i = 2$ and 4, by Lemma 3.6. So

$1 \leq |N(N_2)| \leq 2$,

$1 \leq |N(N_4)| \leq 2$.

We have the following subcases: $|\overline{N}(\overline{N})| = 0$, $|\overline{N}(\overline{N})| = 2$, $|\overline{N}(\overline{N})| = 3$, and $|\overline{N}(\overline{N})| = 1$.

Case 1.1: $|\overline{N}(\overline{N})| = 0$.

Case 1.1.1: $|N(N_2)| = 2$, $|N(N_4)| = 2$.

Then $x \in \mathcal{U} - F$.

Case 1.1.2: $|N(N_2)| = 1$ or $|N(N_4)| = 1$.

Then $x \in F$.

Case 1.2: $|\overline{N}(\overline{N})| = 2$.

Then $x \in U - F$.

Case 1.3: $|\overline{N}(\overline{N})| = 3$.

Then $x \in U - F$.

Case 1.4: $|\overline{N}(\overline{N})| = 1$.

Then $x \in U - F$.

Case 2: $N = \{z_1, z_2, z_3\}$.

We know that $|N(N_i)| \geq [d - t] = 1$, for $i = 2$ and 3, by Lemma 3.6. So

$1 \leq |N(N_2)| \leq 2$,

$1 \leq |N(N_3)| \leq 2$.

We have the following subcases: $|\overline{N}(\overline{N})| = 0$, $|\overline{N}(\overline{N})| = 2$, $|\overline{N}(\overline{N})| = 3$, and $|\overline{N}(\overline{N})| = 1$.

Case 2.1: $|\overline{N}(\overline{N})| = 0$.

Then $x \in F$, otherwise the extended-local fault constraint $t(m)$ is violated. If we assume $x \in U - F$, then we arrive at a contradiction. We know $\{z_1, z_2, z_3, z_{41}, z_{44}, z_{34}\}$ are faulty. However, since $|N(N_2)| \leq 2$, and $|N(N_3)| \leq 2$, both $L(z_2)$ and $L(z_3)$ have one other faulty node, bringing the total number of faulty nodes in the extended-local set of x to eight, which violates the extended-local fault constraint. So $x \in$ F.

Case 2.2: $|\overline{N}(\overline{N})| = 2$.

Then $x \in U$ - F, otherwise the local fault constraint in $L(z_4) \cup \{z_4\}$ is violated.

Case 2.3: $|\overline{N}(\overline{N})| = 3$.

Then $x \in U$ - F, otherwise the local fault constraint in $L(z_4) \cup \{z_4\}$ is violated.

Case 2.4: $|\overline{N}(\overline{N})| = 1$.

Case 2.4.1: $|N(N_2)| = 2$, $|N(N_3)| = 2$.

Then $x \in U$ - F.

Case 2.4.2: $|N(N_2)| = 2$, $|N(N_3)| = 1$.

Case 2.4.2.1: z_2 - z_{12} or z_2 - z_{22} is joined by a 0-link.

Then $x \in$ F.

Case 2.4.2.2: z_2 - z_{23} is joined by a 0-link.

Case 2.4.2.2.1: z_3 - z_{34} is joined by a 0-link.

Then $x \in U$ - F.

Case 2.4.2.2.2: z_3 - z_{34} is joined by a 1-link.

Then $x \in$ F.

Case 2.4.3: $|N(N_2)| = 1$, $|N(N_3)| = 2$.

Then $x \in F$.

Case 2.4.4: $|N(N_2)| = 1$, $|N(N_3)| = 1$.

Then $x \in F$. ❑

Lemma 3.9

Given a mesh graph G and syndrome S, let F be an allowable fault set for S such that $F \neq U$. Let C (or C′) be the set of nodes that have two paths of length 2 with x and are counted twice in F(x) (or F′(x)). Then, the number of faulty nodes in the extended-local set of x, $\forall x \in U$ is given by:

1. $|F(x)| \geq |N| + |\overline{N}(N)| + |N(\overline{N})| - |C|$, if $x \in U - F$

2. $|F'(x)| \geq 1 + |\overline{N}| + |\overline{N}(\overline{N})| + |N(N)| - |C'|$, if $x \in F$ and $|N(N_i)| < t$, $\forall i$.

Proof:

Figure 3.13 shows the extended-local set of x for a mesh. Denote the nodes that have two paths of length 2 with x by z_{12}, z_{23}, z_{34}, and z_{41}, as shown in Figure 3.13.

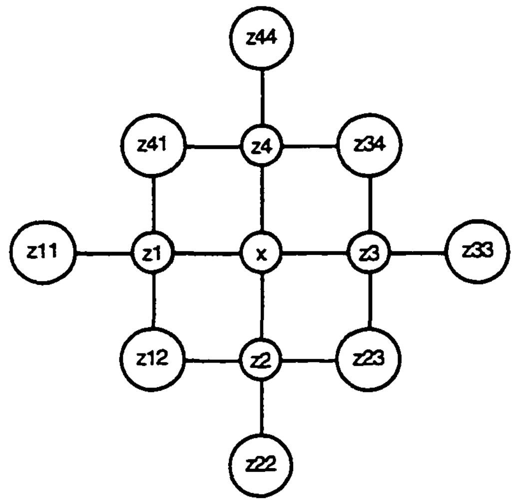


Figure 3.13 Mesh Subgraph

Condition 1: $|F(x)| \geq |N| + |\overline{N}(N)| + |N(\overline{N})| - |C|$.

The derivation of $|F(x)|$ follows from the definition of syndrome decoding using the comparison model. So,

$|F(x)| \geq |N| + |\overline{N}(N)| + |N(\overline{N})|$.

But, the nodes that share two paths of length 2 with x may be counted twice in the computation of F(x), since $|F(x)|$ is the sum of the $\overline{N}(N)$ and $N(\overline{N})$ paths. So, $z_{ij} \in C$ if $z_{ij} \in \{z_{12}, z_{23}, z_{34}, z_{41}\}$ and both paths that join x and z_{ij} are in the computation of F(x). Therefore we must subtract $|C|$ from F(x) so that z_{ij} is not counted twice. So we have

$|F(x)| \geq |N| + |\overline{N}(N)| + |N(\overline{N})| - |C|$, where $|C| \leq 4$ (for a mesh).

(Figure 3.14 shows an example where $|C| = 3$ and x is fault-free.)

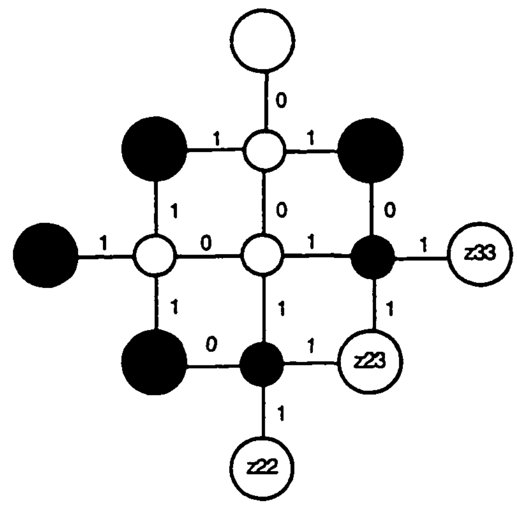


Figure 3.14 F(x) Set for Mesh

Condition 2: $|F'(x)| \geq 1 + |\overline{N}| + |\overline{N}(\overline{N})| + |N(N)| - |C'|$.

In addition to $x \in F$, the nodes in $\overline{N}$ and $\overline{N}(\overline{N})$ are also faulty. This gives the term $1 + |\overline{N}| + |\overline{N}(\overline{N})|$. All the nodes that share a 1-link with x cannot be labeled as faulty or fault-free, since $x \in F$. However, if $z_i \in N$ is fault-free, then all nodes that share a 1-link with z_i are faulty. If $z_i \in N$ is faulty, then all nodes that share a 0-link with it are faulty. Therefore, at least $\min[|N(N_i)|, d - |N(N_i)|]$ nodes in $L(z_i)$ are faulty. Since $|N(N_i)| < t$, $\forall i$, then $\min[|N(N_i)|, d - |N(N_i)|] = |N(N_i)|$, because $t \leq 3$ and $d = 4$. This is true for every $z_i \in N$. Therefore, $F'(x) \geq 1 + |\overline{N}| + |\overline{N}(\overline{N})| + |N(N)|$.

But, as in the F(x) derivation, there are at most four nodes that have two paths of length 2 with x, and therefore may be counted twice in the computation of F'(x), since $|F'(x)|$ is the sum of the $\overline{N}(\overline{N})$ and N(N) paths. So,

$z_{ij} \in C'$ if both paths that join x and z_{ij} are in the computation of F′(x), that is $z_{ij} \in \{z_{12}, z_{23}, z_{34}, z_{41}\}$. Therefore we must subtract $|C'|$ from F′(x) so that z_{ij} is not counted twice. So we have

$$|F'(x)| \geq 1 + |\overline{N}| + |\overline{N}(\overline{N})| + |N(N)| - |C'|, \text{ where } |C'| \leq 4.$$

(Figure 3.15 shows an example where $|C'| = 1$ and $x \in F$. The syndrome in Figure 3.15 is the same as that in Figure 3.14 for comparison.) ❑

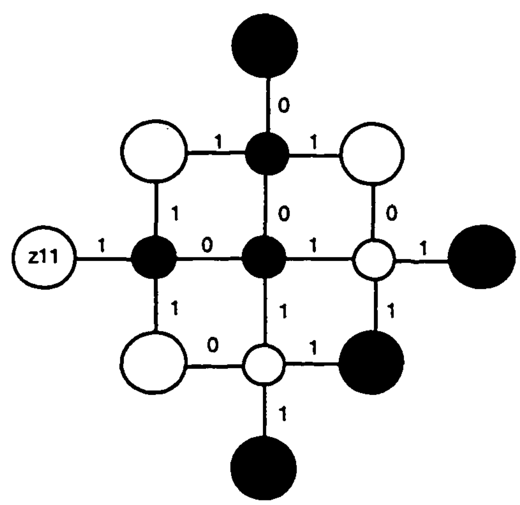


Figure 3.15 F′(x) Set for Mesh

Theorem 3.4

Given a mesh graph G and syndrome S, let F be an allowable fault set for S such that $F \neq U$. If there are no more than $t = d - m$ faults in $L(x) \cup \{x\}$,

$\forall x \in U$, where m = 1 (for mesh only), and no more than t(m) = 6 faults in $L(L(x)) \cup L(x) \cup \{x\}$, $\forall x \in U$, then G is t-in-L_2 diagnosable.

Proof:

To prove the t-in-L_2 diagnosability of a mesh graph G, we must show that if the local and extended-local fault constraints are not violated, then the fault set F is unique for any given syndrome. We show that $x \in F$ if and only if at least one of the following conditions are met:

1. $|N| > t$
2. $\exists z_i$, such that $|N(N_i)| < [d - t]$
3. $|F(x)| > t(m)$, where $F(x) = N \cup \overline{N}(N) \cup N(\overline{N})$.

Let C denote the set of nodes that have two paths of length 2 with x and are counted twice in the computation of F(x). Similarly, let C′ denote the set of nodes that have two paths of length 2 with x and are counted twice in the computation of F′(x).

First, we show that if any one of the conditions is true, then $x \in F$. If condition 1 or 2 is true, then $x \in F$ by Lemma 3.5 or Lemma 3.6, respectively. If condition 3 is true, and $x \in U - F$, then there are more than t(m) faulty nodes in the extended-local set of x. But this is a contradiction since the elfc requires no more than t(m) faulty nodes in the extended-local set of x, so $x \in F$.

Next we show by contradiction that if none of the conditions is satisfied, then $x \notin F$. Since G is a mesh, the number of nodes in the extended-local set of x is

$$d^2 - 3 = 1 + |N| + |\overline{N}| + |\overline{N}(N)| + |N(\overline{N})| + |\overline{N}(\overline{N})| + |N(N)| - 4 \tag{9}$$

Substituting $|F(x)|$ and $|F'(x)|$, from Lemma 3.9, into equation 9 yields

$$d^2 - 3 \le |F(x)| + |C| + |F'(x)| + |C'| - 4 \tag{10}$$

Since condition 3 is not true, $|F(x)| \le t(m)$. We also know that $|F'(x)| \le t(m)$, otherwise the extended-local fault constraint is violated. Substituting $|F'(x)| \le t(m)$ and $|F(x)| \le t(m)$ into equation 10 yields

$$d^2 + 1 \le t(m) + t(m) + |C| + |C'|$$

Simplifying and rearranging this equation yields

$$2 \bullet t(m) \ge d^2 + 1 - (|C| + |C'|) \tag{11}$$

Now, there are only 4 nodes in the set $\{z_{12}, z_{23}, z_{34}, z_{41}\}$. Since $C \cap C' = \emptyset$ (by the definition of F(x) and F′(x)), $|C| + |C'| \le 4$. Substituting this into equation 11 yields

$$2 \bullet t(m) \ge d^2 + 1 - 4$$

which simplifies to

$$t(m) \ge \frac{d^2 - 3}{2}$$

Substituting $d = 4$ and $t(m) = 6$ yields

$$6 \geq \frac{13}{2}$$

which is a contradiction. ❑

3.2.1.4. Special Case - Hypercube

As with the t-in-L_2 diagnosis of the general graph and the mesh graph, we first provide a supporting lemma for the t-in-L_2 diagnosis of a hypercube graph. The analysis of the t-in-L_2 diagnosis of a hypercube follows directly from the general case. We provide the detailed proof here for completeness.

Lemma 3.10

Given a hypercube graph G, with 2^n nodes and syndrome S, let F be an allowable fault set for S such that $F \neq U$. Let C (or C′) be the set of nodes that have two paths of length 2 with x and are counted twice in the computation of F(x) (or F′(x)). Then, the number of faulty nodes in the extended-local set of x, $\forall x \in U$ is given by:

1. $|F(x)| \geq |N| + |\overline{N}(N)| + |N(\overline{N})| - |C|$, if $x \in U - F$

2. $|F'(x)| \geq 1 + |\overline{N}| + |\overline{N}(\overline{N})| + \sum_{i=1}^{|N|} \min\left[|N(N_i)|, n - |N(N_i)|\right] - |C'|$, if $x \in F$.

Proof:

Figure 3.16 shows the extended-local set of x for a hypercube. Denote the nodes that share two paths of length 2 with x by z_{12}, z_{13}, z_{23}, z_{34}, z_{41}, and z_{42}, as shown in Figure 3.16.

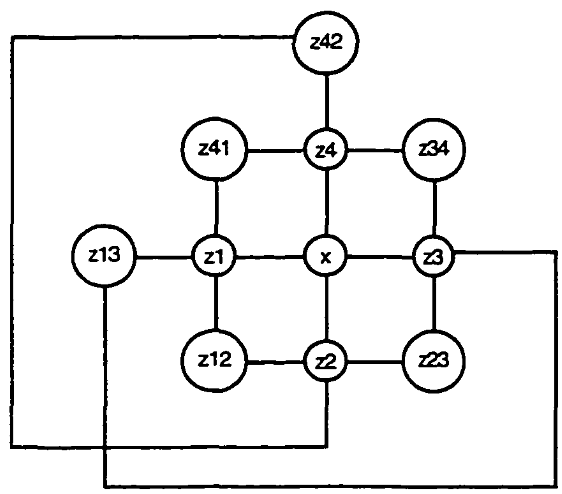


Figure 3.16 Hypercube Subgraph

Condition 1: $|F(x)| \geq |N| + |\overline{N}(N)| + |N(\overline{N})| - |C|$

The derivation of $|F(x)|$ follows from the definition of syndrome decoding using the comparison model. So,

$|F(x)| \geq |N| + |\overline{N}(N)| + |N(\overline{N})|$.

But, there are nodes that share two paths of length 2 with x, and therefore may be counted twice in the computation of F(x), since $|F(x)|$ is the sum of the $\overline{N}(N)$ and $N(\overline{N})$ paths. So, $z_{ij} \in C$ if both paths that join x and z_{ij} are in the computation of F(x). Therefore we must subtract $|C|$ from F(x) so that z_{ij} is not counted twice. So we have

$$|F(x)| \geq |N| + |\overline{N}(N)| + |N(\overline{N})| - |C|.$$

Condition 2: $|F'(x)| \geq 1 + |\overline{N}| + |\overline{N}(\overline{N})| + \sum_{i=1}^{|N|} \min\left[|N(N_i)|, n - |N(N_i)|\right] - |C'|$

In addition to $x \in F$, we know the nodes in $\overline{N}$ and $\overline{N}(\overline{N})$ are also faulty. This gives the term $1 + |\overline{N}| + |\overline{N}(\overline{N})|$. All the nodes that share a 1-link with x cannot be labeled as faulty or fault-free, since $x \in F$. However, if $z_i \in N$ is fault-free then all nodes that share a 1-link with z_i are faulty (this is the term $|N(N_i)|$ inside the 'min' function). If $z_i \in N$ is faulty then all nodes that share a 0-link with z_i are faulty (this is the term $[n - |N(N_i)|]$ inside the 'min' function). Therefore, at least *min*$[|N(N_i)|, n - |N(N_i)|]$ *nodes in* $L(z_i)$ are faulty.

But, as with the F(x) derivation, there are nodes that have two paths of length 2 with x, and therefore may be counted twice in the computation of F′(x), since $|F'(x)|$ is the sum of the $\overline{N}(\overline{N})$ and $N(N_i)$ paths. So, $z_{ij} \in C'$ if both paths that join x and z_{ij} are in the computation of F′(x). Therefore we must subtract $|C'|$ from F′(x) so that z_{ij} is not counted twice. So we have

$$|F'(x)| \geq 1 + |\overline{N}| + |\overline{N}(\overline{N})| + \sum_{i=1}^{|N|} \min\left[|N(N_i)|, n - |N(N_i)|\right] - |C'|. \square$$

Theorem 3.5

Given a hypercube graph G, with 2^n nodes and syndrome S, let F be an allowable fault set for S such that $F \neq U$. If there are no more than $t = n - m$ faults in $L(x) \cup \{x\}$, $\forall x \in U$, where $1 \le m \le \frac{n}{2} - 1$, and no more than $t(m) = \frac{(3+m) \cdot n}{4} + \left\lfloor \frac{m^2}{2} \right\rfloor$ faults in $L(L(x)) \cup L(x) \cup \{x\}$, $\forall x \in U$, then G is t-in-L_2 diagnosable.

Proof:

To prove the t-in-L_2 diagnosability of a hypercube graph G, we must show that if the local and extended-local fault constraints are not violated, then the fault set F is unique for any given syndrome. We show that $x \in F$ if and only if at least one of the following conditions are met:

1. $|N| > t$
2. $\exists z_i$, such that $|N(N_i)| < [n - t]$
3. $|F(x)| > t(m)$, where $F(x) = N \cup \overline{N}(N) \cup N(\overline{N})$.

Let C denote the set of nodes that have two paths of length 2 with x and are counted twice in the computation of F(x). Similarly, let C′ denote the set of nodes that have two paths of length 2 with x and are counted twice in the computation of F′(x).

First, we show that if any one of the conditions is true, then $x \in F$. If condition 1 or 2 is true, then $x \in F$ by Lemma 3.5 or Lemma 3.6, respectively. If condition 3 is true, and $x \in \mathcal{U} - F$, then there are more than t(m) faulty nodes in the extended-local set of x. But this is a contradiction since the elfc requires no more than t(m) faulty nodes in the extended-local set of x, so $x \in F$.

Next, we show by contradiction that if none of the conditions is satisfied, then $x \notin F$. For a graph with minimum cycle length of 5, we know there are d^2 nodes in $L(L(x)) \cup L(x)$, so there are $d^2 - d$ nodes in $L(L(x))$. But, since G is a hypercube, there are exactly 2 paths of length 2 with x and every $z_{ij} \in L(L(x))$. So there are $\frac{n(n-1)}{2}$ nodes in L(L(x)) that are counted twice. Therefore, letting $d = n$, there are $n^2 - n - \frac{n(n-1)}{2}$ nodes in L(L(x)). Also, we know there are $n + 1$ nodes in $L(x) \cup \{x\}$, so the number of nodes in the extended-local set of x is

$$n + 1 + n^2 - n - \frac{n(n-1)}{2} =$$

$$1 + |N| + |\overline{N}| + |\overline{N}(N)| + |N(\overline{N})| + |\overline{N}(\overline{N})| + |N(N)| - \frac{n(n-1)}{2} \tag{12}$$

Substituting |F(x)| and |F′(x)|, from Lemma 3.10, into equation 12 yields

$$n^2 + 1 \leq |F(x)| + |C| + |F'(x)| + |C'| + |N(N)| - \sum_{i=1}^{|N|} \min\left[|N(N_i)|, n - |N(N_i)|\right]. \tag{13}$$

Since condition 3 is not true, $|F(x)| \le t(m)$. We also know that $|F'(x)| \le t(m)$, otherwise the extended-local fault constraint is violated. Substituting $|F'(x)| \le t(m)$ and $|F(x)| \le t(m)$ into equation 13 yields

$$n^2 + 1 \le t(m) + t(m) + |C| + |C'| + |N(N)| - \sum_{i=1}^{|N|} \min\left[|N(N_i)|, n - |N(N_i)|\right].$$

Simplifying and rearranging this equation yields

$$n^2 - 2 \bullet t(m) + 1 \le |N(N)| - \left\{\sum_{i=1}^{|N|} \min\left[|N(N_i)|, n - |N(N_i)|\right]\right\} + |C| + |C'|.$$

Rewriting $|N(N)|$ as $\sum_{i=1}^{|N|} |N(N_i)|$, we can rewrite this equation as

$$n^2 - 2 \bullet t(m) + 1 \le \left\{\sum_{i=1}^{|N|} \left[|N(N_i)| - \min\left\{|N(N_i)|, n - |N(N_i)|\right\}\right]\right\} + |C| + |C'|. \qquad (14)$$

Now, since $z_{ij} \in C$ only if z_{ij} is counted twice in the F(x) computation, and z_{ij} is distance two from x, then $z_{ij} \in N(\overline{N})$ or $z_{ij} \in \overline{N}(N)$, $\forall i, j$. Then

$$|C| \le \frac{|N(\overline{N})| + |\overline{N}(N)|}{2}.$$

Similarly, C′ is computed, so that

$$|C'| \le \frac{|\overline{N}(\overline{N})| + \sum_{i=1}^{|N|} \min\left[|N(N_i)|, n - |N(N_i)|\right]}{2},$$

and then

$$|C| + |C'| \leq \frac{|N(\overline{N})| + |\overline{N}(N)| + |\overline{N}(\overline{N})| + \sum_{i=1}^{|N|} \min[|N(N_i)|, n - |N(N_i)|]}{2}. \qquad (15)$$

But, we know that $|\overline{N}(N)| + |N(\overline{N})| + |\overline{N}(\overline{N})| + |N(N)| = n^2 - n$ (for a hypercube). Substituting this into equation 15, and simplifying yields

$$|C| + |C'| \leq \frac{n^2}{2} - \frac{n}{2} - \frac{\sum_{i=1}^{|N|} \left[|N(N_i)| - \min\{|N(N_i)|, n - |N(N_i)|\}\right]}{2}.$$

Substituting this into equation 14, and simplifying yields

$$\frac{n^2}{2} + \frac{n}{2} - 2 \cdot t(m) + 1 \leq \frac{\sum_{i=1}^{|N|} |N(N_i)| - \min[|N(N_i)|, n - |N(N_i)|]}{2}. \qquad (16)$$

To maximize the summation on the right side of the inequality, $|N(N_i)| > \frac{n}{2}$, since if $|N(N_i)| \leq \frac{n}{2}$, then the 'min' function returns $|N(N_i)|$, which will cancel with the first term in the summation. So $\forall i$, the 'min' function returns $n - |N(N_i)|$ since

$$\min\left\{\frac{n}{2} + 1, n - \left(\frac{n}{2} + 1\right)\right\} = n - \left(\frac{n}{2} + 1\right) = n - |N(N_i)|.$$

The summation is reduced to

$$\sum_{i=1}^{|N|} \left[|N(N_i)| - (n - |N(N_i)|)\right]$$

and so equation 16 is reduced to

$$\frac{n^2}{2}+\frac{n}{2}-2\bullet t(m)+1 \le \left[\sum_{i=1}^{|N|}|N(N_i)|\right]-\frac{n\cdot|N|}{2}. \tag{17}$$

Consider two cases: $|N| \ge n - t + 2$ and $|N| < n - t + 2$. From Lemma 3.6, if $|N| \ge n - t + 2$ then we have

$|N(N_i)| = t - 1$, for $i = 1$ to $[n - t + 1]$ and

$|N(N_i)| = n - 1$, for $i = [n - t + 2]$ to $|N|$,

so

$$\sum_{i=1}^{|N|}|N(N_i)|_{\text{case 1}} = (t-1)\bullet(n - t + 1) + (n-1)\bullet(|N| - [n - t + 1]). \tag{18}$$

If $|N| < n - t + 2$ then from Lemma 3.6 we have

$|N(N_i)| = t - 1$, for $i = 1$ to $|N|$

so

$$\sum_{i=1}^{|N|}|N(N_i)|_{\text{case 2}} = (t-1)\bullet|N|. \tag{19}$$

Case 1: $|N| \ge n - t + 2$.

Equation 18 simplifies to

$$\sum_{i=1}^{|N|}|N(N_i)| = -n^2 - n + 2tn - t^2 + t - |N| + n|N|.$$

Substituting this into equation 17, letting $t = n - m$ and $|N| = t$ (from Lemma 3.5, $|N| \le t$, so the maximum value for $|N|$ is t) yields

$$t(m) \ge \frac{(3+m)\cdot n}{4}+\frac{m^2}{2}+\frac{1}{2}. \tag{20}$$

But,

$$t(m) = \frac{(3+m)\cdot n}{4} + \left\lfloor \frac{m^2}{2} \right\rfloor.$$

Substituting this into equation 20, and simplifying yields

$$\left\lfloor \frac{m^2}{2} \right\rfloor \geq \frac{m^2}{2} + \frac{1}{2}.$$

If the numerator inside the floor function is even, then

$$\frac{m^2}{2} \geq \frac{m^2}{2} + \frac{1}{2},$$

which is a contradiction. If the numerator inside the floor function is odd, then

$$\frac{m^2 - 1}{2} \geq \frac{m^2}{2} + \frac{1}{2},$$

which is also a contradiction.

Case 2: $|N| < n - t + 2$.

Substituting equation 19 into equation 17, letting $t = n - m$ and $|N| = n - t + 1$ (which is the maximum value for $|N|$) yields

$$t(m) \geq \frac{n^2}{4} - \frac{nm}{4} + \frac{m^2}{2} + m + 1. \tag{21}$$

But,

$$t(m) = \frac{(3+m)\cdot n}{4} + \left\lfloor \frac{m^2}{2} \right\rfloor.$$

Substituting this value for t(m) into equation 21 yields

$$\frac{(3+m)\cdot n}{4}+\left\lfloor \frac{m^2}{2} \right\rfloor \geq \frac{n^2}{4}-\frac{nm}{4}+\frac{m^2}{2}+m+1. \tag{22}$$

If the numerator inside the floor function is even, then

$$\frac{(3+m)\cdot n}{4}+\frac{m^2}{2} \geq \frac{n^2}{4}-\frac{nm}{4}+\frac{m^2}{2}+m+1.$$

which simplifies to

$$m \geq \frac{n^2-3n+4}{2n-4}.$$

But, $m \leq \frac{n}{2}-1$. Substituting this for m and simplifying yields

$$n^2 - 4n + 4 \geq n^2 - 3n + 4$$

which is a contradiction. If the numerator inside the floor function in equation 22 is odd, then we have

$$\frac{(3+m)\cdot n}{4}+\frac{m^2-1}{2} \geq \frac{n^2}{4}-\frac{nm}{4}+\frac{m^2}{2}+m+1.$$

which simplifies to

$$m \geq \frac{n^2-3n+6}{2n-4}.$$

But, $m \leq \frac{n}{2}-1$. Substituting this for m yields

$$n^2 - 4n + 4 \geq n^2 - 3n + 6$$

which is also a contradiction. ❑

So we have shown the t-in-L_2 diagnosis of a general graph of regular degree, a mesh graph, and a hypercube graph of degree n. Now, we provide a simple example of the t-in-L_2 diagnosis algorithm by applying it to a hexagon graph.

3.2.1.5. Hexagon Example

We provide an example of the t-in-L_2 diagnosis algorithm. We use a hexagon structure, shown in Figure 3.17. This structure satisfies the requirements of Theorem 3.3. We have provided a syndrome and labeled the nodes A through T. The nodes along the perimeter of the graph are not considered in this example. These nodes have been darkened in Figure 3.17.

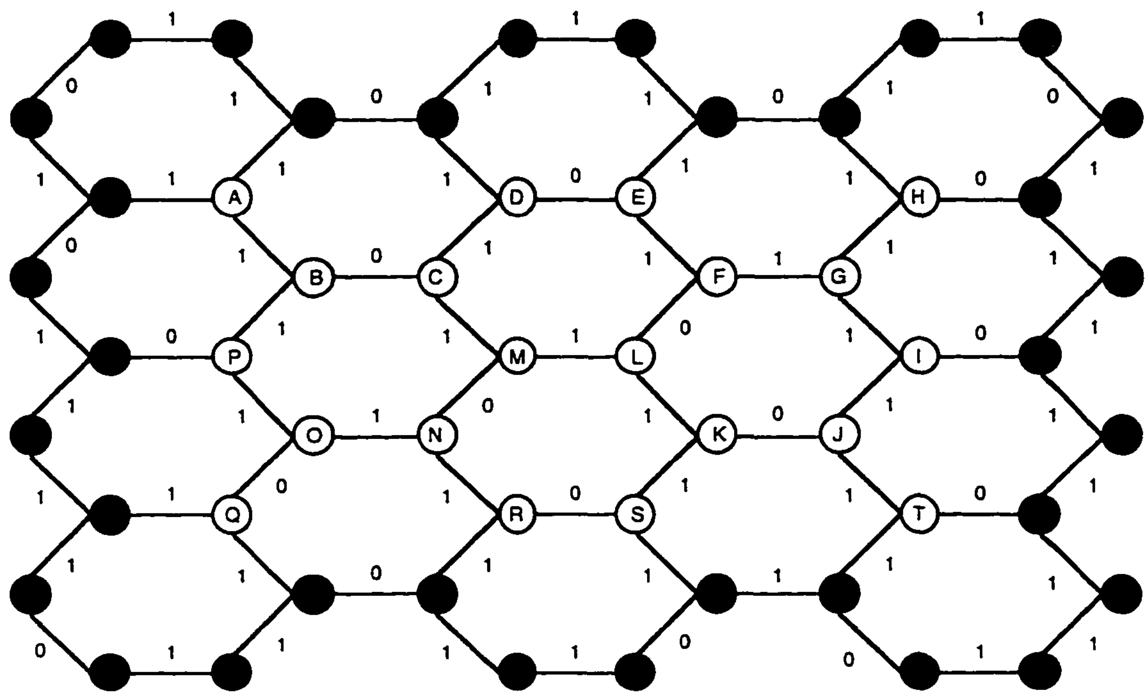


Figure 3.17 Hexagon Example

In this example, $t = d - 1$ so the conditions used to test if x is faulty are:

1. $|N| > 2$

2. $\exists z_i$, such that $|N(N_i)| < 1$

3. $|F(x)| > 5$, where $F(x) = N \cup N(\overline{N}) \cup \overline{N}(N)$.

Node A has $|N| = 3$, so $A \in F$. Node B has $|N| = 2$, $|N(N_i)| \geq 1$, and $|F(x)| = 5$, so $B \in U - F$. Every node in the graph performs these tests in parallel.

3.3. Summary

In this chapter, we developed the t-in-L_2 diagnosis theory. In this theory, we improved on the t-in-L_1 diagnosis theory by eliminating the need for: (1) the global fault constraint of fewer than $|U|/2$ faulty nodes in G, (2) the constraint of having two paths of length two between all nodes, x and y, with d(x,y)=2, and (3) having $x,y \in U$ be in a subgraph defined in [DAS93]. We also relaxed the local fault constraint by introducing an extended-local fault constraint.

It should also be noted that the local fault constraint in the t-in-L_1 diagnosability theory, which is no more than $t = \left\lfloor \frac{d}{2} \right\rfloor + 1$ faults, is the same as the local fault constraint in the t-in-L_2 diagnosability theory when $m = \left\lfloor \frac{d}{2} \right\rfloor - 1$, see Table 3.1.

m	lfc t=d - m	general graph t(m)	mesh t(m)	hypercube t(m)
1	d - 1	$\left\lfloor \frac{3 \cdot d}{2} \right\rfloor + 1$	6	n
2	d - 2	$\left\lfloor \frac{4 \cdot d}{2} \right\rfloor + 4$	N/A	$\frac{5 \cdot n}{4} + 2$
3	d - 3	$\left\lfloor \frac{5 \cdot d}{2} \right\rfloor + 9$	N/A	$\frac{6 \cdot n}{4} + 4$
.	.	.	.	.
.	.	.	.	.
m	d - m	$\left\lfloor \frac{(2+m) \cdot d}{2} \right\rfloor + m^2$	6, m = 1	$\frac{(3+m) \cdot n}{4} + \left\lfloor \frac{m^2}{2} \right\rfloor$
$m = \frac{d}{2} - 1$	$\frac{d}{2} + 1$	$\frac{d^2 - d + 1}{2}$	N/A	$\frac{n^2}{4}$

Table 3.1 t and t(m) for the general graph, mesh, and hypercube

We know that $t \geq \left\lfloor \frac{d}{2} \right\rfloor + 1$ in the t-in-L_2 diagnosability theory, which is consistent with a majority voting scheme, since a majority voting scheme could be used to diagnose the graph if $t \leq \left\lfloor \frac{d}{2} \right\rfloor$, that is, if fewer than half the nodes in L(x) are faulty, then more than half must be fault-free. Then, a voting scheme, where the majority of nodes that have the same test result "wins", could be used and system-level diagnosis is not required.

4. Applications to the Global Navigation Satellite System

Having developed the t-in-L_2 diagnosis theory, we want to show an application to a system that is not a typical multiprocessor system. We also want to show an application of the theory to a system which is not regular, as was the case in all the previous examples.

4.1. Introduction

The Global Positioning System (GPS) is a space-based positioning, velocity, and time system that has three major segments: space, control, and the user. The GPS Space Segment is composed of 24 space vehicles (SVs), i.e. satellites, in six orbital planes. The SVs operate in circular 10,900 nautical mile orbits with a 12-hour period. The GPS Control Segment has five ground-based monitor stations, three of which have uplink capabilities, and a Control Station operated by the Department of Defense. The monitor stations use a GPS receiver to passively track all satellites in view and thus accumulate ranging data from the satellite signals. The information from the monitor stations is processed by the Control Station to determine satellite orbits and to update the navigation message of each satellite. The GPS User Segment consists of receivers that provide positioning, velocity, and precise timing to the user. [SAM95]

4.2. Problem Description

The Global Navigation Satellite System (GNSS), as described here, is comprised of GPS SVs and other augmentations that are added to provide the required accuracy, integrity, and availability for aircraft navigation. We use System-Level Diagnosis (SLD) [PRE67] to determine the diagnosability of the Global Navigation Satellite System (GNSS). The Department of Transportation (DoT) and the Federal Aviation Administration (FAA) in the United States, along with similar agencies in other countries, have a vested interest in how well the GNSS will perform for both civil aviation and for military aircraft systems.

The GNSS, as shown in Figure 4.1, is composed of the following components: GPS SVs, the Wide Area Augmentation System (WAAS), Air Traffic Control, and the aircraft. Each block in Figure 4.1 represents a subsystem of the GNSS. The system is modeled using nodes to represent individual components and directed edges to represent the flow of information.

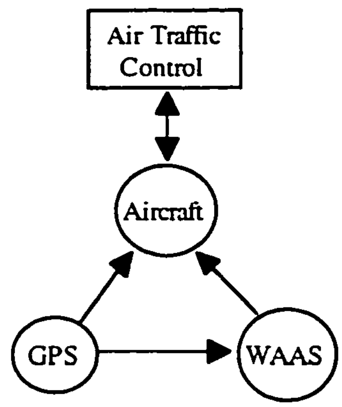


Figure 4.1 Graphical Representation of GNSS

A graph G = (*U*, *E*) represents the entire GNSS, which includes the blocks in Figure 4.1 and their subsystems, where *U* is the set of nodes in G and *E* is the set of edges of G. A subgraph of G, $G_{t,p} \subset G$, represents the GNSS as seen by the aircraft at some time t and some position p. $G_{t,p} \neq G$ since at any given time or place, the aircraft cannot "see" all the GPS SVs, and some SVs may not be in service (e.g. for scheduled maintenance). We are interested in the diagnosability of $G_{t,p}$ for any t, p.

After generating $G_{t,p}$ we create the test graph for $G_{t,p}$, called $G_{t,p}'$. Then, we perform our diagnosis and determine the diagnosability of the GNSS. We investigate the worst case scenario for $G_{t,p}$ which is the situation where the aircraft can "see" the minimum number of GNSS elements. Further investigations should look at the other situations, and provide an overall assessment of the system, e.g. moving averages or standard deviations.

4.2.1. Generating the Test Graph $G_{t,p}'$

Another issue in modeling the GNSS is to identify the failure modes for each node. There are several differences between the graph considered here and the graphs in traditional multiprocessor systems, for which SLD is traditionally applied. All nodes are identical in graphs using traditional SLD, whereas they are not in this application. Furthermore, the edges of G may be faulty, whereas in traditional SLD the edges are assumed fault-free. Figure 4.2 shows the different types of nodes, and their associated edges, for the GNSS subsystems. We model a link failure in the GNSS as a failure of the sending node, to simplify our model. In the GNSS, there are several different types of links: coaxial cabling, fiber optic cabling, and atmospheric transmissions.

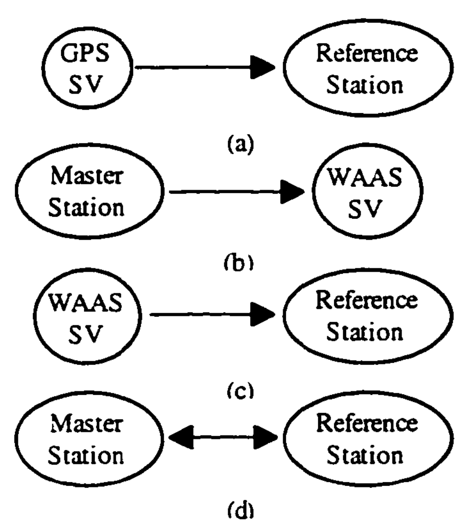


Figure 4.2 Components of Graph $G_{.,p}$

Since we are using the comparison model, our graph must have bi-directional edges between all nodes. In Figure 4.2a, the GPS SV is monitored by the Reference Station. The Reference Station is connected to the Master Station (shown in Figure 4.2d), and the Master Station can determine the status of the GPS SV. Therefore, we can model this as a bi-directional edge in the test graph $G_{t,p}'$ to connect the GPS SV nodes and the Reference Station nodes (in Figure 4.2, the direction of the edges are consistent with the graph $G_{t,p}$, not the test graph $G_{t,p}'$).

In Figure 4.2b, the Master Station sends information to the WAAS SV, the WAAS SV sends information to the Reference Station (shown in Figure 4.2c), which then sends information back to the Master Station (Figure 4.2d). Therefore, we can model this as a bi-directional edge in the test graph $G_{t,p}'$ to connect the Master Station node with the WAAS SV node. Similarly the WAAS SV node and the Reference Station node can share a bi-directional edge in the test graph $G_{t,p}'$. In Figure 4.2d, the Master Station and the Reference Station are already connected by a bi-directional edge since they can send information back and forth.

4.3. Diagnosability

Figure 4.3 shows the test graph $G_{t,p}'$ for the node pairs in $G_{t,p}$. We are now able to use the comparison model in the test graph $G_{t,p}'$, since all nodes are joined by bi-directional edges.

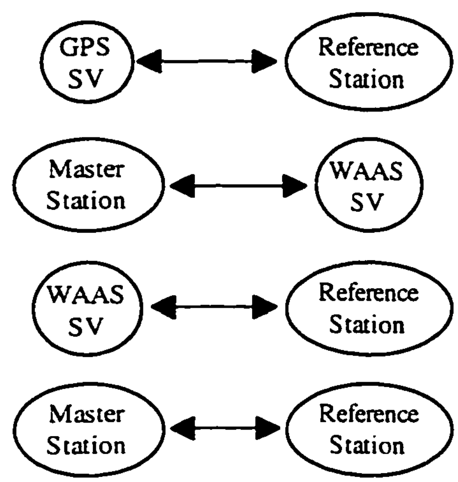


Figure 4.3 Components of Test Graph $G_{t,p}'$

We can simplify the test graph $G_{t,p}'$ by removing the Master Station node since it is inherently redundant and assumed to be fault-free. We combine the WAAS SV node in Figure 4.3 with the Master Station node, and refer to this combined node as the WAAS node. At this point, we introduce the aircraft node in the test graph. The resulting components of the test graph $G_{t,p}'$ are shown in Figure 4.4.

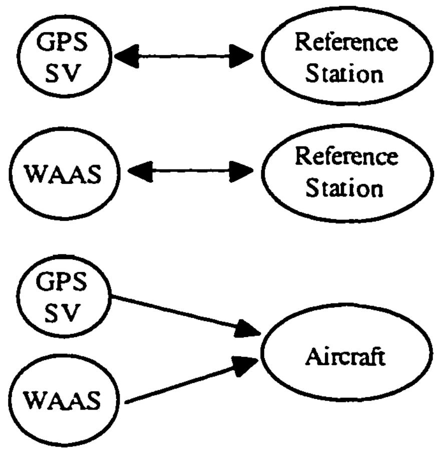
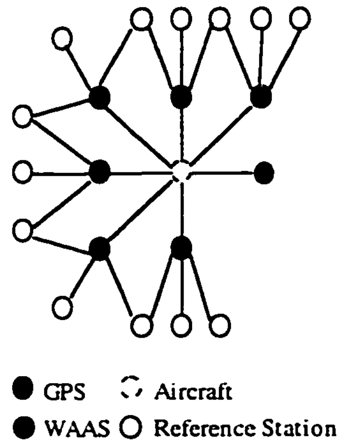


Figure 4.4 Components of the Modified Test Graph $G_{t,p}'$

To simplify the model of the GNSS, we treat each node in the GNSS as identical nodes in the test graph $G_{t,p}'$. Next we need to examine the test graph $G_{t,p}'$. We know from [SAM96] that there are at least six GPS SV nodes and from [POO95] there is at least one WAAS node, $\forall t,p$. Also from [POO95], there are at least three Reference Station nodes for each GPS SV node, with some of those being shared. Thus, we have the graph in Figure 4.5. (We connect these reference stations in a "regular" manner to simply the generation of $G_{t,p}'$. The actual system connections will undoubtedly vary from this, but the same approach can be used.) We have omitted the connections between the WAAS node and the Reference Station nodes to simplify the drawing; however, these connections exist in the analysis.



Note: connections between the WAAS node and the Reference Station nodes omitted for clarity

Figure 4.5 Worst Case $G_{t,p}'$

Before we proceed with the diagnosability of the GNSS, we review the terms and notation used in the t-in-L_2 diagnosability criteria. Let the nodes in L(x) be identified as the set $\{z_1, z_2, ..., z_{deg(x)}\}$ and the nodes in $L(z_i)$ be identified as the set $\{x, z_{i1}, z_{i2}, ..., z_{i(deg(x)-1)}\}$, where i=1, 2, ..., deg(x). We call the set $L(x)\cup\{x\}$, the *local set of x*, and the set $L(L(x))\cup L(x)\cup\{x\}$, the *extended-local set of x*.

Let $\overline{N}$ be the set of nodes in L(x) that share a 0-link with x and N be the set of nodes in L(x) that share a 1-link with x. Similarly, $\overline{N}(\overline{N})$ is the set of nodes that share a 0-link with the nodes in $\overline{N}$ (not including the nodes in $\overline{N}$), $\overline{N}(N)$ is the set of nodes that share a 0-link with the nodes in N (not including the nodes in N), $N(\overline{N})$ is the set of nodes that share a 1-link with the nodes $\overline{N}$ (not including the nodes in $\overline{N}$), and N(N) is the set of nodes that share a 1-link with the nodes in N (not including the nodes in N).

Let N_i be the node $z_i \in N$ that shares a 1-link with x and $N(N_i)$ be the set of nodes that share a 1-link with z_i, not including the node z_i. Similarly $\overline{N}(N_i)$ is the set of nodes that share a 0-link with z_i.

The local fault constraint t is the number of faulty nodes in the local set of x that are allowed. The extended-local fault constraint t(m) is the number of faulty nodes in the extended-local set of x that are allowed. Also, in the figures that follow, the faulty nodes are depicted by dark circles, fault-free nodes by light circles, and nodes labeled using the notation z_{ij} are undetermined.

The study of the GNSS provides an assessment of the t-in-L_2 diagnosability criteria for systems that are not regular. Furthermore, the GNSS has a node (the WAAS node) that is critical to the operation of the system. If this node fails, then the system cannot function as it was intended to. So, we show how this affects the diagnosability of the GNSS

Lemma 4.1

Given the GNSS test graph $G_{t,p}'$ and syndrome S, let F be an allowable fault set for S such that $F \neq U$. With a local fault constraint (lfc) of no more than $t_1 = deg(x) - 1$ faults in $L(x) \cup \{x\}$, where x = aircraft node, and no more than $t_2 = deg(z_i)-1$ faults in $L(z_i) \cup \{z_i\}$, where $z_i \in L(x)$, let F be the set of faulty nodes in $G_{t,p}'$, then

1. If $|N| \leq deg(x) - t_1 + 1$ and $\exists z_i$, such that $|N(N_i)| \geq t_2$, then $x \in U - F$.

2. If $\deg(x) - t_1 + 2 \leq |N| \leq t_1$ and $|N(N_i)| \geq t_2$, for i = 1 to $[|N| - (\deg(x) - t_1)]$, then $x \in U - F$.

3. If $\exists z_i$, such that $|N(N_i)| < [\deg(z_i) - t_2]$, then $x \in F$.

Proof:

Condition 1: $|N| \leq \deg(x) - t_1 + 1$ and $\exists z_i$, such that $|N(N_i)| \geq t_2$.

Suppose $x \notin U - F$ (i.e., $x \in F$). Then, clearly $z_i \in F$ by Lemma 3.5. We know the nodes in $\overline{N}$ are faulty, since x is faulty, and $|\overline{N}| = \deg(x) - |N|$. So there are at least

$$1 + 1 + (\deg(x) - |N|) = 2 + \deg(x) - (\deg(x) - t_1 + 1) = t_1 + 1$$

faulty nodes in the local set of x, a contradiction. So $x \in U - F$. (This is illustrated in Figure 4.6 for $\deg(x) = 7$, $\deg(z_i) = 4$ for i = 1 to 6, $\deg(z_7) = 13$, $t_1 = \deg(x) - 1$, $t_2 = \deg(z_i) - 1$, and $|N| = t_1$).

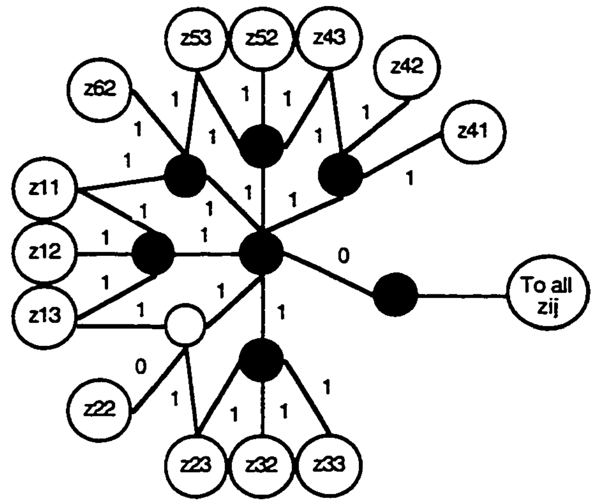


Figure 4.6 $G_{t,p}'$ **with** $|N|=t_1$, $|N(N_1)|=t_2$, $|N(N_i)|=deg(z_i)-1$

Condition 2: $deg(x) - t_1 + 2 \leq |N| \leq t_1$ and $|N(N_i)| \geq t_2$, for i = 1 to $[|N|-(deg(x) - t_1)]$.

Suppose $x \notin U - F$ (i.e., $x \in F$). Then, clearly $z_i \in F$ by Lemma 3.5, for i = 1 to $[|N| - (deg(x) - t_1)]$. The nodes in $\overline{N}$ are faulty, since x is faulty, and $|\overline{N}| = deg(x) - |N|$. So there are at least

$$1 + [|N| - (deg(x) - t_1)] + (deg(x) - |N|) = t_1 + 1$$

faulty nodes in the local set of x, a contradiction. So $x \in U - F$.

Condition 3: $\exists z_i$, $|N(N_i)| < [deg(z_i) - t_2]$.

Suppose $x \notin F$ (i.e., $x \in U - F$). So x and z_i share a 1-link. The number of 0-links incident on z_i is at least

$$(deg(z_i) - 1) - (deg(z_i) - t_2 - 1) = t_2.$$

We know $z_i \in$ F since $x \in U$ - F, and so the nodes that share a 0-link with z_i are also faulty. But then at least $t_2 + 1$ nodes in the local set of z_i are faulty, a contradiction. So $x \in$ F. (This is shown in Figure 4.7, with $t_2 = \deg(z_i) - 1$.) ❑

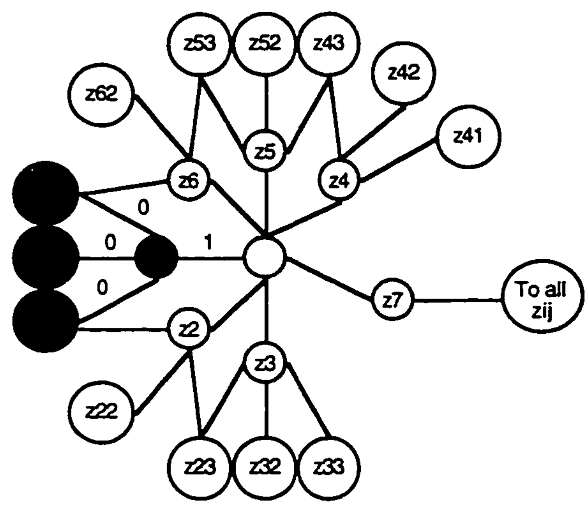


Figure 4.7 $G_{t,p}'$ **with** $|N(N_1)| < \deg(z_1) - t_2$, $t_2 = \deg(z_1) - 1$

Lemma 4.2

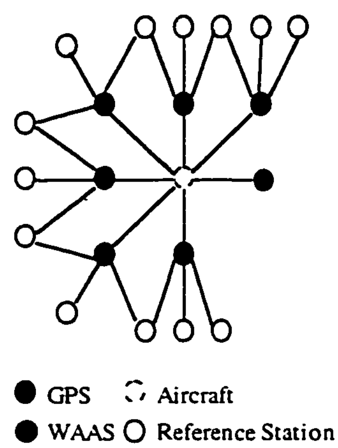
Given the GNSS test graph $G_{t,p}'$ and syndrome S, let F be an allowable fault set for S such that F $\neq U$. Let C (or C′) be the set of nodes that have multiple paths of length 2 with x (the aircraft node) and are counted more than once in the computation of F(x) (or F′(x)). Then, the number of faulty nodes in the extended-local set of x is:

1. $|F(x)| \geq |N| + |\overline{N}(N)| + |N(\overline{N})| - |C|$, if $x \in U$ - F

2. $|F'(x)| \geq 1 + |\overline{N}| + |\overline{N}(\overline{N})| + \sum_{i=1}^{|N|} \min\left[|N(N_i)|, \deg(x) - |N(N_i)|\right]$ - $|C'|$, if $x \in F$.

Proof:

Figure 4.8 shows the extended-local set of x for the GNSS test graph $G_{t,p}'$.



Note: connections between the WAAS node and the Reference Station nodes omitted for clarity

Figure 4.8 GNSS Subgraph $G_{t,p}'$

Condition 1: $|F(x)| \geq |N| + |\overline{N}(N)| + |N(\overline{N})| - |C|$

The derivation of $|F(x)|$ follows from the definition of syndrome decoding using the comparison model. So,

$|F(x)| \geq |N| + |\overline{N}(N)| + |N(\overline{N})|$.

But, there are nodes that share two paths of length 2 with x, and therefore may be counted twice in the computation of F(x), since $|F(x)|$ is the sum of the $\overline{N}(N)$ and $N(\overline{N})$ paths. So, $z_{ij} \in C$ if both paths that join x and z_{ij} are in the computation of F(x). Therefore we must subtract $|C|$ from F(x) so that z_{ij} is not counted twice. So we have

$$|F(x)| \geq |N| + |\overline{N}(N)| + |N(\overline{N})| - |C|.$$

Condition 2: $|F'(x)| \geq 1 + |\overline{N}| + |\overline{N}(\overline{N})| + \sum_{i=1}^{|N|} \min\left[|N(N_i)|, \deg(z_i) - |N(N_i)|\right] - |C'|$

In addition to $x \in F$, we know the nodes in $\overline{N}$ and $\overline{N}(\overline{N})$ are also faulty. This gives the term $1 + |\overline{N}| + |\overline{N}(\overline{N})|$. All the nodes that share a 1-link with x cannot be labeled as faulty or fault-free, since $x \in F$. However, if $z_i \in N$ is fault-free then all nodes that share a 1-link with z_i are faulty (this is the term $|N(N_i)|$ inside the 'min' function). If $z_i \in N$ is faulty then all nodes that share a 0-link with z_i are faulty (this is the term $[\deg(z_i) - |N(N_i)|]$ inside the 'min' function). Therefore, at least $\min[|N(N_i)|, \deg(z_i) - |N(N_i)|]$ nodes in $L(z_i)$ are faulty.

But, as with the F(x) derivation, there are nodes that have two paths of length 2 with x, and therefore may be counted twice in the computation of F'(x), since $|F'(x)|$ is the sum of the $\overline{N}(\overline{N})$ and $N(N_i)$ paths. So, $z_{ij} \in C'$ if both paths that join x and z_{ij} are in the computation of F'(x). Therefore we must subtract $|C'|$ from F'(x) so that z_{ij} is not counted twice. So we have

$$|F'(x)| \geq 1 + |\overline{N}| + |\overline{N}(\overline{N})| + \sum_{i=1}^{|N|} \min\left[|N(N_i)|, \deg(z_i) - |N(N_i)|\right] - |C'|. \square$$

Theorem 4.1

Given the GNSS test graph $G_{t,p}'$ and syndrome S, let F be an allowable fault set for S such that $F \neq U$. If there are no more than $t_1 = 6$ faults in $L(x) \cup \{x\}$, where x = aircraft node, no more than $t_2 = \deg(z)-1$ faults in $L(z) \cup \{z\}$, where $z \in L(x)$, and no more than t(m) = 8 faults in $L(L(x)) \cup L(x) \cup \{x\}$, then $G_{t,p}'$ is t-in-L_2 diagnosable.

> Note: In the GNSS, the WAAS node z_w is a critical element of the system. If it fails, then the system cannot perform to the standards required by aviation applications. Therefore, if we diagnose $z_w \in F$, we can stop our diagnosis of the rest of the graph. Taking this approach, then t(m) = 14 faults in $L(L(x)) \cup L(x) \cup \{x\}$ with $t_1 = 6$ and $t_2 = \deg(z)-1$. See Case 1.2.2 in the following proof.

<u>Proof:</u>

To prove the t-in-L_2 diagnosability of the GNSS test graph $G_{t,p}'$, we must show that if the local and extended-local fault constraints are not violated, then the fault set F is unique for any given syndrome. We show that $x \in F$ if and only if at least one of the following conditions are met:

1. $|N| > t_1$
2. $\exists z_i$, such that $|N(N_i)| < [\deg(z_i) - t_2]$
3. $|F(x)| > t(m)$, where $F(x) = N \cup \overline{N}(N) \cup N(\overline{N})$.

Let C denote the set of nodes that have multiple paths of length 2 with x and are counted twice in the computation of F(x). Similarly, let C′ denote the set of nodes that have multiple paths of length 2 with x and are counted twice in the computation of F′(x).

First, we show that if any one of the conditions is true, then $x \in F$. If condition 1 or 2 is true, then $x \in F$ by Lemma 3.5 or Lemma 4.1, respectively. If condition 3 is true, and $x \in U - F$, then there are more than t(m) faulty nodes in the extended-local set of x. But this is a contradiction since the elfc requires no more than t(m) faulty nodes in the extended-local set of x, so $x \in F$.

Next we show by contradiction that if none of the conditions is satisfied, then $x \notin F$. Since $G_{t,p}'$ is the GNSS, we know there are 8 nodes in $L(x) \cup \{x\}$, and there are 13 nodes in L(L(x)). We also know there are 31 edges in L(L(x)) but only 13 nodes, so there are 18 edges in L(L(x)) that are redundant. So the number of nodes in the extended-local set of x is

$$21 = 1 + |N| + |\overline{N}| + |\overline{N}(N)| + |N(\overline{N})| + |\overline{N}(\overline{N})| + |N(N)| - 18 \qquad (23)$$

Substituting $|F(x)|$ and $|F'(x)|$, from Lemma 4.2, into equation 23 yields

$$39 \leq |F(x)|+|C| + |F'(x)|+|C'| + |N(N)| - \sum_{i=1}^{|N|} \min\left[|N(N_i)|, \deg(z_i) - |N(N_i)|\right]. \quad (24)$$

Since condition 3 is not true, $|F(x)| \leq t(m)$. We also know that $|F'(x)| \leq t(m)$, otherwise the extended-local fault constraint is violated. Substituting $|F'(x)| \leq t(m)$ and $|F(x)| \leq t(m)$ into equation 24 yields

$$39 \leq t(m) + t(m) + |C| + |C'| + |N(N)| - \sum_{i=1}^{|N|} \min\left[|N(N_i)|, \deg(z_i) - |N(N_i)|\right].$$

Simplifying and rearranging this equation yields

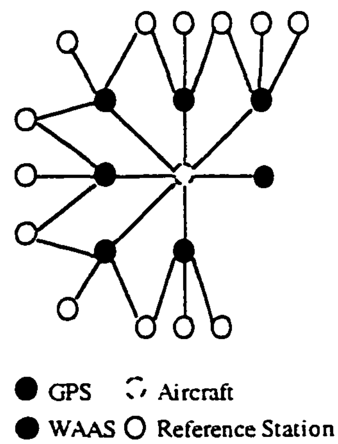
$$39 - 2 \bullet t(m) \leq |N(N)| - \sum_{i=1}^{|N|} \min\left[|N(N_i)|, \deg(z_i) - |N(N_i)|\right] + |C| + |C'|.$$

Rewriting $|N(N)|$ as $\sum_{i=1}^{|N|} |N(N_i)|$, we can rewrite this equation as

$$39 - 2 \bullet t(m) \leq \sum_{i=1}^{|N|} \left[|N(N_i)| - \min\left\{|N(N_i)|, \deg(z_i) - |N(N_i)|\right\}\right] + |C| + |C'|. \quad (25)$$

Now, $z_{ij} \in C$ only if z_{ij} is counted more than once in the $F(x)$ computation, and since z_{ij} is two away from x, then $z_{ij} \in N(\overline{N})$ or $z_{ij} \in \overline{N}(N)$, $\forall i,j$. Similarly, $z_{ij} \in C'$ only if z_{ij} is counted more than once in the $F'(x)$ computation, and since z_{ij} is two away from x, then $z_{ij} \in \overline{N}(\overline{N})$ or $z_{ij} \in \sum_{i=1}^{|N|} \min\left[|N(N_i)|, \deg(z_i) - |N(N_i)|\right], \forall i,j$.

From Figure 4.9 we know there are 8 nodes in $L(L(x))$ with two paths of length 2 with x and 5 nodes with three paths of length 2 with x.



Note: connections between the WAAS node and the Reference Station nodes omitted for clarity

Figure 4.9 Test Graph $G_{t,p}'$

Then

$$|C| + |C'| \le \frac{8}{13} \cdot \frac{|N(\overline{N})| + |\overline{N}(N)| + |\overline{N}(\overline{N})| + \sum_{i=1}^{|N|} \min[|N(N_i)|, \deg(z_i) - |N(N_i)|]}{2} +$$

$$\frac{5}{13} \cdot \frac{|N(\overline{N})| + |\overline{N}(N)| + |\overline{N}(\overline{N})| + \sum_{i=1}^{|N|} \min[|N(N_i)|, \deg(z_i) - |N(N_i)|]}{3}. \quad (26)$$

But, we know that $|\overline{N}(N)| + |N(\overline{N})| + |\overline{N}(\overline{N})| + |N(N)| = 31$ (for the GNSS). Substituting this into equation 26, and simplifying yields

$$|C| + |C'| \le \frac{17}{39}\left[31 - \sum_{i=1}^{|N|}\left[|N(N_i)| - \min\{|N(N_i)|, \deg(z_i) - |N(N_i)|\}\right]\right].$$

Substituting this into equation 25, and simplifying yields

$$t(m) \geq \frac{915}{39} - \frac{22}{39} \cdot \sum_{i=1}^{N} |N(N_i)|. \tag{27}$$

Consider two cases: $|N| \geq deg(x) - t_1 + 2$ and $|N| < deg(x) - t_1 + 2$. From Lemma 4.1, if $|N| \geq deg(x) - t_1 + 2$ then

$$|N(N_i)| = t_2 - 1, \text{ for } i = 1 \text{ to } [deg(x) - t_1 + 1] \text{ and}$$

$$|N(N_i)| = deg(z_i) - 1, \text{ for } i = [deg(x) - t_1 + 2] \text{ to } |N|. \tag{28}$$

If $|N| < deg(x) - t_1 + 2$ then from Lemma 4.1

$$|N(N_i)| = t_2 - 1, \text{ for } i = 1 \text{ to } |N|. \tag{29}$$

Each reference station node has $deg(z_i) = 4$, while the WAAS node, call it node z_w, has $deg(z_w) = 14$.

Case 1: $|N| \geq deg(x) - t_1 + 2$.

There are two subcases, since $G_{t,p}'$ is not a regular graph: the WAAS node shares a 1-link with x or a 0-link with x.

Case 1.1: $x - z_w$ share 0-link.

Equation 28 simplifies to

$$\sum_{i=1}^{|N|} |N(N_i)| = -t_1 t_2 + 8t_2 + 4t_1 + 3N - 32.$$

Substituting this into equation 27, letting $t_1 = deg(x) - 1 = 6$, $t_2 = deg(z_i) - 1 = 3$, and $|N| = t_1$ (from Lemma 3.5) yields

$$t(m) \geq 14.4$$

which is a contradiction since $t(m) = 8$.

Case 1.2: $x - z_w$ share 1-link.

There are two subcases: $|N(N_w)| = t_2 - 1 = deg(z_w) - 2$ and $|N(N_w)| = deg(z_w) - 1$.

Case 1.2.1: $|N(N_w)| = deg(z_w) - 2$.

From equation 28

$|N(N_i)| = deg(z_i) - 1$, for $i = [deg(x) - t_1 + 2]$ to $|N|$.

Since $|N(N_w)| = deg(z_w) - 2$, there are $|N| - (deg(x)-t_1+1)$ nodes in N that are faulty. Letting $t_1 = deg(x) - 1 = 6$ and $|N| = t_1$ (from Lemma 3.5) yields

$$\begin{aligned} F &= \{x, z_i \text{ (for } i = [deg(x) - t_1 + 2] \text{ to } |N|), |\overline{N}|\} \\ &= 1 + |N| - (deg(x) - t_1 + 1) + |\overline{N}| = t_1 = 6 \end{aligned}$$

faulty nodes in the local set of x. Since $deg(z_w) = 14$, $|N(N_w)| = 12$. Therefore $z_w \notin U - F$, otherwise the extended-local fault constraint is violated. So $z_w \in F$. But then there are $t_1 + 1$ faulty nodes in L(x), which is a contradiction.

Case 1.2.2: $|N(N_w)| = deg(z_w) - 1$.

Equation 28 simplifies to

$$\begin{aligned} \sum_{i=1}^{|N|} |N(N_i)| &= \sum_{i=1}^{|N-1|} |N(N_i)| + |N(N_w)| \\ &= -t_1t_2 + 8t_2 + 4t_1 + 3N - 22. \end{aligned}$$

Substituting this into equation 27, letting $t_1 = \deg(x) - 1 = 6$, $t_2 = \deg(z_i) - 1 = 3$, and $|N| = t_1$ (from Lemma 3.5) yields

$$t(m) \geq 8.8$$

which is a contradiction since $t(m) = 8$.

However, if the WAAS node, z_w, has all 1-links adjacent to it, as in this case, then without performing any additional diagnosis on the system we know $z_w \in F$. In the GNSS, if $z_w \in F$ then the system cannot perform its intended function, since this node is a critical element of the system. So in the GNSS, it is sufficient to diagnose $z_w \in F$.

If we consider this artifact of the GNSS in our diagnosis of the aircraft node, then the extended-local fault constraint for $G_{t,p}'$ is increased from $t(m) = 8$ to $t(m) = 14$. This is a significant improvement, which is due to the large degree of the WAAS node relative to the small degree of the reference station nodes.

Case 2: $|N| < \deg(x) - t_1 + 2$.

There are two subcases: the WAAS node shares a 1-link or a 0-link with x.

Case 2.1: x - z_w share 0-link.

Substituting equation 29 into equation 27, letting $t_1 = \deg(x) - 1 = 6$, $t_2 = \deg(z_i) - 1 = 3$, and $|N| = \deg(x) - t_1 + 1 = 2$ (from Lemma 3.5) yields

$t(m) \geq 21.2.$

which is a contradiction since $t(m) = 8$.

Case 2.2: $x - z_w$ share 1-link.

Substituting equation 29 into equation 27, yields

$$\sum_{i=1}^{|N|} |N(N_i)| = [deg(z_i) - 2][|N| - 1] + [deg(z_w) - 2] = (4 - 2)(|N| - 1) + 12.$$

Letting $t_1 = deg(x) - 1 = 6$, $t_2 = deg(z_i) - 1 = 3$, and $|N| = deg(x) - t_1 + 1 = 2$ (from Lemma 3.5) yields

$t(m) \geq 15.6.$

which is also a contradiction since $t(m) = 8$. ❑

4.4. Summary

In summary, the GNSS is the future of aircraft navigation throughout the world. Augmentation systems for GPS are being designed in the United States, Australia, Canada, Europe, Japan, Russia, and other countries. Therefore, there is an immediate need for an evaluation of their performance. To date, there have been very few studies on the performance of the entire system, only subsystem performance evaluations [ASH95][POO95]. This research evaluates the combined GNSS [SAM96], rather than evaluating the parts individually, using a deterministic approach for system-level diagnosis.

We have shown the GNSS, for the worst case scenario, can have up to 6 faults in the local set of x, and up to 8 faults in the extended-local set of x, and still accurately diagnose all nodes. Since there are only 7 nodes adjacent to x, a local fault constraint = 6 is a significant percentage of those nodes. If we consider other cases where there are more than 7 nodes adjacent to x, the t-in-L_2 diagnosis theory allows us to have deg(x) - 1 faulty nodes, so the benefit is not lost as the degree of x increases.

We have also shown that if we can diagnose the WAAS node, z_w, as faulty, then we can stop the diagnosis of the graph since this node is a critical element of the system. Without this node in the real GNSS system, the system would not function. Therefore, there is no reason to diagnose the system with respect to the aircraft node without a fault-free WAAS node. Using this fact, the new extended-local fault constraint for $G_{t,p}'$ is t(m) = 14.

5. Conclusions and Future Work

We have shown a new diagnosability criterion in this research, t-in-L_2 diagnosis. This theory is applied to structures that are sparse, with examples shown for the mesh, hypercube, and hexagon structures. We improved on existing algorithms in local diagnosis [DAS93, SOM92], while generalizing the applicable structures. We eliminated the constraints imposed by those local diagnosis approaches, while allowing more faults in the system. We have shown the framework for applying this theory to all types of structures, regular or not regular.

The development of the t-in-L_2 diagnosis theory originated with a linear programming model. We developed a set of constraint equations, based on the local fault constraint, and an objective function, and used a linear program to optimize our objective, which was to maximize the extended-local fault constraint. Once we optimized our constraint equations, based on the linear program, we developed the mathematical framework for the theory.

In the t-in-L_2 diagnosis theory, we require no more than $t \leq d - m$ faults in $L(x) \cup \{x\}$ and no more than $t(m) = \left\lfloor \frac{(2+m) \cdot d}{2} \right\rfloor + m^2$ faults in $L(L(x)) \cup L(x) \cup \{x\}$, $\forall x \in U$, for a regular graph with minimum cycle length of 5 (see Table 5.1). We noted that the local fault constraint for the t-in-L_1 diagnosis theory is the same as the local fault constraint for the t-in-L_2 diagnosis theory when $m = \frac{d}{2} - 1$, which is the lower bound on m. We know that $t \geq \left\lfloor \frac{d}{2} \right\rfloor + 1$ in the t-in-L_2 diagnosis theory, which corresponds to $m \leq \frac{d}{2} - 1$. This value for t is consistent with a majority voting scheme. In a majority voting scheme, $t \leq \left\lfloor \frac{d}{2} \right\rfloor$ so that the majority of the nodes in L(x) cannot be faulty, which shows that if $t \leq \left\lfloor \frac{d}{2} \right\rfloor$ then we do not need an extended-local fault constraint t(m).

m	lfc t=d - m	general graph t(m)	mesh t(m)	hypercube t(m)
1	d - 1	$\left\lfloor \frac{3 \cdot d}{2} \right\rfloor + 1$	6	n
2	d - 2	$\left\lfloor \frac{4 \cdot d}{2} \right\rfloor + 4$	N/A	$\frac{5 \cdot n}{4} + 2$
3	d - 3	$\left\lfloor \frac{5 \cdot d}{2} \right\rfloor + 9$	N/A	$\frac{6 \cdot n}{4} + 4$
.	.	.	.	.
.	.	.	.	.
m	d - m	$\left\lfloor \frac{(2+m) \cdot d}{2} \right\rfloor + m^2$	6, m = 1	$\frac{(3+m) \cdot n}{4} + \left\lfloor \frac{m^2}{2} \right\rfloor$
$m = \frac{d}{2} - 1$	$\frac{d}{2} + 1$	$\frac{d^2 - d + 1}{2}$	N/A	$\frac{n^2}{4}$

Table 5.1 t and t(m) for the general graph, mesh, and hypercube

The local fault constraint t can be as large as d - 1, thereby allowing for a very large number of faults in $L(x) \cup \{x\}$, $\forall x \in U$. This has not been allowed in any other deterministic approach. In the t-diagnosis approach, for example, there can be d - 1 faults in $L(x)\cup\{x\}$, for only one such $x\in U$, and for the t-in-L_1 approach there can be at most $\left\lfloor \frac{d}{2} \right\rfloor + 1$ faults in $L(x)\cup\{x\}$, $\forall x\in U$.

In Chapter 4 we applied the t-in-L_2 diagnosis theory to a system that is not a multiprocessor system, the Global Navigation Satellite System. Unlike traditional systems to which system-level diagnosis is applied, this system is not regular, the nodes in the system are not identical, and the edges of the graph can fail. However, we have successfully applied the theory to this system. This infers that system-level diagnosis may be applicable to many other "non-traditional" multiprocessor systems. Future investigations into other structures can be made using the t-in-L_2 diagnosis theory.

Future work will look at another application of interest: graphs that are not regular. An ASIC (Application Specific Integrated Circuit), for example, can be partitioned into "nodes" to allow the testing of areas within the device using system-level diagnosis. We insert compare logic between the nodes in the ASIC. The compare logic would have to accommodate the functional differences between nodes, since one node in the ASIC may be an ALU (Arithmetic Logic Unit) and another node may be memory. The partitioning of the design is also a design for testability issue, since the ASIC designer must allow for a mechanism to test the design. Therefore the partitioning of the design is required even if system-level diagnosis is not applied.

The same philosophy can be applied to PCBs (Printed Circuit Boards), partitioning the design, at the board level, into nodes to use system-level diagnosis. This modeling may prove to be simpler than that at the ASIC level, since the elements of an ASIC are generally very different from one another, increasing the complexity of the compare logic needed to implement system-level diagnosis.

Additional work on the GNSS model is needed. The use of "real" data from flight testing should be folded into the model, rather than modeling the underlying components. This would complicate the model, but provide a more accurate analysis of the system. Additional research is needed to evaluate the GNSS with additional elements, like GLONASS (the Russian Global Navigation Satellite System) and EGNOS (the European system). Using the approach shown here, the addition of other augmentations requires little more than adding nodes to the test graph.

Another area of future work is apply the t-in-L_2 diagnosis theory using a weighting scheme for the nodes or edges in the system. There are systems that have certain nodes or edges that are more critical, and thus may require special attention. One example of this is the WAAS node in the GNSS model. We have shown the GNSS will not function as required without this node. Therefore it could be weighted more than the other nodes in the system.

References

[ALL75] F. J. Allan, T. Kameda, S. Toida, "An Approach to the Diagnosability Analysis of a System," *IEEE Trans. on Computers*, vol. C-24, pp. 1040-1042, 1975.

[AND70] S.S. Anderson, *Graph Theory and Finite Combinatorics*, Markham Publishing, Chicago, 1970.

[ASH95] V. Ashkenazi, et al., "Design of a GNSS: Coverage, Accuracy and Integrity," *Proceedings of ION GPS-95*, pp. 463-472, 1995.

[BAR76] F. Barsi, Grandoni, P. Maestrini, "A Theory of Diagnosability of Digital Systems," *IEEE Trans. on Computers*, vol. C-25, pp. 585-593, 1976.

[BER82] J. C. Bermond, C. Delorme, G. Farhi, "Large Graphs with Degree and Diameter III," *Proceedings Coll. Cambridge* 1981. Ann. Discr. Math. 13 North-Holland, pp. 23-32, 1982

[BER86] J. C. Bermond, C. Delorme, J. J. Quisquater, "Strategies for Interconnection Networks: Some Methods from Graph Theory," *Journal of Parallel and Distributed Computing*, vol. 3, pp. 433-449, 1986.

[BER89] J. C. Bermond and C. Peyrat, de Bruijn and Kautz Networks: A Competitor for the Hypercube?," *Proceedings on a Conference on the Hypercube and Distributed Computers*, pp. 279-294, 1989.

[BIA90] R. Bianchini, Jr., K. Goodwin, and D. Nydick, "Practical Application and Implementation of Distributed System-Level Diagnosis Theory," *Proc. of 20th Fault Tolerant Computing Symposium*, pp. 332-339, 1990.

[BIA91] R. Bianchini and R. Buskens, "An Adaptive System-Level Diagnosis Algorithm and its Implementation," *Proc. of 21st Fault Tolerant Computing Symposium*, pp. 222-229, 1991.

[BLO88] D.M. Blough, "Fault Detection and Diagnosis in Multiprocessor Systems," *Ph.D. Thesis*, Johns Hopkins University, 1988.

[BLO89] D.M. Blough, G.F. Sullivan, G.M. Masson, "Fault Diagnosis for Sparsely Interconnected Multiprocessor Systems," *Proc. of 19th Fault Tolerant Computing Symposium*, pp. 62-69, 1989.

[BLO77] M. Blount, "Probabilistic Treatment of Diagnosis in Digital Systems," *Proc. of 7th Fault Tolerant Computing Symposium*, pp. 72-77, 1977.

[BON76] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Elsevier Science Publishing, New York, 1976.

[BRA77] S. P. Bradley, A. C. Hax, T. L. Magnanti, *Applied Mathematical Programming*, Addison-Wesley Publishing, Reading, MA, 1977.

[BRO80] A.E. Brouwer, A.M. Cohen, A. Neumaier, *Distance-Regular Graphs, A Series of Modern Surveys in Mathematics*, Springer-Verlag, 1980.

[CHW81] K. Y. Chwa and S. L. Hakimi, "Schemes for Fault-Tolerant Computing: A Comparison of Modularly Redundant and t-Diagnosable Systems," *Inform. Contr.*, vol. 49, pp. 212-238, 1981.

[CUL95] P. Cull and S. M. Larson, "The Möbius Cubes," *IEEE Trans. on Computers*, vol. 44, pp. 647-659, 1995.

[DAH84] A. T. Dahbura and G. M. Masson, "An O ($n^{2.5}$) Fault Identification Algorithm for Diagnosable Systems," *IEEE Trans. on Computers*, vol. C-33, pp. 486-492, June 1984.

[DAH87] A. T. Dahbura, K. K. Sabnani, and L. L. King, "The Comparison Approach to Multiprocessor Fault Diagnosis," *IEEE Trans. on Computers*, vol. 36, pp. 373-378, March 1987.

[DAS89] Anindya Das, "Diagnosability and Diagnosis of Sparsely Interconnected Multiprocessor Systems," *Ph. D. Thesis*, Concordia University, 1989.

[DAS93] A. Das, K. Thulasiraman, V. K. Agarwal, K. B. Lakshmanan, "Multiprocessor Fault Diagnosis Under Local Constraints," *IEEE Trans. on Computers*, vol. 42, pp. 984-988, Aug. 1993.

[DEC89] Angel L. DeCegama, *The Technology of Parallel Processing, Parallel Processing Architectures and VLSI Hardware*, Volume 1, Prentice-Hall, 1989.

[DEN86] P. J. Denning, "Parallel Computing and Its Evolution," *Comm. of the ACM*, pp. 1163-1167, Dec. 12, 1986.

[FRI75] A. D. Friedman, "A New Measure of Digital System Diagnosis," *Proc. of 5th Fault Tolerant Computing Symposium*, pp. 167-170, 1975.

[FUS89] D. Fussell and S. Rangarajan, "Probabilistic Diagnosis of Multiprocessor Systems with Arbitrary Connectivity," *Proc. of 19th Fault Tolerant Computing Symposium*, pp. 560-565, 1989.

[HAK74] S. L. Hakimi and A. Amin, "Characterization of the Connection Assignment of Diagnosable Systems," *IEEE Trans. on Computers*, vol. C-23, pp. 86-88, Jan.1974.

[HAR69] Frank Harary, *Graph Theory*, Addison-Wesley Publishing, Reading, MA, 1969.

[HOC81] R. W. Hockney and C. R. Jesshope, *Parallel Computers*, Hilger Publishing, 1981.

[HWA93] Kai Hwang, *Advanced Computer Architecture Parallelism, Scalability, Programmability*, McGraw Hill, 1993.

[KAM75] T. Kameda, S. Toida, F. J. Allan, "A Diagnosis Algorithm for Networks," *Inform. Contr.*, pp. 141-148, 1975.

[KOH78] T. Kohda, "On One-Step Diagnosable Systems Containing at most t-Faulty Units," *Systems, Computers, Controls*, vol. 9, pp. 30-37, 1978.

[KRI89] V. Krishnamoorthy, K. Thulasiraman, M. N. Swamy, "Minimum Order Graphs with Specific Diameter, Connectivity and Symmetry," *Networks*, vol. 19, pp. 25-46, 1989.

[KUH80] J. G. Kuhl and S. M. Reddy, "Distributed Fault Tolerance for Large Multiprocessor Systems," *Proc. 17th International Symposium on Computer Architecture*, pp. 32-30, 1980.

[LAK93] S. Lakshmivarahan, J. Jwo, S. K. Dhall, "Symmetry in Interconnection Networks Based on Cayley Graphs of Permutation Groups: A Survey," *Parallel Computing*, pp. 361-407, 1993.

[LAM97] C. Lamb, L. DeBrunner, A. Das, K. Thulasiraman, J. Fagan, R. Sexton, "Fault Tolerance of the Global Navigation Satellite System using System-Level Diagnosis," *Proc. of the 31st Asilomar Conference on Signals, Systems, and Computers*, to appear in Nov. 1997.

[MAE86] E. Maehle, K. Moritzen, and K. Wirl, "A Graph Model for Diagnosis and Reconfiguration and its Application to a Fault-Tolerant Multiprocessor System," *Proc. of 16th Fault Tolerant Computing Symposium*, pp. 292-297, 1986.

[MAH76] S. N. Maheshwari and S. L. Hakimi, "On Models for Diagnosable Systems and Probabilistic Fault Diagnosis," *IEEE Trans. on Computers*, vol. C-25, pp. 228-236, March 1976.

[MAL80] M. Malek, "A Comparison Connection Assignment for Diagnosis of Multiprocessor Systems," *Proc. 17th International Symposium on Computer Architecture*, pp. 31-35, 1980.

[MEY84] G. L. Meyer, "A Diagnosis Algorithm for the BGM System-Level Fault Model," *IEEE Trans. on Computers*, vol. C-33, pp. 756-758, Aug. 1984.

[NAR86] J. Narasimhan, K. Nakajima, "An Algorithm for Determining the Fault Diagnosability of a System," *IEEE Trans. on Computers*, vol. C-35, pp. 1004-1008, Nov. 1986.

[PIN90] C.C. Pinter, *A Book of Abstract Algebra*, Second Edition, McGraw-Hill Publishing, New York, 1990.

[POO95] W. Poor, T. Albertson, P. Yen, "A Wide Area Augmentation System (WAAS) Service Volume Model and its Use in Evaluating WAAS Architecture and Design Sensitivities," *Proceedings of ION GPS-95*, pp. 629-637, 1995.

[PRE67] F. P. Preparata, G. Matze, R. T. Chien, "On the Connection Assignment Problem of Diagnosable Systems," *IEEE Trans. Electr. Compt.*, Vol. EC-16, pp. 848-854, 1967.

[RAG89] V. Raghaven, "Diagnosability Issues in Multiprocessor Systems," *Ph. D. Thesis*, University of Minnesota, 1989.

[SAM89] M. R. Samatham and D. Pradhan, "The de Bruijn Multiprocessor Network: A Versatile Parallel Processing and Sorting Network for VLSI," *IEEE Trans. on Computers*, vol. C-38, pp. 567-581, April 1989.

[SAM96] M. Sams, et al., "Optimum Satellite Constellations for Civil Navigation Use," *Proceedings of ION GPS-96*, pp. 1151-1160, 1996.

[SCH87] E.R. Scheinerman, "Almost Sure Fault Tolerance in Random Graphs," *SIAM J. Comput*, vol. 16, pp. 1124-1134, 1987.

[SOM87] A. K. Somani, V. K. Agarwal, D. Avis, "A Generalized Theory for System Level Diagnosis," *IEEE Trans. on Computers*, vol. C-36, pp. 538-546, May 1987.

[SOM89-1] A. K. Somani, D. Avis, V. K. Agarwal, "On Complexity of Diagnosability and Diagnosis Problems in System-Level Diagnosis," *IEEE Trans. on Computers*, vol. C-38, pp. 195-201, Feb. 1989.

[SOM89-2] A. K. Somani and V. K. Agarwal, "Distributed Syndrome Decoding for Regular Interconnected Structures," *Proc. of 19th Fault Tolerant Computing Symposium*, pp. 70-77, June 1989.

[SOM92] A. K. Somani and V. K. Agarwal, "Distributed Diagnosis Algorithms for Regular Interconnected Structures," *IEEE Trans. on Computers*, vol. 41, pp. 899-906, July 1992.

[SUL84] G. F. Sullivan, "A Polynomial Time Algorithm for Fault Diagnosability," *Proc. 25th Annual Symposium on Foundations Computer Science*, pp. 148-156, 1984.

[SUL86] G. F. Sullivan, "The Complexity of System-Level Fault Diagnosis and Diagnosability," *Ph. D. Thesis*, Yale University, 1986.

[SUL88] G. F. Sullivan, "An $O(t^3+|E|)$ Fault Identification Algorithm for Diagnosable Systems," *IEEE Trans. on Computers*, vol. C-37, pp. 388-397, Apr. 1988.

[YAN86] C. Yang, G. M. Masson, R. Leonetti, "On Fault Isolation and Identification in t_1/t_1-Diagnosable Systems," *IEEE Trans. on Computers*, vol. C-35, pp. 639-643, July 1986.

Appendices

Appendix A Linear Programming Model

The development of the t-in-L_2 diagnosis theory originated with a linear programming model. First, we developed a set of constraint equations, based on the local fault constraint. Then, we derived an objective function (described later) and used a linear program to optimize our objective, which was to maximize the extended-local fault constraint.

We can compare the results of Theorem 3.3 for t-in-L_2 diagnosability with a linear programming model. In generating the set of constraint equations for this model, our objective is to find the range of values for the variables N, $\overline{N}$, $\overline{N}(\overline{N})$, $\overline{N}(N)$, $N(\overline{N})$, and N(N), such that the status of the node x is undetermined.

For example, if constraint 1 in Listing A.1.1 is violated, then $|N| > t$ or $|N| < d - t + 1$. If $|N| > t$ and $x \in \mathcal{U} - F$, then there are at least t + 1 1-links incident on node x, and therefore at least t + 1 faults in $L(x) \cup \{x\}$, which is a contradiction, so $x \in F$. Similarly, if $|N| < d - t + 1$ and $x \in F$, then there are at least t 0-links incident on x, and therefore at least t + 1 faults in $L(x) \cup \{x\}$, which is a contradiction, so $x \in \mathcal{U} - F$.

The remaining constraints are similarly derived. Lindo, a linear program problem solver, is the software program we used to solve the set of constraint equations. It uses the simplex method to solve the set of equations. Here is the set of constraint equations, along with the objective function.

1	$d - t + 1$	$\leq \lvert N \rvert$	$\leq t$	
2	$d - \lvert N \rvert$	$= \lvert \overline{N} \rvert$		
3	$d - t + 1$	$\leq \lvert N(\overline{N}_i) \rvert \leq t,$		$\forall i$
4	$(d - 1) - \lvert N(\overline{N}_i) \rvert$	$= \lvert \overline{N}(\overline{N}_i) \rvert$		
5	$d - t$	$\leq \lvert N(N_i) \rvert$	$\leq t - 1,$	for $i=1, \ldots, [d - t + 1]$
6	$d - t$	$\leq \lvert N(N_i) \rvert$	$\leq d - 1,$	for $i=[d - t + 2], \ldots, \lvert N \rvert$
7	$(d - 1) - \lvert N(N_i) \rvert$	$= \lvert \overline{N}(N_i) \rvert$		

Objective: Minimum of [Maximum of (8, 9)]

8 $$\lvert N \rvert + \lvert N(\overline{N}) \rvert + \sum_{i=1}^{\lvert N \rvert} \left| \overline{N}(N_i) \right|$$

9 $$1 + \lvert \overline{N} \rvert + \lvert \overline{N}(\overline{N}) \rvert + \sum_{i=1}^{\lvert N \rvert} \min\left[\lvert N(N_i) \rvert, d - \lvert N(N_i) \rvert \right]$$

Listing A.1 Constraint Equations

These equations (also shown in listing 3.1) are entered into the Lindo program as shown in Listing A.2. We must substitute values for t and d when entering this into Lindo. In Listing A.2, Y is the objective function, and st stands for "subject to." The optimum value of Y coincides with the elfc, extended-local fault constraint t(m), derived in Theorem 3.3. Listing A.3 is a sample output from Lindo for $t = d - 1$ and $d = 6$. From this, we see the optimum value of $Y = 11$ is consistent with that derived in Theorem 3.3. Table A.1 summarizes the comparison between the optimal Y for the various simulations and the value obtained from Theorem 3.3.

The elfc t(m) is the upper bound on the number of faults that can be tolerated in $L(L(x)) \cup L(x) \cup \{x\}$, one more is not allowed. Therefore, the value for Y from the linear program is greater than or equal to t(m). This is the reason for the difference of one between Y and t(m) in Table A.1.

We varied t in Table A.1 from $\left\lfloor \frac{d}{2} \right\rfloor + 1 \leq t \leq d - 1$ because this is the range of interest. For example, if $t = d$ then all the nodes in L(x) could be faulty, and there would be no way to determine the status of the node x. If the $t < \left\lfloor \frac{d}{2} \right\rfloor + 1$ then a simple majority voting scheme could be used to determine the status of x. The interesting cases are when the $\left\lfloor \frac{d}{2} \right\rfloor + 1 \leq t \leq d - 1$.

MIN Y
st
N < t
N + $\overline{N}$ = d
N$\overline{N}$ - (d-t+1)$\overline{N}$ > 0
N$\overline{N}$ - (t)$\overline{N}$ < 0
N$\overline{N}$ - (d-1)$\overline{N}$ + $\overline{N}\,\overline{N}$ = 0
NN-1 > (d-t)
NN-2 > (d-t)
.
.
NN-t > (d-t)
NN-1 < (t-1)
NN-2 < (t-1)
.
.
NN-[d - t + 1] < (t-1)
NN-[d - t + 2] < (d-1)
.
.
NN-t < (d-1)
NN-1 - MIN-1 > 0
NN-1 + MIN-1 > d
NN-2 - MIN-2 > 0
NN-2 + MIN-2 > d
.
.
NN-t - MIN-t > 0
NN-t + MIN-t > d
NN-1 + $\overline{N}$N-1 = (d-1)
NN-2 + $\overline{N}$N-2 = (d-1)
.
.
NN-t + $\overline{N}$N-t = (d-1)
N + N$\overline{N}$ + $\overline{N}$N-1 + $\overline{N}$N-2 + ... + $\overline{N}$N-t - Y < 0
MIN-1 + MIN-2 + ... + MIN-t + $\overline{N}$ + $\overline{N}\,\overline{N}$ - Y < -1
END

Listing A.2 Input File to Lindo

VARIABLE	VALUE	REDUCED COST
Y	11.000000	0.000000
N	5.000000	0.000000
$\bar{N}$	1.000000	0.000000
$N\bar{N}$	3.000000	0.000000
$\bar{N}\bar{N}$	2.000000	1.000000
NN1	4.000000	0.000000
NN2	4.000000	0.000000
NN3	5.000000	0.000000
NN4	5.000000	0.000000
NN5	5.000000	0.000000
MIN1	2.000000	0.000000
MIN2	2.000000	0.000000
MIN3	1.000000	0.000000
MIN4	1.000000	0.000000
MIN5	1.000000	0.000000
$\bar{N}$N1	1.000000	0.000000
$\bar{N}$N2	1.000000	0.000000
$\bar{N}$N3	0.000000	1.000000
$\bar{N}$N4	0.000000	1.000000
$\bar{N}$N5	0.000000	1.000000

ROW	SLACK/SURPLUS	DUAL PRICES
2)	0.000000	1.000000
3)	0.000000	-1.000000
4)	1.000000	0.000000
5)	2.000000	0.000000
6)	0.000000	0.000000
7)	3.000000	0.000000
8)	0.000000	1.000000
9)	3.000000	0.000000
10)	0.000000	1.000000
11)	4.000000	0.000000
12)	0.000000	0.000000
13)	4.000000	0.000000
14)	0.000000	0.000000
15)	4.000000	0.000000
16)	0.000000	0.000000
17)	2.000000	0.000000
18)	0.000000	-1.000000
19)	2.000000	0.000000
20)	0.000000	-1.000000

ROW	SLACK/SURPLUS	DUAL PRICES
21)	4.000000	0.000000
22)	0.000000	-1.000000
23)	4.000000	0.000000
24)	0.000000	-1.000000
25)	4.000000	0.000000
26)	0.000000	-1.000000
27)	0.000000	0.000000
28)	0.000000	0.000000
29)	0.000000	1.000000
30)	0.000000	1.000000
31)	0.000000	1.000000
32)	1.000000	0.000000
33)	0.000000	1.000000

```
NO. ITERATIONS  =     20
BRANCHES        =     1 DETERM.      = 1.000E0
BOUND ON OPTIMUM:     11.00000
DELETE NONO AT LEVEL 1
ENUMERATION COMPLETE. BRANCHES  = 1 PIVOTS     = 20
LAST INTEGER SOLUTION IS THE BEST FOUND
RE-INSTALLING BEST SOLUTION...
```

Listing A.3 Sample Output from Lindo for d = 6, t = d - 1

lfc t	degree d		Y from Lindo	elfc t(m) from Theorem 3.3
t = d - 1	d =4		8	7
	d =5		9	8
	d =6		11	10
t = d - 2	d =5		15	14
	d =6		17	16
	d =7		19	18
t = d - 3	d =8		30	29
	d =9		32	31
	d =10		35	34
t = d - 4	d =10		47	46
	d =11		50	49
	d =12		53	52
. .	. .		. .	. .
$t = \left\lfloor \frac{d}{2} \right\rfloor + 1$	d =4		8	7
	d =5		15	14
	d =6		17	16
	d =7		28	27

*Note: the difference between t(m) and Y is due to the fact that t(m) is the upper bound on the number of faults, and Y is the smallest value that violates the upper bound. So t(m) = Y - 1.

Table A.1 Verification of elfc using Linear Model, Y

Appendix B Index of Lemmas and Theorems

Appendix C Glossary of Terms

Principle of Mathematical Induction:

1 A_1 is true.

2 If A_c is true then A_{c+1} is also true, for any positive integer c.

3 If statements 1 and 2 are true, then A_n is true for any positive integer n.

Adjacency:

Two nodes, $x_i, x_j, \in E$, are *adjacent* if $d(x_i, x_j) = 1$, where $x_i, x_j \in U$.

Allowable Fault Set:

An *allowable fault set*, F, is a permissible fault set that is consistent with the given syndrome, that is, the nodes in F are faulty, the nodes in U - F are fault-free.

Closed Ring:

A *closed ring* or *cutset*, R, in a graph G is a set of nodes in G whose removal disconnects the graph, i.e., there are at least two components in G-R.

Connected:

A graph is *connected* if all pairs of nodes are joined by a path.

Cycle:

A *cycle* in a graph is a path that begins and ends at the same node, i.e., $x_0 = x_n$.

Cycle Length:

The *cycle length* is the distance between x_0 and x_n, namely $d(x_0, x_n) = n$.

Degree:

The *degree* of a node x_i, $deg(x_i)$ or d, in a graph is the number of lines incident on x_i.

Diagnosable:

Given a syndrome S and two distinct allowable fault sets F_i and F_j, a graph G is *diagnosable* iff $F_i = F_j \ \forall i,j$, with $i \neq j$.

Directed or Undirected Graph:

A graph can have interconnects that are unidirectional (*directed graph*) or bi-directional (*undirected graph*).

Distance:

The *distance* between two nodes in a graph, $d(x_i, x_j)$, is the length of the shortest path between x_i and x_j in the graph.

Fault Set:

A *fault set*, F, is a set of nodes in G, all of which are faulty.

Geodesic:

A shortest x_i - x_j path is called a *geodesic*.

Graph:

A *graph* is a set of *nodes* joined by a set of *edges*. We call this graph $G=(U, E)$, where U is the set of nodes in G and E is the set of edges of G.

Minimum or Proper Cutset:

A *minimum cutset* or *proper cutset* is a set of nodes whose removal reduces G to a graph with exactly two components, i.e., $G-R$ has exactly two components. If $x_i \in R$ then the graph $G-[R-\{x_i\}]$, $\forall x_i \in R$, is connected if R is a proper cutset.

Not Diagnosable:

There may be multiple allowable fault sets, $S(F_i)$, that are consistent with a given syndrome S, where $F_i \oplus F_j \neq \varnothing$. If $z \in F_i \oplus F_j$, then z cannot be labeled as faulty or fault-free and the graph is *not diagnosable*, since we cannot uniquely identify every node in the graph.

Path:

A *path* in a graph is a set of distinct nodes that are connected by distinct edges (if the nodes are distinct then necessarily the edges are distinct).

Permissible Fault Set:

A *permissible fault set* is one that does not violate the fault constraints.

Regular Graph:

A graph is called *regular* of degree b if all nodes have the same degree, i.e., if $\deg(x_i) = b$, $\forall x_i \in U$.

Syndrome:

A *syndrome* is the collection of test results between all adjacent processors in a system.

Syndrome Decoding:

Syndrome decoding is the interpretation of the syndrome that is consistent with the requirements of the fault constraints and with the given syndrome S.